Matrix Analysis: Review of linear algebra

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1. Review of linear algebra
   - Some structures
   - Linear independence and basis
   - Matrices
   - Eigenvalues and eigenvectors

Outline

Definition 1

*A **group** is a set G together with an operation $\odot$ such that*

- *G is close under $\odot$: for all $a, b \in G, a \odot b \in G$,*
- *$\odot$ is associative: for all $a, b, c \in G$, $(a \odot b) \odot c = a \odot (b \odot c)$,*
- *G contains an identity element e for $\odot$: for all $a \in G, a \odot e = e \odot a = a$,*
- *G is close by inversion: for all $a \in G$, there exists a $b \in G$ such that $a \odot b = b \odot a = e$. (usually written $-a$ or a^{-1}).*

*If moreover $\odot$ is commutative in G, i.e. for all $a, b \in G$, $a \odot b = b \odot a$, we say that $(G, \odot)$ is **abelian group**.*

Example 1

Show whether the following sets are groups or not. Are they abelian groups?

- $C(\mathbb{R}, \mathbb{R})$ the set of continuous functions on $\mathbb{R}$, together with the usual addition: $f + g$ is the function defined on $\mathbb{R}$ such that $(f + g)(x) = f(x) + g(x)$.
- It is also a *multiplicative* group?
- What if we use the composition?
- For a given $N \geq 2$, let $\mathcal{G}_N := \{\omega \in \mathbb{C} : \omega^N = 1\}$. Is it a multiplicative group with the usual scalar multiplication?

Definition 2

*A **field** is a set G with two operations $\oplus$ (usually called the addition) and $\otimes$ (the multiplication) such that*

- $(G, \oplus)$ *is an abelian group with (additive) identity 0_G,*
- $(G \backslash \{0_G\}, \otimes)$ *is an abelian group with (multiplicative) identity 1_G,*
- *the multiplication is distributive over the addition: for all $a, b, c \in G$, $a \otimes (b \oplus c) = (a \oplus b) \otimes (a \oplus c)$.*

Definition 3

A **vector space** *over a field* $\mathbb{F}$ *(with operations* $\oplus_F$ *and* $\otimes_F$ *and respective identitites* $0_F, 1_F$*) is a set of vectors* V *together with two operations* $\oplus_V$ *(vector addition) and* $\odot_S$ *(the scalar multiplication) such that*

1. $(V, \oplus_V)$ *is an abelian group, with the zero vector* 0_V*,*
2. *for all* $\mathbf{v} \in V$*,* $1_F \odot_S \mathbf{v} = \mathbf{v}$
3. *the scalar multiplication is distributive: for all* $\mathbf{u}, \mathbf{v} \in V$*, for all* $\alpha \in \mathbb{F}, \alpha \odot_S (\mathbf{u} \oplus_V \mathbf{v}) = \alpha \odot_S \mathbf{u} \oplus_V \alpha \odot_S \mathbf{v}$*,*
4. *the scalar multiplication is compatible: for all* $\alpha, \beta \in \mathbb{F}$*, for all* $\mathbf{v} \in V$*,* $\alpha \odot_S (\beta \odot_S \mathbf{v}) = (\alpha \otimes_F \beta) \odot_S \mathbf{v}$*,*
5. *Distributivity of scalar multiplication of the additive field: for all* $\alpha, \beta \in \mathbb{F}$*, and for all* $\mathbf{v} \in V$*,* $(\alpha \oplus_F \beta) \odot_S \mathbf{v} = \alpha \odot_S \mathbf{v} \oplus_V \beta \odot_S \mathbf{v}$*.*

Example 2

- Classical vectors $\mathbb{R}^n$, $\mathbb{C}^n$
- $\mathbb{R}_n[x] := \{f(x) = a_0 + a_1 x + \cdots + a_n x^n ; (a_0, \cdots, a_n) \in \mathbb{R}^{n+1}\}$
- $\mathbb{R}[x]$?
- $\{(x, y, z)^T : ax + by + cz = 0\}$
- $\{(x, y, z)^T : ax + by + cz = 1\}$

Definition 4

A subset $W \subseteq V$ is a subspace of V if

1. $0_V \in W$
2. *for all* $\mathbf{u}, \mathbf{v} \in W$, $\mathbf{u} + \mathbf{v} \in W$
3. *for all* $\mathbf{v} \in W$ *and* $\alpha \in \mathbb{F}$, $\alpha\mathbf{v} \in W$.

Exercise 1

Let U be a vector space and $V, W \subset U$ two subspaces. Are the following sets subspaces of U?

