Localization Properties and Wavelet-Like Orthonormal Bases for the Lowest Landau Level

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ABSTRACT
As an illustration of the phase space localization problem of quantum mechanical systems, we study a two-dimensional electron gas in a magnetic field, such as encountered in the Fractional Quantum Hall Effect (FQHE). We discuss a general procedure for constructing an orthonormal basis for the lowest Landau level (as well as for the other Landau levels), starting from an arbitrary orthonormal basis in $L^2(\mathbb{R})$. After some remarks concerning localization properties of the wave functions of any orthonormal basis in the Landau levels, we discuss in detail some relevant examples stemming from wavelet analysis like the Haar, the Littlewood-Paley, the Journé and the splines bases. We also propose a toy model, which indicates how the use of wavelets in the analysis of the FQHE may help in the search of the correct ground state of the system. Finally, we exhibit an intriguing equivalence between FQHE states generated by the so-called magnetic translations and vectors of an orthonormal basis derived from an arbitrary multiresolution analysis.

10.1 Introduction: Phase Space localization

Uncertainty relations constitute one of the most characteristic features of a quantum system. According to the standard discussion in quantum mechanics textbooks, two observables of a quantum system, represented by self-adjoint operators A and B, obey the uncertainty relation

$$\Delta A.\Delta B \geq \frac{1}{2}|\langle [A,B]\rangle|, \tag{10.1.1}$$

where $\Delta A \equiv \Delta_\phi A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$ denotes the variance of A in the state ϕ, $\langle C\rangle = \langle \phi, C\phi\rangle$ is the average of the operator C in the state ϕ (we ignore domain considerations), and $[A,B] \equiv AB - BA$ is the commutator of A and B. Taking, in particular, the canonical position and momentum operators

Q and P, we get the familiar relation (we put $\hbar = 1$)

$$\Delta Q.\Delta P \geq \frac{1}{2}, \tag{10.1.2}$$

Since the corresponding eigenvalues q, p are phase space coordinates, the relation (**??**) really means that a quantum system cannot be localized with arbitrary precision *in phase space*.

The best one can achieve is a minimal uncertainty state, that is, a state ϕ for which equality holds in (**??**). Among such states one finds the *canonical coherent states(CS)* (for a recent survey of CS, see [**?**]). These latter states may be obtained as eigenvectors of the annihilation operator $a = 2^{-1/2}(Q + iP)$,

$$a|q,p\rangle = z|q,p\rangle, \quad z = 2^{-1/2}(q - ip), \tag{10.1.3}$$

or with the help of a displacement operator,

$$|q,p\rangle = e^{za^\dagger - \overline{z}a}|0,0\rangle, \quad a^\dagger = 2^{-1/2}(Q - iP), \tag{10.1.4}$$

where $|0,0\rangle = \pi^{-1/4}\exp(-\frac{1}{2}q^2)$ is the ground state of the quantum harmonic oscillator. Now, writing (**??**) in the Schrödinger representation, one obtains the coherent states in a form also known as *Gabor states*, namely,

$$\psi_{q,p}(x) = e^{ipx}\,\psi(x - q), \quad \psi \in L^2(\mathbb{R}, dx). \tag{10.1.5}$$

The question that arises naturally, then, is whether one can extract from this (overcomplete) set a discrete subset that constitutes an orthonormal basis, or at least a frame. We recall that a countable family of vectors $\{\psi_j\}$ in a Hilbert space $\mathfrak{H}$ is called a *(discrete) frame* if there are two positive constants m, M, with $0 < \mathsf{m} \leq \mathsf{M} < \infty$, such that

$$\mathsf{m}\,\|\phi\|^2 \;\leq\; \sum_{j=1}^{\infty} |\langle\psi_j, \phi\rangle|^2 \;\leq\; \mathsf{M}\,\|\phi\|^2, \;\forall\,\phi \in \mathfrak{H}. \tag{10.1.6}$$

The two constants m, M are the *frame bounds*. If $\mathsf{m} = \mathsf{M} > 1$, the frame is said to be *tight*. Of course, if $\mathsf{m} = \mathsf{M} = 1$, and $\|\psi_j\| = 1, \forall\, j$, the set $\{\psi_j\}$ is simply an orthonormal basis.

The answer is given by the Gabor frame theorem [**?**, **?**, **?**] (see [**?**] for an up-to-date survey), which runs as follows. Let $\Lambda = \{(q_n, p_m),\, m, n \in \mathbb{Z}\}$ be a rectangular lattice in the (q,p)-plane. Then the set $\{\psi_{nm} \equiv \psi_{q_n,p_m}\}$ is a frame if the density of Λ is larger than a critical value. More precisely, let $\Lambda = q_o\mathbb{Z} \times 2\pi p_o\mathbb{Z}$, with a unit cell of area $S = 2\pi q_o p_o$, and consider the corresponding family $\{\psi_{nm}\}$, namely,

$$\psi_{nm}(x) = e^{2\pi i m p_o x}\,\psi(x - nq_o), \quad m, n \in \mathbb{Z}. \tag{10.1.7}$$

Then,

(1) If $q_o p_o > 1$, the family $\{\psi_{nm}\}$ is not complete.

(2) If $q_o p_o < 1$, the family $\{\psi_{nm}\}$ is overcomplete and remains so if one removes a finite number of points from Λ.

(3) If $q_o p_o = 1$, the family $\{\psi_{nm}\}$ is complete and remains so if one removes any single point from Λ, but becomes noncomplete if one removes two or more points.

Coming back to the language of frames, this result may be reformulated as follows [**?**]: Frames, even tight frames, with good phase space localization exist for $q_o p_o < 1$; no frame exists for $q_o p_o > 1$; in the critical case $q_o p_o = 1$, frames do exist, but with bad localization properties. In particular, orthonormal bases exist *only* in the critical case, and thus are necessarily poorly localized. In both cases, the localization result follows from the celebrated Balian–Low theorem (BLT), namely [**?**, **?**]:

Theorem 10.1.1 (Balian–Low) *Let $\psi \in L^2(\mathbb{R})$ and let $q_o, p_o > 0$ satisfy $q_o p_o = 1$. If $\{\psi_{nm}\}$ is a Gabor frame (in particular, a Gabor orthonormal basis), then*

$$\textit{either} \quad \int_{-\infty}^{\infty} dx\, |x\,\psi(x)|^2 = \infty \quad \textit{or} \quad \int_{-\infty}^{\infty} dk\, |k\,\widehat{\psi}(k)|^2 = \infty. \qquad (10.1.8)$$

The statement (**??**) means that ψ and $\widehat{\psi}$ cannot both have fast decrease simultaneously (remember that fast decrease of $\widehat{\psi}$ means smoothness of ψ). In other words, the Gabor frame $\{\psi_{nm}\}$ cannot be well localized in phase space.

In his original paper [**?**], Gabor used a Gaussian as the basis function ψ, because it minimizes the joint uncertainty in phase space (thus he constructed a discrete lattice of canonical CS). Although this yields a frame in the case $q_o p_o < 1$, it does *not* in the critical case $q_o p_o = 1$ [**?**]. We will see in Section **??** other instances of this choice.

Now it is customary [**?**] in signal processing to treat in parallel the Gabor or canonical CS formalism and wavelet analysis. Indeed, there are quite a number of physical applications where both may be used, with very comparable results [**?**, **?**, **?**]. Also, on the theoretical side, both formalisms are particular instances of the general CS systems associated to square integrable representations of specific groups [**?**], the Weyl–Heisenberg group for the former, the $ax + b$ group for the latter. Therefore, facing the bad localization properties of critical Gabor frames, one is tempted to use wavelets instead. However, the situation does not improve much, because of the following striking result of Battle [**?**]. Let ψ be a wavelet, that is, a function $\psi \in L^2(\mathbb{R}, dx)$ such that the functions $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k),\ j, k \in \mathbb{Z}$ are mutually orthogonal. Then we have:

Theorem 10.1.2 (Battle) *Let $\psi \in L^2(\mathbb{R})$ be a wavelet. Then it cannot have exponential localization in both position space and momentum space.*

Contrary to the BLT, the proof of Theorem **??** is elementary, going as follows. If ψ and $\widehat{\psi}$ have both exponential decay, it follows that ψ is a Schwartz function with all moments vanishing. This means that $\widehat{\psi}(p)$ vanishes at $p = 0$ to infinite order, and so $\widehat{\psi}(p)$ cannot be real analytic. This contradicts the assumption of exponential decay of $\psi(x)$. Notice that the family $\{\psi_{jk}\}$ is not required to be complete, only orthonormality is involved.

Yet, we will see in the sequel that there are good reasons to consider wavelet bases, rather than CS frames. The physical problem we will envisage in that respect is the Fractional Quantum Hall Effect (FQHE). The latter is a fascinating example of quantum behavior in condensed matter physics, still not completely understood theoretically (see [**?**, **?**] for a review and the original references). At the same time, it yields a beautiful illustration of the problems linked to phase space localization. This particular aspect of the FQHE will be the subject matter of this chapter, which relies heavily on [**?**, **?**, **?**, **?**].

10.2 The Fractional Quantum Hall Effect

The system to be considered for the Fractional Quantum Hall Effect (FQHE) is a (quasi)-planar gas of electrons in a strong magnetic field perpendicular to the plane. The first problem to tackle for discussing the static features of the FQHE is to find the ground state of the system, and this is already a very hard problem. Two main methods have been proposed in the literature to that effect. The first one is a Hartree-Fock approach to a system of N two-dimensional electrons, in which the (totally antisymmetric) ground state wave function is taken as a Slater determinant of one-electron wave functions (see, for instance, [**?**, **?**, **?**]). We recall here that, given a one-electron wave function $\psi(\boldsymbol{r})$, the corresponding Slater determinant wave function of the N-electron state is given [**?**] by

$$\psi^{(N)}(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_1(\boldsymbol{r}_1) & \psi_1(\boldsymbol{r}_2) & \ldots & \psi_1(\boldsymbol{r}_N) \\ \psi_2(\boldsymbol{r}_1) & \psi_2(\boldsymbol{r}_2) & \ldots & \psi_2(\boldsymbol{r}_N) \\ \vdots & \vdots & & \vdots \\ \psi_N(\boldsymbol{r}_1) & \psi_N(\boldsymbol{r}_2) & \ldots & \psi_N(\boldsymbol{r}_N) \end{vmatrix} . \tag{10.2.1}$$

This picture gives good energy values for small or high electron densities. In the intermediate range, however, the best results are obtained with the Laughlin wave function [**?**], which is derived by a variational technique based on a non mean-field approach to the same two-dimensional gas of electrons. We will consider here the first method only.

The first step is to select an adequate wave function for a single electron in the magnetic field. As it is well-known [**?**], the energy levels, the so-called *Landau levels*, are infinitely degenerate, and the degeneracy may

be attributed to the fact that the position of the orbit center is not determined. Thus the problem arises of finding a good basis in the corresponding Hilbert subspace. This is crucial for allowing an easy computation of the energy levels of the whole system, in the presence of perturbations. In particular, the ground state we are looking for belongs to the lowest Landau level (LLL). We will discuss in the sequel a general way of obtaining an orthogonal basis for the LLL.

It is a standard result [**?**] that the Hamiltonian of a single electron confined in the xy-plane and subjected to a strong magnetic field in the z-direction can be transformed into that of a harmonic oscillator. In the symmetric gauge we have

$$H_0 = \frac{1}{2}(p_x - y/2)^2 + \frac{1}{2}(p_y + x/2)^2. \tag{10.2.2}$$

Introducing the canonical variables

$$\widetilde{P} = p_x - y/2, \qquad \widetilde{Q} = p_y + x/2, \tag{10.2.3}$$

this can be written in the form

$$H_0 = \frac{1}{2}(\widetilde{P}^2 + \widetilde{Q}^2). \tag{10.2.4}$$

We use units such that $\hbar = M = e|\boldsymbol{H}|/c = 1$, which also implies that the cyclotron frequency $\omega_c = e|\boldsymbol{H}|/Mc$ and the magnetic length $a_0 = (\hbar c/e|\boldsymbol{H}|)^{1/2}$ are both equal to one.

