Linear Canonical Transformations and Their Unitary Representations*

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(Received 22 January 1971)

We show that the group of linear canonical transformations in a $2N$-dimensional phase space is the real symplectic group $Sp(2N)$, and discuss its unitary representation in quantum mechanics when the N coordinates are diagonal. We show that this $Sp(2N)$ group is the well-known dynamical group of the N-dimensional harmonic oscillator. Finally, we study the case of n particles in a q-dimensional oscillator potential, for which $N = nq$, and discuss the chain of groups $Sp(2nq) \supset Sp(2n) \times O(q)$. An application to the calculation of matrix elements is given in a following paper.

1. INTRODUCTION

It is well known that some of the powerful techniques for solving mechanics problems are based on the symmetry group of canonical transformations, i.e., the transformations in phase space that leave the Hamiltonian and the Poisson brackets of coordinates and momenta invariant. In some cases these transformations concern the coordinates alone, i.e., are point transformations, as is the case when there is invariance under translations, rotations, or permutations of the particles. Symmetry groups of point transformations have been discussed extensively in the literature[1–3] both in their applications to classical and quantum mechanics. Groups of canonical transformations have been less extensively applied particularly in quantum mechanics, in which we require their unitary representation in an appropriate Hilbert space.[4] We shall discuss in this note the group of linear canonical transformations in a $2N$-dimensional phase space and their unitary representation when the N coordinates are diagonal. The linear canonical transformations constitute the dynamical group of the N-dimensional harmonic oscillator.[5] Their unitary representation plays a fundamental role in the understanding of the properties of harmonic oscillator states[5] and their use in many-body calculations.[6]

2. THE SYMPLECTIC GROUP OF LINEAR CANONICAL TRANSFORMATIONS

A canonical transformation is a transformation in phase space which leaves invariant the Poisson brackets

$$\{x_i, p_j\} = \delta_{ij}, \quad \{x_i, x_j\} = \{p_i, p_j\} = 0, \quad i,j = 1, \cdots N. \tag{2.1}$$

To understand the nature of the group that these transformations form, we introduce the notation z_α, $\alpha = 1, 2, \cdots, 2N$, for a vector in phase space defined by

$$z_i \equiv x_i, \quad z_{i+N} \equiv p_i, \quad i = 1, \cdots N. \tag{2.2}$$

The Poisson bracket of two observables f, g is then

$$\{f, g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x_i}\right) = \sum_{\alpha,\beta=1}^{2N} \frac{\partial f}{\partial z_\alpha} K_{\alpha\beta} \frac{\partial g}{\partial z_\beta}, \tag{2.3}$$

where the matrix

$$\mathsf{K} = \|K_{\alpha\beta}\| = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.4}$$

has all its submatrices of dimension $N \times N$.

If we now pass from the vector $\{z_\alpha\}$ in phase space to a new one $\{\bar{z}_\alpha\}$ whose components are functions of the previous one, the transformation will be canonical if

$$\sum_{\gamma,\delta} \frac{\partial \bar{z}_\alpha}{\partial z_\gamma} K_{\gamma\delta} \frac{\partial \bar{z}_\beta}{\partial z_\delta} = K_{\alpha\beta}. \tag{2.5}$$

If, in particular, the transformation between the new and the old vectors in phase space is linear, i.e.,

$$\bar{z}_\alpha = \sum_\beta S_{\alpha\beta} z_\beta, \tag{2.6}$$

the transformation will be canonical if

$$\mathsf{S}\mathsf{K}\tilde{\mathsf{S}} = \mathsf{K}, \tag{2.7}$$

where $\mathsf{S} = \|S_{\alpha\beta}\|$ and the tilde stands for transposed. The matrix S will be assumed real so that $\bar{x}_i$ and $\bar{p}_i$ remain Hermitian when x_i and p_i are represented by Hermitian operators.

The matrix K is the one usually associated with the symplectic group.[3,7] Thus the matrices S satisfying (2.7) are elements of a $2N$-dimensional real symplectic group. We shall write these matrices in the form

$$\mathsf{S} = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix}, \tag{2.8}$$

where the real submatrices are all of dimension $N \times N$ with components

$$\mathsf{A} = \|a_{ij}\|, \quad \mathsf{B} = \|b_{ij}\|, \quad \mathsf{C} = \|c_{ij}\|, \quad \mathsf{D} = \|d_{ij}\|. \tag{2.9}$$

The restriction (2.7) leads then to the equations

$$\mathsf{B}\tilde{\mathsf{A}} = \mathsf{A}\tilde{\mathsf{B}}, \tag{2.10a}$$

$$\mathsf{C}\tilde{\mathsf{D}} = \mathsf{D}\tilde{\mathsf{C}}, \tag{2.10b}$$

$$\mathsf{D}\tilde{\mathsf{A}} - \mathsf{C}\tilde{\mathsf{B}} = \mathsf{I}. \tag{2.10c}$$

We first consider the case when the matrix B is nonsingular. We can then use (2.10c) to determine C by

$$\mathsf{C} = (\mathsf{D}\tilde{\mathsf{A}} - \mathsf{I})\tilde{\mathsf{B}}^{-1}. \tag{2.11a}$$

From this equation and (2.10a) and (2.10b) we see that the restrictions on the remaining submatrices A, B, and D are given by

$$\mathsf{B}\tilde{\mathsf{A}} = \mathsf{A}\tilde{\mathsf{B}}, \tag{2.11b}$$

$$\tilde{\mathsf{B}}\mathsf{D} = \tilde{\mathsf{D}}\mathsf{B}. \tag{2.11c}$$

Equations (2.11) are then the ones that determine the general matrices S of the real symplectic group when

$$\det \mathsf{B} \neq 0. \tag{2.12}$$

When B is singular, we proceed to show that it is always possible to find a nonsingular diagonal matrix B′ such that

$$\begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix} = \begin{pmatrix} \mathsf{I} & \mathsf{B}' \\ 0 & \mathsf{I} \end{pmatrix} \begin{pmatrix} \mathsf{A} - \mathsf{B}'\mathsf{C} & \mathsf{B} - \mathsf{B}'\mathsf{D} \\ \mathsf{C} & \mathsf{D} \end{pmatrix} \tag{2.13}$$

and

$$\det(\mathsf{B} - \mathsf{B}'\mathsf{D}) \neq 0. \tag{2.14}$$

To prove (2.14), let us denote the components of B′ by

$$\mathsf{B}' = \|b_i'\delta_{ij}\|, \quad b_i' \neq 0. \tag{2.15}$$

If, for all choices of the b_i', the matrix

$$\mathsf{B} - \mathsf{B}'\mathsf{D} \tag{2.16}$$

is singular, then it is possible to find a set of real γ_k, not all zero, such that

$$\gamma_k(b_{ik} - b_i'd_{ik}) = 0, \tag{2.17}$$

where in what follows all repeated Latin indices are summed from 1 to N. As this relation must be valid for any nonzero value of b_i', we can conclude that it implies the existence of a set γ_k, not all zero, for which

$$\gamma_k b_{ik} = \gamma_k d_{ik} = 0. \tag{2.18}$$

But this indicates that a linear combination of columns in the right-hand side of (2.8) gives zero, which implies that in this case S is singular. We have though from (2.7) that

$$(\det \mathsf{S})^2 = 1, \tag{2.19}$$

and thus we are led to a contradiction. We conclude, therefore, that it is possible to find a diagonal nonsingular matrix B′ for which (2.14) holds when B itself is singular.

