# Average Chord Length

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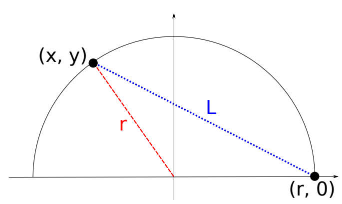
## Problem

Consider a circle of radius $r$: A chord length is defined by the distance between two points on the circumference of the circle. Imagine we chose two points randomly from a uniform and continuous distribution of positions along the perimeter, what is the expectation value of the chord length?

## Solution




First note that picking two points randomly will give the same average length as holding one point still and varying the other randomly. In polar coordinates we choose the point $\theta = 0$ corresponding to $(r, 0)$ as the starting point for all the chords. The endpoint, $(x, y)$, will then be defined by $(\cos\theta, \sin\theta)$ where $\theta$ is a uniform random number between zero and $2\pi$. Actually, due to the symmetry of the system we only need to examine the interval $[0, \pi)$.

We may calculate the length of the chord by taking the norm of the difference of the two points.

$$
\begin{aligned}
L &= \sqrt{(r-x)^2 + (0-y)^2} = r\sqrt{(1-\cos\theta)^2 + \sin^2\theta} \\
&= r\sqrt{2 - 2\cos\theta} = 2r\sqrt{\frac{1-\cos\theta}{2}} = 2r|\sin\frac{\theta}{2}|.
\end{aligned}
$$

Now we can get the expectation value by simply integrating our expression for $L$ over the domain of $\theta$ and dividing by its size.

$$
\begin{aligned}
\langle L \rangle &= \frac{1}{\pi}\int_0^\pi L d\theta = \frac{1}{\pi}\int_0^\pi 2r|\sin\frac{\theta}{2}|d\theta = \frac{2r}{\pi}\int_0^\pi \sin\frac{\theta}{2}d\theta = \\
&= \frac{4r}{\pi}\int_0^{\pi/2}\sin(u)du = \frac{4r}{\pi}[-\cos(u)]_0^{\pi/2} = \frac{4r}{\pi}
\end{aligned}
$$

So the average chord length on a circle of radius $r$ is is $\langle L \rangle = 4r/\pi$.