1. $V \cap W := \{\mathbf{u} : \mathbf{u} \in V \text{ and } \mathbf{u} \in W\}$
2. $V \cup W := \{\mathbf{u} : \mathbf{u} \in V \text{ or } \mathbf{u} \in W\}$
3. $V + W := \{\mathbf{u} : \exists \mathbf{v} \in V, \mathbf{w} \in W : \mathbf{u} = \mathbf{v} + \mathbf{w}\}$

Definition 5

Let $V \subset U$ be a subset of U (not necessarily a subspace). We define its **span** *has the intersection of all subsets of U which contain V. We write $W = \mathrm{span}(V)$. W is a subspace of U (verify this).*

Proposition 1

Let $V \subset U$. $\text{span}(V) = \{\sum_{k=1}^{n} \alpha_k \mathbf{v}_k, k = 1, \cdots\}$.

Exercise 2

Let $\mathbf{u}$ and $\mathbf{v}$ be two linearly independent vectors. Show that $\mathrm{span}\{\mathbf{u}, \mathbf{v}, \mathbf{u}+\mathbf{v}\} = \mathrm{span}\{\mathbf{u}, \mathbf{v}\} = \mathrm{span}\{\mathbf{u}, \mathbf{u}+\mathbf{v}\}$.

Definition 6

Let V be a vector space and $\mathcal{F} = (\mathbf{v}_1, \cdots, \mathbf{v}_n)$ be a family of n vectors in V. We say that the family $\mathcal{F}$ is a **linearly independent** *set of vectors if*

$$\sum_{i=1}^{n} \alpha_i \mathbf{v}_i = 0 \Leftrightarrow \alpha_1 = \cdots = \alpha_n = 0.$$

A family which is not linearly independent is said to be a linearly dependent.

Exercise 3

Write down the definition of what it means to be linearly dependent.

Example 3

- $((1,0),(0,1))$
- $((1,0),(1,1))$
- $((1,0),(0,1),(1,1))$
- $\big((x \mapsto \cos(x)),(x \mapsto \cos(2x)),(x \mapsto \cos^2(x))\big)$

Exercise 4

Consider $V = \mathbb{R}_n[x]$. Are the following families linearly dependent?

- $(1, x, \cdots, x^n)$
- $(1, 1+x, 1+x+x^2, \cdots, 1+x+\cdots+x^{n-1}+x^n)$
- $(1, 1+x, 1+x^2, \cdots, 1+x^n)$
- $(1+x, x+x^2, x^2+x^3, \cdots, x^{n-1}+x^n, x^n+1)$

Definition 7

A family $\mathcal{F} = (\mathbf{v}_1, \cdots, \mathbf{v}_n) \subset V$ *is a* **generating family** *or* **spanning set** *if for all* $\mathbf{v} \in V$*, there exists scalars* $\alpha_1, \cdots, \alpha_n \in \mathbb{F}$ *such that* $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$.

Definition 8

A family $\mathcal{F}$ of vectors is a basis if it is a linearly independent spanning set.

Exercise 5

Are the following families generating? Linearly independent? Basis?

- $(1, x, \cdots, x^n)$
- $(1, 1+x, 1+x+x^2, \cdots, 1+x+\cdots+x^{n-1}+x^n)$
- $(1, 1+x, 1+x^2, \cdots, 1+x^n)$
- $(1+x, x+x^2, x^2+x^3, \cdots, x^{n-1}+x^n, x^n+1)$

Theorem 1

Let V be a vector space and $\mathcal{F} = \{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$ be a basis for V. Then for all $\mathbf{v} \in V$, there exists unique scalars $\alpha_1, \cdots, \alpha_n \in \mathbb{F}$ such that

$$\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i.$$

This unique representation gives rise to the notion of **coordinates** *of a vector with respect to a certain basis.*

Theorem 2

Let V be a vector space and B and C two basis. Then B and C have the same number of vectors.