In the particular case when S is a point transformation, i.e.,

$$\mathsf{S} = \begin{pmatrix} \mathsf{A} & 0 \\ 0 & \tilde{\mathsf{A}}^{-1} \end{pmatrix} = \begin{pmatrix} \mathsf{I} & \mathsf{I} \\ 0 & \mathsf{I} \end{pmatrix} \begin{pmatrix} \mathsf{A} & -\tilde{\mathsf{A}}^{-1} \\ 0 & \tilde{\mathsf{A}}^{-1} \end{pmatrix}, \tag{2.20}$$

we can express it in the form (2.13)–(2.14) with B = I.

As the matrix

$$\begin{pmatrix} \mathsf{I} & \mathsf{B}' \\ 0 & \mathsf{I} \end{pmatrix} \tag{2.21}$$

satisfies the conditions (2.11), we have decomposed the matrix S of (2.8) when B is singular, into the product of two matrices belonging to the symplectic group for both of which the submatrix in the upper right corner is nonsingular. We later show that we can obtain in an elementary fashion the unitary representation of the matrix S when B is nonsingular. Thus the development (2.13) allows us to obtain the unitary representation of the matrix S when B is singular, as a product of two unitary representations of matrices for which the submatrix in the upper right corner is nonsingular.

3. THE UNITARY REPRESENTATION OF LINEAR CANONICAL TRANSFORMATIONS

We wish to find in the quantum mechanical picture the unitary representation of the linear canonical transformations discussed in the previous section. We denote the states in which the coordinates x_i are diagonal by the bras and kets $\langle \mathbf{x}'|$ and $|\mathbf{x}''\rangle$, where whenever something is characterized by the eigenvalues of *all* the coordinates x_i, $i = 1, \cdots, N$, we suppress the index i. The matrix elements of the operators x_i and p_i with respect to this basis are then[4]

$$\langle \mathbf{x}'| x_i |\mathbf{x}''\rangle = x_i'\delta(\mathbf{x}' - \mathbf{x}''), \tag{3.1a}$$

$$\langle \mathbf{x}'| p_i |\mathbf{x}''\rangle = -\frac{1}{i}\frac{\partial}{\partial x_i''}\delta(\mathbf{x}' - \mathbf{x}''), \tag{3.1b}$$

where $\hbar$ is taken as 1 and

$$\delta(\mathbf{x}' - \mathbf{x}'') = \prod_{i=1}^{N} \delta(x_i' - x_i''). \tag{3.1c}$$

If we now pass to another set of coordinates and momenta,

$$\bar{x}_i = a_{ij}x_j + b_{ij}p_j, \tag{3.2a}$$

$$\bar{p}_i = c_{ij}x_j + d_{ij}p_j, \tag{3.2b}$$

which are canonical, i.e., for which the matrices A, B, C, and D satisfy the relations (2.10), then it is clear[4] that the matrix elements of $\bar{x}_i$ and $\bar{p}_i$ between the bras and kets $(\bar{\mathbf{x}}'|$ and $|\bar{\mathbf{x}}'')$ will have the same form as (3.1). We use a round bracket notation for states in which $\bar{x}_i$ is diagonal to distinguish them from the states in which x_i is diagonal, in case we take the same numerical value for the eigenvalues of $\bar{x}_i$ and x_i.

Clearly we could make the development

$$|\bar{\mathbf{x}}') = \int |\mathbf{x}'\rangle \, d\mathbf{x}' \, \langle \mathbf{x}' \mid \bar{\mathbf{x}}'), \tag{3.3}$$

where $d\mathbf{x}' = dx_1' \cdots dx_N'$ and we have an N-dimensional integration. The transformation bracket in (3.3) satisfies the equations[4]

$$\left(a_{ij}x_j' - ib_{ij}\frac{\partial}{\partial x_j'}\right)\langle \mathbf{x}' \mid \bar{\mathbf{x}}') = \bar{x}_i' \langle \mathbf{x}' \mid \bar{\mathbf{x}}'), \quad i = 1, \cdots, N. \tag{3.4}$$

It is important to notice that Eqs. (3.4) do not completely determine the transformation bracket $\langle \mathbf{x}' \mid \bar{\mathbf{x}}')$. In particular, if we multiply $\langle \mathbf{x}' \mid \bar{\mathbf{x}}')$ by an arbitrary function of $\bar{\mathbf{x}}'$, it continues to satisfy Eqs. (3.4). We could, though, fully determine the transformation bracket $\langle \mathbf{x}' \mid \bar{\mathbf{x}}')$ up to a constant phase by the further requirements that the matrix elements of $\bar{x}_i$ and $\bar{p}_i$ with respect to the states $(\bar{\mathbf{x}}'|$ and $|\bar{\mathbf{x}}'')$ have the form (3.1), i.e., that the transformation be canonical, which implies[4]

$$(\bar{\mathbf{x}}'| \bar{x}_i |\bar{\mathbf{x}}'') = \int (\bar{\mathbf{x}}' \mid \mathbf{x}'\rangle\left(a_{ij}x_j' - ib_{ij}\frac{\partial}{\partial x_j'}\right)\langle \mathbf{x}' \mid \bar{\mathbf{x}}'') \, d\mathbf{x}' = \bar{x}_i'\delta(\bar{\mathbf{x}}' - \bar{\mathbf{x}}''), \tag{3.5a}$$

$$(\bar{\mathbf{x}}'| \bar{p}_i |\bar{\mathbf{x}}'') = \int (\bar{\mathbf{x}}' \mid \mathbf{x}'\rangle\left(c_{ij}x_j' - id_{ij}\frac{\partial}{\partial x_j'}\right)\langle \mathbf{x}' \mid \bar{\mathbf{x}}'') \, d\mathbf{x}' = -\frac{1}{i}\frac{\partial}{\partial \bar{x}_i''}\delta(\bar{\mathbf{x}}' - \bar{\mathbf{x}}''), \tag{3.5b}$$

where[4]

$$(\bar{\mathbf{x}}' \mid \mathbf{x}'\rangle = \langle \mathbf{x}' \mid \bar{\mathbf{x}}')^*. \tag{3.5c}$$

Once we have these transformation brackets, we can easily identify them with the unitary representation of the canonical transformation (3.2). For this purpose, we note, for example, that the matrix elements of $\bar{x}_i$ with respect to the basis in which the x_i are diagonal can be written as

$$\langle \mathbf{x}'| \bar{x}_i |\mathbf{x}''\rangle = \int \langle \mathbf{x}' \mid \bar{\mathbf{x}}') \, d\bar{\mathbf{x}}'(\bar{\mathbf{x}}'| \bar{x}_i |\bar{\mathbf{x}}'') \, d\bar{\mathbf{x}}''(\bar{\mathbf{x}}'' \mid \mathbf{x}''\rangle = \int \langle \mathbf{x}' \mid \bar{\mathbf{x}}') \, d\bar{\mathbf{x}}'\bar{x}_i'\delta(\bar{\mathbf{x}}' - \bar{\mathbf{x}}'') \, d\bar{\mathbf{x}}''(\bar{\mathbf{x}}'' \mid \mathbf{x}''\rangle. \tag{3.6}$$

