Discretely entropy stable discontinuous Galerkin methods

Jesse Chan

[1]Department of Computational and Applied Math

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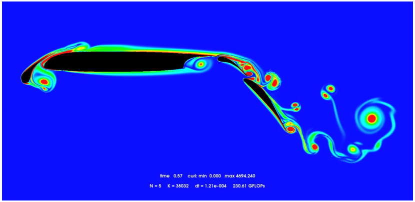
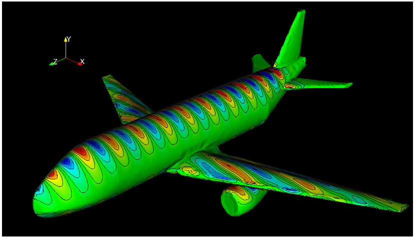
High order methods for hyperbolic PDEs

- Time-dependent solutions of wave and fluid PDEs.
- Low numerical dissipation and dispersion (waves and vortices).
- High order approximations: more accurate per unknown.
- Many-core architectures (matrix free explicit time-stepping).





Figures courtesy of T. Warburton.

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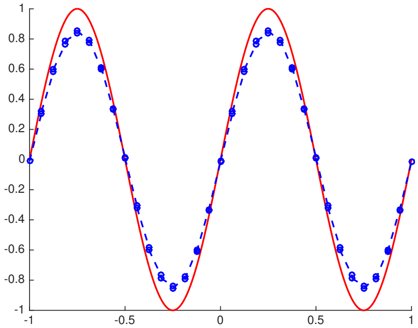

Fine linear approximation.

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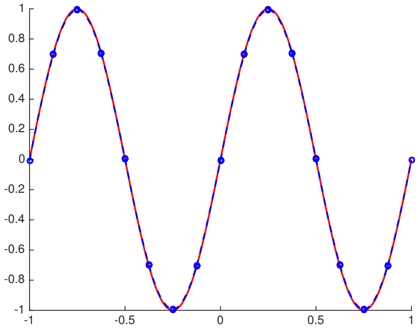

Coarse quadratic approximation.

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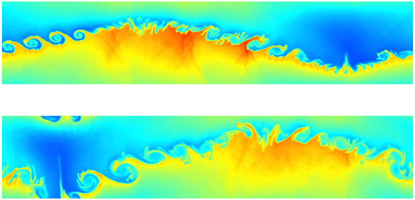

Figure courtesy of Per-Olof Persson.

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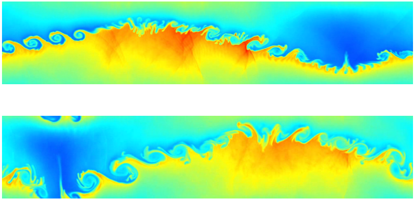

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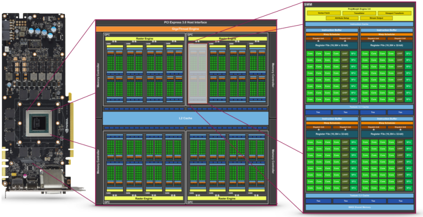

A graphics processing unit (GPU).

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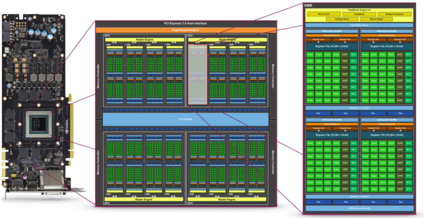


A graphics processing unit (GPU).

Goal: address inherent instability of high order methods!

Talk outline

1. High order DG methods
2. Nonlinear stability for conservation laws
3. Continuous entropy stable formulations
4. Numerical experiments, higher dimensions, curved meshes

Talk outline

1 High order DG methods

2 Nonlinear stability for conservation laws

3 Continuous entropy stable formulations

4 Numerical experiments, higher dimensions, curved meshes

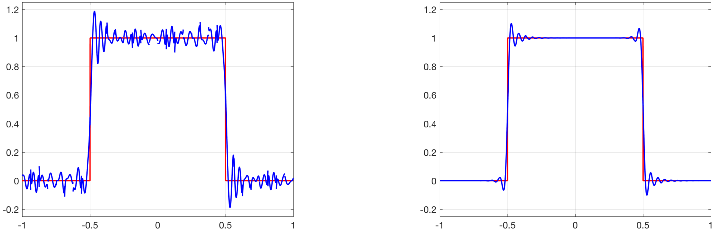
High order DG methods for linear problems

