

# Applications of Wavelets to Quantum Mechanics: a Pedagogical Example

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## Abstract

We discuss in many details two quantum mechanical models of planar electrons which are very much related to the Fractional Quantum Hall Effect.

In particular, we discuss the localization properties of the trial ground states of the models starting from considerations on the numerical results on the energy. We conclude that wavelet theory can be conveniently used in the description of the system.

Finally we suggest applications of our results to the Fractional Quantum Hall Effect.

# 1 Introduction

In these recent years a great effort has been done to find a wave function which minimizes the energy of a two-dimensional system of electrons subjected to a strong constant magnetic field applied perpendicularly to the sample, independently of the electron density. This is, in fact, the first step to understand the main features of the Fractional Quantum Hall Effect (FQHE). Many trial ground states have been proposed so far, none of which has revealed to explain all the experimental data: the most successful is the one proposed by Laughlin, [1] and [2], which describes an incompressible fluid (which therefore carries current without losing energy) whose static energy is very low.

Totally different is the wave function proposed by Morchio, Strocchi and the author in reference [3]. The authors, following the same line of other authors, [4]-[5], consider the system of electron essentially as a two-dimensional crystal. This crystal is built by first considering a gaussian (a coherent state) centered at the origin, and then by "moving" this gaussian along the sites of a triangular lattice. The wave function of the finite volume system is the Slater determinant of the single electron wave functions centered in the relevant lattice sites. The details can be found in [3] where it is also shown that, for low electron densities, the energy of this state is lower than the one obtained by Laughlin's state. However the theoretical value of the 'critical density' at which the crystal phase appears favored with respect to the liquid one is slightly different from the value given by the experiments, see [6]. Therefore, even if a crystal phase is expected, its wave function must be refined.

In this paper we discuss a pedagogical model which suggests in which way one can modify the wave function in [3] to lower the energy, so to explain the experimental data. The idea essentially consists in modifying the single electron wave function trying to achieve a better electron localization. In fact, we expect that the most localized the electron wave function is, the lowest is the result obtained for the Coulomb energy if the electrons are originally localized around different spatial points. This claim follows both

from classical and quantum considerations, see [7] and [3].

The paper is organized as follows:

in Section 2 we introduce a physical model whose ground level is infinitely degenerate (like the one of FQHE). In this way the ground state is not fixed a priori. We construct different trial ground states using the Haar, the Littlewood-Paley and the harmonic oscillator bases. We also discuss their localization properties.

In Section 3 we slightly modify the model previously introduced by fixing the mean positions of the electrons around lattice sites. Then we construct the new basis and we discuss how to compute the energies of the Coulomb interaction in these different bases for both the models considered.

In Section 4, we give and comment the numerical results.

In the Appendix, we introduce other models which can be treated with analogous techniques, and in particular, we show that the FQHE belongs to this class of models.

## 2 The model

Let us consider a system of  $N$  electrons living in a two-dimensional device. We divide the hamiltonian in a single body contribution plus a two-body term:

$$H^{(N)} = \sum_{i=1}^N H_0(i) + \frac{1}{2} \sum_{i \neq j}^N \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}. \quad (2.1)$$

We observe that no background subtraction is considered in  $H^{(N)}$ . This is meaningful only if we restrict to a finite number of electrons, that is, if we keep  $N$  to be finite, otherwise the Coulomb energy diverges when  $N \rightarrow \infty$ . Since we are not going to compute the true energy of the system, but only a two-body contribution, we do not need at this stage to introduce the positive background.

The form of  $H_0(i)$  is chosen here in a convenient way. In fact, we are

more interested in the analogies of the model in (2.1) with the FQHE than to its physical relevance. Therefore, for reasons that will appear clear in the following, we take each  $H_0(i)$  of the form

$$H_0 = \frac{1}{2}(p_x^2 + x^2) + \frac{1}{2}p_y^2 + p_x p_y. \quad (2.2)$$

Therefore, each electron behaves like an harmonic oscillator in  $x$ , is free in  $y$ , and is also subjected to a 'strange' potential which is proportional to the momentum of the electron. We also see that the  $z$ -component does not appear in  $H_0(i)$  (which reflects the fact that the device is two-dimensional).

It is easy to verify that the following canonical transformation

$$\begin{aligned} Q &\equiv p_x + p_y; & P &= -x; \\ Q' &\equiv p_y; & P' &= x - y, \end{aligned} \quad (2.3)$$

preserves the commutation relations,

$$[Q, P] = [Q', P'] = i \quad [Q, P'] = [Q, Q'] = [P, P'] = [P, Q'] = 0,$$

and in this sense it is *canonical*, see [8], and that in the new variables  $H_0$  take the form

$$H_0 = \frac{1}{2}(Q^2 + P^2), \quad (2.4)$$

so that  $Q'$  and  $P'$  disappear from the definition of  $H_0$ .

To discuss the ground energy of the hamiltonian (2.1) we follow the same steps as in [3]: first, we find the ground state of the single electron unperturbed hamiltonian  $H_0$ . Then, we built up the trial ground state of the  $N$ -electrons unperturbed hamiltonian,  $\sum_i H_0(i)$ , as a Slater determinant of these single-electron wave functions. Finally, we compute the matrix element of the Coulomb interaction in this state. As widely discussed in [3], this procedure seems to be justified at least for small electron densities.

