Matrix Analysis: Review of linear algebra

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Definition 1

Let V be a finite dimensional vector space. The mapping $\|\cdot\| : V \to \mathbb{R}$ is called a **vector norm** *if*

1. $\|\mathbf{v}\| \geq 0$, *for all* $\mathbf{v} \in V$ *(positivity),*
2. $\|\mathbf{v}\| = 0 \Leftrightarrow \mathbf{v} = 0_V$ *(definition),*
3. $\|\alpha \mathbf{v}\| = |\alpha| \|\mathbf{v}\|$ *for all* $\alpha \in \mathbb{K}$ *and* $\mathbf{v} \in V$ *(homogeneity),*
4. $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ *for all* $\mathbf{u}, \mathbf{v} \in V$ *(triangle inequality).*

Example 1

Let $V = \mathbb{R}^n$. The following define the traditional Minkowski p norms, for a real number $p \geq 1$:

$$\|\mathbf{x}\|_p = \left(\sum |x_i|^p \right)^{1/p}.$$

Some people call this also Hölder's norm.
Particular examples include:

- $p = 2$: Euclidean norm
- $p = 1$: Manhattan or Taxicab norm
- As $p \to \infty$, we define the *infinity norm* as $\|\mathbf{x}\|_\infty = \max_i |x_i|$.

Proposition 1

Let $\infty \geq q \geq p \geq 1$. *It holds*

$$\|\mathbf{x}\|_p \leq \|\mathbf{x}\|_q \leq n^{1/q-1/p}\|\mathbf{x}\|_p.$$

Definition 2

Two norms N_1 and N_2 are said to be **equivalent** *if there exist two constants α and β such that*

$$\alpha N_1(\mathbf{v}) \leq N_2(\mathbf{v}) \leq \beta N_1(\mathbf{v}), \text{ for all } \mathbf{v} \in V.$$

Proposition 2

Assume $(\mathbf{x}^{(k)})_k$ *is a convergence sequence with respect to a norm* N_1*. If* N_2 *is equivalent to* N_1 *then* $(\mathbf{x}^{(k)})_k$ *is also convergence with respect to* N_2*.*

Proposition 3

On a finite dimensional vector space, all norms are equivalent.

Example 2

Let N be defined as

$$N(\mathbf{u}) = \left(|2u_1 + 3u_2|^2 + |u_2|^2\right)^{1/2}.$$

Does N define a norm?

Proposition 4

Let $A : V \to W$ be a linear function where $\dim(V) = n$ and let $\| \cdot \|$ define a norm on W. If $rk(A) = n$ then $\|A(\mathbf{x})\|$ is a norm.

Proposition 5

Let $\mathbf{u}$ *and* $\mathbf{v}$ *be two* n*-dimensional vectors. Then* **Hölder's inequality** *holds*

$$\sum_{i=1}^{n} |u_i v_i| \leq \|\mathbf{u}\|_p \|\mathbf{v}\|_q,$$

where p *and* q *are such that* $1/p + 1/q = 1$.

Lemma 1 (Young's inequality for product)

Let a and b be non-negative real numbers and $1 < p \leq q < \infty$. It holds

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Definition 3

A vector space $(V, \|\cdot\|)$ is said to be a **normed vector space** *if*

- *V is a vector space over $\mathbb{K}$ and*
- *$\|\cdot\|$ is a norm.*

If moreover V is complete (every Cauchy sequence in V converge in V) we call it a **Banach space**.

Definition 4

Let V be a vector space over the field $\mathbb{K}$. The binary function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{K}$ is called an **inner product** *if for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$*

1. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$,
2. $\langle \mathbf{u}, \mathbf{u} \rangle = 0 \Leftrightarrow \mathbf{v} = 0$,
3. $\langle \alpha\mathbf{u}, \mathbf{v} \rangle = \alpha\langle \mathbf{u}, \mathbf{v} \rangle$, *for all scalar $\alpha \in \mathbb{K}$,*
4. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$,
5. $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$.

One may say that the inner product is a positive definite sesquilinear form.

