Journal of Physics A: Mathematical and General

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To cite this article: J -P Antoine and F Bagarello 1994 *J. Phys. A: Math. Gen.* **27** 2471

J. Phys. A: Math. Gen. 27 (1994) 2471–2481. Printed in the UK

Wavelet-like orthonormal bases for the lowest Landau level

J-P Antoine† and F Bagarello†‡§

Institut de Physique Théorique, Université Catholique de Louvain, B-1348 Louvain-la-Neuve, Belgium

Received 16 July 1993

Abstract. As a first step in the description of a two-dimensional electron gas in a magnetic field, such as encountered in the fractional quantum Hall effect, we discuss a general procedure for constructing an orthonormal basis for the lowest Landau level, starting from an arbitrary orthonormal basis in $L^2(\mathbb{R})$. We discuss in detail two relevant examples coming from wavelet analysis, the Haar and the Littlewood–Paley bases.

1. Introduction

The fractional quantum Hall effect (FQHE) has received much attention in recent years, essentially for its potential practical applications, but this strong interest has not been sufficient so far for producing a theory capable of explaining *all* the experimental data (see [1, 2] for a review and the original references).

The system to be considered is a (quasi)-planar gas of electrons in a strong magnetic field perpendicular to the plane. The first problem to tackle for discussing the static features of the FQHE is to find the ground state of the system, and this is already a very hard problem. Two main methods have been proposed in the literature to that effect. The first one is a Hartree–Fock approach to a system of N two-dimensional electrons (see for instance [3–5]). This picture gives good energy values for small or high electron densities. In the intermediate range, however, the best results are obtained with the Laughlin wavefunction [6], which is derived by a variational technique based on a non-mean-field approach to the same two-dimensional gas of electrons. We will consider here the first method only.

The first step is to select an adequate wavefunction for a single electron in the magnetic field. As is well known [2], the energy levels, the so-called Landau levels, are infinitely degenerate, and there arises the problem of finding a good basis in the corresponding Hilbert subspace. This is crucial for allowing an easy computation of the energy levels of the whole system, in the presence of perturbations. In particular, the ground state we are looking for belongs to the lowest Landau level (LLL). The aim of this paper is to discuss a general way of obtaining an orthogonal basis for the LLL.

It is a standard result [7] that the Hamiltonian of a single electron confined in the xy-plane and subjected to a strong magnetic field in the z-direction can be transformed into that of a harmonic oscillator. In the symmetric gauge we have

$$H_0 = \tfrac{1}{2}\left(p_x - \tfrac{1}{2}y\right)^2 + \tfrac{1}{2}\left(p_y + \tfrac{1}{2}x\right)^2 . \tag{1.1}$$

† E-mail address: Antoine@fyma.ucl.ac.be; bagarello@ipamat.cres.it

‡ Work supported by Fondazione Angelo Della Riccia.

§ Permanent address: Dipartimento di Matematica ed Applicazioni, Fac. Ingegneria, Universitá di Palermo, Viale delle Scienze, I-90128 Palermo, Italy.

Introducing the canonical variables

$$P' = p_x - \tfrac{1}{2}y \qquad Q' = p_y + \tfrac{1}{2}x \tag{1.2}$$

this can be written in the form

$$H_0 = \tfrac{1}{2}(Q'^2 + P'^2). \tag{1.3}$$

Here and in the following, we will use units such that $\hbar = M = e|\boldsymbol{H}|/c = 1$, which also implies that the cyclotron frequency $\omega_c = e|\boldsymbol{H}|/Mc$ and the magnetic length $a_0 = (\hbar c/e|\boldsymbol{H}|)^{1/2}$ are both equal to one.

The eigenstates of the Hamiltonian (1.1) can be found explicitly [8], and they have the following form:

$$\Phi_{mn}(x, y) = (2^{m+n+1}\pi m!n!)^{-1/2}\, e^{(x^2+y^2)/4}(\partial_x + i\partial_y)^m\, (\partial_x - i\partial_y)^n\, e^{-(x^2+y^2)/4} \tag{1.4}$$

corresponding to the eigenvalues

$$E_{mn} \equiv E_n = n + \tfrac{1}{2}. \tag{1.5}$$

Thus we see that the energy levels are all degenerate in m, so that the ground level (LLL) is spanned by the set $\{\Phi_{m0}(x, y)\}$, which forms an orthonormal basis in the LLL. For these wavefunctions, the mean value of the distance from the origin, $r \equiv \sqrt{x^2 + y^2}$, increases with m [7, 8], so that the functions $\Phi_{m0}(x, y)$ are not very well localized. Yet the physics of the problem requires that the wavefunctions be fairly well localized, in particular for approaching the limit of the celebrated Wigner crystal [5]. Thus arises the LLL basis problem that we now discuss.