Before going on, let us show the relation between our pedagogical model and the FQHE. The FQHE is described by the same hamiltonian as in (2.1)

with a different  $H_0$ ,

$$H_0^F = \frac{1}{2}(p_x - y/2)^2 + \frac{1}{2}(p_y + x/2)^2,$$

which describes an electron subjected to a constant magnetic field perpendicular to the device. A canonical transformation similar to the one in (2.3), see [3] and references therein, transforms  $H_0^F$  in the same  $H_0$  (2.4). Therefore, even if  $H_0$  in (2.2) and  $H_0^F$  are different, they are both 'projected' into the same harmonic oscillator (2.4) by different canonical transformations. This difference appears explicitly in the integral transformation rule which relates the expression of the wave functions in the variables  $(x, y)$  and  $(Q, Q')$ , see [8]. The reason why we discuss this pedagogical example and not directly the FQHE is that this transformation rule is very simple for our  $H_0$ , while it is much more difficult for  $H_0^F$ , see [3]. However, a first real application of our approach in the context of FQHE can be found in [9].

From [8] we can easily find that, if  $\Psi(x, y)$  and  $\Phi(Q, Q')$  are respectively the wave functions in the ordinary space and in the 'canonically transformed' space, they are related by

$$\Psi(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dQ \int_{-\infty}^{\infty} dQ' \Phi(Q, Q') \exp^{i[Q'(y-x)+Qx]}. \quad (2.5)$$

To find out the single electron ground state of our model,  $\Psi_0(x, y)$ , it is therefore sufficient (actually equivalent) to find the ground state  $\Phi_0(Q, Q')$  of (2.4). Due to the particular form of this  $H_0$  we see that  $\Phi_0(Q, Q')$  can be factorized and that the dependence on  $Q$  is fixed: regarding the Coulomb interaction as a perturbation of  $H_0$ , it is reasonable to put

$$\Phi(Q, Q') = \frac{1}{\pi^{1/4}} \exp^{-Q^2/2} \phi(Q'). \quad (2.6)$$

Here the function  $\phi(Q')$  is totally free because the variable  $Q'$  does not appear in the hamiltonian (2.4) and the energy of the unperturbed system only depends on  $Q$ . Only the perturbation fix its form. An analogous phenomenon,

known as the *degeneracy of the Landau levels*, takes place for  $H_0^F$ , for the same reason. This explains why the ground state of the FQHE is not fixed.

We are now ready to discuss different choices of  $\phi(Q')$  and their projections in the configuration space via (2.5), comparing the resulting energies and localizations.

We start by considering the Littlewood-Paley orthonormal basis of wavelet. We refer to [10] and [11] for all what concerns the wavelet theory used in this paper.

The mother wavelet of this set is

$$L(x) = (\pi x)^{-1}(\sin(2\pi x) - \sin(\pi x)),$$

which, using the well known definition  $L_{mn}(x) \equiv 2^{-m/2}L(2^{-m}x - n)$ , generates an orthonormal set in  $L^2(\mathbf{R})$ . In particular, we will be interested in the subset  $\{L_m(x)\} \equiv \{L_{m0}(x)\}$ . The functions of this set obviously satisfy the orthonormality condition

$$\langle L_m, L_n \rangle = \delta_{mn}$$

and, when used in (2.5) as different choices for the function  $\phi(Q')$ , they give the following set of wave functions in the configuration space:

$$\Psi_m^{(LP)}(x, y) = \frac{2^{m/2}}{\sqrt{2\pi^{3/4}}} e^{-x^2/2} \chi_{D_x}(y). \quad (2.7)$$

Here  $\chi_{D_x}(y)$  is the characteristic function of the set  $D_x$ , which is equal to one if  $y \in D_x$  and zero otherwise. We have defined

$$D_x = [x - \frac{2\pi}{2^m}, x - \frac{\pi}{2^m}] \cup [x + \frac{\pi}{2^m}, x + \frac{2\pi}{2^m}]. \quad (2.8)$$

Due to the canonicity of the transformation (2.3) the functions of the set  $\{\Psi_m^{(LP)}(x, y) : m \in \mathbf{Z}\}$  are obviously mutually orthonormal. Moreover we see from (2.7) that they are very well localized in both  $x$  and  $y$ .

Another possible choice for the function  $\phi(Q')$  is any function belonging to the Haar wavelet set. The mother wavelet of this set is the function

$$H(x) = \begin{cases} 1, & \text{if } 0 \leq x < 1/2 \\ -1, & \text{if } 1/2 \leq x < 1 \\ 0, & \text{otherwise} \end{cases}$$

and the relevant set is defined in the usual way:  $\{H_m(x)\} \equiv \{H_{m0}(x)\} = \{2^{-m/2}H(2^{-m}x) : m \in \mathbf{Z}\}$ . This set is again orthonormal but the localization of each wave function is rather poor. From (2.5) we get

$$\Psi_m^{(H)}(x, y) = \frac{2^{-m/2}i}{\sqrt{2\pi^{3/4}}} \frac{e^{-x^2/2}}{(y-x)} (e^{i2^{m-1}(y-x)} - 1)^2 \quad (2.9)$$

which again decreases exponentially in  $x$  but goes like  $1/y$  in  $y$ . We do not expect therefore that the set  $\{\Psi_m^{(H)}(x, y) : m \in \mathbf{Z}\}$  can play a relevant role in the energy computation.

