Chapter 3

Equilibrium States of Classical Crystals

One of the basic industries of mathematical physics is to exploit the mathematical similarities between phenomena that are physically unrelated. The cross-fertilization between Euclidean field theory and classical equilibrium statistical mechanics is a case in point. Even before the evolution of quantum field theory into Euclidean field theory, there were parallel developments in these two different areas of physics. After all, one of the fundamental problems in both subjects is to control an infinite-volume limit, and the issue of stability is important to each area, both technically and conceptually. Another similarity lies in the basic quantities to be analyzed. Wightman distributions were not unlike the correlation functions of classical spin systems in thermal equilibrium, although the space-time cluster properties of the former were necessarily different from the spatial cluster properties of the latter. Only when the Wightman distributions were analytically continued to Schwinger functions did the similarities become striking.

Classical equilibrium statistical mechanics, while far less fundamental than quantum field theory, is also far more ambitious in scope. The aim is to derive the bulk properties of matter from the microscopic interactions of its constituent particles. For a magnetic crystal, the constituent particles are atoms fixed in space, with internal degrees of freedom, and the bulk properties involve specific heat, magnetic susceptibility, regime of spontaneous magnetization, etc. For a gas, the constituent particles are molecules moving in space independently, with potentials defining the interaction, and the bulk properties involve specific heat, compressibility, condensation point, etc. Such diversity means that the task of rigorous formulation cannot be unified, so a great variety of essentially different mathematical models is called for.

The earliest work approaching mathematical rigor was on various expansion methods in the analysis of thermodynamic functions for dilute gases. The grand canonical ensemble is a power series in the activity, and Mayer wrote the density and pressure in powers of the activity with rules for computing the coefficients systematically [M22]. The *virial expansion* is the expansion of the pressure in powers of the density, and it is older than the Mayer series. It was first introduced by Ursell [U4], and Mayer him-

self later simplified it by using graph techniques [M21]. Indeed, it was Mayer's heavy reliance on graphs for both virial and activity expansions that later inspired Feynman to use graphs so extensively in quantum field theory [F36–F39].

Convergence of these expansions (uniform in volume) for small values of the expansion parameters was not immediately established at the time – in part, because it certainly depended on the intermolecular forces assumed for the gas. As an alternate approach, systems of integral equations were formulated to characterize the correlation functions. The Mayer–Montroll system was the first of this type to be introduced [M23], and it is useful for microscopic interactions that are essentially repulsive. The Kirkwood–Salzburg equations were introduced later, but have a greater range of application with regard to the potentials considered [K13]. Like the Mayer–Montroll equations, they comprise an infinite, linear, inhomogeneous system, and finite truncations have been accurate in low-activity calculations.

It was Ruelle who employed the Kirkwood–Salzburg equations, with rigorous estimation, to prove volume-uniform convergence of the Mayer series for small activity and a large class of pair potentials characterized only by regularity and decay conditions [R17–R20]. Indeed, Ruelle and Fisher independently gave the first complete proofs that the infinite-volume limit of the pressure exists for arbitrary activity [F46]. For a special class of interactions, a proof had been given by Lee and Yang years before [L16], but there had been limited interest in mathematically rigorous statistical mechanics until it was vigorously promoted by Ruelle and others. Subsequently, Lebowitz and Penrose proved uniform convergence (in volume) of the virial expansion for low density [L13] and this marked the beginning of a renewed interest in investigating a variety of models.

Our focus is on classical crystals – especially magnetic crystals, which have a distinctive history of their own. Since the crystalline state of matter is far from understood, the magnetic interactions between atoms at different lattice sites are not derived from first principles. The modeling is empirical in the sense that one investigates various interactions in a search for the model from which one derives the correct bulk properties of the magnetic material, but the derivation itself is mathematical. For a ferromagnet, one may believe the magnetic interactions to be the fundamental long-range coupling for magnetic dipoles fixed at the lattice sites by an unspecified force, but such an assumption only provides yet another input model from the empirical point of view. The earliest model proposed for ferromagnets was the *Weiss theory* [W2], which is essentially a mean field approximation. The calculations are simple and explicit, and some of the predictions are in qualitative agreement with experiment, but it was immediately clear that a more complicated model was needed. The *Ising model* – originally proposed and developed by Ising [I9] – proved to be such a model, although its basic description is deceptively simple. Peierls gave an ingenious argument (which was not quite a proof) that the two-dimensional Ising model undergoes a phase transition as the temperature is lowered [P13]. A few years later, Onsager [O5] solved the same model explicitly! His solution is difficult, but it yields a wealth of information about the model, including the occurrence of a phase transition at a critical temperature and zero external magnetic field. For the Ising model in arbitrary dimension, it was suspected that a phase transition cannot occur unless the external magnetic field is zero. Lee and Yang extended this wisdom to all ferromagnetic pair interactions with

integrable range, provided the space of spin configurations was still the space of Ising configurations [L17]. Meanwhile, a rigorous study of the thermodynamic limit had been initiated by van Hove [H44].

Both the Onsager solution and the Lee–Yang Theorem were mathematically rigorous as well – a rare virtue at the time. Not until the time of Ruelle's basic work did rigorous theorems on the classical spin models of crystals began to appear rapidly. Griffiths closed the loop-holes in the Peierls argument [G83] at about that time. In spite of the Onsager solution, this was important because the argument could be adapted to higher dimensions. A couple of years later, Griffiths also proved his celebrated correlation inequalities assuming arbitrary ferromagnetic interactions for Ising configurations [G84–G86]. At the same time, Gallavotti and Miracle-Sole established existence and convexity of the pressure in the thermodynamic limit for a very general class of interactions and spin configurations [G8, G9].

There was also a great deal of interest in the mathematical structures associated with classical spin systems. As long as the space of a single-spin distribution is compact, the entire space of spin configurations for the infinite lattice of spin sites is also compact. The C^*-algebra of continuous functions on this space is a complete set of classical observables for the system, while the states of the system are the normalized positive linear functionals on the C^*-algebra. By the Riesz–Markov Theorem, this means that a state can also be given by a probability measure on the compact space of configurations. To determine whether an equilibrium state is unique for a given interaction and temperature, one must be able to characterize equilibrium states in the first place. The starting point is to characterize it as the Gibbs state for a finite system (so it is unique in that case), and it is reasonable to define an infinite-volume equilibrium state as the limit of the sequence of Gibbs states associated with some sequence of sets of lattice sites approaching the whole lattice in the sense of set inclusion or perhaps in a stronger sense. However, the basic theory of pure and mixed phases in a regime of multiple phases dictates that the set of equilibrium states be convex, and it is not clear how to obtain the geometry of a set of limit points, except in very simple cases. Dobrushin, Lanford, and Ruelle characterized equilibrium states with equations that link the finite-volume Gibbs states to conditionings of the infinite-volume state, and the set of solutions is automatically convex. Ruelle himself gave another characterization of equilibrium as a solution of a variational problem involving pressure and entropy [R23]. Equilibrium states with lattice-translational symmetry have an elegant formulation in terms of this functional analysis. This convex subset is a *simplex* whose extreme points are *ergodic states*, which are the pure phases into which a lattice-translation-invariant mixed phase is uniquely decomposed. Lanford and Ruelle applied the Choquet theory of abstract integral decompositions on convex sets to establish this theoretical result [L2]. This picture also provides a framework for the phenomenon of internal symmetry-breaking. If the single-spin distribution has a group symmetry of its own and if the interaction is invariant with respect to this group as well as the lattice-translation group, then a mixed phase may have both symmetries, but its integral decomposition includes ergodic states which are not invariant with respect to this internal group. The occurrence of spontaneous magnetization in the Ising model is the simplest example of this.

The Griffiths inequalities inspired the derivation of other correlation inequalities

for spin systems in general and the Ising model in particular. Lebowitz proved strong correlation inequalities of Griffiths type for the Ising model, using properties more peculiar to that model [L6]. These more powerful inequalities provided bounds on arbitrary correlations in terms of bounds on two-point correlations. On the other hand, Ginibre generalized the Griffiths inequalities considerably – especially to scalar spins where the single-spin distribution is even [G39]. For another large class of scalar spin systems, Fortuin, Kastelyn, and Ginibre showed that any two random variables that are monotone increasing in the spin variables are also positively correlated by the Gibbs state [F52].

At the same time the rigorous results in equilibrium statistical mechanics began to appear so rapidly, Symanzik was promoting Euclidean field theory [S94–S99]. He was also the first to emphasize the formal resemblance between the thermodynamic limit of the expectations associated with the statistical mechanics of a classical field and the infinite-volume limit in space-time of vacuum expectation values associated with the perturbation of the free Euclidean field by an interaction. His idea was to use the methods of classical statistical mechanics to obtain control over the Euclidean field theory, and specifically, he used equations of Kirkwood–Salzburg type [S95]. Unfortunately, no one at that time knew how to analytically continue a Euclidean field theory to a relativistic field theory in the time differences.

When Nelson isolated and exploited the Markov property of the free Euclidean field [N4–N7], Symanzik's program attracted much more serious attention. Guerra, Rosen, and Simon obtained some control over interacting Euclidean scalar fields through correlation inequalities such as the Griffiths inequalities [G103, G104]. They proved the Griffiths inequalities in the continuum limit of a lattice approximation in which the free part of the Euclidean field action is actually the Ising interaction and the interacting part defines an even single-spin distribution for a scalar spin system. They also proved the FKG inequalities with this lattice approximation, where the field interaction is not necessarily even. At the same time, Glimm, Jaffe, and Spencer developed a *cluster expansion* for controlling the infinite-volume limit of two-dimensional Euclidean scalar field theories with weak interaction [G76]. Although it was the first expansion of its kind, this expansion was very much in the spirit of statistical mechanics. Actually, it involved no lattice approximation, since the expansion was designed for the infinite-volume limit of the ultraviolet limit, but the systematic decoupling of random variables in distant regions was still the idea. Glimm, Jaffe, and Spencer were also able to combine a phase boundary expansion with the Peierls argument to show that the quartic self-interacting scalar field has two pure phases for strong coupling in the continuum – i.e., with no lattice approximation [G78–G80].

The connection between the two subjects attracted even more interest when ideas in quantum field theory found application to problems in statistical mechanics as well. An elegant example of this was the question of phase transitions for the *classical Heisenberg ferromagnet*, which was more difficult than the question had been for the Ising model. In two dimensions it was believed to have a unique phase for all temperatures because Mermin had proven that it had no spontaneous magnetization in that case [M35]. (The classical Heisenberg model does not satisfy an abundance of correlation inequalities, so one could not infer a unique phase from Mermin's result.) In higher dimensions it was believed to have a continuum of phases at low temperature – one for each orientation

of a single spin – but the Peierls argument could not be adapted to such a model. The input from Euclidean field theory was Osterwalder–Schrader positivity, or rather the lattice version of this reflection positivity property. Although the classical Heisenberg ferromagnet does not involve scalar spins, it shares the reflection-positivity property of the Ising ferromagnet. Frohlich, Simon, and Spencer actually derived from this property an infrared bound on the Fourier series whose coefficients are the two-point spin correlations [F77]. This implies existence of a phase transition at sufficiently low temperature in dimension greater than two. A short time later, Bricmont, Fontaine, and Landau proved that in two dimensions the equilibrium state is indeed unique at all temperatures [B54, B55].

An outstanding example of the impact of Euclidean field theory on classical statistical mechanics is the rigorous proof of Debeye screening. It had been shown by Edwards and Lenard that the grand canonical partition function for the lattice approximation of the classical Coulomb gas can be written as the partition function associated with the lattice Euclidean sine-Gordon field [E5]. This identification is called the sine-Gordon transformation. Brydges and Federbush combined this transformation with the phase boundary expansion methods of Glimm, Jaffe, and Spencer to rigorously prove that the high-temperature, dilute, Coulomb gas effectively screens the interaction of test charges [B65, B70]. This phenomenon was an experimental fact for which the only previous explanation had been the self-consistent field intuition of Debeye and Hückel [D9]. The long-distance analysis of Brydges and Federbush is very difficult, but the intuition of the sine-Gordon representation is that the quadratic contribution to the cosine interaction is an effective mass term.

This chapter is devoted to methods of analysis for classical spin models of crystals, with particular emphasis on the analysis of reflection-positive interactions on one hand, and high-temperature expansions for general lattice systems on the other. We confine our attention to three dimensions, not because we have to, but because in this chapter we are not concerned with the dimensional parameter, and dimension $d = 3$ is certainly the realistic value. In §3.1 we describe a ferromagnetic model where each spin is fixed in both magnitude and position on a cubic lattice, and the interaction is the fundamental one among magnetic dipoles – a pair interaction whose long-distance decay marginally fails to be integrable. We use the most naive high-temperature expansion (powers of the inverse temperature) to illustrate how a long-distance problem develops when one tries to control correlations in the infinite-volume limit. This is an example of an *infrared problem*, whose solution lies beyond our discussion. Then we shift our focus to lattice models with pair interactions that are integrable over distance and review the basic mathematical structure of equilibrium statistical mechanics in this context. In §3.2 we continue this description from a functional-analytic point of view, where states are realized as positive linear functionals on commutative C^*-algebras. The Choquet simplex theory applied to the convex space of states provides an excellent framework for the ergodic decomposition of a lattice-translation-invariant equilibrium state into pure phases. In §3.3 we introduce the property of reflection positivity for lattices and show how it implies the infrared bound of Frohlich, Simon, and Spencer. Actually, there are two slightly different types of reflection positivity for lattices: one involves reflection through a plane of lattice sites, while the other involves reflection through a plane lying between two consecutive planes of lattice sites. The latter property is

the more important one, but one needs both properties to establish monotonicity in the long-distance approach of the two-point correlation to a (possibly non-zero) limit. This rules out pathologically singular behavior in the infrared estimation. In §3.4 we shift our attention to the Ising model and review its low-temperature phase structure and its critical properties. In §3.5 we introduce the Ginzburg–Landau model and review its similarity to the Ising model. In §3.6 we review a standard version of the abstract theory of polymer expansions in the interests of self-containment. The work of Glimm, Jaffe, and Spencer triggered a whole industry of rigorous cluster expansions for a host of models, each one different in some non-trivial way from every other. It was gradually recognized, however, that all of these expansions – including the Mayer–Montroll expansion and its contemporaries in early statistical mechanics – have basic combinatorial features in common. A unified framework for automating control over a given expansion by the convenient insertion of estimates peculiar to that expansion was developed and promoted through the efforts of Glimm, Jaffe, Malyshev, Seiler, and others (see [G7, G73, G99, M12, S35] and references given therein).

In §3.7 we review the most basic example of a high temperature expansion in the polymer formalism. The classical Heisenberg model and the Ginzburg–Landau model are the cases considered there. In §3.8 we introduce inductive interpolation as another expansion method cast in the polymer formalism. In §3.9 we consider long-range pair interactions (defined by a decay condition) for fixed-magnitude spins because it calls for a generalization of the polymer estimation given in §3.6. We examine the wholesale expansion of the exponential in all couplings and prove convergence at high temperature. Then in §3.10 we show how an expansion based on inductive interpolation yields a better polymer expansion in the sense that the decay condition on the interaction is weakened as a sufficient condition for high-temperature convergence.

We study the combinatorics associated with inductive interpolation as well. For example, an *Nth-degree ordered tree graph* is a mapping η: $\{2,\dots,N\} \to \{1,\dots,N-1\}$ such that $\eta(i) < i$. For interpolation parameters $t_1,\dots,t_N$ there is a monomial associated with η given by

$$f_\eta(t) = \prod_{i=3}^{N} \left(\prod_{k=\eta(i)}^{i-2} t_k \right) \tag{3.1}$$

The oldest combinatorial inequality involving such monomials is an exponential bound on the sum over ordered tree graphs of the integrals:

$$\sum_{N\text{th-degree } \eta} \left(\prod_{i=1}^{N} \int_0^1 dt_i \right) f_\eta(t) \le e^{N-1}. \tag{3.2}$$

A standard application is to prove convergence of the Mayer series for a sufficiently dilute gas. It was later discovered by Robinson that this inequality can actually be replaced by an identity [B21] which shows how sharp this inequality is. We have

$$\sum_{N\text{th-degree } \eta} \left(\prod_{i=1}^{N} \int_0^1 dt_i \right) f_\eta(t) = \frac{N^{N-1}}{N!}. \tag{3.3}$$

This combinatorial estimation has also been replaced by

$$\sum_{N\text{th-degree } \eta} \prod_{j=1}^{N-1} (d_\eta(j)!) \left(\prod_{i=1}^{N} \int_0^1 dt_i \right) f_\eta(t) \leq 4^{N-1}, \tag{3.4}$$

where $d_\eta(j)$ denotes the cardinality of $\eta^{-1}\{j\}$. This refinement has no effect on the convergence of some expansions, such as the Mayer series, but it makes all the difference in the world for other expansions, such as the phase cell cluster expansion in Chap. 5. Shortly after (3.4) was established, it was shown by Speer that this inequality can be replaced by an identity as well [S66]. We have

$$\sum_{N\text{th-degree } \eta} \prod_{j=1}^{N-1} (d_\eta(j)!) \left(\prod_{i=1}^{N} \int_0^1 dt_i \right) f_\eta(t) = \frac{1}{N} \binom{2N-2}{N-1}. \tag{3.5}$$

Again, the identity adds nothing to the application of the corresponding estimate, but it lends a great deal of elegance to the combinatorics.

This combinatorial interlude is the content of §3.11. In §3.12 we consider unbounded scalar spins with long-range pair interactions and a large-spin damping condition on the single-spin distribution. We apply exactly the same wholesale expansion used in §3.9 and show how the large-spin estimation burns up some of the long-distance decay of the interaction. In §3.13 we apply the inductive interpolation and show how the combinatoric estimate (3.4) controls the large-spin estimation without burning up any of the long-distance decay. This means convergence of the latter expansion requires no more long-distance decay than it required in the bounded-spin case.

3.1 Classical Spin Systems

From the standpoint of modeling the critical behavior of macroscopic systems in terms of the equilibrium statistical mechanics of their microscopic structure, perhaps no examples have been more heavily studied than the various classical models for the ferromagnet. The atoms are arranged in a cubic lattice of fixed positions, so the only dynamical variables are internal. Each iron atom is electrically neutral, but has a magnetic moment proportional to its angular momentum $\vec{s}$, whose magnitude is assumed to be unity, for example. Such a model is based on the assumption that each atom is a constantly spinning gyroscope whose inertial axis is affected by the magnetic field. Accordingly, every element $\vec{n}$ of a finite set $\Lambda \subset \mathbb{Z}^3$ has the *spin variable* $\vec{s}_{\vec{n}}$ assigned to it, and one possible assumption for this crystal is that the Hamiltonian has the form

$$H_\Lambda = J \sum_{\vec{n},\vec{n}' \in \Lambda, \vec{n}' \neq \vec{n}'} \frac{3(\vec{s}_{\vec{n}} \cdot (\vec{n} - \vec{n}'))(\vec{s}_{\vec{n}'} \cdot (\vec{n} - \vec{n}')) - |\vec{n} - \vec{n}'|^2 \, \vec{s}_{\vec{n}} \cdot \vec{s}_{\vec{n}'}}{|\vec{n} - \vec{n}'|^5} + \vec{h} \cdot \sum_{\vec{n} \in \Lambda} \vec{s}_{\vec{n}}, \tag{3.1.1}$$

where $\vec{h}$ is a uniform external magnetic field and J is a constant. This expression for the interaction between atoms at different lattice sites is based on the elementary field of a magnetic dipole. We omit the kinetic terms because they will cancel out in the normalization of a classical equilibrium state, as the momentum variables are independent.

For inverse temperature β, the *equilibrium state* of this system is the probability measure on the space of spin configurations defined as the normalization of the measure.

$$e^{-\beta H_\Lambda(\vec{s})} \prod_{\vec{n}} d\sigma(\vec{s}_{\vec{n}}), \tag{3.1.2}$$

where $d\sigma$ denotes normalized Lebesgue measure on the sphere $\mathbb{S}^2$ of possible spin values. In general, $d\sigma$ will denote the *single-spin distribution* for the systems we study. The exponential factor is the *Gibbs factor*, and it is the solution of the elementary variational problem that is involved in deriving one probability measure from the constrained statistics of another. In this case, the original measure is just the product of single-spin distributions, the constraint is that the statistical average of the energy is fixed, and the inverse temperature β is the Lagrange multiplier.

The *partition function* for a finite set Λ is the mass of the unnormalized measure (3.1.2) given by the Gibbs factor – i.e.,

$$Z_\Lambda(\beta, \vec{h}) = Z(\beta, H_\Lambda) = \left(\prod_{\vec{n}} \int d\sigma(\vec{s}_{\vec{n}}) \right) e^{-\beta H_\Lambda(\vec{s})}. \tag{3.1.3}$$

The probability measure $d\mu_\Lambda^{\beta,\vec{h}}$ for the Gibbs state is just

$$d\mu_\Lambda^{\beta,\vec{h}}(\vec{s}) = Z_\Lambda(\beta, \vec{h})^{-1} e^{-\beta H_\Lambda(\vec{s})} \prod_{\vec{n}} d\sigma(\vec{s}_{\vec{n}}), \tag{3.1.4}$$

and so the expectation of an arbitrary product $\prod_{j=1}^{M} s_{\vec{n}_j \iota_j}$ of spin components is given by

$$\left\langle \prod_{j=1}^{M} s_{\vec{n}_j \iota_j} \right\rangle_\Lambda = Z_\Lambda(\beta, \vec{h})^{-1} \left(\prod_{\vec{n}} \int d\sigma(\vec{s}_{\vec{n}}) \right) e^{-\beta H_\Lambda(\vec{s})} \prod_{j=1}^{M} s_{\vec{n}_j \iota_j} \tag{3.1.5}$$

A standard problem is to control the $\Lambda = \mathbb{Z}^3$ limit of such expectations for high temperature (small β). From the limiting expectations one can recover a probability measure by applying some functional analysis – the Riesz–Markov Theorem in this case, where the space of spin configurations over $\mathbb{Z}^3$ is compact, or Minlos' Theorem in the case where the space of spin configurations is linear instead of compact. Such a measure is an *infinite-volume equilibrium state*.

To control the $\Lambda = \mathbb{Z}^3$ limit for small β – i.e., the *thermodynamic limit* – one might try to expand a given expectation in powers of β. This is the most naive example of

a high-temperature expansion, but in the case of this long-range interaction, it is only asymptotic – not convergent. Let $\langle\cdot\rangle_0$ denote the expectation functional of the product measure (on the compact space $\prod_{\vec{n}\in\mathbb{Z}^3} \mathbb{S}^2$) of the single-spin distribution, and set $\vec{h}=0$ for simplicity. To first order in the inverse temperature we have the expansion

$$\left\langle \prod_{j=1}^{M} s_{\vec{n}_j\iota_j} \right\rangle_\Lambda = \left\langle \prod_{j=1}^{M} s_{\vec{n}_j\iota_j} \right\rangle_0 - \beta \sum_{\vec{n},\vec{n}'\in\{\vec{n}_1,\ldots,\vec{n}_M\}} \left\langle (J_{\vec{n}-\vec{n}'}\,\vec{s}_{\vec{n}'})\cdot\vec{s}_{\vec{n}} \prod_{j=1}^{M} s_{\vec{n}_j\iota_j} \right\rangle_0 + O(\beta^2), \quad (3.1.6)$$

where $J_{\vec{n}}$ denotes the 3×3 matrix defined by

$$(J_{\vec{n}})_{\iota\nu} = \begin{cases} 0, & \vec{n}=0, \\ J\dfrac{3n_\iota n_\nu - \delta_{\nu\iota}|\,\vec{n}\,|^2}{|\,\vec{n}\,|^5}, & \vec{n}\neq 0. \end{cases} \quad (3.1.7)$$

Of the spin variables that are initially differentiated down from the exponent, only those $\vec{s}_{\vec{n}}$ for which $\vec{n}$ occurs in $\{\vec{n}_1,\ldots,\vec{n}_M\}$ can appear, because the $\langle\cdot\rangle_0$-expectation is zero if any spin component at any lattice site has an odd power. This also implies

$$(H_\Lambda)_0 = 0, \quad (3.1.8)$$

so the β-derivative of the partition function in the denominator has contributed nothing here. It is significant that in (3.1.6) the first-order contribution does not depend on Λ at all. The second-order contribution is a little more involved:

$$\begin{aligned} \frac{d^2}{d\beta^2}\left\langle \prod_{j=1}^{M} s_{\vec{n}_j\iota_j} \right\rangle_\Lambda \Big|_{\beta=0} &= \left\langle H_\Lambda^2 \prod_{j=1}^{M} s_{\vec{n}_j\iota_j} \right\rangle_0 - (H_\Lambda^2)_0 \left\langle \prod_{j=1}^{M} s_{\vec{n}_j\iota_j} \right\rangle_0 \\ = \sum_{\vec{n},\vec{n}'\in\{n_1,\ldots,n_M\}} \sum_{\vec{n}''\in\Lambda} &\left\langle ((J_{\vec{n}-\vec{n}''}\,\vec{s}_{\vec{n}''})\,\vec{s}_{\vec{n}})((J_{\vec{n}'-\vec{n}''}\,\vec{s}_{\vec{n}''})\cdot\vec{s}_{\vec{n}'}) \prod_{j=1}^{M} \vec{s}_{\vec{n}_j\iota_j} \right\rangle_0 \\ + \sum_{\substack{\vec{n},\vec{n}',\vec{m},\vec{m}'\in\{n_1,\ldots,n_M\} \\ \vec{m}'\neq\vec{m}}} &\left\langle ((J_{\vec{n}-\vec{m}}\,\vec{s}_{\vec{m}})\cdot\vec{s}_{\vec{n}})((J_{\vec{n}'-\vec{m}'}\,\vec{s}_{\vec{m}'})\cdot\vec{s}_{\vec{n}'}) \prod_{j=1}^{M} \vec{s}_{\vec{n}_j\iota_j} \right\rangle_0 \\ - \sum_{\substack{\vec{n},\vec{n}',\vec{m},\vec{m}'\in\{n_1,\ldots,n_M\} \\ \vec{m}\neq\vec{m}'}} &(((J_{\vec{n}'-\vec{m}}\,\vec{s}_{\vec{m}})\cdot\vec{s}_{\vec{n}})((J_{\vec{n}'-\vec{m}'}\,\vec{s}_{\vec{m}'})\cdot\vec{s}_{\vec{n}'}))_0 \left\langle \prod_{j=1}^{M} s_{\vec{n}_j\iota_j} \right\rangle_0 \\ - \sum_{\vec{n},\vec{n}'\in\{n_1,\ldots,n_M\}} \sum_{\vec{n}''\in\Lambda} &(((J_{\vec{n}-\vec{n}''}\,\vec{s}_{\vec{n}''})\cdot\vec{s}_{\vec{n}})((J_{\vec{n}'-\vec{n}''}\,\vec{s}_{\vec{n}''})\cdot\vec{s}_{\vec{n}'}))_0 \\ \times \left\langle \prod_{j=1}^{M} \vec{s}_{\vec{n}_j\iota_j} \right\rangle_0 & \end{aligned} \quad (3.1.9)$$

because, for example,

$$\left\langle ((J_{\vec{n}-\vec{m}}\ \vec{s}_{\vec{m}})\cdot \vec{s}_{\vec{n}})^2 \prod_{j=1}^{M} s_{\vec{n}_j \iota_j} \right\rangle_0$$

$$= \langle ((J_{\vec{n}-\vec{m}}\ \vec{s}_{\vec{m}})\cdot \vec{s}_{\vec{n}})^2 \rangle_0 \left\langle \prod_{j=1}^{M} s_{\vec{n}_j \iota_j} \right\rangle_0 , \quad \vec{m}, \vec{n} \in \Lambda \backslash \{n_1, \ldots, n_M\}. \tag{3.1.10}$$

Now the second and third double sums are clearly independent of Λ, while the first and fourth are not. On the other hand, the sum over Λ converges absolutely as $\Lambda \to \mathbb{Z}^3$ because

$$|J_{\vec{n}-\vec{n}''}| \leq \frac{c}{|\vec{n}-\vec{n}''|^3}, \tag{3.1.11}$$

and therefore

$$\sum_{\vec{n}''} |J_{\vec{n}-\vec{n}''}|\,|J_{\vec{n}'-\vec{n}''}| \leq c \frac{\ln(2+|\vec{n}'-\vec{n}|)}{|\vec{n}''-\vec{n}|^3}. \tag{3.1.12}$$

Thus, the thermodynamic limit is controlled to second order in β. Similar estimation controls this limit to all orders in β, but this control is only perturbative. Does the power series in β have a nonzero radius of convergence independent of Λ?

Actually, the number of products of expectations of powers of H_Λ contributing to the Nth-order β-derivative at $\beta = 0$ is bounded by $c^N N!$, so the Taylor coefficient has a bound of the form c^N, provided each such product of expectations has such a bound. On the other hand, an example of the multiple sums to be estimated is

$$\sum_{\vec{n}', \vec{n}'', \ldots, \vec{n}^{(N)}} \prod_{\nu=0}^{N} |J_{\vec{n}^{(\nu)}-\vec{n}^{(\nu+1)}}| \leq c^N \frac{(\ln(2+|\vec{n}^{(N+1)}-\vec{n}|))^N}{|\vec{n}^{(N+1)}-\vec{n}|^3}. \tag{3.1.13}$$

Although we have a bound of the form

$$(\ln(2+|\vec{n}'-\vec{n}|))^N \leq c_{N,\alpha}(1+|\vec{n}'-\vec{n}|)^\alpha \tag{3.1.14}$$

for every positive exponent α, we also have the dependence

$$c_{N,\alpha} = O(N!) \tag{3.1.15}$$

for any choice of α. This is where convergence of the power series fails, and this type of divergence is known as an *infrared divergence*. The point is that this problem would not arise if $J_{\vec{n}-\vec{n}'}$ had just a little more long-distance decay – i.e., if $J_{\vec{n}}$ were summable.

The infinite-volume expectations can be controlled nonperturbatively for high temperature by the application of renormalization group methods, but such technology did not exist at the time that rigorous work on ferromagnetic spin systems was initiated. Moreover, magnetic crystals represent a very complicated state of matter from the most fundamental point of view, so it is not at all clear that the most basic interaction between magnetic dipoles is the correct one here. For these reasons, the traditional effort has been a search for shorter-range spin interactions that yield the correct bulk

properties of ferromagnets. *From this point on, we make the convenient assumption that $J_{\vec{n}-\vec{n}'}$ is a scalar and that*

$$\sum_{\vec{n}'} |J_{\vec{n}-\vec{n}'}| = \sum_{\vec{n}'} |J_{\vec{n}'}| < \infty. \tag{3.1.16}$$

The Hamiltonian now has the form

$$H_\Lambda = \sum_{\vec{n},\vec{n}' \in \Lambda} J_{\vec{n}-\vec{n}'} \vec{s}_{\vec{n}} \vec{s}_{\vec{n}'} + \vec{h} \cdot \sum_{\vec{n} \in \Lambda} \vec{s}_{\vec{n}} \tag{3.1.17}$$

for an external magnetic field $\vec{h}$. Boundary conditions may be modifications of the coupling coefficient $J_{\vec{n}-\vec{n}'}$ or constraints on the possible spin values at the boundary. As far as high-temperature expansions are concerned, it will be proven later that the power series in β does indeed have a nonzero radius of convergence independent of Λ and its boundary conditions for this class of spin models.

The study of the spin product expectations is closely related to the analysis of *thermodynamic functions*, which are generated by the *free energy*, defined by

$$f_\Lambda(\beta, \vec{h}) = \frac{1}{|\Lambda|} \ln Z_\Lambda(\beta, \vec{h}). \tag{3.1.18}$$

First, it is a trivial consequence of (3.1.16) that

$$Z_\Lambda(\beta, \vec{h}) \leq e^{\beta(c+|\vec{h}|)|\Lambda|}, \tag{3.1.19}$$

so the logarithm has a bound linear in $|\Lambda|$. Such a bound is a *stability bound*, and it plays a central role in the context of Euclidean field theory, as we have already seen in Chap. 1. Its proof was difficult in the field context, but is a triviality in this context because the spin variables are bounded. It obviously provides a Λ-independent bound on the free energy. In addition, the infinite-volume free energy

$$f(\beta, \vec{h}) = \lim_{\Lambda \to \mathbb{Z}^3} f_\Lambda(\beta, \vec{h}) \tag{3.1.20}$$

is actually unique for arbitrary values of β if the limit is taken *in the sense of van Hove*. For every positive integer r, define $N_r^+(\Lambda)$ (resp. $N_r^-(\Lambda)$) as the number of $\vec{n}$ for which $\{1, 2, \ldots, r\}^3 + r\vec{n}$ intersects with (resp. is contained in) the set Λ. A sequence $\{\Lambda_j\}$ converging to $\mathbb{Z}^3$ in the sense of set inclusion is said to converge in the sense of van Hove if and only if

$$\lim_{j \to \infty} \frac{N_r^+(\Lambda_j)}{N_r^-(\Lambda_j)} = 1 \tag{3.1.21}$$

for every r. The uniqueness of the van Hove limit of $f_\Lambda(\beta, \vec{h})$ is a general type of result. The elementary inequality

$$Z_{\Lambda \cup \Lambda'}(\beta, \vec{h}) \leq Z_\Lambda(\beta, \vec{h}) Z_{\Lambda'}(\beta, \vec{h}) \exp\left(2\beta \sum_{\vec{n} \in \Lambda} \sum_{\vec{n}' \in \Lambda'} |J_{\vec{n}-\vec{n}'}|\right), \quad \Lambda \cap \Lambda' = \emptyset, \tag{3.1.22}$$

lies at the heart of the proof, which we do not pursue. However, this convergence was one of the early important theorems proven in the original program to develop mathematically rigorous statistical mechanics. Notice also that the free energy has the convexity property on the space of interactions – i.e., with

$$f(\beta, \vec{h}, \{J_{\vec{n}}\}) = \lim_{\Lambda \to \mathbb{Z}^3 \text{(van Hove)}} \frac{1}{|\Lambda|} \ln Z(\beta, H_\Lambda), \tag{3.1.23}$$

we have the inequality

$$\begin{aligned} & f(\beta, t\vec{h} + (1-t)\vec{h}', \{tJ_{\vec{n}} + (1-t)J'_{\vec{n}}\}) \\ \leq\ & tf(\beta, \vec{h}; \{J_{\vec{n}}\}) + (1-t)f(\beta, \vec{h}'; \{J'_{\vec{n}}\}), \qquad 0 \leq t \leq 1, \end{aligned} \tag{3.1.24}$$

which follows from the Hölder inequality

$$Z(\beta, tH_\Lambda + (1-t)H'_\Lambda) \leq Z(\beta, H_\Lambda)^t Z(\beta, H'_\Lambda)^{1-t}, \qquad 0 \leq t \leq 1. \tag{3.1.25}$$

The *thermodynamic functions* are defined to be the partial derivatives of the free energy with respect to the components of $\vec{h}$. The $\Lambda = \mathbb{Z}^3$ limit of either the free energy or any one of these partial derivatives is the *thermodynamic limit* of the given function.

For example, consider the first-order partial derivatives of the free energy:

$$\frac{\partial}{\partial h_\iota} f_\Lambda(\beta, \vec{h}) = \frac{-\beta}{|\Lambda|} \sum_{\vec{n} \in \Lambda} \langle s_{\vec{n}\iota} \rangle_\Lambda. \tag{3.1.26}$$

The thermodynamic limit of such a function is an infinite-volume average of single-spin expectations and is therefore a bulk property of the lattice spin system. Its regularity properties are important to the study of critical behavior with respect to $\vec{h}$ at low temperature (large β). Closely related examples of thermodynamic functions are the second-order partial derivatives:

$$\frac{\partial^2}{\partial h_\iota \partial h_{\iota'}} f_\Lambda(\beta, \vec{h}) = \frac{\beta^2}{|\Lambda|} \sum_{\vec{n}, \vec{n}' \in \Lambda} (\langle s_{\vec{n}\iota} s_{\vec{n}'\iota'} \rangle_\Lambda - \langle s_{\vec{n}\iota} \rangle_\Lambda \langle s_{\vec{n}'\iota'} \rangle_\Lambda). \tag{3.1.27}$$

The existence of the thermodynamic limit of such a derivative obviously depends on the behavior of the *truncated expectation*

$$\langle s_{\vec{n}\iota} s_{\vec{n}'\iota'} \rangle_\Lambda^T = \langle s_{\vec{n}\iota} s_{\vec{n}'\iota'} \rangle_\Lambda - \langle s_{\vec{n}\iota} \rangle_\Lambda \langle s_{\vec{n}'\iota'} \rangle_\Lambda. \tag{3.1.28}$$

If the long-distance decay is summable uniformly in $\langle \cdot \rangle_\Lambda$, then this second partial derivative exists in the $\Lambda = \mathbb{Z}^3$ limit.

How does one characterize an infinite-volume equilibrium state? In particular, the limiting state for any convergent sequence of Gibbs states based on a sequence $\{\Lambda_j\}$ of finite subsets converging to $\mathbb{Z}^3$ in the sense of van Hove should qualify as an equilibrium state, and the existence of at least one such sequence is guaranteed by a compactness argument. Moreover, there should be only one equilibrium state in the

case where these sequences of Gibbs states all converge to the same state. This case occurs for high temperature (small β), where one may regard the Gibbs factor as a small multiplicative perturbation of the product of single-spin distributions. In the case where uniqueness fails, infinite-volume characterizations are very important.

Consider an *arbitrary* probability measure on $\prod_{\vec{n}\in\mathbb{Z}^3} \mathbb{S}^2$ of the form $\rho(\vec{s}) \prod_{\vec{n}} d\sigma(\vec{s}_{\vec{n}})$, where the density

$$\rho_\Lambda(\vec{s}_{\vec{n}}|\vec{n}\in\Lambda) = \int \rho(\vec{s}) \prod_{\vec{n}\notin\Lambda} d\sigma(\vec{s}_n) \tag{3.1.29}$$

has the property that $\rho_\Lambda \ln \rho_\Lambda$ is integrable with respect to $\prod_{\vec{n}\in\Lambda} d\sigma(\vec{s}_{\vec{n}})$. The *entropy* of the state ρ on Λ is defined by

$$S_\Lambda(\rho) = -\frac{1}{|\Lambda|} \int \rho_\Lambda \ln \rho_\Lambda \prod_{\vec{n}\in\Lambda} d\sigma(\vec{s}_{\vec{n}}), \tag{3.1.30}$$

while the *internal energy density* of this state for a given interaction is defined by

$$E_\rho^\Lambda(\vec{h}, \{J_{\vec{n}}\}) = \frac{1}{|\Lambda|} \int \rho_\Lambda H_\Lambda \prod_{\vec{n}\in\Lambda} d\sigma(\vec{s}_{\vec{n}}) \tag{3.1.31}$$

with H_Λ given by (3.1.17). It is yet another basic result that the infinite-volume limits

$$S(\rho) = \lim_{\Lambda\to\mathbb{Z}^3}(\text{van Hove})\, S_\Lambda(\rho), \tag{3.1.32}$$

$$E_\rho(\vec{h}, \{J_{\vec{n}}\}) = \lim_{\Lambda\to\mathbb{Z}^3}(\text{van Hove})\, E_\rho^\Lambda(\vec{h}, \{\vec{J}_{\vec{n}}\}) \tag{3.1.33}$$

exist. Moreover, the entropy $S(\rho)$ is actually an affine function on the convex set of states – i.e.,

$$S(t\rho + (1-t)\rho') = tS(\rho) + (1-t)S(\rho'), \qquad 0 \le t \le 1. \tag{3.1.34}$$

This property is an easy consequence of the concavity and subadditivity of the function

$$\xi(x) = -x \ln x. \tag{3.1.35}$$

Indeed,

$$\xi(tx + (1-t)x') \ge t\xi(x) + (1-t)\xi(x'), \qquad 0 \le t \le 1, \tag{3.1.36}$$

because $\xi''(x) < 0$, while

$$\xi(x + x') \le \xi(x) + \xi(x') \tag{3.1.37}$$

follows from

$$(x + x')^{x+x'} \ge x^x x'^{x'}, \qquad x, x' > 0. \tag{3.1.38}$$

First, it is clear that

$$S_\Lambda(t\rho + (1-t)\rho') \ge tS_\Lambda(\rho) + (1-t)S_\Lambda(\rho'), \qquad 0 \le t \le 1, \tag{3.1.39}$$

is a result of (3.1.36). On the other hand, (3.1.37) implies

$$\begin{aligned} & \xi(t\rho_\Lambda(\vec{s}) + (1-t)\rho'_\Lambda(\vec{s})) \\ \leq\ & \xi(t\rho_\Lambda(\vec{s})) + \xi((1-t)\rho'_\Lambda(\vec{s})) \\ =\ & -t\rho_\Lambda(\vec{s})\ln(t\rho_\Lambda(\vec{s})) - (1-t)\rho'_\Lambda(\vec{s})\ln((1-t)\rho'_\Lambda(\vec{s})) \\ =\ & -t\rho_\Lambda(\vec{s})\ln\rho_\Lambda(\vec{s}) - (1-t)\rho'_\Lambda(\vec{s})\ln\rho'_\Lambda(\vec{s}) - t\rho_\Lambda(\vec{s})\ln t \\ & -(1-t)\rho'_\Lambda(\vec{s})\ln(1-t), \end{aligned} \tag{3.1.40}$$

so we have

$$\begin{aligned} \int \xi(t\rho_\Lambda(\vec{s}) + (1-t)\rho'_\Lambda(\vec{s})) \prod_{\vec{n}\in\Lambda} d\sigma(\vec{s}_{\vec{n}}) &\leq t\int \xi(\rho_\Lambda(\vec{s})) \prod_{\vec{n}\in\Lambda} d\sigma(\vec{s}_{\vec{n}}) \\ &+ (1-t)\int \xi(\rho'_\Lambda(\vec{s})) \prod_{\vec{n}\in\Lambda} d\sigma(\vec{s}_{\vec{n}}) \\ &- t\ln t - (1-t)\ln(1-t), \end{aligned} \tag{3.1.41}$$

from which it follows that

$$\begin{aligned} S_\Lambda(t\rho + (1-t)\rho') \leq\ & tS_\Lambda(\rho) + (1-t)S_\Lambda(\rho') \\ & -\frac{1}{|\Lambda|}t\ln t - \frac{1}{|\Lambda|}(1-t)\ln(1-t). \end{aligned} \tag{3.1.42}$$

If we combine this with (3.1.39), it is clear that Eq. (3.1.34) holds in the $\Lambda = \mathbb{Z}^3$ limit.