2. The LLL basis problem

While the solutions (1.4) can be found very easily directly in the configuration space, it is not easy at all to find another basis, orthogonal or not, spanning the same energy level. An efficient and elegant method, based on a technique introduced in [9], has been discussed in some detail in [5] and [7], and we will use it here. The transformation (1.2) can be seen as a part of a canonical transformation from the variables x, y, p_x, p_y into the new ones Q, P, Q', P', where

$$P = p_y - \tfrac{1}{2}x \qquad Q = p_x + \tfrac{1}{2}y. \tag{2.1}$$

These operators satisfy the following commutation relations:

$$[Q, P] = [Q', P'] = i \qquad [Q, P'] = [Q', P] = [Q, Q'] = [P, P'] = 0. \tag{2.2}$$

It is shown in [7, 9] that a wavefunction in the (x, y)-space is related to its PP'-expression by the formula

$$\Psi(x, y) = \frac{e^{ixy/2}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(xP'+yP+PP')}\Psi(P, P')\, dP\, dP'. \tag{2.3}$$

The usefulness of the PP'-representation stems from the expression (1.3) of H_0. Indeed, in this representation, the Schrödinger equation admits eigenvectors $\Psi(P, P')$ of H_0 of the form $\Psi(P, P') = f(P')h(P)$. Thus the ground-state wavefunction of (1.3) must have the form $f_0(P')h(P)$, where

$$f_0(P') = \pi^{-1/4}\, e^{-P'^2/2} \tag{2.4}$$

and the function $h(P)$ is arbitrary, which manifests the degeneracy of the LLL.

Choosing $h(P) = f_0(P)$, the authors of [5] show, using (2.3), that the corresponding wavefunction in the (x, y)-space is nothing but $\Phi_{00}(x, y)$ as defined in (1.4). They also construct a complete set of functions of the LLL with Gaussian localization, centred on the sites of a regular two-dimensional lattice. However, this basis is not orthogonal, and, in addition, each vector has a well defined, fixed (essential) support, so that there is no possibility of modifying the mutual overlap for fixed electron density. This complete set of functions of the LLL is obtained simply by acting on $\Phi_{00}(x, y)$ itself with the translation operators $T(a_i)$ defined by

$$T(a_i) \equiv \exp(\mathrm{i}\Pi_c \cdot a_i) \qquad i = 1, 2 \tag{2.5}$$

where $\Pi_c \equiv (Q, P)$ and a_i are the lattice basis vectors. The set obtained in this way is still in the LLL since the operators $T(a_i)$ commute with H_0, by virtue of the commutation relations (2.2). Moreover, $[T(a_1), T(a_2)] = 0$ if the area of the cell of the lattice is such that $a_{1_x}a_{2_y} - a_{1_y}a_{2_x} = 2\pi$. Completeness of the set is proven by showing its unitary equivalence with the set of coherent states constructed as eigenstates of the operator $A \equiv 2^{-1/2}(Q + \mathrm{i}P)$ with minimal lattice cell area, π, and by using the well known completeness of coherent states [10] (this equivalence, which is already clear on the expression (1.4), was first discussed by Boon [11]). Orthogonality, however, has to be enforced, since coherent states are in general not mutually orthogonal, and this spoils much of the simplicity of the basis functions, and in particular the localization properties for intermediate fillings.

Another approach, whose aim is to preserve the latter, and some sort of translation invariance, is due to Ferrari [12], who has constructed an orthonormal basis for the LLL by taking infinite superpositions of the above (coherent) states. The resulting basis vectors are Bloch functions, which may be made translation invariant over the nodes of a given lattice, typically triangular or hexagonal (remember that the Wigner crystal is a triangular lattice). Clearly this basis describes very well the two-dimensional low-density system of electrons of the FQHE, but its construction is rather involved and *ad hoc*.

