Split form DG methods

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1 Notes

Integration by parts gives

$$\sum_{D^k} (\nabla u, \boldsymbol{v})_{L^2(D^k)} = \sum_{D^k} (-u, \nabla \cdot \boldsymbol{v})_{L^2(D^k)} + \langle u, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{L^2(\partial D^k)}.$$

We replace the values of u with $\{\{u\}\}$ on each element boundary to define the global DG gradient operator ∇_h

$$\sum_{D^k} (\nabla_h u, \boldsymbol{v})_{L^2(D^k)} \coloneqq \sum_{D^k} (-u, \nabla \cdot \boldsymbol{v})_{L^2(D^k)} + \langle \{\{u\}\}, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{L^2(\partial D^k)}.$$

Integrating by parts again and the introduction of the lift operator shows that

$$\nabla_h u = \nabla u + \frac{1}{2} L\left([\![u]\!] \boldsymbol{n}\right)$$

where L is the lift operator.

The DG divergence operator is similarly defined as

$$\nabla_h \cdot u = \nabla \cdot u + \frac{1}{2} L\left([\![\boldsymbol{u}]\!] \cdot \boldsymbol{n}\right)$$

and it can be shown that, for a periodic mesh,

$$(\nabla_h u, \boldsymbol{v}) = (-u, \nabla_h \cdot \boldsymbol{v}).$$

Need to incorporate boundary conditions.

We also have that, for interior elements,

$$(\nabla_h \cdot \boldsymbol{u}, v \mathbb{1}_{D^k}) = (\boldsymbol{u}, \nabla \cdot v \mathbb{1}_{D^k}) + \langle \{\{\boldsymbol{u}\}\} \cdot \boldsymbol{n}, v\rangle_{\partial D^k}.$$

and as a result when $v = 1$

$$(\nabla_h \cdot \boldsymbol{u}, \mathbb{1}_{D^k}) = \int_{\partial D^k} \{\{\boldsymbol{u}\}\} \cdot \boldsymbol{n}$$

2 Variable advection

A split formulation for advection is

$$\left(\frac{\partial u}{\partial t}, v\right) + \frac{1}{2}\left(\nabla_h \cdot \Pi_N\left(\boldsymbol{\beta} u\right), v\right) + \frac{1}{2}\left(\boldsymbol{\beta}\cdot\nabla_h u, v\right) + \frac{1}{2}\left(\left(\nabla\cdot\boldsymbol{\beta}\right) u, v\right) = 0.$$

Taking $v = u$ yields and using $(\nabla_h u, \boldsymbol{v}) = (-u, \nabla_h \cdot \boldsymbol{v})$ yields the energy statement

$$\frac{1}{2}\left\|u\right\|^2 + \frac{1}{2}\left(\nabla_h \cdot \Pi_N\left(\boldsymbol{\beta} u\right), u\right) - \frac{1}{2}\left(u, \nabla_h \cdot \Pi_N\left(\boldsymbol{\beta} u\right)\right) = \frac{1}{2}\left(-\left(\nabla\cdot\boldsymbol{\beta}\right) u, u\right),$$

implying that $\frac{1}{2}\left\|u\right\|^2 = 0$ if $\nabla \cdot \boldsymbol{\beta} = 0$, or that the method is energy conserving. The only difference in this formulation is the introduction of Π_N, which can be defined at a discrete level using any quadrature scheme for which a discrete projection is well-defined.

3 Local conservation

Writing this in non-conservative form raises the question of local conservation. Integrating the original equation over D^k and using Gauss' theorem gives

$$\int_{D^k} \frac{\partial u}{\partial t} + \int_{\partial D^k} \beta_n u = 0.$$

Taking $v = 1$ on D^k yields

$$\int_{D^k} \frac{\partial u}{\partial t} + \frac{1}{2} \left(\nabla_h \cdot \Pi_N \left(\boldsymbol{\beta} u \right), \mathbb{1}_{D^k} \right) + \frac{1}{2} \left(\boldsymbol{\beta} \cdot \nabla_h u, \mathbb{1}_{D^k} \right) + \frac{1}{2} \left(\left(\nabla \cdot \boldsymbol{\beta} \right) u, \mathbb{1}_{D^k} \right) = 0.$$

The first term gives

$$\left(\nabla_h \cdot \Pi_N \left(\boldsymbol{\beta} u \right), \mathbb{1}_{D^k} \right) = \int_{\partial D^k} \{\{ \Pi_N \left(\boldsymbol{\beta} u \right) \}\} \cdot \boldsymbol{n}.$$

The second term gives

$$\left(\boldsymbol{\beta} \cdot \nabla_h u, \mathbb{1}_{D^k} \right) = \left(\nabla u, \boldsymbol{\beta} \right)_{D^k} + \frac{1}{2} \langle [\![u]\!], \boldsymbol{\beta} \cdot \boldsymbol{n} \rangle = \left(u, -\nabla \cdot \boldsymbol{\beta} \right)_{D^k} + \langle \{\{ u \}\}, \boldsymbol{\beta} \cdot \boldsymbol{n} \rangle$$

through integration by parts and an assumption that $\boldsymbol{\beta} \cdot \boldsymbol{n}$ is periodic. Cancelling volume terms, we end up with the statement of local conservation

$$\int_{D^k} \frac{\partial u}{\partial t} + \frac{1}{2} \int_{\partial D^k} \left(\{\{ \Pi_N \left(\boldsymbol{\beta} u \right) \}\} + \left(\Pi_N \left(\boldsymbol{\beta} \right) \{\{ u \}\} \right) \right) \cdot \boldsymbol{n} = 0$$

which is a discrete version of the continuous statement of local conservation.

4 Penalization

Penalization can be added by adding any positive-definite stabilization term (upwind, penalty, Lax-Friedrichs) through the regular divergence flux.

5 Extension to other hyperbolic problems

Example: acoustic wave equation, simply discretize by replacing $\nabla, \nabla \cdot$ with the DG versions. Automatically skew symmetric and energy stable.

Example: Burgers' equation

Example: Entropy splitting of Buckley-Leverett.