What is the motivation for introducing these additional thermodynamic quantities? First, it is obvious that

$$S_\Lambda(\rho) - \beta E^\Lambda_\rho(\vec{h}, \{J_{\vec{n}}\}) = \beta f_\Lambda(\beta, \vec{h}, \{J_{\vec{n}}\}) \tag{3.1.43}$$

in the special case

$$\rho_\Lambda = Z(\beta, H_\Lambda)^{-1} e^{-\beta H_\Lambda}. \tag{3.1.44}$$

In the $\Lambda = \mathbb{Z}^3$ limit this means that

$$S(\rho) - \beta E_\rho(\vec{h}, \{J_{\vec{n}}\}) = \beta f(\beta, \vec{h}, \{J_{\vec{n}}\}) \tag{3.1.45}$$

for any state ρ realized as the limit of a van Hove sequence of Gibbs states. Moreover:

(a) $S(\rho)$ is upper semi-continuous with respect to the weak topology on the convex set of states.

(b) For an arbitrary density ρ,

$$S(\rho) - \beta E_\rho(\vec{h}, \{J_{\vec{n}}\}) \leq \beta f(\beta, \vec{h}, \{J_{\vec{n}}\}). \tag{3.1.46}$$

The *Gibbs variational principle* characterizes the equilibrium states as those densities ρ for which (3.1.45) holds. For our class of spin systems one can show that this equation

is equivalent to the Dobrushin–Lanford–Ruelle Equations, which are probably the most popular characterization of equilibrium states. We shall not pursue this subject, as it would take us too far afield.

An interaction $\{J_{\vec{n}}\}$ is *ferromagnetic* if and only if $J_{\vec{n}} \leq 0$ for all $\vec{n}$, and this class of interactions – aside from qualitatively describing magnetic interactions – has advantages that make the models more amenable to analysis. One such advantage – at least for our 3-component spin variables $\vec{s}_{\vec{n}}$ – is a *Lee–Yang Theorem*. Originally proven in the case of the Ising model – which we discuss in §3.4 – such a theorem states that if the components of an external field are replaced by complex variables, then the analytic partition function still has no zeros, except possibly where the real parts are zero. The Lee–Yang Theorem has been established for a small number of ferromagnetic models, including our class of spin models with the components of $\vec{h}$ as the variables for analytic continuation. The significance of this result is that $\ln Z_\Lambda(\beta, \vec{h})$ has a natural extension to a function of $\vec{h} + i\vec{k}$ that is analytic everywhere except possibly where $\vec{h} = 0$. This extension should not be confused with the principal log, as the property $Z_\Lambda(\beta, \vec{h} + i\vec{k}) \neq 0$ does not exclude the "winding" of complex values about the origin. The free energy $f_\Lambda(\beta, \vec{h} + i\vec{k})$ is simply the unique analytic function for which $f_\Lambda(\beta, \vec{h})$ is given by (3.1.18).

Without describing the details, we simply state that for any van Hove sequence $\{\Lambda_j\}$ of finite subsets of $\mathbb{Z}^3$, the sequence $\{f_{\Lambda_j}(\beta, \vec{h} + i\vec{k})\}$ of analytic functions converges uniformly on compact subsets disjoint from $\{\vec{h} = 0\}$. Thus, the limiting function $f(\beta, \vec{h} + i\vec{k})$ has the same domain of analyticity and is the analytic continuation of the free energy $f(\beta, \vec{h})$ in the infinite volume. Therefore, all of the thermodynamic functions have this property as well.

It is generally believed that the thermodynamic functions of a given spin system in the class under study are piecewise real-analytic in $\vec{h}$ whether the model is ferromagnetic or not, but such a general result has never been established. Another conjecture is that infinite-volume equilibrium states are unique at every point of real $\vec{h}$-analyticity in $(\beta, \vec{h})$-space. After all, the partial derivatives have the form

$$\left(\prod_{i=1}^{3} \frac{\partial^{N_i}}{\partial h_i^{N_i}}\right) f(\beta, \vec{h}) = \lim_{\Lambda \to \mathbb{Z}^3} \frac{1}{|\Lambda|} \sum_{\vec{n} \in \Lambda} \sum_{k} \langle A_{k\vec{n}} \rangle_\Lambda \tag{3.1.47}$$

where each $A_{k\vec{n}}$ depends on only a finite number of spin variables $\vec{s}_{\vec{m}}$. However, it is not clear that the limiting state is uniquely determined by the existence of these derivatives, even assuming translation invariance, which yields the reduction $\sum_k \langle A_{k0} \rangle$. In general, one can imagine different states yielding the same sums. On the other hand, both uniqueness of the state and analyticity of the free energy for $\vec{h} \neq 0$ have been shown to follow from convergence and exponential decay of truncated correlations for various ferromagnetic spin systems.

At any point $(\beta, \vec{h})$ where real analyticity of the free energy fails, the model is said to have a *phase transition*. The occurrence of a phase transition can reflect the existence of more than one infinite-volume equilibrium state. In §3.3 we consider a special class of models where this is actually the case.

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3.2 Phase Transitions and Ergodic Decomposition

The occurrence of a phase transition is often accompanied by a breakdown in uniqueness of the infinite-volume equilibrium state. In this event, one is interested in classifying these equilibrium states or *phases*. Sometimes this multiplicity of phases is accompanied by the breakdown of some group symmetry respected by the unique phase one had in the case of high temperature (small β). Such a phenomenon is referred

to as *symmetry-breaking*, and certainly the most obvious candidate would involve the group of discrete translations on the lattice. After all, the class of spin interactions under study are translation-invariant. However, this kind of symmetry-breaking is beyond the scope of our concerns. *We shall consider $\mathbb{Z}^3$-translation-invariant equilibrium states only.*

We are interested in the possibility of symmetry-breaking with respect to the internal symmetry group 0(3) acting on the spin variables $\vec{s}_{\vec{n}}$ simultaneously. Clearly, the dot product in the interaction is invariant with respect to this symmetry. For $\vec{h} \neq 0$ the internal symmetry is obviously broken, but it goes without saying that *spontaneous* symmetry breaking is the interesting issue, so the case $\vec{h} = 0$ is the focus here. Indeed, for ferromagnetic spin systems, such symmetry-breaking is easily described by *spontaneous magnetization*. For a given unit vector $\vec{u}$, consider the translation-invariant phase associated with the inverse temperature β and the external magnetic field $\vec{h} = h\vec{u}$, $h > 0$. The uniqueness of this phase in the ferromagnetic case is *assumed* here, and the spontaneous magnetization $M(\beta) \geq 0$ is given by

$$M(\beta) = \lim_{h \downarrow 0} (\vec{s}_0)(\beta, h\,\vec{u}) \cdot \vec{u}\ . \tag{3.2.1}$$

By convexity of the free energy, this limit is monotone decreasing, and the idea is that if the model has a phase transition, then $M(\beta) > 0$ for sufficiently large β. Notice that the lattice site at which the single-spin expectation is evaluated is irrelevant because the equilibrium state is translation-invariant – i.e.,

$$\left\langle \prod_{j=1}^{N} s_{\vec{n}_j + \vec{n}, \iota_j} \right\rangle (\beta, \vec{h}) = \left\langle \prod_{j=1}^{N} s_{\vec{n}_j, \iota_j} \right\rangle (\beta, \vec{h}) \tag{3.2.2}$$

for all $\vec{n} \in \mathbb{Z}^3$. The physical meaning of the case $M(\beta) > 0$ is that at sufficiently low temperature the ferromagnet remains magnetized in the direction of the external magnetic field after it is switched off. This also means that the model has a continuum of special equilibrium states parametrized by $\vec{u} \in S^2$ when $\vec{h} = 0$.

Let $\mathcal{S}_{\beta, \vec{h}}$ be the set of translation-invariant equilibrium states for inverse temperature β and external magnetic field $\vec{h}$. By the characterization (3.1.45), $\mathcal{S}_{\beta, \vec{h}}$ is a convex set because E_ρ is linear in ρ and $S(\rho)$ is affine in ρ. $\mathcal{S}_{\beta, \vec{h}}$ is obviously compact in the topology of pointwise convergence of linear functionals on the commutative C^*-algebra of continuous functions on $\prod_{\vec{m} \in \mathbb{Z}^3} \mathbb{S}^2$. An *extreme point* of a given convex set cannot be written as a non-trivial convex combination of other elements of the set. The extreme points of $\mathcal{S}_{\beta, \vec{h}}$ are said to be the *pure phases* or the *ergodic phases*, and we first consider the problem of decomposing an arbitrary element of $\mathcal{S}_{\beta, \vec{h}}$ into these pure phases. Let $d\mu$ denote the probability measure on $\prod_{\vec{m} \in \mathbb{Z}^3} \mathbb{S}^2$ associated with a given translation-invariant equilibrium state ρ and consider the Hilbert space $\mathcal{H} = L^2(d\mu)$. Let $\{U(\vec{n})\colon\ \vec{n} \in \mathbb{Z}^3\}$

be the unitary group induced by the group of $d\mu$-preserving homeomorphisms defined by translation on the lattice – i.e.,

$$(U(\vec{n})\varphi)(\vec{s}_{\vec{m}}|\vec{m}\in\mathbb{Z}^3) = \varphi(\vec{s}_{\vec{m}-\vec{n}}|\vec{m}\in\mathbb{Z}^3). \tag{3.2.3}$$

Let P denote the orthogonal projection onto the subspace of vectors invariant with respect to this unitary group. Moreover, let $\mathcal{A}$ denote the commutative C^*-algebra of operators on $\mathcal{H}$ defined as multiplication by the continuous functions on $\prod_{\vec{m}\in\mathbb{Z}^3} \mathbb{S}^2$. Then the C^*-algebra $\mathcal{B}$ generated by $P\mathcal{A}P$ is also commutative, so the positive linear functional

$$B \mapsto (B1,1)_{\mathcal{H}}, \qquad B \in \mathcal{B}, \tag{3.2.4}$$

can be represented by a measure $d\nu$ on the compact space X of characters (multiplicative *linear functionals) on $\mathcal{B}$. Thus

$$(B1,1)_{\mathcal{H}} = \int_X \chi(B)d\nu(\chi), \qquad B \in \mathcal{B}. \tag{3.2.5}$$

Since $d\mu$ is a probability measure on $\prod_{\vec{m}\in\mathbb{Z}^3} \mathbb{S}^2$ and $P1 = 1$, it follows that $d\nu$ is a probability measure on X. In the special case $B = PA_\zeta P$, where A_ζ denotes multiplication by some continuous function ζ on $\prod_{\vec{m}\in\mathbb{Z}^3} \mathbb{S}^2$, we have

$$\begin{aligned} \rho(\zeta) &= \int \zeta\, d\mu \\ &= (A_\zeta 1,1)_{\mathcal{H}} \\ &= (PA_\zeta P1,1)_{\mathcal{H}} \\ &= \int_X \chi(PA_\zeta P)d\nu(\chi). \end{aligned} \tag{3.2.6}$$

On the other hand, we may define the state

$$\rho_\chi(\zeta) = \chi(PA_\zeta P), \tag{3.2.7}$$

so we have a convex decomposition of the arbitrary translation-invariant phase:

$$\rho(\zeta) = \int_X \rho_\chi(\zeta)d\nu(\chi). \tag{3.2.8}$$

X is realized as the index set for the states into which ρ has been decomposed. How do we know this decomposition essentially includes only the states in $\mathcal{S}_{\beta,\vec{h}}$ – i.e., how do we show that ρ_χ is a translation-invariant phase for $d\nu$-almost all $\chi \in X$?

First, the translation-invariance is obvious. Indeed, if ζ is a continuous function on $\prod_{\vec{m}\in\mathbb{Z}^3} \mathbb{S}^2$ and if we introduce the notation

$$\zeta_{\vec{n}}(\vec{s}_{\vec{m}}|\vec{m}\in\mathbb{Z}^3) = \zeta(\vec{s}_{\vec{m}-\vec{n}}|\vec{m}\in\mathbb{Z}^3), \tag{3.2.9}$$

then

$$\begin{aligned}\rho_\chi(\zeta_{\vec{n}}) &= \chi(PA_{\zeta_{\vec{n}}}P)\\ &= \chi(PU(\vec{n})^{-1}A_\zeta U(\vec{n})P)\\ &= \chi(PA_\zeta P)\\ &= \rho_\chi(\zeta).\end{aligned} \tag{3.2.10}$$

Next, we verify that ρ_χ is an equilibrium state for $d\nu$-almost all χ, using the variational principle. Since ρ satisfies Eq. (3.1.45), we have

$$\int_X S(\rho_\chi)d\nu(\chi) - \beta\int_X E_{\rho_\chi}(\vec{h},\{J_{\vec{m}}\})d\nu(\chi) = \beta f(\beta,\vec{h},\{J_{\vec{m}}\}) \tag{3.2.11}$$

because $S(\rho)$ is an affine function of ρ and E_ρ is linear in ρ. Thus

$$\int_X d\nu(\chi)[\beta f(\beta,\vec{h},\{J_{\vec{m}}\}) + \beta E_{\rho_\chi}(\vec{h},\{J_{\vec{m}}\}) - S(\rho_\chi)] = 0, \tag{3.2.12}$$

while in any case

$$\beta f(\beta,\vec{h},\{J_{\vec{m}}\}) + \beta E_{\rho_\chi}(\vec{h},\{J_{\vec{m}}\}) - S(\rho_\chi) \geq 0 \tag{3.2.13}$$

for all χ by the Gibbs variational inequality. Hence, the integrand vanishes for $d\nu$-almost all χ, but this vanishing condition is again the characterization adopted for equilibrium states.

Notice that this last argument does not use the special nature of $d\nu$ – i.e., it applies to *any* probability measure that decomposes ρ into other states. This means that if $\mathcal{S}$ is the convex set of *all* states, then the subset of equilibrium states is not "interior" to $\mathcal{S}$, but rather a "face" of $\mathcal{S}$. As far as the measure $d\nu$ is concerned, we have yet to show that it decomposes ρ into *pure* translation-invariant phases – i.e., extreme points of $\mathcal{S}_{\beta,\vec{h}}$ – i.e., ergodic phases.

Consider an arbitrary convex decomposition $\rho = \alpha\rho_1 + (1-\alpha)\rho_0$ of ρ into two translation-invariant phases ρ_0 and ρ_1. Obviously, the corresponding probability measures $d\mu_0$ and $d\mu_1$ on $\prod_{\vec{m}\in\mathbb{Z}^3}\mathbb{S}^2$ are absolutely continuous with respect to $d\mu$, so there exist positive $d\mu$-integrable functions η^0 and η^1 such that

$$d\mu_\iota = \eta^\iota\, d\mu, \tag{3.2.14}$$

$$\left.\begin{array}{l}(1-\alpha)\eta^0 + \alpha\eta^1 = 1\\ \eta^2_{\vec{n}} = \eta^2, \quad \vec{n}\in\mathbb{Z}^3,\end{array}\right\} \quad d\mu\text{-almost everywhere,} \tag{3.2.15}$$

where we have used the notation (3.2.9). If $A_\iota = A_{\eta^\iota}$, the translation-invariance property implies

$$A_\iota U(\vec{n}) = U(\vec{n})A_\iota, \qquad \vec{n}\in\mathbb{Z}^3, \tag{3.2.16}$$

so both A_ι commute with P as well. Thus

$$\begin{aligned}\rho_\iota(\zeta) &= \int \zeta\eta^\iota \, d\mu \\ &= (A_\zeta A_\iota 1, 1) \\ &= (PA_\zeta PA_\iota P1, 1)_{\mathcal{H}} \\ &= \int_X \chi(PA_\zeta PA_{\eta_\iota} P) d\nu(\chi) \\ &= \int_X \chi(PA_\zeta P)\chi(PA_{\eta_\iota} P) d\nu(\chi) \\ &= \int_X \rho_\chi(\zeta)\rho_\chi(\eta_\iota) d\nu(\chi) \end{aligned} \tag{3.2.17}$$

because χ ranges over the characters of $\mathcal{B}$. Now there is a partial ordering of probability measures on convex compact sets known as *Bishop–de Leeuw ordering*, denoted by $\prec$. Given two such probability measures $d\lambda$ and $d\lambda'$,

$$d\lambda \prec d\lambda' \Leftrightarrow \int \zeta \, d\lambda \leq \int \zeta \, d\lambda', \tag{3.2.18}$$

for convex, continuous ζ. What we have essentially shown above is that the measure $d\tilde{\nu}$ defined by

$$\tilde{\nu}(\mathcal{E}) = \nu\{\chi \in X\colon \rho_\chi \in \mathcal{E}\} \tag{3.2.19}$$

is not only a maximal probability measure on $\mathcal{S}_{\beta,\vec{h}}$ in the Bishop-de-Leeuw sense, but also *the maximum* probability measure on $\mathcal{S}_{\beta,\vec{h}}$. This implies that $\mathcal{S}_{\beta,\vec{h}}$ is a *Choquet simplex* and that ρ_χ is an extreme point – thus an ergodic phase – for $d\nu$-almost all $\chi \in X$.

We now examine alternate characterizations of extremal translation-invariant phases. For a fixed arbitrary state $\rho \in \mathcal{S}_{\beta,\vec{h}}$ we have the associated construction $(d\mu, \mathcal{H}, U(\vec{n}), P, \mathcal{B}, X, d\nu)$ described above. For arbitrary $\varphi \in \mathcal{H}$,

$$P\varphi = \lim_{j\to\infty} \frac{1}{|\Lambda_j|} \sum_{\vec{n}\in\Lambda_j} U(\vec{n})\varphi \tag{3.2.20}$$

for every nested sequence $\{\Lambda_j\}$ of finite rectangular sets of lattice sites converging to $\mathbb{Z}^3$ in the sense of set inclusion. This fact is a special case of the *Mean Ergodic Theorem*, the proof of which we omit. The implication here is that

$$\begin{aligned}\lim_{j\to\infty} \frac{1}{|\Lambda_j|} \sum_{\vec{n}\in\Lambda_j} \rho(\xi\zeta_{\vec{n}}) &= \lim_{j\to\infty} \frac{1}{|\Lambda_j|} \sum_{\vec{n}\in\Lambda_j} (A_\xi A_{\zeta_{\vec{n}}} 1, 1)_{\mathcal{H}} \\ &= \lim_{j\to\infty} \frac{1}{|\Lambda_j|} \sum_{\vec{n}\in\Lambda_j} (A_\xi U(\vec{n}) A_\zeta 1, 1)_{\mathcal{H}} \\ &= (A_\xi PA_\zeta 1, 1)_{\mathcal{H}}. \end{aligned} \tag{3.2.21}$$

If ρ is a non-trivial convex combination of $\rho_0, \rho_1 \in \mathcal{S}_{\beta,\vec{h}}$ as above, then – using the same notation – we have

$$(A_\xi P A_\zeta 1, 1)_{\mathcal{H}} = (1-\alpha)(A_\xi P A_0 P A_\zeta 1, 1)_{\mathcal{H}} + \alpha(A_\xi P A_1 P A_\zeta 1, 1)_{\mathcal{H}}, \tag{3.2.22}$$

which implies

$$P = (1-\alpha)PA_0 + \alpha PA_1. \tag{3.2.23}$$

Since both A_ι commute with P, *this decomposition can be nontrivial only if* $\dim P > 1$. Thus

$$\rho \text{ is extremal} \Leftrightarrow \text{ran } P = \mathbb{C}1. \tag{3.2.24}$$

On the other hand, if ran $P = \mathbb{C}1$, then

$$\begin{aligned}(A_\xi P A_\zeta 1, 1)_{\mathcal{H}} &= (A_\xi 1, 1)_{\mathcal{H}} (A_\zeta 1, 1)_{\mathcal{H}} \\ &= \rho(\xi)\rho(\zeta).\end{aligned} \tag{3.2.25}$$

In summary,

$$\rho \text{ is extremal} \Leftrightarrow \lim_{j\to\infty} \frac{1}{|\Lambda_j|} \sum_{\vec{n}\in\Lambda_j} \rho(\xi\zeta_{\vec{n}}) = \rho(\xi)\rho(\eta), \tag{3.2.26}$$

and this equivalence motivates the synonym "ergodic" for the extreme points of $\mathcal{S}_{\beta,\vec{h}}$.

This basic theory of ergodic decomposition provides a strategy for proving the existence of multiple phases – and therefore a phase transition – for those values of the parameters at which this phenomenon is expected. To prove that $\mathcal{S}_{\beta,\vec{h}}$ is not just a singleton, one needs only to construct a state $\rho \in \mathcal{S}_{\beta,\vec{h}}$ such that

$$\lim_{j\to\infty} \frac{1}{|\Lambda_j|} \sum_{\vec{n}\in\Lambda_j} \rho(\xi\zeta_{\vec{n}}) \neq \rho(\xi)\rho(\zeta) \tag{3.2.27}$$

for some pair of continuous ξ, ζ. Suppose we have constructed such a state ρ with functions ζ and ξ where $\xi = \zeta^*$. In this case, the property may be written as the strict inequality

$$\lim_{j\to\infty} \rho\left(\frac{1}{|\Lambda_j|^2} \left| \sum_{\vec{n}\in\Lambda_j} \zeta_{\vec{n}} \right|^2 \right) > |\rho(\zeta)|^2 \tag{3.2.28}$$

because ρ is translation-invariant. In the $\vec{h} = 0$ case, this inequality actually implies that *the multiplicity of phases is accompanied by a spontaneous breakdown of* $SO(3)$ *symmetry*. Indeed, consider the probability measure $d\tilde{\nu}$ on $\mathcal{S}_{\beta,0}$ that decomposes an $SO(3)$-invariant ρ into ergodic phases. For every $\omega \in \mathcal{S}_{\beta,0}$,

$$\omega\left(\frac{1}{|\Lambda_j|^2} \left| \sum_{\vec{n}\in\Lambda_j} \zeta_{\vec{n}} \right|^2 \right) \leq \omega(|\zeta|^2) \tag{3.2.29}$$

by the Schwarz inequality and the translation-invariance of ω. It follows from dominated convergence and ergodicity of $\omega \in \text{supp } \tilde{\nu}$ that

$$\lim_{j\to\infty} \rho\left(\frac{1}{|\Lambda_j|^2}\left|\sum_{\vec{n}\in\Lambda_j}\zeta_{\vec{n}}\right|^2\right) = \int d\tilde{\nu}(\omega)\lim_{j\to\infty}\omega\left(\frac{1}{|\Lambda_j|^2}\left|\sum_{\vec{n}\in\Lambda_j}\zeta_{\vec{n}}\right|^2\right)$$
$$= \int d\tilde{\nu}(\omega)|\omega(\zeta)|^2; \tag{3.2.30}$$

so the strict inequality (3.2.28) implies the existence of ergodic states ω for which $\omega(\zeta) \neq 0$. What does this have to do with $SO(3)$ symmetry? Let g range over $SO(3)$ rotations and dg denote Haar measure in this group variable. Let us suppose that ζ has the additional property

$$\int(\zeta \circ \mathcal{R}_g)dg = 0, \tag{3.2.31}$$

where $\mathcal{R}_g$ denotes the product action of g on $\prod_{\vec{n}\in\mathbb{Z}^3} \mathbb{S}^2$. Then the states ω for which $\omega(\zeta) \neq 0$ cannot be $SO(3)$-invariant.

We close this section with a discussion of the $\vec{h} \neq 0$ case for certain ferromagnetic spin systems. We see from (3.1.27) that

$$\frac{\partial^2}{\partial h_\iota^2} f(\beta, \vec{h}) = \beta^2 \sum_{\vec{n}\in\mathbb{Z}^3} (\rho(s_{\vec{n}_\iota} s_{\vec{0}_\iota}) - \rho(s_{\vec{0}_\iota})^2) \tag{3.2.32}$$

for every translation-invariant equilibrium state ρ arising from the given values of β and $\vec{h}$. Now the Lee–Yang Theorem implies existence of this derivative, and in the next section we discuss a special class of spin systems for which the ρ shall be proven to satisfy the condition

$$\rho(s_{\vec{n}_\iota} s_{\vec{0}_\iota}) \leq \rho(s_{\vec{m}_\iota} s_{\vec{0}_\iota}), \qquad |n_\kappa| \geq |m_\kappa| \text{ for } \kappa = 1, 2, 3. \tag{3.2.33}$$

This can only mean that $\rho(s_{\vec{n}_\iota} s_{\vec{0}_\iota}) - \rho(s_{\vec{0}_\iota})^2$ is summable, but this condition, in turn, implies the ergodicity condition

$$\lim_{j\to\infty} \frac{1}{|\Lambda_j|} \sum_{\vec{n}\in\Lambda_j} \rho(s_{\vec{n}_\iota} s_{\vec{0}_\iota}) = \rho(s_{\vec{0}_\iota})^2 \tag{3.2.34}$$

The point is that if one could extend this last condition to an arbitrary pair of observables, this would show that *every* state $\rho \in \mathcal{S}_{\beta,\vec{h}}$ is ergodic, *in which case* $\mathcal{S}_{\beta,\vec{h}}$ *could have only one element for* $h \neq 0$. The desired result has never been established for $\mathbb{S}^2$-valued spin systems in three dimensions to our knowledge – even for the class satisfying (3.2.33). Nevertheless, a unique phase is believed to exist for every $\vec{h} \neq 0$. In any case, the uniqueness of the expectation value in (3.2.1) has now been verified for the spin systems having this special monotonicity.

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3.3 Phase Transitions and Reflection Positivity

We have already encountered reflection positivity in the context of Euclidean field theory, where the property was called OS positivity. The most beautiful application of reflection positivity in the context of statistical mechanics is the Fröhlich–Simon–Spencer argument, which produces an $SO(3)$-invariant state $\rho_\beta \in \mathcal{S}_{\beta,0}$ such that for large β (low temperature),

$$\lim_{j\to\infty} \frac{1}{|\Lambda_j|} \sum_{\vec{n}\in\Lambda_j} \rho_\beta(s_{\vec{n}\iota} s_{\vec{0}\iota}) = c \neq 0, \qquad \iota = 1,2,3, \tag{3.3.1}$$

for a certain class of spin systems, where $\{\Lambda_j\}$ is an arbitrary sequence of finite rectangular sets of lattice sites converging to $\mathbb{Z}^3$ in the sense of set inclusion. On the other hand,

$$\int \sum_{\iota'} g_{\iota\iota'} s_{\vec{n}\iota'} dg = 0, \qquad \iota = 1,2,3, \tag{3.3.2}$$

and since ρ_β is $SO(3)$-invariant, this implies

$$\rho_\beta(s_{\vec{n}\iota}) = 0, \qquad \iota = 1,2,3. \tag{3.3.3}$$

It follows that the strict inequality (3.2.28) holds with

$$\zeta(\vec{s}_{\vec{m}} \mid \vec{m} \in \mathbb{Z}^3) = s_{\vec{0}\iota}, \tag{3.3.4}$$

and so we have a phase transition if we can find the desired state.

We begin by defining the class of spin systems. For $\iota = 1, 2, 3$ we define the mappings $\Gamma^{(\iota)}, \widetilde{\Gamma}^{(\iota)}\colon \mathbb{Z}^3 \to \mathbb{Z}^3$ by

$$\Gamma^{(\iota)}(\vec{m})_\kappa = m_\kappa, \qquad \kappa \neq \iota, \tag{3.3.5.0}$$

$$\Gamma^{(\iota)}(\vec{m})_\iota = 1 - m_\iota, \tag{3.3.5.1}$$

$$\widetilde{\Gamma}^{(\iota)}(\vec{m})_\kappa = m_\kappa, \qquad \kappa \neq \iota, \tag{3.3.6.0}$$

$$\widetilde{\Gamma}^{(\iota)}(\vec{m})_\iota = -m_\iota. \tag{3.3.6.1}$$

$\Gamma^{(\iota)}$ is reflection of the lattice through the $x_\iota = 1/2$ plane, while $\widetilde{\Gamma}^{(\iota)}$ is reflection through the $x_\iota = 0$ plane. We stipulate that the interaction has the following properties:

$$J_{\pi(\vec{m})} = J_{\vec{m}}, \qquad \pi \in \mathrm{Perm}\{1, 2, 3\}, \tag{3.3.7}$$

$$J_{\widetilde{\Gamma}^{(\iota)}(\vec{m})} = J_{\vec{m}}, \tag{3.3.8}$$

$$\sum_{\substack{\vec{m},\vec{n} \\ m_\iota, n_\iota > 0}} J_{\vec{n} - \Gamma^{(\iota)}(\vec{m})} \gamma^*_{\vec{m}} \gamma_{\vec{n}} \leq 0 \tag{3.3.9}$$

for each value of ι and arbitrary complex numbers $\gamma_{\vec{m}}$ indexed by $\mathbb{Z}^3$. Next, we realize the desired state as an infinite-volume limit of the Gibbs states with *periodic boundary conditions* – *an* infinite-volume limit, because uniqueness is not guaranteed for low temperature; existence is guaranteed by a compactness argument. Periodic boundary conditions are chosen because they admit translation in a finite volume. Any sequence of states that are translation-invariant in this sense on their respective finite rectangular sets of lattice sites – with the sequence of sets converging to $\mathbb{Z}^3$ in the sense of set inclusion – has only $\mathbb{Z}^3$-translation-invariant limit states, even in the absence of uniqueness. The limit states are also isotropic by virtue of (3.3.7) if the cutoff is isotropic. Accordingly, we consider Gibbs states defined by the Hamiltonians

$$H_\Lambda = \sum_{\vec{m},\vec{n} \in \Lambda} J^\Lambda_{\vec{m}-\vec{n}}\, \vec{s}_{\vec{m}} \cdot \vec{s}_{\vec{n}} \tag{3.3.10}$$

for finite sets of the form

$$\Lambda = \{-2^{N-1} + 1, \ldots, -1, 0, 1, \ldots, 2^{N-1} - 1, 2^{N-1}\}^3, \tag{3.3.11}$$

where $J^\Lambda_{\vec{n}}$ is Λ-periodic – i.e.,

$$J^\Lambda_{\vec{n}+2^N\vec{q}} = J^\Lambda_{\vec{n}} \qquad \vec{q} \in \mathbb{Z}^3. \tag{3.3.12}$$

For each ι we now define the Λ-modified mappings $\Gamma_\Lambda^{(\iota)}$, $\widetilde{\Gamma}_\Lambda^{(\iota)}$: $\Lambda \to \Lambda$ by

$$\Gamma_\Lambda^{(\iota)}(\vec{n}) = \Gamma^{(\iota)}(\vec{n}), \tag{3.3.13}$$

$$\widetilde{\Gamma}_\Lambda^{(\iota)}(\vec{n}) = \widetilde{\Gamma}^{(\iota)}(\vec{n}), \qquad n_\iota \neq 2^{N-1}, \tag{3.3.14.0}$$

$$\widetilde{\Gamma}_\Lambda^{(\iota)}(\vec{n}) = \vec{n}, \qquad n_\iota = 2^{N-1}. \tag{3.3.14.1}$$

In the periodic ι-direction $\Gamma_\Lambda^{(\iota)}$ is the reflection illustrated by Fig. 3.3.1 (where $N = 4$),

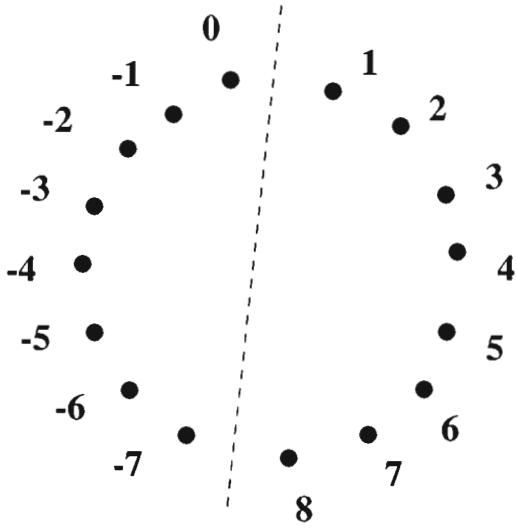

Figure 3.3.1:

while $\widetilde{\Gamma}_\Lambda^{(\iota)}$ is the reflection illustrated by Fig. 3.3.2. We assume that the Λ-modified interaction has the following properties:

$$J^\Lambda_{\pi(\vec{m})} = J^\Lambda_{\vec{m}}, \qquad \pi \in \mathrm{Perm}\{1,2,3\}, \tag{3.3.15}$$

$$J^\Lambda_{\widetilde{\Gamma}_\Lambda^{(\iota)}(\vec{m})} = J^\Lambda_{\vec{m}}, \tag{3.3.16}$$

$$\sum_{\substack{\vec{m},\vec{n}\in\Lambda \\ m_\iota, n_\iota > 0}} J^\Lambda_{\vec{n}-\Gamma_\Lambda^{(\iota)}(\vec{m})} \gamma^*_{\vec{m}} \gamma_{\vec{n}} \leq 0. \tag{3.3.17}$$

Finally, we require that

$$\lim_{\Lambda \nearrow \mathbb{Z}^3} J^\Lambda_{\vec{n}} = J_{\vec{n}}, \tag{3.3.18}$$

so any limit of the Gibbs states

$$d\mu^\beta_\Lambda(\vec{s}) \;=\; Z_\Lambda(\beta)^{-1} e^{-\beta H_\Lambda(\vec{s})} \prod_{\vec{m}\in\Lambda} d\sigma(\vec{s}_{\vec{m}}), \tag{3.3.19}$$

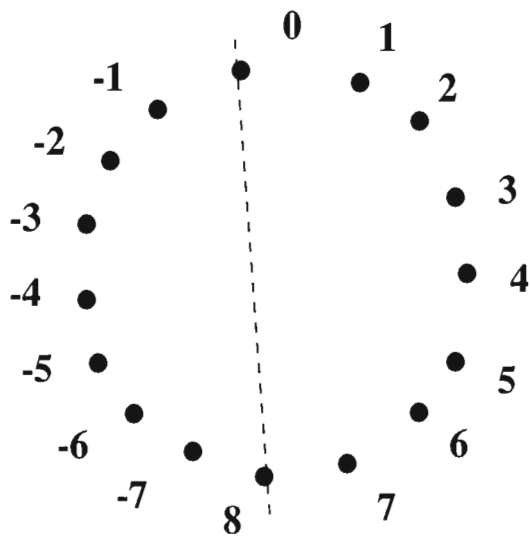

Figure 3.3.2:

$$Z_\Lambda(\beta) = \left(\prod_{\vec{m}\in\Lambda}\int d\sigma(\vec{s}_{\vec{m}})\right) e^{-\beta H_\Lambda(\vec{s})}, \tag{3.3.20}$$

is an isotropic translation-invariant equilibrium state for the interaction $\{J_{\vec{n}}\}$. The inequality (3.3.17) actually implies the inequality (3.3.9), and this property of the interaction is *reflection positivity* adapted to spin systems.

Actually, on a lattice we are dealing with two slightly different kinds of reflection positivity – both of which we use. Reflection transformations $\theta_\Lambda^{(\iota)}$, $\tilde{\theta}_\Lambda^{(\iota)}$: $\prod_{\vec{n}\in\Lambda}\mathbb{S}^2 \to \prod_{\vec{n}\in\Lambda}\mathbb{S}^2$ are defined by

$$\theta_\Lambda^{(\iota)}(\vec{s})_{\vec{m}} = \vec{s}_{\Gamma_\Lambda^{(\iota)}(\vec{m})}, \tag{3.3.21}$$

$$\tilde{\theta}_\Lambda^{(\iota)}(\vec{s})_{\vec{m}} = \vec{s}_{\tilde{\Gamma}_\Lambda^{(\iota)}(\vec{m})} \tag{3.3.22}$$

First consider an arbitrary continuous function $F(\vec{s})$ that depends on only those spin variables $\vec{s}_{\vec{m}}$ for which $\vec{m}\in\Lambda$ and $m_\iota > 0$. In terms of Gibbs expectations, we have:

$$\mu_\Lambda^\beta(\theta_\Lambda^{(\iota)}(\mathcal{E})) = \mu_\Lambda^\beta(\mathcal{E}), \qquad \mathcal{E}\subset\prod_{\vec{n}\in\Lambda}\mathbb{S}^2, \tag{3.3.23}$$

$$\int (F\circ\theta_\Lambda^{(\iota)})F^* d\mu_\Lambda^\beta \geq 0. \tag{3.3.24}$$

The reflection-invariance is a direct consequence of the $J^{\Lambda}_{\vec{n}-\vec{m}}$ translation-invariance combined with (3.3.16), while the positivity condition follows from applying (3.3.17) to a wholesale power series expansion of the exponential factor

$$\exp\left(-2\beta \sum_{\substack{\vec{m},\vec{n}\in\Lambda \\ m_\iota>0 \\ n_\iota\le 0}} J^{\Lambda}_{\vec{n}-\vec{m}}\, \vec{s}_{\vec{m}} \cdot \vec{s}_{\vec{n}}\right)$$

$$= \exp\left(-2\beta \sum_{\substack{\vec{m},\vec{n}\in\Lambda \\ m_\iota,n_\iota>0}} J^{\Lambda}_{\vec{n}-\Gamma^{(\iota)}_{\Lambda}(\vec{m})}\, \vec{s}_{\Gamma^{(\iota)}_{\Lambda}(\vec{m})} \cdot \vec{s}_{\vec{n}}\right) \tag{3.3.25}$$

that contributes to the Gibbs measure. The residual exponential factor is

$$\exp(-\beta H_{\Lambda^-_\iota} - \beta H_{\Lambda^+_\iota}) = (e^{-\beta H_{\Lambda^+_\iota}} \circ \theta^{(\iota)}_{\Lambda}) e^{-\beta H_{\Lambda^+_\iota}}, \tag{3.3.26}$$

$$\Lambda^-_\iota = \{\vec{m}\in\Lambda\colon\ m_\iota \le 0\}, \tag{3.3.27$-$}$$

$$\Lambda^+_\iota = \{\vec{m}\in\Lambda\colon\ m_\iota > 0\}. \tag{3.3.27$+$}$$

This type of reflection positivity plays the major role in establishing the existence of a phase transition.