In the sequel of this paper, we will discuss other choices for the function $h(P)$, leading to very different orthonormal bases for the LLL. The key observation is that one wants basis wavefunctions which are both well localized *and* orthogonal. Then obvious candidates are orthogonal wavelets, as discussed at length in [13]. Not only do they enjoy good localization properties, but the latter are easily controlled by varying the scale parameter, in contrast to the Gaussian-like functions of [5]. In addition, wavelets seem well adapted to a physical problem which has an intrinsic hierarchical structure [14, 15]. In particular, the relevant parameter, namely the filling factor, may take arbitrary rational values [16], and this suggests some sort of fractal behavior, which again points to wavelets.

More precisely, we will construct bases for the lowest Landau level, via the transformation (2.3), starting from a general (orthogonal) basis in $L^2(\mathbb{R})$, and then particularize to the case where that basis is taken to be an orthonormal (ON) basis of wavelets. Finally we will discuss in some detail the LLL bases corresponding to two standard wavelet bases, namely the Haar and the Littlewood–Paley bases [13]. For the convenience of the reader, we sketch in the appendix the essential aspects of the wavelet transform, and in particular the construction of an ON wavelet basis from a multiresolution analysis.

We begin by briefly discussing some general ideas about the construction of an ON basis in the LLL, taking spline functions as an example, without going into details. A detailed discussion of the Haar and the Littlewood–Paley bases will be carried out in the next section.

In the PP'-representation, restriction to the LLL forces the dependence on P' of the wavefunction to be that of (2.4), so that the Gaussian integration on P' in (2.3) can be performed exactly. Then, starting from a wavefunction $\Psi_n(P, P') = f_0(P')h_n(P)$, where

$\{h_n(P)\}$ is an arbitrary ON basis in $L^2(\mathbb{R})$, we define a new set of functions by

$$h_n^{(2)}(x, y) = \frac{e^{ixy/2}}{\sqrt{2}\pi^{3/4}} \int_{-\infty}^{\infty} e^{iyP} e^{-(x+P)^2/2} h_n(P)\,dP\,. \tag{2.6}$$

Then the set $\{h_n^{(2)}(x, y)\}$ is a basis for the LLL since any wavefunction of the LLL can be written in the PP'-representation as the tensor product $f_0(P')h(P)$, and the set $\{h_n(P)\}$ is a basis in $L^2(\mathbb{R})$. Orthogonality of the wavefunctions $h_n^{(2)}(x, y)$ follows from the canonicity of the change of variables given in (1.2), (2.1) or simply by an explicit calculation of the matrix element $\langle h_n|h_m\rangle$, using the integral (2.6).

We can conclude, therefore, that using the transformation (2.6), any (ON) basis in $L^2(\mathbb{R})$ can be transformed into an (ON) basis for the LLL.

Remark. The PP'-representation can be used in the same way to construct an (ON) basis of any Landau level, not necessarily the lowest one. This is achieved simply by replacing the Gaussian in (2.4) by the correct wavefunction for $f_0(P')$. Of course one must face the increasing difficulty of computing the integrals in (2.3).

Example: linear spline. The linear spline function $\theta(x)$ is defined in the following way:

$$\theta(x) = \begin{cases} 1 - |x| & -1 \leqslant x \leqslant 1 \\ 0 & \text{otherwise}. \end{cases} \tag{2.7}$$

As is well known [13], this function generates an orthonormal wavelet basis in $L^2(\mathbb{R})$, using the technique of multiresolution analysis, which is summarized in the appendix. First one constructs a function $\phi(x)$ whose integer tranlates $\{\phi(x-n)\}$ are orthogonal for different integer values of n. Then one derives from ϕ a function $\psi(x)$ which is the mother wavelet. In the present case one gets for the Fourier transform of $\psi(x)$:

$$\widehat{\psi}(\omega) = \sqrt{\frac{3}{2\pi}}\, e^{i\omega/2} \sin^2\frac{\omega}{4} \sqrt{\frac{1+2\sin^2(\omega/4)}{(1+2\cos^2(\omega/2))(1+2\cos^2\omega/4)}} \left(\frac{\sin\omega/4}{\omega/4}\right)^2 . \tag{2.8}$$

Then the set $\{\psi_{m,n}(x)\} \equiv \{2^{-m/2}\psi(2^{-m}x - n), m, n \in \mathbb{Z}\}$ is an ON basis in $L^2(\mathbb{R})$ [13], so that, using (2.6), we could obtain an ON basis for the LLL. The same steps can be repeated, for instance, for a general spline function, discussed again in [13] or [19]. Here too, one may use multiresolution analysis in order to get an ON basis in $L^2(\mathbb{R})$, and then the integral (2.6) will transform it into an ON basis for the LLL.