Definition 9

The **dimension** *of a vector space is the number of vectors in any of its basis. We write* $\dim(V) = n$*. A vector space can be*

- *Finite dimensional if* $\dim(V) < \infty$*, or*
- *Infinite dimensional if* $\dim(V) = \infty$*.*

Exercise 6

What is the dimension of the following vector spaces:

- $\mathbb{R}_n[x]$
- $\mathbb{R}[x]$
- $\mathbb{R}^n$
- $\mathbb{C}^n$

Theorem 3

Let V be a finite dimensional vector space with $\dim(V) = n < \infty$ and let $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$. The following statements are equivalent:

1. *S is a basis for V.*
2. *S is a spanning set.*
3. *S is linearly independent.*

Definition 10

Let U and V be two vector spaces over the same field $\mathbb{F}$. A map $f : U \to V$ is said to be a **linear map** *if*

- *for all $\mathbf{u}, \mathbf{v} \in U$, $f(\mathbf{u} +_U \mathbf{v}) = f(\mathbf{u}) +_V f(\mathbf{v})$,*
- *for all $\alpha \in \mathbb{F}$ and $\mathbf{u} \in U$, $f(\alpha\mathbf{u}) = \alpha f(\mathbf{u})$.*

Example 4

- $x \mapsto 2x, \alpha x$
- For a given vector $\mathbf{a} \in \mathbb{K}^n$, the map $T_{\mathbf{a}} : \mathbb{K}^n \to \mathbb{K}, \mathbf{x} \mapsto \mathbf{a}^T \mathbf{x} = \sum a_i x_i$ is linear.

Exercise 7

Let $C^1(\mathbb{R})$ be the set of continuously differentiable functions. Verify that $T : C^1 \to C^0, f \mapsto f'$ is a linear map.

Exercise 8

Prove that for any vector spaces V, W and any linear map $f : V \to W$, $f(0) = 0$.

Definition 11

A **matrix** *is a table of numbers. We denote the set of matrices of size m times n over the field $\mathbb{F}$ as $\mathbb{F}^{m\times n}$.*

Proposition 2

Let V and W be two finite dimensional vectors spaces with $\dim(U) = n$ and $\dim(V) = m$ and let $f : V \to W$ be a linear map. Let $S = (\mathbf{v}_1, \cdots, \mathbf{v}_n)$ be a basis for V. Then f is completely determined by the values of $f(\mathbf{v}_i)$.

Exercise 9

Let $f : U = \mathbb{R}_3[x] \to V = \mathbb{R}_3[x]$ be defined as the differentiation operator. Compute the matrices associated to f given the following basis

- $U = \mathrm{span}(1, x, x^2, x^3)$ and $V = \mathrm{span}(1, x, x^2, x^3)$.
- $U = \mathrm{span}(1, x, x^2, x^3)$ and $V = \mathrm{span}(1, 1 + x, 1 + x^2, 1 + x^3)$.
- $U = \mathrm{span}(1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3)$ and $V = \mathrm{span}(1, 1 + x, 1 + x + x^2, 1 + x + x^2 + x^3)$.

Definition 12

Let V and W be two vector spaces and $\phi : V \to W$ a linear transformation.
The **range** *or* **image** *of ϕ is the subspace*
$R(\phi) = Im(\phi) = \{\mathbf{w} \in W : \exists \mathbf{v} \in V$ *with* $\mathbf{w} = \phi(\mathbf{v})\} \subset W$.

Definition 13

Let V and W be two vector spaces and $\phi : V \to W$ a linear transformation. The **nullspace** *or* **kernel** *of ϕ is the subspace*
$N(\phi) = Ker(\phi) = \phi^{-1}(0) = \{\mathbf{v} \in V : \phi(\mathbf{v}) = 0\} \subset V$.

Exercise 10

Prove that the range and kernel of a linear mapping are indeed subspaces.

Exercise 11

Let $f : V \to W$, $S = (\mathbf{v}_1, \mathbf{v}_k)$ and $T = (f(\mathbf{v}_i))_i$. What can be said about T if

- S is a spanning set?
- S is linearly dependent?
- S is linearly independent?
- S is a basis?

Definition 14

The **rank** *of a linear application is the dimension of its range:*
$rk(f) = \dim(f(V))$.

Theorem 4 (Rank-nullity theorem)

Let V and W be two vector spaces with $\dim(V) = n < \infty$ and let $f : V \to W$ be a linear map. It holds

$$\dim(ker(f)) + rk(f) = \dim(V).$$

Definition 15

Let $A \in \mathbb{F}^{m\times m}$*. Its* **trace** *is defined as the sum of its diagonal entries:*

$$tr : \begin{array}{ccc} \mathbb{F}^{m\times m} & \to & \mathbb{F} \\ A & \mapsto & tr(A) = \sum_{i=1}^{m} a_{i,i} \end{array}$$

Exercise 12

Show that the trace is linear and prove the following identity:

$$tr(AB) = tr(BA), \text{ for any } A \in \mathbb{F}^{m\times n}, B \in \mathbb{F}^{n\times m}.$$