We end this section discussing another class of trial ground states of the hamiltonian  $H_0$ . This time we will take non-wavelet functions. In particular, we consider for  $\phi(Q')$  the first three eigenstates of the hamiltonian  $H_0 = \frac{1}{2}(Q'^2 + P'^2)$ , which are orthonormal, and compute the projections in the configuration space of the complete function  $\Phi(Q, Q')$  using the transformation rule (2.5). We easily find the following results:

$$\begin{aligned} \Psi_0^{(HO)}(x, y) &= \frac{1}{\sqrt{\pi}} e^{-(y^2+2x^2-2xy)/2} \\ \Psi_1^{(HO)}(x, y) &= i\sqrt{\frac{2}{\pi}} (y-x) e^{-(y^2+2x^2-2xy)/2} \\ \Psi_2^{(HO)}(x, y) &= \frac{1}{\sqrt{2\pi}} (1 - 2(y-x)^2) e^{-(y^2+2x^2-2xy)/2} \end{aligned} \quad (2.10)$$

All these wave functions, still mutually orthogonal, have a rather good localization in both  $x$  and  $y$ . In particular the best localized is, of course,  $\Psi_0^{(HO)}(x, y)$ .

The reason for considering also these wave functions in this paper is that the choice of the ground state of the harmonic oscillator, which gives here  $\Psi_0^{(HO)}(x, y)$ , is the one which allows the construction of the trial ground state used in the description of the FQHE, see [3]. It is therefore useful, in our opinion, to have a comparison between these different approaches.

### 3 Energy Computation and Wavelet Bases on a Lattice

We start this Section discussing the way in which the energy of the system can be computed. As a matter of fact, we are not going to compute the true energy of the system, since in any case this is not physically very interesting, but only a certain matrix element which is enough to get some relevant informations on the ground state of the model, since it contains the main contribution to the energy. First of all, we fix  $N = 2$  in (2.1) since, in any case, the total energy is essentially a sum of two-body contributions.

In any book of Many-Body Theory it is shown that the computation of the energy of an  $N$ -electrons system, in the Hartree-Fock approximation, is a (summation of the) difference of two contributions, called respectively the direct and the exchange terms. We should add to this difference also the ground energy of the kinetic Hamiltonian  $\sum_{i=1}^N H_0(i)$ . However, in our model, as well as in the FQHE, this contribution is constant, in the sense that it does not depend on the particular choice of the function  $\phi$  in (2.6). Therefore it will be neglected in all the future considerations.

For our two-electrons system the wave function is the following Slater determinant

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2!}}(\Psi_1(\mathbf{r}_1)\Psi_2(\mathbf{r}_2) - \Psi_2(\mathbf{r}_1)\Psi_1(\mathbf{r}_2)) \quad (3.1)$$

where  $\Psi_i(\mathbf{r}_j)$ ,  $i, j = 1, 2$ , are the single electron wave functions obtained in the previous Section, see (2.7), (2.9) and (2.10). The Coulomb energy  $E_c$  of



the system is therefore

$$E_c \equiv \int d^2\mathbf{r}_1 \int d^2\mathbf{r}_2 \frac{\Psi(\mathbf{r}_1, \mathbf{r}_2)^* \Psi(\mathbf{r}_1, \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} = V_d - V_{ex}$$

where

$$V_d = \int d^2\mathbf{r}_1 \int d^2\mathbf{r}_2 \frac{|\Psi_1(\mathbf{r}_1)|^2 |\Psi_2(\mathbf{r}_2)|^2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

and

$$V_{ex} = \int d^2\mathbf{r}_1 \int d^2\mathbf{r}_2 \frac{\Psi_1(\mathbf{r}_1)^* \Psi_2(\mathbf{r}_2)^* \Psi_1(\mathbf{r}_2) \Psi_2(\mathbf{r}_1)}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

are respectively the direct and the exchange term.

It is well known that, at least for localized wave functions, the exchange contribution is much smaller than the direct one. This feature is explicitly discussed, for instance, in [3], where these contributions are explicitly computed for the FQHE. Therefore in this paper we will not compute  $V_{ex}$ , since it is not expected to change significantly the numerical results. We will focus our attention only on the computation of the direct term  $V_d$ , which, with a little abuse of language, will be still often called the 'energy' of the system.

Before computing the expressions of  $V_d$  for the Littlewood-Paley and for the harmonic oscillator bases we use these same bases as starting points to introduce a 'natural' lattice in our model. The reason for doing this is again that a lattice is a natural structure for the FQHE at least for small electron densities, see [3]. Actually, since we are dealing with only two electrons, we will think of our lattice as two spatially not coincident points. We again refer to [3], and reference therein, for the details concerning the construction of the lattice associated to  $H_0$ .