The eigenstates of the Hamiltonian (**??**) can be found explicitly, and they have the following form [**?**]:

$$\Phi_{lm}(x,y) = (2^{l+m+1}\pi\, l!m!)^{-1/2} e^{(x^2+y^2)/4} (\partial_x + i\partial_y)^m\ (\partial_x - i\partial_y)^l e^{-(x^2+y^2)/2}, \tag{10.2.5}$$

corresponding to the eigenvalues

$$E_{lm} \equiv E_l = l + 1/2,\ l, m = 0, 1, \ldots. \tag{10.2.6}$$

Thus we see that the energy levels are all degenerate in m, so that the ground level (LLL) is spanned by the set $\{\Phi_{0m}(x,y)\}$, which forms an orthonormal basis for the LLL. For these wave functions, the mean value of the distance from the origin, $r \equiv \sqrt{x^2+y^2}$, increases with m [**?**, **?**], so that the functions $\Phi_{0m}(x,y)$ are not very well localized. Yet the physics of the problem requires that the wave functions be fairly well localized, in particular for approaching the low temperature limit of the celebrated Wigner crystal [**?**]. This is the LLL basis problem.

While the solutions (**??**) can be found easily directly in the configuration space, it is not easy at all to find another basis, orthogonal or not, spanning the same energy level. An efficient and elegant method, based on a technique introduced in [**?**], has been discussed in some detail in [**?**] and

[**?**], and we will use it here. The transformation (**??**) can be seen as a part of a canonical transformation from the variables x, y, p_x, p_y into the new ones $Q, P, \widetilde{Q}, \widetilde{P}$, where

$$P = p_y - x/2, \qquad Q = p_x + y/2. \tag{10.2.7}$$

These operators satisfy the following commutation relations:

$$[Q, P] = [\widetilde{Q}, \widetilde{P}] = i, \tag{10.2.8}$$

all other commutators vanishing. Moreover, $(x_o, y_o) \equiv (-P, Q)$ are the coordinates of the center of the magnetic orbit, so that the degeneracy of the energy levels (**??**), corresponding to the fact that the Hamiltonian H_o does not depend on Q, P, may indeed be interpreted as the indeterminacy in the position of the orbit.

It is shown in [**?**, **?**] that a wave function in the (x, y)-space is related to its $P\widetilde{P}$-expression by the formula

$$\Psi(x, y) = \frac{e^{ixy/2}}{2\pi} \int_{-\infty}^{\infty} dP \int_{-\infty}^{\infty} d\widetilde{P}\, e^{i(x\widetilde{P} + yP + P\widetilde{P})}\, \Psi(P, \widetilde{P})\,. \tag{10.2.9}$$

The usefulness of the $P\widetilde{P}$-representation stems from the expression (**??**) of H_0. Indeed, in this representation, the Schrödinger equation admits eigenvectors $\Psi(P, \widetilde{P})$ of H_0 of the form $\Psi(P, \widetilde{P}) = f(\widetilde{P})h(P)$. In particular, the ground state wave function of (**??**) must have the form $f_0(\widetilde{P})h(P)$, where

$$f_0(\widetilde{P}) = \pi^{-1/4}\, e^{-\widetilde{P}^2/2} \tag{10.2.10}$$

is the familiar harmonic oscillator ground state wave function and the function $h(P)$ is arbitrary, which manifests the degeneracy of the LLL.

Let us now come back to the problem of localization of the wave function $\Psi(x, y)$. Writing (**??**) for the case of the LLL, using (**??**), we get

$$\Psi^{\mathrm{L}}(x, y) = \frac{e^{ixy/2}}{\sqrt{2}\pi^{3/4}} \int_{-\infty}^{\infty} dP\, e^{iyP} e^{-(x+P)^2/2}\, h(P)\,. \tag{10.2.11}$$

In order to study the localization properties of the function $\Psi^{\mathrm{L}}(x, y)$, we have to analyze its asymptotic behavior. From (**??**), standard convolution arguments indicate that the behavior of $\Psi^{\mathrm{L}}(x, y)$ for large x is related to the behavior for large P of $h(P)$, while that for large y is related to the behavior for large values of Q of the Fourier transform $\widehat{h}(Q)$ of $h(P)$. This argument extends easily to the other Landau levels: we can show that if $\Psi(x, y)$ belongs to the l-th Landau level, then its asymptotic behavior in x and y is related, respectively, to the behavior of $P^l h(P)$ and $Q^l \widehat{h}(Q)$ for large values of their arguments.

Now, since the transformation (**??**), (**??**) is canonical, it is clear that the transformed family $\{\Psi^{\mathrm{L}}_m(x,y) \in L^2(\mathbb{R}^2)\}$ of any family $\{h_m(P) \in L^2(\mathbb{R})\}$ is orthonormal in $L^2(\mathbb{R}^2)$ if and only if $\{h_m(P)\}$ is orthonormal in $L^2(\mathbb{R})$. Thus, the LLL basis problem boils down to finding suitable functions $h_m(P)$ that constitute a basis with the desirable properties. Since the classical ground state for a two dimensional electron gas is a Wigner triangular lattice with electrons sharply localized on the lattice sites, we expect that the quantum ground state should not be very different from the classical one. For this reason, we would like to find the best localization of the single electron wave functions compatible with the physics of the problem. On the other hand, orthogonality of the basis makes life much easier (otherwise, the norm of the N-electron wave function (**??**) grows as $\mathcal{O}(N)$ when $N \to \infty$), so that one would like to impose it too. The problem is that sharp localization and orthogonality are *not* compatible, as results from the many attempts made in the literature and, more fundamentally, from Theorems **??** and **??**. Let us discuss some of these proposals, in all of which the basis functions $h_m(P)$ are generated from a single function $h(P)$.

10.2.1 Bases Generated with Magnetic Translations

A physically interesting possibility is to generate a basis by acting on a suitable function $\psi_o(x,y)$ with the so-called magnetic translation operators $T(\boldsymbol{a}_i)$ defined by

$$T(\boldsymbol{a}_i) \equiv \exp\left(i\boldsymbol{\Pi}_c \cdot \boldsymbol{a}_i\right), \quad i = 1,2, \tag{10.2.12}$$

where $\boldsymbol{\Pi}_c \equiv (Q,P)$ and $\boldsymbol{a}_i$ are the basis vectors of a suitable lattice Λ. The set obtained in this way is still in the LLL since the operators $T(\boldsymbol{a}_i)$ commute with H_0, in virtue of the commutation relations (**??**). Moreover $[T(\boldsymbol{a}_1), T(\boldsymbol{a}_2)] = 0$ if the area of th e lattice cell is such that $a_{1_x}a_{2_y} - a_{1_y}a_{2_x} = 2\pi$. This geometrical condition on the lattice is known as the *rationality condition*. This technique was pioneered by Boon [**?**], who chose for $\psi_o(x,y)$ a Gaussian, and for Λ a rectangular lattice of unit cell area 2π, corresponding to $q_o = p_o = 1$ in (**??**), thus effectively constructing a basis of canonical coherent states (or Gabor states) at the critical density $q_o p_o = 1$. The resulting basis is complete, but not orthogonal.

In the same way, the authors of [**?**] choose $h(P) = f_0(P)$ and show, using (**??**), that the corresponding wave function in the (x,y)-space is nothing but $\Phi_{00}(x,y)$ as defined in (**??**), thus again a Gaussian. Using magnetic translation operators $T(\boldsymbol{a}_i)$, they also construct a complete set of functions for the LLL with Gaussian localization, centered on the sites of a regular two-dimensional lattice, this time a triangular lattice of unit cell area 2π (that of the Wigner crystal). Completeness of the set is proven by showing its unitary equivalence with the previous set of coherent states. As expected, this basis is not orthogonal either, since coherent states are in

general not mutually orthogonal. In addition, in order to belong to the LLL, each vector must have a well-defined, fixed (essential) support, so that there is no possibility of modifying the mutual overlap for fixed electron density. Orthogonality can be enforced by hand, with a Gram-Schmidt orthogonalization method, for instance, but this leads to increasingly long linear combinations, thus spoiling much of the simplicity of the basis functions, and in particular the localization properties.

Another approach, whose aim is to preserve the latter, and at the same time some sort of translation invariance, is due to Ferrari [**?**], who has constructed an orthonormal basis for the LLL, by taking infinite superpositions of the above (coherent) states (this is not very different from the approach of [**?**]). The resulting basis vectors are Bloch functions, which may be made translation invariant over the nodes of a given lattice, typically triangular or hexagonal. Clearly this basis describes very well the two-dimensional low-density system of electrons of the FQHE, but its construction is rather involved and ad hoc.

Of course, the mutual exclusion between orthogonality and sharp localization follows directly from the Balian-Low theorem (Theorem **??**), as first pointed out in [**?**]. Indeed, the BLT forces any wave function $\Psi(x, y)$ orthogonal to all its magnetic translates

$$\Psi_{m,n}(x, y) = T(\boldsymbol{a}_1)^m \, T(\boldsymbol{a}_2)^n \, \Psi(x, y), \quad m, n \in \mathbb{Z}, \tag{10.2.13}$$

to be poorly localized in x and/or y, at least if $\boldsymbol{a_1}$ and $\boldsymbol{a_2}$ satisfy the rationality condition introduced above. For this reason, for instance, the Gaussian-like wave functions of [**?**] are not orthonormal and an orthonormalization procedure preserving the translation invariance has to be set up. This procedure, however, destroys the exponential localization, in agreement with the Balian-Low theorem.

In order to bypass this limitation, Zak constructs an orthogonal basis using for generating function $h(P)$ the function

$$\delta(P) = \begin{cases} 1, & |P| < 1/2, \\ 0, & |P| > 1/2. \end{cases} \tag{10.2.14}$$

The basis functions, however, are poorly localized. More precisely, the product of the uncertainties in the coordinates x_o, y_o of the center of the electron orbit is infinite. This is again a direct consequence of the Balian-Low theorem, which thus finds here a direct application to the electron dynamics.

10.2.2 Wavelet Bases

As an alternative, one can try to use dilations and translations instead of the magnetic translations (**??**), that is, to turn to wavelets for constructing an orthonormal set in $L^2(\mathbb{R})$ which, via (**??**), produces an orthonormal set for the LLL. Here too, however, a no-go result arises concerning the

localization of the single electron wave-function, namely, Battle's Theorem **??**. Together with our previous result concerning the relation between the asymptotic behaviors of $\Psi_L(x,y)$ and $h(P), \widehat{h}(Q)$, this implies that the use of wavelets in the construction of the ground state for the LLL cannot either produce a wave function with compact (or Gaussian-like) localization in the $\boldsymbol{r}$-space. So one may wonder why wavelets could give energy results lower than the ones obtained using different approaches. Although we do not have numerical results for the LLL, we may expect that the use of wavelets can give a significative insight in the comprehension of the FQHE. This claim is based on several arguments:

- The Battle argument does not prevent $h(P)$ and $\widehat{h}(Q)$ to have, for instance, a Gaussian decay for the first and a fast decay for the second (for instance a decay like Q^{-n} with a large integer n). This behavior should be compared with the one allowed by the Balian-Low theorem.

- Not only do wavelets enjoy such reasonable localization properties, but the latter are easily controlled by varying the scale parameter. In this way, one may modify the mutual overlap between two neighboring wave functions, for fixed electron density, in contrast to the Gaussian-like functions of [**?**].

