WEDGE NOTES

JESSE CHAN

Abstract. Three main contributions: stable DG formulations for general vertex-mapped wedges, and energy stable DG schemes for acoustic-elastic coupling in the presence of arbitrary heterogeneous media.

1. Introduction.

2. Vertex-mapped wedges.

Properties:

1. $\frac{\partial r}{\partial xyz} J, \frac{\partial t}{\partial xyz} J \in P^1(\triangle) \otimes P^1([-1,1])$.
2. $\frac{\partial s}{\partial xyz} J \in P^0(\triangle) \otimes P^2([-1,1])$.
3. On triangular faces, $\boldsymbol{n} J^f \in P^0(\triangle)$.
4. On quadrilateral faces, $\boldsymbol{n} J^f \in P^1(\triangle)$.

Since $\frac{\partial}{\partial r}, \frac{\partial}{\partial t} : P^N(\triangle) \otimes P^N([-1,1]) \to P^{N-1}(\triangle) \otimes P^N([-1,1])$, this implies nodal collocation can be used to apply geometric factors.

Curvilinear wedges treated using quadrature-based skew-symmetric formulation; only need to interpolate over triangles.

3. Nodal elements.

Important note: GQ nodes in the extruded direction.

Metric identities are satisfied for low order mappings of wedges; i.e.

$$\widehat{\nabla} \cdot \begin{pmatrix} r_x J & r_y J & r_z J \\ s_x J & s_y J & s_z J \\ t_x J & t_y J & t_z J \end{pmatrix} = \widehat{\nabla} \cdot (J\boldsymbol{G}) = 0.$$

where $\boldsymbol{G}$ is the matrix of geometric factors. Even with this, GLL is not energy-stable.

3.1. Lift matrices under WADG.

Under WADG, the computation of the RHS uses only reference differentiation and lift matrices.

For triangular faces, the lift matrix consists of $(N+1)$ diagonal sub-matrices. These diagonal entries are all identical.

For quadrilateral faces, the lift matrix consists of block diagonal columns.

4. Acoustic-elastic coupling.

Elastic wave equation (velocity-stress formulation)

$$\boldsymbol{C}^{-1} \frac{\partial \boldsymbol{\sigma}}{\partial t} + \nabla \cdot \boldsymbol{v} = 0$$
$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \operatorname{sym}(\nabla \sigma) = 0.$$

Assuming $\mu = 0$, reduction to the acoustic wave equation with $p = \operatorname{tr}(\boldsymbol{S})$ and wavespeed $c = \sqrt{\frac{\lambda}{\rho}}$.

$$\frac{1}{c^2} \frac{\partial p}{\partial t} = \nabla \cdot \boldsymbol{u}$$
$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = \nabla p.$$

For acoustic-elastic interfaces, continuity conditions are given as

$$\boldsymbol{S}\boldsymbol{n} = p\boldsymbol{n}$$
$$\boldsymbol{v} \cdot \boldsymbol{n} = \boldsymbol{u} \cdot \boldsymbol{n}.$$

We define the numerical flux $\boldsymbol{f}^*$ as the sum of a central flux term (involving the average of left and right hand sides at an interface) and a penalization term.

On the acoustic side of the interface, this flux is

$$\left\langle \left(\boldsymbol{A}_n^T \boldsymbol{\sigma} - p\boldsymbol{n}\right) - \tau_u \left(\left(\boldsymbol{v} - \boldsymbol{u}\right) \cdot \boldsymbol{n}\right) \boldsymbol{n}, \boldsymbol{\tau} \right\rangle_{L^2(\partial D^k)} + \left\langle \left(\boldsymbol{v} - \boldsymbol{u}\right) \cdot \boldsymbol{n} - \tau_p \boldsymbol{n} \cdot \left(\boldsymbol{A}_n^T \boldsymbol{\sigma} - p\boldsymbol{n}\right), v \right\rangle_{L^2(\partial D^k)}$$