Using the results of the previous Section it is easy to prove that the unitary operators

$$T_1 = e^{iQ'a} \quad T_2 = e^{iP'b} \quad (3.2)$$

both commute with  $H_0$  since they do not depend on  $Q$  and  $P$ . Moreover, they also commute with each other if  $ab = 2\pi N$ ,  $\forall N \in \mathbf{Z}$ . From the definition (2.3), one can also observe that, for any function  $f(x, y) \in L^2(\mathbf{R})$ ,

$T_1 f(x, y) = f(x, y + a)$  and  $T_2 f(x, y) = e^{i(x-y)b} f(x, y)$ . Therefore,  $T_2$  is simply a multiplication for a phase while  $T_1$  acts like a shift operator. This is enough for our present aim: we can take the two different sites of our lattice along the  $y$ -axis, with a 'lattice'-distance  $a = 2\pi$ .

In particular, defining

$$\Phi_m^{(LP)}(x, y) \equiv T_1 \Psi_m^{(LP)}(x, y) = \frac{2^{m/2}}{\sqrt{2\pi^{3/4}}} e^{-x^2/2} \chi_{D_{x-a}}(y) \quad (3.3)$$

from the Littlewood-Paley wavelets (2.7), and

$$\Phi_0^{(HO)}(x, y) \equiv T_1 \Psi_0^{(HO)}(x, y) = \frac{1}{\sqrt{\pi}} e^{-((y+a)^2 + 2x^2 - 2x(y+a))/2} \quad (3.4)$$

for the most localized function of the harmonic oscillator states, (2.10), we conclude that both  $\Phi_m^{(LP)}(x, y)$  and  $\Phi_0^{(HO)}(x, y)$  are eigenstates of  $H_0$  belonging to the ground level. This simply follows from the commutation rule  $[T_1, H_0] = 0$ .

It is also easy to verify that  $\langle \Psi_m^{(LP)}, \Phi_m^{(LP)} \rangle = 0$ ,  $\forall m \geq 1$ .

The situation is a bit different for the oscillator wave functions; the scalar product gives  $\langle \Psi_0^{(HO)}, \Phi_0^{(HO)} \rangle = e^{-\pi^2}$ . This implies that the Slater determinant is normalized within an error of  $e^{-2\pi^2} = O(10^{-9})$ . Therefore this extra contribution can safely be neglected here, and we will work with  $\Psi_0^{(HO)}$  and  $\Phi_0^{(HO)}$  as if they were mutually orthogonal.

We continue this Section manipulating  $V_d$  for two different models:

in the first one, which we call the "Non-Lattice Model", the electrons are both localized around the origin but they are described by different wave functions (this is necessary in order not to annihilate the Slater determinant (3.1));

in the second one, the "Lattice Model", the electrons are described by the same wave function localized around different space points. Of course, this is the model which is more similar to the FQHE as already discussed in [3], and in this perspective it has a particular interest.

We will omit the computation of the energy with the Haar basis  $\{\Psi_m^{(H)}(x, y)\}$  since it is not expected to be relevant for understanding the FQHE. This is because the wave functions  $\Psi_m^{(H)}(x, y)$  are the most delocalized function within the ones we have introduced in the previous Section, so that the Coulomb energy is expected to be bigger than the one obtained by the other bases.

### 3.1 Non-Lattice Model

We start with manipulating the expression of  $V_d$  for the basis in (2.7). From now on we will omit the index  $d$  since we will be concerned only with the direct contribution to the energy. Moreover, to explicitate the dependence on the quantum numbers  $m, n$  and on the basis, we put

$$V_{LP}^{m,n} = \int d^2\mathbf{r}_1 \int d^2\mathbf{r}_2 \frac{|\Psi_m^{(LP)}(\mathbf{r}_1)|^2 |\Psi_n^{(LP)}(\mathbf{r}_2)|^2}{|\mathbf{r}_1 - \mathbf{r}_2|} \quad (3.5)$$

where  $m \neq n$  because of the Pauli principle. The integration in  $y_1$  can be easily performed. After some manipulation and change of variables we can also perform the integration in  $x_1$  and we obtain

$$V_{LP}^{m,n} = \frac{2^{m+n}}{4\pi^3} \sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} dx e^{-x^2/2} \int_{x+\pi/2^n}^{x+2\pi/2^n} dt \log[\phi_{mn}(x, t)] \quad (3.6)$$

where we have defined the following function

$$\begin{aligned} \phi_{mn}(x, t) \equiv & \frac{(t - \frac{\pi}{2^m} + \sqrt{x^2 + (t - \frac{\pi}{2^m})^2})(t + \frac{2\pi}{2^m} + \sqrt{x^2 + (t + \frac{2\pi}{2^m})^2})}{(t + \frac{\pi}{2^m} + \sqrt{x^2 + (t + \frac{\pi}{2^m})^2})(t - \frac{2\pi}{2^m} + \sqrt{x^2 + (t - \frac{2\pi}{2^m})^2})} \times \\ & \times \frac{(t - \pi(\frac{3}{2^n} + \frac{1}{2^m}) + \sqrt{x^2 + (t - \pi(\frac{3}{2^n} + \frac{1}{2^m}))^2})}{(t - \pi(\frac{3}{2^n} - \frac{1}{2^m}) + \sqrt{x^2 + (t - \pi(\frac{3}{2^n} - \frac{1}{2^m}))^2})} \times \\ & \times \frac{(t - \pi(\frac{3}{2^n} - \frac{2}{2^m}) + \sqrt{x^2 + (t - \pi(\frac{3}{2^n} - \frac{2}{2^m}))^2})}{(t - \pi(\frac{3}{2^n} + \frac{2}{2^m}) + \sqrt{x^2 + (t - \pi(\frac{3}{2^n} + \frac{2}{2^m}))^2})} \end{aligned}$$

It is possible to see that the above integral is certainly defined for all  $n$  different from  $m \pm 1$ . The integration can be easily performed numerically and the results are discussed in Section 4.