- The use of wavelets is strongly suggested by the intrinsic hierarchical structure of the FQHE [**?**, **?**], and even a possible fractal behavior in the relevant parameter, the filling factor, which may take arbitrary rational values.

- In the next section we will discuss a toy model, closely related to the FQHE, whose numerical results give, in our opinion, a strong indication of the relevance of wavelets as the building block of the many body wave function.

- Although multiresolution analysis is generally thought of as the standard technique for generating orthonormal wavelet bases, we will show in Section **??** that it can be used, via magnetic translation operators, to generate a Gabor orthonormal basis with vectors remarkably similar to the wavelet basis vectors.

We conclude this section with a final remark. It is clear that a good localization is not the only requirement to be imposed on the generating function $h(P)$. In fact, a much better localization could be obtained by taking the orthonormal basis in $L^2(\mathbb{R})$ arising from the quantum harmonic oscillator. Each of these basis vectors has, together with its Fourier transform, a Gaussian decay, which is reflected by an analogous behavior of the corresponding functions in the LLL obtained by the integral transform (**??**). However, this construction has a series of drawbacks. Among others, for instance, we lose completely the lattice structure of the electron gas as well

as the hierarchical features of the effect. Moreover, the basis is no longer constructed using a single wave function. Of course, several other choices could be made (for instance, a Wilson basis [**?**]) that make the construction closer to the one we have in mind. We hope to be able to analyze these and other possibilities in a near future.

10.3 A Toy Model

In this section, we discuss a pedagogical model, originally introduced in [**?**], closely related to the FQHE, of which it can be considered as a simplified version. This model will suggest the way in which the wave function of the degenerate lowest Landau level should be chosen in order to lower the energy and fit the experimental data. The idea consists essentially in modifying the single electron wave function in order to achieve a better electron localization (compatible with the Balian-Low and the Battle constraints). In fact, as for the FQHE, we expect that a better localized single electron wave function gives rise to a lower Coulomb energy for any lattice system.

We begin by introducing a physical model whose ground level is infinitely degenerate (like the one of the Hamiltonian (**??**)). In this way the ground state is not fixed *a priori*. We construct different trial ground states using the Haar, the Littlewood-Paley and the harmonic oscillator bases in $L^2(\mathbb{R})$.

For this model we also define a lattice generated by a pair of magnetic translations. Then we construct different bases for the lattice corresponding to the LLL and we discuss some numerical results which can be safely interpreted as an indication of a better localization of the wavelet orthonormal bases as compared to the harmonic oscillator basis.

In our toy model, we will not introduce the positive background, the reason being that we will compute in the following only a two electron (direct) Coulomb energy. Therefore, since no thermodynamical limit will be considered here, there is no need of subtracting the interaction energy between the electrons and the background, as in [**?**].

The Hamiltonian is

$$H^{(N)} = \sum_{i=1}^{N} H_0'(i) \; + \frac{1}{2} \sum_{i \neq j}^{N} \frac{1}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}, \tag{10.3.1}$$

where $\boldsymbol{r}_i$ denotes the position of the i^{th} electron and each $H_0'(i)$ has the form

$$H_0' = \frac{1}{2}(p_x^2 + x^2) + \frac{1}{2}p_y^2 + p_x p_y. \tag{10.3.2}$$

This peculiar Hamiltonian, in which the z-component does not appear (reflecting the fact that the device is two-dimensional), shares an interesting feature with the original single electron Hamiltonian (**??**). Introduce the

following canonical transformation:

$$\begin{array}{ll} Q \equiv p_x + p_y \ , & P = -x \ , \\ \widetilde{Q} \equiv p_y \ , & \widetilde{P} = x - y \ , \end{array} \tag{10.3.3}$$

which preserves the commutation relations,

$$[Q, P] = [\widetilde{Q}, \widetilde{P}] = i, \qquad [Q, \widetilde{P}] = [Q, \widetilde{Q}] = [P, \widetilde{P}] = [P, \widetilde{Q}] = 0.$$

Then H_0' takes the form

$$H_0' = \frac{1}{2}(P^2 + Q^2), \tag{10.3.4}$$

so that $\widetilde{Q}$ and $\widetilde{P}$ disappear from the definition of H_0', exactly as for the Hamiltonian (**??**). In particular, since in both cases a pair of variables disappears from the Hamiltonian, it is clear that the energy levels, which we still call Landau levels, and in particular the LLL, are all infinitely degenerate. Therefore, for this toy model also, the ground state is not uniquely determined.

As for the FQHE, any wave function in (x, y) can be written as the unitary transformed wave function given in, say, the variables $(Q, \widetilde{Q})$, see [**?**]:

$$\Psi(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dQ \int_{-\infty}^{\infty} d\widetilde{Q}\, e^{i[\widetilde{Q}(y-x)+Qx]}\, \Phi(Q, \widetilde{Q})\,. \tag{10.3.5}$$

Any single electron ground state of our model, $\Psi_0(x, y)$, is therefore uniquely related to the ground state $\Phi_0(Q, \widetilde{Q})$ of (**??**). Because to the particular form of H_0', we see that $\Phi_0(Q, \widetilde{Q})$ can be factorized and that the dependence on Q is fixed. Therefore, in the case of the lowest Landau level, we must have:

$$\Phi(Q, \widetilde{Q}) = \pi^{-1/4} e^{-Q^2/2}\, \varphi(\widetilde{Q}), \tag{10.3.6}$$

Here the function $\varphi(\widetilde{Q})$ is arbitrary, since the variable $\widetilde{Q}$ does not appear in the Hamiltonian (**??**), and the energy of the unperturbed system depends on Q alone. Its form is fixed by the perturbation (Coulomb repulsion) only.

In the following, we will consider different expressions for $\varphi(\widetilde{Q})$, and we will show that the different results of the mean values of H_0' for these different $\varphi(\widetilde{Q})$ allow us to obtain some information concerning the localization features of the corresponding single electron wave functions in configuration space.

We start by considering the Littlewood-Paley orthonormal wavelet basis (we refer to [**?**, **?**] for further details on wavelet analysis). The mother wavelet of this set is

$$\psi(x) = (\pi x)^{-1}(\sin 2\pi x - \sin \pi x), \tag{10.3.7}$$

and it generates an orthonormal basis $\{\psi_{mn},\, m,n \in \mathbb{Z},\}$ in $L^2(\mathbb{R})$ by the standard definition $\psi_{mn}(x) \equiv 2^{-m/2}\psi(2^{-m}x - n)$. We will consider the subset $\{\psi_m(x) \equiv \psi_{m0}(x),\ m \in \mathbb{Z}\}$. The functions of this set obviously satisfy the orthonormality condition $\langle \psi_m, \psi_{m'} \rangle = \delta_{mm'}$, and, when used in (**??**) as different choices for the function $\varphi(\widetilde{Q})$, they yield the following set of wave functions in configuration space:

$$\Psi_m^{(\mathrm{LP})}(x,y) = \frac{2^{m/2}}{\sqrt{2}\,\pi^{3/4}} e^{-x^2/2} \chi_{D_x}(y). \tag{10.3.8}$$

Here $\chi_{D_x}(y)$ is the characteristic function of the set D_x, equal to one if $y \in D_x$ and zero otherwise, and we have defined

$$D_x = [x - \frac{2\pi}{2^m}, x - \frac{\pi}{2^m}] \cup [x + \frac{\pi}{2^m}, x + \frac{2\pi}{2^m}]. \tag{10.3.9}$$

Because of the canonicity of the transformation (**??**), the functions of the family $\{\Psi_m^{(\mathrm{LP})}(x,y),\ m \in \mathbb{Z}\}$ are obviously mutually orthonormal. Moreover, we see from (**??**) that they are very well localized in both x and y. In particular, they present a localization property which was not allowed in the FQHE because of the Battle no-go result, Theorem **??**.

Another possible choice for $\varphi(\widetilde{Q})$ is related to the Haar basis. The mother wavelet is the function

$$h(x) = \begin{cases} 1, & \text{if } 0 \le x < 1/2\ , \\ -1, & \text{if } 1/2 \le x < 1\ , \\ 0, & \text{otherwise}\ \ , \end{cases} \tag{10.3.10}$$

and the corresponding family is defined in the usual way: $\{h_m(x) \equiv h_{m0}(x) = 2^{-m/2}h(2^{-m}x),\, m \in \mathbb{Z}\}$. This set is again orthonormal (as well as its transformed counterpart in configuration space, by (**??**)), but the localization of each wave function is rather poor. From (**??**), we get

$$\Psi_m^{(\mathrm{H})}(x,y) = \frac{i\,2^{-m/2}}{\sqrt{2}\,\pi^{3/4}} \frac{e^{-x^2/2}}{(y-x)} (e^{i2^{m-1}(y-x)} - 1)^2\ , \tag{10.3.11}$$

which again decreases exponentially in x, but goes like $1/y$ in y. For this reason, we do not expect that the set $\{\Psi_m^{(\mathrm{H})}(x,y),\ m \in \mathbb{Z}\}$ can play a relevant role in the energy computation.

Finally we consider a non-wavelet orthonormal set. We take for $\varphi(\widetilde{Q})$ in (**??**) the first three eigenstates of the Hamiltonian $H_0 = \frac{1}{2}(\widetilde{P}^2 + \widetilde{Q}^2)$ and, for each of these functions, we compute its expression in configuration space using (**??**). We easily find the following results:

$$\Psi_0^{(\mathrm{HO})}(x,y) \quad = \frac{1}{\sqrt{\pi}}\, e^{-(y^2+2x^2-2xy)/2}\ ,$$

$$\Psi_1^{(\mathrm{HO})}(x,y) = i\sqrt{\frac{2}{\pi}}\,(y-x)\,e^{-(y^2+2x^2-2xy)/2}\ , \tag{10.3.12}$$
$$\Psi_2^{(\mathrm{HO})}(x,y) = \frac{1}{\sqrt{2\pi}}\,(1-2(y-x)^2)\,e^{-(y^2+2x^2-2xy)/2}\ .$$

All these wave functions, automatically orthogonal to each other, have a rather good localization in both x and y. It is clear that the best localized one is $\Psi_0^{(\mathrm{HO})}(x,y)$.

As already mentioned before, we will only compute the energy for $N = 2$ in (**??**) since, in any case, the total energy is essentially a sum of two-body contributions. As in all the analogous energy computations, the single electron energy can be considered as constant and therefore does not affect the results.

The Slater determinant for the two electron system is

$$\Psi(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{1}{\sqrt{2}}\left[\Psi_1(\boldsymbol{r}_1)\Psi_2(\boldsymbol{r}_2) - \Psi_2(\boldsymbol{r}_1)\Psi_1(f r_2)\right]\ , \tag{10.3.13}$$

where $\Psi_i(\boldsymbol{r}_j)$, $i,j = 1,2$, are single electron wave functions like in (**??**), (**??**) and (**??**). The Coulomb energy E_c of the system is therefore

$$E_c \equiv \int d^2\boldsymbol{r}_1 \int d^2\boldsymbol{r}_2\, \frac{\overline{\Psi(\boldsymbol{r}_1,\boldsymbol{r}_2)}\Psi(\boldsymbol{r}_1,\boldsymbol{r}_2)}{|\boldsymbol{r}_1-\boldsymbol{r}_2|} = V_d - V_{ex}\ ,$$

where

$$V_d = \int d^2\boldsymbol{r}_1 \int d^2\boldsymbol{r}_2\, \frac{|\Psi_1(\boldsymbol{r}_1)|^2|\Psi_2(\boldsymbol{r}_2)|^2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}$$

and

$$V_{ex} = \int d^2\boldsymbol{r}_1 \int d^2\boldsymbol{r}_2\, \frac{\overline{\Psi_1(\boldsymbol{r}_1)}\,\overline{\Psi_2(\boldsymbol{r}_2)}\Psi_1(\boldsymbol{r}_2)\Psi_2(\boldsymbol{r}_1)}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}$$

are the direct and the exchange term, respectively.