Proposition 6

Let V be a vector space and $\langle \cdot, \cdot \rangle$ be an inner product. The mapping $\| \cdot \|$ defined for $\mathbf{u} \in V$ as $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle$ is a norm on V.

Proposition 7 (Cauchy-Schwarz)

Let $\langle \cdot, \cdot \rangle$ *be an inner product on* V*. It holds, for all* $\mathbf{u}, \mathbf{v} \in V$

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\| \|\mathbf{v}\|,$$

where $\| \cdot \|$ *is the norm induced by the inner product.*

Exercise 1

Show that the equality in Cauchy-Schwarz inequality occurs if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.

Definition 5

A vector space equipped with an inner product is called an **inner product space**.
If the space is also complete, we call it a **Hilbert space**.

Exercise 2

Show that the trace defines an inner product on the space of matrices:

$$\langle A, B\rangle = tr(B^* A).$$

The associated norm is called the **Frobenius**, denoted $\|\cdot\|_F$. What is $\|A\|_F^2$?

Proposition 8

An inner product $\langle \cdot, \cdot \rangle$ fulfills the following basic properties (in an vector space V on the field of scalar $\mathbb{K}$):

- *Let $\mathbf{u} \in V$, $T_{\mathbf{u}} : V \to \mathbb{K}$ defined for all $\mathbf{v} \in V$ as $T_{\mathbf{u}}(\mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle$ is a linear map from V to $\mathbb{K}$.*
- $\langle 0, \mathbf{u} \rangle = 0 = \langle \mathbf{u}, 0 \rangle$ *for every* $\mathbf{u} \in V$.
- $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$, *for every* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.
- $\langle \mathbf{u}, \lambda \mathbf{v} \rangle = \overline{\lambda} \langle \mathbf{u}, \mathbf{v} \rangle$, *for every* $\mathbf{u}, \mathbf{v} \in V$ *and* $\lambda \in \mathbb{K}$.

Definition 6

Let $V, \langle \cdot, \cdot \rangle$ be an inner product space. Two vectors $\mathbf{u}, \mathbf{v}$ are called **orthogonal**

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0.$$

Definition 7

Let $V, \langle \cdot, \cdot \rangle$ be an inner product space. Two families of vectors S and T are called **orthogonal** *if*

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0, \text{ for all } \mathbf{u} \in S, \mathbf{v} \in T.$$

Exercise 3

Prove the Pythagorean theorem: if $\mathbf{u}$ and $\mathbf{v}$ are two orthogonal vectors, then

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2,$$

where $\|\cdot\|$ denotes the norm induced by the given scalar product.

Definition 8

A vector is said to be **unit norm** *or* **normalized** *if* $\|\mathbf{u}\| = 1$.
A family of vectors is said to be **orthonormal** *if it is a family of unit-norm vectors and orthogonal.*

Proposition 9

A family of p vectors is orthonormal if and only if the matrix U containing those vectors column-wise is such that $U^T U = I_p$.

Proposition 10 (Gram-Schmidt)

Let $S = (\mathbf{v}_1, \cdots, \mathbf{v}_k)$ *be a linearly independent family vectors. Then there exists an orthonormal family* $(\mathbf{w}_1, \cdots, \mathbf{w}_k)$ *such that* $\operatorname{span}(\mathbf{v}_1, \cdots, \mathbf{v}_j) = \operatorname{span}(\mathbf{w}_1, \cdots, \mathbf{w}_j)$ *for all* $1 \leq j \leq k$.

Definition 9

A square matrix $A \in \mathbb{K}^{n \times n}$ *is called* **unitary** *(resp.* **orthogonal***) if*

$$A^*A = AA^* = I_n \qquad \text{(resp. } A^TA = AA^T = I_n\text{).}$$

Exercise 4

Show that for U a unitary matrix, $|\det(U)| = 1$. What does it mean for a real orthogonal matrix?

Proposition 11

Let $A \in \mathbb{K}^{n \times n}$. The following statements are equivalent

1. A *is unitary.*
2. A *preserves the ℓ^2 norm: $\|A\mathbf{u}\| = \|\mathbf{u}\|$, for all $\mathbf{u} \in \mathbb{K}^n$.*
3. *The columns of A form an orthonormal system.*