B-SPLINE DG NOTES

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Abstract. Achieve extra h-resolution for the same CFL condition by using high order B-splines as a local approximation space on tensor product elements.

Achieve a spectral element-type implementation using quadrature nodes for B-splines.

1. Introduction.

2. Stable timestep restrictions for high order DG methods.

We illustrate this using the advection equation.

$$\begin{aligned} V^T A_h U &= \sum_k (\beta \cdot \nabla u, v)_{L^2(D^k)} + \langle f^*(u), v\rangle_{L^2(\partial D^k)} \\ &\leq \|\beta\|_{L^\infty} \sum_k \|\nabla u\| \|v\| + \|f^*(u)\|_{L^2(\partial D^k)} \|v\|_{L^2(\partial D^k)} \\ &\lesssim \|\beta\|_{L^\infty} \sum_k \left(C_I \|\boldsymbol{J}\boldsymbol{G}\|_{L^\infty(\widehat{D})} + C_N^2 \left\|J^f\right\|_{L^\infty(\widehat{D})}\right) \|1/J\|_{L^\infty(\widehat{D})} \|u\|_{L^2(D^k)} \|v\|_{L^2(D^k)} \end{aligned}$$

The CFL condition is then determined by mesh-dependent geometric factors of $O(h)$ and order-dependent constants in the inverse and trace inequalities C_I, C_N.

2.1. Explicit constants in trace and inverse inequalities for polynomial and B-spline approximations.

For polynomials, $C_I, C_N = O(N^2)$ on the reference element (see, for example [1, 2, 3]).

For B-splines with $K_{\text{sub}} = N$, $C_N = O(N)$ instead of $O(N^2)$.

For sufficiently regular solutions, convergence is in L^2 as $C(h/N)^{N+1}$ while still maintaining an $O(h/N^2)$ CFL. Or, if we measure by $h_N = h/N$, CFL is $O(h_N/N)$. This results from interpolation estimates for one-dimensional B-splines [4]

3. Reduced quadrature using Gauss-Legendre-Lobatto rules.

Can cast into collocation style using skew-symmetric formulation by using a degree $N + K_{\text{sub}} - 1$ Gauss-Legendre or Gauss-Lobatto rule.

Show plots of GLL nodal bases for polynomials vs for B-spline bases - less oscillation for basis functions at nodes on/near the boundary.

4. Future work.

GLL-based spectral elements have other advantages, notably a connection to summation by parts (SBP) operators. Wave propagation problems yield variational formulations which are a-priori stable under inexact quadrature. However, SBP properties are crucial to guaranteeing energy stability for more complex physics, such as nonlinear hyperbolic conservation laws.

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