It is well known that, at least for localized wave functions, the exchange contribution is much smaller than the direct one. This feature is explicitly discussed, for instance, in [**?**], where these contributions are explicitly computed for the FQHE. Hence, we will ignore V_{ex} in the sequel. However, with a little abuse of language, V_d will be still called the 'energy' of the system.

Before computing the expressions of V_d for the Littlewood-Paley and for the harmonic oscillator bases, we use these same bases as building blocks to introduce a *natural lattice* in the toy model. The reason for doing so is obviously that a lattice is a natural structure for the FQHE, at least for small electron densities, see [**?**]. Actually, since we are dealing with two electrons only, we will think of our lattice as two separate points in space. We again refer to [**?**], and reference therein, for the details concerning the construction of the lattice associated to H_0.

In analogy with our previous results, it is easy to prove that the unitary operators

$$T_1 = e^{i\,\widetilde{Q}\,a}, \qquad T_2 = e^{i\,\widetilde{P}\,b} \tag{10.3.14}$$

both commute with H_0' and that they also commute with each other if $ab = 2\pi$. From the definition (**??**), one can also observe that, for any function $f(x,y) \in L^2(\mathbb{R})$, one has $T_1 f(x,y) = f(x,y+a)$ and $T_2 f(x,y) = e^{i(x-y)b} f(x,y)$. Therefore, T_2 is simply a multiplication by a phase, whereas T_1 acts like a shift operator. This is enough for our present aim, we can take the two different sites of our 'lattice' along the y-axis, with a 'lattice'-spacing $a = 2\pi$.

Defining, in particular,

$$\Phi_m^{(\mathrm{LP})}(x,y) \equiv T_1 \Psi_m^{(\mathrm{LP})}(x,y) = \frac{2^{m/2}}{\sqrt{2}\pi^{3/4}} e^{-x^2/2} \chi_{D_{x-a}}(y) \tag{10.3.15}$$

from the Littlewood-Paley wavelets (**??**), and

$$\Phi_0^{(\mathrm{HO})}(x,y) \equiv T_1 \Psi_0^{(\mathrm{HO})}(x,y) = \frac{1}{\sqrt{\pi}} e^{-[(y+a)^2+2x^2-2x(y+a)]/2} \tag{10.3.16}$$

for the most localized harmonic oscillator state, (**??**), we conclude that both $\Phi_m^{(\mathrm{LP})}(x,y)$ and $\Phi_0^{(\mathrm{HO})}(x,y)$ are eigenstates of H_0' belonging to the ground level. This is an easy consequence of the commutation rule $[T_1, H_0'] = 0$. It is also easy to verify that $\langle \Psi_m^{(\mathrm{LP})}, \Phi_m^{(\mathrm{LP})} \rangle = 0$, $\forall m \geq 1$.

The situation is a bit different for the oscillator wave functions. The scalar product gives $\langle \Psi_0^{(\mathrm{HO})}, \Phi_0^{(\mathrm{HO})} \rangle = e^{-\pi^2}$; this implies that the Slater determinant is normalized within an error of $e^{-2\pi^2} = \mathcal{O}(10^{-9})$. From now on, this extra contribution will be neglected, and we will work with $\Psi_0^{(\mathrm{HO})}$ and $\Phi_0^{(\mathrm{HO})}$ as if they were mutually orthogonal.

The conclusions on the localization of the wave function will follow from the computation of V_d in two different models.

- In the first one, which we call the "Non-Lattice Model", the electrons are both localized around the origin, but they are described by different wave functions (this is necessary in order to satisfy the Pauli principle).

- In the second one, the "Lattice Model", the electrons are described by the same wave function localized around different space points. Of course, this model is closer to the FQHE as already discussed in [**?**], and in this perspective it has a particular interest.

Because of their poor localization properties, as compared with that of the Littlewood-Paley and the harmonic oscillator bases, we will omit from our analysis the Haar wave functions $\{\Psi_m^{(\mathrm{H})}(x,y)\}$.

10.3.1 Non-Lattice Model

We start manipulating the expression of V_d for the basis (**??**). Omitting the index d, we put

$$V_{\rm LP}^{(m,n)} = \int d^2\boldsymbol{r}_1 \int d^2\boldsymbol{r}_2 \, \frac{|\Psi_m^{\rm (LP)}(\boldsymbol{r}_1)|^2 |\Psi_n^{\rm (LP)}(\boldsymbol{r}_2)|^2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \tag{10.3.17}$$

where $m \neq n$ because of the Pauli principle. The y_1-integration can be performed easily. After some algebra, we can also perform the x_1-integration and we obtain

$$V_{\rm LP}^{(m,n)} = \frac{2^{m+n}}{4\pi^3}\sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} dx \; e^{-x^2/2} \int_{x+\pi/2^n}^{x+2\pi/2^n} dt \; \log\left[\varphi_{mn}(x,t)\right] \tag{10.3.18}$$

where we have defined

$$\begin{aligned}
\varphi_{mn}(x,t) \;\equiv\; & \frac{(t - \frac{\pi}{2^m} + \sqrt{x^2 + (t - \frac{\pi}{2^m})^2})(t + \frac{2\pi}{2^m} + \sqrt{x^2 + (t + \frac{2\pi}{2^m})^2})}{(t + \frac{\pi}{2^m} + \sqrt{x^2 + (t + \frac{\pi}{2^m})^2})(t - \frac{2\pi}{2^m} + \sqrt{x^2 + (t - \frac{2\pi}{2^m})^2})} \\
& \times \frac{(t - \pi(\frac{3}{2^n} + \frac{1}{2^m}) + \sqrt{x^2 + (t - \pi(\frac{3}{2^n} + \frac{1}{2^m}))^2})}{(t - \pi(\frac{3}{2^n} - \frac{1}{2^m}) + \sqrt{x^2 + (t - \pi(\frac{3}{2^n} - \frac{1}{2^m}))^2})} \\
& \times \frac{(t - \pi(\frac{3}{2^n} - \frac{2}{2^m}) + \sqrt{x^2 + (t - \pi(\frac{3}{2^n} - \frac{2}{2^m}))^2})}{(t - \pi(\frac{3}{2^n} + \frac{2}{2^m}) + \sqrt{x^2 + (t - \pi(\frac{3}{2^n} + \frac{2}{2^m}))^2})} \; .
\end{aligned}$$

The integration can be easily performed numerically, but obviously not analytically.

For computing the energy for the harmonic oscillator wave functions (**??**), we introduce the (two-dimensional) Fourier transform of the Coulomb potential

$$\frac{1}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} = \frac{1}{2\pi} \int \frac{d^2\boldsymbol{k}}{|\boldsymbol{k}|} \, e^{-i\boldsymbol{k}\cdot(\boldsymbol{r}_1 - \boldsymbol{r}_2)} \; .$$

In this way, the integrations on $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in V_d are reduced to Gaussian integrals and therefore can be easily performed. Denoting by $V_{\rm HO}^{(i,j)}$ the 'energies' corresponding to the wave functions (**??**), we find:

$$\begin{aligned}
V_{\rm HO}^{(0,1)} &= \frac{1}{2\pi} \int \frac{d^2\boldsymbol{k}}{|\boldsymbol{k}|} \, (1 - \frac{k_y^2}{2}) \, e^{-(k_x^2 + 2k_y^2 + 2k_x k_y)/2} \\
V_{\rm HO}^{(0,2)} &= \frac{1}{4\pi} \int \frac{d^2\boldsymbol{k}}{|\boldsymbol{k}|} \, k_y^2 \, (2 - \frac{k_y^2}{4}) \, e^{-(k_x^2 + 2k_y^2 + 2k_x k_y)/2} \\
V_{\rm HO}^{(1,2)} &= \frac{1}{4\pi} \int \frac{d^2\boldsymbol{k}}{|\boldsymbol{k}|} \, (1 - \frac{k_y^2}{2}) \, k_y^2 \, (2 - \frac{k_y^2}{4}) \, e^{-(k_x^2 + 2k_y^2 + 2k_x k_y)/2}
\end{aligned}$$

$V_{\rm LP}^{(1,3)} = 0.45784$	$V_{\rm LP}^{(1,6)} = 0.43412$	$V_{\rm LP}^{(1,n)} = 0.43376,\ n \geq 10$
$V_{\rm LP}^{(2,4)} = 0.77328$	$V_{\rm LP}^{(2,7)} = 0.74986$	$V_{\rm LP}^{(2,n)} = 0.74950,\ n \geq 11$
$V_{\rm LP}^{(3,5)} = 1.14544$	$V_{\rm LP}^{(3,8)} = 1.12345$	$V_{\rm LP}^{(3,n)} = 1.12312,\ n \geq 15$
$V_{\rm LP}^{(4,7)} = 1.51843$	$V_{\rm LP}^{(4,11)} = 1.51321$	$V_{\rm LP}^{(4,n)} = 1.51319,\ n \geq 15$
$V_{\rm LP}^{(6,n)} = 2.29705,$ $n \geq 15$	$V_{\rm LP}^{(9,n)} = 3.47063,$ $n \geq 20$	$V_{\rm LP}^{(15,n)} = 5.81705,$ $n \geq 25$

TABLE 10.1. Values of the matrix elements $V_{\rm LP}^{(m,n)}$ in (**??**) for different values of (m, n).

We report in Table **??** the numerical results for $V_{\rm LP}^{(m,n)}$ for several values of (m, n), to be compared with the corresponding energies of the harmonic oscillator, namely,

$$V_{\rm HO}^{(0,1)} = 0.91873\ ; \qquad V_{\rm HO}^{(0,2)} = 0.39019\ ; \qquad V_{\rm HO}^{(1,2)} = 0.11041\ . \quad (10.3.19)$$

A more extensive table of similar results may be found in [**?**].

10.3.2 Lattice Model

Before commenting on these results, we describe the second model, which differs from the first one in that here we compute the energy by using the same wave function centered at different lattice sites. Therefore, the energy will depend on a single quantum number, m.

We denote by $V_{\rm LP}^{(m)}$ the energy

$$V_{\rm LP}^{(m)} = \int d^2\boldsymbol{r}_1 \int d^2\boldsymbol{r}_2\, \frac{|\Psi_m^{\rm (LP)}(\boldsymbol{r}_1)|^2 |\Phi_m^{\rm (LP)}(\boldsymbol{r}_2)|^2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|}\ . \quad (10.3.20)$$

The computation of this matrix element follows the same steps as before, and we get a similar expression:

$$V_{\rm LP}^{(m)} = \frac{2^{2m}}{4\pi^3}\sqrt{\frac{\pi}{2}} \int_{-\infty}^{\infty} dx\, e^{-x^2/2} \int_{x+2\pi+\pi/2^m}^{x+2\pi+2\pi/2^m} dt\, \log\left[\varphi_m(x,t)\right] \quad (10.3.21)$$

where

$$\varphi_m(x,t) \equiv \frac{(t+\frac{2\pi}{2^m}+\sqrt{x^2+(t+\frac{2\pi}{2^m})^2})(t-\frac{4\pi}{2^m}+\sqrt{x^2+(t-\frac{4\pi}{2^m})^2})}{(t+\frac{\pi}{2^m}+\sqrt{x^2+(t+\frac{\pi}{2^m})^2})(t-\frac{5\pi}{2^m}+\sqrt{x^2+(t-\frac{5\pi}{2^m})^2})} \times \frac{(t-\frac{\pi}{2^m}+\sqrt{x^2+(t-\frac{\pi}{2^m})^2})^2}{(t-\frac{2\pi}{2^m}+\sqrt{x^2+(t-\frac{2\pi}{2^m})^2})^2}\ .$$

$V_{\rm LP}^{(1)} = 0.27083$	$V_{\rm LP}^{(2)} = 0.17462$	$V_{\rm LP}^{(3)} = 0.16376$
$V_{\rm LP}^{(4)} = 0.16141$	$V_{\rm LP}^{(5)} = 0.16085$	$V_{\rm LP}^{(6)} = 0.16071$
$V_{\rm LP}^{(7)} = 0.16067$	$V_{\rm LP}^{(m)} = 0.16066,\ m > 7$	

TABLE 10.2. Values of the matrix elements $V_{\rm LP}^{(m)}$ in (**??**) for different values of m.