3. The Haar basis

In this section we will discuss in some detail the LLL basis generated by the Haar basis of wavelets. We start by introducing the mother wavelet

$$h(x) = \begin{cases} 1 & \text{if } 0 \leqslant x < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} \leqslant x < 1 \\ 0 & \text{otherwise}. \end{cases} \tag{3.1}$$

This function gives an explicit example of a wavelet ON basis in $L^2(\mathbb{R})$, obtained via the usual formula

$$h_{mn}(x) = 2^{-m/2} h(2^{-m}x - n)\,. \tag{3.2}$$

Of course, since $h(x)$ is a discontinuous function, its localization in frequency space is poor. However, since the transformation (2.6) is not a Fourier transform, it is not clear *a priori* that the corresponding functions $\{h_{mn}^{(2)}(x, y)\}$ will also have a poor localization in both variables.

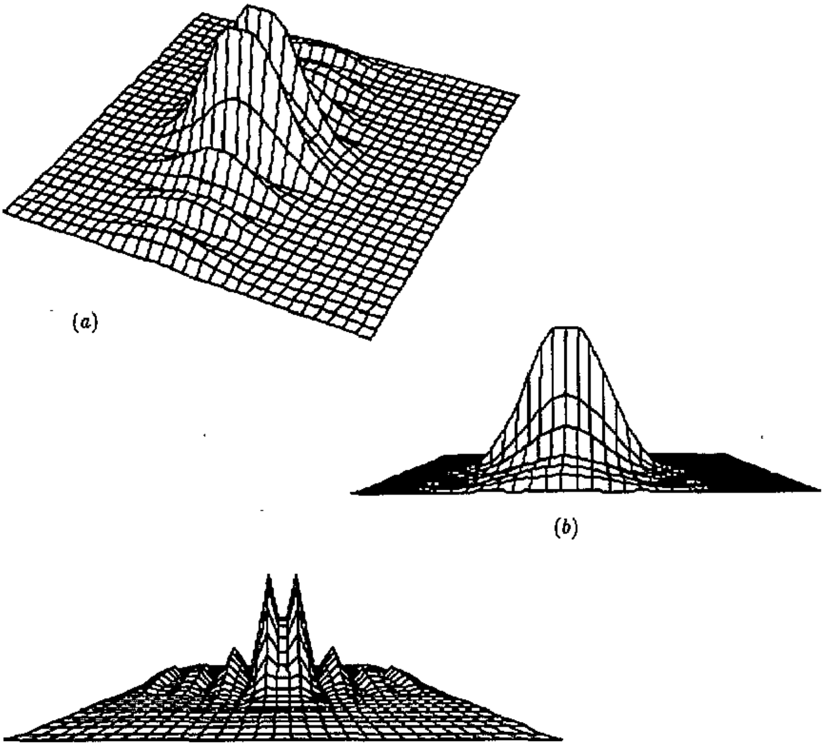

Figure 1. (*a*) Dependence of $|H_{00}(x, y)|$ on x and y. (*b*) Dependence of $|H_{00}(x, y)|$ on x. (*c*) Dependence of $|H_{00}(x, y)|$ on y.

In fact we will see below that it is *not* the case, by investigating the asymptotic behaviour of the basis functions.

From (2.6) we get

$$H_{mn}(x, y) = \frac{e^{ixy/2}}{\sqrt{2}\pi^{3/4}} 2^{-m/2} \left\{ \int_{2^m n}^{2^m(n+1/2)} e^{iyP} e^{-(x+P)^2/2} \, dP - \int_{2^m(n+1/2)}^{2^m(n+1)} e^{iyP} e^{-(x+P)^2/2} \, dP \right\}. \tag{3.3}$$

Using standard results on Gaussian integrals [17, 18], we find that

$$\begin{aligned} H_{mn}(x, y) = \frac{e^{-ixy/2} e^{-y^2/2}}{2\pi^{1/4}} 2^{-m/2} \{ & 2\Xi(x - iy + 2^m n + 2^{m-1}) \\ & - \Xi(x - iy + 2^m n) - \Xi(x - iy + 2^m n + 2^m)\} \end{aligned} \tag{3.4}$$

where $\Xi(z) = \Phi(z/\sqrt{2})$ and the error function $\Phi(z)$ is defined by the integral

$$\Phi(z) \equiv \operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \, dt \qquad z \in \mathbb{C}.$$