Now $\bar{x}_i'$, $\bar{x}_i''$ are not operators but just variables over which we carry out integrations, and thus, using (3.1a), we can write

$$\bar{x}_i'\delta(\bar{\mathbf{x}}' - \bar{\mathbf{x}}'') = \langle \bar{\mathbf{x}}'| x_i |\bar{\mathbf{x}}''\rangle, \tag{3.7}$$

where we stress the angular, rather than round, brackets of the matrix elements. To express then (3.6) as a matrix multiplication, we define the matrix elements of a unitary matrix U in a basis in which the x_i are diagonal[4] as

$$\langle \mathbf{x}'| U |\mathbf{x}''\rangle \equiv \langle \mathbf{x}' \mid \mathbf{x}''), \quad \text{which implies} \quad \langle \mathbf{x}'| U^{-1} |\mathbf{x}''\rangle = (\mathbf{x}' \mid \mathbf{x}''\rangle, \tag{3.8}$$

thus getting

$$\langle \mathbf{x}'| \bar{x}_i |\mathbf{x}''\rangle = \int \langle \mathbf{x}'| U |\bar{\mathbf{x}}'\rangle \, d\bar{\mathbf{x}}' \, \langle \bar{\mathbf{x}}'| x_i |\bar{\mathbf{x}}''\rangle \, d\bar{\mathbf{x}}'' \, \langle \bar{\mathbf{x}}''| U^{-1} |\mathbf{x}''\rangle. \tag{3.9}$$

In operator language we have then[4]

$$\bar{x}_i = Ux_iU^{-1}, \tag{3.10}$$

and it is clear that an entirely similar analysis gives us

$$\bar{p}_i = Up_iU^{-1}. \tag{3.11}$$

If we carry out in succession two canonical transformations that give rise to the unitary representations U and V, the new coordinates and momenta in the quantum mechanical picture are affected by the transformation VU. Thus we have the quantum mechanical equivalent[4] of the classical canonical transformations and therefore also a unitary representation of the general symplectic group of which they form a part.

We now proceed to determine $\langle \mathbf{x}'| U |\mathbf{x}''\rangle$ explicitly when, in the canonical transformation (2.6), the matrix S is given by (2.8), where the submatrix B is nonsingular. We shall use the notation

$$\langle \mathbf{x}' \mid \mathbf{x}'') = \langle \mathbf{x}'| U |\mathbf{x}''\rangle \equiv \phi(\mathbf{x}', \mathbf{x}''), \tag{3.12}$$

and proceed to show that ϕ must have the form

$$\phi(\mathbf{x}', \mathbf{x}'') = \alpha \exp\left[i(\lambda_{ij}x_i'x_j' + \mu_{ij}x_i'x_j'' + \nu_{ij}x_i''x_j'')\right], \tag{3.13}$$

where α is a constant, the $N \times N$ matrices

$$\mathfrak{L} = \|\lambda_{ij}\|, \quad \mathcal{M} = \|\mu_{ij}\|, \quad \mathcal{N} = \|\nu_{ij}\| \tag{3.14}$$

are real, and $\mathfrak{L}$ and $\mathcal{N}$ are symmetric. These parameters can be determined as follows: First, as $\langle \mathbf{x}' \mid \bar{\mathbf{x}}')$ satisfies Eqs. (3.4), we obtain that

$$\left(a_{ij}x_j' - ib_{ij}\frac{\partial}{\partial x_j'}\right)\phi(\mathbf{x}', \mathbf{x}'') = x_i''\phi(\mathbf{x}', \mathbf{x}''), \tag{3.15}$$

which implies that

$$[a_{ij}x_j' + b_{ij}(2\lambda_{jk}x_k' + \mu_{jk}x_k'')]\phi(\mathbf{x}', \mathbf{x}'') = x_i''\phi(\mathbf{x}', \mathbf{x}''), \tag{3.16a}$$

which in matrix notation takes the form

$$\mathsf{A}\mathbf{x}' + 2\mathsf{B}\mathfrak{L}\mathbf{x}' + \mathsf{B}\mathcal{M}\mathbf{x}'' = \mathbf{x}''. \tag{3.16b}$$

As $\mathbf{x}'$ and $\mathbf{x}''$ are arbitrary independent vector and B is nonsingular, this implies that

$$\mathfrak{L} = -\tfrac{1}{2}\mathsf{B}^{-1}\mathsf{A}, \quad \mathcal{M} = \mathsf{B}^{-1}. \tag{3.17}$$

The matrix $\mathfrak{L}$ is symmetric, as it should be, as a consequence of (2.11b). Passing now to Eq. (3.5a), we see from (3.16a), using the notation (3.12), that it implies

$$\int \phi^*(\mathbf{x}', \bar{\mathbf{x}}')\phi(\mathbf{x}', \bar{\mathbf{x}}'')\, d\mathbf{x}' = \delta(\bar{\mathbf{x}}' - \bar{\mathbf{x}}''). \tag{3.18}$$

Thus we must have the restriction

$$\begin{aligned} &|\alpha|^2 \exp[i\nu_{ij}(-\bar{x}_i'\bar{x}_j' + \bar{x}_i''\bar{x}_j'')] \\ &\quad\times \int \exp\{-ix_i'[\mu_{ij}(\bar{x}_j' - \bar{x}_j'')]\}\, d\mathbf{x}' \\ &= |\alpha|^2 \exp\{i\nu_{ij}(-\bar{x}_i'\bar{x}_j' + \bar{x}_i''\bar{x}_j'')\}(2\pi)^N \\ &\quad\times \prod_{i=1}^{N} \delta[\mu_{ij}(\bar{x}_j'' - \bar{x}_j')] \\ &= |\alpha|^2 (2\pi)^N |\det \mathsf{B}|\, \delta(\bar{\mathbf{x}}'' - \bar{\mathbf{x}}'). \end{aligned} \tag{3.19}$$

In (3.19) we made use of the fact that if

$$y_i'' = \mu_{ij}\bar{x}_j'' \tag{3.20a}$$

or

$$\bar{x}_i'' = b_{ij}y_j'', \tag{3.20b}$$

and similarly for $\bar{x}_i'$, then

$$\prod_{i=1}^{N} \delta(y_i'' - y_i') = |J| \prod_{i=1}^{N} \delta(\bar{x}_i'' - \bar{x}_i'), \tag{3.21}$$

where J is the Jacobian of the transformation (3.20b), i.e.,

$$J = \left| \frac{\partial \bar{x}_i''}{\partial y_j''} \right| = \det \mathsf{B}. \tag{3.22}$$

Thus the restriction that $\bar{x}_i$ must have its canonical form (3.5a) leads, up to a phase, to the value

$$\alpha = [(2\pi)^N |\det \mathsf{B}|]^{-\frac{1}{2}}. \tag{3.23}$$

The next step is to make use of Eq. (3.5b), which leads to

$$\begin{aligned} &\int \phi^*(\mathbf{x}', \bar{\mathbf{x}}')\left[\left(c_{ij}x_j' - id_{ij}\frac{\partial}{\partial x_j'}\right)\phi(\mathbf{x}', \bar{\mathbf{x}}'')\right] d\mathbf{x}' \\ &= \int \phi^*(\mathbf{x}', \bar{\mathbf{x}}')[c_{ij}x_j' + d_{ij}(2\lambda_{jk}x_k' + \mu_{jk}\bar{x}_k'')]\phi(\mathbf{x}', \bar{\mathbf{x}}'')\, d\mathbf{x}' \\ &= -\frac{1}{i}\frac{\partial}{\partial \bar{x}_i''}\delta(\bar{\mathbf{x}}' - \bar{\mathbf{x}}''). \end{aligned} \tag{3.24a}$$