The matrix element of the Coulomb energy between the ground state of the harmonic oscillator and the translated state (the most localized among all the wave functions of the harmonic oscillator) can be written as

$$\begin{aligned} V_{\rm HO}^{(0)} &\equiv \int d^2\boldsymbol{r}_1 \int d^2\boldsymbol{r}_2 \, \frac{|\Psi_0^{(\rm HO)}(\boldsymbol{r}_1)|^2 |\Phi_0^{(\rm HO)}(\boldsymbol{r}_2)|^2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \\ &= \frac{1}{2\pi} \int \frac{d^2\boldsymbol{k}}{|\boldsymbol{k}|} \, e^{-2\pi i k_y - k_y^2 - k_x^2/2 - k_x k_y}. \end{aligned} \tag{10.3.22}$$

The different values of $V_{\rm LP}^{(m)}$ are reported in Table **??**, whereas the energy of the harmonic oscillator is

$$V_{\rm HO}^{(0)} = 0.16515. \tag{10.3.23}$$

As already said, all these results can be easily understood in terms of localization properties of the single electron wave functions.

First of all, we observe that, whereas for the first model the energy increases as m and n both increase, for the lattice model it is just the opposite, the energy decreases for increasing m. The reason is clear. From the definitions (**??**), (**??**) and (**??**), we see that, when m increases, both $\Psi_m^{(\rm LP)}(\boldsymbol{r})$ and $\Phi_m^{(\rm LP)}(\boldsymbol{r})$ improve their localization. Since, in the Lattice Model, the electrons are localized at a distance of 2π, which corresponds to the lattice spacing, it is not surprising that the lower bound for the ground state energy is obtained when the electron wave function reaches its best localization, that is, in the classical limit (where these wave functions are nothing but δ functions centered on the lattice sites). This is so because, in this way, the distance between the electrons is maximized, so that the Coulomb interaction reaches its minimum value. These considerations explain very well the results in Table **??**. In fact, we see that, when m increases, the energy decreases from 0.27083 to the asymptotic value 0.16066. This value is already reached for $m = 9$ and stays essentially constant for larger m. This value could also be predicted in a heuristic way. From the definition (**??**) of $\Psi_m^{(\rm LP)}(\boldsymbol{r})$, we deduce that, for very large m, this function behaves like a $\Psi_\infty^{(\rm LP)}(\boldsymbol{r})$, whose square modulus is essentially a classical wave function,

that is, a δ function,

$$|\Psi_\infty^{(\mathrm{LP})}(\boldsymbol{r})|^2 = \frac{1}{\sqrt{\pi}}\, e^{-x^2}\delta(x-y)\ . \tag{10.3.24}$$

In the same way, we have $|\Phi_\infty^{(\mathrm{LP})}(\boldsymbol{r})|^2 = \frac{1}{\sqrt{\pi}}\, e^{-x^2}\delta(x-2\pi-y)$. We can compute the energy E_∞ in this limit and we get

$$\begin{aligned} E_\infty &\equiv \int d^2\boldsymbol{r}_1 \int d^2\boldsymbol{r}_2\, \frac{|\Psi_\infty^{(\mathrm{LP})}(\boldsymbol{r}_1)|^2 |\Phi_\infty^{(\mathrm{LP})}(\boldsymbol{r}_2)|^2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|} \\ &= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dx\, e^{-x^2/2}}{\sqrt{x^2+2\pi x+2\pi^2}} \ = \ 0.16066. \end{aligned}$$

Another interesting remark is that the energy of the harmonic oscillator, $V_{\mathrm{HO}}^{(0)} = 0.16515$, is slightly bigger than almost all the $V_{\mathrm{LP}}^{(m)}$. This difference could be interpreted again by a better localization of the functions $\Psi_m^{(\mathrm{LP})}(\boldsymbol{r})$ with respect to that of $\Psi_0^{(\mathrm{HO})}(\boldsymbol{r})$.

Let us comment now on the results of the Non-Lattice Model. In particular, let us show that the results in Table **??** also suggest the relevance of the wavelets in the analysis of the two-dimensional gas system.

This time, the two electrons are both localized around the origin. We expect that the more localized the wave functions are, the larger is the overlap between them and, therefore, the larger is the energy. In a classical picture, it would be as if we put two pointlike charges at the same point. This system is not stable, of course, and we expect a very high energy for this configuration. This is exactly what happens. If we try to compute the energy $V_{\mathrm{LP}}^{(m,n)}$ for m and n very large, we expect that the result is the same as that obtained by computing

$$V_{\mathrm{LP}}^{\infty} \equiv \int d^2\boldsymbol{r}_1 \int d^2\boldsymbol{r}_2\, \frac{|\Psi_\infty^{(\mathrm{LP})}(\boldsymbol{r}_1)|^2 |\Psi_\infty^{(\mathrm{LP})}(\boldsymbol{r}_2)|^2}{|\boldsymbol{r}_1-\boldsymbol{r}_2|} = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} dx\, \frac{e^{-x^2/2}}{\sqrt{x^2}},$$

which diverges, as expected. More comments on these and other numerical results can be found in [**?**].

Finally, we observe again that the wavelet wave functions appear to be better localized than the oscillator ones. This is deduced from the fact that, for this model, the best localized functions correspond to the maximum of the energy. In fact, we have $V_{\mathrm{HO}}^{(0,1)} = 0.91873$, $V_{\mathrm{HO}}^{(0,2)} = 0.39019$ and $V_{\mathrm{HO}}^{(1,2)} = 0.11041$. We see that the maximum of these values corresponds to the most localized wave function and, yet, it is much smaller than the results obtained with wavelets.

10.4 Wavelet Bases for the LLL

The results just discussed clearly show that orthonormal wavelets can be a good starting point for constructing the ground state of the FQHE. In this section, we will describe several bases for the LLL, mostly following [**?**].

10.4.1 The Haar Basis

Let us begin with the orthonormal bases of the LLL generated by the Haar wavelet basis $\{h_{mn}(x),\, m,n\in\mathbb{Z}\}$, described in (**??**). Of course, since $h(x)$ is a discontinuous function, its localization in frequency space is poor. However, since the transformation (**??**) is not a Fourier transform, it is not clear *a priori* that the corresponding functions $\{H_{mn}(x,y)\}$ will also have a poor localization in both variables. In fact, we will see below that it is *not* the case, by investigating the asymptotic behavior of the basis functions. This should not be a surprise in view of our previous comment on the relation between the asymptotic behaviors of $\Psi^{\rm L}(x,y), h(P)$ and $\widehat{h}(Q)$ in Section **??**.

From (**??**), we get

$$\begin{aligned} H_{mn}(x,y) \;=\; & \frac{e^{ixy/2}}{\sqrt{2}\pi^{3/4}}\,2^{-m/2}\left[\int_{2^m n}^{2^m(n+1/2)} dP\, e^{iyP}e^{-(x+P)^2/2}\right. \\ & \left. -\int_{2^m(n+1/2)}^{2^m(n+1)} dP\, e^{iyP}e^{-(x+P)^2/2}\right] \end{aligned} \tag{10.4.1}$$

Using standard results on Gaussian integrals [**?**, **?**], we find

$$\begin{aligned} H_{mn}(x,y) \;=\; & \frac{e^{-ixy/2}e^{-y^2/2}}{2\pi^{1/4}}\,2^{-m/2}\left[2\,\Xi(x-iy+2^m n+2^{m-1})\right. \\ & \left. -\,\Xi(x-iy+2^m n)-\Xi(x-iy+2^m n+2^m)\right] \end{aligned} \tag{10.4.2}$$

where $\Xi(z)=\Phi(z/\sqrt{2})$ and the error function $\Phi(z)$ is defined by the integral

$$\Phi(z)\equiv\ \mathrm{erf}\, z=\frac{2}{\sqrt{\pi}}\int_0^z\ e^{-t^2}\,dt,\quad z\in\mathbb{C}.$$

For $m=n=0$, in particular, this gives

$$H_{00}(x,y)=\frac{e^{-ixy/2}e^{-y^2/2}}{2\pi^{1/4}}\ \left[2\,\Xi(x-iy+1/2)-\Xi(x-iy)-\Xi(x-iy+1)\right]. \tag{10.4.3}$$

The function $H_{00}(x,y)$ is much better localized in the x variable than in y, as it is shown numerically in [**?**]. This is confirmed by estimating the asymptotic behavior of the function H_{00}, which may be deduced from the

asymptotic expansion of the error function [**?**, **?**], which yields:

$$\Xi(z) \simeq 1 - \sqrt{\frac{2}{\pi}}\,\frac{e^{-z^2/2}}{z}\left(1 + O(\frac{1}{z^2})\right), \quad z \to \infty, |\arg z| < \frac{3\pi}{4}. \quad (10.4.4)$$

Thus we find the following asymptotic expansion for the function $H_{00}(x, y)$:

$$H_{00}(x,y) \simeq \frac{e^{ixy/2}e^{-x^2/2}}{2\pi^{1/4}}\sqrt{\frac{2}{\pi}}\left(\frac{1}{x-iy} + \frac{e^{-1/2-x+iy}}{x-iy+1} - 2\frac{e^{-1/8-(x-iy)/2}}{x-iy+1/2}\right), \quad (10.4.5)$$

which displays the Gaussian localization of the wave function in the variable x and shows the rather poor localization in y.

An analogous behavior can be obtained for the generic function $H_{mn}(x, y)$ given in (**??**), where n indexes the center of the original mother wavelet and $m \in \mathbb{Z}$ is the scale parameter. As we have already discussed after (**??**), the asymptotic behavior of $H_{mn}(x, y)$ in x is governed by the asymptotic behavior of $h_{mn}(P)$, and the one in y by that of the Fourier transform of $h_{mn}(P)$. Since in the present case, $h_{mn}(P)$ has compact support (increasing monotonically with m), we expect $H_{mn}(x, y)$ to be strongly localized in x and delocalized in y, and that its decay in x gets faster for smaller m. This is indeed the case, as may be seen by an explicit computation along the same lines as above [**?**, **?**].

10.4.2 The Littlewood-Paley Basis

Another simple example of an orthonormal wavelet basis of $L^2(\mathbb{R})$ is the Littlewood-Paley basis [**?**], generated from the mother wavelet (**??**). Like the Haar basis, it comes from multiresolution analysis (MRA). The behavior of the function (**??**) is, in a sense, complementary to that of the Haar wavelet: it is very well localized in frequency space (it has a compact support)

$$\widehat{\psi}(\omega) = \begin{cases} (2\pi)^{-1/2}, & \text{if } \pi \le |\omega| \le 2\pi, \\ 0, & \text{otherwise,} \end{cases} \quad (10.4.6)$$

whereas, as one can see from (**??**), it decays like $1/x$ in configuration space.

