

Matrix Analysis: Review of linear algebra

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Definition 1

Let V be a finite dimensional vector space. The mapping $\|\cdot\| : V \rightarrow \mathbb{R}$ is called a **vector norm** if

- ① $\|\mathbf{v}\| \geq 0$, for all $\mathbf{v} \in V$ (positivity),
- ② $\|\mathbf{v}\| = 0 \Leftrightarrow \mathbf{v} = \mathbf{0}_V$ (definition),
- ③ $\|\alpha\mathbf{v}\| = |\alpha|\|\mathbf{v}\|$ for all $\alpha \in \mathbb{K}$ and $\mathbf{v} \in V$ (homogeneity),
- ④ $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ for all $\mathbf{u}, \mathbf{v} \in V$ (triangle inequality).

Example 1

Let $V = \mathbb{R}^n$. The following define the traditional Minkowski p norms, for a real number $p \geq 1$:

$$\|\mathbf{x}\|_p = \left(\sum |x_i|^p \right)^{1/p}.$$

Some people call this also Hölder's norm.

Particular examples include:

- $p = 2$: Euclidean norm
- $p = 1$: Manhattan or Taxicab norm
- As $p \rightarrow \infty$, we define the *infinity norm* as $\|\mathbf{x}\|_\infty = \max_i |x_i|$.

Proposition 1

Let $\infty \geq p \geq q \geq 1$. It holds

$$\|\mathbf{x}\|_p \leq \|\mathbf{x}\|_q \leq n^{1/q-1/p} \|\mathbf{x}\|_p.$$

Definition 2

Two norms N_1 and N_2 are said to be **equivalent** if there exist two constants α and β such that

$$\alpha N_1(\mathbf{v}) \leq N_2(\mathbf{v}) \leq \beta N_1(\mathbf{v}), \text{ for all } \mathbf{v} \in V.$$

Proposition 2

Assume $(\mathbf{x}^{(k)})_k$ is a convergence sequence with respect to a norm N_1 . If N_2 is equivalent to N_1 then $(\mathbf{x}^{(k)})_k$ is also convergence with respect to N_2 .

Proposition 3

On a finite dimensional vector space, all norms are equivalent.

Example 2

Let N be defined as

$$N(\mathbf{u}) = (|2u_1 + 3u_2|^2 + |u_2|^2)^{1/2}.$$

Does N define a norm?

Proposition 4

Let $A : V \rightarrow W$ be a linear function where $\dim(V) = n$ and let $\|\cdot\|$ define a norm on W . If $\text{rk}(A) = n$ then $\|A(\mathbf{x})\|$ is a norm.

Proposition 5

Let \mathbf{u} and \mathbf{v} be two n -dimensional vectors. Then **Hölder's inequality** holds

$$\sum_{i=1}^n |u_i v_i| \leq \|\mathbf{u}\|_p \|\mathbf{v}\|_q,$$

where p and q are such that $1/p + 1/q = 1$.

Lemma 1 (Young's inequality for product)

Let a and b be non-negative real numbers and $1 < p \leq q < \infty$. It holds

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Definition 3

A vector space $(V, \|\cdot\|)$ is said to be a **normed vector space** if

- V is a vector space over \mathbb{K} and
- $\|\cdot\|$ is a norm.

If moreover V is complete (every Cauchy sequence in V converge in V) we call it a **Banach space**.

Definition 4

Let V be a vector space over the field \mathbb{K} . The binary function $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{K}$ is called an **inner product** if for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$

- ① $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$,
- ② $\langle \mathbf{u}, \mathbf{u} \rangle = 0 \Leftrightarrow \mathbf{u} = \mathbf{0}$,
- ③ $\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$, for all scalar $\alpha \in \mathbb{K}$,
- ④ $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$,
- ⑤ $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$.

One may say that the inner product is a positive definite sesquilinear form.

Proposition 6

Let V be a vector space and $\langle \cdot, \cdot \rangle$ be an inner product. The mapping $\| \cdot \|$ defined for $\mathbf{u} \in V$ as $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle$ is a norm on V .