But, making use of (3.18), we see that

$$\begin{aligned} &-\frac{1}{i}\frac{\partial}{\partial \bar{x}_i''}\delta(\bar{\mathbf{x}}' - \bar{\mathbf{x}}'') \\ &= \int \phi^*(\mathbf{x}', \bar{\mathbf{x}})[-\mu_{ji}x_j' - 2\nu_{ji}\bar{x}_j'']\phi(\mathbf{x}', \bar{\mathbf{x}}'')\, d\mathbf{x}'. \end{aligned} \tag{3.24b}$$

From Eqs. (3.24) we obtain the following relation in matrix notation,

$$(\mathsf{C} + 2\mathsf{D}\mathfrak{L})\mathbf{x}' + \mathsf{D}\mathcal{M}\bar{\mathbf{x}}'' = -\tilde{\mathcal{M}}\mathbf{x}' - 2\tilde{\mathcal{N}}\bar{\mathbf{x}}'', \tag{3.25}$$

and, as $\mathbf{x}'$ and $\bar{\mathbf{x}}''$ are arbitrary independent vectors, we obtain

$$\mathsf{C} = -2\mathsf{D}\mathfrak{L} - \tilde{\mathcal{M}} = \mathsf{D}\mathsf{B}^{-1}\mathsf{A} - \tilde{\mathsf{B}}^{-1} = (\tilde{\mathsf{D}}\mathsf{A} - I)\tilde{\mathsf{B}}^{-1}, \tag{3.26a}$$

$$\mathcal{N} = -\tfrac{1}{2}\tilde{\mathcal{M}}\tilde{\mathsf{D}} = -\tfrac{1}{2}\tilde{\mathsf{B}}^{-1}\tilde{\mathsf{D}} = -\tfrac{1}{2}\mathsf{D}\mathsf{B}^{-1}. \tag{3.26b}$$

Equation (3.26a) is automatically satisfied in view of the relation (2.11a), while (3.26b) gives us the explicit form of the matrix $\mathcal{N}$, which from (2.11c) is symmetric as it should be. Thus, finally, up to a constant phase factor, we can write (in matrix and vector notation) the unitary representation of the linear canonical transformation (2.6) as

$$\begin{aligned} &\langle \mathbf{x}' | U | \mathbf{x}'' \rangle \\ &= [(2\pi)^N |\det \mathsf{B}|]^{-\frac{1}{2}} \\ &\quad\times \exp[-\tfrac{1}{2}i(\tilde{\mathbf{x}}'\mathsf{B}^{-1}\mathsf{A}\mathbf{x}' - 2\tilde{\mathbf{x}}'\mathsf{B}^{-1}\mathbf{x}'' + \tilde{\mathbf{x}}''\mathsf{D}\mathsf{B}^{-1}\mathbf{x}'')], \end{aligned} \tag{3.27}$$

where $\tilde{\mathbf{x}}'$ and $\tilde{\mathbf{x}}''$ are the transposed vectors $\mathbf{x}'$ and $\mathbf{x}''$. We recall though that this holds only in the case when B is nonsingular, and thus B^{-1} exists.

When B is singular, we can consider the development (2.13). To obtain the unitary representation in this case, we need to calculate the product of two unitary representations of the form (3.27). We proceed first to discuss the unitary representation of the product of two arbitrary canonical transformations

$$\begin{aligned} \begin{pmatrix} \mathsf{A}_2 & \mathsf{B}_2 \\ \mathsf{C}_2 & \mathsf{D}_2 \end{pmatrix}\begin{pmatrix} \mathsf{A}_1 & \mathsf{B}_1 \\ \mathsf{C}_1 & \mathsf{D}_1 \end{pmatrix} &= \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{pmatrix} \\ &= \begin{pmatrix} \mathsf{A}_2\mathsf{A}_1 + \mathsf{B}_2\mathsf{C}_1 & \mathsf{A}_2\mathsf{B}_1 + \mathsf{B}_2\mathsf{D}_1 \\ \mathsf{C}_2\mathsf{A}_1 + \mathsf{D}_2\mathsf{C}_1 & \mathsf{C}_2\mathsf{B}_1 + \mathsf{D}_2\mathsf{D}_1 \end{pmatrix}, \end{aligned} \tag{3.28}$$

where we assume B_1 and B_2 nonsingular. We shall denote the unitary representations of the matrices in (3.28) by U_2, U_1, and U, respectively. The unitary representation of the product of canonical transformations is then

$$\begin{aligned}\langle \mathbf{x}'|\, U_1U_2\,|\mathbf{x}''\rangle &= \int \langle \mathbf{x}'|\, U_1\,|\mathbf{x}'''\rangle\, d\mathbf{x}'''\langle \mathbf{x}'''|\, U_2\,|\mathbf{x}''\rangle \\ &= (2\pi)^{-N}(|\det \mathsf{B}_1|\cdot|\det \mathsf{B}_2|)^{-\frac{1}{2}} \\ &\quad\times \exp\,[-\tfrac{1}{2}i(\tilde{\mathbf{x}}'\mathsf{B}_1^{-1}\mathsf{A}_1\mathbf{x}' + \tilde{\mathbf{x}}''\mathsf{D}_2\mathsf{B}_2^{-1}\mathbf{x}'')] \\ &\quad\times \int d\mathbf{x}'''\exp\{-\tfrac{1}{2}i[\tilde{\mathbf{x}}'''(\mathsf{D}_1\mathsf{B}_1^{-1} + \mathsf{B}_2^{-1}\mathsf{A}_2)\mathbf{x}''' \\ &\quad - 2\tilde{\mathbf{x}}'''(\tilde{\mathsf{B}}_1^{-1}\mathbf{x}' + \mathsf{B}_2^{-1}\mathbf{x}'')]\}. \end{aligned} \tag{3.29}$$

From (2.11b) and (2.11c) we see that the real matrix $\mathsf{D}_1\mathsf{B}_1^{-1} + \mathsf{B}_2^{-1}\mathsf{A}_2$ is symmetric, and thus there exists an orthogonal transformation

$$\mathbf{x}''' = \mathcal{O}\mathbf{y} \tag{3.30}$$

that diagonalizes this matrix, i.e.,

$$\tilde{\mathcal{O}}(\mathsf{D}_1\mathsf{B}_1^{-1} + \mathsf{B}_2^{-1}\mathsf{A}_2)\mathcal{O} = \boldsymbol{\Delta} = \|\delta_i\delta_{ij}\|. \tag{3.31}$$

We note furthermore that

$$\mathsf{D}_1\mathsf{B}_1^{-1} + \mathsf{B}_2^{-1}\mathsf{A}_2 = \mathsf{B}_2^{-1}\mathsf{B}\mathsf{B}_1^{-1}, \tag{3.32}$$

and thus the number of zeros among the real eigenvalues δ_i will be equal to the dimension minus the rank of the matrix B which we shall denote by p. We can always select the matrix $\mathcal{O}$ in such a way that the first p eigenvalues $\delta_1, \cdots, \delta_p$ are the ones that vanish. If we then denote by $\mathbf{z}$ the vector

$$\mathbf{z} \equiv \tilde{\mathcal{O}}(\tilde{\mathsf{B}}_1^{-1}\mathbf{x}' + \mathsf{B}_2^{-1}\mathbf{x}''), \tag{3.33}$$

the integral in (3.29) takes the form

$$\begin{aligned}\int\cdots\int dy_1\cdots dy_N \exp\left(-\frac{i}{2}\sum_{i=p+1}^{N}\delta_i y_i^2\right)\exp\left(i\sum_{i=1}^{N} y_i z_i\right) &= (2\pi)^N\prod_{i=1}^{p}\delta(z_i)\prod_{j=p+1}^{N}|\delta_j|^{-1} \\ &\quad\times \exp\,(-\tfrac{1}{4}i\pi\operatorname{sgn}\delta_j)\exp\left(\frac{i}{2}\frac{z_j^2}{\delta_j}\right), \end{aligned} \tag{3.34}$$