- Constant linear advection on $[-1,1]$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \qquad u(-1) = u(1).$$

- Semi-discrete form: let $[\![u]\!] = u^+ - u$ and $\tau \geq 0$

$$\sum_k \int_{D^k} \left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \right) v + \int_{\partial D^k} \frac{n_x - \tau \left| n_x \right|}{2} [\![u]\!] \, v = 0, \qquad \forall v \in V_h.$$



(a) Central flux ($\tau = 0$) (b) Upwind flux ($\tau = 1$)

DG energy estimates for linear advection

- Let $V_h = \bigoplus_k P^N\left(D^k\right)$, define global DG derivative $D_h^x : V_h \rightarrow V_h$:

$$\left(D_h^x u, v\right)_\Omega = \sum_{D^k} \left(\frac{\partial u}{\partial x}, v\right)_{D^k} + \frac{1}{2}\left\langle [\![u]\!], v\right\rangle_{\partial D^k}, \qquad v \in V_h,$$

$$\left(D_h^x u, v\right)_\Omega = \langle u, v\rangle_{\partial\Omega} - \left(u, D_h^x v\right)_\Omega, \qquad \text{(integration-by-parts)}.$$

- Advection formulation: semi-definite penalization $s(u,v)$

$$\left(\frac{\partial u}{\partial t} + D_h^x u, v\right)_\Omega + \underbrace{\sum_k \left\langle -\tau \frac{|n_x|}{2} [\![u]\!], v\right\rangle_{\partial D^k}}_{s(u,v),\ \text{pos. semi-def.}}.$$

- Energy method (periodic domains): take $v = u$, integrate by parts

$$\frac{1}{2}\frac{\partial}{\partial t}\|u\|^2_{L^2(\Omega)} = -s(u,u) \leq 0.$$

Talk outline

Entropy stability for nonlinear conservation laws

- System of nonlinear conservation laws, convex entropy $S(\boldsymbol{u})$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial x} = 0, \qquad \boldsymbol{v} = \frac{\partial S}{\partial \boldsymbol{u}}.$$

- Nonlinear entropy inequality: chain rule + entropy potential ψ

$$\boldsymbol{v}^T \left(\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial x} \right) = 0$$
$$\Longrightarrow \frac{\partial S(\boldsymbol{u})}{\partial t} + \left(\boldsymbol{v}^T \boldsymbol{f}(\boldsymbol{u}) - \psi(\boldsymbol{u}) \right) \Big|_{-1}^{1} \leq 0.$$

- Periodic linear advection: square entropy, $\boldsymbol{v} = \boldsymbol{u}$

$$\int_\Omega \frac{\partial S(u)}{\partial t} \leq 0, \qquad S(u) = \frac{u^2}{2}.$$

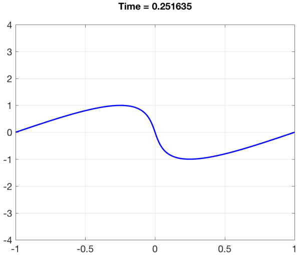
Discrete instability for nonlinear conservation laws

- How to differentiate $f(u)$? Burgers' equation: $f(u) = u^2/2$

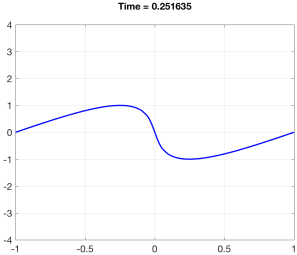
$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0, \qquad u \in V_h, \quad u^2 \notin V_h.$$

- Loss of chain rule with L^2 projection P_N or inexact quadrature.

$$\left(\frac{\partial u}{\partial t} + \frac{1}{2} D_h^x P_N u^2, v\right)_\Omega = 0, \qquad \frac{1}{2}\frac{\partial P_N u^2}{\partial x} \neq P_N\left(u\frac{\partial u}{\partial x}\right)$$