Now consider the slightly different reflection transformation $\tilde{\theta}^{(\iota)}_{\Lambda}$ With the (slightly different) condition that $F(\vec{s})$ depend on only those spin variables $\vec{s}_{\vec{m}}$ for which $\vec{m}\in\Lambda$ and $m_\iota \ge 0$. We have:

$$\mu^{\beta}_{\Lambda}(\tilde{\theta}^{(\iota)}_{\Lambda}(\mathcal{E})) = \mu^{\beta}_{\Lambda}(\mathcal{E}), \tag{3.3.28}$$

$$\int (F \circ \tilde{\theta}^{(\iota)}_{\Lambda}) F^* d\mu^{\beta}_{\Lambda} \ge 0. \tag{3.3.29}$$

This positivity condition does not follow from (3.3.17) at all; instead, we observe that the symmetry (3.3.16) implies

$$\begin{aligned} H_\Lambda &= H_{\Lambda^-_\iota \setminus \Lambda^0_\iota} + H_{\Lambda^0_\iota} + H_{\Lambda^+_\iota} + 2 \sum_{\substack{\vec{m}\in\Lambda^0_\iota \\ \vec{n}\in\Lambda^+_\iota}} J^{\Lambda}_{\vec{n}-\vec{m}}\, \vec{s}_{\vec{n}} \cdot \vec{s}_{\vec{m}} \\ &\quad + 2 \sum_{\substack{\vec{m}\in\Lambda^0_\iota \\ \vec{n}\in\Lambda^-_\iota \setminus \Lambda^0_\iota}} J^{\Lambda}_{\vec{n}-\vec{m}}\, \vec{s}_{\vec{n}}\, \vec{s}_{\vec{m}} \\ &= H_{\Lambda^0_\iota} + H_{\Lambda^+_\iota} + 2 \sum_{\substack{\vec{m}\in\Lambda^0_\iota \\ \vec{n}\in\Lambda^-_\iota}} J^{\Lambda}_{\vec{n}-\vec{m}}\, \vec{s}_{\vec{n}}\, \vec{s}_{\vec{m}} \end{aligned}$$

$$+\tilde{\theta}_\Lambda^{(\iota)}\left(H_{\Lambda_\iota^+}+2\sum_{\substack{\vec{m}\in\Lambda_\iota^0\\ \vec{n}\in\Lambda_\iota^+}} J^\Lambda_{\vec{n}-\vec{m}}\,\vec{s}_{\vec{n}}\cdot\vec{s}_{\vec{m}}\right), \tag{3.3.30}$$

where

$$\Lambda_\iota^0=\{\vec{m}\in\Lambda:\ m_\iota=0 \text{ or } m_\iota=2^{N-1}\}. \tag{3.3.31}$$

The point is that

$$\tilde{\Gamma}_\Lambda^{(\iota)}(\vec{m})=\vec{m},\qquad \vec{m}\in\Lambda_\iota^0, \tag{3.3.32}$$

so the $\vec{s}_{\vec{m}}$ in (3.3.30) may be treated as coefficients with respect to the action of the transformation on the $\vec{s}_{\vec{n}}$-dependence. Thus

$$\begin{aligned}
&Z_\Lambda(\beta)\int(F\circ\tilde{\theta}_\Lambda^{(\iota)})F^*\,d\mu_\Lambda^\beta\\
&=\left(\prod_{\vec{m}\in\Lambda_\iota^0}\int d\sigma(\vec{s}_{\vec{m}})\right)e^{-\beta H_{\Lambda_\iota^0}}\left(\prod_{\vec{n}\in\Lambda\setminus\Lambda_\iota^0}\int d\sigma(\vec{s}_{\vec{n}})\right)\\
&\quad\left(\exp\left(-\beta H_{\Lambda_\iota^+}-2\beta\sum_{\substack{\vec{m}\in\Lambda_\iota^0\\ \vec{n}\in\Lambda_\iota^+}} J^\Lambda_{\vec{n}-\vec{m}}\,\vec{s}_{\vec{n}}\cdot\vec{s}_{\vec{m}}\right)\circ\tilde{\theta}_\Lambda^{(\iota)}\right)\\
&\quad\times\exp\left(-\beta H_{\Lambda_\iota^+}-2\beta\sum_{\substack{\vec{m}\in\Lambda_\iota^0\\ \vec{n}\in\Lambda_\iota^+}} J^\Lambda_{\vec{n}-\vec{m}}\,\vec{s}_{\vec{n}}\cdot\vec{s}_{\vec{m}}\right)(F\circ\tilde{\theta}_\Lambda^{(\iota)})F^*\\
&=\left(\prod_{\vec{m}\in\Lambda_\iota^0}\int d\sigma(\vec{s}_{\vec{m}})\right)e^{-\beta H_{\Lambda_\iota^0}}\left|\left(\prod_{\vec{n}\in\Lambda_\iota^+}\int d\sigma(\vec{s}_{\vec{n}})\right)F\right.\\
&\quad\left.\times\exp\left(-\beta H_{\Lambda_\iota^+}-2\beta\sum_{\substack{\vec{m}\in\Lambda_\iota^0\\ \vec{n}\in\Lambda_\iota^+}} J^\Lambda_{\vec{n}-\vec{m}}\,\vec{s}_{\vec{n}}\cdot\vec{s}_{\vec{m}}\right)\right|^2.
\end{aligned}\tag{3.3.33}$$

Note that $F(\vec{s})$ may depend on the $\vec{s}_{\vec{m}}$ for which $\vec{m}\in\Lambda_\iota^0$ without violating this positivity. This type of reflection positivity plays a relatively minor role in the analysis, but we need it to rule out pathological long-distance behavior.

Let ρ_β denote our choice of limiting state for the sequence – indexed by the volume parameter N – of Gibbs states described above. *Our task is to show that it has the*

property (3.3.1), and the next step in this direction is to establish that the function $\rho_\beta(s_{\vec{n}\kappa}s_{\vec{0}\kappa})$ actually has monotonic behavior. In an infinite volume, the reflection transformations $\theta^{(\iota)}$, $\tilde{\theta}^{(\iota)}$ are obviously defined by

$$\theta^{(\iota)}(\vec{s})_{\vec{m}} = \vec{s}_{\Gamma^{(\iota)}(\vec{m})}, \tag{3.3.34}$$

$$\tilde{\theta}^{(\iota)}(\vec{s})_{\vec{m}} = \vec{s}_{\tilde{\Gamma}^{(\iota)}(\vec{m})}, \tag{3.3.35}$$

and ρ_β has the positivity properties

$$\rho_\beta((F \circ \theta^{(\iota)})F^*) \geq 0, \tag{3.3.36}$$

$$\rho_\beta((F \circ \tilde{\theta}^{(\iota)})F^*) \geq 0, \tag{3.3.37}$$

where $F(\vec{s})$ depends on only those spin variables $\vec{s}_{\vec{m}}$ for which $m_\iota > 0$. Now we may analyze $\rho_\beta(s_{q\vec{e}_\iota,\kappa}s_{\vec{0}\kappa})$ for positive integers q most easily. By translation-invariance and by (3.3.37) we have

$$\begin{aligned} & \rho_\beta((s_{2r\vec{e}_\iota,\kappa} - s_{(2r-1)\vec{e}_\iota,\kappa})(s_{\vec{e}_\iota\kappa} - s_{\vec{0}\kappa})) \\ = \; & \rho_\beta((s_{r\vec{e}_\iota,\kappa} - s_{(r-1)\vec{e}_\iota,\kappa})(s_{(1-r)\vec{e}_\iota,\kappa} - s_{-r\vec{e}_\iota,\kappa})) \\ \leq \; & 0 \end{aligned} \tag{3.3.38}$$

for all positive integers r, because $s_{-r\vec{e}_\iota,\kappa} - s_{(1-r)\vec{e}_\iota,\kappa}$ is the $\tilde{\theta}^{(\iota)}$-reflection of $s_{r\vec{e}_\iota,\kappa} - s_{(r-1)\vec{e}_\iota,\kappa}$. On the other hand, (3.3.36) implies

$$\begin{aligned} & \rho_\beta((s_{(2r+1)\vec{e}_\iota,\kappa} - s_{2r\vec{e}_\iota,\kappa})(s_{\vec{e}_\iota\kappa} - s_{\vec{0}\kappa})) \\ = \; & \rho_\beta((s_{(r+1)\vec{e}_\iota,\kappa} - s_{r\vec{e}_\iota,\kappa})(s_{(1-r)\vec{e}_\iota,\kappa} - s_{-r\vec{e}_\iota,\kappa})) \\ \leq \; & 0 \end{aligned} \tag{3.3.39}$$

because $s_{-r\vec{e}_\iota,\kappa} - s_{(1-r)\vec{e}_\iota,\kappa}$ is the $\theta^{(\iota)}$-reflection of $s_{(r+1)\vec{e}_\iota,\kappa} - s_{r\vec{e}_\iota,\kappa}$. Thus

$$\rho_\beta((s_{(q+1)\vec{e}_\iota,\kappa} - s_{q\vec{e}_\iota,\kappa})(s_{\vec{e}_\iota\kappa} - s_{\vec{0}\kappa})) \leq 0 \tag{3.3.40}$$

for all positive integers q, and this inequality may be written as

$$\rho_\beta(s_{q\vec{e}_\iota,\kappa}s_{\vec{0}\kappa}) \leq \frac{1}{2}\rho_\beta(s_{(q-1)\vec{e}_\iota,\kappa}s_{\vec{0}\kappa}) + \frac{1}{2}\rho_\beta(s_{(q+1)\vec{e}_\iota,\kappa}s_{\vec{0}\kappa}). \tag{3.3.41}$$

Therefore $\rho_\beta(s_{q\vec{e}_\iota,\kappa}s_{\vec{0}\kappa})$ *is a convex function of* q *for* $q > 0$. Since this function also has the bound

$$|\rho_\beta(s_{q\vec{e}_\iota,\kappa}s_{\vec{0}\kappa})| \leq 1, \tag{3.3.42}$$

it can only be monotone decreasing for $q > 0$. Since ρ_β is isotropic, translation-invariant, and $\tilde{\theta}^{(\iota)}$-invariant, it follows that

$$\rho_\beta(s_{\vec{n}\kappa}s_{\vec{0}\kappa}) \leq \rho_\beta(s_{\vec{m}\kappa}s_{\vec{0}\kappa}) \tag{3.3.43}$$

for $|n_\iota| \geq |m_\iota|, \iota = 1,2,3$. Moreover, the monotonicity in each coordinate direction together with these symmetries imply

$$\lim_{|\vec{n}|\to\infty} \rho_\beta(s_{\vec{n}\kappa} s_{\vec{0}\kappa}) = c_\beta \tag{3.3.44}$$

exists, and this long-range condition implies

$$\lim_{j\to\infty} \frac{1}{|\Lambda_j|} \sum_{\vec{n}\in\Lambda_j} \rho_\beta(s_{\vec{n}\kappa} s_{\vec{0}\kappa}) = c_\beta \tag{3.3.45}$$

for every nested sequence $\{\Lambda_j\}$ of finite rectangular sets of lattice sites converging to $\mathbb{Z}^3$ in the sense of set inclusion. We have reduced the problem to showing that the more straightforward limit (3.3.44) is non-zero, in which case c_β is known as a *long-range order parameter*.

Aside from the symmetry and reflection positivity already assumed for the interaction, there is yet another property it must have, if we are to show that $c_\beta \neq 0$ for sufficiently large β. The strategy is to consider the distribution

$$T_\beta(\vec{k}) = \sum_{\vec{n}} (\rho_\beta(s_{\vec{n}\kappa} s_{\vec{0}\kappa}) - c_\beta) e^{i\vec{n}\cdot\vec{k}} \tag{3.3.46}$$

on the torus $[-\pi,\pi]^3$ Since $\rho_\beta(s_{\vec{n}\kappa} s_{\vec{0}\kappa}) - c_\beta$ is a non-negative function on the lattice $\mathbb{Z}^3$ that decreases to the limit zero in every direction, it follows from elementary Fourier analysis that $T_\beta(\vec{k})$ is actually a measurable function; indeed, one can show that $\prod_{\iota=1}^{3} (e^{ik_\iota} - 1) T_\beta(\vec{k})$ is a continuous function. Thus

$$\rho_\beta(\vec{k}) = c_\beta \delta(\vec{k}) + T_\beta(\vec{k}), \qquad T_\beta \text{ measurable}, \tag{3.3.47}$$

is the structure of the Fourier series with coefficients $\rho_\beta(s_{\vec{n}\kappa} s_{\vec{0}\kappa})$. What is the additional property the interaction must have? We shall presently use the reflection-positivity to derive the bound

$$\rho_\beta\left(\exp\left(\beta \sum_{\vec{m},\vec{n}} \widetilde{J}_{\vec{m}-\vec{n}} (2\,\vec{s}_{\vec{m}} - \vec{\alpha}_{\vec{m}}) \cdot \vec{\alpha}_{\vec{n}}\right)\right) \leq 1, \quad \{\vec{\alpha}_{\vec{m}}\} \in (\mathbb{R}^3)^{\mathbb{Z}^3}, \tag{3.3.48}$$

$$\widetilde{J}_{\vec{m}} = J_{\vec{m}} - \delta_{\vec{m}\vec{0}} \sum_{\vec{n}} J_{\vec{n}}, \tag{3.3.49}$$

and this leads to a bound on $\rho_\beta(\vec{k})$ as follows. If we scale $\{\vec{\alpha}_{\vec{m}}\}$ with the replacement $\vec{\alpha}_{\vec{m}} \to \lambda\, \vec{\alpha}_{\vec{m}}$ and expand the exponential in powers of λ, we have

$$1 - \beta\lambda^2 \sum_{\vec{m},\vec{n}} \widetilde{J}_{\vec{m}-\vec{n}}\, \vec{\alpha}_{\vec{m}} \cdot \vec{\alpha}_{\vec{n}} + 2\beta^2\lambda^2 \sum_{\vec{m},\vec{n}} \widetilde{J}_{\vec{m}-\vec{n}} \sum_{\vec{m}',\vec{n}'} \widetilde{J}_{\vec{m}'-\vec{n}'}$$
$$\times \rho_\beta((\vec{s}_{\vec{m}} \cdot \vec{\alpha}_{\vec{n}})(\vec{s}_{\vec{m}'}\ \ \vec{\alpha}_{\vec{n}'})) + O(\lambda^3) \leq 1, \tag{3.3.50}$$

where

$$\begin{aligned}\rho_\beta\left(2\beta\sum_{\vec{m},\vec{n}}\widetilde{J}_{\vec{m}-\vec{n}}\ \vec{s}_{\vec{m}}\cdot\vec{\alpha}_{\vec{n}}\right) &= 2\beta\sum_{\vec{m},\vec{n}}\widetilde{J}_{\vec{m}-\vec{n}}\rho_\beta(\vec{s}_{\vec{m}})\ \ \vec{\alpha}_{\vec{n}}\\ &= 2\beta\rho_\beta(\vec{s}_{\vec{0}})\cdot\sum_{\vec{m},\vec{n}}\widetilde{J}_{\vec{m}-\vec{n}}\ \vec{\alpha}_{\vec{n}}\\ &= 0 \qquad (3.3.51)\end{aligned}$$

as a consequence of translation-invariance and the normalization

$$\sum_{\vec{m}}\widetilde{J}_{\vec{m}}=0. \qquad (3.3.52)$$

Now subtract 1 from the inequality, divide by λ^2, and then take the $\lambda=0$ limit:

$$\begin{aligned}&\sum_{\vec{m},\vec{n}}\widetilde{J}_{\vec{m}-\vec{n}}\sum_{\vec{m}',\vec{n}'}\widetilde{J}_{\vec{m}'-\vec{n}'}\rho_\beta((\vec{s}_{\vec{m}}\cdot\vec{\alpha}_{\vec{n}})(\vec{s}_{\vec{m}'}\cdot\vec{\alpha}_{\vec{n}'}))\\ \leq\ &\frac{1}{2}\beta^{-1}\sum_{\vec{m},\vec{n}}\widetilde{J}_{\vec{m}-\vec{n}}\ \vec{\alpha}_{\vec{m}}\cdot\vec{\alpha}_{\vec{n}}\ . \qquad (3.3.53)\end{aligned}$$

If we make the choice $\vec{\alpha}_{\vec{n}}=\vec{e}_\kappa\,\alpha_{\vec{n}}$, then

$$\begin{aligned}\sum_{\vec{m},\vec{m}'}\rho_\beta(s_{\vec{m}\kappa}s_{\vec{m}'\kappa})\gamma_{\vec{m}}\gamma_{\vec{m}'} &\leq \frac{1}{2}\beta^{-1}\sum_{\vec{m}}\alpha_{\vec{m}}\gamma_{\vec{m}}, \qquad (3.3.54)\\ \gamma_{\vec{m}} &= \sum_{\vec{n}}\widetilde{J}_{\vec{m}-\vec{n}}\alpha_{\vec{n}}. \qquad (3.3.55)\end{aligned}$$

Since

$$\rho_\beta(s_{\vec{m}\kappa}s_{\vec{m}'\kappa})=\rho_\beta(s_{\vec{m}-\vec{m}',\kappa}s_{\vec{0}\kappa}), \qquad (3.3.56)$$

this inequality may be written as

$$\begin{aligned}\left(\prod_{\iota=1}^{3}\int_{-\pi}^{\pi}dk_\iota\right)|\hat{\gamma}(\vec{k})|^2\rho_\beta(\vec{k}) &\leq \frac{1}{2}\beta^{-1}\left(\prod_{\iota=1}^{3}\int_{-\pi}^{\pi}dk_\iota\right)\hat{\alpha}(\vec{k})^*\hat{\gamma}(\vec{k}), \qquad (3.3.57)\\ \hat{\gamma}(\vec{k}) &= (\widehat{J}(\vec{k})-\widehat{J}(0))\hat{\alpha}(\vec{k}). \qquad (3.3.58)\end{aligned}$$

As $\hat{\alpha}(\vec{k})$ is arbitrary, this implies

$$(\widehat{J}(\vec{k})-\widehat{J}(0))^2\rho_\beta(\vec{k})\leq\frac{1}{2}\beta^{-1}(\widehat{J}(\vec{k})-J(0)), \qquad (3.3.59)$$

and so from (3.3.47) an *infrared bound* is derived:

$$T_\beta(\vec{k})\leq\frac{1}{2}\beta^{-1}(\widehat{J}(\vec{k})-\widehat{J}(0))^{-1}. \qquad (3.3.60)$$

The additional condition on the interaction is that $(\widehat{J}(\vec{k}) - \widehat{J}(0))^{-1}$ *be integrable.* Indeed, since

$$\rho_\beta(\vec{k}) \leq c_\beta \delta(\vec{k}) + \frac{1}{2}\beta^{-1}(\widehat{J}(\vec{k}) - \widehat{J}(0))^{-1}, \tag{3.3.61}$$

we have

$$\begin{aligned} c_\beta &\geq \left(\prod_{\iota=1}^{3}\int_{-\pi}^{\pi} dk_\iota\right)\rho_\beta(\vec{k}) - \frac{1}{2}\beta^{-1}\left(\prod_{\iota=1}^{3}\int_{-\pi}^{\pi} dk_\iota\right)(\widehat{J}(\vec{k}) - \widehat{J}(0))^{-1} \\ &= (2\pi)^3\rho_\beta(s^2_{\vec{0}\,\kappa}) - \frac{1}{2}\beta^{-1}\left(\prod_{\iota=1}^{3}\int_{-\pi}^{\pi} dk_\iota\right)(\widehat{J}(\vec{k}) - \widehat{J}(0))^{-1} \\ &= \frac{1}{3}(2\pi)^3 - \frac{1}{2}\beta^{-1}\left(\prod_{\iota=1}^{3}\int_{-\pi}^{\pi} dk_\iota\right)(\widehat{J}(\vec{k}) - \widehat{J}(0))^{-1} \end{aligned} \tag{3.3.62}$$

because

$$\sum_{\kappa=1}^{3}\rho_\beta(s^2_{\vec{0}\,\kappa}) = \rho_\beta(\vec{s}_{\vec{0}} \quad \vec{s}_{\vec{0}}) = 1 \tag{3.3.63}$$

and ρ_β is invariant with respect to the internal symmetry group $SO(3)$. Thus $c_\beta > 0$ for sufficiently large β if the integral is finite. At last we have a class of interactions for which a phase transition occurs.

It remains to verify the bound (3.3.48), and to this end, we return to the finite box Λ of lattice sites with periodic boundary conditions. For a technical reason we also approximate $d\sigma(\vec{s}_{\vec{m}})$ with $\varphi(\vec{s}_{\vec{m}})d\vec{s}_{\vec{m}}$, where $\varphi(\vec{s}_{\vec{m}})$ is a strictly positive, continuous, bounded function on $\mathbb{R}^3$ that is sharply peaked at the surface $\mathbb{S}^2$ Let $\langle\cdot\rangle_{\Lambda,\varphi}$ denote the Gibbs expectation functional under these circumstances. Then shifts in the $d\vec{s}_{\vec{m}}$-integrations yield

$$\begin{aligned} &\left\langle \exp\left(\beta\sum_{\vec{m},\vec{n}\in\Lambda}\widetilde{J}^{\Lambda}_{\vec{m}-\vec{n}}\right)(2\vec{s}_{\vec{m}} - \vec{\alpha}_{\vec{m}})\cdot\vec{\alpha}_{\vec{m}}\right\rangle_{\Lambda,\varphi} \\ &= \left\langle \exp\left(-\beta\left(\sum_{\vec{n}\in\Lambda}J^{\Lambda}_{\vec{n}}\right)\sum_{\vec{m}\in\Lambda}(2\vec{s}_{\vec{m}} + \vec{\alpha}_{\vec{m}})\cdot\alpha_{\vec{m}}\right)\prod_{\vec{m}\in\Lambda}\frac{\varphi(\vec{s}_{\vec{m}} + \vec{\alpha}_{\vec{m}})}{\varphi(s_{\vec{m}})}\right\rangle_{\Lambda,\varphi} \\ &= \left\langle\prod_{\vec{m}\in\Lambda}G_{\vec{m}}(\vec{s}_{\vec{m}})\right\rangle_{\Lambda,\varphi}, \end{aligned} \tag{3.3.64}$$

$$\begin{aligned} G_{\vec{\ell}}(\vec{s}_{\vec{m}}) &= \exp\left(-\beta\left(\sum_{\vec{n}\in\Lambda}J^{\Lambda}_{\vec{n}}\right)(2\vec{s}_{\vec{m}} + \vec{\alpha}_{\vec{\ell}})\cdot\vec{\alpha}_{\vec{\ell}}\right) \\ &\quad \frac{\varphi(\vec{s}_{\vec{m}} + \vec{\alpha}_{\vec{\ell}})}{\varphi(\vec{s}_{\vec{m}})}. \end{aligned} \tag{3.3.65}$$

At this point, we apply a *chessboard estimate*:

$$\left\langle \prod_{\vec{m}\in\Lambda} G_{\vec{m}}(s_{\vec{m}}) \right\rangle_{\Lambda,\varphi} \leq \prod_{\vec{\ell}\in\Lambda} \left\langle \prod_{\vec{m}\in\Lambda} G_{\vec{\ell}}(\vec{s}_{\vec{m}}) \right\rangle_{\Lambda,\varphi}^{1/|\Lambda|} . \tag{3.3.66}$$

Before deriving this, note that if we now reverse the shift in $d\,\vec{s}_{\vec{m}}$-integrations for each new expectation, we obtain

$$\begin{aligned} &\left\langle \prod_{\vec{m}\in\Lambda} G_{\vec{\ell}}(s_{\vec{m}}) \right\rangle_{\Lambda,\varphi} \\ &= \left\langle \exp\left(-\beta \left(\sum_{\vec{n}\in\Lambda} J_{\vec{n}}^{\Lambda} \right) \sum_{\vec{m}\in\Lambda} (2\,\vec{s}_{\vec{m}} + \vec{\alpha}_{\vec{\ell}})\cdot\vec{\alpha}_{\vec{\ell}} \right) \frac{\varphi(\vec{s}_{\vec{m}} + \vec{\alpha}_{\vec{\ell}})}{\varphi(\vec{s}_{\vec{m}})} \right\rangle_{\Lambda,\varphi} \\ &= \left\langle \exp\left(\beta \sum_{\vec{m},\vec{n}\in\Lambda} \tilde{J}_{\vec{m}-\vec{n}}^{\Lambda} (2\,\vec{s}_{\vec{m}} - \vec{\alpha}_{\vec{\ell}})\cdot\vec{\alpha}_{\vec{\ell}} \right) \right\rangle_{\Lambda,\varphi} \\ &= 1 \end{aligned} \tag{3.3.67}$$

because the sum over $\vec{n}\in\Lambda$ now annihilates the exponent by virtue of the normalization (3.3.49). This verifies the bound (3.3.48) in the limit as the regularization of $d\sigma$ is removed.

How does one derive the chessboard estimate? The input is the reflection-positivity property

$$\langle F^*(F\circ\theta_{\Lambda}^{(\iota)})\rangle_{\Lambda,\varphi} \geq 0, \qquad \iota = 1,2,3, \tag{3.3.68}$$

where F depends on only those spin variables $\vec{s}_{\vec{m}}$ for which $\vec{m}\in\Lambda_\iota^+$. It is easy to check that the φ-modification of the single-spin distribution has not destroyed this property. Now, clearly, this kind of expectation is actually an inner product on the space of such random variables, and so the Schwarz inequality applies:

$$\begin{aligned} &|\langle F^*(\vec{s}_{\vec{m}} \mid \vec{m}\in\Lambda_\iota^+) G(\vec{s}_{\vec{m}} \mid \vec{m}\in\Lambda_\iota^-)\rangle_{\Lambda,\varphi}| \\ &\leq \langle F^*(F\circ\theta_{\Lambda}^{(\iota)})\rangle_{\Lambda,\varphi}^{1/2} \langle G^*\langle G\circ\theta_{\Lambda}^{(\iota)})\rangle_{\Lambda,\varphi}^{1/2} \\ &= \langle F^*(\vec{s}_{\vec{m}} \mid \vec{m}\in\Lambda_\iota^+) F(\vec{s}_{\Gamma^{(\iota)}(\vec{m})} \mid \vec{m}\in\Lambda_\iota^+)\rangle_{\Lambda,\varphi}^{1/2} \\ &\quad \times \langle G^*(\vec{s}_{\vec{m}} \mid \vec{m}\in\Lambda_\iota^-) G(\vec{s}_{\Gamma^{(\iota)}(\vec{m})} \mid \vec{m}\in\Lambda_\iota^-)\rangle_{\Lambda,\varphi}^{1/2}. \end{aligned} \tag{3.3.69}$$

If we apply this estimation to the expectation (3.3.64) we obtain

$$\left\langle \prod_{\vec{m}\in\Lambda} G_{\vec{m}}(s_{\vec{m}}) \right\rangle_{\Lambda,\varphi} \leq \left\langle \prod_{\vec{m}\in\Lambda} G_{\vec{m}}^0(s_{\vec{m}}) \right\rangle_{\Lambda,\varphi}^{1/2} \left\langle \prod_{\vec{m}\in\Lambda} G_{\vec{m}}^1(s_{\vec{m}}) \right\rangle_{\Lambda,\varphi}^{1/2}, \tag{3.3.70}$$

$$G^0_{\vec{m}} = \begin{cases} G_{\vec{m}}, & \vec{m} \in \Lambda_\iota^+, \\ G_{\Gamma^{(\iota)}(\vec{m})}, & \vec{m} \in \Lambda_\iota^-, \end{cases} \tag{3.3.71.0}$$

$$G^1_{\vec{m}} = \begin{cases} G_{\Gamma^{(\iota)}(\vec{m})}, & \vec{m} \in \Lambda_\iota^+, \\ G_{\vec{m}}, & \vec{m} \in \Lambda_\iota^-. \end{cases} \tag{3.3.71.1}$$

What is different about the new products? Notice that each product still includes all of the spin variables but only half of the functions $G_{\vec{m}'}$ – e.g., $G^0_{\vec{m}}$ can only be those $G_{\vec{m}'}$ for which $\vec{m}\,' \in \Lambda_\iota^+$ – so this inequality is certainly a step in the direction of (3.3.66). How do we iterate this estimation? First, exploit the discrete translation-invariance of the state to write

$$\begin{aligned} \left\langle \prod_{\vec{m}\in\Lambda} G^0_{\vec{m}}(\vec{s}_{\vec{m}}) \right\rangle_{\Lambda,\varphi} &= \left\langle \prod_{\vec{m}\in\Lambda} G^0_{\vec{m}}(\vec{s}_{\vec{m}+2^{N-2}\vec{e}_\iota}) \right\rangle_{\Lambda,\varphi} \\ &= \left\langle \prod_{\vec{m}\in\Lambda} G^0_{\vec{m}-2^{N-2}\vec{e}_\iota}(\vec{s}_{\vec{m}}) \right\rangle_{\Lambda,\varphi}, \end{aligned} \tag{3.3.72}$$

where the addition is understood to be periodic – e.g.,

$$\Lambda_\iota^- - 2^{N-2}\,\vec{e}_\iota = \{\vec{m} \in \Lambda\colon m_\iota \le -2^{N-2} \text{ or } m_\iota > 2^{N-2}\}, \tag{3.3.73}$$

$$\Lambda - 2^{N-2}\,\vec{e}_\iota = \Lambda. \tag{3.3.74}$$

Next, apply (3.3.69) to this expectation to obtain

$$\left\langle \prod_{\vec{m}\in\Lambda} G^0_{\vec{m}-2^{N-2}\vec{e}_\iota}(\vec{s}_{\vec{m}}) \right\rangle_{\Lambda,\varphi} \le \left\langle \prod_{\vec{m}\in\Lambda} G^{00}_{\vec{m}}(s_{\vec{m}}) \right\rangle^{1/2}_{\Lambda,\varphi} \left\langle \prod_{\vec{m}\in\Lambda} G^{01}_{\vec{m}}(\vec{s}_{\vec{m}}) \right\rangle^{1/2}_{\Lambda,\varphi}, \tag{3.3.75}$$

$$G^{00}_{\vec{m}} = \begin{cases} G^0_{\vec{m}-2^{N-2}\vec{e}_\iota}, & \vec{m} \in \Lambda_\iota^+, \\ G^0_{\Gamma^{(\iota)}(\vec{m})-2^{N-2}\vec{e}_\iota}, & \vec{m} \in \Lambda_\iota^-, \end{cases} \tag{3.3.76.0}$$

$$G^{01}_{\vec{m}} = \begin{cases} G^0_{\Gamma^{(\iota)}(\vec{m})-2^{N-2}\vec{e}_\iota}, & \vec{m} \in \Lambda_\iota^+, \\ G^0_{\vec{m}-2^{N-1}\vec{e}_\iota}, & \vec{m} \in \Lambda_\iota^-. \end{cases} \tag{3.3.76.1}$$

Applying this same idea to the $G^1_{\vec{m}}$-product and combining the result with (3.3.75), we see that (3.3.70) becomes

$$\begin{aligned} \left\langle \prod_{\vec{m}\in\Lambda} G_{\vec{m}}(\vec{s}_{\vec{m}}) \right\rangle_{\Lambda,\varphi} \le & \left\langle \prod_{\vec{m}\in\Lambda} G^{00}_{\vec{m}}(\vec{s}_{\vec{m}}) \right\rangle^{1/4}_{\Lambda,\varphi} \left\langle \prod_{\vec{m}\in\Lambda} G^{01}_{\vec{m}}(\vec{s}_{\vec{m}}) \right\rangle^{1/4}_{\Lambda,\varphi} \\ & \times \left\langle \prod_{\vec{m}\in\Lambda} G^{10}_{\vec{m}}(\vec{s}_{\vec{m}}) \right\rangle^{1/4}_{\Lambda,\varphi} \left\langle \prod_{\vec{m}\in\Lambda} G^{11}_{\vec{m}}(\vec{s}_{\vec{m}}) \right\rangle^{1/4}_{\Lambda,\varphi}, \end{aligned} \tag{3.3.77}$$

where $G^{00}_{\vec{m}}$ can only be those $G_{\vec{m}'}$ for which $\vec{m}\,' \in \Lambda^+_\iota \cap (\Lambda^+_\iota - 2^{N-2}\,\vec{e}_\iota)$, the $G^{01}_{\vec{m}}$ only those $G_{\vec{m}'}$ for which $\vec{m}\,' \in \Lambda^+_\iota \cap (\Lambda^-_\iota - 2^{N-2}\,\vec{e}_\iota)$, the $G^{10}_{\vec{m}}$ those $G_{\vec{m}'}$ such that $\vec{m}\,' \in \Lambda^-_\iota \cap (\Lambda^+_\iota - 2^{N-2}\,\vec{e}_\iota)$, and $G^{11}_{\vec{m}}$ those for which $\vec{m}\,' \in \Lambda^-_\iota \cap (\Lambda^-_\iota - 2^{N-2}\,\vec{e}_\iota)$. The next step in the iteration is applied to each of these four products and involves the translation by $2^{N-3}\,\vec{e}_\iota$. Eventually, we get

$$\left\langle \prod_{\vec{m}\in\Lambda} G_{\vec{m}}(\vec{s}_{\vec{m}}) \right\rangle_{\Lambda,\varphi} \leq \prod_{q=-2^{N-1}+1}^{2^{N-1}} \left\langle \prod_{r=-2^{N-1}+1}^{2^{N-1}} \prod_{\vec{m}\in\Lambda^0_\iota} G_{\vec{m}+q\vec{e}_\iota}(\vec{s}_{\vec{m}+r\vec{e}_\iota}) \right\rangle^{1/2^N}_{\Lambda,\varphi}, \tag{3.3.78}$$

and if we apply this same multiple reflection scheme for all three of the coordinate directions indexed by ι, we finally have the chessboard estimate because $\bigcap_{\iota=1}^{3} \Lambda^0_\iota = \{0\}$.

Having described the theory, we conclude this section with an example of an interaction that satisfies all of the above requirements. Let

$$J_{-\vec{e}_\iota} = J_{\vec{e}_\iota} = -1, \qquad \iota = 1,2,3, \tag{3.3.79}$$

$$J_0 = 6, \tag{3.3.80}$$

$$J_{\vec{n}} = 0, \qquad \text{all other } \vec{n} \tag{3.3.81}$$

This is clearly ferromagnetic, but in the present context, it is more to the point to observe that this interaction is isotropic, reflection-invariant, and satisfies the reflection-positivity condition. Moreover,

$$\widehat{J}(\vec{k}) = 2\sum_{\iota=1}^{3}(1-\cos k_\iota), \tag{3.3.82}$$

so $\widehat{J}(0) = 0$ and

$$\widehat{J}(\vec{k})^{-1} \leq c|\,\vec{k}\,|^{-2}, \quad \text{small } \vec{k}\,. \tag{3.3.83}$$

This singularity is integrable in 3 dimensions, and so we have a phase transition. This specific model is the *classical Heisenberg ferromagnet*. The ferromagnetic property should not be confused with the reflection positivity property; this model happens to have both.

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3.4 The Ising Model

Perhaps no model in statistical mechanics has been analyzed more extensively than the Ising model. It has a deceptively elementary mathematical description, and yet its behavior with respect to parameter changes has provided an interesting laboratory for the rigorous study of critical phenomena. Originally proposed as a crude approximation of the quantum Heisenberg ferromagnet, its experimental predictions are actually less accurate than those of the classical Heisenberg ferromagnet, and yet it has contributed a great deal to an understanding of cooperative behavior at the macroscopic level. This model also has an interesting relationship to a class of Euclidean lattice field theories.

At each site on the cubic lattice in three dimensions, we replace $\mathbb{S}^2$ with $\mathbb{S}^0 = \{-1, 1\}$ and focus on the interaction between nearest neighbors introduced at the end of the previous section. For finite $\Lambda \subset \mathbb{Z}^3$.

$$H_\Lambda = -\sum_{\langle \vec{n}, \vec{n}' \rangle \subset \Lambda} s_{\vec{n}} s_{\vec{n}'} + h \sum_{\vec{n} \in \Lambda} s_{\vec{n}}, \qquad s_{\vec{n}} = \pm 1, \tag{3.4.1}$$

where $\langle \vec{n}, \vec{n}\,' \rangle$ is notation for two-element sets of lattice sites that are nearest neighbors. Although this model is classical in the sense that the C^*-algebra of observables is commutative, the spin values have been quantized into "spin up" and "spin down." The external magnetic field h is effectively a scalar, and for a $\mathbb{Z}^3$-translation-invariant equilibrium state at a given inverse temperature β, the spontaneous magnetization $M(\beta) \geq 0$ is given by

$$M(\beta) = \lim_{h \downarrow 0} \langle s_{\vec{0}} \rangle(\beta, h). \tag{3.4.2}$$

For sufficiently large β, this model exhibits spontaneous magnetization (i.e., $M(\beta) > 0$), which is a breakdown of the global internal symmetry $s_{\vec{n}} \leftrightarrow -s_{\vec{n}}$ of the expectations – a symmetry of the Hamiltonian at $h = 0$.

The Lee–Yang Theorem applies to the Ising model – i.e., the partition function

$$Z_\Lambda(\beta, h) = \sum_{s \in \{-1,1\}^\Lambda} e^{-\beta H_\Lambda(s)} \tag{3.4.3}$$

has no zeros if analytically continued to $h + ik$, except possibly on the imaginary axis $h = 0$. This means the free energy

$$f_\Lambda(\beta, h) = \frac{1}{|\Lambda|} \ln Z_\Lambda(\beta, h) \tag{3.4.4}$$

has an analytic continuation to all $h + ik$ in a neighborhood of the real axis except possibly a set on the imaginary axis. As in the case of $\mathbb{S}^2$-valued spins, this leads to convergence of these analytic continuations of $\{f_{\Lambda_j}(\beta, h)\}$ – for every van Hove sequence $\{\Lambda_j\}$ – to an analytic continuation of $f(\beta, h)$ to the same domain. Since the thermodynamic functions are h-derivatives of the free energy, they have the same domain of analyticity.

If $\langle\cdot\rangle(\beta, h)$ denotes some translation-invariant, equilibrium state for inverse temperature β and scalar magnetic field h, then

$$\frac{\partial}{\partial h} f(\beta, h) = -\beta \langle s_{\vec{0}} \rangle(\beta, h) \tag{3.4.5}$$

at those parameter values. We infer this from the observation that

$$\frac{\partial}{\partial h} f_\Lambda(\beta, h) = \frac{-\beta}{|\Lambda|} \sum_{\vec{m} \in \Lambda} \langle s_{\vec{m}} \rangle_\Lambda(\beta, h) \tag{3.4.6}$$

for every finite set $\Lambda \subset \mathbb{Z}^3$, where $\langle\cdot\rangle_\Lambda(\beta, h)$ is the corresponding Gibbs state. This average is the Λ-cutoff on the magnetization, so in the thermodynamic limit it is uniquely defined for $h \neq 0$. If there is a discontinuity at $h = 0$ for a given value of β, this is precisely where spontaneous magnetization occurs. In this case, one can see two thermodynamic phases $\langle\cdot\rangle_\pm(\beta, 0)$ – distinguished by

$$\langle s_{\vec{0}} \rangle_\pm(\beta, 0) = \pm M(\beta). \tag{3.4.7}$$

Naturally, every convex combination of these two equilibrium states is also an equilibrium state.