where we used the notation

$$\delta_i = (\operatorname{sgn}\delta_i)\,|\delta_i|. \tag{3.35}$$

Introducing then (3.34) into (3.29), we obtain explicitly the unitary representation of the product of two canonical transformations. In particular, if we want the unitary representation of a canonical transformation where B is singular, we just have to consider the product (2.13), i.e.,

$$\begin{aligned}&\mathsf{A}_2 = \mathsf{D}_2 = \mathsf{I},\quad \mathsf{B}_2 = \mathsf{B}',\quad \mathsf{C}_2 = 0,\quad \mathsf{A}_1 = \mathsf{A} - \mathsf{B}'\mathsf{C}, \\ &\mathsf{B}_1 = \mathsf{B} - \mathsf{B}'\mathsf{D},\quad \mathsf{C}_1 = \mathsf{C},\quad \mathsf{D}_1 = \mathsf{D}.\end{aligned} \tag{3.36}$$

We note from (3.34) that the unitary representation will contain a product of as many δ functions as the nullity (i.e., dimension minus rank) of the matrix B. In particular, if we are dealing with a point transformation in which from (2.20) we have

$$\mathsf{B}' = \mathsf{I},\quad \mathsf{D} = \tilde{\mathsf{A}}^{-1},\quad \mathsf{B} = \mathsf{C} = 0, \tag{3.37}$$

then we obtain

$$\langle \mathbf{x}'|\, U_1U_2\,|\mathbf{x}''\rangle = |\det \mathsf{A}|^{\frac{1}{2}}\,\delta(\mathsf{A}\mathbf{x}' - \mathbf{x}''). \tag{3.38}$$

This last result is to be expected, as for a unimodular A (i.e., $\det \mathsf{A} = 1$) we see that the unitary representation transforms àn arbitrary wavefunction $\psi(\mathbf{x}')$ into

$$\int \psi(\mathbf{x}')\, d\mathbf{x}'\,\langle \mathbf{x}'|\, U\,|\mathbf{x}''\rangle = \psi(\mathsf{A}^{-1}\mathbf{x}''). \tag{3.39}$$

We now return to the product of two canonical transformations (3.28) but assume that B is nonsingular. We note then that the integral (3.34) takes the form

$$(2\pi)^N\left(\prod_{i=1}^{N}|\delta_i|^{-1}\right) \times \exp\left(-\tfrac{1}{4}i\pi\sum_i \operatorname{sgn}\delta_i\right)\exp\left(\tfrac{1}{2}i\sum_i (z_i^2/\delta_i)\right). \tag{3.40}$$

But from (3.33) we obtain

$$\begin{aligned}\sum_{i=1}^{N}\frac{z_i^2}{\delta_i} &= \sum_{i=1}^{N}[(\tilde{\mathbf{x}}'\mathsf{B}_1^{-1} + \tilde{\mathbf{x}}''\tilde{\mathsf{B}}_2^{-1})\mathcal{O}]_i\frac{1}{\delta_i}[\tilde{\mathcal{O}}(\tilde{\mathsf{B}}_1^{-1}\mathbf{x}' + \mathsf{B}_2^{-1}\mathbf{x}'')]_i \\ &= (\tilde{\mathbf{x}}'\mathsf{B}_1^{-1} + \tilde{\mathbf{x}}''\tilde{\mathsf{B}}_2^{-1})\mathcal{O}\boldsymbol{\Delta}^{-1}\tilde{\mathcal{O}}(\tilde{\mathsf{B}}_1^{-1}\mathbf{x}' + \mathsf{B}_2^{-1}\mathbf{x}'') \\ &= (\tilde{\mathbf{x}}'\mathsf{B}_1^{-1} + \tilde{\mathbf{x}}''\tilde{\mathsf{B}}_2^{-1})\mathsf{B}_1\mathsf{B}^{-1}\mathsf{B}_2(\tilde{\mathsf{B}}_1^{-1}\mathbf{x}' + \mathsf{B}_2^{-1}\mathbf{x}''), \end{aligned} \tag{3.41}$$

so that the unitary representation of the product of two canonical transformations for which B_1 and B_2 are nonsingular becomes in this case

$$\begin{aligned}\langle \mathbf{x}'|\, U_1U_2\,|\mathbf{x}''\rangle &= \exp\left(-\tfrac{1}{4}i\pi\sum_i \operatorname{sgn}\delta_i\right) \\ &\quad\times [(2\pi)^N|\det \mathsf{B}_1|\cdot|\det \mathsf{B}_2|\cdot|\det\boldsymbol{\Delta}|]^{-\frac{1}{2}} \\ &\quad\times \exp\{-\tfrac{1}{2}i[\tilde{\mathbf{x}}'\mathsf{B}_1^{-1}\mathsf{A}_1\mathbf{x}' + \tilde{\mathbf{x}}''\mathsf{D}_2\mathsf{B}_2^{-1}\mathbf{x}'' \\ &\quad - (\tilde{\mathbf{x}}'\mathsf{B}_1^{-1} + \tilde{\mathbf{x}}''\tilde{\mathsf{B}}_2^{-1})\mathsf{B}_1\mathsf{B}^{-1}\mathsf{B}_2(\tilde{\mathsf{B}}_1^{-1}\mathbf{x}' + \mathsf{B}_2^{-1}\mathbf{x}'')]\}. \end{aligned} \tag{3.42}$$

Making use of some of the relations between the submatrices A, B, C, and D discussed in Sec. 2, as well as of the definition (3.28), we finally obtain

$$\langle \mathbf{x}'|\, U_1U_2\,|\mathbf{x}''\rangle = \exp\left(-\tfrac{1}{4}i\pi\sum_i \operatorname{sgn}\delta_i\right)\langle \mathbf{x}'|\, U\,|\mathbf{x}''\rangle. \tag{3.43}$$

The matrix elements of U_1, U_2, and U, in a representation in which the coordinates are diagonal, are given by (3.27) when we substitute in it the corresponding canonical transformation.

As there is no way of making the phase factor disappear in (3.43) by multiplying the unitary representation (3.27) by an appropriate constant phase factor, this representation then constitutes a ray representation[8] of the $2N$-dimensional symplectic group. We have even explicitly obtained in (3.43) the phase of this ray representation in the case when the submatrices B of all the symplectic matrices (2.8) are nonsingular.

