Entropy stable notes

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1 Curved meshes

Non-affine geometric factors

Let $\boldsymbol{F}(\boldsymbol{U})$ denote the flux matrix whose rows are

$$\boldsymbol{F}(\boldsymbol{U}) = \begin{pmatrix} \boldsymbol{F}_x \\ \boldsymbol{F}_y \end{pmatrix}.$$

The conservation law we're interested in is the following

$$\frac{\partial \boldsymbol{U}}{\partial t} + \nabla \cdot \boldsymbol{F}(\boldsymbol{U}) = 0,$$

where the divergence is taken over each column of $\boldsymbol{F}(\boldsymbol{U})$.

Let $\boldsymbol{G}$ denote the Jacobian of the geometric mapping

$$\boldsymbol{G}_{ij} = \frac{\partial \widehat{\boldsymbol{x}}_j}{\partial \boldsymbol{x}_i},$$

and let J denote the determinant of $\boldsymbol{G}$. One can show (Kopriva?) that

$$\widehat{\nabla} \cdot (J\boldsymbol{G}) = 0.$$

At the continuous level, the physical gradient and divergence satisfy

$$(\nabla u, \boldsymbol{v})_{D^k} = \left(J\boldsymbol{G}\widehat{\nabla} u, \boldsymbol{v}\right)_{\widehat{D}}, \qquad (\nabla \cdot \boldsymbol{u}, v)_{D^k} = \left(\widehat{\nabla} \cdot \left(J\boldsymbol{G}^T u\right), \boldsymbol{v}\right)_{\widehat{D}},$$

as well as a corresponding integration by parts property.

2 Flux differencing

Replace the flux derivative with

$$\left(\nabla \cdot \boldsymbol{F}_S(\boldsymbol{U}(\boldsymbol{x}), \boldsymbol{U}(\boldsymbol{x}'))|_{\boldsymbol{x}'=\boldsymbol{x}}\right)_{D^k}.$$

It is possible to remove the effect of geometric aliasing by evaluating the above term via

$$\left(\widehat{\nabla} \cdot \boldsymbol{F}^k_S(\boldsymbol{U}(\boldsymbol{x}), \boldsymbol{U}(\boldsymbol{x}'))\Big|_{\boldsymbol{x}'=\boldsymbol{x}}\right)_{\widehat{D}}, \qquad \boldsymbol{F}^k_S = \{\{J\boldsymbol{G}\}\}\, \boldsymbol{F}_S\left(\boldsymbol{U}(\boldsymbol{x}), \boldsymbol{U}(\boldsymbol{x}')\right)$$