(a) $N = 7, K = 8$



(b) $N = 7, K = 9$

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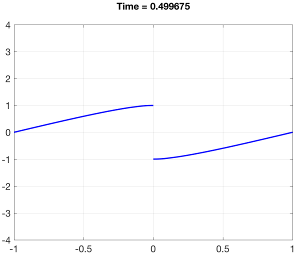

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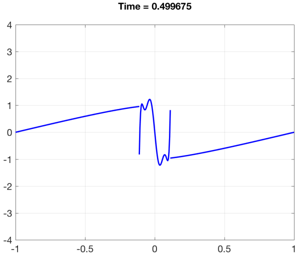

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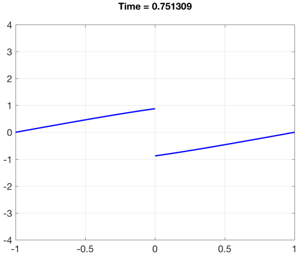

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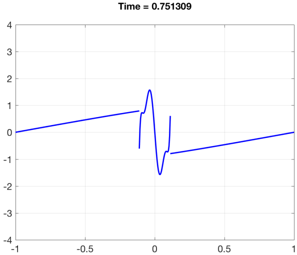

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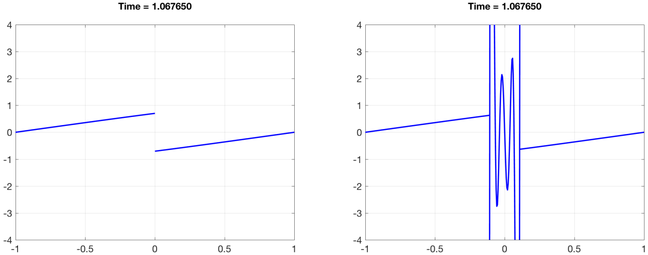

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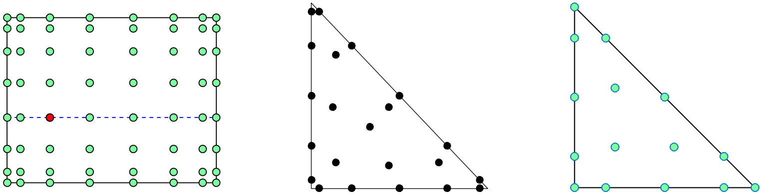
Entropy stable (ES) summation-by-parts (SBP) methods

- Entropy conservation: needs mass lumping, nodal collocation!

$$\left(\boldsymbol{I}_n \otimes \boldsymbol{M}\right)\frac{\partial \boldsymbol{u}}{\partial t} + \left(2\left(\boldsymbol{I}_n \otimes \boldsymbol{S}\right) \circ \boldsymbol{F}_S\right)\mathbf{1} = 0, \quad \text{(Semi-discrete form)}$$

$$\Longrightarrow \mathbf{1}^T\left(\boldsymbol{M}\frac{\partial S}{\partial t} + \boldsymbol{B}\left(\boldsymbol{v}^T\boldsymbol{f} - \psi\right)\right) = 0, \quad \text{(Entropy conservation)}$$



(c) Tensor product SBP nodes (GLL quadrature)

(d) Diag-norm SBP nodes ($N = 4$, 22 pts)

(e) Non-SBP nodes ($N = 4$, 15 pts)

Fisher and Carpenter (2013). *High-order ES finite difference schemes for nonlinear conservation laws: Finite domains.*
Gassner, Winters, and Kopriva (2016). *Split form nodal DG schemes with SBP property for the comp. Euler equations.*
Chen and Shu (2017). *ES high order DG methods with suitable quadrature rules for hyperbolic conservation laws.*

Talk outline

Burgers' equation: energy stable formulations

- Split formulation (replace $\frac{\partial}{\partial x}$ with some D_h^x + stabilization for DG).

$$\frac{\partial u}{\partial t} + \frac{1}{3}\left(\frac{\partial u^2}{\partial x} + u\frac{\partial u}{\partial x}\right) \Longrightarrow \frac{\partial u}{\partial t} + \frac{1}{3}\left(D_h^x P_N u^2 + P_N\left(u D_h^x u\right)\right) = 0$$