The second-order h-derivative also sheds light on this kind of critical behavior. The *magnetic susceptibility* is given by

$$\chi(\beta, h) \equiv \frac{\partial^2}{\partial h^2} f(\beta, h) = \beta^2 \sum_{\vec{m}} (\langle s_{\vec{m}} s_{\vec{0}} \rangle - \langle s_{\vec{0}} \rangle^2). \tag{3.4.8}$$

All of the arguments for the OS-positive class of $\mathbb{S}^2$-valued spin systems apply to the Ising model as well, so the monotonicity condition

$$\langle s_{\vec{n}} s_{\vec{0}} \rangle \leq \langle s_{\vec{m}} s_{\vec{0}} \rangle, \qquad |n_\kappa| \geq |m_\kappa| \text{ for } \kappa = 1, 2, 3, \tag{3.4.9}$$

holds. Combining this with the Lee–Yang Theorem, one can infer that $\langle s_{\vec{m}} s_{\vec{0}} \rangle - \langle s_{\vec{0}} \rangle^2$ is summable for $h \neq 0$. When $h = 0$, does the sum diverge when a phase transition occurs? Actually, translation-invariant states realized as limits of $h = 0$ Gibbs states

with periodic boundary conditions were investigated in the last section. Although not unique in general, such an equilibrium state is invariant with respect to the global internal symmetry (in this case $s_{\vec{n}} \mapsto -s_{\vec{n}}$) so the magnetization is zero in such a limit. On the other hand, the OS-positivity of the nearest-neighbor ferromagnetic interaction was also shown to imply

$$\lim_{\vec{m}\to\infty} \langle s_{\vec{m}} s_{\vec{0}} \rangle = c_\beta > 0 \tag{3.4.10}$$

for sufficiently large β. Obviously, the magnetic susceptibility is infinite in this parameter range.

Correlation inequalities have played a major role in establishing properties of the Ising model, since the special structure admits a wealth of useful monotonicity relations. We briefly mention some of these inequalities with no attempt at completeness.

1) *Griffiths Inequalities*. Let A and B be arbitrary subsets of an arbitrary finite $\Lambda \subset \mathbb{Z}^3$ Then for $h \leq 0$,

$$\langle s_A \rangle_\Lambda \geq 0, \tag{3.4.11}$$

$$\langle s_A s_B \rangle_\Lambda \geq \langle s_A \rangle_\Lambda \langle s_B \rangle_\Lambda, \tag{3.4.12}$$

where

$$s_A = \prod_{\vec{n}\in A} s_{\vec{n}}. \tag{3.4.13}$$

Moreover, for arbitrary finite $\Lambda' \subset \mathbb{Z}^3$ containing Λ,

$$\langle s_A \rangle_\Lambda \leq \langle s_A \rangle_{\Lambda'} \tag{3.4.14}$$

provided $\langle \cdot \rangle_\Lambda$ is defined with no boundary conditions. This last inequality implies existence of the $\Lambda = \mathbb{Z}^3$ limit for such a sequence of Gibbs states, since the uniform bound

$$|\langle s_A \rangle_\Lambda| \leq 1 \tag{3.4.15}$$

obviously holds. The Griffiths inequalities are also useful in addressing the issue of phase transitions. If a given ferromagnetic interaction yields spontaneous magnetization at a given temperature, so does any interaction that is "more ferromagnetic".

2) *Lebowitz Inequalities*. Introduce duplicate Ising spin variables $t_{\vec{m}}$ and set

$$q_{\vec{m}} = \frac{1}{\sqrt{2}}(s_{\vec{m}} + t_{\vec{m}}), \tag{3.4.16}$$

$$r_{\vec{m}} = \frac{1}{\sqrt{2}}(s_{\vec{m}} - t_{\vec{m}}). \tag{3.4.17}$$

Then for $h \leq 0$,

$$\langle\langle q_A q_B \rangle\rangle_\Lambda \geq \langle\langle q_A \rangle\rangle_\Lambda \langle\langle q_B \rangle\rangle_\Lambda, \tag{3.4.18}$$

$$\langle\langle r_A r_B \rangle\rangle_\Lambda \geq \langle\langle r_A \rangle\rangle_\Lambda \langle\langle r_B \rangle\rangle_\Lambda, \tag{3.4.19}$$

$$\langle\langle q_A r_B \rangle\rangle_\Lambda \geq \langle\langle q_A \rangle\rangle_\Lambda \langle\langle r_B \rangle\rangle_\Lambda, \tag{3.4.20}$$

where the expectation $\langle\langle\cdot\rangle\rangle_\Lambda$ is defined by

$$\langle\langle s_A t_B\rangle\rangle_\Lambda = \langle s_A\rangle_\Lambda \langle t_B\rangle'_\Lambda \tag{3.4.21}$$

with $\langle\cdot\rangle'_\Lambda$ the Gibbs expectation in the duplicate spin variables. These inequalities have significant consequences. The last inequality, for example, enables one to bound the expectation of a product of spin variables with a sum of products of expectations of products of fewer spin variables. Such bounds can be used to derive the long-distance decay of general correlations from the decay of the two-point function.

3) *Fortuin–Kastelyn–Ginibre Inequality.* If $F(s)$ and $G(s)$ are monotone increasing functions with respect to each spin variable $s_{\vec{m}}$, then

$$\langle F(s)G(s)\rangle_\Lambda \geq \langle F(s)\rangle_\Lambda \langle G(s)\rangle_\Lambda. \tag{3.4.22}$$

One application of this correlation inequality is that the expectation of such an observable is monotone increasing in the external magnetic field. Since uniqueness of the equilibrium state can be established for large values of the external field by an expansion, this implies uniqueness of the equilibrium state for values of h where the free energy is analytic – i.e., all $h \neq 0$, by the Lee–Yang Theorem.

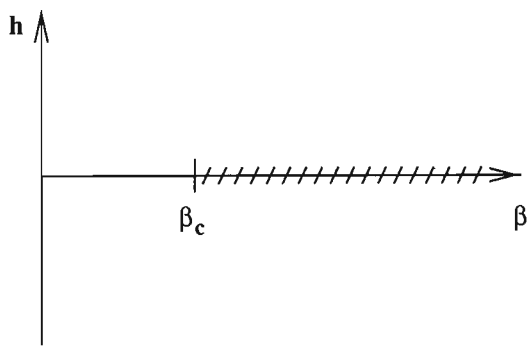

Figure 3.4.1:

The phase diagram in β and h for the three-dimensional Ising model is simple. The model has a unique phase for all parameter values other than $(\beta, 0)$, $\beta \geq \beta_c$, where β_c is the *critical value* of the inverse temperature β. On this half-line, the model has equilibrium states only of the form

$$\langle\cdot\rangle_\alpha(\beta) = \alpha\langle\cdot\rangle_+(\beta,0) + (1-\alpha)\langle\cdot\rangle_-(\beta,0), \qquad 0 \leq \alpha \leq 1. \tag{3.4.23}$$

For the $\alpha = 1/2$ state, the spontaneous magnetization is obviously zero, but

$$\langle s_{\vec{m}} s_{\vec{0}}\rangle_{1/2}(\beta) = \langle s_{\vec{m}} s_{\vec{0}}\rangle_+(\beta,0) \tag{3.4.24}$$

because

$$\langle s_{\vec{m}} s_{\vec{0}}\rangle_-(\beta,0) = \langle s_{\vec{m}} s_{\vec{0}}\rangle_+(\beta,0) \tag{3.4.25}$$

by the $s_{\vec{n}} \mapsto -s_{\vec{n}}$ change of variables. Hence

$$c_\beta = \lim_{\vec{m}\to\infty} \langle s_{\vec{m}} s_{\vec{0}} \rangle_{1/2}(\beta) = \lim_{\vec{m}\to\infty} \langle s_{\vec{m}} s_{\vec{0}} \rangle_+(\beta, 0). \tag{3.4.26}$$

Since $\langle\cdot\rangle_+(\beta,0)$ is a pure phase,

$$\begin{aligned} \lim_{\vec{m}\to\infty} \langle s_{\vec{m}} s_{\vec{0}} \rangle_+(\beta,0) &= (\langle s_{\vec{0}} \rangle_+(\beta,0))^2 \\ &= M(\beta)^2, \end{aligned} \tag{3.4.27}$$

so we have the relation

$$c_\beta = M(\beta)^2. \tag{3.4.28}$$

For $\beta > \beta_c$, the spontaneous magnetization is non-zero, but it is also continuous in β, so $M(\beta_c) = 0$. This means that there is a unique phase at $\beta = \beta_c$ and that the long-range order parameter c_β vanishes there as well.

The properties of the critical point are governed by *critical exponents*. Consider the expectation $\langle s_{\vec{m}} s_{\vec{0}} \rangle(\beta_c, 0)$ for example. The large-$\vec{m}$ decay implicit in the equation $c_{\beta_c} = 0$ is actually very weak, and indeed, the decay is not exponential in this special case. Instead, it is a power law decay – namely,

$$\langle s_{\vec{m}} s_{\vec{0}} \rangle \sim c |\, \vec{m} \,|^{-1+\eta}, \qquad \vec{m} \to \infty, \tag{3.4.29}$$

where η is an example of a critical exponent. In the next section we shall see the similarity of the Ising ferromagnet to scalar Euclidean field theory on a lattice. By comparison to (3.4.29), the two-point correlation of the free massless field on a lattice is given by

$$\frac{1}{(2\pi)^3} \left(\prod_{\iota=1}^{3} \int_{-\pi}^{\pi} dp_\iota \right) e^{-i\vec{m}\cdot\vec{p}} \frac{1}{2\sum_{\iota=1}^{3}(1-\cos p_\iota)} \sim c |\, \vec{m} \,|^{-1}, \qquad \vec{m} \to \infty, \tag{3.4.30}$$

where this lattice function is the Green's function of the lattice Laplacian. For this reason, $3 - \eta$ is called the *anomalous dimension* of the critical state. The value of η cannot be computed in closed form, nor is there any known method that determines the value to an arbitrary degree of accuracy, but it is believed to lie between 0.04 and 0.06. In any case, the magnetic susceptibility

$$\chi(\beta, 0) = \beta^2 \sum_{\vec{m}} \langle s_{\vec{m}} s_{\vec{0}} \rangle(\beta, 0), \tag{3.4.31}$$

is clearly infinite at $\beta = \beta_c$.

Another exponent γ governs the divergence of the magnetic susceptibility at $\beta = \beta_c$:

$$\chi(\beta, 0) \sim c(\beta_c - \beta)^{-\gamma}, \qquad \beta \nearrow \beta_c. \tag{3.4.32}$$

Yet another exponent ν controls the divergence of the *correlation length* $\xi(\beta)$ defined by

$$(s_{\vec{m}} s_{\vec{0}})(\beta, 0) \sim c \exp(-|\, \vec{m} \,|/\xi(\beta)), \qquad \vec{m} \to \infty, \tag{3.4.33}$$

for $\beta < \beta_c$. Its divergence has the form

$$\xi(\beta) \sim c(\beta_c - \beta)^{-\nu}, \qquad \beta \nearrow \beta_c, \tag{3.4.34}$$

and the relation

$$\gamma = (2 - \eta)\nu \tag{3.4.35}$$

is a consequence of the celebrated *Widom scaling hypothesis*. In contrast to the magnetic susceptibility, the spontaneous magnetization has the property

$$M(\beta) = 0, \qquad \beta \leq \beta_c, \tag{3.4.36}$$

and is continuous for all β. For $\beta > \beta_c$, the dependence has the behavior

$$M(\beta) \sim c(\beta - \beta_c)^{\delta}, \qquad \beta \searrow \beta_c, \tag{3.4.37}$$

where δ is another critical exponent. The

$$2\delta = 3\nu - \gamma \tag{3.4.38}$$

is another consequence of the scaling hypothesis.

The values of these exponents are known to be *universal*. This means that each exponent has the same value for large families of spin systems that include the Ising model. The point is that each critical exponent is independent of the short-distance details of a model. They reflect long-distance properties shared by many systems. The renormalization group sheds a great deal of light on this universality, as the values of the critical exponents are actually associated with a fixed point of the RG transformation.

One of the earliest achievements in the study of the Ising model was the proof that spontaneous magnetization indeed occurs for sufficiently low temperature. (Originally applied to two dimensions, it is adaptable to three dimensions.) This proof did not depend on reflection-positivity methods (which were unknown) and the strategy was to construct each of the two pure phases for $h = 0$ by taking the thermodynamic limit with boundary conditions that favor the chosen phase. Outside the finite region Λ of lattice sites, one may specify all spin values to be $+1$, for example. This was the *Peierls argument*, and the representation is the earliest example of a *low temperature expansion*, which we shall not pursue.

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3.5 The Ginzburg–Landau Model

One of the fundamental connections in mathematical physics is the close resemblance of the Ising model to the lattice approximation of the Euclidean scalar field theory with quartic interaction – more commonly known as the *Ginzburg–Landau model*. The latter is given by a probability measure on the space of all real-valued configurations on $\mathbb{Z}^3$, but the nearest-neighbor coupling is the same. From the point of view of classical statistical mechanics, there is no longer any constraint on the scalar spin values – i.e., $S^0 = \{-1, 1\}$ is replaced by $\mathbb{R}$ at each lattice site. On the other hand, the form of the field interaction assigns a strong probability to "spin-up or spin-down" behavior in a certain parameter range.

The lattice approximation of the Φ^4_3 Euclidean field theory is similar in spirit to the standard ultraviolet cutoff, but the starting point is a lattice field ϕ with a discrete Gaussian covariance:

$$C^{lat}_{\vec{n}\,\vec{n}'} = \frac{1}{(2\pi)^3} \left(\prod_{\iota=1}^{3} \int_{-\pi}^{\pi} dp_\iota \right) e^{-i\vec{p}\cdot(\vec{n}-\vec{n}')} \left(m_0^2 + 2\sum_{\iota=1}^{3} (1 - \cos p_\iota) \right)^{-1} \tag{3.5.1}$$

As an infinite matrix, C^{lat} is the following inverse matrix:

$$C^{lat} = (-\Delta^{lat} + m_0^2)^{-1}, \tag{3.5.2}$$

$$\Delta^{lat}_{\vec{n}\,\vec{n}'} = \begin{cases} -6, & \vec{n}\,' = \vec{n}, \\ 1, & \vec{n}, \vec{n}\,' \text{ nearest neighbors}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.5.3}$$

Δ^{lat} is the lattice Laplacian, since the quadratic form is given by

$$(-\Delta^{lat}\phi, \phi) = \sum_{\langle \vec{n}, \vec{n}' \rangle} (\phi(\vec{n}\,') - \phi(\vec{n}))^2. \tag{3.5.4}$$

We define $d\mu_0^{lat}$ as the Gaussian measure on $\mathbb{R}^{\mathbb{Z}^3}$ with mean zero and covariance C^{lat} Formally, $d\mu_0^{lat}$ is the normalization of the measure

$$\exp\left(-\frac{1}{2}((-\Delta^{lat} + m_0^2)\phi, \phi)\right) \prod_{\vec{n}} d\phi(\vec{n}).$$

For an arbitrary finite set $\Lambda \subset \mathbb{Z}^3$ define χ_Λ as the characteristic function of Λ of $\mathbb{Z}^3$ The interacting probability measure on $\mathbb{R}^\Lambda$ is given by

$$d\mu_\Lambda^{\lambda,\tau}(\phi) = \widehat{Z}_\Lambda(\lambda,\tau)^{-1} \exp(-I^{lat}(\chi_\Lambda\phi)) d\mu_0^{lat}(\chi_\Lambda\phi) \tag{3.5.5}$$

$$\widehat{Z}_\Lambda(\lambda,\tau) = \int \exp(-I^{lat}(\chi_\Lambda\phi)) d\mu_0^{lat}(\chi_\Lambda\phi) \tag{3.5.6}$$

$$I^{lat}(\phi) = \sum_{\vec{m}} V(\phi(\vec{m})), \tag{3.5.7}$$

$$V(z) = \lambda z^4 + \tau z^2, \qquad \lambda > 0. \tag{3.5.8}$$

Naturally, the continuum limit involves the lattice spacing parameter, which we set equal to unity here. Also the restriction of the Gaussian measure to $\chi_\Lambda\phi$ reflects a choice of boundary conditions.

How do we recognize $d\mu_\Lambda$ as the Gibbs state of a spin Hamiltonian for inverse temperature β? First we identify the spin variable:

$$s_{\vec{n}} = \beta^{-1/2}\phi(\vec{n}). \tag{3.5.9}$$

Thus

$$\begin{aligned} &\frac{1}{2}((-\Delta^{lat} + m_0^2)\chi_\Lambda\phi, \chi_\Lambda\phi) \\ &= -\beta \sum_{\langle \vec{n}, \vec{m} \rangle \subset \Lambda} s_{\vec{n}} s_{\vec{m}} + \beta \sum_{\vec{m} \in \Lambda} \left(3 + \frac{1}{2}m_0^2\right) s_{\vec{m}}^2, \end{aligned} \tag{3.5.10}$$

$$V(\phi(\vec{m})) = \lambda\beta^2 s_{\vec{m}}^4 + \beta\tau s_{\vec{m}}^2. \tag{3.5.11}$$

Second, we give λ and τ the β-dependence that makes the single-spin distribution

$$d\sigma(z) = \exp\left[-\beta\left(3 + \frac{1}{2}m_0^2 + \tau(\beta)\right) z^2 - \lambda(\beta)\beta^2 z^4\right] dz \tag{3.5.12}$$

independent of β. Accordingly, we set

$$\tau(\beta) = \beta^{-1}\hat{\tau} - \frac{1}{2}m_0^2 - 3, \tag{3.5.13}$$

$$\lambda(\beta) = \beta^{-2}\hat{\lambda}, \tag{3.5.14}$$

with $\hat{\lambda}$ and $\hat{\tau}$ the independent parameters, so

$$d\mu_\Lambda^{\lambda(\beta),\tau(\beta)}(\phi) = Z_\Lambda(\beta)^{-1} \exp\left(\beta \sum_{\langle \vec{n},\vec{m}\rangle \subset \Lambda} s_{\vec{n}} s_{\vec{m}}\right) \prod_{\vec{n}\in\Lambda} d\sigma(s_{\vec{n}}), \tag{3.5.15}$$

$$Z_\Lambda(\beta) = \beta^{-(\text{card }\Lambda)/2} \widehat{Z}_\Lambda(\lambda(\beta),\tau(\beta)), \tag{3.5.16}$$

How do we discern anything resembling the behavior of the Ising model in this Gibbs state? In terms of the independent parameters, we write the single-spin distribution as follows:

$$\begin{aligned} d\sigma(z) &= \exp(-\hat{\tau}z^2 - \hat{\lambda}z^4)dz \\ &= \exp(-\hat{\lambda}(z^2 + \hat{\tau}/2\hat{\lambda})^2 + \hat{\tau}^2/4\hat{\lambda})dz. \end{aligned} \tag{3.5.17}$$

If $\hat{\tau}$ is negative, the completed square has a local maximum at $z = 0$ and local minima at

$$z = \pm\sqrt{-\hat{\tau}/2\hat{\lambda}}. \tag{3.5.18}$$

In this case the graph of the completed square is a double well. The point is that at a given lattice site the single-spin distribution has its *maximum* probability density at the spin values $\pm\sqrt{-\hat{\tau}/2\hat{\lambda}}$. This is a major reason why the lattice field theory has the same critical behavior as the Ising model for sufficiently negative $\hat{\tau}$. Another, rather different reason is that this theory is approximated by the Griffiths–Simon superposition of Ising models, which we do not describe here. The Lee–Yang Theorem applies to the Ginzburg–Landau model as a consequence.

For fixed $\hat{\lambda}$ and $\hat{\tau}$, one expects to have uniqueness of the $\Lambda = \mathbb{Z}^3$ limit of $d\mu_\Lambda^{\lambda,\tau}$ in the case where β is small. This uniqueness breaks down for negative $\hat{\tau}$ if β is sufficiently large relative to $\hat{\lambda}/\hat{\tau}^2$, but every possible measure is still a convex combination of two pure phases characterized by the spontaneous magnetization, which is defined exactly as it was for the Ising model. Indeed, if $Z_\Lambda(\lambda,\tau,h)$ and $d\mu_\Lambda^{\lambda,\tau,h}$ are defined by replacing $V(z)$ with $V(z) + hz$, then

$$\begin{aligned} \left\langle \prod_{j=1}^N \phi(\vec{m}_j) \right\rangle (\lambda,\tau,h) &= \lim_{\Lambda\to\mathbb{Z}^3} \left\langle \prod_{j=1}^N \phi(\vec{m}_j) \right\rangle_\Lambda (\lambda,\tau,h) \\ &= \lim_{\Lambda\to\mathbb{Z}^3} \int \prod_{j=1}^N \phi(\vec{m}_j) d\mu_\Lambda^{\lambda,\tau,h}(\phi) \end{aligned} \tag{3.5.19}$$

exists for $h \neq 0$ – i.e., the infinite-volume state is unique for $h \neq 0$ by the same reasoning that was applied to the Ising model. The spontaneous magnetization is given by

$$M(\beta) = \langle s_{\vec{0}} \rangle_+(\beta) = \beta^{-1/2}\langle \phi(\vec{0}) \rangle_+(\lambda(\beta),\tau(\beta))$$

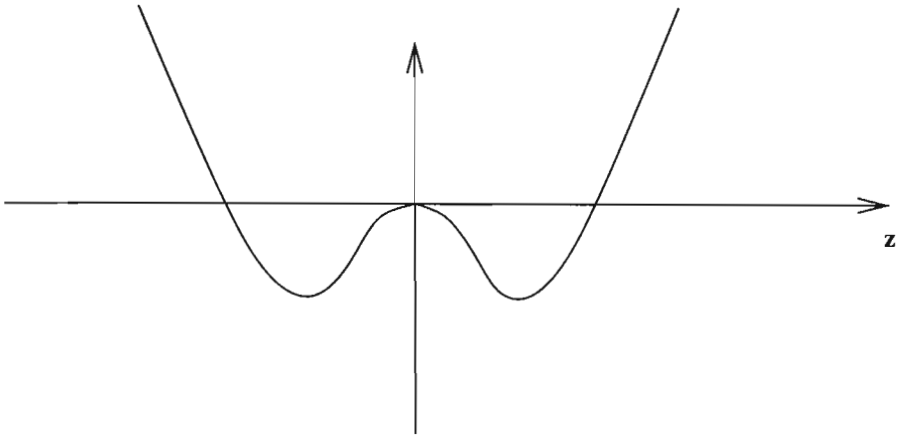

Figure 3.5.1:

$$= \lim_{h \searrow 0} \beta^{-1/2} \langle \phi(\vec{0}) \rangle (\lambda(\beta), \tau(\beta), h), \tag{3.5.20}$$

where this expectation is $\mathbb{Z}^3$-translation-invariant. Also,

$$\begin{aligned} \langle s_{\vec{0}} \rangle_-(\beta) &= \beta^{-1/2} \langle \phi(\vec{0}) \rangle_- (\lambda(\beta), \tau(\beta)) \\ &= \lim_{h \nearrow 0} \beta^{-1/2} \langle \phi(\vec{0}) \rangle (\lambda(\beta), \tau(\beta), h) = -M(\beta) \end{aligned} \tag{3.5.21}$$

by the $z \leftrightarrow -z$ symmetry of $V(z)$. If $d\mu$ ranges over convex combinations of probability measures realized as limits of the net $\{d\mu_\Lambda^{\lambda,\tau}\}$, then $d\mu$ is parametrized by the unit interval as the convex combinations

$$d\mu = \alpha d\mu_+ + (1 - \alpha) d\mu_-. \tag{3.5.22}$$

If $\alpha = \frac{1}{2}$, then the $\phi(\vec{n}) \leftrightarrow -\phi(\vec{n})$ symmetry is unbroken for $d\mu$ and the long-range order parameter is given by

$$c(\beta) = \lim_{\vec{m} \to \infty} \beta^{-1} \int \phi(\vec{m}) \phi(\vec{0}) d\mu(\phi). \tag{3.5.23}$$

On the other hand, the $\phi(\vec{n}) \mapsto -\phi(\vec{n})$ change of variables yields

$$\int \phi(\vec{m}) \phi(\vec{0}) d\mu_-(\phi) = \int \phi(\vec{m}) \phi(\vec{0}) d\mu_+(\phi), \tag{3.5.24}$$

so we have the same conclusion as in the case of the Ising model:

$$\begin{aligned} c(\beta) &= \lim_{\vec{m} \to \infty} \beta^{-1} \int \phi(\vec{m}) \phi(\vec{0}) d\mu_+(\phi) \\ &= \beta^{-1} \left(\int \phi(\vec{0}) d\mu_+(\phi) \right)^2 = M(\beta)^2. \end{aligned} \tag{3.5.25}$$

There is long-range order for those parameter values that yield spontaneous magnetization.

Now consider the phase diagram of the Ginzburg–Landau model in the parameters λ and τ *for fixed inverse temperature* β *and variable* $\hat{\lambda}$, $\hat{\tau}$. In the $h = 0$ plane, this system has been proven to have the regimes for a unique phase given by Fig. 3.5.2. Actually, there is only one critical curve. We have drawn the boundaries of parameter regions for convergent expansions, where $\varepsilon > 0$ is understood to be sufficiently small. The region $\{\lambda \leq \varepsilon\sqrt{\tau}\}$ is the regime of the Glimm–Jaffe–Spencer cluster expansion, while in the region

$$\{\varepsilon\lambda \geq 1\} \cap \{\varepsilon^2\lambda \geq \tau^2\}$$

there is a convergent cluster expansion that decouples the Hamiltonian directly – an expansion that will be introduced in §3.7. Both expansions are often referred to as high-temperature expansions because they establish the properties of a unique phase. In the two-phase regime there is a low-temperature expansion based on the *Peierls argument*, but we shall not pursue it. The actual phase diagram is given by Fig. 3.5.3, where the boundary is called the *critical manifold*.

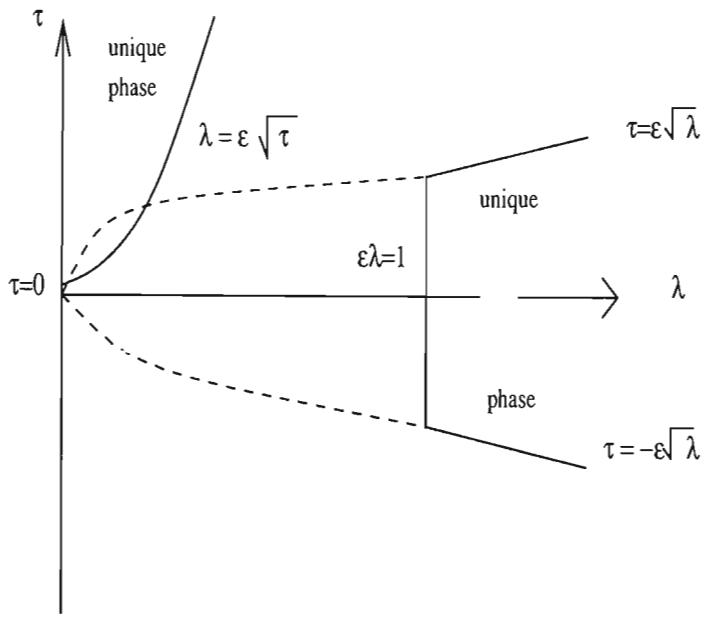

Figure 3.5.2:

The Ginzburg–Landau model has the same critical exponents as the Ising model. The spontaneous magnetization is a continuous function of the parameters, and

$$M(\lambda, \tau) \sim c(\tau_c(\lambda) - \tau)^{\delta}, \qquad \tau \nearrow \tau_c(\lambda), \tag{3.5.26.0}$$

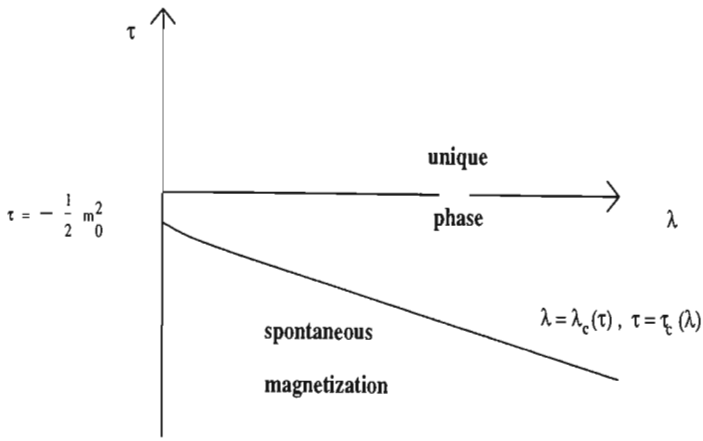

Figure 3.5.3:

$$M(\lambda,\tau) \sim c(\lambda_c(\tau)-\lambda)^{\delta}, \qquad \lambda \nearrow \lambda_c(\tau), \tag{3.5.26.1}$$

with $\delta \approx 0.31$. The magnetic susceptibility has the divergent behavior

$$\chi(\lambda,\tau) \sim c(\tau-\tau_c(\lambda))^{-\gamma}, \qquad \tau \searrow \tau_c(\lambda), \tag{3.5.27.0}$$

$$\chi(\lambda,\tau) \sim c(\lambda-\lambda_c(\tau))^{-\gamma}, \qquad \lambda \searrow \lambda_c(\tau), \tag{3.5.27.1}$$

with $\gamma \approx 1.24$, where

$$\chi(\lambda,\tau,h) = \frac{\partial^2}{\partial h^2} f(\lambda,\tau,h), \tag{3.5.28}$$

$$f(\lambda,\tau,h) = \lim_{\lambda\to\mathbb{Z}^3} \frac{1}{|\Lambda|} \ln Z_\Lambda(\lambda,\tau,h). \tag{3.5.29}$$

Clearly,

$$\chi(\lambda,\tau,h) = \beta^{-1} \sum_{\vec{m}} (\langle\phi(\vec{m})\phi(\vec{0})\rangle_{\lambda\tau h} - (\langle\phi(\vec{0})\rangle_{\lambda\tau h})^2) \tag{3.5.30}$$

is well-defined even for $h = 0$ in the unique phase region arbitrarily close to the critical manifold. The summand is an example of a truncated correlation (already defined in §1.12) and such correlations are known to decay exponentially when this particular one does, and with the same correlation length $\xi(\lambda,\tau,h)$. This is a result of the Lebowitz inequalities for the Euclidean lattice ϕ_3^4 field theory. In the case $h = 0$ and $\lambda > \lambda_c(\tau)$, the picture is:

$$\langle\phi(\vec{0})\rangle = 0, \tag{3.5.31}$$

$$\langle\phi(\vec{m})\phi(\vec{0})\rangle \sim c\exp(-|\vec{m}|/\xi(\lambda,\tau)), \tag{3.5.32}$$

$$\xi(\lambda,\tau) \sim c(\tau - \tau_c(\lambda))^{-\nu}, \qquad \tau \searrow \tau_c(\lambda), \tag{3.5.33.0}$$

$$\xi(\lambda,\tau) \sim c(\lambda - \lambda_c(\tau))^{-\nu}, \qquad \lambda \searrow \lambda_c(\tau), \tag{3.5.33.1}$$

with $\nu \approx 0.63$. Actually, one has the relation $2\delta = 3\nu - \gamma$ by the Widom scaling hypothesis. At $\lambda = \lambda_c(\tau)$, the two-point function $(\phi(\vec{m})\phi(\vec{0}))$ still decays for large $\vec{m}$, but it obeys an inverse power law. The $\tau = -\frac{1}{2}m_0^2$, $\lambda = 0$ case is special:

$$\begin{aligned}(\phi(\vec{m})\phi(\vec{0}))_{\lambda=\tau=0} &= \frac{1}{(2\pi)^3}\left(\prod_{\iota=1}^{3}\int_{-\pi}^{\pi} dp_\iota\right)\frac{e^{i\vec{m}\cdot\vec{p}}}{2\sum_{\iota=1}^{3}(1-\cos p_\iota)} \\ &\sim c|\,\vec{m}\,|^{-1}, \qquad \vec{m}\to\infty, \end{aligned} \tag{3.5.34}$$

because in this case the model is just the free scalar lattice field with zero mass – i.e., the Gaussian whose quadratic form is given by the lattice Laplacian (and therefore whose covariance is the lattice Green's function). For every other point on the critical manifold, however,

$$(\phi(\vec{m})\phi(\vec{0}))_{\lambda=\lambda_c(\tau)} \sim c|\,\vec{m}\,|^{-1+\eta}, \qquad \vec{m}\to\infty, \tag{3.5.35}$$

where η is believed to lie between 0.04 and 0.06. The scaling hypothesis yields the relation $\gamma = (2-\eta)\nu$.

The correlation inequalities mentioned in the last section generalize to the lattice ϕ_3^4 field theory, and it is worthwhile to state the generalizations. We suppress the dependence on h, λ and τ and write $d\mu_\Lambda$ for the measure.

1) *Griffiths Inequalities*. For finite $\Lambda \subset \mathbb{Z}^3$ and $h \leq 0$,

$$\int \prod_{\vec{m}} \phi(\vec{m})^{\kappa_{\vec{m}}}\, d\mu_\Lambda(\phi) \geq 0, \tag{3.5.36}$$

$$\int \prod_{\vec{m}} \phi(\vec{m})^{\kappa_{\vec{m}}+\kappa'_{\vec{m}}}\, d\mu_\Lambda(\phi) \leq \int \prod_{\vec{m}} \phi(\vec{m})^{\kappa_{\vec{m}}}\, d\mu_\Lambda(\phi) \int \prod_{\vec{m}} \phi(\vec{m})^{\kappa'_{\vec{m}}}\, d\mu_\Lambda(\phi), \tag{3.5.37}$$

where κ and κ' are arbitrary multi-indices. Such inequalities hold for all scalar-valued spin systems whose single-spin distributions are even and whose Hamiltonians are ferromagnetic. The restriction $h \leq 0$ is necessary because the external magnetic field term must be included in the Hamiltonian instead of in the single-spin distribution.

2) *Lebowitz Inequalities*. Introduce duplicate variables $\psi(\vec{m})$ and set

$$q(\vec{m}) = \frac{1}{\sqrt{2}}(\phi(\vec{m}) + \psi(\vec{m})), \tag{3.5.38}$$

$$r(\vec{m}) = \frac{1}{\sqrt{2}}(\phi(\vec{m}) - \psi(\vec{m})). \tag{3.5.39}$$

Then for $h \leq 0$,

$$\left\langle\left\langle \prod_{\vec{m}} q(\vec{m})^{\kappa_{\vec{m}}+\kappa'_{\vec{m}}} \right\rangle\right\rangle_\Lambda \geq \left\langle\left\langle \prod_{\vec{m}} q(\vec{m})^{\kappa_{\vec{m}}} \right\rangle\right\rangle_\Lambda \left\langle\left\langle \prod_{\vec{m}} q(\vec{m})^{\kappa'_{\vec{m}}} \right\rangle\right\rangle_\Lambda \quad (3.5.40)$$

$$\left\langle\left\langle \prod_{\vec{m}} r(\vec{m})^{\kappa_{\vec{m}}+\kappa'_{\vec{m}}} \right\rangle\right\rangle_\Lambda \geq \left\langle\left\langle \prod_{\vec{m}} r(\vec{m})^{\kappa_{\vec{m}}} \right\rangle\right\rangle_\Lambda \left\langle\left\langle \prod_{\vec{m}} r(\vec{m})^{\kappa'_{\vec{m}}} \right\rangle\right\rangle_\Lambda \quad (3.5.41)$$

$$\left\langle\left\langle \prod_{\vec{m}} (q(\vec{m})^{\kappa_{\vec{m}}} r(\vec{m})^{\kappa'_{\vec{m}}}) \right\rangle\right\rangle_\Lambda \leq \left\langle\left\langle \prod_{\vec{m}} q(\vec{m})^{\kappa_{\vec{m}}} \right\rangle\right\rangle_\Lambda \left\langle\left\langle \prod_{\vec{m}} r(\vec{m})^{\kappa'_{\vec{m}}} \right\rangle\right\rangle_\Lambda \quad (3.5.42)$$

for arbitrary multi-indices κ and κ', where

$$\langle\langle F(\phi,\psi)\rangle\rangle_\Lambda = \int d\mu_\Lambda(\phi) \int d\mu_\Lambda(\psi) F(\phi,\psi). \quad (3.5.43)$$

3) *Fortuin–Kastelyn–Ginibre Inequality.* If $F(\phi)$ and $G(\phi)$ are monotone increasing functions with respect to each variable $\phi(\vec{m})$, then

$$\int F(\phi)G(\phi)d\mu_\Lambda(\phi) \geq \int F(\phi)d\mu_\Lambda(\phi) \int G(\phi)d\mu_\Lambda(\phi). \quad (3.5.44)$$

Indeed, such an inequality holds for any probability measure of the form

$$e^{W(\phi)} \prod_m d\phi(\vec{m}),$$

where $W(\phi)$ is class C^2 and

$$\frac{\partial^2 W}{\partial\phi(\vec{m})\partial\phi(\vec{n})} \geq 0, \qquad \vec{m}\neq\vec{n} \quad (3.5.45)$$

The Ginzburg–Landau model obviously satisfies this condition.

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3.6 Polymer Expansions

There is a fairly unified view of expansion methods in statistical mechanics and quantum field theory embodied in the abstract theory of polymer expansions. Historically, the various expansions that were developed over the years to deal with different problems were eventually perceived to have common elements. The abstract formalism that was subsequently developed is quite elegant and neatly separates the technical issues in a specific expansion method from those common elements. Since some of the expansions are very complicated, this axiomatic framework has been very convenient. On the other hand, the formulation that is usually advertised in the literature does not cover all of our needs. Before discussing this, we review the standard estimation strategy for proving convergence.

One begins with a set $\mathcal{P}$ of objects called *polymers*. There is a notion of *intersection* defined by a set of unordered pairs of polymers. Two polymers *intersect* if there is an unordered pair in which they both appear. Otherwise, they are *disjoint*. An unordered pair is not to be confused with a two-element set, as a polymer may have a pairing with itself. On the other hand, it may not, in which case the polymer is disjoint from itself. Every polymer ζ has a complex number $z(\zeta)$ assigned to it called the *activity* of

ζ. The *partition function* for this structure is given by

$$Z = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{\zeta_1,\ldots,\zeta_n \in \mathcal{P} \\ \text{mutually disjoint}}} \prod_{\nu=1}^{n} z(\zeta_\nu), \tag{3.6.1}$$

so to express the usual partition function of a concrete model in this formalism, one must identify polymers and their activities for that model.

An *n-polymer* is an n-tuple of polymers, and the activity of an n-polymer $P = (\zeta_1, \ldots, \zeta_n)$ is defined as

$$z(P) = \prod_{\nu=1}^{n} z(\zeta_\nu). \tag{3.6.2}$$

A *line* of P is any distinct pair $\{\mu, \nu\}$ of indices such that ζ_μ intersects ζ_ν. The *graph* of P is the set of all lines of P, while the indices $\{1, \ldots, n\}$ of P are the *vertices of the graph*. Note here that distinct vertices may index the same polymer, and therefore distinct lines may arise from the same intersecting pair of polymers. Two vertices are *connected* if there is a path of lines from one to the other. This equivalence relation partitions $\{1, \ldots, n\}$ into equivalence classes called the *connected components of the graph*. The graph is *disconnected* if the connected components are the singletons $\{\nu\}$, and the graph is *connected* if the only connected component is $\{1, \ldots, n\}$. An n-polymer P is connected (resp. disconnected) if its graph $G_P \equiv G(\zeta_1, \ldots, \zeta_n)$ is.

For an *arbitrary* graph G on $\{1, \ldots, n\}$, an *n-vertex subgraph* is just a subset of G – as a set of lines – with $\{1, \ldots, n\}$ as its set of vertices, whether they are connected or not. If G is connected, one defines

$$n(G) = \sum_{\substack{\text{connected } n\text{-vertex} \\ \text{subgraphs } \tilde{G} \text{ of } G}} (-1)^{\text{card } \tilde{G}} \tag{3.6.3}$$

as the *index* of G. The *index* of a connected n-polymer P is defined as

$$n(P) = n(G_P). \tag{3.6.4}$$

The fundamental algebraic property of a partition function is an abstraction of what has long been known as the "Linked Cluster Theorem" in different contexts. In short,

$$\ln Z = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{P \in \mathcal{K}_n} n(P) z(P), \tag{3.6.5}$$

where $\mathcal{K}_n$ denotes the set of connected n-polymers.