For $m = n = 0$, in particular, this gives

$$H_{00}(x, y) = \frac{e^{-ixy/2} e^{-y^2/2}}{2\pi^{1/4}} \left\{ 2\Xi\left(x - iy + \tfrac{1}{2}\right) - \Xi(x - iy) - \Xi(x - iy + 1) \right\}. \tag{3.5}$$

The modulus of $H_{00}(x, y)$ is plotted in figures 1(*a*), 1(*b*) and 1(*c*). The difference between the three figures is in the point of view of the observer. In particular, figure 1(*b*) shows the x-axis, while in figure 1(*c*) only the y-axis is shown. Therefore figure 1(*b*) yields the behaviour of $|H_{00}(x, y)|$ in x, while figure 1(*c*) gives the behaviour of the function in y. Clearly the function $H_{00}(x, y)$ is much better localized in the x variable than in y.

It is interesting to compare these graphical results with the asymptotic behaviour of the function H_{00}, which may be deduced from the asymptotic expansion of the error function given in [17, 18]:

$$\Phi(z) \simeq 1 - \frac{1}{\sqrt{\pi}} \frac{\mathrm{e}^{-z^2}}{z}\left(1 + \mathrm{O}\left(\frac{1}{z^2}\right)\right) \qquad z \to \infty \qquad |\arg z| < \frac{3\pi}{4}.$$

Thus we find the following asymptotic expansion for the function $H_{00}(x, y)$:

$$H_{00}(x, y) \simeq \frac{\mathrm{e}^{\mathrm{i}xy/2}\,\mathrm{e}^{-x^2/2}}{2\pi^{1/4}}\sqrt{\frac{2}{\pi}}\left(\frac{1}{x - \mathrm{i}y} + \frac{\mathrm{e}^{-1/2-x+\mathrm{i}y}}{x - \mathrm{i}y + 1} - 2\frac{\mathrm{e}^{-1/8-(x-\mathrm{i}y)/2}}{x - \mathrm{i}y + \frac{1}{2}}\right) \tag{3.6}$$

which displays the Gaussian localization of the wavefunction in the variable x and shows the rather poor localization in y.

An analogous behaviour can be obtained for the generic function $H_{mn}(x, y)$ given in (3.4), where n indexes the centre of the original mother wavelet and $m \in \mathbb{Z}$ is the scale parameter. Using (2.6), it is easily seen that the asymptotic behaviour of $h_n^{(2)}(x, y)$ in x is governed by the asymptotic behaviour of $h_n(P)$, and the one in y by that of the Fourier transform of $h_n(P)$. Since in the present case, $h_n(P)$ has compact support (increasing monotonically with m), we expect $H_{mn}(x, y)$ to be strongly localized in x and delocalized in y, and that its decay in x gets faster for smaller m. This is indeed the case, as may be seen by an explicit computation along the same lines as above. We omit the details since they do not add much to the previous computation.

4. The Littlewood–Paley basis

One can find in the literature [13] another simple example of an ON set of wavelets which forms a basis of $L^2(\mathbb{R})$ coming from multiresolution analysis. This is the Littlewood–Paley basis, generated from the mother wavelet

$$\Psi(x) = (\pi x)^{-1}(\sin 2\pi x - \sin \pi x). \tag{4.1}$$

The behaviour of this function is, in a sense, complementary to that of the Haar wavelet: it is very well localized in frequency space (it has a compact support)

$$\widehat{\Psi}(\omega) = \begin{cases} (2\pi)^{-1/2} & \text{if } \pi \leqslant |\omega| \leqslant 2\pi \\ 0 & \text{otherwise} \end{cases}$$

while, as one can see from (4.1), it decays like $1/x$ in configuration space.

We will see that an analogous complementary behaviour is also found for the wavefunctions in the LLL. We will show, in fact, that they are exponentially localized in the y-variable, while in the other variable they will behave like $1/x$.