We will see that an analogous complementary behavior is found also in the form of the wave functions in the LLL [**?**]. We will show, in fact, that they are exponentially localized in the y-variable, while in the other variable they will behave like $1/x$.

In order to perform the integration in (**??**), it is convenient to use the Fourier transform of $\psi_{mn}(x) = 2^{-m/2}\psi(2^{-m}x - n)$. We have:

$$\Psi_{mn}(x,y) = \frac{e^{ixy/2}}{\sqrt{2}\pi^{3/4}}\int_{-\infty}^{\infty} dP\, e^{iyP}e^{-(x+P)^2/2} \times \frac{2^{-m/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, \widehat{\psi}(\omega)\, e^{i\omega(2^{-m}P-n)}$$

$$= \frac{e^{ixy/2}2^{-m/2}}{2\pi^{7/4}\sqrt{2}} \int_{\mathcal{D}} d\omega\, e^{-i\omega n} \int_{-\infty}^{\infty} dP\, e^{iP(y+2^{-m}\omega)} e^{-(x+P)^2/2}, \tag{10.4.7}$$

where we have defined the set $\mathcal{D} = [-2\pi, -\pi] \cup [\pi, 2\pi]$ and the order of integrations can be exchanged by Fubini's theorem [**?**].

Performing the simple Gaussian integration in P, we find

$$\Psi_{mn}(x,y) = \frac{e^{ixy/2}2^{-m/2}}{2\pi^{5/4}} \int_{\mathcal{D}} d\omega\, e^{-i\omega n} e^{-ix(y+2^{-m}\omega)}\, e^{-(y+2^{-m}\omega)^2/2}, \tag{10.4.8}$$

which can again be explicitly computed, in terms of the modified error integral used in the previous section. We get:

$$\begin{aligned}\Psi_{mn}(x,y) = \frac{2^{(m-3)/2}}{\pi^{3/4}}\, e^{ixy/2}\, e^{iy2^m n}\, e^{-(x+2^m n)^2/2} \\ \times \left[\Xi(2^{1-m}\pi - (y+ix) - in2^m) - \Xi(2^{-m}\pi - (y+ix) - in2^m)\right. \\ \left. + \Xi(2^{1-m}\pi + (y+ix) + in2^m) - \Xi(2^{-m}\pi + (y+ix) + in2^m)\right].\end{aligned} \tag{10.4.9}$$

This expression is rather similar to (**??**). For $m = n = 0$, in particular, we obtain

$$\begin{aligned}\Psi_{00}(x,y) &= \frac{e^{ixy/2}e^{-x^2/2}}{2\sqrt{2}\pi^{3/4}} \left[\Xi(2\pi - y - ix) - \Xi(\pi - y - ix)\right. \\ &\quad \left. + \Xi(2\pi + y + ix) - \Xi(\pi + y + ix)\right]\end{aligned} \tag{10.4.10}$$

Graphs of this function are displayed in [**?**]. Using again the asymptotic formula (**??**), this yields

$$\begin{aligned}\Psi_{00}(x,y) \simeq \frac{e^{-ixy/2}e^{-y^2/2}}{2\pi^{5/4}} \left[-\frac{e^{2\pi(y+ix)}e^{-2\pi^2}}{|2\pi - y - ix|} + \frac{e^{\pi(y+ix)}e^{-\pi^2/2}}{|\pi - y - ix|}\right. \\ \left. - \frac{e^{-2\pi(y+ix)}e^{-2\pi^2}}{|2\pi + y + ix|} + \frac{e^{-\pi(y+ix)}e^{-\pi^2/2}}{|\pi + y + ix|}\right],\end{aligned} \tag{10.4.11}$$

which displays the exponential decay of $|\Psi_{00}(x,y)|$ in y and the slow decay in x.

This is the announced complementarity with respect to the Haar basis: the first one is better localized in x, the other one in y.

10.4.3 The Journé Basis

The Journé basis is an example of orthonormal wavelet basis of $L^2(\mathbb{R})$ which is not constructed from a multiresolution analysis [**?**]. It is generated

by a mother wavelet very similar to the Littlewood-Paley function (**??**),

$$\widehat{\psi}_{\rm J}(\omega) = \begin{cases} (2\pi)^{-1/2}, & \text{if } \frac{4\pi}{7} \leq |\omega| \leq \pi \text{ and } 4\pi \leq |\omega| \leq \frac{32\pi}{7} \\ 0, & \text{otherwise.} \end{cases} , \tag{10.4.12}$$

Thus it has compact support in the Fourier variable and is therefore delocalized in configuration space. The computation of the wave functions corresponding to this set in the LLL is very similar to the one of the Littlewood-Paley basis and gives the following result [**?**]:

$$\begin{aligned} \Psi^{(\rm J)}_{mn}(x,y) = \frac{2^{(m-3)/2}}{\pi^{3/4}}\, e^{ixy/2}\, e^{iy2^m n}\, e^{-(x+2^m n)^2/2} \\ \times \big[\Xi(i2^m n - 4\pi 2^{-m} + ix + y) - \Xi(2^m n - \pi 2^{-m} + ix + y) \\ +\Xi(i2^m n + \pi 2^{-m} + ix + y) - \Xi(i2^m n + 4\pi 2^{-m} + ix + y) \\ -\Xi(i2^m n - \tfrac{32\pi}{7} 2^{-m} + ix + y) + \Xi(i2^m n - \tfrac{4\pi}{7} 2^{-m} + ix + y) \\ -\Xi(i2^m n - \tfrac{4\pi}{7} 2^{-m} + ix + y) + \Xi(i2^m n + \tfrac{32\pi}{7} 2^{-m} + ix + y)\big] . \end{aligned} \tag{10.4.13}$$

We see that, apart from the complication arising from the different support of the mother wavelet, the above expression is quite close to the one for the Littlewood-Paley basis. As far as the asymptotic behavior of the above set is concerned, all the considerations made for the Littlewood-Paley basis can be repeated. The speed of decay in x is essentially unchanged by the value of m, except for the usual normalization considerations. The value of m, however, directly affects the behavior in y. For instance, we observe that $\Psi^{(\rm J)}_{-6,0}(x,y)$ is essentially zero for x, y not too large, whereas it begins to be different from zero for larger values only (this follows from the increasing delocalization in y when m decreases). We expect an analogous effect for $\Psi^{(\rm LP)}_{-6,0}(x,y)$ also, although the range of values considered in [**?**] is still not sufficient to display this delocalization.

10.4.4 Spline Bases and Final Remarks

Finally, we discuss the projection in the LLL of some orthonormal spline bases of $L^2(\mathbb{R})$. We discuss in particular the linear, quadratic, cubic and quartic splines. The extension of the results to other splines is straightforward.

The essential ingredient for computing the basis is the mother wavelet which generates the orthonormal basis of the LLL via the usual integral transform (**??**). It is explained in detail in [**?**] how to build up these functions. The computations are somewhat tedious and become more and more difficult with increasing order of the splines ($N = 1$ for linear splines, $N = 2$ for quadratic splines, and so on). We therefore omit the details and show

only the first mother wavelet in the Fourier variable:

$$\widehat{h^{(1)}}(\omega) = \sqrt{\frac{3}{2\pi}}\, e^{i\omega/2}\, \sin^2(\omega/4)\, \left(\frac{\sin\,\omega/4}{\omega/4}\right)^2\, \sqrt{\frac{2-\cos\,\omega/2}{(2+\cos\,\omega/2)(2+\cos\,\omega)}} \tag{10.4.14}$$

5 The other mother functions $h^{(N)}$, for $N = 2, 3, 4$, are of the same type, only more complicated. Hence we omit their analytical form, which may be found in [**?**], but show in Figure **??** the moduli of all four functions. It turns out that the different functions $h^{(N)}$ look very similar to each other, so that we do not expect big differences in their projections $\Psi^{(N)}_{mn}(x, y)$ in the LLL. This is indeed what is observed, and so we will consider the function $\Psi^{(1)}_{mn}(x, y)$ only. First, $\Psi^{(1)}_{3,0}(x, y)$ appears to be the most localized function in both variables. On the contrary, the most delocalized one is that with $m = -6$. A strong delocalization, expecially in x, is also evident for $\Psi^{(1)}_{-3,0}(x, y)$. However, both $\Psi^{(1)}_{-3,0}(x, y)$ and $\Psi^{(1)}_{-6,0}(x, y)$ oscillate very much and it is not clear at all from the plots how they do behave at infinity. Why is this so? As it is well known [**?**, Section 5.4], the original spline mother wavelets of order N ($N = 1, 2, 3, ...$) have compact support in space, but the set generated by scaling and translation consists of mutually nonorthogonal functions. In order to obtain an orthogonal basis, one has to modify the original mother wavelets, losing in this way the original compactness of the support and obtaining the functions $h^{(N)}$ above. Therefore both the mother wavelet and its Fourier transform have noncompact support. Therefore, there is no *a priori* reason for having well localized functions in the LLL if one starts from orthogonal splines.

We end these considerations with a brief comparison between the spline wave functions and the other functions previously introduced. First of all we notice that the spline wave functions seem to be more localized in x than the Journé and the Littlewood-Paley basis for positive m. Because of the oscillations discussed above, the opposite situation holds for negative m. Moreover, they also appear to be less localized in x with respect to the Haar basis. This is expected from the discussion of Section **??**, since this basis is generated by an x-compactly supported mother wavelet. Similar conclusions could be drawn for the behavior of all these functions in y.

What is still missing in this analysis is the computation of the matrix elements of the Coulomb potential in the bases discussed here. This would give, in fact, a final answer about the relevance for the analysis of the FQHE of various classes of orthonormal bases and, in particular, of the orthonormal wavelet bases. Of course, contrary to what has been done in [**?**], it seems hardly possible to compute these matrix elements analytically. Nevertheless, taking also into account the results of the toy model, it is reasonable to expect that the numerical results will be good enough to suggest a deeper analysis of this subject.

(a) (b)

(c) (d)

FIGURE 10.1. Modulus of the first four splines: (a) $\widehat{h^{(1)}}(\omega)$; (b) $\widehat{h^{(2)}}(\omega)$; (c) $\widehat{h^{(3)}}(\omega)$; (d) $\widehat{h^{(4)}}(\omega)$.

10.5 Magnetic Translations and Multiresolution Analysis

This is, however, not the end of the story. Indeed, recent work by one of us [**?**] has revealed a deeper, yet unexpected, connection between multiresolution analysis (MRA) and the FQHE. Whereas wavelets — mostly generated from an MRA — were used in the previous sections for producing nicely localized orthonormal bases, here one establishes, via the so-called kq-representation, a direct connection between FQHE states generated by magnetic translations (thus Gabor states) and the coefficients of the low-pass filter m_o of an MRA. Thus we end up using MRA for constructing physically meaningful *Gabor* bases! In order to streamline the discussion, the relevant information concerning the MRA and the kq-representation is postponed to the Appendix (Section **??**).

10.5.1 Lattices of Magnetic Translations and the Filling Factor

Let us go back to the magnetic translations $T(\boldsymbol{a}_i)$ defined in (**??**), and let Λ be the lattice generated by $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. For simplicity, we take first for Λ a square lattice with unit cell of area 2π:

$$\boldsymbol{a}_1 = a(1,0),\ \boldsymbol{a}_2 = a(0,1),\ a^2 = 2\pi. \tag{10.5.1}$$

This is not a real limitation, as far as the Coulomb energy is not computed explicitly, and it is quite useful to keep the notation simple. Moreover, the generalization to lattices of arbitrary shape is straightforward.