The proof is quite standard by now, and we sketch it right here for the convenience of the reader. Notice that

$$\exp\left(\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{P \in \mathcal{K}_n} n(P) z(P)\right) = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{n_1,\ldots,n_N=0}^{\infty} \left(\prod_{k=1}^{N} \frac{1}{n_k!}\right) \sum_{P^1 \in \mathcal{K}_{n_1}} \cdots \sum_{P^N \in \mathcal{K}_{n_N}} \prod_{k=1}^{N} n(P^k) \prod_{k=1}^{N} z(P^k) \tag{3.6.6}$$

must somehow reduce to

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{\zeta_1,\ldots,\zeta_n \in \mathcal{P} \\ \text{mutually disjoint}}} \prod_{\nu=1}^{n} z(\zeta_\nu) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\zeta_1,\ldots,\zeta_n \in \mathcal{P}} \prod_{\mu<\nu} (1 - \chi(\zeta_\mu, \zeta_\nu)) \prod_{\nu} z(\zeta_\nu), \quad (3.6.7)$$

where

$$\chi(\zeta, \zeta') = \begin{cases} 0, & \zeta \text{ and } \zeta' \text{ disjoint,} \\ 1, & \zeta \text{ and } \zeta' \text{ intersect.} \end{cases} \quad (3.6.8)$$

On one hand,

$$\prod_{k=1}^{N} n(P^k) = \sum_{\substack{\text{connected} \\ n_1\text{-vertex subgraphs } G_1 \text{ of } G_{P1}}} \cdots \sum_{\substack{\text{connected } n_N\text{-vertex} \\ \text{subgraphs } G_N \text{ of } G_{PN}}} (-1)^{\sum_{k=1}^{N} \text{card } G_k}, \quad (3.6.9)$$

while on the other hand,

$$\begin{aligned} \prod_{\mu<\nu} (1 - \chi(\zeta_\mu, \zeta_\nu)) &= \sum_{\substack{\text{graphs } G \\ \text{on } \{1,\ldots,n\}}} \prod_{\{\mu,\nu\} \in G} (-\chi(\zeta_\mu, \zeta_\nu)) \\ &= \sum_{\substack{n\text{-vertex subgraphs } G \\ \text{of } G(\zeta_1,\ldots,\zeta_n)}} (-1)^{\text{card } G}. \end{aligned} \quad (3.6.10)$$

In this sum, $(\zeta_1, \ldots, \zeta_n)$ is an arbitrary n-polymer, while G is an arbitrary subgraph, not necessarily connected.

Now in the expression (3.6.6) define $\mathcal{R}(n_1, \ldots, n_N)$ as the set of all indexed partitions of $\left\{1, \ldots, \sum_{k=1}^{N} n_k\right\}$ where the subsets have the cardinalities $n_1, \ldots, n_N$. Given $\Gamma \in \mathcal{R}(n_1, \ldots, n_N)$, define $P^\Gamma(P^1, \ldots, P^N)$ as the $\sum_{k=1}^{N} n_k$-polymer defined such that if μ is the νth integer in the subset indexed by k,

$$P^\Gamma_\mu(P^1, \ldots, P^N) = P^k_\nu. \quad (3.6.11)$$

The key observation is that for *any* $\Gamma \in \mathcal{R}(n_1, \ldots, n_N)$, the identity (3.6.9) may be written as

$$\prod_{k=1}^{N} n(P^k) = \sum_{\substack{\sum_k n_k\text{-vertex subgraphs } G \text{ of } G_\Gamma(P^1,\ldots,P^N) \text{ such that} \\ \Gamma \text{ is the set of } G\text{-components}}} (-1)^{\text{card } G} \quad (3.6.12)$$

where $G_\Gamma(P^1,\dots,P^N)$ denotes the graph of $P^\Gamma(P^1,\dots,P^N)$ and *G-components* are defined by *G-connectedness*. Two vertices are G-connected if there is a path of lines in G from one to the other. Now since

$$\prod_{k=1}^{N} z(P^k) = z(P^\Gamma(P^1,\dots,P^N)), \tag{3.6.13}$$

$$\text{card } \mathcal{R}(n_1,\dots,n_N) = \prod_{k=1}^{N} \frac{1}{n_k!}\left(\sum_{k=1}^{N} n_k\right)!, \tag{3.6.14}$$

it follows that we have the formula

$$\exp\left(\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{P\in\mathcal{K}_n} n(P)z(P)\right) = \sum_{N=0}^{\infty}\frac{1}{N!}\sum_{n_1,\dots,n_N=0}^{\infty}\frac{1}{\left(\sum_{k=1}^{N} n_k\right)!}$$
$$\times \sum_{\Gamma\in\mathcal{R}(n_1,\dots,n_N)}\ \sum_{P^1\in\mathcal{K}_{n_1}}\cdots\sum_{P^N\in\mathcal{K}_{n_N}} z(P^\Gamma(P^1,\dots,P^N))$$
$$\times \sum_{\substack{\sum_k n_k\text{-vertex subgraphs } G \text{ of} \\ G_\Gamma(P^1,\dots,P^N) \text{ such that} \\ \Gamma \text{ is the set of } G\text{-components,}}} (-1)^{\text{card } G} \tag{3.6.15}$$

which may, in turn, be written as

$$\exp\left(\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{P\in\mathcal{K}_n} n(P)z(P)\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{N=0}^{\infty}\frac{1}{N!}$$
$$\times \sum_{\substack{n_1,\dots,n_N\geq 0 \\ \sum_k n_k = n}}\ \sum_{\Gamma\in\mathcal{R}(n_1,\dots,n_N)}\ \sum_{n_1\text{-polymers } P^1}\cdots\sum_{n_N\text{-polymers } P^N} z(P^\Gamma(P^1,\dots,P^N))$$
$$\sum_{\substack{n\text{-vertex subgraphs } G \text{ of } G_\Gamma(P^1,\dots,P^N) \text{ such that} \\ \Gamma \text{ is the set of } G\text{-components}}} (-1)^{\text{card } G} \tag{3.6.16}$$

because the restriction on the innermost sum makes the restrictions $P^k \in \mathcal{K}_{n_k}$ superfluous. However, this yields the further reduction:

$$\exp\left(\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{P\in\mathcal{K}_n} n(P)z(P)\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{n\text{-polymers}} z(P)$$
$$\times \sum_{N=0}^{\infty}\frac{1}{N!}\sum_{\substack{n_1,\dots,n_N\geq 0 \\ \sum_k n_k = n}}\ \sum_{\Gamma\in\mathcal{R}(n_1,\dots,n_N)}$$
$$\sum_{\substack{n\text{-vertex subgraphs } G \text{ of } G_P \text{ such that} \\ \Gamma \text{ is the set of } G\text{-components}}} (-1)^{\text{card } G} \tag{3.6.17}$$

By the summation identification

$$\sum_{\substack{n_1,\dots,n_N\geq 0\\ \sum_k n_k=n}} \sum_{\Gamma\in\mathcal{R}(n_1,\dots,n_N)} = \sum_{\substack{\text{all indexed partitions } \Gamma\\ \text{of } \{1,\dots,n\} \text{ using } N \text{ subsets}}}$$

$$= N! \sum_{\substack{\text{all partitions of } \{1,\dots,n\}\\ \text{using } N \text{ subsets}}}, \tag{3.6.18}$$

the desired identity now follows from the observation that the resulting inner sums, starting with the N-summation, are just a decomposition of the sum over all n-vertex subgraphs of G_P.

With the algebraic identity established, we turn to the issue of convergence, which is crucial to the application of polymer expansions. Actually, we need to be a little more concrete at this point. In addition to an activity, a polymer must now have a *support* and a *volume*, so the additional structure is a set $\mathcal{A}$ that the polymers live in. In applications, this set can be the lattice $\mathbb{Z}^d$, the partition of $\mathbb{R}^d$ into unit cubes, or even a wavelet basis in a multiscale analysis of a continuum model. For a given polymer ζ, a finite subset supp $\zeta \subset \mathcal{A}$ is assigned to it, and we refer to it as the support of ζ. We define the volume $|\zeta|$ of the polymer as simply the cardinality of supp ζ. The intersection properties of the polymers are assumed to be consistent with this notion of support in the obvious sense: they are disjoint as polymers if and only if their supports are disjoint as sets. The estimates that one often assumes for the abstract theory – and verifies for a given model – are:

$$|z(\zeta)| \leq C_0 e^{-c_0|\zeta|}, \tag{3.6.19}$$

$$\text{card}\{\zeta\in\mathcal{P}:\ \alpha\in\text{supp }\zeta, |\zeta| = M\} \leq C_1 e^{c_1 M}, \tag{3.6.20}$$

with c_0, c_1, C_0, C_1 all positive constants, whose relative sizes will be an issue. The first bound is referred to as an *energy estimate*, while the second is an *entropy estimate*. Now for every finite $\mathcal{S}\subset\mathcal{A}$, consider

$$Z(\mathcal{S}) = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\substack{\zeta_1,\dots,\zeta_n \text{ mutually}\\ \text{disjoint, supp } \zeta_\nu\subset\mathcal{S}}}\prod_{\nu=1}^{n} z(\zeta_\nu), \tag{3.6.21}$$

$$\ln Z(\mathcal{S}) = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\substack{P\in\mathcal{K}_n\\ \text{supp } P\subset\mathcal{S}}} n(P)z(P), \tag{3.6.22}$$

where it is understood that for any n-polymer $P=(\zeta_1,\dots,\zeta_n)$

$$\text{supp } P = \bigcup_{\nu=1}^{n}\text{supp }\zeta_\nu. \tag{3.6.23}$$

The problem is to control the behavior of these expansions as $\mathcal{S}$ approaches $\mathcal{A}$ in the sense of set inclusion.

We first consider the expansion of $Z(\mathcal{S})$, as it is easier to control from a combinatorial standpoint. Fix $n \geq 0$ and consider the inner sum: since $\zeta_1, \ldots, \zeta_n$ are disjoint, we may select for each such sequence of polymers a sequence $(\alpha_1, \ldots, \alpha_n)$ of points in $\mathcal{S}$, *all distinct*, such that α_ν lies in the support of ζ_ν, for each ν. The disjointness also implies

$$\sum_{\nu=1}^{n} |\zeta_\nu| \leq \text{card } \mathcal{S} \equiv m, \tag{3.6.24}$$

so we over-sum as follows:

$$\sum_{\substack{\zeta_1,\ldots,\zeta_n \text{ mutually} \\ \text{disjoint, supp } \zeta_\nu \subset \mathcal{S}}} \prod_{\nu=1}^{n} |z(\zeta_\nu)| \leq \sum_{\substack{\alpha_1,\ldots,\alpha_n \in \mathcal{S} \\ \text{all distinct}}} \sum_{\substack{\zeta_1,\ldots,\zeta_n \in \mathcal{P} \\ \alpha_\nu \in \text{supp } \zeta_\nu \\ \sum_\nu |\zeta_\nu| \leq m}} \prod_{\nu=1}^{n} |z(\zeta_\nu)|. \tag{3.6.25}$$

The energy estimate implies

$$\prod_{\nu=1}^{n} |z(\zeta_\nu)| \leq C_0^n e^{-c_0 \sum_\nu |\zeta_\nu|} \leq C_0^n, \tag{3.6.26}$$

while the entropy estimate implies

$$\text{card}\{(\zeta_1, \ldots, \zeta_n) \in \mathcal{P}^n\colon\ \alpha_\nu \in \text{supp } \zeta_\nu \text{ and } |\zeta_\nu| = m_\nu\} \leq C_1^n e^{c_1 \sum_\nu m_\nu}. \tag{3.6.27}$$

Hence

$$\begin{aligned}
&\sum_{\substack{\zeta_1,\ldots,\zeta_n \text{ mutually} \\ \text{disjoint, supp } \zeta_\nu \subset \mathcal{S}}} \prod_{\nu=1}^{n} |z(\zeta_\nu)| \\
&\leq C_0^n C_1^n \sum_{\substack{\alpha_1,\ldots,\alpha_n \in \mathcal{S} \\ \text{all distinct}}} \sum_{\substack{\sum_{\nu=1}^{n} m_\nu \leq m \\ m_\nu \geq 0}} e^{c_1 \sum_\nu m_\nu} \\
&\leq C_0^n C_1^n e^{c_1 m} \text{card}\left\{(m_1, \ldots, m_n)\colon \sum_{\nu=1}^{n} m_\nu \leq m, m_\nu \geq 0\right\} \\
&\quad \times \text{card}\{(\alpha_1, \ldots, \alpha_n) \in \mathcal{S}^n\colon \text{ all distinct}\}.
\end{aligned} \tag{3.6.28}$$

The first cardinality counts the number of points with integer coordinates in the positive octant of n-dimensional Cartesian space enclosed by the simplicial face with vertices $m\vec{e}_1, \ldots, m\vec{e}_n$. This number is roughly the volume $m^n/n!$ of that region of space. Allowing for boundary effects, one has

$$\text{card}\left\{(m_1, \ldots, m_n)\colon\ m_\nu \geq 0, \sum_{\nu=1}^{n} m_\nu \leq m\right\} \leq \sum_{\nu=0}^{n} \frac{m^\nu}{\nu!}. \tag{3.6.29}$$

The second cardinality is obviously the number

$$\text{Perm}(m, n) = \frac{m!}{(m-n)!} = \binom{m}{n} n! \tag{3.6.30}$$

of permutations of m objects taken n at a time. *Thus*

$$\sum_{\substack{\zeta_1,\dots,\zeta_n \text{ mutually} \\ \text{disjoint, supp } \zeta_\nu \subset \mathcal{S}}} \prod_{\nu=1}^{n} |z(\zeta_\nu)| \leq C_0^n C_1^n e^{c_1 m} \binom{m}{n} n! \sum_{\nu=0}^{n} \frac{m^\nu}{\nu!}$$

$$\leq 2^m C_0^n C_1^n e^{c_1 m} n! \sum_{\nu=0}^{n} \frac{m^\nu}{\nu!}, \tag{3.6.31}$$

so if $C_0 C_1 < 1$, *then*

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{\zeta_1,\dots,\zeta_n \text{ mutually} \\ \text{disjoint, supp } \zeta_\nu \subset \mathcal{S}}} \prod_{\nu=1}^{n} |z(\zeta_\nu)| \leq 2^m e^{c_1 m}$$

$$\times \sum_{n=0}^{\infty} C_0^n C_1^n \sum_{\nu=0}^{n} \frac{m^\nu}{\nu!} \leq 2^m e^{c_1 m} \frac{e^m}{1 - C_0 C_1}. \tag{3.6.32}$$

This is an abstract stability bound on the partition function $Z(\mathcal{S})$, and it obviously implies

$$\frac{1}{\text{card } \mathcal{S}} \ln Z(\mathcal{S}) \leq c, \qquad \mathcal{S} \subset \mathcal{A}, \tag{3.6.33}$$

provided $C_0 C_1 < 1$. In spite of the algebraic connection between the polymer expansions of $Z(\mathcal{S})$ and $(\text{card } \mathcal{S})^{-1} \ln Z(\mathcal{S})$, this estimation does *not* control the expansion of the latter quantity uniformly in $\mathcal{S}$ – for the very simple reason that $\ln(1+z)$ has radius of convergence equal to 1. One must control the latter expansion directly. The issue of uniform convergence will now depend on the relative sizes of the constants c_0 and c_1 as well as on the absolute size of c_0, which we did not use in the above estimation. The combinatoric estimation is substantially deeper as well.

One can no longer sum over n-tuples $(\alpha_1, \dots, \alpha_n)$ of points as we have just done, because the estimation must avoid the occurrence of powers of card $\mathcal{S}$. The connectivity properties of the n-polymers summed over must be exploited, and a tree structure is needed to obtain good control over the n-fold sum. To this end, one normally defines a *tree graph* on the vertices $\{1, \dots, n\}$ to be a simply connected graph on $\{1, \dots, n\}$ and applies the standard graph-theoretic result that the index of an arbitrary connected graph on $\{1, \dots, n\}$ is bounded by the number of connected subgraphs that are tree graphs. We omit the proof, as the result is widely known outside of mathematical physics. Thus, for an arbitrary connected graph G on $\{1, \dots, n\}$,

$$|n(G)| \leq \sum_{\substack{\text{tree graphs } T \text{ that are} \\ \text{connected subgraphs of } G}} 1 \tag{3.6.34}$$

and so if we insert this estimation in the nth term (with $n > 1$) of the polymer expansion of $\ln Z(\mathcal{S})$ and interchange the sums, we get

$$\sum_{P \in \mathcal{K}_n} |n(G_P)|\, |z(P)|$$

$$\leq \sum_{\substack{\text{tree graphs } T \\ \text{on } \{1,\dots,n\}}} \sum_{m_1,\dots,m_n=0}^{\infty} \sum_{\substack{P=(\zeta_1,\dots,\zeta_n)\in\mathcal{K}_n \\ T \text{ is a subgraph of } G_P \\ |\zeta_\nu|=m_\nu}} \prod_{\nu=1}^{n} |z(\zeta_\nu)|, \qquad (3.6.35)$$

where we have further decomposed the sum with respect to volumes of polymers. Controlling the inner-most sum is the crucial part of the estimation. If one tries to count the connected graphs of which T is a connected subgraph, the resulting over-summation is disastrous. It is important to note that one is summing over all connected *n-polymers* – not n-vertex graphs – on which T lives. As one uses the structure of T to sum over all such n-polymers, the polymers P for which G_P is not simply connected are automatically included in the summation.

To control the sum with T, one finds it convenient to use the standard notion of *coordination number*. For a tree graph T on $\{1,\dots,n\}$, the coordination number $d_\nu(T)$ of a vertex ν is just the number of lines in T that meet ν. Every tree graph can be reduced to a single vertex by the following iterative procedure. Let A be the subset of vertices ν for which $d_\nu(T) > 1$ and let T' be the tree graph on A obtained by removing the single lines meeting the other vertices. Obviously, there will now be vertices $\nu \in A$ such that $d_\nu(T') = 1$, so now remove those single lines to obtain the tree graph T'' on the subset A' obtained by removing those vertices from A. This procedure can be iterated for as long as there are distinct vertices.

The summation for fixed T is carried out as follows. First apply the energy estimate to obtain

$$\prod_{\nu=1}^{n} |z(\zeta_\nu)| \leq C_0^n e^{-c_0 \sum_\nu m_\nu}, \qquad (3.6.36)$$

and pull this bound outside the inner-most sum, which now becomes (with $\mathcal{S}_T^P$ to be defined)

$$\sum_{\substack{P=(\zeta_1,\dots,\zeta_n)\in\mathcal{K}_n \\ T \text{ is a subgraph of } G_P \\ |\zeta_\nu|=m_\nu}} 1 \leq C_1^n e^{c_1 \sum_\nu m_\nu} \operatorname*{supremum}_{\substack{P=(\zeta_1,\dots,\zeta_n)\in\mathcal{K}_n \\ T \text{ is a subgraph of } G_P \\ |\zeta_\nu|=m_\nu}} \sum_{(\alpha_1,\dots,\alpha_n)\in\mathcal{S}_T^P} 1. \qquad (3.6.37)$$

Here we have applied the entropy estimate to each sum over polymers ζ_ν such that $\alpha_\nu \in \operatorname{supp} \zeta_\nu$. This replaces the sum over polymers with a supremum over polymers, but one still has to sum over the possible α_ν determined by the T-connectivity property of a given n-polymer. This vital control is implicit in the definition of $\mathcal{S}_T^P$, which reflects the tree reduction just described. First, for every vertex ν such that $d_\nu(T) = 1$, $\alpha_\nu \in \operatorname{supp} \zeta_{\hat{\nu}}$, where $\{\nu, \hat{\nu}\}$ is that single line. Then for every ν' such that $d_{\nu'}(T') = 1$, $\alpha_{\nu'} \in \operatorname{supp} \zeta_{\hat{\nu}'}$, where $\{\nu', \hat{\nu}'\}$ is *that* single line, and so on. It is easy to see that this definition of $\mathcal{S}_T^P$ is the correct one for the bound (3.6.37) and that, since the multiplicity of $\mu = \hat{\nu}^{\#}$ is $d_\mu(T) - 1$,

$$\sum_{(\alpha_1,\dots,\alpha_n)\in\mathcal{S}_T^P} 1 \leq \operatorname{card}(\mathcal{S}) \prod_{\mu=1}^{n} m_\mu^{d_\mu(T)-1}, \qquad (3.6.38)$$

where the last polymer in the tree reduction is not connected to anything. This means that $\alpha_\nu \in \mathcal{S}$ is the only restriction for the last remaining vertex ν, so $\mathrm{card}(\mathcal{S})$ appears as a factor.

Combining (3.6.36–3.3.38) with (3.6.35), one obtains interesting tree graph estimation of the nth term (with $n > 1$) of the polymer expansion:

$$\sum_{P\in\mathcal{K}_n} |n(G_P)|\,|z(P)| \le \sum_{\substack{\text{tree graphs } T \\ \text{on } \{1,\dots,n\}}} \sum_{m_1,\dots,m_n=0}^{\infty}$$

$$C_0^n C_1^n e^{(c_1-c_0)\sum_\nu m_\nu}\,\mathrm{card}(\mathcal{S}) \prod_{\mu=1}^{n} m_\mu^{d_\mu(T)-1} \tag{3.6.39}$$

The next step is to decompose the sum over all tree graphs with respect to coordination numbers, and then apply the celebrated Cayley Theorem, which states that the number of tree graphs on $\{1,\dots,n\}$ with coordination numbers $d_1,\dots,d_n$ is just the ratio $(n-2)!/\prod_{\mu=1}^{n}(d_\mu-1)!$. Hence

$$\begin{aligned}
&\sum_{P\in\mathcal{K}_n} |n(G_P)|\,|z(P)| \\
&\le (n-2)!C_0^nC_1^n|\mathcal{S}| \sum_{\substack{d_1,\dots,d_n\ge 1 \\ \sum_\mu d_\mu = 2n-2}} \sum_{m_1,\dots,m_n=0}^{\infty} e^{(c_1-c_0)\sum_\nu m_\nu} \prod_{\mu=1}^{n} \frac{m_\mu^{d_\mu-1}}{(d_\mu-1)!} \\
&\le (n-2)!C_0^nC_1^n\mathrm{card}(\mathcal{S}) \sum_{m_1,\dots,m_n=0}^{\infty} e^{(c_1-c_0)\sum_\nu m_\nu} \prod_{\mu=1}^{n} e^{m_\mu} \\
&= (n-2)!C_0^nC_1^n\mathrm{card}(\mathcal{S})(1-e^{c_1-c_0+1})^{-n},
\end{aligned} \tag{3.6.40}$$

provided $c_1 < c_0 - 1$. This yields the bound

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{P\in\mathcal{K}_n} |n(G_P)|\,|z(P)| \\
&\le 1 + \sum_{\zeta\in\mathcal{P}} |z(\zeta)| + \mathrm{card}(\mathcal{S}) \sum_{n=2}^{\infty} \frac{1}{n(n-1)} C_0^nC_1^n(1-e^{c_1-c_0+1})^{-n}
\end{aligned} \tag{3.6.41}$$

on the polymer expansion of $\ln Z(\mathcal{S})$. Clearly, the $n = 1$ contribution has the bound given by

$$\sum_{\zeta\in\mathcal{P}} |z(\zeta)| \le C_0C_1\mathrm{card}(\mathcal{S}) \sum_{n=0}^{\infty} e^{(c_1-c_0)m} = \frac{C_0C_1}{1-e^{c_1-c_0}}\mathrm{card}(\mathcal{S}), \tag{3.6.42}$$

where we have summed over volumes and over $\alpha \in \mathcal{S}$, applying the entropy estimate in summing over $\zeta \in \mathcal{P}$ with fixed volume and supports containing fixed α. In other words,

the basic estimation is the same as in the $n > 1$ case, but there is no combinatoric estimation to do. In summary,

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{P \in \mathcal{K}_n} |n(G_P)| \, |z(P)| \leq 1 + \mathrm{card}(\mathcal{S}) \frac{C_0 C_1}{1 - e^{c_1 - c_0}}$$
$$+ \mathrm{card}(\mathcal{S}) \sum_{n=2}^{\infty} \frac{1}{n(n-1)} C_0^n C_1^n (1 - e^{c_1 - c_0 + 1})^{-n}, \tag{3.6.43}$$

and the dominating series converges if $C_0 C_1 \leq 1 - e^{c_1 - c_0 + 1}$

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3.7 Expansion for Nearest-Neighbor Interactions

As applications of the theory of polymer expansions, the high-temperature expansions of the classical Heisenberg model and of the lattice ϕ_3^4 model are probably the most elementary. This section is devoted to describing these two specific examples as part of our progression toward more difficult expansion methods. We consider the high-temperature classical Heisenberg ferromagnet first.

For a finite set Λ of lattice sites, the space of configurations is $\prod_{\vec{m} \in \Lambda} \mathbb{S}^2$ and the single-spin distribution is surface measure $d\sigma$ on $\mathbb{S}^2$ For zero external magnetic field,

the Hamiltonian is just

$$H_\Lambda = -J \sum_{\langle \vec{m}, \vec{n} \rangle \subset \Lambda} \vec{s}_{\vec{m}} \quad \vec{s}_{\vec{n}}, \tag{3.7.1}$$

and our goal is to control the $\Lambda = \mathbb{Z}^3$ limit of the Gibbs expectations

$$\left\langle \prod_{j=1}^{N} s_{\vec{n}_j \iota_j} \right\rangle_\Lambda = Z_\Lambda^{-1} \int \prod_{j=1}^{N} s_{\vec{n}_j \iota_j} e^{-\beta H_\Lambda(\vec{s})} \prod_{\vec{\ell} \in \Lambda} d\sigma(\vec{s}_{\vec{\ell}}), \tag{3.7.2}$$

$$Z_\Lambda = \int e^{-\beta H_\Lambda(\vec{s})} \prod_{\vec{\ell} \in \Lambda} d\sigma(\vec{s}_{\vec{\ell}}) \tag{3.7.3}$$

for small β (high temperature). These quantities are generated by the *generalized partition function*

$$Z(\vec{h}_{\vec{\ell}}: \ \vec{\ell} \in \Lambda) = \int \exp\left(\sum_{\vec{\ell} \in \Lambda} \vec{h}_{\vec{\ell}} \cdot \vec{s}_{\vec{\ell}} + \beta J \sum_{\langle \vec{m}, \vec{n} \rangle \subset \Lambda} \vec{s}_{\vec{m}} \quad \vec{s}_{\vec{n}} \right) \prod_{\vec{\ell} \in \Lambda} d\sigma(\vec{s}_{\vec{\ell}}) \tag{3.7.4}$$

and our task is to develop a disconnected polymer expansion of

$$\widehat{Z}(\vec{h}_{\vec{\ell}}: \ \vec{\ell} \in \Lambda) = \frac{Z(\vec{h}_{\vec{\ell}}: \ \vec{\ell} \in \Lambda)}{\prod_{\vec{\ell} \in \Lambda} Z(\vec{h}_{\vec{\ell}})}, \tag{3.7.5}$$

where we set

$$Z(\vec{h}_{\vec{\ell}}) = \int \exp(\vec{h}_{\vec{\ell}} \quad \vec{s}_{\vec{\ell}}) d\sigma(\vec{s}_{\vec{\ell}}). \tag{3.7.6}$$

Once the expansion is developed, we establish the energy estimate and the entropy estimate for Λ-uniform convergence of the corresponding polymer expansion of

$$\frac{1}{\text{card } \Lambda} \ln \widehat{Z}(\vec{h}_{\vec{\ell}}: \ \vec{\ell} \in \Lambda) = \frac{1}{\text{card } \Lambda} \left(\ln Z(\vec{h}_{\vec{\ell}}: \ \vec{\ell} \in \Lambda) - \sum_{\vec{\ell} \in \Lambda} \ln Z(\vec{h}_{\vec{\ell}}) \right) \tag{3.7.7}$$

in connected n-polymers in the high-temperature regime.

In this example, the most obvious expansion method works. Simply expand $Z(\vec{h}_{\vec{\ell}}: \ \vec{\ell} \in \Lambda)$ in powers of β:

$$Z(\vec{h}_{\vec{\ell}}: \ \vec{\ell} \in \Lambda) = \sum_{N=0}^{\infty} \frac{1}{N!} \beta^N J^N \sum_{\langle \vec{m}_1, \vec{n}_1 \rangle \subset \Lambda} \cdots \sum_{\langle \vec{m}_N, \vec{n}_N \rangle \subset \Lambda}$$

$$\int \exp\left(\sum_{\vec{\ell} \in \Lambda} \vec{h}_{\vec{\ell}} \cdot \vec{s}_{\vec{\ell}} \right) \prod_{j=1}^{N} (\vec{s}_{\vec{m}_j} \cdot \vec{s}_{\vec{n}_j}) \prod_{\vec{\ell} \in \Lambda} d\sigma(\vec{s}_{\vec{\ell}}). \tag{3.7.8}$$

However, a power series in β is not the most natural point of view, even in this case. We first write this in terms of multi-indices on nearest-neighbor bonds. Let $\mathcal{B}_\Lambda$ be the set of such bonds contained in Λ and $\mathcal{A}_N^\Lambda$ the set of all multi-indices κ: $\mathcal{B}_\Lambda \to \{0, 1, 2, \ldots\}$ such that

$$|\kappa| = \sum_{\langle \vec{m}, \vec{n} \rangle \subset \Lambda} \kappa(\langle \vec{m}, \vec{n} \rangle) = N. \tag{3.7.9}$$

The element of $\mathcal{A}_N^\Lambda$ associated with a sequence $(\langle \vec{m}_1, \vec{n}_1 \rangle, \ldots, \langle \vec{m}_N, \vec{n}_N \rangle)$ is defined by

$$\kappa(\langle \vec{m}, \vec{n} \rangle) = \text{card}\{j\colon \langle \vec{m}_j, \vec{n}_j \rangle = \langle \vec{m}, \vec{n} \rangle\}, \tag{3.7.10}$$

and there are

$$\frac{N!}{\prod_{\langle \vec{m}, \vec{n} \rangle} \kappa(\langle \vec{m}, \vec{n} \rangle)!}$$

such sequences with which a given multi-index κ is associated. Since the terms in the expansion depend only on the associated multi-index, we have

$$\begin{aligned}
&Z(\vec{h}_{\vec{\ell}}\colon \vec{\ell} \in \Lambda) \\
&= \sum_{N=0}^{\infty} (\beta J)^N \sum_{\kappa \in \mathcal{A}_N^\Lambda} \frac{1}{\prod_{\langle \vec{m}, \vec{n} \rangle} \kappa(\langle \vec{m}, \vec{n} \rangle)!} \left(\prod_{\vec{\ell} \in \Lambda \setminus \text{supp } \kappa} Z(\vec{h}_{\vec{\ell}}) \right) \\
&\quad \times \int \exp\left(\sum_{\vec{\ell} \in \text{ supp } \kappa} \vec{h}_{\vec{\ell}} \cdot \vec{s}_{\vec{\ell}} \right) \prod_{\langle \vec{m}, \vec{n} \rangle} (\vec{s}_{\vec{m}} \cdot \vec{s}_{\vec{n}})^{\kappa(\langle \vec{m}, \vec{n} \rangle)} \prod_{\vec{\ell} \in \text{ supp } \kappa} d\sigma(\vec{s}_{\vec{\ell}}) \\
&= \sum_{\kappa \in \mathcal{A}^\Lambda} \prod_{\langle \vec{m}, \vec{n} \rangle} \frac{(\beta J)^{\kappa(\langle \vec{m}, \vec{n} \rangle)}}{\kappa(\langle \vec{m}, \vec{n} \rangle))!} \prod_{\vec{\ell} \in \Lambda \setminus \text{supp } \kappa} Z(\vec{h}_{\vec{\ell}}) \int \exp\left(\sum_{\vec{\ell} \in \text{ supp } \kappa} \vec{h}_{\vec{\ell}} \cdot \vec{s}_{\vec{\ell}} \right) \\
&\quad \times \prod_{\langle \vec{m}, \vec{n} \rangle} (\vec{s}_{\vec{m}} \cdot \vec{s}_{\vec{n}})^{\kappa(\langle \vec{m}, \vec{n} \rangle)} \prod_{\vec{\ell} \in \text{ supp } \kappa} d\sigma(\vec{s}_{\vec{\ell}}),
\end{aligned} \tag{3.7.11}$$

where $\mathcal{A}^\Lambda = \bigcup_{N=0}^{\infty} \mathcal{A}_N^\Lambda$ and supp κ is the set of lattice sites joined by those bonds to which κ assigns non-zero weights.

We define a polymer to be any connected set of nearest-neighbor bonds $\langle \vec{m}, \vec{n} \rangle$. The support of a polymer is just the set of lattice sites joined by the bonds, while the activity of a polymer is to be inferred from a reorganization of this power series expansion. For a given multi-index κ, there is a uniquely determined set $\{\zeta_1, \ldots, \zeta_n\}$ of polymers with mutually disjoint supports which partitions supp κ. There are $n!$ permutations of this set of polymers, and each permutation is an element of $\mathcal{D}_n$. Thus

$$Z(\vec{h}_{\vec{\ell}}\colon \vec{\ell} \in \Lambda)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(\zeta_1,\ldots,\zeta_n)\in\mathcal{D}_n} \left(\prod_{\vec{\ell}\in\Lambda\setminus \bigcup_{k=1}^{n} \operatorname{supp}\zeta_k} Z(\vec{h}_{\vec{\ell}}) \right) \sum_{\substack{\kappa\in\mathcal{A}^{\Lambda} \\ \operatorname{supp}\kappa = \bigcup_{k=1}^{n} \operatorname{supp}\zeta_k}}$$

$$\left(\prod_{\langle\vec{m},\vec{n}\rangle} \frac{(\beta J)^{\kappa(\langle\vec{m},\vec{n}\rangle)}}{\kappa(\langle\vec{m},\vec{n}\rangle)!} \right) \left(\prod_{\vec{\ell}\in\bigcup_{k=1}^{n} \operatorname{supp}\zeta_k} \int d\sigma(\vec{s}_{\vec{\ell}}) \right)$$

$$\prod_{\vec{\ell}\in\bigcup_{k=1}^{n} \operatorname{supp}\zeta_k} \exp(\vec{h}_{\vec{\ell}} \cdot \vec{s}_{\vec{\ell}}) \prod_{\langle\vec{m},\vec{n}\rangle} (\vec{s}_{\vec{m}} \quad \vec{s}_{\vec{n}})^{\kappa(\langle\vec{m},\vec{n}\rangle)}, \tag{3.7.12}$$

and it is apparent that both the multiple integration and the multiple summation factorize over the mutually disjoint polymer supports:

$$Z(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell}\in\Lambda)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(\zeta_1,\ldots,\zeta_n)\in\mathcal{D}_n} \left(\prod_{\vec{\ell}\in\Lambda\setminus \bigcup_{k=1}^{n} \operatorname{supp}\zeta_n} Z(\vec{h}_{\vec{\ell}}) \right)$$

$$\times \prod_{k=1}^{n} \left(\sum_{\substack{\kappa\in\mathcal{A}^{\Lambda} \\ \operatorname{supp}\kappa = \operatorname{supp}\zeta_k}} \prod_{\langle\vec{m},\vec{n}\rangle} \frac{(\beta J)^{\kappa(\langle\vec{m},\vec{n}\rangle)}}{\kappa(\langle\vec{m},\vec{n}\rangle)!} \left(\prod_{\vec{\ell}\in\operatorname{supp}\zeta_k} \int d\sigma(\vec{s}_{\vec{\ell}}) \right) \right.$$

$$\left. \prod_{\vec{\ell}\in\operatorname{supp}\zeta_k} \exp(\vec{h}_{\vec{\ell}} \cdot \vec{s}_{\vec{\ell}}) \prod_{\langle\vec{m},\vec{n}\rangle} (\vec{s}_{\vec{m}} \quad \vec{s}_{\vec{n}})^{\kappa(\langle\vec{m},\vec{n}\rangle)} \right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(\zeta_1,\ldots,\zeta_n)\in\mathcal{D}_n} \left(\prod_{\vec{\ell}\in\Lambda\setminus \bigcup_{k=1}^{n} \operatorname{supp}\zeta_k} Z(\vec{h}_{\vec{\ell}}) \right) \prod_{k=1}^{n} \left(\prod_{\vec{\ell}\in\operatorname{supp}\zeta_k} \int d\sigma(\vec{s}_{\vec{\ell}}) \right)$$

$$\prod_{\langle\vec{m},\vec{n}\rangle\in\zeta_k} \left(\sum_{\nu=1}^{\infty} \frac{(\beta J)^{\nu}}{\nu!} (\vec{s}_{\vec{m}} \quad \vec{s}_{\vec{n}})^{\nu} \right) \prod_{\vec{\ell}\in\operatorname{supp}\zeta_k} \exp(\vec{h}_{\vec{\ell}} \quad \vec{s}_{\vec{\ell}}) \tag{3.7.13}$$

because the multi-index α assigns a non-zero weight to precisely those bonds lying in ζ_k. In this way, we have identified the polymer activity:

$$\widehat{Z}(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell}\in\Lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(\zeta_1,\ldots,\zeta_n)\in\mathcal{D}_n} \prod_{k=1}^{n} z(\zeta_k), \tag{3.7.14}$$

$$z(\zeta) = \left(\prod_{\vec{\ell}\in \text{ supp } \zeta} Z(\vec{h}_{\vec{\ell}})^{-1}\right)\left(\prod_{\vec{\ell}\in \text{ supp } \zeta}\int d\sigma(\vec{s}_{\vec{\ell}})\right)\prod_{\vec{\ell}\in \text{ supp } \zeta}\exp(\vec{h}_{\vec{\ell}}\cdot\vec{s}_{\vec{\ell}})$$
$$\times\prod_{\langle\vec{m},\vec{n}\rangle\in\zeta}(\exp(\beta J\,\vec{s}_{\vec{m}}\cdot\vec{s}_{\vec{n}})-1). \tag{3.7.15}$$

This is the reason why a power series in β is not the most natural high-temperature expansion in any case. The difference between unity and a nearest-neighbor exponential is a more natural high-temperature quantity in this context. Estimating the activity is simple enough in this case. By Jensen's inequality

$$Z(\vec{h}_{\vec{\ell}})\geq 4\pi, \tag{3.7.16}$$

and so $|\vec{s}_{\vec{\ell}}|\leq 1$ implies

$$\begin{aligned}|z(\zeta)| &\leq (4\pi)^{-|\zeta|}\prod_{\vec{\ell}\in \text{ supp } \zeta}e^{|\vec{h}_{\vec{\ell}}|}\left(\prod_{\vec{\ell}\in\text{supp }\zeta}\int d\sigma(\vec{s}_{\vec{\ell}})\right)\\ &\quad\prod_{\langle\vec{m},\vec{n}\rangle\in\zeta}|\exp(\beta J\,\vec{s}_{\vec{m}}\cdot\vec{s}_{\vec{n}})-1|\\ &\leq (4\pi)^{-|\zeta|}\prod_{\vec{\ell}\in \text{ supp } \zeta}e^{|\vec{h}_{\vec{\ell}}|}\left(\prod_{\vec{\ell}\in\text{supp }\zeta}\int d\sigma(\vec{s}_{\vec{\ell}})\right)\\ &\quad\prod_{\langle\vec{m},\vec{n}\rangle\in\zeta}|\beta J\,\vec{s}_{\vec{m}}\;\;\vec{s}_{\vec{n}}\exp(\beta J\,\vec{s}_{\vec{m}}\;\;\vec{s}_{\vec{n}})|\\ &\leq \left(\prod_{\vec{\ell}\in \text{ supp } \zeta}e^{|\vec{h}_{\vec{\ell}}|}\right)(\beta Je^{\beta J})^{\text{card }\zeta}\end{aligned} \tag{3.7.17}$$

For β sufficiently small, $\beta J+\ln(\beta J)$ is sufficiently negative for this bound to be the desired energy estimate, since card $\zeta\geq\frac{1}{2}|\zeta|$.

The entropy estimate is simple for the kind of polymer defined here. If M is the given volume (= cardinality of support) for all polymers whose supports contain a fixed lattice site $\vec{n}$, then such a polymer can have no more than $3M$ bonds. On the other hand, it follows from the solution of the Königsberg bridge problem that a connected set of bonds can be covered by a path of bonds such that no bond occurs more than twice. Therefore, the number of polymers is bounded by the number of paths originating at $\vec{n}$ of length no greater than $6M$. Since the number of all paths of r bonds originating at a fixed site is bounded by 6^r, we have the desired entropy bound – specifically,

$$\text{card}\{\text{polymers }\zeta\colon\ \vec{n}\in\text{ supp }\zeta,|\zeta|=M\}\leq 6^{6M}. \tag{3.7.18}$$

Given these input estimates, the abstract theory of polymers guarantees the Λ-uniform convergence of the associated expansion of (3.7.7) in connected n-polymers for sufficiently small β.

