Modern Regression

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1. Matrices: what you need to know

Outline

Definition 2.1

A two dimensional array of number is called a **matrix**. *Typically, if A is a matrix which has 3 rows and 4 columns, we write $A \in \mathbb{R}^{3\times 4}$.*

Proposition 2.1

Let $A \in \mathbb{R}^{m\times n}$ *and* $B \in \mathbb{R}^{p\times q}$*. Assuming the results exist, we have*

- $(A+B)_{i,j} = A_{i,j} + B_{i,j}$.
- $(AB)_{i,j} = \sum_{k=1}^{n} A_{i,k}B_{k,j}$.
- *The transpose is such that* $(A^T)_{j,i} = A_{i,j}$

Definition 2.2

We say that a square matrix $A \in \mathbb{R}^{n\times n}$ is invertible if there exists a matrix $B \in \mathbb{R}^{n\times n}$ such that

$$AB = BA = I$$

where I is a diagonal matrix with only 1's along the diagonal and 0 elsewhere.

Proposition 2.2

Assuming a square matrix A is invertible, then its inverse is unique.
In this case we say that B is the inverse of A and we may write $B = A^{-1}$.

Definition 2.3

A matrix A is said to be symmetric if and only if $A^T = A$. It is antisymmetric if $A^T = -A$.

Proposition 2.3

Manipulating matrices is not just as trivial as manipulating numbers. Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$. The following rules apply

- $A + B$ *exists if and only if* $m = p$ *and* $n = q$.
- AB *exists if and only if* $p = n$.
- AB *exists does not imply* BA *exists (let alone* $AB = BA$*)*
- *if the product exists,* $(AB)^T = B^T A^T$.
- *if the inverses exist and the product exists, we have* $(AB)^{-1} = B^{-1} A^{-1}$.

Proposition 2.4

Let A be any square matrix. Then $B = \frac{A+A^T}{2}$ is symmetric.

Definition 2.4

Let A be an $n \times n$ square matrix. Its **trace** *is defined as*

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{i,i}.$$

Definition 2.5

Let $A \in \mathbb{R}^{n\times n}$ be any square matrix. We define its **determinant** *recursively via Laplace's formula:*

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{i,j} \det(A^{-i,-j}) = \sum_{j=1}^{n} (-1)^{i+j} A_{i,j} \det(A^{-i,-j})$$

where the formula holds for any j in the first expression or any i in the second. We have also defined the submatrix $A^{-i,-j}$ as the matrix extracted from A by deleting row i and column j.
The recursion is set by the case $n = 1$ as $\det\begin{pmatrix} a \end{pmatrix} = a$.

Proposition 2.5

The determinant of a 2×2 *matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is given by

$$\det(A) = ad - bc.$$

Proposition 2.6

Let $A \in \mathbb{R}^{3\times 3}$. *We have*

$$\det(A) = A_{1,1}A_{2,2}A_{3,3} + A_{1,2}A_{2,3}A_{3,1} + A_{1,3}A_{2,1}A_{3,2} - A_{1,3}A_{2,2}A_{3,1} - A_{1,2}A_{2,1}A_{3,3} - A_{1,1}A_{2,3}A_{3,2}.$$

Proposition 2.7

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$. *If* A *is invertible, is inverse is given by*

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Exercise 2.1

Find the inverses of the following matrices:

- $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$
- $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$
- $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$