Proposition 7 (Cauchy-Schwarz)

Let $\langle \cdot, \cdot \rangle$ be an inner product on V . It holds, for all $\mathbf{u}, \mathbf{v} \in V$

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|,$$

where $\|\cdot\|$ is the norm induced by the inner product.

Exercise 1

Show that the equality in Cauchy-Schwarz inequality occurs if and only if \mathbf{u} and \mathbf{v} are linearly dependent.

Definition 5

*A vector space equipped with an inner product is called an **inner product space**.*

*If the space is also complete, we call it a **Hilbert space**.*

Exercise 2

Show that the trace defines an inner product on the space of matrices:

$$\langle A, B \rangle = \text{tr}(B^* A).$$

The associated norm is called the **Frobenius**, denoted $\|\cdot\|_F$. What is $\|A\|_F^2$?

Proposition 8

An inner product $\langle \cdot, \cdot \rangle$ fulfills the following basic properties (in an vector space V on the field of scalar \mathbb{K}):

- *Let $\mathbf{u} \in V$, $T_{\mathbf{u}} : V \rightarrow \mathbb{K}$ defined for all $\mathbf{v} \in V$ as $T_{\mathbf{u}}(\mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle$ is a linear map from V to \mathbb{K} .*
- *$\langle 0, \mathbf{u} \rangle = 0 = \langle \mathbf{u}, 0 \rangle$ for every $\mathbf{u} \in V$.*
- *$\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$, for every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.*
- *$\langle \mathbf{u}, \lambda \mathbf{v} \rangle = \bar{\lambda} \langle \mathbf{u}, \mathbf{v} \rangle$, for every $\mathbf{u}, \mathbf{v} \in V$ and $\lambda \in \mathbb{K}$.*

Definition 6

Let $V, \langle \cdot, \cdot \rangle$ be an inner product space. Two vectors \mathbf{u}, \mathbf{v} are called **orthogonal**

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0.$$

Definition 7

Let $V, \langle \cdot, \cdot \rangle$ be an inner product space. Two families of vectors S and T are called **orthogonal** if

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0, \text{ for all } \mathbf{u} \in S, \mathbf{v} \in T.$$

Exercise 3

Prove the Pythagorean theorem: if \mathbf{u} and \mathbf{v} are two orthogonal vectors, then

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2,$$

where $\|\cdot\|$ denotes the norm induced by the given scalar product.

Definition 8

A vector is said to be **unit norm** or **normalized** if $\|\mathbf{u}\| = 1$.

A family of vectors is said to be **orthonormal** if it is a family of unit-norm vectors and orthogonal.

Proposition 9

A family of p vectors is orthonormal if and only if the matrix U containing those vectors column-wise is such that $U^T U = I_p$.

Proposition 10 (Gram-Schmidt)

Let $S = (\mathbf{v}_1, \dots, \mathbf{v}_k)$ be a linearly independent family vectors. Then there exists an orthonormal family $(\mathbf{w}_1, \dots, \mathbf{w}_k)$ such that $\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_j) = \text{span}(\mathbf{w}_1, \dots, \mathbf{w}_j)$ for all $1 \leq j \leq k$.

Definition 9

A square matrix $A \in \mathbb{K}^{n \times n}$ is called **unitary** (resp. **orthogonal**) if

$$A^* A = A A^* = I_n \quad (\text{resp. } A^T A = A A^T = I_n).$$

Exercise 4

Let U be a unitary matrix and λ one of its eigenvalues. Show that $|\lambda| = 1$. What can be said about $|\det(U)|$? What does it mean for a real orthogonal matrix?

Proposition 11

Let $A \in \mathbb{K}^{n \times n}$. The following statements are equivalent

- 1 A is unitary.
- 2 A preserves the ℓ^2 norm: $\|A\mathbf{u}\| = \|\mathbf{u}\|$, for all $\mathbf{u} \in \mathbb{K}^n$.
- 3 The columns of A form an orthonormal system.

Exercise 5

Are sums and product of unitary matrices also unitary?

Exercise 6

Let U be a unitary matrix. Show that two vectors \mathbf{x} and \mathbf{y} are orthogonal if and only if $U\mathbf{x}$ and $U\mathbf{y}$ are orthogonal.

Exercise 7

Let U be a unitary matrix. Show that $\text{adj}(U)$ is unitary.