We now consider the high-temperature expansion of the lattice ϕ_3^4 model discussed in §3.5. For this model, the generalized partition function is

$$\begin{aligned} Z(h_{\vec{\ell}}\colon \vec{\ell} \in \Lambda) &= \int \exp\left(\sum_{\vec{\ell} \in \Lambda} h_{\vec{\ell}} \phi(\vec{\ell}) + \sum_{\langle \vec{m}, \vec{n} \rangle \subset \Lambda} \phi(\vec{m}) \phi(\vec{n}) \right) \\ &\quad \times \prod_{\vec{\ell} \in \Lambda} \left(\exp\left(-\lambda \phi(\vec{\ell})^4 - \left(\tau + \frac{1}{2} m_0^2 \right. \right. \right. \\ &\quad \left. \left. \left. + \frac{1}{2} \iota_\Lambda(\vec{\ell}) \right) \phi(\vec{\ell})^2 \right) d\phi(\vec{\ell}) \right), \end{aligned} \tag{3.7.19}$$

where $\iota_\Lambda(\vec{\ell})$ is the number of nearest neighbors of $\vec{\ell}$ in Λ. The inverse temperature β does not appear explicitly unless we make the change of variable (3.5.9), but the expansion corresponding to (3.7.13) should certainly have the required activity estimate without this scaling. We have

$$\begin{aligned} Z(h_{\vec{\ell}}\colon \vec{\ell} \in \Lambda) &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(\zeta_1, \ldots, \zeta_n) \in \mathcal{D}_n} \left(\prod_{\vec{\ell} \in \Lambda \setminus \bigcup_{k=1}^{n} \zeta_k} Z(h_{\vec{\ell}}) \right) \\ &\quad \times \prod_{k=1}^{n} \left(\int \prod_{\langle \vec{m}, \vec{n} \rangle \in \zeta_k} \left(\sum_{\nu=1}^{\infty} \frac{1}{\nu!} \phi(\vec{m})^\nu \phi(\vec{n})^\nu \right) \prod_{\vec{\ell} \in \operatorname{supp} \zeta_k} \exp(h_{\vec{\ell}} \phi(\vec{\ell})) \right. \\ &\quad \times \prod_{\vec{\ell} \in \operatorname{supp} \zeta_k} \left(\exp\left(-\lambda \phi(\vec{\ell})^4 - \left(\tau + \frac{1}{2} m_0^2 \right. \right. \right. \\ &\quad \left. \left. \left. \left. + \frac{1}{2} \iota_\Lambda(\vec{\ell}) \right) \phi(\vec{\ell})^2 \right) d\phi(\vec{\ell}) \right) \right), \end{aligned} \tag{3.7.20}$$

$$Z(h_{\vec{\ell}}) = \int_{-\infty}^{\infty} \exp\left(h_{\vec{\ell}} \omega - \lambda \omega^4 - \left(\tau + \frac{1}{2} m_0^2 + \frac{1}{2} \iota_\Lambda(\vec{\ell}) \right) \omega^2 \right) d\omega. \tag{3.7.21}$$

Since the variables are unbounded, it may appear that the expression

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu!} \phi(\vec{m})^\nu \phi(\vec{n})^\nu = \exp(\phi(\vec{m}) \phi(\vec{n})) - 1 \tag{3.7.22}$$

is a problem, but the single-spin distribution for each variable assigns a small probability to large values. Moreover, large values of the parameter λ provide small values of the hidden expansion parameter. Following the example of (3.7.13)–(3.7.15), we

identify our polymer activity:

$$z(\zeta) = \prod_{\vec{\ell}\in\mathrm{supp}\ \zeta} Z(h_{\vec{\ell}})^{-1} \int \prod_{\langle\vec{m},\vec{n}\rangle\in\zeta} (\exp(\phi(\vec{m})\phi(\vec{n})) - 1)$$
$$\times \prod_{\vec{\ell}\in\mathrm{supp}\ \zeta} \exp(h_{\vec{\ell}}\phi(\vec{\ell})) \prod_{\vec{\ell}\in\mathrm{supp}\ \zeta} \left(\exp\left(-\lambda\phi(\vec{\ell})^4 - \left(\tau + \frac{1}{2}m_0^2 + \frac{1}{2}\iota_\Lambda(\vec{\ell})\right)\phi(\vec{\ell})^2\right) d\phi(\vec{\ell})\right). \quad (3.7.23)$$

Since

$$|\exp(\phi(\vec{m})\phi(\vec{n})) - 1| \leq |\phi(\vec{m})\phi(\vec{n})| \exp|\phi(\vec{m})\phi(\vec{n})|$$
$$\leq |\phi(\vec{m})\phi(\vec{n})| \exp\left(\frac{1}{2}\phi(\vec{m})^2\right) \exp\left(\frac{1}{2}\phi(\vec{n})^2\right), \quad (3.7.24)$$

we have

$$\prod_{\langle\vec{m},\vec{n}\rangle\in\zeta} |\exp(\phi(\vec{m})\phi(\vec{n})) - 1| \leq \prod_{\vec{\ell}\in\mathrm{supp}\ \zeta} \left(|\phi(\vec{\ell})|^{\iota_\Lambda(\vec{\ell})} \exp\left(\frac{1}{2}\iota_\Lambda(\vec{\ell})\phi(\vec{\ell})^2\right)\right). \quad (3.7.25)$$

On the other hand, Jensen's inequality implies

$$Z(h_{\vec{\ell}}) \geq \int_{-\infty}^{\infty} \exp\Big(-\lambda\omega^4 - (\tau + \frac{1}{2}m_0^2 + \frac{1}{2}\iota_\Lambda(\vec{\ell}))\omega^2\Big) d\omega$$
$$\geq \int_{-\infty}^{\infty} \exp\Big(-\lambda\omega^4 - (\tau + \frac{1}{2}m_0^2 + 3)\omega^2\Big) d\omega \equiv \sigma_0(\lambda,\tau), \quad (3.7.26)$$

so the activity estimate is

$$|z(\zeta)| \leq \sigma_0(\lambda,\tau)^{-|\zeta|} \prod_{\vec{\ell}\in\mathrm{supp}\ \zeta} \int_{-\infty}^{\infty} |\omega|^{\iota_\Lambda(\vec{\ell})} \exp\left(h_{\vec{\ell}}\omega - \lambda\omega^4 - \tau\omega^2 - \frac{1}{2}m_0^2\omega^2\right) d\omega$$
$$= \sigma_0(\lambda,\tau)^{-|\zeta|} \prod_{\vec{\ell}\in\mathrm{supp}\ \zeta} \left(\lambda^{-\frac{1}{4}-\frac{1}{4}\iota_\Lambda(\vec{\ell})} \int_{-\infty}^{\infty} |\omega'|^{\iota_\Lambda(\vec{\ell})} \right.$$
$$\left. \times \exp\left(\lambda^{-1/4}h_{\vec{\ell}}\omega' - \omega'^4 - \lambda^{-1/2}\tau\omega'^2 - \frac{1}{2}m_0^2\lambda^{-1/2}\omega'^2\right) d\omega'\right). \quad (3.7.27)$$

For arbitrary constant $c > 0$, consider the regime

$$\tau \leq c\sqrt{\lambda}, \qquad \lambda \geq 1. \quad (3.7.28)$$

We have the reductions

$$\sigma_0(\lambda,\tau) = \lambda^{-1/4} \int_{-\infty}^{\infty} \exp\Big(-\omega'^4 - \lambda^{-1/2}(\frac{1}{2}m_0^2 + 3 + \tau)\omega'^2\Big) d\omega' \geq c\lambda^{-1/4}, \quad (3.7.29)$$

$$\begin{aligned} & \exp\left(\lambda^{-1/4} h_{\vec{\ell}} \omega' - \omega'^4 - \lambda^{-1/2} \tau \omega'^2 - \frac{1}{2} m_0^2 \lambda^{-1/2} \omega'^2\right) \\ \leq\ & \exp(\lambda^{-1/4} h_{\vec{\ell}} \omega' - \omega'^4) \\ \leq\ & \exp(|h_{\vec{\ell}}|\, |\omega'| - \omega'^4) \end{aligned} \tag{3.7.30}$$

so the activity estimate becomes

$$|z(\zeta)| \leq c^{|\zeta|} \prod_{\vec{\ell} \in \operatorname{supp} \zeta} \left(\lambda^{-\iota_\Lambda(\vec{\ell})/4} \int_{-\infty}^{\infty} |\omega'|^{\iota_\Lambda(\vec{\ell})} d\omega'\right) \tag{3.7.31}$$

for the parameter restrictions. Since $\iota_\Lambda(\vec{\ell}) \geq 1$ for $\vec{\ell} \in \operatorname{supp} \zeta$, we therefore have

$$|z(\zeta)| \leq c^{|\zeta|} \lambda^{-|\zeta|/4} \prod_{\vec{\ell} \in \operatorname{supp} \zeta} c(h_{\vec{\ell}}) \tag{3.7.32}$$

in this regime. *It is now clear that we have the desired activity bound for sufficiently large λ, relative to the arbitrary constant in $\tau \leq c\sqrt{\lambda}$.*

To see how this regime can be interpreted as a high-temperature regime, we recall that the scaling (3.5.9) of integration variables induces the parameter transformations

$$\tau = \beta^{-1} \hat{\tau} - \frac{1}{2} m_0^2 - 3, \tag{3.7.33}$$

$$\lambda = \beta^{-2} \hat{\lambda} \tag{3.7.34}$$

with $\hat{\tau}$ and $\hat{\lambda}$ fixed independently of β. Note that (3.7.28) is equivalent to

$$\hat{\tau} \leq c\sqrt{\hat{\lambda}} + 3\beta + \frac{3}{2} \beta m_0^2, \qquad \hat{\lambda} \geq \beta^2, \tag{3.7.35}$$

while large λ is equivalent to small β for fixed $\hat{\lambda}$. The size of $\hat{\tau}$ does not affect β if we choose $c = \hat{\tau}/\sqrt{\hat{\lambda}}$.

References

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2. D. Ruelle, *Statistical Mechanics*, Benjamin, Reading, 1969.

3.8 Inductive Interpolation

Our next example of a polymer expansion is another analysis of the lattice ϕ_3^4 model with mass m_0 and coupling constant λ that is similar in spirit to the expansion presented in the last section in the sense that it attempts to decouple the quadratic form

$(-\Delta^{lat}\phi, \phi)$. Recall that the partition function is given by the integral

$$Z_\Lambda = \int \exp\left(-\frac{1}{2} \sum_{\langle \vec{m}, \vec{n} \rangle \subset \Lambda} (\phi(\vec{m}) - \phi(\vec{n}))^2 \right.$$
$$\left. -\frac{1}{2} m_0^2 \sum_{\vec{n} \in \Lambda} \phi(\vec{n})^2 - I^{lat}(\chi_\Lambda \phi) \right) \prod_{\vec{\ell} \in \Lambda} d\phi(\vec{\ell}) \tag{3.8.1}$$

$$I^{lat}(\phi) = \sum_{\vec{n}} V(\phi(\vec{n})) = \tau \sum_{\vec{n}} \phi(\vec{n})^2 + \lambda \sum_{\vec{n}} \phi(\vec{n})^4, \tag{3.8.2}$$

for every finite $\Lambda \subset \mathbb{Z}^3$. The generalized partition function is given by

$$Z(h_{\vec{\ell}} : \vec{\ell} \in \Lambda) = \int \exp\left(\sum_{\vec{\ell} \in \Lambda} h_{\vec{\ell}} \phi(\vec{\ell}) - \frac{1}{2} \sum_{\langle \vec{m}, \vec{n} \rangle \subset \Lambda} (\phi(\vec{m}) - \phi(\vec{n}))^2 \right.$$
$$\left. -\frac{1}{2} m_0^2 \sum_{\vec{n} \in \Lambda} \phi(\vec{n})^2 - I^{lat}(\chi_\Lambda \phi) \right) \prod_{\vec{\ell} \in \Lambda} d\phi(\vec{\ell}). \tag{3.8.3}$$

The polymer expansion we develop in this section is based on *inductive interpolation*, which shall be the basis of the phase cell cluster expansion introduced in Chap. 5. In the statistical-mechanical context of this chapter, however, it is just a refinement. Indeed, for this particular model, it is superfluous. Its introduction at this level is useful because of its sophistication relative to the wholesale expansion discussed in the last section.

Accordingly, we introduce a linear ordering of Λ and base an inductive expansion on it - an expansion where each term reflects a history of interpolations. The first step is an attempt to decouple $\phi(\vec{n}^{\,1})$ from the other variables, where $\vec{n}^{\,1}$ is the first element of Λ. By the fundamental theorem of calculus

$$\exp\left(\sum_{\langle \vec{n}, \vec{n}^1 \rangle \subset \Lambda} \phi(\vec{n}) \phi(\vec{n}^{\,1}) \right) = 1 + \sum_{\langle \vec{m}, \vec{n}^1 \rangle \subset \Lambda} \phi(\vec{m}) \phi(\vec{n}^{\,1}) \int_0^1 dt_1$$
$$\exp\left(t_1 \sum_{\langle \vec{n}, \vec{n}^1 \rangle \subset \Lambda} \phi(\vec{n}) \phi(\vec{n}_{\,1}) \right), \tag{3.8.4}$$

so this first interpolation yields

$$Z(h_{\vec{\ell}} : \vec{\ell} \in \Lambda)$$
$$= Z(h_{\vec{\ell}} : \vec{\ell} \in \Lambda \backslash \{\vec{n}^{\,1}\}) \int \exp(h_{\vec{n}^1} \phi(\vec{n}^{\,1}) - \frac{1}{2} \iota_\Lambda(\vec{n}^{\,1}) \phi(\vec{n}^{\,1})^2 - \frac{1}{2} m_0^2 \phi(\vec{n}^{\,1})^2$$
$$-V(\phi(\vec{n}^{\,1}))) d\phi(\vec{n}^{\,1}) + \sum_{\langle \vec{m}, \vec{n}^1 \rangle \subset \Lambda} \int \phi(\vec{m}) \phi(\vec{n}^{\,1}) \int_0^1 ds_1 \exp\left(\sum_{\vec{\ell} \in \Lambda} h_{\vec{\ell}} \phi(\vec{\ell}) \right.$$

$$-\frac{1}{2}m_0^2\sum_{\vec{\ell}\in\Lambda}\phi(\vec{\ell})^2 - I^{lat}(\chi_\Lambda\phi) + t_1\sum_{\langle\vec{n}^{\,1},\vec{n}\rangle\subset\Lambda}\phi(\vec{n}^{\,1})\phi(\vec{n}) - \frac{1}{2}\sum_{\langle\vec{m},\vec{n}\rangle\subset\Lambda\setminus\{\vec{n}^{\,1}\}}$$

$$(\phi(\vec{m})-\phi(\vec{n}))^2 - \frac{1}{2}\sum_{\langle\vec{n},\vec{n}^{\,1}\rangle\subset\Lambda}(\phi(\vec{n}^{\,1})^2+\phi(\vec{n})^2)\Bigg)\prod_{\vec{\ell}\in\Lambda}d\phi(\vec{\ell}). \qquad (3.8.5)$$

This leading term is a *decoupled term*, while the other terms are *remainder terms*. The second step in the inductive expansion is applied to each term, but the interpolation now varies with the term. For the decoupled term we apply the same formula to $Z(h_{\vec{\ell}}\colon\ \vec{\ell}\in\Lambda\setminus\{\vec{n}^{\,1}\})$, except $\phi(\vec{n}^{\,2})$ is now the variable we try to decouple from the other variables, where $\vec{n}^{\,2}$ is the successor of $\vec{n}^{\,1}$ in Λ. For a given remainder term labeled by $\langle\vec{n},\vec{n}^{\,1}\rangle$, we decompose the $d\phi$-integral with the interpolation that attempts to decouple both $\phi(\vec{n})$ and $\phi(\vec{n}^{\,1})$ from the other variables.

In this way, a sum over possible sequences of interpolations is induced by an algorithm that dictates the next step for a given term based on the sequence of past interpolations that developed it. Consider an arbitrary term that has developed at an arbitrary stage in the expansion. Since each previous step was an interpolation followed by a choice of new term, a product $\phi(\vec{m})\phi(\vec{n})$ associated with a nearest-neighbor bond appeared in an integral with each expansion step selecting a remainder term, so sets of variables have also developed. If the term under consideration happens to be a decoupled term, it has the form

$$Z\left(h_{\vec{\ell}}\colon\ \vec{\ell}\in\Lambda\setminus\bigcup_\gamma\Lambda_\gamma\right)\prod_\gamma K_\gamma(h_{\vec{\ell}}\colon\ \vec{\ell}\in\Lambda_\gamma),$$

where K_γ is some complicated $\prod_{\vec{\ell}\in\Lambda_\gamma}d\phi(\vec{\ell})$-integral developed by the γth segment of successive expansion steps. The sets Λ_γ of lattice sites are mutually disjoint, and each Λ_γ labels the variables differentiated down by the ιth succession of expansion steps. Given this decoupled term, the next step is to decompose $Z\left(h_{\vec{\ell}}\colon\ \vec{\ell}\in\Lambda\setminus\bigcup_\gamma\Lambda_\gamma\right)$ exactly as $Z(h_{\vec{\ell}}\colon\ \vec{\ell}\in\Lambda)$ was, except that if $\vec{n}\,'$ is the first lattice site in $\Lambda\setminus\bigcup_\gamma\Lambda_\gamma$ with respect to the linear ordering on Λ, then $\phi(\vec{n}\,')$ is now the random variable one tries to decouple from the others.

Now suppose the term under consideration is a remainder term instead. Let $\{t_i\}$ denote the set of parameters for past interpolations. The term has the form

$$\mathcal{K} = \prod_\gamma K_\gamma(h_{\vec{\ell}}\colon\ \vec{\ell}\in\Lambda_\gamma)\int\prod_{\vec{n}\in\Lambda\setminus\bigcup_\gamma\Lambda_\gamma}\phi(\vec{n})^{N_{\vec{n}}}\left(\prod_i\int_0^1 dt_i\right)f(t)$$

$$\times \quad \exp\left(\sum_{\vec{\ell} \in \Lambda \setminus \bigcup_\gamma \Lambda_\gamma} h_{\vec{\ell}} \phi(\vec{\ell}) - I^{lat}\left(\chi_{\Lambda \setminus \bigcup_\gamma \Lambda_\gamma} \phi \right) + W_{\Lambda \setminus \bigcup_\gamma \Lambda_\gamma}(\phi; t) \right.$$

$$\left. - \frac{1}{2} \sum_{\vec{\ell} \in \Lambda \setminus \bigcup_\gamma \Lambda_\gamma} (m_0^2 + \iota_\Lambda(\vec{\ell})) \phi(\vec{\ell})^2 \right) \prod_{\vec{\ell} \in \Lambda \setminus \bigcup_\gamma \Lambda_\gamma} d\phi(\vec{\ell}), \tag{3.8.6}$$

where $W_{\Lambda \setminus \bigcup_\gamma \Lambda_\gamma}(\phi; t)$ denotes the interpolation of

$$W_{\Lambda \setminus \bigcup_\gamma \Lambda_\gamma}(\phi) = \sum_{\langle \vec{m}, \vec{n} \rangle \subset \Lambda \setminus \bigcup_\gamma \Lambda_\gamma} \phi(\vec{m}) \phi(\vec{n}) \tag{3.8.7}$$

that has developed and $f(t)$ is the product of powers of the t_i having grown as old t-dependencies have been differentiated down by new interpolations. In this case, the next step is to base the interpolation on the attempt to decouple the variables in Λ' from the rest of the variables in $\Lambda \setminus \bigcup_\gamma \Lambda_\gamma$, where Λ' includes those $\phi(\vec{n})$ that have been differentiated down – i.e.,

$$\Lambda' = \left\{ \vec{n} \in \Lambda \setminus \bigcup_\gamma \Lambda_\gamma \colon N_{\vec{n}} \neq 0 \right\}. \tag{3.8.8}$$

The interpolation formula yields the form

$$\begin{aligned} \mathcal{K} \quad = \quad & Z\left(h_{\vec{\ell}} \colon \vec{\ell} \in \Lambda \setminus \left(\Lambda' \cup \bigcup_\gamma \Lambda_\gamma \right) \right) K'(h_{\vec{\ell}} \colon \vec{\ell} \in \Lambda') \prod_\gamma K_\gamma(h_{\vec{\ell}} \colon \vec{\ell} \in \Lambda_\gamma) \\ & + \prod_\gamma K_\gamma(h_{\vec{\ell}} \colon \vec{\ell} \in \Lambda_\gamma) \int_0^1 dt' \sum_{\substack{\langle \vec{m}', \vec{m} \rangle \subset \Lambda \setminus \bigcup_\gamma \Lambda_\gamma \\ \vec{m}' \in \Lambda', \vec{m} \notin \Lambda'}} \int \phi(\vec{m}\,') \phi(\vec{m}) \\ & \times \prod_{\vec{n} \in \Lambda'} \phi(\vec{n})^{N_{\vec{n}}} \int_0^1 dt' \left(\prod_i \int_0^1 dt_i \right) f(t) f_{\vec{m}', \vec{m}}(t) \\ & \times \exp\left[\sum_{\vec{\ell} \in \Lambda \setminus \bigcup_\gamma \Lambda_\gamma} h_{\vec{\ell}} \phi(\vec{\ell}) - I^{lat}\left(\chi_{\Lambda \setminus \bigcup_\gamma \Lambda_\gamma} \phi \right) + t' W_{\Lambda \setminus \bigcup_\gamma \Lambda_\gamma}(\phi, t) \right. \\ & \left. + (1 - t')(W_{\Lambda'}(\phi; t) + W_{\Lambda \setminus (\Lambda' \cup \bigcup_\gamma \Lambda_\gamma)}(\phi; t) - \frac{1}{2} \sum_{\vec{\ell} \in \Lambda \setminus \bigcup_\gamma \Lambda_\gamma} (m_0^2 + \iota_\Lambda(\vec{\ell})) \phi(\vec{\ell})^2 \right] \end{aligned}$$

$$\times \prod_{\vec{\ell} \in \Lambda \setminus \bigcup_\gamma \Lambda_\gamma} d\phi(\vec{\ell}). \tag{3.8.9}$$

$f_{\vec{m}',\vec{m}}(t)$ is just the product of the old interpolation parameters t_i associated with decoupling those complementary pairs of sets that separated $\vec{m}\,'$ from $\vec{m}$. This means

$$W_{\Lambda \setminus \bigcup_\gamma \Lambda_\gamma}(\phi; t) = \sum_{\langle \vec{m}', \vec{m} \rangle \subset \Lambda \setminus \bigcup_\gamma \Lambda_\gamma} f_{\vec{m}'\vec{m}}(t) \phi(\vec{m}\,') \phi(\vec{m}) \tag{3.8.10}$$

$W_{\Lambda'}(\phi; t)$ denotes the restriction of the sum to those nearest-neighbor pairs both of whose lattice sites lie in Λ', and

$$\begin{aligned} K'(h_{\vec{\ell}}\colon \vec{\ell} \in \Lambda') \;=\; & \int \prod_{\vec{n} \in \Lambda'} \phi(\vec{n})^{N_{\vec{n}}} \left(\prod_i \int_0^1 dt_i \right) f(t) \\ & \times \exp \left(\sum_{\vec{\ell} \in \Lambda'} h_{\vec{\ell}} \phi(\vec{\ell}) - I^{lat}(\chi_{\Lambda'} \phi) + W_{\Lambda'}(\phi; t) \right. \\ & \left. - \frac{1}{2} \sum_{\vec{\ell} \in \Lambda'} (m_0^2 + \iota_\Lambda(\vec{\ell})) \phi(\vec{\ell})^2 \right) \prod_{\vec{\ell} \in \Lambda'} d\phi(\vec{\ell}). \end{aligned} \tag{3.8.11}$$

The next step in the development of a history is to pick either the decoupled term or a remainder term in (3.8.9). This completes our inductive definition of the expansion. Since Λ is a finite set, it is clear that every possible history of steps eventually terminates with a decoupled term of the form

$$K = \prod_\gamma K_\gamma(h_{\vec{\ell}}\colon \vec{\ell} \in \Lambda_\gamma), \qquad \bigcup_\gamma \Lambda_\gamma = \Lambda. \tag{3.8.12}$$

Such a decoupled term is a *completed term*. The Λ_γ are disjoint, and the factorization over this partition of Λ is due to the history of choices. In the γth segment of successive expansion steps, every step except the last one chooses a remainder term. The last step chooses the decoupled term, and so one starts all over again with the residual set of variables, initiating the next sequence of expansion steps. We have now expanded the original partition function in these completed terms, and our task is to control the expansion by organizing the terms and writing the K_γ more explicitly for estimation.

Let x_0 be a distinguished element of an N-element set Γ. An *ordered connectivity graph* on (Γ, x_0) is a mapping $\mathfrak{G}$ from $\{2, \ldots, N\}$ into $\Gamma \times \Gamma$ such that with $\mathfrak{G}(i) = (g_0(i), g_1(i))$,

$$g_1(i) \neq g_1(j), \quad i \neq j, \tag{3.8.13}$$

$$g_0(i) \in \{x_0\} \cup \{g_1(j)\colon\, j < i\}. \tag{3.8.14}$$

The point here is that the γth sequence of expansion steps is naturally described by an ordered connectivity graph $\mathfrak{G}^\gamma$ on $(\Lambda_\gamma, \vec{n}^{\,\gamma})$ where $g_0(i)$ and $g_1(i)$ are nearest-neighbors and $\vec{n}^{\,\gamma}$ is the lattice site that this succession of steps initially tries to decouple from other lattice sites. Every time a remainder term is chosen for an interpolation step, an additional product $\phi(g_0^\gamma(i))\phi(g_1^\gamma(i))$ appears in the $d\phi$-integral, where $\phi(g_0^\gamma(i))$ is a variable that has already been "differentiated down" from the exponent and is among the variables the interpolation is attempting to decouple from the variables that appear only in the exponent thus far – among them, the variable $\phi(g_1^\gamma(i))$, which is now "differentiated down" for the first time. This variable cannot occur as a $\phi(g_1^\gamma(i))$ for any other i – only as a $\phi(g_0^\gamma(i))$ for later i. This γth history of expansion steps has the tree structure described by (3.8.13) and (3.8.14) above. The K_γ-factor corresponding to this history in the completed term has the form

$$K_\gamma(h_{\vec{\ell}}\colon\ \vec{\ell} \in \Lambda_\gamma) = \left(\prod_i \int_0^1 dt_i\right) f_{\mathfrak{G}^\gamma}(t) \int \prod_i (\phi(g_0^\gamma(i))\phi(g_1^\gamma(i)))$$

$$\times \exp\left(\sum_{\vec{\ell}\in\Lambda_\gamma} h_{\vec{\ell}}\phi(\vec{\ell}) - I^{lat}(\chi_{\Lambda_\gamma}\phi) + W_{\Lambda_\gamma}(\phi;t)\right.$$

$$\left. - \frac{1}{2}\sum_{\vec{\ell}\in\Lambda_\gamma} (m_0^2 + \iota_\Lambda(\vec{\ell}))\phi(\vec{\ell})^2\right) \prod_{\vec{\ell}\in\Lambda_\gamma} d\phi(\vec{\gamma}), \tag{3.8.15}$$

where the monomial $f_{\mathfrak{G}^\gamma}(t)$ is yet to be determined.

It is important to note that a factor K_γ *can* be an integral in just *one* variable. After all, the initial interpolation in a new sequence of expansion steps attempts to decouple the first of the remaining lattice variables from all the other variables. If that first step chooses the decoupled term, then the sequence is terminated already, and the corresponding o.c.g. (ordered connectivity graph) is *empty*. In this case, $N = 1$ and

$$K_\gamma = Z(h_{\vec{n}_1^\gamma}) = \int \exp(h_{\vec{n}_1^\gamma}\phi(\vec{n}^{\,\gamma}) - V(\phi(\vec{n}^{\,\gamma}))$$

$$-\frac{1}{2}(m_0^2 + \iota_\Lambda(\vec{n}^{\,\gamma}))\phi(\vec{n}^{\,\gamma})^2)d\phi(\vec{n}^{\,\gamma}) \tag{3.8.16}$$

occurs with no interpolation parameters at all. The point is that the polymers we define below do not index such factors, and indeed, the single term in the whole expansion consisting of only those factors – i.e., the term developed by the series of decouplings that never "differentiates down" anything – must correspond to unity in the polymer expansion of an abstract partition function. As in the previous section, the abstract partition function in this case is actually the normalization

$$\widehat{Z}_\Lambda = Z(h_{\vec{\ell}}\colon\ \vec{\ell} \in \Lambda) \prod_{\vec{\ell}\in\Lambda} Z(h_{\vec{\ell}})^{-1} \tag{3.8.17}$$

of the generalized partition function.

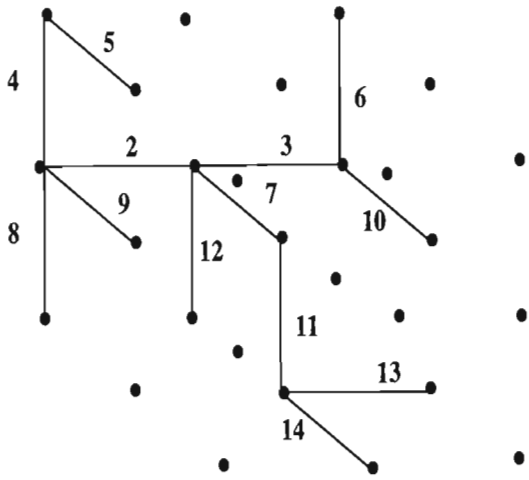

Figure 3.8.1:

Now from the standpoint of realizing $f_{\mathfrak{G}^\gamma}(t)$ explicitly, it is clear from the inductive step described above that (if we drop the superscript γ)

$$f_{\mathfrak{G}}(t) = \prod_i f_{g_0(i),g_1(i)}(t) \tag{3.8.18}$$

so it is a matter of examining the $f_{\vec{m}',\vec{m}}(t)$ introduced there. By way of illustration, suppose the ordered connectivity graph $\mathfrak{G}$ is given by the tree-like development of nearest-neighbor bonds on the cubic lattice $\mathbb{Z}^3$ given by Fig. 3.8.1. The i-labeled line segment is associated with the ordered pair $(g_0(i), g_1(i))$ of lattice sites as well as with the interpolation parameter t_{i-1}. Moreover, the $(i-1)$st interpolation attempts to decouple $\{\phi(g_0(2)), \phi(g_0(3)), \ldots, \phi(g_0(i))\}$ from the other variables in Λ_t. Thus, the product of parameters "differentiated down" for the nearest-neighbor pair $(g_0(i), g_1(i)) = (g_1(i'), g_1(i))$ is given by

$$f_{i'i}(t) \equiv f_{g_1(i')g_1(i)}(t) = \prod_{i'-1<j<i-1} t_j \tag{3.8.19}$$

because the $(i'-1)$st interpolation "differentiates down" $\phi(g_1(i'))$ for the first time, while the subsequent interpolation steps attempt – in particular – to decouple $\phi(g_1(i'))$ from $\phi(g_1(i))$ without "differentiating down" the product $\phi(g_1(i))\phi(g_1(i'))$ until the $(i-1)st$ interpolation. In our concrete example,

$$\begin{aligned} f_{\mathfrak{G}}(t) &= f_{12}(t)f_{13}(t)f_{17}(t)f_{1,12}(t)f_{24}(t)f_{28}(t) \\ &\quad \times f_{29}(t)f_{36}(t)f_{3,10}(t)f_{45}(t)f_{7,11}(t)f_{11,13}(t)f_{11,14}(t) \\ &= 1 \times t_1 \times (t_1t_2t_3t_4t_5) \times (t_1t_2t_3t_4t_5t_6t_7t_8t_9t_{10}) \times t_2 \\ &\quad \times (t_2t_3t_4t_5t_6) \times (t_2t_3t_4t_5t_6t_7) \times (t_3t_4) \times (t_3t_4t_5t_6t_7t_8) \end{aligned}$$

$$\begin{aligned} & \times 1 \times (t_7 t_8 t_9) \times t_{11} \times (t_{11} t_{12}) \\ = \; & t_1{}^3\, t_2{}^5\, t_3{}^6\, t_4{}^6\, t_5{}^5\, t_6{}^4\, t_7{}^4\, t_8{}^3\, t_9{}^2 t_{10}\, t_{11}{}^2\, t_{12}. \end{aligned} \tag{3.8.20}$$

In general,

$$g_1(i') = g_0(i) \Rightarrow i' = \eta_{\mathfrak{G}}(i), \tag{3.8.21}$$

where $\eta_{\mathfrak{G}}$ is defined by

$$\eta_{\mathfrak{G}}(i) = g_1^{-1}(g_0(i)) \tag{3.8.22}$$

with the convention that $g_1^{-1}(g_0(2)) = 1$. We combine (3.8.18) with (3.8.19) to obtain the formula

$$f_{\mathfrak{G}}(t) = \prod_i (t_{\eta_{\mathfrak{G}}(i)} \cdots t_{i-3} t_{i-2}), \tag{3.8.23}$$

and this completes our description of an interpolated factor in a completed term.

The product of lattice field amplitudes that has developed in the $d\phi$-integral associated with a given ordered connectivity graph $\mathfrak{G}$ is just

$$\prod_i (\phi(g_0(i))\phi(g_1(i))).$$

Since every $(g_0(i), g_1(i))$ must appear as a nearest-neighbor pair for this model, and since a lattice site can occur no more than once as some $g_1(i)$, notice that a given lattice site $\vec{n}$ can occur no more than seven times as some $g_0(i)$. With $0 \le t_i \le 1$, we have the trivial estimation

$$0 \le f_{\vec{m}',\vec{m}}(t) \le 1 \tag{3.8.24}$$

in the formula (3.8.10), so

$$\begin{aligned} |W_{\Lambda'}(\phi; t)| &\le \sum_{\langle \vec{m}', \vec{m} \rangle \subset \Lambda'} |\phi(\vec{m}')\phi(\vec{m})| \\ &\le \frac{1}{2} \sum_{\langle \vec{m}', \vec{m} \rangle \subset \Lambda'} (\phi(\vec{m}')^2 + \phi(\vec{m})^2) \\ &= \frac{1}{2} \sum_{\vec{\ell} \in \Lambda'} \iota_{\Lambda'}(\vec{\ell}) \phi(\vec{\ell})^2 \end{aligned} \tag{3.8.25}$$

Since $\iota_{\Lambda'}(\vec{\ell}) \le \iota_{\Lambda}(\vec{\ell})$, it follows that

$$\begin{aligned} & \left| \int \prod_i (\phi(g_0(i))\phi(g_1(i))) \exp\left(\sum_{\vec{\ell} \in \Lambda'} h_{\vec{\ell}} \phi(\vec{\ell}) - I^{lat}(\chi_{\Lambda'}\phi) \right.\right. \\ & \left.\left. + W_{\Lambda'}(\phi; t) - \frac{1}{2} \sum_{\vec{\ell}\,' \in \Lambda'} (m_0^2 + \iota_{\Lambda}(\vec{\ell}))\phi(\vec{\ell}\,')^2 \right) \prod_{\vec{\ell} \in \Lambda'} d\phi(\vec{\ell}) \right| \\ \le \; & \prod_{\vec{\ell} \in \Lambda'} \int_{-\infty}^{\infty} |\omega|^{\langle \text{card } g_1^{-1}\langle \vec{\ell} \rangle \rangle + 1} \exp\left(h_{\vec{\ell}}\omega - V(\omega) - \frac{1}{2} m_0^2 \omega^2 \right) d\omega. \end{aligned} \tag{3.8.26}$$

The single-spin estimation is now done in the same manner as in the previous section. The combinatorial obstacle in the expansion is to sum over the ordered connectivity graphs.

Once again, consider an N-element set Γ with a distinguished element x_0. An *unordered connectivity graph* on (Γ, x_0) is simply the range of an ordered connectivity graph on (Γ, x_0). In the combinatorics literature, the synonym for this term is "tree" – not to be confused with "tree graph," already defined. An ordered connectivity graph may be regarded as a history of tree growth. Now to apply the abstract theory of polymer expansions, we specify our *polymers* to be unordered connectivity graphs on sets of lattice sites, where only nearest-neighbor bonds belong to it. The support of a polymer is just the set of lattice sites involved, so the volume is the number of these lattice sites. The entropy estimate is easy in this case. Indeed, if M is the given volume for all polymers whose supports contain a fixed lattice site $\vec{n}^{\,0}$, then such a polymer consists of $M-1$ bonds. The number of these polymers is therefore bounded by the number of paths (consisting of nearest-neighbor bonds) of length $\leq 2(M-1)$ originating at $\vec{n}^{\,0}$. It is obvious that a path can cover a tree of bonds without using any bond more than twice. On the other hand, the number of all paths of r bonds originating at $\vec{n}^{\,0}$ is bounded by 8^r, so the bound (3.6.20) is established in this case.

How do we realize $\widehat{Z}_\Lambda$ as an expansion in activities of these polymers? The expansion we have developed is actually a sum over all finite sequences $(\mathfrak{G}^1, \ldots, \mathfrak{G}^n)$, $n = 1, 2, 3, \ldots$, of *ordered* connectivity graphs on disjoint pointed sets $(\Lambda_\gamma, \vec{n}^{\,\gamma})$ such that:

(a) $\vec{n}^{\,\gamma+1}$ succeeds $\vec{n}^{\,\gamma}$ with respect to the linear ordering of Λ,

(b) $\vec{n}^{\,\gamma}$ is the first element in Λ_γ with respect to that ordering.

With each such sequence is associated a completed term $\prod_{\gamma=1}^{n} K_\gamma(h_{\vec{\ell}}\colon\ \vec{\ell} \in \Lambda_\gamma)$, where each K_γ-factor is entirely determined by the corresponding history of interpolation steps described by $\mathfrak{G}^\gamma$. In particular, there is a mapping

$$\mathfrak{G} \mapsto K_{\mathfrak{G}}(h_{\vec{\ell}}\colon\ \vec{\ell} \in \Lambda_{\mathfrak{G}})$$

from ordered connectivity graphs to the interpolated factors that occur in completed terms. Now for an *unordered* connectivity graph T consisting of nearest-neighbor bonds on the pointed set $(\Lambda', \vec{n}\,')$ with $\vec{n}\,'$ the first site occurring in Λ', specify the *activity* of this polymer T to be

$$z(T) = \sum_{\substack{\text{ordered connectivity} \\ \text{graphs } \mathfrak{G} \text{ on } (\Lambda, \vec{n}\,') \\ T_{\mathfrak{G}} = T}} K_{\mathfrak{G}}(h_{\vec{\ell}}\colon\ \vec{\ell} \in \Lambda') \prod_{\vec{\ell} \in \Lambda'} Z(h_{\vec{\ell}})^{-1}, \tag{3.8.27}$$

where $T_{\mathfrak{G}}$ is the range of $\mathfrak{G}$. Thus, we have the expansion

$$\widehat{Z}_\Lambda = \sum_{n=0}^{\infty} \sum_{(T_1, \ldots, T_n) \in \mathcal{D}'_n} \prod_{i=1}^{n} z(T_i), \tag{3.8.28}$$

where $\mathcal{D}'_n$ denotes the set of all n-tuples $(T_1, \ldots, T_n)$ of mutually disjoint polymers such that the first element in supp T_{i+1} succeeds the first element in supp T_i. If $\mathcal{D}_n$ denotes the set of disjoint n-tuples of polymers *without* the order condition, then

$$\widehat{Z}_\Lambda = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(T_1,\ldots,T_n)\in\mathcal{D}_n} \prod_{i=1}^{n} z(T_i), \tag{3.8.29}$$

so control of the induced expansion of

$$|\Lambda|^{-1} \ln \widehat{Z}_\Lambda = |\Lambda|^{-1} (\ln Z(h_{\vec{\ell}}\colon\ \vec{\ell} \in \Lambda) - \sum_{\vec{\ell}\in\Lambda} \ln Z(h_{\vec{\ell}})) \tag{3.8.30}$$

in connected n-tuples of polymers is guaranteed by the general theory in the range of parameters for which we have an activity estimate

$$|z(T)| \le C_0 e^{-c_0|T|} \tag{3.8.31}$$

with c_0 sufficiently large. We infer the desired bound from (3.7.26), (3.7.29), (3.8.26), and a fundamental combinatorial result. For every unordered connectivity graph T on a pointed set $(\Lambda', \vec{n}\,')$,

$$\sum_{\substack{\text{ordered connectivity} \\ \text{graphs } \mathfrak{G} \text{ on } (\Lambda', \vec{n}')}} f_{\mathfrak{G}}(s) = 1. \tag{3.8.32}$$

We prove this identity in §3.11.