In order to perform the integration in (2.6), it is convenient to use the Fourier transform of $\Psi_{mn}(x) = 2^{-m/2}\Psi(2^{-m}x - n)$. We have

$$\begin{aligned}\Psi_{mn}(x, y) &= \frac{\mathrm{e}^{\mathrm{i}xy/2}}{\sqrt{2}\pi^{3/4}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}yP}\,\mathrm{e}^{-(x+P)^2/2}\,\mathrm{d}P\,\frac{2^{-m/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{\Psi}(\omega)\,\mathrm{e}^{\mathrm{i}\omega(2^{-m}P-n)}\,\mathrm{d}\omega \\ &= \frac{\mathrm{e}^{\mathrm{i}xy/2}2^{-m/2}}{2\pi^{7/4}\sqrt{2}}\int_{\mathcal{D}}\mathrm{d}\omega\,\mathrm{e}^{-\mathrm{i}\omega n}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}P(y+2^{-m}\omega)}\,\mathrm{e}^{-(x+P)^2/2}\,\mathrm{d}P\end{aligned} \tag{4.2}$$

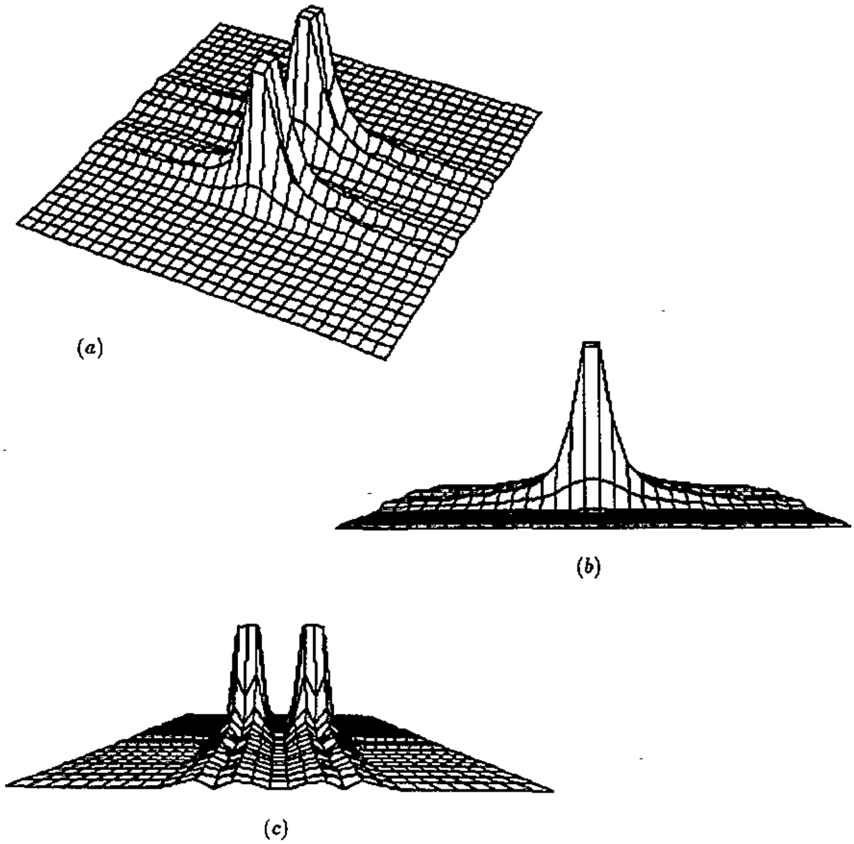

Figure 2. (a) Dependence of $|\Psi_{00}(x, y)|$ on x and y. (b) Dependence of $|\Psi_{00}(x, y)|$ on x. (c) Dependence of $|\Psi_{00}(x, y)|$ on y.

where we have defined the set $\mathcal{D} = [-2\pi, -\pi] \cup [\pi, 2\pi]$. The order of integration can be exchanged as one can see easily by using Fubini's theorem, see [20].

Performing the simple Gaussian integration in P, we find

$$\Psi_{mn}(x, y) = \frac{e^{ixy/2} 2^{-m/2}}{2\pi^{5/4}} \int_{\mathcal{D}} d\omega\, e^{-i\omega n} e^{-ix(y+2^{-m}\omega)}\, e^{-(y+2^{-m}\omega)^2/2} \tag{4.3}$$

which can again be explicitly computed, in terms of the error integral already used in the previous section. We get

$$\begin{aligned}\Psi_{mn}(x, y) = {} & \frac{e^{ixy/2}\, e^{iy2^m n}\, e^{-(x+2^m n)^2/2}}{\pi^{3/4}}\, 2^{(m-3)/2} \\ & \times \left\{ \Xi(2^{1-m}\pi - (y + ix) - in2^m) - \Xi(2^{-m}\pi - (y + ix) - in2^m) \right. \\ & \left. + \Xi(2^{1-m}\pi + (y + ix) + in2^m) - \Xi(2^{-m}\pi + (y + ix) + in2^m) \right\} .\end{aligned} \tag{4.4}$$

