# Matrix Analysis: Review of linear algebra

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- Application to systems of differential equations

## Example 1.1

Study the characteristic polynomials, trace, determinants, eigenvalues of the two following matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Compute also $A^2$ and $B^2$ and conclude that, although these matrices share common characteristics, they cannot be equivalent!

### Definition 1.1

*A Jordan block of order $k$ and value $\lambda$ is defined as the matrix*

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix}, k \geq 2$$

*and $J_1(\lambda) = \lambda$.*

## Exercise 1.1

Compute the following

1. $J_k(0)J_k(0)^T$.
2. $J_k(0)^T J_k(0)$.
3. $I - J_k(0)^T J_k(0)$.

**Definition 1.2**

*We call a matrix a* **Jordan matrix***, a matrix that block-diagonal where each block is itself a Jordan block.*

### Theorem 1.1

*Let $A \in \mathbb{K}^{n\times n}$. There exists a non-singular matrix $S$ such that*

$$A = S\,\mathrm{diag}\left(J_{n_1}(\lambda_1), \cdots, J_{n_k}(\lambda_k)\right),$$

*where $n_1 + \cdots + n_k = n$ and $\lambda_1, \cdots, \lambda_k$ are not necessarily distinct eigenvalues of $A$. The Jordan form is unique, up to permutation of the blocks.*

## Proposition 1.1

*To compute a Jordan canonical form of a matrix $A$ it suffices to follow these steps:*

1. *Compute the distinct eigenvalues of $A$: $\lambda_1, \cdots, \lambda_r$. They have algebraic multiplicities $p_1, \cdots, p_r$.*
2. *Compute $n_i^{(k)} = rk(A - \lambda_i I)^k$ for $1 \leq i \leq r$ and $0 \leq k \leq p_i$ (you can actually stop before $p_i$: as soon as $n_i^{(k)} = p_i$)*
3. *For each eigenvalue $\lambda_i$, they are $n_i^{(k-1)} - n_i^{(k)}$ Jordan-blocks of size $\geq k$. (hence, in the computation above, you can stop once you reach $n_i^{(k)} = n_i^{(k-1)}$)*
4. *Working backwards gives the number of blocks of a given size.*

### Example 1.2

Find the Jordan canonical form of the matrix $A$, given the following information:

$$p_A(\lambda) = (\lambda - 2)^7 (\lambda - 3)^3,$$

$$rk(A-2I) = 7, \quad rk(A-2I)^2 = 4, \quad rk(A-2I)^3 = 3, \quad rk(A-2I)^4 = 3$$

$$rk(A-3I) = 8, \quad rk(A-3I)^2 = 7, \quad rk(A-3I)^3 = 7.$$

## Exercise 1.2

Find the Jordan canonical form of the following matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ -1 & -1 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

### Remark 1.1

General strategy for finding the *Jordanizing matrix* (this is far from rigorous ... understanding why this work is no easy thing)

1. Find a set of linearly independent eigenvectors – this gives the number of Jordan blocks
2. Find $\mathbf{v}_i^{(2)}$ such that $(A - \lambda_i I)\mathbf{v}_i^{(2)} = \mathbf{v}_i^{(1)}$ (i.e. find a way to create your eigenvector in the range of $A - \lambda_i$). We say that $\mathbf{v}_i^{(2)}$ is a **(generalized) eigenvector of order** $2$ with eigenvalue $\lambda_i$.
3. Iterate until you're happy (i.e. you have enough linearly independent vectors, with the appropriate property)

### Remark 1.2

They are many different use case: practice as much as you can until you have encountered all of them: e.g.:

- $n$ distinct eigenvalues
- $1$ eigenvalue of (algebraic) multiplicity $n$
- 1 eigenvalue with algebraic multiplicity 3 but geometric multiplicity $2$ (how to pick the eigenvectors?)
- algebraic multiplicity 4, geometric multiplicity 2: blocks of size 1 and 3, or 2 and 2 ?
- etc ...

## Exercise 1.3

Find the Jordan canonical form and the (generalized) eigenvectors of the following matrices

$$A = \begin{bmatrix} -2 & 2 & 1 \\ -7 & 4 & 2 \\ 5 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 8 & 0 & 1 \\ 3 & 7 & 3 \\ -1 & 0 & 6 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Regarding $C$: Make sure to find the generalized eigenvector of order 2 as linear combination of the traditional eigenvectors.

**Exercise 1.4**

Let $P$ be a matrix such that $P^2 = P$. Compute its Jordan canonical form.

**Definition 1.3**

*Let $A$ be a $n \times n$ square matrix. The monic polynomial of minimum degree $m_A(x)$ such that $m_A(x) = 0$ is called its* **minimal polynomial***.*

### Remark 1.3

- This polynomial exists: the set of (monic) polynomials which annihilate $A$ is non-void.
- It has degree at most $n$.
- It is unique!

### Theorem 1.2

*Let $A \in \mathbb{K}^{n\times n}$ and $m_A$ its minimal polynomials. Let $q$ be a polynomial such that $q(A) = 0$. Then $m_A$ is a divisor of $q$.*

### Exercise 1.5

Find the minimal polynomial of the following matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

## Exercise 1.6

Find the minimal and characteristic polynomials of the following matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

### Exercise 1.7

Let $A \in \mathbb{K}^{n\times n}$ be such that $A^2 = A$. Give its minimal polynomial and compute its Jordan form.

**Theorem 1.3**

*For any $n^{th}$ degree (monic) polynomial*

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0 x^0$$

*there exists an $n \times n$ matrix $A$ such that $m_A(x) = p(x)$.*

**Definition 1.4**

*The $A$ matrix introduced in the previous theorem is called the* **companion matrix** *of the monic polynomial.*

## Proposition 1.2

*Let $X' = AX$ be a linear first order differential system. The general solution is given by*

$$X(t) = \sum_{i=1}^{k} X_i(t),$$

*where $k$ is the number of Jordan blocks of the matrix $A$ and $X_i$ is the solution restricted to the $i^{th}$ block. It is given by*

$$X_i(t) = \sum_{j=1}^{n_i} e^{\lambda t} C_j \sum_{\ell=1}^{j} \frac{t^{j-\ell}}{(j-\ell)!} \mathbf{v}_\ell,$$

*where $\mathbf{v}_i$ are the Jordanizing vector of the current block.*
*For instance, we have:*

$$X(t) = C_1 e^{\lambda t} \mathbf{v}_1 + C_2 e^{\lambda t} (t\mathbf{v}_1 + \mathbf{v}_2) + C_3 e^{\mu t} \mathbf{v}_3,$$

*for a $3 \times 3$ problem where one of the eigenvalue has a lack of geometric multiplicity.*

## Exercise 1.8

Find solutions to the following problems:

$$\begin{cases} x'(t) & = & 2x - 3y \\ y'(t) & = & -x + 4y \end{cases}$$

$$\begin{cases} x'(t) & = & 2x - y \\ y'(t) & = & x + 4y \end{cases}$$

$$y'' = -3y' - 2y.$$