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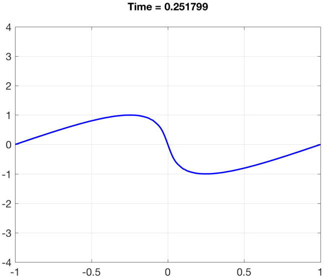
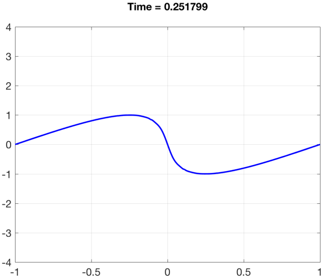


(a) $\tau = 0$



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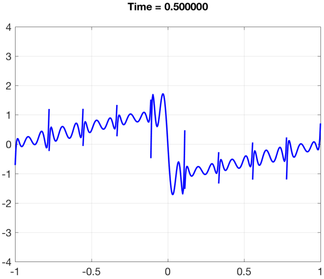

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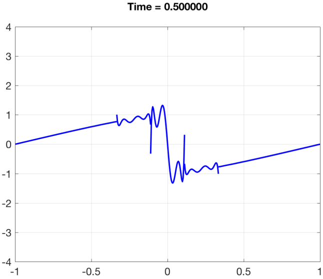

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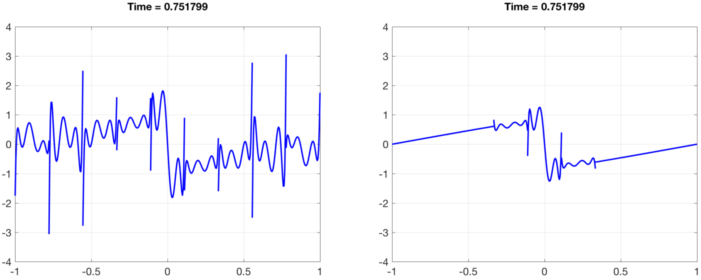

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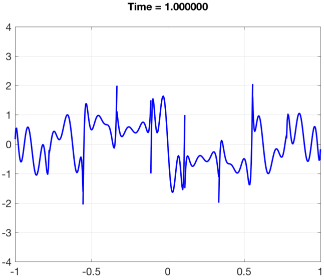

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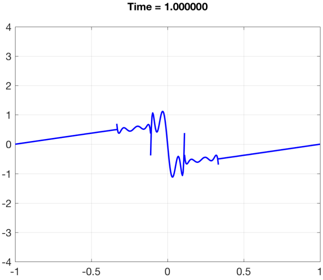

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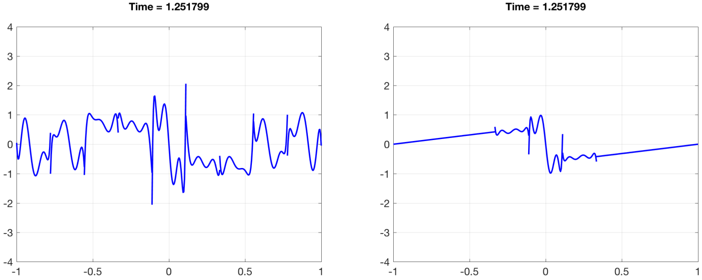

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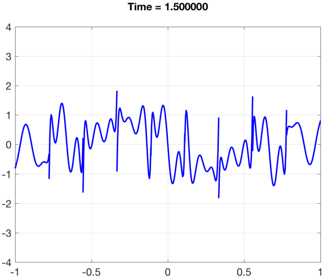

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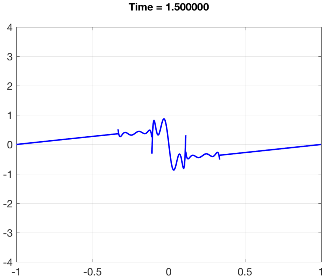

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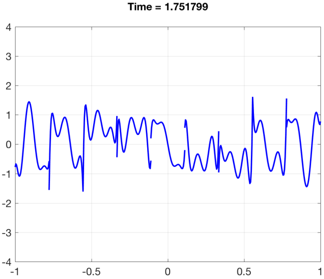