Definition 16

The **determinant** *of a matrix is defined in one of the following ways:*

1. *It is the only function* $f : \mathbb{F}^n \times \cdots \mathbb{F}^n \to \mathbb{F}$ *that is linear with respect to each column, alternating* $f(\cdots, \mathbf{u}, \cdots, \mathbf{v}, \cdots) = -f(\cdots, \mathbf{v}, \cdots, \mathbf{u}, \cdots)$ *and normalized such that* $f(I) = 1$*.*
2. $\det(A) = \sum_{\sigma \in P_n} \operatorname{sign}(\sigma) a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$ *where* P_n *is the set of permutations of* $\{1, \cdots, n\}$ *and* $\operatorname{sign}(\sigma) = (-1)^s$ *where* s *is the number of pairwise interchanges in* σ*.*
3. $\det(A) = \sum_{j=1}^n a_{i,j} \det(A_{i,j})$ *where* $A_{i,j}$ *is the matrix obtained from* A *by deleting the row* i *and column* j*.*

Exercise 13

Prove or compute the following results:

- $\det(AB) = \det(A)\det(B)$
- Computations for 2×2 matrices and Sarrus' rule for 3×3.
- $\det(A^T) =?$
- $A adj(A) = adj(A)A = \det(A)I$, where $adj(A)_{i,j} = (-1)^{i+j} A_{j,i}$ is the adjunct or adjugate matrix.

Definition 17

A matrix A is said to be **diagonal** *if $a_{i,j} = 0$ for $i \neq j$.*

Definition 18

A matrix A is said to be **upper triangular** *if $a_{i,j} = 0$ for $i > j$.*

Definition 19

A matrix A is said to be **lower triangular** *if $a_{i,j} = 0$ for $i < j$.*

Definition 20

A matrix A is said to be **symmetric** *if $A^T = A$.*

Definition 21

A matrix A is said to be **skew-symmetric** *if $A^T = -A$.*

Definition 22

A matrix A is said to be **Hermitian** *if $A^* := \bar{A}^T = A$.*

Definition 23

A matrix A is said to be **invertible** *if there exists a matrix B such that $AB = BA = I$. We write $B = A^{-1}$.*
If it is not invertible, it is said to be **singular**.

Exercise 14

Are all sets of these particular matrices subspaces of the vector space of matrices? In case of vector subspaces, what are their dimensions and give some basis.

Exercise 15

Which kind of structure does the set of symmetric matrices have?

Exercise 16

Prove that A is invertible if and only if $\det(A) \neq 0$ and give a formula for its inverse.

Exercise 17

Let T be an upper triangular matrix. Show that $\det(T) = \prod t_{ii}$.

Proposition 3

Given a square matrix A, the following statements are equivalent

1. A *is invertible.*
2. $ker(A) = \{0\}$.
3. $R(A) = \mathbb{K}^n$.

Definition 24

We say that a matrix A is **similar** *to a matrix B and write $A \sim B$ if there exists an invertible matrix P such that $A = PBP^{-1}$.*

Exercise 18

Let f be the differential operator on the set of degree 2 polynomials. Let $S = (1, x, x^2)$ and $T = (1, 1 + x, 1 + x + x^2)$. Furthermore, let A be the representation of f in the basis S and B the matrix representing f in T. Show that $A \sim B$. What does P represent?

Definition 25

$V = S \oplus T$ is the **direct sum** *of the subspaces S and T if*

1. *$S \cap T = \{0\}$ and*
2. *$V = S + T$.*

Exercise 19

Let S be the set of symmetric matrices and T the set of skew-symmetric matrices. Show that $\mathbb{K}^{n\times n} = S \oplus T$.

Definition 26

Given a square matrix $A \in \mathbb{K}^{n \times n}$. A pair of vector and scalar $(\mathbf{x}, \lambda) \in \mathbb{K} \times \mathbb{K}^n$ is called an **eigenpair** *if*

- $\mathbf{x} \neq 0$,
- $A\mathbf{x} = \lambda \mathbf{x}$.

$\mathbf{x}$ *is called an* **eigenvector** *with* **eigenvalue** λ.
The set of all eigenvectors corresponding to an eigenvalue λ is called the **eigenspace** *corresponding to λ*

Exercise 20

Verify that the eigenspaces are indeed vector spaces.

