

Matrix Analysis: Spectral theorems

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1 Spectral theorems

Outline

1 Spectral theorems

Definition 2.1

A matrix $A \in \mathbb{K}^{n \times n}$ is called **normal** if it commutes with its conjugate transpose: $A^* A = A A^*$.

Example 2.1

This concept generalizes known matrices:

- Symmetric matrices
- Unitary and orthogonal matrices
- Skew-symmetric matrices

Exercise 2.1

Is the following matrix normal? Unitary? Orthogonal? Symmetric? Skew-symmetric?

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Conclude that set of normal matrices is a strictly bigger set than the previous examples.

Proposition 2.1

Show that a square matrix A is triangular and normal if and only if A is diagonal.

Theorem 2.1 (Spectral theorem for normal matrices)

Let A be a squared normal matrix. Then there exists an orthonormal basis of eigenvectors.

Remark 2.1

The existence of an orthonormal basis of eigenvectors is equivalent to the diagonalizability of A by a unitary matrix. Indeed, let $V = [\mathbf{v}_1, \dots, \mathbf{v}_n]$ be the matrix containing the n orthogonal (and normalized!) eigenvectors with respective eigenvalues $\lambda_1, \dots, \lambda_n$. Then

- ① V is unitary (see theorem before Schur's triangularization)
- ② $AV = \text{diag}(\lambda_1, \dots, \lambda_n)V$.

Note that the converse of the previous theorem is obviously true.

Definition 2.2

A is said to be unitarily diagonalizable if it is diagonalizable via a unitary matrix.

Exercise 2.2

Are all diagonalizable matrices unitary diagonalizable?

Corollary 2.1

Let A be a squared normal matrix and let \mathbf{u} and \mathbf{v} be two eigenvectors with distinct eigenvalues $\lambda \neq \mu$. Then $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ (where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product).

Remark 2.2

Finding the spectral decomposition of a normal matrix is *easy*:

- 1 Find the eigenvalues of the matrix
- 2 Find an orthogonal basis of the eigenspaces $\text{Ker}(A - \lambda)$ for all eigenvalues λ (this can be done by finding linearly independent eigenvectors, and then applying Gram-Schmidt)
- 3 Stack all the vectors together, keeping the diagonal entries of the diagonal matrix in the right order!

Exercise 2.3

Find an orthogonal matrix U such that $U^T A U$ is diagonal, where

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

Theorem 2.2

Special normal matrices are easy to recognize: Let A be a normal matrix. The followings hold

- *A is Hermitian ($A^* = A$) if and only if $\sigma(A) \subset \mathbb{R}$.*
- *A is skew-symmetric ($A^* = -A$) if and only if $\sigma(A) \subset i\mathbb{R}$ (i.e. for all $\lambda \in \sigma(A)$, $\Re(\lambda) = 0$)*
- *A is unitary ($AA^* = I$) if and only if $|\lambda| = 1$ for all $\lambda \in \sigma(A)$.*

Exercise 2.4

Characterize all the normal matrices which are also nilpotent.

Theorem 2.3

Let $A, B \in \mathbb{K}^{n \times n}$ be two commuting square matrices and assume that A is normal. Then A^ commutes with B .*

Corollary 2.2

Assume that A and B are two normal commuting matrices. Then AB is also normal.

Exercise 2.5

Let $A \in \mathbb{R}^{n \times n}$. Define two quadratic forms $Q_1, Q_2 : \mathbb{R}^n \rightarrow \mathbb{R}$ as

$$Q_1(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$$

$$Q_2(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T (A + A^T) \mathbf{x}.$$

Show that $Q_1(\mathbf{x}) = Q_2(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$.

Definition 2.3

Let $A \in \mathbb{K}^{n \times n}$ be a Hermitian matrix. The scalar function

$$Q : \begin{array}{ccc} \mathbb{K}^n & \rightarrow & \mathbb{K} \\ \mathbf{x} & \mapsto & \langle A\mathbf{x}, \mathbf{x} \rangle \end{array}$$

is called a **complex quadratic form**.

Theorem 2.4

Let $A \in \mathbb{K}^{n \times n}$. The following statements are equivalent:

- ① A is Hermitian.
- ② $\mathbf{x}^* A \mathbf{x} \in \mathbb{R}$, for all $\mathbf{x} \in \mathbb{K}^n$.
- ③ $A^* A = A A^*$ and $\sigma(A) \subset \mathbb{R}$.
- ④ $S^* A S$ is Hermitian for all $S \in \mathbb{K}^{n \times n}$.

Definition 2.4

Let A be a diagonal matrix. The complex quadratic form defined as $Q_A(\mathbf{x}) = \langle A\mathbf{x}, \mathbf{x} \rangle$ is called a **diagonal complex quadratic form**.

Exercise 2.6

What are the general formula (based on the coordinates of the vectors and entries of the matrices) for a complex quadratic form and a diagonal form?

Proposition 2.2

Every complex quadratic form can be transformed into a diagonal quadratic form.

Proposition 2.3

Let $A \in \mathbb{K}^{n \times n}$ and let $\lambda_1, \dots, \lambda_n$ be its eigenvalues enumerated such that $\lambda_1 \leq \dots \leq \lambda_n$. (Note: This ordering is possible since the eigenvalues are real!) Then (see homework)

$$\lambda_1 \|\mathbf{x}\|_2^2 \leq Q_A(\mathbf{x}) \leq \lambda_n \|\mathbf{x}\|_2^2.$$

Exercise 2.7

Let A be a Hermitian matrix. Prove the inequality

$$\lambda_1 \|\mathbf{x}\|_2^2 \leq Q_A(\mathbf{x}) \leq \lambda_n \|\mathbf{x}\|_2^2.$$

Exercise 2.8

When do you get equalities in the previous proposition?

Theorem 2.5 (Courant-Fisher)

Let $A \in \mathbb{K}^{n \times n}$ be a Hermitian matrix and $1 \leq k \leq n$. Then

$$\lambda_k = \min_{\dim(V)=k} \max_{\mathbf{x} \in V; \|\mathbf{x}\|_2=1} \langle A\mathbf{x}, \mathbf{x} \rangle = \max_{\dim(V)=n-k+1} \min_{\mathbf{x} \in V; \|\mathbf{x}\|_2=1} \langle A\mathbf{x}, \mathbf{x} \rangle.$$

Theorem 2.6 (Weyl)

Let A, B be two $n \times n$ Hermitian matrices. Then

$$\lambda_k(A) + \lambda_1(B) \leq \lambda_k(A + B) \leq \lambda_k(A) + \lambda_n(B).$$