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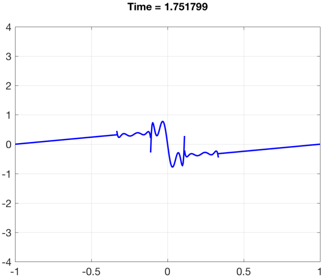

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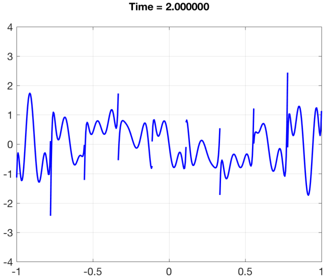

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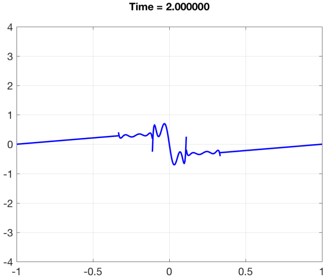

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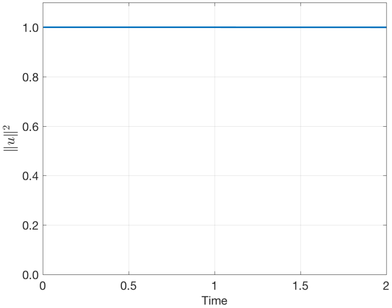
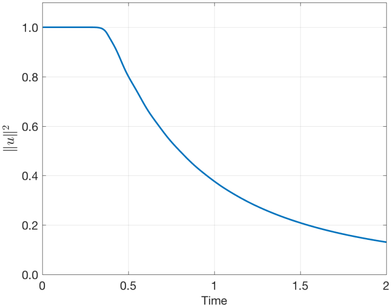


(a) Energy conservative ($\tau = 0$)



(b) Energy stable ($\tau = 1$)

Flux differencing: split formulations and beyond

- Tadmor's entropy conservative finite volume flux

$$
\begin{aligned}
\boldsymbol{f}_S(\boldsymbol{u},\boldsymbol{u}) &= \boldsymbol{f}(\boldsymbol{u}), && \text{(consistency)}\\
\boldsymbol{f}_S(\boldsymbol{u},\boldsymbol{v}) &= \boldsymbol{f}_S(\boldsymbol{v},\boldsymbol{u}), && \text{(symmetry)}\\
(\boldsymbol{v}_L - \boldsymbol{v}_R)^T \boldsymbol{f}(\boldsymbol{u}_L,\boldsymbol{u}_R) &= \psi_L - \psi_R, && \text{(conservation).}
\end{aligned}
$$

- Easy example: Burgers' equation, let $u_L = u(x), u_R = u(y)$

$$
f_S(u_L, u_R) = \frac{1}{6}\left(u(x)^2 + u(x)u(y) + u(y)^2\right),
$$

$$
\frac{\partial f(u)}{\partial x} \Longrightarrow 2\frac{\partial f_S(u_x, u_y)}{\partial x}\bigg|_{y=x} = \frac{1}{3}\frac{\partial u^2}{\partial x} + \frac{1}{3}u\frac{\partial u}{\partial x} + \frac{1}{3}u^2\frac{\cancel{\partial 1}}{\cancel{\partial x}}.
$$

- Harder example: compressible Euler (entropy conservative mass flux)

$$
f_S^{\rho}(\boldsymbol{u}_L,\boldsymbol{u}_R) = \{\{\rho\}\}^{\log}\{\{u\}\}, \qquad \{\{u\}\}^{\log} = \frac{u_L - u_R}{\log u_L - \log u_R}.
$$

Chandrashekar (2013). *Kinetic energy preserving and entropy stable FV schemes for compressible Euler and NS equations.*