To compute the energy for the harmonic oscillator wave functions (2.10) it is better to use the following equality:

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{1}{2\pi} \int \frac{d^2k}{|\mathbf{k}|} e^{-i\mathbf{k} \cdot (\mathbf{r}_1 - \mathbf{r}_2)}$$

In this way the integrations in  $\mathbf{r}_1$  and  $\mathbf{r}_2$  in  $V_d$  are reduced to gaussian integrals and therefore can be easily performed. Calling  $V_{ho}^{i,j}$  the 'energies' related to the wave functions in (2.10), we find:

$$V_{ho}^{0,1} = \frac{1}{2\pi} \int \frac{d^2k}{|\mathbf{k}|} \left(1 - \frac{k_y^2}{2}\right) e^{-(k_x^2 + 2k_y^2 + 2k_x k_y)/2} \quad (3.7)$$

$$V_{ho}^{0,2} = \frac{1}{4\pi} \int \frac{d^2k}{|\mathbf{k}|} k_y^2 \cdot \left(2 - \frac{k_y^2}{4}\right) e^{-(k_x^2 + 2k_y^2 + 2k_x k_y)/2} \quad (3.8)$$

$$V_{ho}^{1,2} = \frac{1}{4\pi} \int \frac{d^2k}{|\mathbf{k}|} \left(1 - \frac{k_y^2}{2}\right) \cdot k_y^2 \cdot \left(2 - \frac{k_y^2}{4}\right) e^{-(k_x^2 + 2k_y^2 + 2k_x k_y)/2} \quad (3.9)$$

### 3.2 Lattice Model

In this Subsection we use the wave functions (3.3) and (3.4) obtained using the shift operator  $T_1$ . We start considering the Littlewood-Paley basis.

As we have already said this time the energy is computed using the same wave function centered in different lattice sites. Therefore it depends only on a quantum number,  $m$ .

We call  $V_{LP}^{(m)}$  this energy

$$V_{LP}^{(m)} = \int d^2\mathbf{r}_1 \int d^2\mathbf{r}_2 \frac{|\Psi_m^{(LP)}(\mathbf{r}_1)|^2 |\Phi_m^{(LP)}(\mathbf{r}_2)|^2}{|\mathbf{r}_1 - \mathbf{r}_2|}. \quad (3.10)$$

The computation of this matrix element follows the same steps of the calculus of the analogous contribution in (3.5), and one get a similar expression:

$$V_{LP}^{(m)} = \frac{2^{2m}}{4\pi^3} \sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} dx e^{-x^2/2} \int_{x+2\pi+\pi/2^m}^{x+2\pi+2\pi/2^m} dt \log [\phi_m(x, t)] \quad (3.11)$$

where

$$\begin{aligned} \phi_m(x, t) \equiv & \frac{(t + \frac{2\pi}{2^m} + \sqrt{x^2 + (t + \frac{2\pi}{2^m})^2})(t - \frac{4\pi}{2^m} + \sqrt{x^2 + (t - \frac{4\pi}{2^m})^2})}{(t + \frac{\pi}{2^m} + \sqrt{x^2 + (t + \frac{\pi}{2^m})^2})(t - \frac{5\pi}{2^m} + \sqrt{x^2 + (t - \frac{5\pi}{2^m})^2})} \times \\ & \times \frac{(t - \frac{\pi}{2^m} + \sqrt{x^2 + (t - \frac{\pi}{2^m})^2})^2}{(t - \frac{2\pi}{2^m} + \sqrt{x^2 + (t - \frac{2\pi}{2^m})^2})^2}. \end{aligned}$$

It is possible to prove that the integral surely exists for  $m > 1$ , which is a constraint satisfied in our conditions since we are interested in studying the behavior of the wave functions and of the energy for large values of  $m$ . The reason of this interest is that for big  $m$  the wave functions are more localized even in the variable  $y$ , as one can see from (2.7) and (2.8).

We see that the result is very similar to the one for  $V_{LP}^{m,n}$ , as expected. However we will show in Section 4 that the numerical outputs are very different.

We end this Section simplifying the expression of the matrix element of the Coulomb energy within the ground state of the harmonic oscillator and its translated. Using the integral formula for the Coulomb potential, we get

$$\begin{aligned} V_{ho}^0 & \equiv \int d^2\mathbf{r}_1 \int d^2\mathbf{r}_2 \frac{|\Psi_0^{(HO)}(\mathbf{r}_1)|^2 |\Phi_0^{(HO)}(\mathbf{r}_2)|^2}{|\mathbf{r}_1 - \mathbf{r}_2|} = \\ & = \frac{1}{2\pi} \int \frac{d^2k}{|\mathbf{k}|} e^{-2\pi i k_y - k_y^2 - k_x^2/2 - k_x k_y} \end{aligned} \quad (3.12)$$

The reason why only  $\Psi_0^{(HO)}(\mathbf{r})$  is considered here is essentially that this is the most localized function among all the harmonic oscillator wave functions.