This expression is rather similar to the one in (3.4). For $m = n = 0$, in particular, we obtain

$$\begin{aligned}\Psi_{00}(x, y) = {} & \frac{e^{ixy/2}\, e^{-x^2/2}}{2\sqrt{2}\pi^{3/4}} \{ \Xi(2\pi - y - ix) - \Xi(\pi - y - ix) \\ & + \Xi(2\pi + y + ix) - \Xi(\pi + y + ix) \} .\end{aligned} \tag{4.5}$$

The modulus of the function Ψ_{00} is plotted in figures 2(*a*), 2(*b*) and 2(*c*). Again in the three pictures we use different points of view in order to show the different decay properties of $|\Psi_{00}(x, y)|$ in the two variables. We see that $|\Psi_{00}(x, y)|$ goes to zero very rapidly in y, whereas its decay is rather slow in x.

This result can be made rigorous by using again the asymptotic formula for the error function. We find

$$\Psi_{00}(x, y) \simeq \frac{e^{-ixy/2}e^{-y^2/2}}{2\pi^{5/4}}\left\{-\frac{e^{2\pi(y+ix)}\,e^{-2\pi^2}}{|2\pi - y - ix|} + \frac{e^{\pi(y+ix)}\,e^{-\pi^2/2}}{|\pi - y - ix|} - \frac{e^{-2\pi(y+ix)}\,e^{-2\pi^2}}{|2\pi + y + ix|} + \frac{e^{-\pi(y+ix)}\,e^{-\pi^2/2}}{|\pi + y + ix|}\right\} \tag{4.6}$$

which displays the exponential decay of $|\Psi_{00}(x, y)|$ in y and the slow decay in x, as observed on figure 2.

We see here the announced complementarity with respect to the Haar basis: the first one is better localized in x, the other one in y.

5. Conclusion

We have discussed general new ON bases for the LLL and we have given some details on two particular examples of these bases.

Since the basis functions in our examples have a rather slow asymptotic decay, we do not expect them to be a good choice for single electron wavefunctions in a Hartree–Fock computation of the Coulomb interaction for intermediate electron density. However, they can give interesting results for low density, where the electrons are supposed to be far enough from each other, so that their wavefunctions overlap very little. Again the scale parameter m may be used for controlling precisely the size of that overlap.

Moreover, the technique introduced here may be exploited for constructing a basis better adapted for describing a system like ours, which is symmetric with respect of the exchange of the variables.

Appendix A. Orthonormal bases of wavelets

The wavelet transform (WT) is by now a well established tool in many branches of physics, such as acoustics, spectroscopy, geophysics, astrophysics, fluid mechanics (turbulence), medical imagery,... (see [21] for a survey of the present status). Basically it is a time-scale representation, which allows a fine analysis of non-stationary signals and a good reconstruction of a signal from its WT, both in one and in two dimensions (image processing).

The basic formula for the (continuous) WT of a one-dimensional signal $s \in L^2(\mathbb{R})$ reads

$$S(a, b) = a^{-1/2}\int \overline{\psi}\left(\frac{x-b}{a}\right) s(x)\,\mathrm{d}x \tag{A.1}$$

where $a > 0$ is a scale parameter and $b \in \mathbb{R}$ a translation parameter. Both the function $\psi(x)$, called the *analysing wavelet*, and its Fourier transform $\widehat{\psi}(\omega)$ must be well localized, and in addition ψ is assumed to have zero mean:

$$\int \psi(x)\,\mathrm{d}x = 0\,. \tag{A.2}$$

Combined with the localization properties, this relation makes the WT (A.1) into a local filter and ensures its efficiency in signal analysis and reconstruction.

However, in practice, one often uses a *discretized* WT, obtained by restricting the parameters a and b in (A.1) to the points of a lattice, typically a dyadic one:

$$S_{j,k} = 2^{-j/2} \int \overline{\psi}(2^{-j}x - k)\, s(x)\, \mathrm{d}x \qquad j, k \in \mathbb{Z}. \tag{A.3}$$

Very general functions ψ satisfying the admissibility conditions described above will yield a good WT, but then the functions $\{\psi_{j,k}(x) \equiv 2^{j/2}\psi(2^j x - k),\ j, k \in \mathbb{Z}\}$ are in general not orthogonal to each other! One of the successes of the WT was the discovery that it is possible to construct functions ψ for which $\{\psi_{jk},\ j, k \in \mathbb{Z}\}$ is indeed an orthonormal basis of $L^2(\mathbb{R})$. In addition, such a basis still has the good properties of wavelets, including space *and* frequency localization. This is the key to their usefulness in many applications, including the present one. In the rest of this appendix, we will briefly sketch the construction of these ON bases of wavelets. The full story may be found, for instance, in [13].