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3.9 A Generalization of the Polymer Estimation

We now recall the infinite-volume problem brought up in §3.1 (for high temperature). Returning to the lattice system whose single-spin distribution is Lebesgue measure $d\sigma$ on $\mathbb{S}^2$, we consider the long-range Hamiltonian in a zero external magnetic field given by

$$H_\Lambda = \sum_{\vec{n},\vec{n}'\in\Lambda} J_{\vec{n}-\vec{n}'}\, \vec{s}_{\vec{n}} \cdot \vec{s}_{\vec{n}'}, \tag{3.9.1}$$

where in this section we assume only the decay condition

$$\sum_{\vec{n}} |J_{\vec{n}}| < \infty, \tag{3.9.2}$$

which is just summability. This spin model is treated differently from the classical Heisenberg model, as the theory of polymer expansions presented in §3.6 must be generalized in this context, and the Königsberg bridge-island estimation is too crude as well. The algebraic structure of the expansion is the same, however. One expands

$$\widehat{Z}(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell}\in\Lambda) = \frac{Z(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell}\in\Lambda)}{\prod\limits_{\vec{\ell}\in\Lambda} Z(\vec{h}_{\vec{\ell}})}, \tag{3.9.3}$$

$$Z(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell}\in\Lambda) = \int \exp\left(\sum_{\vec{\ell}\in\Lambda} \vec{h}_{\vec{\ell}} \cdot \vec{s}_{\vec{\ell}} - \beta \sum_{\vec{n},\vec{n}'\in\Lambda} J_{\vec{n}-\vec{n}'}\, \vec{s}_{\vec{n}}\ \ \vec{s}_{\vec{n}'}\right) \times \prod_{\vec{\ell}\in\Lambda} d\sigma(\vec{s}_{\vec{\ell}}) \tag{3.9.4}$$

in all couplings. We have already learned from the nearest-neighbor case that a straight power series in β may converge for small β, but from the point of view of polymers it is more natural to recombine terms into an expansion of the form (3.7.15). Accordingly,

$$\exp\left(-\beta \sum_{\vec{n},\vec{n}'\in\Lambda} J_{\vec{n}-\vec{n}'}\, \vec{s}_{\vec{n}} \cdot \vec{s}_{\vec{n}'}\right) = \sum_{\Gamma\subset\Lambda\times\Lambda} \prod_{(\vec{n},\vec{n}')\in\Gamma} [\exp(-\beta J_{\vec{n}-\vec{n}'}\, \vec{s}_{\vec{n}} \cdot \vec{s}_{\vec{n}'}) - 1] \tag{3.9.5}$$

is the identity we use in this case. Clearly, our polymers will be subsets of $\Lambda \times \Lambda$ that connect the lattice sites in their own supports, but we already have a problem with the entropy estimate. If we define the volume of a polymer to be the cardinality of the set of lattice sites supporting it, then – since arbitrarily distant lattice sites in Λ are directly coupled by a long-range interaction – the bound on the number of polymers with fixed volume and having a fixed lattice site in their supports depends on Λ and diverges as $\Lambda \nearrow \mathbb{Z}^3$.

An alternative that might seem tempting is to define the volume of a polymer to be the sum of ℓ^1-lengths of the line segments associated with the ordered pairs of lattice

sites. The ℓ^1-length of a line segment $\{\vec{m}, \vec{n}\}$ is defined to be $\sum_\iota |m_\iota - n_\iota|$, where the motivation here is to give integer values to the polymer volumes. With this understanding, we fix an integer $L > 0$ and a lattice site $\vec{n} \in \mathbb{Z}^3$ and try to count all polymers with volume L and $\vec{n}$ in their supports – i.e., all connected sets of line segments having these constraints. By the Königsberg bridge-island solution, every connected set of line segments has a path consisting of precisely those line segments, originating at any chosen point, and traversing each line segment no more than twice. Therefore, it is enough to bound the number of all possible paths of line segments originating at $\vec{n}$ and having total length $\hat{L}$ with $L \le \hat{L} \le 2L$. Now classify such paths according to the number N of line segments they consist of and the sequence $(L_1, \dots, L_N)$ of lengths according to the sequence of line segments. With N and $(L_1, \dots, L_N)$ fixed, it is clear that the number of such paths originating at $\vec{n}$ is bounded by

$$c_0^N \prod_{k=1}^N L_k^2 \le c_0^N \left(\frac{1}{N} \sum_{k=1}^N L_k \right)^{2n} \tag{3.9.6}$$

On the other hand, we know that the number of sequences $(L_1, \dots, L_N)$ of *positive integers* with $\sum_{k=1}^N L_k = \hat{L}$ is bounded by $e^{\hat{L}-1}$ Summing over all such sequences and over all N, we bound the number of paths originating at $\vec{n}$ with length $\hat{L}$ by the sum

$$\begin{aligned} \sum_{N=1}^\infty e^{\hat{L}-1} c_0^N \left(\frac{\hat{L}}{N} \right)^{2N} &\le e^{\hat{L}-1} \left(\sum_{N=1}^\infty c_0^{N/2} \left(\frac{\hat{L}}{N} \right)^N \right)^2 \\ &\le e^{\hat{L}-1} \left(\sum_{N=1}^\infty \frac{(\hat{L}\sqrt{c_0})^N}{N!} \right)^2 \\ &\le \exp(2\hat{L}\sqrt{c_0} + \hat{L} - 1). \end{aligned} \tag{3.9.7}$$

In this way, we obtain the desired entropy estimate. Can we balance this with an energy estimate in terms of the polymer volume defined here? Clearly, the activity of a polymer ζ is given by

$$\begin{aligned} z(\zeta) &= \frac{1}{\prod_{\vec{\ell} \in \mathrm{supp}\ \zeta} Z(\vec{h}_{\vec{\ell}})} \int \prod_{\{\vec{n}, \vec{n}'\} \in \zeta} [\exp(-\beta J_{\vec{n}-\vec{n}'} \vec{s}_{\vec{n}} \vec{s}_{\vec{n}'}) - 1] \\ &\times \left(\prod_{\vec{\ell} \in \mathrm{supp}\ \zeta} e^{\vec{h}_{\vec{\ell}} \cdot \vec{s}_{\vec{\ell}}} \right) \prod_{\vec{\ell} \in \mathrm{supp}\ \zeta} d\sigma(\vec{s}_{\vec{\ell}}), \end{aligned} \tag{3.9.8}$$

so by (3.7.16) we have the bound

$$|z(\zeta)| \le \prod_{\{\vec{n}, \vec{n}'\} \in \zeta} (\beta |J_{\vec{n}-\vec{n}'}| e^{\beta |J_{\vec{n}-\vec{n}'}|}). \tag{3.9.9}$$

This is inadequate because the number of factors can be much smaller than the volume of ζ – i.e., the number of line segments has nothing to do with the sum of their ℓ^1-lengths because the line segments can be arbitrarily long. One can still extract a bound that has exponential decay in the volume of ζ if the pair interaction $J_{\vec{n}-\vec{n}'}$ has sufficiently fast exponential decay in $|\vec{n} - \vec{n}'|$, but such an interaction is essentially "nearest-neighbor" in character – a property much too restrictive.

These two failed alternatives hardly represent a crisis, however. After all, high-temperature expansions for long-range interactions were developed long before the abstract polymer formalism was around. Moreover, the latter has a trivial generalization that is ideal for this high-temperature problem.

Accordingly, we return to defining the volume of a polymer as the cardinality of its support and re-examine the abstract theory. Instead of insisting on both an energy estimate and an entropy estimate, we note that it is sufficient to require only the estimate

$$\sum_{\substack{\zeta:\ \alpha\in \text{ supp } \zeta \\ |\zeta|=M}} |z(\zeta)| \leq C_2 e^{-c_2 M} \tag{3.9.10}$$

for $c_2 > 0$ large enough. Recall that the estimation of the inner-most sum

$$\sum_{\substack{P=(\zeta_1,\ldots,\zeta_n)\in\mathcal{K}_n \\ T \text{ is a subgraph of } G_P \\ |\zeta_\nu|=m_\nu}} \prod_{\nu=1}^{n} |z(\zeta_\nu)|$$

in (3.6.35) was done by combining the energy estimate with the entropy estimate. The resulting bound was

$$C_0^n C_1^n e^{(c_1-c_0)\sum_\nu m_\nu} (\text{card } \mathcal{S}) \prod_{\mu=1}^{n} m_\mu^{d_\mu(T)-1},$$

where C_0 and c_0 (resp. C_1 and c_1) were given in the energy estimate (3.6.19) (resp. the entropy estimate (3.6.20)). On the other hand, the estimate (3.6.37) was necessary only because (3.6.36) estimated the product of activities right out of the summation. If we apply (3.9.10) to each sum of activities over polymers ζ_ν such that $\alpha_\nu \in \text{supp } \zeta_\nu$, the bound on the multiple sum simply becomes

$$C_2^n e^{-c_2\sum_\nu m_\nu} (\text{card } \mathcal{S}) \prod_{\nu=1}^{n} m_\mu^{d_\mu(T)-1},$$

which is all that is needed for the convergence theory.

In the present context, one has to establish (3.9.10) by using (3.9.9), but the Königsberg estimation applied in the nearest-neighbor context is no longer sharp enough. Indeed, consider a fixed lattice site $\vec{n}^{\,0}$ and a polymer ζ containing $\vec{n}^{\,0}$ with $|\zeta| =$ card supp $\zeta = M$. Once again, by the Königsberg bridge-island solution, there is a path of line segments consisting of precisely those line segments that are in ζ, originating at $\vec{n}^{\,0}$, and traversing no element of ζ more than twice. Let π_ζ denote such a path

for each ζ. Then

$$\sum_{\substack{\zeta:\ \vec{n}^0 \in \text{ supp } \zeta \\ |\zeta|=M}} |z(\zeta)| \le \beta^M \sum_{\substack{\zeta:\ \vec{n}^0 \in \text{ supp } \zeta \\ |\zeta|=M}} c^{M-D_{\pi_\zeta}} \prod_{(\vec{m},\vec{m}')\in\pi_\zeta} (|J_{\vec{m}-\vec{m}'}|^{1/2} e^{\beta|J_{\vec{m}-\vec{m}'}|}), \tag{3.9.11}$$

where D_{π_ζ} is the number of line segments that have been traversed twice and we have applied

$$|J_{\vec{m}-\vec{m}'}| \le c|J_{\vec{m}-\vec{m}'}|^{1/2} \tag{3.9.12}$$

to those line segments $\{\vec{n}, \vec{n}\,'\}$ that have *not* been traversed twice. Thus

$$\sum_{\substack{\zeta:\ \vec{n}^0 \in \text{ supp } \zeta \\ |\zeta|=M}} |z(\zeta)| \le c^M \beta^M \sum_{M'=M}^{2M} \sum_{\vec{m}^1,\dots,\vec{m}^{M'}} \prod_{i=1}^{M'} (|J_{\vec{m}^i-\vec{m}^{i-1}}|^{1/2} e^{\beta|J_{\vec{m}^i-\vec{m}^{i-1}}|}) \tag{3.9.13}$$

because the lengths of the sequences π_ζ range from M to $2M$. The idea is to estimate the multiple sum from the inside on out, applying a summability condition when summing over $\vec{n}_i$ for fixed $\vec{n}_{i-1}$, but this iteration depends on the summability condition.

$$\sum_{\vec{n}} |J_{\vec{n}}|^{1/2} < \infty, \tag{3.9.14}$$

which is stronger than condition (3.9.2).

One can still establish convergence of this polymer expansion without this restriction on the long-range interaction. Indeed, the required combinatorics is standard and predates the polymer formalism. We can label these line-segment polymers with connected graphs and control the multiple summation with those subgraphs which are tree graphs. This control is quite similar to the key estimation for convergence of the abstract polymer expansion of the abstract free energy in §3.6. We shall not pursue this, however. Instead, we devote the next section to the polymer expansion based on applying inductive interpolation to this long-range model. The point of this alternative is that the tree structure is already there in an individual polymer.

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3.10 Unit-Vector Spins with Long-Range Coupling

We continue our analysis of the lattice system whose single-spin distribution is Lebesgue measure $d\sigma$ on $\mathbb{S}^2$ and whose Hamiltonian in a finite region $\Lambda \subset \mathbb{Z}^3$ is given by

$$H_\Lambda = \sum_{\vec{n},\vec{m}\in\Lambda} J_{\vec{n}-\vec{m}}\, \vec{s}_{\vec{n}} \cdot \vec{s}_{\vec{m}} \tag{3.10.1}$$

We now describe an expansion to control the $\Lambda = \mathbb{Z}^3$ limit of the expectations for high temperature, assuming only that

$$\sum_{\vec{n}} |J_{\vec{n}}| < \infty, \qquad J_0 = 0. \tag{3.10.2}$$

This condition is the weaker of the two conditions mentioned in the previous section. The stronger condition seemed to be called for, but there are various ways to remedy this. Here we generate the products of exponential differences in a more controlled way – using inductive interpolation, as we did in §3.8 – only now we use the more general polymer estimation.

Introduce a linear ordering of Λ and attempt to decouple $\vec{s}_{\vec{n}^1}$ from all other spin variables in the integral expression for $Z(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell} \in \Lambda)$ by interpolation, where $\vec{n}^{\,1}$ is the first lattice site in that ordering. We insert the identity

$$\begin{aligned}
&\exp\left(-\beta \sum_{\vec{n},\vec{m}} J_{\vec{n}-\vec{m}}\ \vec{s}_{\vec{n}}\ \ \vec{s}_{\vec{m}}\right) \\
= \ &\exp\left(-\beta \sum_{\vec{n},\vec{m}\neq\vec{n}^1} J_{\vec{n}-\vec{m}}\ \vec{s}_{\vec{n}}\ \ \vec{s}_{\vec{m}}\right) \\
&+ \int_0^1 dt_1 \left(-\beta \sum_{\vec{n}=\vec{n}^1 \text{ or } \vec{m}=\vec{n}^1} J_{\vec{n}-\vec{m}}\ \vec{s}_{\vec{n}} \cdot \vec{s}_{\vec{m}}\right) \\
&\times \exp\left(-t_1\beta \sum_{\vec{n},\vec{m}} J_{\vec{n}-\vec{m}}\ \vec{s}_{\vec{n}} \cdot \vec{s}_{\vec{m}} - (1-t_1)\beta\right. \\
&\left.\times \sum_{\vec{n},\vec{m}\neq\vec{n}^1} J_{\vec{n}-\vec{m}}\ \vec{s}_{\vec{n}} \cdot \vec{s}_{\vec{m}}\right)
\end{aligned} \tag{3.10.3}$$

in the integral to obtain

$$\begin{aligned}
&Z(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell} \in \Lambda) = Z(\vec{h}_{\vec{n}^1}) Z(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell} \in \Lambda \backslash \{\vec{n}^{\,1}\}) \\
&- \beta \sum_{\vec{n}=\vec{n}^1 \text{ or } \vec{m}=\vec{n}^1} J_{\vec{n}-\vec{m}} \int \vec{s}_{\vec{n}} \cdot \vec{s}_{\vec{m}} \\
&\times \int_0^1 dt_1 \exp\left(-t_1\beta \sum_{\vec{n},\vec{m}} J_{\vec{n}-\vec{m}}\ \vec{s}_{\vec{n}} \cdot \vec{s}_{\vec{m}} - (1-t_1)\beta \sum_{\vec{n},\vec{m}\neq\vec{n}^1} J_{\vec{n}-\vec{m}}\ \vec{s}_{\vec{n}} \cdot \vec{s}_{\vec{m}}\right) \\
&\times \prod_{\vec{\ell}\in\Lambda} d\sigma(\vec{s}_{\vec{\ell}}).
\end{aligned} \tag{3.10.4}$$

We use the same terminology as before: the factorized term is called a decoupled term, while the other terms are called remainder terms. The second stage in the development

of the expansion is applied to each term, with the variables we attempt to decouple dictated by the term. In the decoupled term, the same interpolation is applied to the factor $Z(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell}\in\Lambda\backslash\{\vec{n}^{\,1}\})$, but $\vec{s}_{\vec{n}^{\,2}}$ is the variable to be decoupled from the other variables, where $\vec{n}^{\,2}$ is the first element in $\Lambda\backslash\{\vec{n}^{\,1}\}$. For a given remainder term – labeled by $(\vec{n},\vec{m})$ with either $\vec{n}=\vec{n}^{\,1}$ or $\vec{m}=\vec{n}^{\,1}$ – the interpolation is based on the attempt to decouple both $\vec{s}_{\vec{n}}$ and $\vec{s}_{\vec{m}}$ from the other variables.

The iteration of this procedure develops a sum over histories of interpolations, with each interpolation branching out that development in many possible directions for every partial history previously developed. We still refer to a *step* as an interpolation followed by a choice of term, so every term that has developed at a given stage is labeled by a history of previous steps. The nature of the next interpolation depends on the term. If a given term is a decoupled term, it has the form

$$Z\left(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell}\in\Lambda\backslash\bigcup_\gamma\Lambda_\gamma\right)\prod_\gamma K_\gamma(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell}\in\Lambda_\gamma),$$

where K_γ is an integral in only those spin variables $\vec{s}_{\vec{\ell}}$ for which $\vec{\ell}\in\Lambda_\gamma$. As in §3.8, K_γ is called a completed factor, and it is developed by the γth succession of steps, where "succession" has the same meaning here as it did there. For such a term, the next interpolation is applied to the factor $Z\left(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell}\in\Lambda\backslash\bigcup_\gamma\Lambda_\gamma\right)$ only, and in exactly the same manner as it is applied to $Z(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell}\in\Lambda)$ – as an attempt to decouple the spin $\vec{s}_{\vec{n}'}$ in the integral from the other spins $\vec{s}_{\vec{n}}$, where $\vec{n}\,'$ is the first lattice site in the given set of sites $\vec{n}$ with respect to the linear ordering.

On the other hand, if the given term happens to be a remainder term, then it has the form

$$\begin{aligned}&\prod_\gamma K_\gamma(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell}\in\Lambda_\gamma)\left(\prod_i\int_0^1 dt_i\right)\\ &f_{\mathfrak{G}}(t)\int\prod_{(\vec{n},\vec{m})\in\mathcal{U}_{\mathfrak{G}}}(-\beta J_{\vec{n}-\vec{m}}\,\vec{s}_{\vec{n}}\cdot\vec{s}_{\vec{m}})\\ \times\quad&\exp\left(\sum_{\vec{\ell}\in\Lambda\backslash\bigcup_\gamma\Lambda_\gamma}\vec{h}_{\vec{\ell}}\ \vec{s}_{\vec{\ell}}-\beta H^{\mathfrak{G}}_{\Lambda\backslash\bigcup_\gamma\Lambda_\gamma}(t)\right)\prod_{\vec{\ell}\in\Lambda\backslash\bigcup_\gamma\Lambda_\gamma}d\sigma(\vec{s}_{\vec{\ell}}).\end{aligned}$$

Let Λ' denote that set our previous interpolation was trying to decouple from the other lattice sites in $\Lambda\backslash\bigcup_\gamma\Lambda_\gamma$. This remainder term represents a failure to decouple, and it contains a factor $J_{\vec{n}-\vec{n}'}\ \vec{s}_{\vec{n}}\ \vec{s}_{\vec{n}'}$ in the integrand where $\vec{n}\,'\notin\Lambda'$. $\mathfrak{G}$ is an ordered connectivity graph supported by Λ' and this additional lattice site $\vec{n}\,'$, while

$H^{\mathfrak{G}}_{\Lambda\setminus\bigcup_\gamma \Lambda_\gamma}(t)$ is the interpolated Hamiltonian that has developed so far. $f_{\mathfrak{G}}(t)$ is the product of interpolation parameters determined by $\mathfrak{G}$ as in §3.8, although $\mathfrak{G}$ represents a partial history of steps at this stage. The K_γ are still called completed factors, and the next interpolation is applied to the integral described here. It is based on trying to decouple $\{\vec{n}\,'\} \cup \Lambda'$ from the other lattice sites and generates new terms – new remainder terms, and one decoupled term which has the form

$$Z\left(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell} \in \Lambda\setminus((\Lambda' \cup \{\vec{n}\,'\}) \cup \bigcup_\gamma \Lambda_\gamma)\right) K'(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell} \in \Lambda' \cup \{\vec{n}\,'\})$$
$$\times \prod_\gamma K_\gamma(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell} \in \Lambda_\gamma).$$

Every branch in this inductive process eventually terminates.

Although the model to be analyzed here is somewhat different from the model considered in §3.8, the combinatorial structure of the expansion is exactly the same. We have expanded $Z(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell} \in \Lambda)$ in terms of the form

$$\prod_\gamma K_{\mathfrak{G}^\gamma}(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell} \in \Lambda_\gamma), \qquad \mathfrak{G}^\gamma \text{ an o.c.g. on } (\Lambda_\gamma, \vec{n}^{\,\gamma}),$$

where $\vec{n}^{\,\gamma}$ is the first lattice site in Λ_γ, $\Lambda = \bigcup_\gamma \Lambda_\gamma$, the Λ_γ are disjoint, and

$$K_{\mathfrak{G}^\gamma}(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell} \in \Lambda_\gamma) = \left(\prod_i \int_0^1 dt_i\right) f_{\mathfrak{G}^\gamma}(t)$$
$$\times \int \prod_{(\vec{n},\vec{n}\,')\in \mathcal{U}_{\mathfrak{G}^\gamma}} (-\beta J_{\vec{n}-\vec{n}\,'}\ \vec{s}_{\vec{n}}\ \ \vec{s}_{\vec{n}\,'})$$
$$\times \exp\left(\sum_{\vec{\ell}\in\Lambda_\gamma} \vec{h}_{\vec{\ell}}\ \ \vec{s}_{\vec{\ell}} - \beta H^{\mathfrak{G}^\gamma}_{\Lambda_\gamma}(t)\right) \prod_{\vec{\ell}\in\Lambda_\gamma} d\sigma(\vec{s}_{\vec{\ell}}). \tag{3.10.5}$$

For a given o.c.g. $\mathfrak{G}$ on a pointed set $(\Lambda', \vec{\ell}\,')$, the interpolated Hamiltonian is given by

$$H^{\mathfrak{G}}_{\Lambda'}(t) = \sum_{\vec{m},\vec{m}'\in\Lambda'} J_{\vec{m}-\vec{m}'} \left(\prod_{i^-_{\vec{m},\vec{m}'} \le j < i^+_{\vec{m}\vec{m}'}} t_j\right) \vec{s}_{\vec{m}} \cdot \vec{s}_{\vec{m}'}, \tag{3.10.6}$$

$$i^-_{\vec{m}\vec{m}'} = \min\{g_2^{-1}(\vec{m}), g_2^{-1}(\vec{m}\,')\}, \tag{3.10.7–}$$

$$i^+_{\vec{m}\vec{m}'} = \max\{g_2^{-1}(\vec{m}), g_2^{-1}(\vec{m}\,')\}. \tag{3.10.7+}$$

The only difference between the ordered connectivity graphs here and those considered before is that the ordered pairs of lattice sites are no longer nearest-neighbor bonds.

The summability property of the coupling $J_{\vec{n}-\vec{n}'}$ must be exploited in two ways – to control the expansion itself and to bound the interpolated Hamiltonian from below. The latter estimation is obvious:

$$H^{\mathfrak{G}}_{\Lambda'}(t) \geq -c|\Lambda'| \tag{3.10.8}$$

because the spins are unit vectors. Thus

$$\begin{aligned} &|K_{\mathfrak{G}}(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell} \in \Lambda')| \\ &\leq\ c^{\beta|\Lambda'|} \prod_{\vec{\ell}\in\Lambda'} Z(\vec{h}_{\vec{\ell}}) \prod_{(\vec{n},\vec{n}')\in\mathcal{U}_{\mathfrak{G}}} |\beta J_{\vec{n}-\vec{n}'}| \left(\prod_i \int_0^1 dt_i\right) f_{\mathfrak{G}}(t). \end{aligned} \tag{3.10.9}$$

The formalism for polymers is actually the expansion of

$$\widehat{Z}(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell} \in \Lambda) = \frac{Z(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell} \in \Lambda)}{\prod\limits_{\vec{\ell}\in\Lambda} Z(\vec{h}_{\vec{\ell}})}, \tag{3.10.10}$$

and – as in §3.8 – a polymer is defined as an unordered connectivity graph for the expansion we have developed. For a given u.c.g. $\mathcal{U}$ on $(\Lambda', \vec{\ell}\,')$, the activity of $\mathcal{U}$ is given by

$$z(\mathcal{U}) = \sum_{\substack{\text{u.c.g. } \mathfrak{G} \text{ on } (\Lambda', \vec{\ell}\,') \\ \mathcal{U}_{\mathfrak{G}}=\mathcal{U}}} K_{\mathfrak{G}}(\vec{h}_{\vec{\ell}}\colon\ \vec{\ell} \in \Lambda') \prod_{\vec{\ell}\in\Lambda'} Z(\vec{h}_{\vec{\ell}})^{-1}, \tag{3.10.11}$$

and we may group the terms of our expansion into the form (3.8.29). Given (3.10.9) we obtain a bound on the activity:

$$\begin{aligned} |z(\mathcal{U})| \ &\leq\ c^{\beta|\Lambda'|} \prod_{(\vec{n},\vec{n}')\in\mathcal{U}} |\beta J_{\vec{n}-\vec{n}'}| \sum_{\substack{\text{o.c.g. } \mathfrak{G} \text{ on } (\Lambda', \vec{\ell}\,') \\ \mathcal{U}_{\mathfrak{G}}=\mathcal{U}}} \left(\prod_i \int_0^1 dt_i\right) f_{\mathfrak{G}}(t) \\ &=\ c^{\beta|\Lambda'|} \prod_{(\vec{n},\vec{n}')\in\mathcal{U}} |\beta J_{\vec{n}-\vec{n}'}|, \end{aligned} \tag{3.10.12}$$

where we have applied the u.c.g. identity (3.8.32). Our point of departure from §3.8 is that we now have to apply generalized polymer estimation, as a standard activity bound is no longer enough for the number of polymers we are dealing with. Accordingly, we observe that

$$\sum_{\substack{\text{u.c.g. } \mathcal{U}\colon\ \vec{m}_1\in \text{ supp } \mathcal{U} \\ \text{card supp } \mathcal{U}=M}} |z(\mathcal{U})| \leq c^{\beta M} \beta^{M-1} \sum_{\substack{\text{u.c.g. } \mathcal{U}\colon\ \vec{m}_1\in\text{supp } \mathcal{U} \\ \text{card supp } \mathcal{U}=M}} \prod_{(\vec{n},\vec{n}')\in\mathcal{U}} |J_{\vec{n}-\vec{n}'}| \tag{3.10.13}$$

for fixed $\vec{m}_1 \in \Lambda$. On the other hand, we may write the summation as

$$\sum_{\substack{\text{u.c.g. } \mathcal{U}\text{: } \vec{m}_1 \in \text{supp } \mathcal{U} \\ \text{card supp } \mathcal{U} = M}} = \frac{1}{(M-1)!} \sum_{\substack{\text{u.c.g. } U \\ \text{supp } U = \{1,\dots,M\}}} \sum_{\substack{\vec{m}_2,\dots,\vec{m}_M \in \Lambda \\ \vec{m}_1,\vec{m}_2,\dots,\vec{m}_M \text{ distinct}}}, \tag{3.10.14}$$

and since $J_{\vec{n}-\vec{n}'} = J_{\vec{n}'-\vec{n}}$, the terms are the same for those U having the same tree graph. The tree graph associated with U is given by

$$T_U = \{\{i,j\}\colon (i,j) \in U\}, \tag{3.10.15}$$

and for a given tree graph T on $\{1,\dots,M\}$ there are exactly M unordered connectivity graphs U such that $T_U = T$, because the directedness of the line segments in a u.c.g. are determined by the point of origin only. Thus

$$\sum_{\substack{\text{u.c.g. } \mathcal{U}\text{: } \vec{m}_1 \in \text{supp } \mathcal{U} \\ \text{card supp } \mathcal{U} = M}} |z(\mathcal{U})| \le \frac{c^{\beta M}}{(M-1)!}(2\beta)^{M-1}$$
$$\times \sum_{\substack{\text{tree graphs } T \\ \text{on } \{1,\dots,M\}}} \sum_{\vec{m}_2,\dots,\vec{m}_M \in \Lambda} \prod_{\{i,j\} \in T} |J_{\vec{m}_i - \vec{m}_j}|, \tag{3.10.16}$$

where we have also removed the summation restriction that $\vec{m}_1, \vec{m}_2, \dots, \vec{m}_M$ be distinct. We now have the familiar situation where the tree graph controls the multiple sum. The rule at each stage of the summation is to sum out those $\vec{m}_i$ for which i is connected to only one j, and this enables us to use

$$\sum_{\vec{n}'} |J_{\vec{n}-\vec{n}'}| \le c \tag{3.10.17}$$

to obtain

$$\sum_{\substack{\text{u.c.g. } \mathcal{U}\text{: } \vec{m}_1 \in \text{supp } \mathcal{U} \\ \text{card supp } \mathcal{U} = M}} |z(\mathcal{U})|$$
$$\le \frac{c^{\beta M}}{(M-1)!}(2\beta)^{M-1} c^{M-1} \text{card}\{\text{tree graphs } T \text{ on } \{1,\dots,M\}\}. \tag{3.10.18}$$

By Cayley's Theorem, the number of such tree graphs is M^{M-2}, so now we have a bound of the generalized type (3.9.10) if β is small enough.

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3.11 Combinatorial Properties of Interpolation Weights

We have already defined a number of graph-theoretic objects, and it is appropriate at this point to collect the purely combinatoric results that are relevant to inductive interpolation. We begin by proving the u.c.g. identity (3.8.32), which we have applied twice already.

Let $\mathcal{U}$ be a fixed unordered connectivity graph on a finite set with distinguished point x_0 and associate to each $(p,q) \in \mathcal{U}$ a complex variable w_{pq}. The idea is to apply inductive interpolation to the function

$$\exp \sum_{(p,q)\in\mathcal{U}} w_{pq}$$

based on attempts to eliminate those variables w_{pq} for which p is "old" and q is "new". In that sense, we first try to "decouple" x_0 from the other points and obtain

$$\exp \sum_{(p,q)\in\mathcal{U}} w_{pq} = \exp \sum_{\substack{(p,q)\in\mathcal{U} \\ p\neq x_0}} w_{pq} + \sum_{(x_0,q')\in\mathcal{U}} w_{x_0q'} \int_0^1 dt_1 \exp\left(t_1 \sum_{(x_0,q)\in\mathcal{U}} w_{x_0q} + \sum_{\substack{(p,q)\in\mathcal{U} \\ p\neq x_0}} w_{pq} \right), \quad (3.11.1)$$

where the first step is to pick one of these terms. The first term is the "decoupled" term, and a term labeled by (x_0,q') is a "remainder" term. If the first step chooses the decoupled term, then we are "done". If the remainder term for a given q' is chosen, then the second interpolation is based on trying to decouple $\{x_0,q'\}$ from the other points. Thus

$$\exp\left(t_1 \sum_{(x_0,q)\in\mathcal{U}} w_{x_0q} + \sum_{\substack{(p,q)\in\mathcal{U} \\ p\neq x_0}} w_{pq} \right) = \exp\left(t_1 w_{x_0q'} + \sum_{\substack{(p,q)\in\mathcal{U} \\ p\neq q', x_0}} w_{pq} \right)$$

$$+\left(t_1 \sum_{\substack{(x_0,q'')\in\mathcal{U}\\ q''\neq q'}} w_{x_0q''} + \sum_{(q',q'')\in\mathcal{U}} w_{q'q''}\right)\int_0^1 dt_2 \exp V_{x_0q'}(w;t_1,t_2), \quad (3.11.2)$$

$$V_{x_0q'}(w;t_1,t_2) = t_1 w_{x_0q'} + \sum_{\substack{(p,q)\in\mathcal{U}\\ p\neq q'}} w_{pq}$$

$$+ t_2\left(t_1 \sum_{\substack{(x_0,q)\in\mathcal{U}\\ q\neq q',x_0}} w_{x_0q} + \sum_{(q',q)\in\mathcal{U}} w_{q'q}\right), \quad (3.11.3)$$

where the first term is – once again – the "decoupled" term, and there are now two types of remainder terms. The second step is to choose a term in (3.11.2), and the decoupled term is regarded as a completed history. If either type of remainder term associated with a given q' is chosen, then the third interpolation attempts to decouple $\{x_0, q', q''\}$ from the other points. More generally, at any given stage in a history of steps, the set that the next interpolation tries to decouple from its complement is the set of all points met by the variables w_{pq} that have already been "differentiated down" from the exponent. Clearly, the next step either completes a history by choosing a decoupled term or introduces a new point to that set by choosing a remainder term. Therefore, every branch of the expansion eventually terminates because supp $\mathcal{U}$ is a finite set. Once the expansion is developed, the only remainder terms are exactly those terms where *all* of the variables w_{pq} appear. A decoupled term will always have at least one variable missing from the exponent that has not been previously "differentiated down" from the exponent. Hence

$$\exp \sum_{(p,q)\in\mathcal{U}} w_{pq} = \prod_{(p',q')\in\mathcal{U}} \frac{\partial}{\partial w_{p'q'}} \exp \sum_{(p,q)\in\mathcal{U}} w_{pq}$$

$$= \prod_{(p',q')\in\mathcal{U}} \frac{\partial}{\partial w_{p'q'}} \begin{pmatrix} \text{sum of all} \\ \text{remainder} \\ \text{terms} \end{pmatrix}, \quad (3.11.4)$$

and these remainder terms are labeled by those ordered connectivity graphs $\mathfrak{G}$ for which $\mathcal{U}_{\mathfrak{G}} = \mathcal{U}$. For a given o.c.g. $\mathfrak{G}$ representing a history of attempts to decouple points, the differentiation with respect to t_{i-1} associated with the $(i-1)$st interpolation brings down the product

$$z_{\mathfrak{G}(i)} \prod_{k=\eta_{\mathfrak{G}}(i)}^{i-2} t_k, \qquad \eta_{\mathfrak{G}}(i) = g_1^{-1}(g_0(i)), \quad (3.11.5)$$

from the exponent. Therefore, we may write the sum of all remainder terms as

$$\prod_{(p,q)\in\mathcal{U}} w_{pq} \sum_{\mathfrak{G}:\ \mathcal{U}_{\mathfrak{G}}=\mathcal{U}} \left(\prod_i \int_0^1 dt_i\right) f_{\mathfrak{G}}(t) \exp V_{\mathfrak{G}}(w;t),$$

where $V_{\mathfrak{G}}(w;t)$ is the final interpolated form of the exponent for the history represented by $\mathfrak{G}$, and

$$f_{\mathfrak{G}}(t) = \prod_i \left(\prod_{k=\eta_{\mathfrak{G}}(i)}^{i-2} t_k \right). \tag{3.11.6}$$

On the other hand,

$$\exp V_{\mathfrak{G}}(w;t) = \prod_{(p,q)\in\mathcal{U}} (1 + O(w_{pq})), \tag{3.11.7}$$

so the only contribution having only one power of each w_{pq} is

$$\prod_{(p,q)\in\mathcal{U}} w_{pq} \sum_{\mathfrak{G}:\ \mathcal{U}_{\mathfrak{G}}=\mathcal{U}} \left(\prod_i \int_0^1 dt_i \right) f_{\mathfrak{G}}(t),$$

and therefore the constant contribution to the r.h.s. of (3.11.4) is just

$$\sum_{\mathfrak{G}:\ \mathcal{U}_{\mathfrak{G}}=\mathcal{U}} \left(\prod_i \int_0^t dt_i \right) f_{\mathfrak{G}}(t).$$

Since the constant contribution to the l.h.s. is unity, we have the desired identity (3.8.32).

There are other combinatorial identities, which are based on the notion of *ordered tree graph*. An Nth-degree ordered tree graph is defined to be any mapping $\eta\colon \{2,\dots,N\} \to \{1,\dots,N-1\}$ such that $\eta(i) < i$. Clearly, for an o.c.g. $\mathfrak{G}$, $\eta_{\mathfrak{G}}$ is an ordered tree graph, and the point is that the quantity $f_{\mathfrak{G}}(t)$ depends only on $\eta_{\mathfrak{G}}$ – i.e., $f_\eta(t)$ is well-defined:

$$f_\eta(t) = \prod_i \left(\prod_{k=\eta(i)}^{i-2} t_k \right). \tag{3.11.8}$$

There is an ordered tree graph identity known as the *Robinson identity*:

$$\sum_{N\text{th-degree } \eta} \left(\prod_{i=1}^{N} \int_0^1 dt_i \right) f_\eta(t) = \frac{N^{N-1}}{N!}, \tag{3.11.9}$$

and the u.c.g. identity may be used to prove it. To this end, we first consider the effect of permutations of underlying sets on ordered and unordered connectivity graphs.

Let (Γ, x_0) be a pointed set with cardinality N and consider an arbitrary o.c.g. $\mathfrak{G}$ on (Γ, x_0). Let h be a permutation of Γ with $h(x_0) = x_0$. Clearly, $(h \times h) \circ \mathfrak{G}$ is an o.c.g. on (Γ, x_0) and

$$\mathcal{U}_{(h\times h)\circ\mathfrak{G}} = (h \times h)(\mathcal{U}_{\mathfrak{G}}), \tag{3.11.10}$$

$$\begin{aligned} \eta_{(h\times h)\circ\mathfrak{G}}(i) &= (h \circ g_1)^{-1}(h(g_0(i))) \\ &= g_1^{-1}(g_0(i)) \\ &= \eta_{\mathfrak{G}}(i) \end{aligned} \tag{3.11.11}$$

Indeed, if $\mathfrak{G}$ and $\mathfrak{G}'$ are any two ordered connectivity graphs on (Γ, x_0), then we have the equivalence:

$$\eta_{\mathfrak{G}'} = \eta_{\mathfrak{G}} \Leftrightarrow \mathfrak{G}' = (h \times h) \circ \mathfrak{G} \text{ for some permutation } h \text{ of } (\Gamma, x_0). \tag{3.11.12}$$

An *automorphism* of an unordered connectivity graph $\mathcal{U}$ on (Γ, x_0) is a permutation h of Γ such that $h(x_0) = x_0$ and

$$(h \times h)(\mathcal{U}) = \mathcal{U}. \tag{3.11.13}$$

In the case $\mathcal{U}_{\mathfrak{G}} = \mathcal{U}_{\mathfrak{G}}$, (3.11.12) becomes:

$$\eta_{\mathfrak{G}'} = \eta_{\mathfrak{G}} \Leftrightarrow \mathfrak{G}' = (h \times h) \circ \mathfrak{G} \text{ for some automorphism } h. \tag{3.11.14}$$

Finally, if $\mathcal{U}$ and $\mathcal{U}'$ are unordered connectivity graphs on (Γ, x_0), we say that $\mathcal{U} \sim \mathcal{U}'$ if and only if there is a permutation h of Γ with $h(x_0) = x_0$ and

$$\mathcal{U}' = (h \times h)(\mathcal{U}). \tag{3.11.15}$$

Let $\nu(\mathcal{U})$ denote the number of $\mathcal{U}' \sim \mathcal{U}$ and $\alpha(\mathcal{U})$ the number of automorphisms of $\mathcal{U}$ (including the identity). Since there is a one-to-one correspondence between the $\mathcal{U}' \sim \mathcal{U}$ and the left cosets of the automorphism group of $\mathcal{U}$ regarded as a subgroup of the group of permutations h of Γ with $h(x_0) = x_0$, we have

$$\alpha(\mathcal{U})\nu(\mathcal{U}) = (N-1)! \tag{3.11.16}$$

We are now ready to prove the Robinson identity.