## 4 Numerical Results and Comments

In this Section we will discuss the numerical results for both the models proposed previously and we comment these results, giving particular attention to the localization properties of the wave functions.

We start by considering the Non-Lattice model. We report in Table 1 the results for  $V_{LP}^{m,n}$  for various values of  $(m, n)$ .

The energies of the harmonic oscillator are easily computed:

$$V_{ho}^{0,1} = 0.91873; \quad V_{ho}^{0,2} = 0.39019; \quad V_{ho}^{1,2} = 0.11041 \quad (4.1)$$

For the Littlewood-Paley basis in the contest of the Lattice model the situation is resumed in Table 2, while the energy of the harmonic oscillator is, in this case,

$$V_{ho}^0 = 0.16515. \quad (4.2)$$

We can now comment these results. The first obvious consideration is that, while for the first model the energy increases as much as  $m$  and  $n$  both increase, for the lattice the situation is just the opposite: the energy decreases for  $m$  increasing. Let us try to explain this different behaviour. From the definitions (2.7), (2.8) and (3.3) we see that when  $m$  increases the supports in  $y$  of the functions decrease. Therefore both  $\Psi_m^{(LP)}(\mathbf{r})$  and  $\Phi_m^{(LP)}(\mathbf{r})$  improve their localization for  $m$  increasing. Since the electrons are localized at a distance of  $2\pi$ , we return to a situation similar to the one of the FQHE. It is well known that the lower bound for the ground energy is obtained if the electrons are punctually localized on the lattice sites, that is, in the classical limit. This is because in this way the distance between the electron is maximized, and therefore the Coulomb interaction gets its minimum value. These considerations explain very well the results in Table 2. We see, in fact, that when  $m$  increases the energy decreases from 0.27083 to the asymptotic value 0.16066. This value is already reached for  $m = 9$  and stay essentially unchanged even for bigger  $m$ . This value could also be predicted in an heuristic way. From the definition of  $\Psi_m^{(LP)}(\mathbf{r})$  we deduce that, for  $m$  very large, this function behaves like a  $\Psi_\infty^{(LP)}(\mathbf{r})$  whose square modulus is

$$|\Psi_\infty^{(LP)}(\mathbf{r})|^2 = \frac{1}{\sqrt{\pi}} e^{-x^2} \delta(x - y). \quad (4.3)$$

An analogous formula holds for  $\Phi_\infty^{(LP)}(\mathbf{r})$ ,  $|\Phi_\infty^{(LP)}(\mathbf{r})|^2 = \frac{1}{\sqrt{\pi}} e^{-x^2} \delta(x - 2\pi - y)$ . We can compute the energy  $E_\infty$  in this limit and we get

$$\begin{aligned} E_\infty &\equiv \int d^2\mathbf{r}_1 \int d^2\mathbf{r}_2 \frac{|\Psi_\infty^{(LP)}(\mathbf{r}_1)|^2 |\Phi_\infty^{(LP)}(\mathbf{r}_2)|^2}{|\mathbf{r}_1 - \mathbf{r}_2|} = \\ &= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dx e^{-x^2/2}}{\sqrt{x^2 + 2\pi x + 2\pi^2}} = 0.16066 \end{aligned}$$

We observe that this result exactly coincides with the one obtained making  $m$  increase. We can conclude that the wave functions are really more and more localized since, in fact, their square modulus converges to an exponential function in  $x$  times a  $\delta(x - y)$ .

We furthermore observe that the energy of the harmonic oscillator,  $V_{ho}^0 = 0.16515$ , is slightly bigger than almost all the  $V_{LP}^{(m)}$ . This difference could be interpreted again as a better localization of the functions  $\Psi_m^{(LP)}(\mathbf{r})$  with respect to  $\Psi_0^{(HO)}(\mathbf{r})$ . This is the reason why we have only computed the energy for this wave function and not, say, for  $\Psi_1^{(HO)}(\mathbf{r})$  or  $\Psi_2^{(HO)}(\mathbf{r})$  whose localization is a bit worse. These numerical results strongly suggest to use wavelet instead of oscillator functions for computing the energy even in the FQHE, in the attempt of explaining better the phase transition between the Wigner and the Laughlin phases, as discussed in the Introduction.

Analogous conclusions can be obtained considering the results in Table 1. We have left these results for the end since they are less directly connected with the picture of the FQHE, since no lattice is present in this model.

We first add an extra information to Table 1: all the results turn out to be symmetric under the exchange  $m \leftrightarrow n$ , as they must.