Proposition 4

$\lambda \in \mathbb{K}$ *is an eigenvalue for* $A \in \mathbb{K}^{n\times n}$ *if and only if*

$$\det(A - \lambda I) = 0.$$

Definition 27

For a given square matrix $A \in \mathbb{K}^{n\times n}$*, its characteristic polynomial* $p_A(x)$ *is defined as*

$$p_A(x) = \det(A - xI).$$

Hence, the zeros of the characteristic polynomial corresponds to the eigenvalues!

Definition 28

The set $\sigma(A) = \{x \in \mathbb{K} : p_A(x) = 0\}$ *is called the* **spectrum** *of* A.

Exercise 21

Show that

$$p_A(x) = (-1)^n x^n + (-1)^{n-1} tr(A) + \cdots + \det(A)$$

and show that

$$tr(A) = \sum_{i=1}^{n} \lambda_i \quad \det(A) = \prod_{i=1}^{n} \lambda_i$$

where the λ_i are the n (possibly complex and repeated eigenvalues of A). Conclude that A is invertible $\Leftrightarrow 0 \notin \sigma(A)$.

Exercise 22

Let A and B be two square matrices such that $A \sim B$. It holds

$$tr(A) = tr(B)$$
$$\det(A) = \det(B).$$

Theorem 5 (Invertible matrix theorem)

Let $A \in \mathbb{K}^{n\times n}$. The following statements are equivalent

1. A *is non-singular*
2. A^{-1} *exists*
3. $rk(A) = n$
4. *the columns of* A *are linearly independent*
5. *the rows of* A *are linearly independent*
6. $\det(A) \neq 0$
7. *the dimension of the range of* A *is* n
8. *the nullity of* A *is* 0
9. $A\mathbf{x} = \mathbf{y}$ *is consistent (= admits at least one solution) for each* $\mathbf{y} \in \mathbb{K}^n$
10. *if* $A\mathbf{x} = \mathbf{y}$ *is consistent then the solution is unique*
11. $A\mathbf{x} = \mathbf{y}$ *has a unique solution for each* $\mathbf{y} \in \mathbb{K}^n$
12. *the only solution to* $A\mathbf{x} = 0$ *is* $\mathbf{x} = 0$
13. 0 *is not an eigenvalue of* A

Proposition 5

Let $\mathbf{u}$ and $\mathbf{v}$ be two eigenvectors associated to the two different eigenvalues λ and μ respectively. Then $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

Exercise 23

Show the following: there exists a non-singular matrix V and a diagonal matrix D such that $A = VDV^{-1}$ if and only if there exists n linearly independent eigenvectors $\mathbf{v}_i$ with respective eigenvalues λ_i.

Definition 29

We say that a matrix A is **diagonalizable** *if there exists a non-singular matrix P and a diagonal matrix D such that*

$$A = PDP^{-1}.$$

Definition 30

Let $p_A(x) = (-1)^n(x-\lambda_1)^{p_1}\cdots(x-\lambda_r)^{p_r}$ *with* $\sum p_i = n$*, be written in its (complex) factorized form. Then*

- p_i *is the* **algebraic multiplicity** *of the eigenvalue* λ_i
- $\dim(ker(A-\lambda_i)) = n - rk(A-\lambda_i) =: q_i$ *is the* **geometric multiplicity** *of the eigenvalue* λ_i*.*

Proposition 6

Let $A \in \mathbb{K}^{n\times n}$ and let $\lambda_1, \cdots, \lambda_r$ be r distinct eigenvalues with respective geometric multiplicities $q_1, \cdots, q_r$. Let furthermore $\mathbf{v}_i^j$ be the j^{th} eigenvector with eigenvalue λ_i, $1 \leq i \leq r$, $1 \leq j \leq q_i$. Then the family $\{\mathbf{v}_i^j\}_{i,j}$ is a linearly independent family of vectors.

Theorem 6

Let $A \in \mathbb{K}^{n\times n}$*.* A *is diagonalizable if and only if* $q_i = p_i$ *for all* r *distinct eigenvalues.*

Corollary 1

If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Example 5

The process of diagonalizing a matrix is always the same:

1. Compute the characteristic polynomial
2. Find the eigenvalues and their respective algebraic multiplicities
3. For each eigenvalue, find a basis of the eigenspaces
4. Side: if you find less eigenvectors than the total dimension, the matrix is not diagonalizable
5. Define the matrix $V = [\mathbf{v}_1, \dots, \mathbf{v}_n]$ containing all the eigenvectors
6. Define the matrix $D = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$
7. You obtain the diagonalization $A = VDV^{-1}$.

Apply this to

$$A = \begin{bmatrix} -1 & 3 & -5 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{bmatrix}.$$