SUGGESTED EXERCISES: MODERN REGRESSION

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1. Simple linear regression

Homework 1. Prove the following statement

Proposition 1.1 (Prove at home)**.** *Under the conditions that the LINE assumptions are valid, the following partition of errors hold*

$$SSTO = SSR + SSE.$$

Homework 2 (Optional). Try to prove the following relation

Proposition 1.2. *Pearson's correlation coefficient can be computed as*

$$r = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2 \sum_{i=1}^n (y_i - \bar{y})^2}}.$$

Homework 3. Getting some practice with t distributions. You are trying to find the t multiplier in the definition of a confidence interval for a (under certain assumptions on your data) t distributed variable.

(1) Recall the expression of a confidence interval for a t-distributed random variable.
(2) What is the t multiplier if you have 15 samples and ask for a 95% confidence interval?
(3) What is the t multiplier if you have 26 samples and ask for a 95% confidence interval?
(4) What is the t multiplier if you have 15 samples and ask for a 91% confidence interval?

Homework 4. Suppose that you produce dragon fruits and you've noticed that you produce 55% of the red kind and 45% of the white kind. You have in front of you 100 dragon fruits 53 of which are red. Can you conclude that the sample is representative of your production?

2. Multiple linear regression: more dimensions

3. Matrices: what you need to know

Homework 5. What happens to inverses if the matrix A is not square? Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2/3 & -1/3 & 1/3 \\ -1/3 & 2/3 & 1/3 \end{pmatrix}$. Show that BA is some identity (which one?) but AB clearly is not. This shows that we may have left inverses which are not right inverses.

Homework 6. Let A be any square matrix. Show that the matrix $C = \frac{A - A^T}{2}$ is antisymmetric.

Homework 7. Prove the following using Laplace formula.

Proposition 3.1. *Let $A \in \mathbb{R}^{3\times 3}$. We may use Sarrus' rule to compute a 3×3 determinant:*

$$\begin{aligned} \det(A) = {} & A_{1,1}A_{2,2}A_{3,3} + A_{1,2}A_{2,3}A_{3,1} + A_{1,3}A_{2,1}A_{3,2} \\ & - A_{1,3}A_{2,2}A_{3,1} - A_{1,2}A_{2,1}A_{3,3} - A_{1,1}A_{2,3}A_{3,2}. \end{aligned}$$

Homework 8. Show that the equivalent of Sarrus rule (sum of positive diagonals minus sum of negative diagonals) doesn't work for dimension 4.

Homework 9. Show the following result on inverses of 2 dimensional matrices

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Proposition 3.2. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$. *If* A *is invertible, its inverse is given by*

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Homework 10. Let $D, U, L \in \mathbb{R}^{n\times n}$ where D is a diagonal matrix, U is an upper triangular matrix (i.e. $U_{i,j} = 0$ if $i > j$) and L is a lower triangular matrix (i.e. $L_{i,j} = 0$ if $j > i$). Find $\det(D), \det(L), \det(U)$.

Homework 11. Determine whether the following vectors are orthogonal or not

- $\mathbf{x} = (6, 1, 4)^T$ and $\mathbf{y} = (2, 0, -3)^T$.
- $\mathbf{x} = (0, 0, -1)^T$ and $\mathbf{y} = (1, 1, 1)^T$.
- $\mathbf{x} = (0, 0, -1)^T$ and $\mathbf{y} = (-1, -1, 0)^T$.
- $\mathbf{x} = (a, 0, 0)^T$ and $\mathbf{y} = (0, 0, b)^T$ for some numbers a and b.

Homework 12. Let $\mathbf{x} = (-7, -18, \alpha)^T$ and $\mathbf{y} = (0, -10, -7)^T$. Find the value(s) of α for which $\mathbf{x}$ and $\mathbf{y}$ are perpendicular.

Homework 13. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Show that A is positive definite.

Homework 14. Let L be a one-dimensional subspace (i.e. a line) in an n dimensional space $\mathbb{R}^n$. For any vector $x \in \mathbb{R}^n$, find the projection of this x onto the line L.

Homework 15. Let $A \in \mathbb{R}^{m\times d}$ and $b \in \mathbb{R}^m$. Find the gradient of the function

$$f(x) = \|Ax - b\|^2.$$

What are the critical points of the function f?

Homework 16. Let $f(x, y, z) = xy + yz + zx$. Using the formula for bilinear forms, compute the gradient of f.

Homework 17. Let $f(x) = \|Ax-b\|_2^2 + \lambda\|x\|_2^2$ for some matrix $A \in \mathbb{R}^{m\times d}$, some vector $b \in \mathbb{R}^m$ and a constant $\lambda \geq 0$. Find the critical points of f.

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