This time the two electrons are localized both around the origin. We expect that the most localized the wave functions are, the maximum is the overlap between them and, therefore, the maximum is the energy. In a classical picture it would be like if we put two pointlike charges in the same point. Of course this system is not stable and we expect a very high energy for this configuration. This is exactly what happens. If we try to compute

the energy  $V_{LP}^{m,n}$  for  $m$  and  $n$  very large, we expect that the result is the same obtained by computing

$$V_{LP}^{\infty} \equiv \int d^2\mathbf{r}_1 \int d^2\mathbf{r}_2 \frac{|\Psi_{\infty}^{(LP)}(\mathbf{r}_1)|^2 |\Psi_{\infty}^{(LP)}(\mathbf{r}_2)|^2}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dx e^{-x^2/2}}{\sqrt{x^2}}$$

which diverges, as expected.

Moreover, we see from Table 1 also that for each  $m$  fixed, when  $n$  increases,  $V_{LP}^{m,n}$  converges toward an asymptotic value. These values could be predicted with great precision by considering the following quantities:

$$\begin{aligned} V_{LP}^{(m,\infty)} &\equiv \int d^2\mathbf{r}_1 \int d^2\mathbf{r}_2 \frac{|\Psi_m^{(LP)}(\mathbf{r}_1)|^2 |\Psi_{\infty}^{(LP)}(\mathbf{r}_2)|^2}{|\mathbf{r}_1 - \mathbf{r}_2|} = \\ &= \frac{2^{m-1}}{\pi^2} \sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} dx e^{-x^2/2} \log[\Phi_m(x)] \end{aligned} \quad (4.4)$$

where we have defined

$$\Phi_m(x) = \frac{(x + \frac{2\pi}{2^m} + \sqrt{x^2 + (x + \frac{2\pi}{2^m})^2})(x - \frac{\pi}{2^m} + \sqrt{x^2 + (x - \frac{\pi}{2^m})^2})}{(x + \frac{\pi}{2^m} + \sqrt{x^2 + (x + \frac{\pi}{2^m})^2})(x - \frac{2\pi}{2^m} + \sqrt{x^2 + (x - \frac{2\pi}{2^m})^2})}$$

The numerical results are reported in Table 3. We see that this results are extremely good, in the sense that they coincide with the asymptotic values of  $V_{LP}^{m,n}$ , for any  $m$  fixed.

We observe also that, for any  $m$  fixed, the energy decreases when  $n$  increases. This can be understood using the usual picture since, modifying only one wave function, the overlap between the two decreases.

Finally we observe again that the wavelets wave functions appear to be better localized than the oscillator ones. This is deduced, this time, since the best localized functions correspond to the maximum in energy. In fact, we have  $V_{ho}^{0,1} = 0.91873$ ,  $V_{ho}^{0,2} = 0.39019$  and  $V_{ho}^{1,2} = 0.11041$ . We see that the maximum of these values corresponds to the most localized wave function and, however, is much less than the results one gets using wavelets.

We conclude the analysis of these results again with the conviction that wavelets can have a strong utility in the problem of finding the ground state of the FQHE.



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## Appendix: Other Possible Models

In this Appendix we want to discuss briefly the main features of other physical models which could be treated with analogous techniques. We only discuss the situation in two spatial dimensions.

The essential ingredient to define the model is the single electron hamiltonian  $H_0$  in (2.1). This operator must satisfy certain constraints. In fact it must be so that it exists a canonical transformation, generalizing the one in (2.3), which transforms the original  $H_0$  in  $x, y, p_x$  and  $p_y$  in an hamiltonian depending only on a couple of conjugate variables. The most general linear transformation is the following

$$\begin{aligned}\tilde{x}_i &= \sum_j (a_{ij}x_j + b_{ij}p_j) \\ \tilde{p}_i &= \sum_j (c_{ij}x_j + d_{ij}p_j)\end{aligned}$$

where  $x_j$  and  $p_j$  are the original canonically conjugate variables and  $\tilde{x}_i$  and  $\tilde{p}_i$  are the new ones. In [8] it is discussed this kind of transformations and, in particular, it is shown how the wave function transforms under this change of variables. In this paper, we have applied the results in [8] only to the hamiltonian in (2.2). Similar changes of variables can also be applied to other hamiltonians, like the following ones

$$\begin{aligned}H_1 &= \frac{1}{2}(p_x^2 + x^2) + \frac{1}{2}y^2 + xy \\ H_2 &= \frac{1}{2}(p_x^2 + p_y^2 + x^2 + y^2 + 2xy + 2yp_x + (p_x x + xp_x)) \\ H_3 &= \frac{1}{2}(p_x - y/2)^2 + \frac{1}{2}(p_y + x/2)^2.\end{aligned}$$

In particular, the last one is the Hamiltonian of the FQHE. All these hamiltonians can be transformed into the one of an harmonic oscillator with a suitable canonical transformation. Of course the link between the wave functions in configuration space and in the variables  $(Q, Q')$  is different depending

on the coefficients  $a_{ij}$ ,  $b_{ij}$ ,  $c_{ij}$  and  $d_{ij}$  and it is often not so easy as in formula (2.5).

The analysis of the FQHE will be discussed in a future paper.