For a given unordered connectivity graph $\mathcal{U}$ on (Γ, x_0), we first decompose the sum in the u.c.g. identity (3.8.32) as follows:

$$\sum_{\eta \in \mathcal{N}(\mathcal{U})} \sum_{\substack{\mathfrak{G}:\ \eta_{\mathfrak{G}} = \eta \\ \mathcal{U}_{\mathfrak{G}} = \mathcal{U}}} \left(\prod_i \int_0^1 dt_i \right) f_{\mathfrak{G}}(t) = 1, \tag{3.11.17}$$

$$\mathcal{N}(\mathcal{U}) = \{\eta_{\mathfrak{G}}:\ \mathcal{U}_{\mathfrak{G}} = \mathcal{U}\}. \tag{3.11.18}$$

Since $f_{\mathfrak{G}}(t) = f_{\eta_{\mathfrak{G}}}(t)$, the identity reduces to:

$$\sum_{\eta \in \mathcal{N}(\mathcal{U})} \operatorname{card}\{\mathfrak{G}:\ \eta_{\mathfrak{G}} = \eta, \mathcal{U}_{\mathfrak{G}} = \mathcal{U}\} \left(\prod_i \int_0^1 dt_i \right) f_\eta(t) = 1, \tag{3.11.19}$$

but (3.11.14) implies that the set cardinality is precisely $\alpha(\mathcal{U})$. If we combine this observation with (3.11.16), we obtain:

$$\sum_{\eta \in \mathcal{N}(\mathcal{U})} \left(\prod_i \int_0^1 dt_i \right) f_\eta(t) = \frac{\nu(\mathcal{U})}{(N-1)!}. \tag{3.11.20}$$

Now (3.11.12) implies $\mathcal{U}' \sim \mathcal{U} \Leftrightarrow \mathcal{N}(\mathcal{U}') = \mathcal{N}(\mathcal{U})$, so choose a representative u.c.g. from each equivalence class defined by $\sim$ and index this selection: $\mathcal{U}_1, \ldots, \mathcal{U}_m$. Since every

Nth-degree ordered tree graph η can be realized as $\eta_{\mathfrak{G}}$ for some o.c.g. $\mathfrak{G}$ on (Γ, x_0), it follows that the disjoint $\mathcal{N}(\mathcal{U}_1), \ldots, \mathcal{N}(\mathcal{U}_m)$ comprise the set of *all* Nth-degree ordered tree graphs. Thus

$$\sum_{N\text{th-degree } \eta} \left(\prod_{i=1}^{N} \int_0^t dt_i \right) f_\eta(t) = \frac{1}{(N-1)!} \sum_{k=1}^{m} \nu(\mathcal{U}_k). \tag{3.11.21}$$

On the other hand, the disjoint equivalence classes $\{\mathcal{U} \sim \mathcal{U}_1\}, \ldots, \{\mathcal{U} \sim \mathcal{U}_m\}$ comprise the set of all unordered connectivity graphs on (Γ, x_0). These objects are actually trees whose line segments are directed, and the directedness is determined by regarding x_0 as the point of origin. It follows from Cayley's Theorem that

$$\sum_{k=1}^{m} \nu(\mathcal{U}_k) = N^{N-2}, \tag{3.11.22}$$

and so (3.11.9) is finally verified.

It is interesting that in statistical mechanics this sum of interpolation weights arose in expansions long before the Robinson identity was known. Instead, the inequality

$$\sum_{N\text{th-degree } \eta} \left(\prod_{i=1}^{N} \int_0^t dt_i \right) f_\eta(t) \leq e^{N-1} \tag{3.11.23}$$

was verified and used. This bound now follows from the identity.

This is not the end of the story. If η is an Nth-degree ordered tree graph, define the weight function d_η such that $d_\eta(j) + 1$ is the coordination number at j for the tree graph associated with η – i.e., define

$$d_\eta(j) = \text{card}\{i\colon\ \eta(i) = j\}. \tag{3.11.24}$$

Then we have the *Speer identity*:

$$\sum_{N\text{th-degree } \eta} \left(\prod_j d_\eta(j)! \right) \left(\prod_i \int_0^1 dt_i \right) f_\eta(t) = \frac{1}{N} \binom{2N-2}{N-1}, \tag{3.11.25}$$

and an immediate consequence is the bound

$$\sum_{N\text{th-degree } \eta} \left(\prod_j d_\eta(j)! \right) \left(\prod_i \int_0^1 dt_i \right) f_\eta(t) \leq 4^{N-1} \tag{3.11.26}$$

The introduction of the factorial growths associated with coordination numbers of a tree graph do not change the exponential character of the bound! The key to the Speer identity is the interpolation weight identification

$$\left(\prod_{i=1}^{N} \int_0^t dt_i \right) f_\eta(t) = \frac{1}{\pi(\eta)}, \tag{3.11.27}$$

where $\pi(\eta)$ is the number of permutations p of $\{2,\dots,N\}$ such that $\eta \circ p$ is still an ordered tree graph.

To prove this identity, we first calculate the interpolation weight directly. If we set

$$\text{card}\{i\colon\ \eta(i) \le k \le i-2\} = \omega_\eta(k), \tag{3.11.28}$$

then (3.11.8) may be written as

$$f_\eta(t) = \prod_{k=1}^{N-1} t_k^{\omega_\eta(k)}, \tag{3.11.29}$$

from which we obtain

$$\left(\prod_{i=1}^{N}\int_0^1 dt_i\right) f_\eta(t) = \prod_{k=1}^{N-1} \frac{1}{\omega_k(k)+1}. \tag{3.11.30}$$

Therefore, the problem is to show that

$$\pi(\eta) = \prod_{k=1}^{N-1} (\omega_\eta(k)+1), \tag{3.11.31}$$

and the best approach is to count the possibilities at each step as we build a permutation p such that $\eta \circ p$ is an ordered tree graph. We may determine the values $p(2),\dots,p(N)$ in order, where the only constraint is that $\eta(p(i)) < i$. The candidates for $p(2)$ clearly comprise the set of i' such that $\eta(i') = 1$. More generally, the candidates for $p(i)$ are those $i' \ne p(2),\dots,p(i-1)$ such that $\eta(i') < i$. Note that for $j < i$, the requirement $\eta(p(j)) < j < i$ governing the previous selections implies

$$\{p(2),\dots,p(i-1)\} \subset \{i'\colon\ \eta(i') < i\}, \tag{3.11.32}$$

so there are precisely

$$\text{card}\{i'\colon\ \eta(i') < i\} - (i-2) = \text{card}\{i'\colon\ i' > i-1 \text{ and } \eta(i') < i\} \tag{3.11.33}$$

candidates for $p(i)$. The latter set has the same cardinality as the set of candidates because

$$\{2,\dots,i-1\} \subset \{i'\colon\ \eta(i') < i\}. \tag{3.11.34}$$

On the other hand,

$$\begin{aligned}\{i'\colon\ i' > i-1 \text{ and } \eta(i') < i\} &= \{i'\colon\ \eta(i') < i < i'+1\}\\ &= \{i'\colon\ \eta(i') \le i-1 \le i'-1\},\end{aligned} \tag{3.11.35}$$

so it follows from (3.11.28) that the number of candidates for $p(i)$ is $\omega_\eta(i-1)+1$. Therefore, the number of possible p is just

$$\prod_{i=2}^{N} (\omega_\eta(i-1)+1),$$

and this completes the proof of (3.11.27).

The Speer identity now develops as follows. For any two ordered tree graphs η and η', we can find a permutation p of $\{2,\ldots,N\}$ with $\eta' = \eta \circ p$ if and only if $d_{\eta'}(j) = d_\eta(j)$ for all j. Pick a representative η from each of these equivalence classes and index these ordered tree graphs: $\eta_1,\ldots,\eta_M$. Then

$$\sum_{N\text{th-degree } \eta} \left(\prod_{j=1}^{N-1} d_\eta(j)!\right) \left(\prod_{i=1}^{N} \int_0^1 dt_i\right) f_\eta(t)$$

$$= \sum_{k=1}^{M} \left(\prod_{j=1}^{N-1} d_{\eta_k}(j)!\right) \sum_{\eta:\ d_\eta = d_{\eta_k}} \left(\prod_{i=1}^{N} \int_0^1 dt_i\right) f_\eta(t). \tag{3.11.36}$$

On the other hand, the permutations p of $\{2,\ldots,N\}$ for which $\eta \circ p = \eta$ are precisely those p that permute the sets $\eta^{-1}(\{j\})$ independently, so the number of such permutations is $\prod_{j=1}^{N-1} d_\eta(j)!$ Hence

$$\begin{aligned} \pi(\eta) &= \left(\prod_{j=1}^{N-1} d_\eta(j)!\right) \text{card}\{\eta':\ \eta' = \eta \circ p \text{ for some permutation } p\} \\ &= \left(\prod_{j=1}^{N-1} d_\eta(j)!\right) \text{card}\{\eta':\ d_{\eta'} = d_\eta\}. \end{aligned} \tag{3.11.37}$$

Combining this with the interpolation weight identity (3.11.27), we reduce (3.11.36) to:

$$\begin{aligned} &\sum_{N\text{th-degree } \eta} \left(\prod_{j=1}^{N-1} d_\eta(j)!\right) \left(\prod_{i=1}^{N} \int_0^1 dt_i\right) f_\eta(t) \\ &= \sum_{k=1}^{M} \sum_{\eta:\ d_\eta = d_{\eta_k}} \frac{1}{\text{card}\{\eta':\ d_{\eta'} = d_{\eta_k}\}} \\ &= \sum_{k=1}^{M} 1 \\ &= M, \end{aligned} \tag{3.11.38}$$

so the dependence of M on N is the answer. M enumerates the equivalence classes defined by the equations $d_\eta(j) = d_{\eta'}(j)$, so M is the cardinality of the set

$$\mathcal{W}_N = \{(n_1,\ldots,n_{N-1}):\ n_j = d_\eta(j) \text{ for some ordered tree graph } \eta\}. \tag{3.11.39}$$

On the other hand, integer sequences $(n_1,\ldots,n_{N-1})$ arising from ordered tree graphs are characterized by the constraints $n_j \geq 0$ and

$$k \leq \sum_{j=1}^{k} n_j \leq N-1. \tag{3.11.40}$$

Given this alternative definition of $\mathcal{W}_N$, it is a standard combinatoric fact that

$$M = \text{card}\mathcal{W}_N = \frac{1}{N}\binom{2N-2}{N-1}, \tag{3.11.41}$$

and so (3.11.25) is proven.

We conclude our discussion of pure combinatorics by stating the *Anghel-Tataru identity* without proof. For complex variables $w_1, \ldots, w_N$,

$$\sum_{\substack{p \text{ permutes } \{1,\ldots,N\} \\ p(1)=1}} \sum_{N\text{th-degree } \eta} \left(\prod_{i=3}^{N} w_{p(\eta(i))}\right) \left(\prod_{i=1}^{N} \int_0^1 dt_i\right) f_\eta(t) = \left(\sum_{i=1}^{N} w_i\right)^{N-2} \tag{3.11.42}$$

This result is a master identity for interpolation weights. It reduces to the Robinson identity if we set $w_i = 1$ for all i, while we can obtain the Speer identity by applying the integration

$$\left(\prod_{i=1}^{N} \int_0^\infty dw_i\right) \exp\left(-\sum_{i=1}^{N} w_i\right)$$

to this equation.

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3.12 Scalar Spins with Long Range Coupling

We return to the case of scalar spins with assumptions defining a rather large class of models. Recall that the generalized Ising model associated with the Euclidean lattice ϕ_3^4 theory (i.e., the Ginzburg–Landau model) had only nearest-neighbor couplings. We now complicate this unbounded spin case with the introduction of long-range couplings as we have already done in the $\mathbb{S}^2$-valued spin case – and consider once again the wholesale expansion in couplings. Let

$$H_\Lambda = \sum_{\vec{m},\vec{n}\in\Lambda} J_{\vec{m}-\vec{n}} s_{\vec{m}} s_{\vec{n}}, \tag{3.12.1}$$

where $J_{\vec{n}'-\vec{n}}$ satisfies the decay condition

$$\sum_{\vec{n}} |J_{\vec{n}}|^{1/3} < \infty. \tag{3.12.2}$$

This condition is stronger than condition (3.9.14), let alone condition (3.9.2), but we reserve refinements for the next section. The non-inductive expansion together with the Königsberg estimation is the most convenient context in which to describe how the extra long-distance decay can be used to deal with the large spin values whose uncoupled probabilities are quenched by a single-spin distribution

$$d\sigma(z) = e^{-U(z)} dz \tag{3.12.3}$$

with $U(z)$ a continuous function such that

$$U(z) \geq \hat{\tau} z^2 - c, \qquad \hat{\tau} > 0. \tag{3.12.4}$$

The Ginzburg–Landau model meets this single-spin requirement for arbitrary κ.

The transcription of the non-inductive expansion formula (3.9.5) to the present model is obvious. The polymers themselves are exactly the same – subsets of $\Lambda \times \Lambda$ whose supports are connected by their elements. We have

$$\begin{aligned}
&\exp\left(-\beta \sum_{\vec{n},\vec{m}\in\Lambda} J_{\vec{m}-\vec{n}} s_{\vec{m}} s_{\vec{n}}\right) \\
&= \sum_{\Gamma\subset\Lambda\times\Lambda} \prod_{(\vec{n},\vec{m})\in\Gamma} \{\exp(-\beta J_{\vec{m}-\vec{n}} s_{\vec{m}} s_{\vec{n}}) - 1\} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{\zeta_1,\ldots,\zeta_n\in\mathcal{P} \\ \text{mutually disjoint}}} \\
&\quad \prod_{i=1}^{n} \left[\prod_{(\vec{n},\vec{m})\in\zeta_i} \{\exp(-\beta J_{\vec{m}-\vec{n}} s_{\vec{m}} s_{\vec{n}}) - 1\} \right],
\end{aligned} \tag{3.12.5}$$

and the $\prod_{\vec{\ell}\in\Lambda} \int d\sigma(s_{\vec{\ell}})$-integral of the product over i is the product over i of the $\prod_{\vec{\ell}\in \mathrm{supp}\ \zeta_i} \int d\sigma(s_{\vec{\ell}})$-integrals. This yields the polymer expansion of

$$\widehat{Z}_\Lambda = Z(h_{\vec{\ell}}\colon \vec{\ell}\in\Lambda) \prod_{\vec{\ell}\in\Lambda} Z_{\{\vec{\ell}\}}(h_{\vec{\ell}})^{-1}, \tag{3.12.6}$$

$$Z(h_{\vec{\ell}}\colon \vec{\ell}\in\Lambda) = \int \exp\left(\sum_{\vec{\ell}\in\Lambda} h_{\vec{\ell}} s_{\vec{\ell}} - \beta \sum_{\vec{n},\vec{m}\in\Lambda} J_{\vec{m}-\vec{n}} s_{\vec{m}} s_{\vec{n}}\right) \times \prod_{\vec{\ell}\in\Lambda} d\sigma(s_{\vec{\ell}}), \tag{3.12.7}$$

where the activity of a polymer is given by

$$z(\zeta) = \frac{1}{\prod_{\vec{\ell}\ \mathrm{supp}\ \zeta} Z_{\{\vec{\ell}\}}(h_{\vec{\ell}})} \int \prod_{(\vec{n},\vec{m})\in\zeta} \{\exp(-\beta J_{\vec{m}-\vec{n}} s_{\vec{n}} s_{\vec{m}}) - 1\} \times \exp\left(\sum_{\vec{\ell}\in\mathrm{supp}\ \zeta} h_{\vec{\ell}} s_{\vec{\ell}}\right) \prod_{\vec{\ell}\in\mathrm{supp}\ \zeta} d\sigma(s_{\vec{\ell}}). \tag{3.12.8}$$

Clearly, the preliminary bound on the activity follows from the basic estimate

$$\begin{aligned} |\exp(-\beta J_{\vec{m}-\vec{n}} s_{\vec{n}} s_{\vec{m}}) - 1| &\le \beta |J_{\vec{m}-\vec{n}} s_{\vec{m}} s_{\vec{n}}| \int_0^1 dt\, \exp(-t\beta J_{\vec{m}-\vec{n}} s_{\vec{n}} s_{\vec{m}}) \\ &\le \beta |s_{\vec{n}} s_{\vec{m}} J_{\vec{m}-\vec{n}}| \exp\left\{\frac{1}{2}\beta |J_{\vec{m}-\vec{n}}| (s_{\vec{n}}^2 + s_{\vec{m}}^2)\right\} \end{aligned} \tag{3.12.9}$$

and the summability

$$\sum_{\vec{n}} |J_{\vec{n}}| < \infty, \tag{3.12.10}$$

which is an obvious consequence of (3.12.2). We also apply the Schwarz inequality as follows:

$$\begin{aligned} &\int \left(\prod_{(\vec{n},\vec{m})\in\zeta} |s_{\vec{n}} s_{\vec{m}}|\right) \exp\left\{\frac{1}{2}\beta \sum_{(\vec{n},\vec{m})\in\zeta} |J_{\vec{m}-\vec{n}}| (s_{\vec{n}}^2 + s_{\vec{m}}^2)\right\} \\ &\quad \times \exp\left(\sum_{\vec{\ell}\in\mathrm{supp}\ \zeta} h_{\vec{\ell}} s_{\vec{\ell}}\right) \prod_{\vec{\ell}\in\ \mathrm{supp}\ \zeta} d\sigma(s_{\vec{\ell}}) \\ &\le \left(\int \prod_{(\vec{n},\vec{m})\in\zeta} (s_{\vec{n}}^2 s_{\vec{m}}^2) \prod_{\vec{\ell}\in\ \mathrm{supp}\ \zeta} d\sigma(s_{\vec{\ell}})\right)^{1/2} \end{aligned}$$

$$\times \left(\int \exp \left\{ \beta \sum_{(\vec{n},\vec{m})\in\zeta} |J_{\vec{m}-\vec{n}}|(s^2_{\vec{n}} + s^2_{\vec{m}}) \right\} \right.$$

$$\left. \times \exp \left(2 \sum_{\vec{\ell}\in \operatorname{supp}\zeta} h_{\vec{\ell}} s_{\vec{\ell}} \right) \prod_{\vec{\ell}\in \operatorname{supp}\zeta} d\sigma(s_{\vec{\ell}}) \right)^{1/2} \tag{3.12.11}$$

By (3.12.4) we have

$$\int \prod_{(\vec{n},\vec{m})\in\zeta} (s^2_{\vec{n}} s^2_{\vec{m}}) \prod_{\vec{\ell}\in \operatorname{supp}\zeta} d\sigma(s_{\vec{\ell}}) \le e^{c|\zeta|} \hat{\tau}^{-2 \operatorname{card}\zeta} \prod_{\vec{\ell}\in \operatorname{supp}\zeta} N_\zeta(\vec{\ell})!, \tag{3.12.12}$$

where $N_\zeta(\vec{\ell})$ denotes the number of line segments in ζ that meet $\vec{\ell}$. The other $\prod_{\vec{\ell}\in\operatorname{supp}\zeta} \int d\sigma(s_{\vec{\ell}})$-integral is bounded by

$$e^{c|\zeta|} \hat{\tau}(\beta)^{-|\zeta|} \exp \left(-\hat{\tau}(\beta)^{-1} \sum_{\vec{\ell}\in\operatorname{supp}\zeta} h^2_{\vec{\ell}} \right)$$

for sufficiently small β, where

$$\hat{\tau}(\beta) = \hat{\tau} - \beta \sum_{\vec{n}} |J_{\vec{n}}|. \tag{3.12.13}$$

If we combine all these estimates with the Jensen inequality

$$Z_{\{\vec{\ell}\}}(h_{\vec{\ell}}) \ge Z_{\{\vec{\ell}\}} \exp \left(-h_{\vec{\ell}} Z^{-1}_{\{\vec{\ell}\}} \int_{-\infty}^{\infty} z e^{-\beta J_0 z^2} d\sigma(z) \right), \tag{3.12.14}$$

we obtain the following bound on the activity:

$$|z(\zeta)| \le c^{|\zeta|} \beta^{\operatorname{card}\zeta} \hat{\tau}^{-\operatorname{card}\zeta} \hat{\tau}(\beta)^{-\frac{1}{2}|\zeta|} \prod_{\ell\in\operatorname{supp}\zeta} (N_\zeta(\vec{\ell})!)^{1/2} \prod_{(\vec{n},\vec{m})\in\zeta} |J_{\vec{m}-\vec{n}}|$$

$$\times \exp \sum_{\vec{\ell}\in\operatorname{supp}\zeta} \left(c(\beta) h_{\vec{\ell}} + \frac{1}{2} \hat{\tau}(\beta)^{-1} h^2_{\vec{\ell}} \right). \tag{3.12.15}$$

Before we try to sum over polymers, we must find something in this product bound against which to cancel the factorial growths. Since powers of small numbers cannot accomplish this, the product of varying coupling strengths is the only candidate. Accordingly, one writes

$$\prod_{(\vec{n},\vec{m})\in\zeta} |J_{\vec{m}-\vec{n}}| = \prod_{\vec{\ell}\in\operatorname{supp}\zeta} \left(\prod_{\substack{(\vec{n},\vec{m})\in\zeta \\ \vec{\ell}=\vec{n} \text{ or } \vec{\ell}=\vec{m}}} |J_{\vec{m}-\vec{n}}|^{1/2} \right) \tag{3.12.16}$$

and observes that for $\alpha_1, \ldots, \alpha_N > 0$,

$$\ln\left(\prod_{j=1}^{N} \alpha_j\right) = \sum_{j=1}^{N} \ln \alpha_j \leq N \ln\left(\frac{1}{N}\sum_{j=1}^{N} \alpha_j\right) \tag{3.12.17}$$

by log concavity. This means

$$\prod_{j=1}^{N} \alpha_j \leq \frac{1}{N^N}\left(\sum_{j=1}^{N} \alpha_j\right)^N \leq \frac{1}{N!}\left(\sum_{j=1}^{N} \alpha_j\right)^N, \tag{3.12.18}$$

and in our context we get

$$\prod_{\substack{(\vec{n},\vec{m})\in\zeta \\ \vec{\ell}=\vec{n} \text{ or } \vec{\ell}=\vec{m}}} |J_{\vec{m}-\vec{n}}|^{1/6} \leq \frac{1}{(N_\zeta(\vec{\ell})!)^{1/2}} \left(\sum_{\substack{(\vec{n},\vec{m})\in\zeta \\ \vec{\ell}=\vec{n} \text{ or } \vec{\ell}=\vec{m}}} |J_{\vec{m}-\vec{n}}|^{1/3}\right)^{1/2}. \tag{3.12.19}$$

The point is that (3.12.2) bounds the sum with a universal constant. Thus

$$\begin{aligned} |z(\zeta)| \quad &\leq \quad c^{|\zeta|}\beta^{\text{ card } \zeta}\hat{\tau}^{-\text{card } \zeta}\hat{\tau}(\beta)^{-\frac{1}{2}|\zeta|} \prod_{(\vec{n},\vec{m})\in\zeta} |J_{\vec{m}-\vec{n}}|^{2/3} \\ &\quad \times \exp \sum_{\vec{\ell}\in \text{supp } \zeta} \left(c(\beta)h_{\vec{\ell}} + \frac{1}{2}\hat{\tau}(\beta)^{-1}h_{\vec{\ell}}^2\right), \end{aligned} \tag{3.12.20}$$

and we are ready to sum over polymers.

The estimation is now done exactly as in §3.9, because the bound no longer differs in any essential way from that which occurred in the $\mathbb{S}^2$-valued spin case. To be sure, a fractional power of each $|J_{\vec{n}'-\vec{n}}|$ has been burned up, and the Königsberg bridge-island estimation will cut the remaining power by half. This is the reason why the assumption (3.12.2) is convenient.

Once again, there are ways to refine the expansion technique. Although estimation such as (3.12.19) is important to a phase cell cluster expansion, we show in the next section that this consumption of small factors is unnecessary for the class of unbounded scalar spin systems considered here.

References

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3.13 Extra $1/N!$ Factors in the Inductive Expansion

We now replace the wholesale expansion of the previous section with the inductive expansion already applied to the $\mathbb{S}^2$-valued spin case in §3.10. Compared with the inductive expansion in that case, the expansion rules have not changed, but the estimation on the polymer activity has changed, since there are factorial growth bounds on single-spin expectations in this case. The point to be made in this section is that the combinatorial properties of the interpolation weights enables us to cancel the factorial growths without using any of the long-distance decay of the interaction! Thus we can replace the long-range requirement

$$\sum_{\vec{n}} |J_{\vec{n}}|^{1/3} < \infty \tag{3.13.1}$$

of the less refined expansion and estimation with the weaker requirement

$$J \equiv \sum_{\vec{n}} |J_{\vec{n}}| < \infty. \tag{3.13.2}$$

The challenge in this section is to find that alternative cancellation of the factorial growths.

The inductive interpolation described in §3.10 generates the desired polymer expansion of a generalized partition function:

$$\widehat{Z}(h_{\vec{\ell}}\colon\ \vec{\ell} \in \Lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(\mathcal{U}_1,\dots,\mathcal{U}_n)\in\mathcal{D}_n} \prod_{i=1}^{n} z(\mathcal{U}_i) \tag{3.13.3}$$

where the polymers are unordered connectivity graphs. The activity is given by

$$\begin{aligned} z(\mathcal{U}) = {} & \prod_{\vec{\ell}\in \text{ supp } \mathcal{U}} Z_{\{\vec{\ell}\}}(h_{\vec{\ell}})^{-1} \sum_{\substack{\text{o.c.g. } \mathfrak{G}\colon \mathcal{U}_{\mathfrak{G}} = \mathcal{U} \\ g_0(2) = \text{ first site in supp } \mathcal{U}}} \\ & \left(\prod_i \int_0^1 dt_i\right) f_{\mathfrak{G}}(t) \prod_{(\vec{n},\vec{m})\in\mathcal{U}} (-\beta J_{\vec{m}-\vec{n}}) \int \prod_{(\vec{n},\vec{m})\in\ \mathcal{U}} (s_{\vec{n}} s_{\vec{m}}) \\ & \times \exp\left(\sum_{\vec{\ell}\in \text{ supp } \mathcal{U}} h_{\vec{\ell}} s_{\vec{\ell}} - \beta H^{\mathfrak{G}}_{\text{supp } \mathcal{U}}(t)\right) \prod_{\vec{\ell}\in \text{ supp } \mathcal{U}} d\sigma(s_{\vec{\ell}}), \end{aligned} \tag{3.13.4}$$

where "first site" refers to the fixed linear ordering chosen for Λ, and $H^{\mathfrak{G}}_{\text{supp } \mathcal{U}}(t)$ is the interpolated Hamiltonian for the history of attempted decouplings associated with $\mathfrak{G}$. It is easy to establish by induction that it has the convex form:

$$H^{\mathfrak{G}}_{\text{supp } \mathcal{U}}(t) = \sum_{\mathcal{L} \text{ partitions supp } \mathcal{U}} \omega^{\mathfrak{G}}_{\mathcal{L}}(t) \sum_{\Lambda'\in\mathcal{L}} H_{\Lambda'}, \tag{3.13.5}$$

$$\sum_{\mathcal{L} \text{ partitions supp } \mathcal{U}} \omega^{\mathfrak{G}}_{\mathcal{L}}(t) = 1, \qquad \omega^{\mathfrak{G}}_{\mathcal{L}}(t) \geq 0. \tag{3.13.6}$$

On the other hand, (3.13.2) implies

$$H_{\Lambda'} \geq -J \sum_{\vec{\ell} \in \Lambda'} s_{\vec{\ell}}^2 \tag{3.13.7}$$

for an arbitrary restriction of the Hamiltonian. Thus

$$H^{\mathfrak{G}}_{\text{supp } \mathcal{U}}(t) \geq -J \sum_{\vec{\ell} \in \text{supp } \mathcal{U}} s_{\vec{\ell}}^2, \tag{3.13.8}$$

and so we have the activity bound:

$$\begin{aligned} |z(\mathcal{U})| \;\leq\; & \prod_{\vec{\ell} \in \text{ supp } \mathcal{U}} Z_{\{\vec{\ell}\}}(h_{\vec{\ell}})^{-1} \beta^{\text{card } \mathcal{U}} \prod_{(\vec{n},\vec{m}) \in \mathcal{U}} |J_{\vec{m}-\vec{n}}| \\ & \times \prod_{\vec{\ell} \in \text{ supp } \mathcal{U}} \int_{-\infty}^{\infty} |z|^{N_{\mathcal{U}}(\vec{\ell})} e^{h_{\vec{\ell}} z + 2\beta J z^2} d\sigma(z), \end{aligned} \tag{3.13.9}$$

where we have also used the u.c.g. identity (3.8.22). $N_{\mathcal{U}}(\vec{\ell})$ denotes the number of $(\vec{n},\vec{m}) \in \mathcal{U}$ meeting $\vec{\ell}$ – as in the previous section – and (3.12.4) implies

$$\begin{aligned} & \int_{-\infty}^{\infty} |z|^{N_{\mathcal{U}}(\vec{\ell})} e^{h_{\vec{\ell}} z + \beta c z^2} d\sigma(z) \\ \leq\; & \left(\int_{-\infty}^{\infty} z^{2N_{\mathcal{U}}(\vec{\ell})} d\sigma(z) \right)^{1/2} \left(\int_{-\infty}^{\infty} e^{2h_{\vec{\ell}} z + 2\beta J z^2} d\sigma(z) \right)^{1/2} \\ \leq\; & (\hat{\tau} - 2\beta J)^{-1/4} c\hat{\tau}^{-\frac{1}{2} N_{\mathcal{U}}(\vec{\ell})} \exp^{\left(\frac{1}{2}(\hat{\tau} - 2\beta J)^{-1} h_{\vec{\ell}}^2\right)} (N_{\mathcal{U}}(\vec{\ell})!)^{1/2} \end{aligned} \tag{3.13.10}$$

for small β. Combining this with (3.12.14), we reduce (3.13.9) to:

$$\begin{aligned} |z(\mathcal{U})| \;\leq\; & c^{\,|\mathcal{U}|} \beta^{\text{ card } \mathcal{U}} \hat{\tau}^{-\text{card } \mathcal{U}} \prod_{\vec{\ell} \in \text{ supp } \mathcal{U}} (N_{\mathcal{U}}(\vec{\ell})!)^{1/2} \prod_{(\vec{n},\vec{m}) \in \mathcal{U}} |J_{\vec{m}-\vec{n}}| \\ & \times \exp \sum_{\vec{\ell} \in \text{ supp } \mathcal{U}} \left(c(\beta) h_{\vec{\ell}} + \frac{1}{2} (\hat{\tau} - 2\beta J)^{-1} h_{\vec{\ell}}^2 \right), \end{aligned} \tag{3.13.11}$$

which is certainly more refined than (3.12.15).

Following the general polymer estimation scheme, we seek to reduce the obvious bound (with card $\mathcal{U} \geq |\mathcal{U}| - 1$)

$$\begin{aligned} & \sum_{\substack{\text{u.c.g. } \mathcal{U}:\ \vec{n}^0 \in \text{ supp } \mathcal{U} \\ |\mathcal{U}| = M}} |z(\mathcal{U})| \\ \leq\; & c^M (\hat{\tau}^{-1}\beta)^{M-1} \sum_{\substack{\text{u.c.g. } \mathcal{U}:\ \vec{n}^0 \in \text{ supp } \mathcal{U} \\ |\mathcal{U}| = M}} \prod_{(\vec{n},\vec{m}) \in \mathcal{U}} |J_{\vec{m}-\vec{n}}| \end{aligned}$$

$$\times \prod_{\vec{\ell} \in \text{ supp } \mathcal{U}} ((N_{\mathcal{U}}(\vec{\ell})!)^{1/2} X(\beta, h_{\vec{\ell}})), \tag{3.13.12}$$

$$X(\beta, h) = \exp\left(c(\beta)h + \frac{1}{2}(\hat{\tau} - 2\beta J)^{-1} h^2 \right), \tag{3.13.13}$$

to a bound of the form (3.9.10). We cannot throw away the combinatorial advantage of summing over unordered connectivity graphs if we wish to control these factorial growths without burning up fractional powers of the factors $|J_{\vec{m}-\vec{n}}|$. The point is that we were indeed throwing away that advantage in §3.10. To see this, we write the sum in terms of tree graphs on $\{1, \dots, M\}$ with line segments directed by choosing 1 as the root. First, we have the summation identity

$$\sum_{\substack{\text{u.c.g. } \mathcal{U}\text{: } \vec{n}^0 \in \text{ supp } \mathcal{U} \\ |\mathcal{U}| = M}} = M \sum_{\substack{\text{u.c.g. } \mathcal{U}\text{: } |\mathcal{U}| = M \\ \vec{n}^0 = \text{ root of } \mathcal{U}}} \tag{3.13.14}$$

because the summand is symmetric in $(\vec{n}, \vec{m})$ and the root only orders the individual pairs. Second,

$$\sum_{\substack{\text{u.c.g. } \mathcal{U}\text{: } |\mathcal{U}| = M \\ \vec{n}^0 = \text{ root of } \mathcal{U}}} = \sum_{\substack{\text{tree graphs } T \text{ on } \{1,\dots,M\} \\ \text{rooted at } 1}} \frac{1}{(M-1)!} \sum_{\substack{Q\text{: } \{1,\dots,M\} \to \Lambda \\ \text{injective, } Q(1) = \vec{n}^0}}, \tag{3.13.15}$$

where the summand is altered as follows:

$$\prod_{(\vec{n},\vec{m}) \in \mathcal{U}} |J_{\vec{m}-\vec{n}}| \longmapsto \prod_{(i,j) \in T} |J_{Q(j)-Q(i)}|,$$

$$\prod_{\ell \in \text{supp } \mathcal{U}} X(\beta, h_{\vec{\ell}}) \longmapsto \prod_{i=1}^{M} X(\beta, h_{Q(i)}),$$

$$\prod_{\vec{\ell} \in \text{ supp } \mathcal{U}} (N_{\mathcal{U}}(\vec{\ell}))^{1/2} \longmapsto \prod_{i=1}^{M} (d_i(T)!)^{1/2}.$$

The $(M-1)!$ in the denominator compensates for the over-counting of subsets of Λ due to the permutation of $\{2, \dots, M\}$ corresponding to different injections with the same range. Hence (with $|h_{\vec{\ell}}| \le c$)

$$\begin{aligned} & \sum_{\substack{\text{u.c.g. } \mathcal{U}\text{: } \vec{n}^0 \in \text{ supp } \mathcal{U} \\ |\mathcal{U}| = M}} |z(\mathcal{U})| \\ \le\ & \frac{c^M (\kappa^{-1}\beta)^{M-1} M}{(M-1)!} \sum_{\substack{\text{tree graphs } T \text{ on } \{1,\dots,M\} \\ \text{rooted at } 1}} \prod_{i=1}^{M} (d_i(T)!)^{1/2} \\ & \times \sum_{\substack{Q\text{: } \{1,\dots,M\} \to \Lambda \\ Q(1) = \vec{n}^0}} \prod_{(i,j) \in T} |J_{Q(j)-Q(i)}|, \end{aligned} \tag{3.13.16}$$

where we have also over-estimated by dropping the requirement that Q be injective. For fixed T, the sum over Q is just a multi-index summation, with the iterative application of

$$\sum_{Q(i)\in\Lambda} |J_{Q(j)-Q(i)}| \le J \tag{3.13.17}$$

indicated by the structure of T. At each stage, only an index that occurs in only one of the remaining factors is summed over. We described this in the context of abstract polymer expansions in §3.6. Thus

$$\sum_{\substack{\text{u.c.g. } \mathcal{U}:\ \vec{n}^0\in \text{ supp } \mathcal{U} \\ |\mathcal{U}|=M}} |z(\mathcal{U})| \le \frac{c^M(\kappa^{-1}\beta)^{M-1}M}{(M-1)!}$$

$$\times \sum_{\substack{\text{tree graphs } T \text{ on } \{1,\dots,M\} \\ \text{rooted at } 1}} \prod_{i=1}^{M} (d_i(T)!)^{1/2}, \tag{3.13.18}$$

so we may now apply Cayley's Theorem, which states that the number of tree graphs on $\{1,\dots,M\}$ with specified root and coordination numbers $d_1,\dots,d_M$ is just

$$(M-2)!/\prod_{i=1}^{M}(d_i-1)!$$

The sum over T is bounded by

$$(M-2)! \sum_{\substack{d_1,\dots,d_M>0 \\ \sum_i d_i=2M-1}} \prod_{i=1}^{M} (d_i!)^{-1/2} d_i \le 2^{M/2}(M-2)!$$

$$\times \text{card}\left\{(d_1,\dots,d_M):\ d_1,\dots,d_M>0 \text{ and } \sum_{i=1}^{M} d_i = 2M-1\right\}, \tag{3.13.19}$$

but the cardinality of the set is known to be bounded by 4^M, so we finally have the desired bound for convergence of the polymer expansion at high temperature (small β).

The reciprocal factorials were essentially hidden in the summation over unordered connectivity graphs, which, in turn, arose from the u.c.g. identity (3.8.32). That combinatorial identity exploits the smallness of the interpolation weights to a degree comparable to the estimate (3.11.26) rather than (3.11.23). One can derive the required polymer estimate by using ordered tree graphs instead of the polymers (u.c.g.) themselves. To this end, we write

$$\sum_{\substack{\text{u.c.g. } \mathcal{U}:\ |\mathcal{U}|=M \\ \vec{n}^0=\text{ root of } \mathcal{U}}} |z(\mathcal{U})| \le c^M(\hat{\tau}^{-1}\beta)^{M-1} \sum_{\substack{\text{o.c.g. } \mathfrak{G}:\ \deg \mathfrak{G}=M \\ g_0(2)=\vec{n}^0}}$$

$$\prod_{i=2}^{M} |J_{g_1(i)-g_0(i)}| \left(\prod_{i=1}^{M}\int_0^1 dt_i\right) f_{\eta_{\mathfrak{G}}}(t)$$

$$\times \prod_{j=1}^{M-1} ((d_{\eta_{\mathfrak{S}}}(j)+1)!)^{1/2}, \tag{3.13.20}$$

where the previously defined coordination numbers are related by

$$N_{\mathcal{U}_{\mathfrak{S}}}(g_1(j)) = d_{\eta_{\mathfrak{S}}}(j) + 1. \tag{3.13.21}$$

Thus

$$\sum_{\substack{\text{u.c.g. } \mathcal{U}:\ |\mathcal{U}|=M \\ \vec{n}^0 = \text{ root of } \mathcal{U}}} |z(\mathcal{U})| \le c^M (\hat{\tau}^{-1}\beta)^{M-1} \sum_{M\text{th-degree } \eta} \left(\prod_{i=1}^{M} \int_0^1 dt_i \right) f_\eta(t)$$

$$\times \prod_{j=1}^{M-1} ((d_\eta(j)+1)!)^{1/2} \sum_{\substack{\text{o.c.g. } \mathfrak{S}:\ \eta_{\mathfrak{S}}=\eta \\ g_0(2)=\vec{n}^0}} \prod_{i=1}^{M} |J_{g_1(i)-g_0(i)}|, \tag{3.13.22}$$

and we estimate the inner most sum by the iterative application of (3.13.2) dictated by the ordered tree graph η, which specifies which site variable $\vec{\ell}_j$ α given site variable $\vec{\ell}_i$ is "connected back to." This reduces the bound to:

$$\sum_{\substack{\text{u.c.g. } \mathcal{U}:\ |\mathcal{U}|=M \\ \vec{n}^0 = \text{ root of } \mathcal{U}}} |z(\mathcal{U})| \le c^M (\hat{\tau}^{-1}\beta)^{M-1} \sum_{M\text{th-degree } \eta} \left(\prod_{i=1}^{M} \int_0^1 dt_i \right) f_\eta(t)$$

$$\times \prod_{j=1}^{M-1} ((d_\eta(j)+1)!)^{1/2}, \tag{3.13.23}$$

and this is the point where (3.11.26) may be directly applied. The more traditional but weaker estimate (3.11.23) would require the use of some of the long-distance decay of the interaction to cancel the factorial growths in the manner described in the previous section. This, in turn, would require the condition (3.9.2) on the interaction, in spite of our use of inductive interpolation.

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