With this choice, (**??**) becomes

$$\begin{aligned} \Psi_{m,n}(x,y) &= T(\boldsymbol{a}_1)^m T(\boldsymbol{a}_2)^n \Psi(x,y) \\ &= (-1)^{mn} e^{i\frac{1}{2}a(my-nx)} \Psi(x+ma, y+na). \end{aligned} \tag{10.5.2}$$

Taking now $\Psi(x,y)$ in the form (**??**), we obtain after some algebra

$$\Psi_{m,n}(x,y) = \frac{e^{i\frac{xy}{2}+iamy}}{\sqrt{2}\pi^{3/4}} \int_{-\infty}^{\infty} dP\, e^{i(y+na)P-(x+ma+P)^2/2}\, h(P)\,. \tag{10.5.3}$$

We are interested now in finding conditions on $h(P)$ that guarantee the orthogonality of the resulting wave functions, viz.

$$S_{m,n} \equiv \langle \Psi_{0,0}, \Psi_{m,n} \rangle = \delta_{m,0}\, \delta_{n,0}. \tag{10.5.4}$$

Inserting (**??**), we find

$$S_{m,n} = \int_{-\infty}^{\infty} dp\, e^{inap}\, \overline{h(p+ma)}\, h(p). \tag{10.5.5}$$

Following [**?**], we introduce now the kq-representation. To that effect, it is convenient to split the integral over $\mathbb{R}$ in an infinite sum of integrals restricted to $[ra, (r+1)a)$, $r \in \mathbb{Z}$, use the kq-representation and then write everything in terms of a single integral over the unit cell $\Box$ in the (k, q)-plane. Using (**??**) and the well-known equality

$$\sum_{l\in\mathbb{Z}} e^{ixl2\pi/c} = c \sum_{l\in\mathbb{Z}} \delta(x - cl), \tag{10.5.6}$$

we get

$$\begin{aligned} S_{m,n} &= \sum_{r\in\mathbb{Z}} \int_{ra}^{(r+1)a} dp\, e^{inap}\, \overline{h(p+ma)}\, h(p) \\ &= \sum_{r\in\mathbb{Z}} e^{inra^2} \int_0^a dp\, e^{inap}\, \overline{h(p+(r+m)a)}\, h(p+ra) \\ &= \sum_{r\in\mathbb{Z}} \frac{1}{a} \int_0^a dq \int_0^a dk \int_0^a dk'\, e^{ir(k-k')a}\, e^{inaq-ik'ma}\, h(k,q)\, \overline{h(k',q)}, \end{aligned}$$

so that

$$S_{m,n} = \int_\Box dk\, dq\, e^{inaq-ikma}\, |h(k,q)|^2. \tag{10.5.7}$$

By the completeness of the set $\{e^{inaq-ikma},\ n, m \in \mathbb{Z}\}$ in the unit cell $\Box$, we conclude that $S_{m,n} = \delta_{m,0}\, \delta_{n,0}$ if and only if $h(k,q)$ is a pure phase, so that $|h(k,q)|$ is independent of k and q. This result can be considered as a slight generalization of the procedure discussed in [**?**] to the FQHE for filling factor $\nu = 1$.

It is easy to generalize this result to a filling factor $\nu = \frac{1}{2}$, using the following idea. A filling factor $\nu = 1$ corresponds to our square lattice Λ (of spacing $a = \sqrt{2\pi}$) with all the sites occupied. Therefore, a $\nu = \frac{1}{2}$ 2-D electron gas can be associated to the same lattice, but only half occupied: one lattice site is free, but the one on the left and the one on the right are occupied. If we require the orthonormality of the related set of single electron wave functions, it is enough to ask that $S_{m,2n} = \delta_{m,0}\, \delta_{n,0}$. This is equivalent also to choosing a different lattice Λ_2, with a unit cell twice as large as before and basis vectors $a(1,0)$ and $2a(0,1)$ (of course, we could as well choose any other lattice with unit cell of area 4π). Equation (**??**) then gives

$$S_{m,2n} = \int_\Box dk\, dq\, e^{i2naq-ikma}\, |h(k,q)|^2 = \delta_{m,0}\, \delta_{n,0}, \tag{10.5.8}$$

which can be rewritten as

$$\frac{1}{2} \int_\Box dk\, dq\, e^{inaq-ikma} \left(\left| h(k, \frac{q}{2}) \right|^2 + \left| h(k, \frac{q+a}{2}) \right|^2 \right) = \delta_{m,0}\, \delta_{n,0}. \tag{10.5.9}$$

Using again the completeness of the functions $e^{inaq-ikma}$, $n, m \in \mathbb{Z}$ in $\square$, this implies that

$$J_2(k,q) \equiv \left|h(k,\frac{q}{2})\right|^2 + \left|h(k,\frac{q+a}{2})\right|^2 = \frac{1}{\pi}, \quad \text{for almost all } k,q \in \square. \tag{10.5.10}$$

The generalization to $\nu = \frac{1}{M}$, $M \in \mathbb{N}$, is straightforward. We simply require the orthonormality of the wave functions located at a (e.g. horizontal) distance of M sites:

$$S_{m,Mn} = \int_{\square} dk\, dq\, e^{iMnaq-ikma}\, |h(k,q)|^2 = \delta_{m,0}\, \delta_{n,0},$$

and, proceeding as above, we deduce that $h(k,q)$ must satisfies the equality

$$J_M(k,q) \equiv \left|h(k,\frac{q}{M})\right|^2 + \left|h(k,\frac{q+a}{M})\right|^2 + \ldots + \left|h(k,\frac{q+(M-1)a}{M})\right|^2 = \frac{M}{2\pi}, \tag{10.5.11}$$

almost everywhere for $k, q \in \square$.

The extension to a filling $\nu = \frac{L}{M}$, with L and M relatively prime, can be performed by imposing that condition $S_{m,n} = \delta_{m,0}\, \delta_{n,0}$ holds only for those (m,n) corresponding to a rectangular lattice in which only L among M lattice sites are occupied. We will not consider this extension here.

10.5.2 From MRA to FQHE and Back

Using these results, we will now establish the correspondence between FQHE states and MRA.

As recalled in Section **??**, an MRA of $L^2(\mathbb{R})$ is determined by the sequence $\{h_n\}_{n\in\mathbb{Z}}$ of the Fourier coefficients of the 2π-periodic function $m_o(\omega)$, the low-pass filter of the MRA.

Now we use the sequence $\{h_n\}_{n\in\mathbb{Z}}$ to define the following function, which strongly resembles $m_o(\omega)$:

$$T_2(\omega) = \begin{cases} \frac{1}{\sqrt{a}} \sum_{l\in\mathbb{Z}} h_l e^{-il\omega a}, & \omega \in [0,a), \\ 0, & \text{otherwise.} \end{cases} \tag{10.5.12}$$

It is clear that $T_2(\omega)$ is square integrable and nonperiodic. In particular, by the normalization condition (**??**), we have

$$\|T_2\|_2^2 = \int_{\mathbb{R}} |T_2(\omega)|^2 d\omega = 1.$$

Therefore the kq-transform of this function, $t_2(k,q) = (ZT_2)(k,q)$, is well defined in $L^2(\square)$. ¿From (**??**), we find

$$t_2(k,q) = \frac{1}{\sqrt{a}} \sum_{n\in\mathbb{Z}} e^{-ikna} T_2(q+na). \tag{10.5.13}$$

The boundary conditions (**??**) are obviously satisfied: $t_2(k+a,q) = t_2(k,q)$ and $t_2(k,q+a) = e^{ika}t_2(k,q)$. It is easy to check that $t_2(k,q)$ satisfies also the orthonormality conditions (**??**). In fact, since we are interested to the value of $t_2(k,q)$ only in $\square$, and since $T_2(\omega)$ is different from zero only for $\omega \in [0,a[$, we conclude that, for $(k,q) \in \square$,

$$\begin{aligned} J_2(k,q) &= \frac{1}{a}\left(\left|T(\frac{q}{2})\right|^2 + \left|T(\frac{q+a}{2})\right|^2\right) \\ &= \frac{1}{a^2}\sum_{l,s\in\mathbb{Z}} h_l\,\overline{h_s}\,e^{i(s-l)qa/2}\,(1+(-1)^{l+s}), \end{aligned}$$

which is equal to $1/\pi$ a.e. in $k,q \in \square$, by (**??**). This implies that $t_2(k,q)$ gives rise to a family of functions $\Psi_{m,n}(x,y)$ in the LLL, mutually orthonormal and corresponding to $\nu = 1/2$. The explicit form of these $\Psi_{m,n}(x,y)$ is given in the next section, where we will also compare these results with the ones obtained in [**?**].

The above procedure can be easily extended to fillings $\nu = \frac{1}{2L}$ (the extension to an odd denominator is not so straightforward). Instead of (**??**), we define now the function

$$T_{2L}(\omega) = \begin{cases} \frac{1}{\sqrt{a}}\sum_{l\in\mathbb{Z}} h_l e^{-il\omega La}, & \omega \in [0,a)\ , \\ 0, & \text{otherwise.} \end{cases} \tag{10.5.14}$$

This is again a square integrable function satisfying $\|T_{2L}\|^2 = 1$. Defining $t_{2L}(k,q) = (ZT_{2L})(k,q)$ we have, for $k,q \in \square$, $t_{2L}(k,q) = \frac{1}{\sqrt{a}}T_{2L}(q) = \frac{1}{a}\sum_{l\in\mathbb{Z}} h_l e^{-ilqLa}$. Also $t_{2L}(k,q)$ satisfies the correct boundary conditions. With these definitions, using the rationality condition $a^2 = 2\pi$ and collecting separately the 'even' contributions, namely

$$\left|t_{2L}\left(k,\frac{q}{2L}\right)\right|^2,\ \left|t_{2L}\left(k,\frac{q+2a}{2L}\right)\right|^2,\ldots,$$

and the 'odd' ones ,

$$\left|t_{2L}\left(k,\frac{q+a}{2L}\right)\right|^2,\ \left|t_{2L}\left(k,\frac{q+3a}{2L}\right)\right|^2,\ldots,$$

we get

$$\begin{aligned} &J_{2L}(k,q) \\ &= \left|t_{2L}\left(k,\frac{q}{2L}\right)\right|^2 + \left|t_{2L}\left(k,\frac{q+a}{2L}\right)\right|^2 + \ldots + \left|t_{2L}\left(k,\frac{q+(2L-1)a}{2L}\right)\right|^2 \\ &= L\left(\left|t_{2L}\left(k,\frac{q}{2L}\right)\right|^2 + \left|t_{2L}\left(k,\frac{q+a}{2L}\right)\right|^2\right) \\ &= \frac{L}{a^2}\sum_{l,s\in\mathbb{Z}} h_l\,\overline{h_s}\,e^{i(s-l)qa/2}(1+(-1)^{l+s}), \end{aligned}$$

which is again independent of k and q, since it equals L/π a.e. in $\Box$, by condition (**??**). Finally, equation (**??**) is a consequence of the equality $\nu^{-1} = M = 2L$. We conclude that $t_{2L}(k,q)$ produces, in configuration space, a set of mutually orthonormal wave functions spanning the LLL for $\nu = \frac{1}{2L}$.