Table 1

$V_{LP}^{(1,3)} = 0.45784$	$V_{LP}^{(1,4)} = 0.43955$	$V_{LP}^{(1,5)} = 0.43520$
$V_{LP}^{(1,6)} = 0.43412$	$V_{LP}^{(1,7)} = 0.43385$	$V_{LP}^{(1,8)} = 0.43378$
$V_{LP}^{(1,9)} = 0.43377$	$V_{LP}^{(1,m)} = 0.43376 \ m \geq 10$	
$V_{LP}^{(2,4)} = 0.77328$	$V_{LP}^{(2,5)} = 0.75524$	$V_{LP}^{(2,6)} = 0.75093$
$V_{LP}^{(2,7)} = 0.74986$	$V_{LP}^{(2,8)} = 0.74959$	$V_{LP}^{(2,9)} = 0.74953$
$V_{LP}^{(2,10)} = 0.74951$	$V_{LP}^{(2,m)} = 0.74950 \ m \geq 11$	
$V_{LP}^{(3,5)} = 1.14544$	$V_{LP}^{(3,6)} = 1.12849$	$V_{LP}^{(3,7)} = 1.12445$
$V_{LP}^{(3,8)} = 1.12345$	$V_{LP}^{(3,9)} = 1.12320$	$V_{LP}^{(3,10)} = 1.12314$
$V_{LP}^{(3,11)} = 1.12313$	$V_{LP}^{(3,m)} = 1.12312 \ m \geq 15$	
$V_{LP}^{(4,7)} = 1.51843$	$V_{LP}^{(4,8)} = 1.51449$	$V_{LP}^{(4,9)} = 1.51352$
$V_{LP}^{(4,10)} = 1.51327$	$V_{LP}^{(4,11)} = 1.51321$	$V_{LP}^{(4,m)} = 1.51319 \ m \geq 15$
$V_{LP}^{(5,8)} = 1.91054$	$V_{LP}^{(5,9)} = 1.90662$	$V_{LP}^{(5,10)} = 1.90565$
$V_{LP}^{(5,11)} = 1.90541$	$V_{LP}^{(5,m)} = 1.90533 \ m \geq 15$	
$V_{LP}^{(6,8)} = 2.31870$	$V_{LP}^{(6,9)} = 2.30226$	$V_{LP}^{(6,10)} = 2.29835$
$V_{LP}^{(6,11)} = 2.29738$	$V_{LP}^{(6,m)} = 2.29705 \ m \geq 15$	
$V_{LP}^{(7,9)} = 2.71004$	$V_{LP}^{(7,10)} = 2.69358$	$V_{LP}^{(7,11)} = 2.68966$
$V_{LP}^{(7,m)} = 2.68838 \ m \geq 15$		
$V_{LP}^{(8,10)} = 3.10119$	$V_{LP}^{(8,11)} = 3.08475$	$V_{LP}^{(8,15)} = 3.07956$
$V_{LP}^{(8,m)} = 3.07954 \ m \geq 20$		
$V_{LP}^{(9,11)} = 3.49229$	$V_{LP}^{(9,12)} = 3.47584$	$V_{LP}^{(9,13)} = 3.47192$
$V_{LP}^{(9,15)} = 3.47071$	$V_{LP}^{(9,m)} = 3.47063 \ m \geq 20$	
$V_{LP}^{(10,12)} = 3.88337$	$V_{LP}^{(10,13)} = 3.86691$	$V_{LP}^{(10,15)} = 3.86202$
$V_{LP}^{(10,m)} = 3.86171 \ m \geq 20$		
$V_{LP}^{(15,20)} = 5.81737$	$V_{LP}^{(15,m)} = 5.81705 \ m \geq 25$	

Table 1.– Values of the matrix elements in (3.6) for different values of  $(m, n)$ .

Table 2

$V_{LP}^{(1)} = 0.27083$	$V_{LP}^{(2)} = 0.17462$	$V_{LP}^{(3)} = 0.16376$
$V_{LP}^{(4)} = 0.16141$	$V_{LP}^{(5)} = 0.16085$	$V_{LP}^{(6)} = 0.16071$
$V_{LP}^{(7)} = 0.16067$	$V_{LP}^{(m)} = 0.16066 \ m > 7$	

Table 2.– Values of the matrix elements in (3.11) for different values of  $m$ .

Table 3

$V_{LP}^{(1,\infty)} = 0.43376$	$V_{LP}^{(2,\infty)} = 0.74950$	$V_{LP}^{(3,\infty)} = 1.12312$
$V_{LP}^{(4,\infty)} = 1.51319$	$V_{LP}^{(5,\infty)} = 1.90533$	$V_{LP}^{(6,\infty)} = 2.29705$
$V_{LP}^{(7,\infty)} = 2.68838$	$V_{LP}^{(8,\infty)} = 3.07954$	$V_{LP}^{(9,\infty)} = 3.47063$
$V_{LP}^{(10,\infty)} = 3.86171$	$V_{LP}^{(15,\infty)} = 5.81704$	$V_{LP}^{(20,\infty)} = 7.77238$
$V_{LP}^{(25,\infty)} = 9.72772$	$V_{LP}^{(30,\infty)} = 11.68307$	$V_{LP}^{(50,\infty)} = 19.50437$
$V_{LP}^{(100,\infty)} = 39.05772$		

Table 3.– Values of the matrix elements in (4.4) for different values of  $m$ .

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