4. LINEAR CANONICAL TRANSFORMATIONS AND THE DYNAMICAL GROUP OF THE OSCILLATOR

The set of real $2N$-dimensional matrices (2.8) that satisfy the conditions (2.10) constitute the symplectic group $Sp(2N)$. A subgroup of $Sp(2N)$ is formed by the matrices (2.8) that are also orthogonal, which besides (2.10) satisfy

$$\begin{pmatrix}\tilde{\mathsf{A}} & \tilde{\mathsf{C}}\\ \tilde{\mathsf{B}} & \tilde{\mathsf{D}}\end{pmatrix}\begin{pmatrix}\mathsf{A} & \mathsf{B}\\ \mathsf{C} & \mathsf{D}\end{pmatrix}=\begin{pmatrix}\tilde{\mathsf{A}}\mathsf{A}+\tilde{\mathsf{C}}\mathsf{C} & \tilde{\mathsf{A}}\mathsf{B}+\tilde{\mathsf{C}}\mathsf{D}\\ \tilde{\mathsf{B}}\mathsf{A}+\tilde{\mathsf{D}}\mathsf{C} & \tilde{\mathsf{B}}\mathsf{B}+\tilde{\mathsf{D}}\mathsf{D}\end{pmatrix}=\begin{pmatrix}\mathsf{I} & 0\\ 0 & \mathsf{I}\end{pmatrix}. \tag{4.1}$$

The relations (2.10) and (4.1) can also be expressed in terms of the complex matrices

$$\mathcal{U}\equiv\mathsf{A}+i\mathsf{B},\quad \mathcal{U}^*\equiv\mathsf{A}-i\mathsf{B},\quad \mathcal{V}\equiv\mathsf{D}-i\mathsf{C},\quad \mathcal{V}^*\equiv\mathsf{D}+i\mathsf{C}, \tag{4.2}$$

and we showed in another publication[9] that they are satisfied if

$$\mathcal{V}=\mathcal{U}\text{ and }\mathcal{U}\text{ unitary, i.e., }\mathcal{U}^*\mathcal{U}=\mathsf{I}. \tag{4.3}$$

Thus the subgroup of orthogonal linear canonical transformations is actually a representation of the N-dimensional unitary group $\mathcal{U}(N)$ whose elements are

$$\begin{pmatrix}\frac{1}{2}(\mathcal{U}+\mathcal{U}^*) & -\frac{1}{2}i(\mathcal{U}-\mathcal{U}^*)\\ \frac{1}{2}i(\mathcal{U}-\mathcal{U}^*) & \frac{1}{2}(\mathcal{U}+\mathcal{U}^*)\end{pmatrix}. \tag{4.4}$$

We can construct the elements of the Lie algebra of $Sp(2N)$ and its subgroup $\mathcal{U}(N)$ in terms of bilinear expressions in coordinates and momenta.[5] It is more convenient to do this in terms of the creation and annihilation operators defined as usual by

$$\eta_i\equiv 2^{-\frac{1}{2}}(x_i-ip_i),\quad \xi_i\equiv 2^{-\frac{1}{2}}(x_i+ip_i),\quad i=1,\cdots,N, \tag{4.5}$$

whose commutation relations are

$$[\eta_i,\eta_j]=[\xi_i,\xi_j]=0,\quad [\xi_i,\eta_j]=\delta_{ij}. \tag{4.6}$$

We consider the $N(2N+1)$ bilinear operators

$$H_i=\tfrac{1}{2}(\eta_i\xi_i+\xi_i\eta_i)\equiv\mathcal{C}_{ii}+\tfrac{1}{2},\quad i=1,\cdots,N, \tag{4.7a}$$

$$\eta_i\xi_j\equiv\mathcal{C}_{ij},\quad i\neq j,\quad i,j=1,\cdots,N, \tag{4.7b}$$

$$\eta_i\eta_j,\quad i\leq j=1,\cdots,N, \tag{4.7c}$$

$$\xi_i\xi_j,\quad i\leq j=1,\cdots,N. \tag{4.7d}$$

From (4.6), these operators close under commutation, and, if we obtain their root vectors with respect to the set of commuting operators H_i, we find that they are the generators[5] of the group $Sp(2N)$. The set of operators $\mathcal{C}_{ij}$, $i,j=1,\cdots,N$, of (4.7a) and (4.7b) also close under commutation, and their root vectors indicate that they are the generators of the $\mathcal{U}(N)$ group. A subset of this last set given by

$$\mathcal{C}_{ij}-\mathcal{C}_{ji}=i(x_ip_j-x_jp_i)=x_i\frac{\partial}{\partial x_j}-x_j\frac{\partial}{\partial x_i} \tag{4.8}$$

clearly[10] gives the generators of the $\mathcal{O}(N)$ subgroup of $\mathcal{U}(N)$.

We note that the H_i of (4.7a) are just

$$H_i=\tfrac{1}{2}(\eta_i\xi_i+\xi_i\eta_i)=\tfrac{1}{2}(p_i^2+x_i^2), \tag{4.9}$$

and the Hamiltonian of the N-dimensional harmonic oscillator is given by

$$H=\sum_{i=1}^{N}H_i=\tfrac{1}{2}\sum_{i=1}^{N}(p_i^2+x_i^2). \tag{4.10}$$

Thus the group $Sp(2N)$, whose generators are the operators (4.7), is the dynamical group of the N-dimensional oscillator. The group $\mathcal{U}(N)$, whose generators are the $\mathcal{C}_{ij}$ of (4.7a) and (4.7b), is the symmetry group of the harmonic oscillator, as can be seen directly because

$$[\mathcal{C}_{ij},H]=0 \tag{4.11}$$

and also from the fact that the orthogonal group of canonical transformations clearly leaves H invariant.

In applications[6] we usually deal with several particles, say n, in an oscillator potential of definite number of dimensions q so that $N=nq$. Denoting by $\mu=1,\cdots,q$ the component and by $s=1,\cdots,n$, the particle indices, we can now express the coordinates and momenta as well as the creation and annihilation operators in the following notation:

$$\mathbf{x}_s=\{x_{\mu s}\},\quad \mathbf{p}_s=\{p_{\mu s}\},\quad \boldsymbol{\eta}_s=\{\eta_{\mu s}\},\quad \boldsymbol{\xi}_s=\{\xi_{\mu s}\}. \tag{4.12}$$

Clearly we can in (4.7) substitute i by μs and contract with respect to the component index μ, thus getting the operators

$$\mathcal{H}_s = \tfrac{1}{2}(\boldsymbol{\eta}_s \cdot \boldsymbol{\xi}_s + \boldsymbol{\xi}_s \cdot \boldsymbol{\eta}_s), \quad s = 1, \cdots, n, \tag{4.13a}$$

$$\boldsymbol{\eta}_s \cdot \boldsymbol{\xi}_t, \quad s \neq t, \quad s, t = 1, \cdots, n, \tag{4.13b}$$

$$\boldsymbol{\eta}_s \cdot \boldsymbol{\eta}_t, \quad s \leq t = 1, \cdots, n, \tag{4.13c}$$

$$\boldsymbol{\xi}_s \cdot \boldsymbol{\xi}_t, \quad s \leq t = 1, \cdots, n. \tag{4.13d}$$

From (4.6) these operators close under commutation and have the same type of root vectors with respect to the set of commuting operators $\mathcal{H}_s$ as we had previously for the operators (4.7) with respect to H_i. Thus the operators (4.13) are the generators of a group $Sp(2n)$ which is a subgroup of $Sp(2N) = Sp(2nq)$.