On the elastic side of the interface, the flux is

$$\left\langle \left(p\boldsymbol{n} - \boldsymbol{A}_n^T \boldsymbol{\sigma}\right) - \tau_v \left(\left(\boldsymbol{u} - \boldsymbol{v}\right) \cdot \boldsymbol{n}\right) \boldsymbol{n}, \boldsymbol{w} \right\rangle_{L^2(\partial D^k)} + \left\langle \boldsymbol{A}_n \left(\boldsymbol{u} - \boldsymbol{v}\right) - \tau_\sigma \boldsymbol{A}_n \left(p\boldsymbol{n} - \boldsymbol{A}_n^T \boldsymbol{\sigma}\right), \boldsymbol{q} \right\rangle_{L^2(\partial D^k)}$$

After integrating by parts the v and $\boldsymbol{w}$ equations, taking $(v, \boldsymbol{\tau}) = (p, \boldsymbol{u})$ and $(\boldsymbol{w}, \boldsymbol{q}) = (\boldsymbol{v}, \boldsymbol{\sigma})$ yields flux terms which cancel.

Acoustic side central flux terms

$$\left\langle \boldsymbol{u} \cdot \boldsymbol{n} + \boldsymbol{v} \cdot \boldsymbol{n}, p \right\rangle_{L^2(\partial D^k)} + \left\langle \boldsymbol{A}_n^T \boldsymbol{\sigma} - p\boldsymbol{n}, \boldsymbol{u} \right\rangle_{L^2(\partial D^k)}$$

Elastic central flux terms

$$\left\langle \boldsymbol{A}_n^T \boldsymbol{\sigma} + p\boldsymbol{n}, \boldsymbol{v} \right\rangle_{L^2(\partial D^k)} + \left\langle \boldsymbol{A}_n \left(\boldsymbol{u} - \boldsymbol{v}\right), \boldsymbol{\sigma} \right\rangle_{L^2(\partial D^k)}$$

The terms

$$\left\langle \boldsymbol{u} \cdot \boldsymbol{n}, p \right\rangle_{L^2(\partial D^k)}, \qquad \left\langle \boldsymbol{A}_n^T \boldsymbol{\sigma}, \boldsymbol{v} \right\rangle_{L^2(\partial D^k)}$$

are cancelled locally over each element. The remaining terms cancel after summing contributions from both sides of the acoustic-elastic interface.

Dissipative penalization terms can then be added to provide a stabilization. For the acoustic domain,

$$\left\langle \boldsymbol{n}^T \boldsymbol{S} \boldsymbol{n} - p, p \right\rangle_{L^2(\partial D^k)}, \qquad \left\langle \boldsymbol{v} - \boldsymbol{u}, \boldsymbol{u} \right\rangle_{L^2(\partial D^k)}$$

Note: penalization does not require exact integration. It shouldn't even require explicit quadrature...

5. Numerical experiments. Test wedges, tets, wedge-tet hybrid meshes with other waves for due diligence.

5.1. Stoneley wave. Test acoustic-elastic coupling.

6. Future work. Need to load $O(N^3)$ geofacs ($O(N^2)$ geofacs per wedge). Can reduce triangular costs using Bernstein-Bezier elements

- Volume kernel: reduce from $O(N^5)$ to $O(N^4)$ computational complexity ($O(N^2)$ per triangle, cheaper application of geofacs using Bernstein polynomial multiplication, $O(N^2)$ in extruded direction but this cost is near-negligible due to the use of fast shared memory).
- Surface kernel: still $O(N^4)$. 1D interpolation operators are the same cost whether BB or nodal. Lift matrix cost can be reduced, but doesn't change asymptotic cost.
- Update kernel: for vertex-mapped wedges, can reduce from $O(N^5)$ to $O(N^4)$: $J \in P^1(\triangle)$ over each element, so can be applied using polynomial multiplication and projection down.