Flux differencing: split formulations and beyond

- Tadmor's entropy conservative finite volume flux

$$
\begin{aligned}
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\boldsymbol{f}_S(\boldsymbol{u},\boldsymbol{v}) &= \boldsymbol{f}_S(\boldsymbol{v},\boldsymbol{u}), && \text{(symmetry)}\\
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$$\begin{aligned} \boldsymbol{f}_S(\boldsymbol{u},\boldsymbol{u}) &= \boldsymbol{f}(\boldsymbol{u}), && \text{(consistency)} \\ \boldsymbol{f}_S(\boldsymbol{u},\boldsymbol{v}) &= \boldsymbol{f}_S(\boldsymbol{v},\boldsymbol{u}), && \text{(symmetry)} \\ (\boldsymbol{v}_L-\boldsymbol{v}_R)^T\,\boldsymbol{f}\,(\boldsymbol{u}_L,\boldsymbol{u}_R) &= \psi_L-\psi_R, && \text{(conservation).} \end{aligned}$$

- Easy example: Burgers' equation, let $u_L = u(x), u_R = u(y)$

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Chandrashekar (2013). *Kinetic energy preserving and entropy stable FV schemes for compressible Euler and NS equations.*

Discrete entropy conservation: a continuous formulation

Theorem (Chan 2017)

Let $\boldsymbol{u}_x = \boldsymbol{u}\left((P_N \boldsymbol{v})(x)\right), \boldsymbol{u}_y = \boldsymbol{u}\left((P_N \boldsymbol{v})(y)\right)$*, and let* $\boldsymbol{u}$ *solve*

$$\left(\frac{\partial \boldsymbol{u}}{\partial t} + \left(2D_h^x \boldsymbol{f}_S(\boldsymbol{u}_x, \boldsymbol{u}_y)\right)\big|_{y=x}, \boldsymbol{w}\right)_\Omega = 0, \qquad \forall \boldsymbol{w} \in V_h.$$

Then $\boldsymbol{u}$ *satisfies* $\int_\Omega \frac{\partial S(\boldsymbol{u})}{\partial t} + \int_{\partial\Omega} P_N \boldsymbol{v}^T \boldsymbol{f}(\boldsymbol{u}_x) - \psi(\boldsymbol{u}_x) = 0.$

Chan (2017). *On discretely entropy conservative and entropy stable discontinuous Galerkin methods.*

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Then $\boldsymbol{u}$ *satisfies* $\int_\Omega \frac{\partial S(\boldsymbol{u})}{\partial t} + \int_{\partial\Omega} P_N \boldsymbol{v}^T \boldsymbol{f}(\boldsymbol{u}_x) - \psi(\boldsymbol{u}_x) = 0.$

Sketch of proof

Step 1 (time term): take $\boldsymbol{w} = P_N \boldsymbol{v}(\boldsymbol{u})$*. For method of lines,* $\frac{\partial \boldsymbol{u}}{\partial t} \in V_h$*.*

$$\left(\frac{\partial \boldsymbol{u}}{\partial t}, P_N \boldsymbol{v}\right)_\Omega = \left(\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v}\right)_\Omega = \left(\frac{\partial S}{\partial \boldsymbol{u}}\frac{\partial \boldsymbol{u}}{\partial t}, 1\right)_\Omega = \left(\frac{\partial S(\boldsymbol{u})}{\partial t}, 1\right)_\Omega.$$

Chan (2017). *On discretely entropy conservative and entropy stable discontinuous Galerkin methods.*

Discrete entropy conservation: a continuous formulation

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Let $\boldsymbol{u}_x = \boldsymbol{u}\left((P_N \boldsymbol{v})(x)\right), \boldsymbol{u}_y = \boldsymbol{u}\left((P_N \boldsymbol{v})(y)\right)$, *and let* $\boldsymbol{u}$ *solve*

$$\left(\frac{\partial \boldsymbol{u}}{\partial t} + \left(2D_h^x \boldsymbol{f}_S(\boldsymbol{u}_x, \boldsymbol{u}_y)\right)\Big|_{y=x}, \boldsymbol{w}\right)_{\Omega} = 0, \qquad \forall \boldsymbol{w} \in V_h.$$

Then $\boldsymbol{u}$ *satisfies* $\int_{\Omega} \frac{\partial S(\boldsymbol{u})}{\partial t} + \int_{\partial\Omega} P_N \boldsymbol{v}^T \boldsymbol{f}(\boldsymbol{u}_x) - \psi(\boldsymbol{u}_x) = 0.$

Sketch of proof

Step 2 (spatial term): integrate by parts.