We can also contract the operators (4.7) with respect to index s, obtaining, for example,

$$\mathcal{C}_{\mu\nu} \equiv \sum_{s=1}^{n} \eta_{\mu s}\xi_{\nu s}, \tag{4.14}$$

which by a similar reasoning as above will be the generators of a group $\mathcal{U}(q)$. The antisymmetrized part of these generators, i.e.,

$$\Lambda_{\mu\nu} \equiv \mathcal{C}_{\mu\nu} - \mathcal{C}_{\nu\mu} \quad \mu, \nu = 1, \cdots, q, \tag{4.15}$$

constitute as before the generators of an orthogonal group $\mathcal{O}(q)$. It is clear that the $\Lambda_{\mu\nu}$ and the generators of $Sp(2n)$ commute, as the latter are by construction invariant under rotations in the q-dimensional space. Thus, for the problem of n particles in a q-dimensional harmonic oscillator potential, we have the following chain of groups:

$$Sp(2nq) \supset Sp(2n) \times \mathcal{O}(q). \tag{4.16}$$

It is interesting to see in which way the subgroups $\mathcal{O}(q)$ and $Sp(2n)$ act on the coordinates $x_{\mu s}$ and momenta $p_{\mu s}$. Clearly for $\mathcal{O}(q)$ we have the orthogonal transformation $\|\mathcal{O}_{\mu\nu}\|$,

$$\bar{x}_{\mu s} = \sum_{\nu} \mathcal{O}_{\mu\nu} x_{\nu s}, \quad \bar{p}_{\mu s} = \sum_{\nu} \mathcal{O}_{\mu\nu} p_{\nu s}, \tag{4.17}$$

while for $Sp(2n)$ we obtain

$$\bar{x}_{\mu s} = \sum_{t=1}^{n} a_{st} x_{\mu t} + \sum_{t=1}^{n} b_{st} p_{\mu t},$$
$$\bar{p}_{\mu s} = \sum_{t=1}^{n} c_{st} x_{\mu t} + \sum_{t=1}^{n} d_{st} p_{\mu t}, \tag{4.18}$$

with the $2n$-dimensional matrix of the type (2.8) satisfying again the restrictions (2.10).

The set of all states of an N-dimensional harmonic oscillator belong to one of two irreducible representations of the group $Sp(2N)$. This can be seen as follows: First all states can be expressed as homogeneous polynomials of degree $r = 0, 1, \cdots$ in the creation operators η_i acting on the ground state $|0\rangle$. Using the generators $\eta_i\xi_j$, $i > j$, repeatedly, we can transform these states into

$$(r!)^{-\frac{1}{2}}\eta_N^r |0\rangle. \tag{4.19}$$

Applying then the generator ξ_N^2, we finally convert them either in

$$|0\rangle \; r \text{ even or } \eta_N |0\rangle \; r \text{ odd}. \tag{4.20}$$

The two IR's are then characterized by the eigenvalues of the N weight generators (4.7a) corresponding to the minimum weight states (4.20), i.e.,

$$[\tfrac{1}{2} \cdots \tfrac{1}{2}] = [(\tfrac{1}{2})^N] \quad \text{or} \quad [\tfrac{1}{2}\tfrac{1}{2} \cdots \tfrac{3}{2}] = [(\tfrac{1}{2})^{N-1}\tfrac{3}{2}]. \tag{4.21}$$

This result continues to hold when we have n particles in a q-dimensional oscillator in which case $N = nq$.

Now the chain of groups (4.16), which characterizes states defined by boson creation operators acting on the ground state, looks very similar to the corresponding problem for fermions[11] of spin $\frac{1}{2}$ in a given shell of angular momentum l. We briefly review the fermion case to establish the parallelism in detail. Assume that we have n different types of fermions (e.g., $n = 2$ if we have both protons and neutrons) in a shell of orbital angular momentum l. We can then define the indices

$$\mu \equiv m\sigma, \quad m = l, \cdots, -l, \quad \sigma = \tfrac{1}{2}, -\tfrac{1}{2},$$
$$s = 1, \cdots, n, \tag{4.22}$$

in which μ can take $4l + 2$ values. The fermion creation and annihilation operators can then be denoted respectively by

$$b^+_{\mu s}, \quad b^{\mu s}. \tag{4.23}$$

The bilinear operators

$$b^+_{\mu s} b^{\mu' s'}, \tag{4.24a}$$

$$b^+_{\mu s} b^+_{\mu' s'}, \tag{4.24b}$$

$$b^{\mu s} b^{\mu' s'} \tag{4.24c}$$

then constitute the generators of an $\mathcal{O}[2n(4l + 2)]$ group.[11] If we contract with respect to the μ index [keeping in mind a phase factor $(-1)^{l+m+\frac{1}{2}+\sigma}$ for (4.24b) and (4.24c)], we get the generators of an $Sp_u(2n)$ group.[12] The u index here means we are dealing with the compact symplectic group (a subgroup of a unitary rather than of a linear group) since all representations we have for a fermion system are finite dimensional due to the fact that the Pauli principle limits the total number of states[13] to 2^{4l+2}.

Had we, on the other hand, contracted the operators (4.24a) with respect to the s index, we would have obtained the generators of a $(4l + 2)$-dimensional unitary group $\mathcal{U}(4l + 2)$, whose compact symplectic subgroup $Sp_u(4l + 2)$ has the generators

$$\sum_s [b^+_{m\sigma s}b^{m'\sigma' s} + (-1)^{m+m'+\sigma+\sigma'}b^+_{-m'-\sigma' s}b^{-m-\sigma s}]. \quad (4.25)$$

The generators of $Sp_u(2n)$ and $Sp_u(4l + 2)$ clearly commute with each other since the former by construction are invariant under the transformations of the latter. Thus in the fermion case we have the chain of groups

$$\mathcal{O}(2nq) \supset Sp_u(2n) \times Sp_u(q), \quad q = 4l + 2. \quad (4.26)$$

If we have just one type of particles, i.e., $n = 1$, the group $Sp_u(2)$ is identical to $SU(2)$ and the group $O(3)$, which is homeomorphic to it, is the usual quasispin group. For two types of particles, e.g., protons and neutrons, $Sp_u(4)$, which is isomorphic to $\mathcal{O}(5)$, is the generalized quasispin discussed by Hecht[14] and others.

In the fermion case all states belong to one of two IR's of $\mathcal{O}^+(2nq)$,

$$[\tfrac{1}{2} \cdots \tfrac{1}{2}\tfrac{1}{2}] \quad \text{or} \quad [\tfrac{1}{2} \cdots \tfrac{1}{2} - \tfrac{1}{2}], \quad (4.27)$$

which parallels the result (4.21) for bosons. The IR's of the subgroups $Sp_u(2n)$ and $Sp_u(q)$ for a given IR (4.27) of $\mathcal{O}^+(2nq)$ are complementary in the sense that they are in one-to-one correspondence as defined in Ref. 13. In particular this correspondence for $n = 1$ gives the relation between quasispin and seniority. In the boson chain (4.16) the IR's of $\mathcal{O}(q)$ are characterized by the partitions

$$[\lambda_1\lambda_2 \cdots \lambda_{\frac{1}{2}q}] \quad \text{or} \quad [\lambda_1\lambda_2 \cdots \lambda_{\frac{1}{2}(q-1)}], \quad (4.28)$$

depending on whether q is even or odd. As was shown by Chacón,[15] these partitions also characterize completely the IR's of $Sp(2n)$. Thus the groups $Sp(2n)$ and $\mathcal{O}(q)$ are complementary in the same sense as $Sp_u(2n)$ and $Sp_u(q)$ were complementary in the fermion problem. We note though that in the boson problem $Sp(2n)$ is a noncompact group and so its IR's are infinite dimensional, as we have an infinite number of harmonic oscillator states corresponding to a definite IR of $\mathcal{O}(q)$.

The group $Sp(2n)$ plays then, with respect to the boson operators associated with particles in an harmonic oscillator potential, a role similar to the one the generalized quasispin has with respect to Fermi operators. In the following paper we use this quasispin for bosons[16] in the evaluation of one-particle matrix elements with harmonic oscillator states, in a way that parallels the use of the fermion quasispin by Lawson and MacFarlane[17] for a similar problem. Later we plan to extend this viewpoint to more than one particle.