The construction is based on two facts: first, almost all examples of orthonormal bases of wavelets can be derived from a multiresolution analysis, and then the whole construction may be transcripted into the language of quadrature mirror filters (QMF), familiar in the signal processing literature.

A *multiresolution analysis* of $L^2(\mathbb{R})$ is an increasing sequence of closed subspaces

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \tag{A.4}$$

with $\bigcup_{j\in\mathbb{Z}} V_j$ dense in $L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, and such that:

(1) $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$
(2) there exists a function $\phi \in V_0$, called a *scaling* function, such that $\{\phi(x - k), k \in \mathbb{Z}\}$ is an ON basis of V_0.

Combining (1) and (2), one gets an ON basis of V_j, namely $\{\phi_{j,k}(x) \equiv 2^{j/2}\phi(2^j x - k),\ k \in \mathbb{Z}\}$.

Each V_j can be interpreted as an approximation space: the approximation of $f \in L^2(\mathbb{R})$ at the resolution 2^j is defined by its projection onto V_j. The additional details needed for increasing the resolution from 2^j to 2^{j+1} are given by the projection of f onto the orthogonal complement W_j of V_j in V_{j+1}:

$$V_j \oplus W_j = V_{j+1} \tag{A.5}$$

and we have

$$\bigoplus_{j\in\mathbb{Z}} W_j = L^2(\mathbb{R}). \tag{A.6}$$

Then the theory asserts the existence of a function ψ, called the *mother* of the wavelets, explicitly computable from ϕ, such that $\{\psi_{j,k}(x) \equiv 2^{j/2}\psi(2^j x - k),\ j, k \in \mathbb{Z}\}$ constitutes an orthonormal basis of $L^2(\mathbb{R})$: these are the *orthonormal wavelets*.

The construction of ψ proceeds as follows. First, the inclusion $V_0 \subset V_1$ yields the relation

$$\phi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} h_n \phi(2x - n) \qquad h_n = \langle \phi_{1,n} | \phi \rangle. \tag{A.7}$$

Taking Fourier transforms, this gives

$$\widehat{\phi}(\omega) = m_0(\omega/2)\widehat{\phi}(\omega/2) \tag{A.8}$$

where

$$m_0(\omega) = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} h_n\, \mathrm{e}^{-\mathrm{i}n\omega} \tag{A.9}$$

is a 2π-periodic function. Iterating (A.8), one gets the scaling function as the (convergent!) infinite product

$$\widehat{\phi}(\omega) = (2\pi)^{-1/2} \prod_{j=1}^{\infty} m_0(2^{-j}\omega) . \tag{A.10}$$

Then one defines the function $\psi \in W_0 \subset V_1$ by the relation

$$\widehat{\psi}(\omega) = e^{i\omega/2}\, \overline{m_0(\omega/2+\pi)}\, \widehat{\phi}(\omega/2) \tag{A.11}$$

or, equivalently

$$\psi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} (-1)^{n-1} h_{-n-1} \phi(2x-n) \tag{A.12}$$

and proves that the function ψ indeed generates an ON basis with all the required properties.

Various additional conditions may be imposed on the function ψ (hence on the basis wavelets): arbitrary regularity, several vanishing moments (in any case, ψ always has mean zero), fast decrease at infinity, even compact support. The technique consists in translating the multiresolution structure into the language of QMF filters, and putting suitable constraints on the filter coefficients h_n. For instance, ψ has compact support if only finitely many h_n differ from zero (in technical terms, $\{h_n\}$ is a FIR filter). Notice that the correspondence $h_n \Leftrightarrow (-1)^{n-1} h_{-n-1}$ between the Fourier coefficients of ϕ and ψ in (A.7) and (A.12) expresses precisely the fact that the pair (ϕ, ψ) generates a QMF.

The simplest example of this construction is the Haar basis discussed in section 3, which comes from the scaling function $\phi(x) = 1$ for $0 \leqslant x < 1$ and 0 otherwise. Similarly, the linear spline function $\theta(x)$ given in (2.7) yields the wavelet (2.8). Other explicit examples may be found in [13].

Acknowledgments

We thank F Strocchi and G Morchio for many stimulating discussions on the PP′-representation and on coherent states.

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