Conversely, we show now that a function $h(k,q)$ which produces an orthonormal set of translated functions in the LLL determines a sequence $\{h_n\}$ satisfying condition (**??**) and, therefore, generating an MRA. Let indeed $h(k,q) \in L^2(\Box)$, satisfying the boundary conditions (**??**) and such that

$$|h(k,q/2)|^2 + |h(k,(q+a)/2)|^2 = \frac{1}{\pi}\,, \qquad \text{a.e. in } \Box\,. \tag{10.5.15}$$

This means that, in configuration space, the corresponding set $\{\Psi_{m,n}(x,y)\}$ is orthonormal for $\nu = 1/2$. Define now

$$h_n(k) = \int_0^a dq\, e^{inaq}\, h(k,q), \quad k \in [0,a). \tag{10.5.16}$$

Although $h_n(k)$ is a function of k, it is straightforward to check that $\sum_{n\in\mathbb{Z}} h_n(k)\,\overline{h_{n+2l}(k)}$ actually does not depend on k for *any* choice of $h(k,q)$, if the equality (**??**) is satisfied. Indeed, using (**??**) and (**??**), we find

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} h_n(k)\,\overline{h_{n+2l}(k)} &= a\int_0^a dq\,|h(k,q)|^2\, e^{-2ilaq} \\
&= \frac{a}{2}\int_0^a dq\, e^{-ilaq}\,(|h(k,q/2)|^2 + |h(k,(q+a)/2)|^2) \\
&= \frac{a}{2\pi}\int_0^a dq\, e^{-ilaq} \;=\; \delta_{l,0}.
\end{aligned}$$

This result shows that any orthonormal basis in the LLL for a filling factor $\nu = \frac{1}{2}$ produces an MRA of $L^2(\mathbb{R})$ which, in general, depends on an external parameter $k \in [0,a)$. Again, the extension to a filling factor $\nu = \frac{1}{2L}$, $L \in \mathbb{N}$, is straightforward.

In conclusion, there exists a complete equivalence between an MRA and an orthonormal set of single electron wave functions in the LLL (for $\nu = \frac{1}{2L}$, $L \in \mathbb{N}$).

10.5.3 Comparison with Wavelet Orthonormal bases

It remains to analyze the relation between the approach just described and the one originally proposed in [**?**] and described in Section **??**. On one hand the method developed here applies only to orthonormal bases generated by an MRA, whereas that based on the transform (**??**) works for any orthonormal basis. More important, whereas Section **??** leads to wavelet bases,

whose elements are obtained by acting on the mother function by dilation and translation operators, here one acts by the magnetic translations (**??**), that is, one obtains a *Gabor* basis!

Nevertheless, since the two methods have much in common, we expect that the resulting wave functions should not be very different. This is indeed true, as we shall see now, at least in the case of the Haar basis.

For this case, the set $\{h_n\}_{n\in\mathbb{Z}}$ reduces to $h_0 = h_1 = \frac{1}{\sqrt{2}}$, and all the other coefficients vanish. The corresponding ground state wave function $H_{00}(x, y)$ is given in (**??**), and its asymptotic behavior (**??**), displays the Gaussian localization of the wave function in the variable x and the rather poor localization in y.

Let us now proceed as above. For a filling factor $\nu = \frac{1}{2}$ and a generic MRA, the function T_2 which produces an orthonormal set of translates in the LLL is given in (**??**). Using the transformation rule (**??**), we get

$$\begin{aligned} T_2(x,y) &= \frac{e^{ixy/2}}{\sqrt{2}\pi^{3/4}} \int_{-\infty}^{\infty} dQ\, e^{iyQ-(x+Q)^2/2}\, T_2(Q) \\ &= \frac{\sqrt{a}e^{ixy/2}}{2\pi^{3/4}} \sum_{l\in\mathbb{Z}} h_l \int_0^a dQ\, e^{iQ(y-la)-(x+Q)^2/2}. \end{aligned}$$

Specializing to the coefficients corresponding to the Haar wavelet, this gives

$$T_2(x,y) = \frac{\sqrt{a}e^{ixy/2}}{2^{3/2}\pi^{5/4}} \int_0^a dQ\, e^{iQy-(x+Q)^2/2}(1 + e^{-iQa}). \qquad (10.5.17)$$

This function can again be written in terms of the modified error function $\Xi(z)$:

$$\begin{aligned} T_2(x,y) &= \frac{\sqrt{a}\, e^{-ixy/2-y^2/2}}{4\pi^{3/4}} \left[\Xi(x+a-iy) + \Xi(x+a-i(y-a))\right. \\ &\qquad \left. -\Xi(x-iy) - \Xi(x-i(y-a))\right], \qquad (10.5.18) \end{aligned}$$

$\mathrm{e}^{-ixy/2-y^2/2}4\pi^{3/4}$ $[\Xi(x+a-iy)+\Xi(x+a-i(y-a))and its asymptotic behavior follows from (**??**) : to$

$$T_2(x,y) \simeq \frac{\sqrt{a}e^{ixy/2-x^2/2}}{2^{3/2}\pi^{5/4}} \left[\frac{1}{x-iy} + \frac{e^{\pi-ia(x-iy)}}{x-i(y-a)}\right.$$

$$\left. -\frac{e^{-\pi-a(x-iy)}}{x+a-iy)} - \frac{e^{-a(x-iy)(1+i)}}{x+a-i(y-a)}ght\right].$$

Thus, although two procedures yield different wave functions, H_{00} and T_2, the latter have the same asymptotic behavior, that is, they lead to the same localization properties of the electrons. This result can be considered as a consequence of the Balian-Low Theorem **??** applied to the present situation, on one hand, and of the Battle Theorem **??** for the wavelet case. Indeed, both theorems give severe constraints on the localization properties of a wave function when orthonormality is imposed.

As a final comment, we may remark that the whole procedure goes over to higher Landau levels [**?**], simply because the wave functions of the latter have a very simple form in the $P\widetilde{P}$-representation as discussed after (**??**).

10.6 Conclusion

The most obvious outcome of this review is that more work is needed. On one hand, wavelet orthonormal bases seem useful for describing the LLL one electron wave functions, as suggested by the toy model of Section **??**. On the other hand, the equivalence between FQHE states and basis vectors stemming from an MRA is intriguing and certainly deserves more scrutiny. In particular, in the context of the FQHE, the perennial question [**?**], wavelets or Gabor?, has the unexpected answer: Both! More precisely, MRA and magnetic translations conspire to yield a physically significant analysis tool.

Now there remains to determine which approach gives the best wave functions. Clearly, the touchstone will be the (numerical) computation of Coulomb energies in realistic models, but this is another story!.

0.1 Appendix: Two Mathematical Tools

0.1.1 The kq-Representation and the Zak Transform

Originally introduced in many-body physics [**?**, **?**], the kq-representation has become a mathematically interesting object. We give here only a few definitions and refer to [**?**, **?**, **?**] for further reading and for applications.

The genesis of the kq-representation consists in the well-known possibility of a simultaneous diagonalization of any two commuting operators. In [**?**], it is shown that the following distributions

$$\psi_{kq}(x) = \sqrt{\frac{2\pi}{a}} \sum_{n\in\mathbb{Z}} e^{ikna}\delta(x-q-na), \quad k \in [0,a),\ q \in [0,2\pi/a) \quad (0.1.1)$$

are (generalized) eigenstates of both $T(a) = e^{ipa}$ and $\tau(\frac{2\pi}{a}) = e^{ix2\pi/a}$. Here a is a positive real number which plays the role of a lattice spacing. As discussed in [**?**], these $\psi_{kq}(x)$ are Bloch-like functions corresponding to infinitely localized Wannier functions. They also satisfy orthogonality and closure properties. Thus they can be used to define a new representation of the wave functions by means of the integral (Zak) transform $Z : L^2(\mathbb{R}) \to L^2(Q)$, where $Q = [0,a) \times [0,2\pi/a)$, defined as follows:

$$h(k,q) \equiv (ZH)(k,q) \equiv \int_{\mathbb{R}} dx\, \overline{\psi_{kq}(x)}\, H(x), \quad H \in L^2(\mathbb{R}),\ (k,q) \in Q. \quad (0.1.2)$$

¿From now on, we will assume that $a^2 = 2\pi$, which corresponds to fixing the spacing of the lattice underlying the 2-D electron gas.

Replacing $\psi_{kq}(x)$ by its explicit expression (**??**), we obtain

$$h(k,q) = (ZH)(k,q) = \frac{1}{\sqrt{a}} \sum_{n\in\mathbb{Z}} e^{-iknq} H(q+na), \tag{0.1.3}$$

which can be inverted and gives the x-representation $H(x) \in L^2(\mathbb{R})$ of a function $h(k,q) \in L^2(Q)$ as follows:

$$H(x) = (Z^{-1}h)(x) = \int_Q dk\, dq\, \psi_{kq}(x)\, h(k,q). \tag{0.1.4}$$

By (**??**), this equation gives:

$$H(x+na) = \frac{1}{\sqrt{a}} \int_0^a dk\, e^{ikna}\, h(k,x), \quad \forall x \in [0,a),\, \forall n \in \mathbb{Z}. \tag{0.1.5}$$

While originally defined in the kq-cell Q, the Zak transform $h(k,q) \in L^2(Q)$ may be extended to the whole kq-plane ($k,q \in \mathbb{R}$), provided it satisfies the following boundary conditions:

$$\begin{aligned} h(k+a,q) &= h(k,q), \\ h(k,q+a) &= e^{ika} h(k,q). \end{aligned} \tag{0.1.6}$$

0.1.2 The Low-Pass Filter of an MRA

The concept of multiresolution analysis is well-known [**?**], so we have only to fix our notations. Let $\{V_j,\, j \in \mathbb{Z}\}$ be an MRA, generated by the scaling function $\varphi \in V_0$ (we use the ordering $V_j \subset V_{j+1}$). As a consequence of the inclusion $V_0 \subset V_1$, the scaling function satisfies the familiar two-scale relation

$$\varphi(x) = \sqrt{2} \sum_{n\in\mathbb{Z}} h_n\, \phi(2x-n). \tag{0.1.7}$$

Taking Fourier transforms, this gives

$$\widehat{\varphi}(\omega) = m_o(\omega/2)\, \widehat{\varphi}(\omega/2), \tag{0.1.8}$$

where

$$m_o(\omega) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} h_n\, e^{-in\omega} \tag{0.1.9}$$

is a 2π–periodic function (and a low-pass filter). Then one defines the mother wavelet $\psi \in W_0 \subset V_1$ by the relation

$$\widehat{\psi}(\omega) = e^{i\omega/2}\, \overline{m_o(\omega/2+\pi)}\, \widehat{\varphi}(\omega/2), \tag{0.1.10}$$

or, equivalently

$$\psi(x) = \sqrt{2} \sum_{n\in\mathbb{Z}} (-1)^{n-1} h_{-n-1}\, \varphi(2x-n), \tag{0.1.11}$$

and proves that the function ψ indeed generates an orthonormal basis with all the required properties. Thus the low-pass filter m_o indeed generates the MRA. Eqivalently, one can start from the Fourier coefficients $\{h_n\}$ defining the two-scale relation (**??**), properly normalized:

$$\sum_{n\in\mathbb{Z}} |h_n|^2 = 1. \tag{0.1.12}$$

Furthermore, from the 2π-periodicity of the function $m_o(\omega)$, together with the orthogonality of the set $\{\varphi(x-k)\}$, $k\in\mathbb{Z}$, it follows that

$$|m_o(\omega)|^2 + |m_o(\omega+\pi)|^2 = 1 \qquad \text{a.e.} \tag{0.1.13}$$

This equation can be rewritten in two equivalent forms where the coefficients h_n appear explicitly:

$$\sum_{n\in\mathbb{Z}} h_n\,\overline{h_{n+2k}} = \delta_{k,0}, \qquad \forall k\in\mathbb{Z} \tag{0.1.14}$$

or

$$\sum_{n,k\in\mathbb{Z}} h_n\,\overline{h_{n+2k}} e^{2ik\omega} = 1, \qquad \text{a.e.} \tag{0.1.15}$$

or yet, in a more convenient form,

$$\frac{1}{2}\sum_{n,l\in\mathbb{Z}} h_n\,\overline{h_l}\,e^{i(l-n)\omega}(1+(-1)^{l+n}) = 1, \qquad \text{a.e.} \tag{0.1.16}$$

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