When dealing with the problem of n particles in a q-dimensional oscillator potential, we are also interested in the unitary representation of the group $Sp(2n)$ of canonical transformations (4.18). Clearly, in the case when $\mathsf{B} = \|b_{st}\|$ is nonsingular, we can immediately generalize the reasoning of Sec. 3, and obtain that the unitary representation in the basis in which the coordinates $\mathbf{x}_s$ are diagonal is, in the vector notation (4.12), given by

$$\begin{aligned}\langle \mathbf{x}' | \, U \, | \mathbf{x}'' \rangle &= [(2\pi)^n \, |\det \mathsf{B}|]^{-\frac{1}{2}} \exp \Big(-\tfrac{1}{2}i \sum_{s,t} [\tilde{\mathbf{x}}'_s(\mathsf{B}^{-1}\mathsf{A})_{st}\mathbf{x}'_t \\ &\quad - 2\tilde{\mathbf{x}}'_s(\mathsf{B}^{-1})_{st}\mathbf{x}''_t + \tilde{\mathbf{x}}''_s(\mathsf{D}\mathsf{B}^{-1})_{st}\mathbf{x}''_t]\Big). \quad (4.29)\end{aligned}$$

A particular case is that of a single particle ($n = 1$) in a three-dimensional ($q = 3$) harmonic oscillator. Designating by $\mathbf{r} = (x_1x_2x_3)$ the position vector, we have that the unitary representation of $Sp(2)$ in a scheme where the coordinates are diagonal is

$$\begin{aligned}\langle \mathbf{r}' | \, U \, | \mathbf{r}'' \rangle = (2\pi \, |b|)^{-\frac{3}{2}} \exp [-(i/2b)(ar'^2 - 2\mathbf{r}' \cdot \mathbf{r}'' \\ + dr''^2)]. \quad (4.30)\end{aligned}$$

If we want the matrix U in a scheme in which the Hamiltonian H, angular momentum L^2, and projection L_z are diagonal, we can obtain it from (4.30) with the help of the relation

$$\begin{aligned}&\langle n'l'm' | \, U \, | n''l''m'' \rangle \\ &= \iint \langle n'l'm' \mid \mathbf{r}' \rangle \, d\mathbf{r}' \langle \mathbf{r}' | \, U \, | \mathbf{r}'' \rangle \, d\mathbf{r}'' \langle \mathbf{r}'' \mid n''l''m'' \rangle, \quad (4.31)\end{aligned}$$

where $\langle \mathbf{r} \mid nlm \rangle$ is the three-dimensional harmonic oscillator wavefunction and $\langle nlm \mid \mathbf{r} \rangle$ its conjugate. Using the relations[18,19] for Laguerre polynomials,

$$\begin{aligned}&\int_0^\infty x^{\nu+1}e^{-\beta x^2}L_n(\alpha x^2)J_\nu(xy)\,dx \\ &\quad = 2^{-\nu-1}\beta^{-\nu-n-1}(\beta - \alpha)^n y^\nu e^{-y^2/4\beta}L_n^{\nu}\left(\frac{\alpha y^2}{4\beta(\alpha - \beta)}\right), \quad (4.32)\end{aligned}$$

$$\begin{aligned}L_n^{l+\frac{1}{2}}(\mu x^2) = \sum_{m=0}^{n} \frac{[\Gamma(n + l + \frac{3}{2})]^2}{m! \, [\Gamma(n - m + l + \frac{3}{2})]^2} \\ \times \mu^{n-m}(1 - \mu)^m L_{n-m}^{l+\frac{1}{2}}(x^2), \quad (4.33)\end{aligned}$$

we obtain straightforwardly that the unitary representation (4.31), which is clearly diagonal in l, m and independent of m, has the explicit form

$$\begin{aligned}\langle n'lm|\, U\, |n''lm\rangle &= i^l[n'!\, n''!\, \Gamma(n' + l + \tfrac{3}{2})\Gamma(n'' + l + \tfrac{3}{2})]^{\frac{1}{2}} \\ &\quad\times (b + ia)^{-n'-l-\frac{3}{2}}(-b + ia)^{n'}\gamma^{-l-\frac{3}{2}} \\ &\quad\times \sum_p \{[p!\,(n' - p)!\,(n'' - p)!\,\Gamma(p + l + \tfrac{3}{2})]^{-1} \\ &\quad\times [\gamma^2(a^2 + b^2)]^{-p}[1 - \gamma^{-1}(a^2 + b^2)^{-1}]^{n'-p} \\ &\quad\times (1 - \gamma^{-1})^{n''-p}\}, \qquad (4.34)\end{aligned}$$

where γ is given by

$$\begin{aligned}\gamma = \{&b(1 + a^2 + b^2) \\ &-i[a - d(a^2 + b^2)]\}\,[2b(a^2 + b^2)]^{-1}. \qquad (4.35)\end{aligned}$$

If we consider an element of $Sp(2)$ that belongs also to the orthogonal subgroup $\mathcal{O}(2)$ of this group, i.e.,

$$a = d = \cos\varphi, \quad b = -c = \sin\varphi, \qquad (4.36)$$

we have $a^2 + b^2 = 1$ and $\gamma = 1$, and thus the unitary representation (4.34) simplifies drastically, taking the form

$$\langle n'lm|\, U\, |n''lm\rangle = \delta_{n'n''}\, i^{-\frac{3}{2}} e^{i(2n'+l+\frac{3}{2})\varphi}. \qquad (4.37)$$

This result we, of course, expect as the state $|nlm\rangle$ can be written as an homogeneous[9] polynomial of degree $2n + l$ in the creation operators $\boldsymbol{\eta}$ acting on the ground state. The linear canonical transformation (4.36) implies then

$$\bar{\boldsymbol{\eta}} = e^{i\varphi}\boldsymbol{\eta}, \qquad (4.38)$$

and thus the state $|nlm\rangle$ transforms, up to a constant phase, in the way indicated by the unitary representation (4.37).

* Work supported by Comisión Nacional de Energía Nuclear (México).

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JOURNAL OF MATHEMATICAL PHYSICS VOLUME 12, NUMBER 8 AUGUST 1971

Canonical Transformations and Matrix Elements*

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(Received 22 January 1971)

We use the ideas on linear canonical transformations developed previously to calculate the matrix elements of the multipole operators between single-particle states in a three-dimensional oscillator potential. We characterize first the oscillator states in the chain of groups $Sp(6) \supset Sp(2) \times \mathcal{O}(3)$, $Sp(2) \supset \mathcal{O}_S(2)$, and $\mathcal{O}(3) \supset \mathcal{O}_L(2)$, and then expand the multipole operators in terms of irreducible tensors with respect to the $Sp(2) \times \mathcal{O}(3)$ group. Their matrix elements are obtained by applying the Wigner–Eckart theorem with respect to both the $Sp(2)$ and $\mathcal{O}(3)$ groups. In this way an explicit expression for the radial integral of r^k, $k > 0$, is obtained.

1. INTRODUCTION

While canonical transformations play a fundamental role in the solution of problems of classical and quantum mechanics, their application, in the latter case, to the evaluation of matrix elements has not been fully explored. In the preceding paper[1] we discussed the linear canonical transformations and showed that the symplectic group which they form is the dynamical group of the harmonic oscillator. We wish in this paper to make use of this group and its subgroups in the evaluation of the matrix elements of the multipole operators

$$\mathcal{Y}_{kt}(\mathbf{r}) \equiv r^k Y_{kt}(\theta, \varphi) \qquad (1.1)$$