$$\left(\left(D_h^x \boldsymbol{f}_S(\boldsymbol{u}_x, \boldsymbol{u}_y)\right)\Big|_{y=x}, P_N \boldsymbol{v}\right)_{\Omega}$$
$$+ \langle \boldsymbol{f}_S\left(\boldsymbol{u}_x, \boldsymbol{u}_x\right), (P_N \boldsymbol{v})\boldsymbol{n}_x\rangle_{\partial\Omega} - \left(D_h^x \left(\boldsymbol{f}_S(\boldsymbol{u}_x, \boldsymbol{u}_y)\left(P_N \boldsymbol{v}\right)(x)\right)\Big|_{y=x}, 1\right)_{\Omega}.$$

Chan (2017). *On discretely entropy conservative and entropy stable discontinuous Galerkin methods.*

Discrete entropy conservation: a continuous formulation

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Let $\boldsymbol{u}_x = \boldsymbol{u}\left((P_N \boldsymbol{v})(x)\right), \boldsymbol{u}_y = \boldsymbol{u}\left((P_N \boldsymbol{v})(y)\right)$, *and let* $\boldsymbol{u}$ *solve*

$$\left(\frac{\partial \boldsymbol{u}}{\partial t} + \left.\left(2 D_h^x \boldsymbol{f}_S(\boldsymbol{u}_x, \boldsymbol{u}_y)\right)\right|_{y=x}, \boldsymbol{w}\right)_\Omega = 0, \qquad \forall \boldsymbol{w} \in V_h.$$

Then $\boldsymbol{u}$ *satisfies* $\int_\Omega \frac{\partial S(\boldsymbol{u})}{\partial t} + \int_{\partial\Omega} P_N \boldsymbol{v}^T \boldsymbol{f}(\boldsymbol{u}_x) - \psi(\boldsymbol{u}_x) = 0.$

Sketch of proof

Step 2 (spatial term): gather volume terms, use conservation and IBP.

$$\left(D_h^x \left.\left(\boldsymbol{f}_S(\boldsymbol{u}_x, \boldsymbol{u}_y)\left((P_N \boldsymbol{v})(x) - (P_N \boldsymbol{v})(y)\right)\right)\right|_{y=x}, 1\right)_\Omega$$

$$= \left(D_h^x \left.\left(\psi(\boldsymbol{u}_x) - \psi(\boldsymbol{u}_y)\right)\right|_{y=x}, 1\right)_\Omega = \left\langle \psi(\boldsymbol{u}_x), 1\boldsymbol{n}_x\right\rangle_{\partial\Omega}.$$

Chan (2017). *On discretely entropy conservative and entropy stable discontinuous Galerkin methods.*

Discrete entropy conservation: a continuous formulation

Theorem (Chan 2017)

Let $\boldsymbol{u}_x = \boldsymbol{u}\left((P_N \boldsymbol{v})(x)\right), \boldsymbol{u}_y = \boldsymbol{u}\left((P_N \boldsymbol{v})(y)\right)$, *and let* $\boldsymbol{u}$ *solve*

$$\left(\frac{\partial \boldsymbol{u}}{\partial t} + \left(2D_h^x \boldsymbol{f}_S(\boldsymbol{u}_x, \boldsymbol{u}_y)\right)\Big|_{y=x}, \boldsymbol{w}\right)_\Omega = 0, \qquad \forall \boldsymbol{w} \in V_h.$$

Then $\boldsymbol{u}$ *satisfies* $\int_\Omega \frac{\partial S(\boldsymbol{u})}{\partial t} + \int_{\partial\Omega} P_N \boldsymbol{v}^T \boldsymbol{f}(\boldsymbol{u}_x) - \psi(\boldsymbol{u}_x) = 0.$

- Difficulty: $\boldsymbol{u} \in V_h$, but $\boldsymbol{v}(\boldsymbol{u}) \notin V_h$! Need $\boldsymbol{u} = \boldsymbol{u}\left(P_N \boldsymbol{v}\right)$ for

$$\left(P_N \boldsymbol{v}_L - P_N \boldsymbol{v}_R\right)^T \boldsymbol{f}\left(\boldsymbol{u}_L, \boldsymbol{u}_R\right) = \psi_L - \psi_R.$$

- Proof requires only (inexact) quadrature-based L^2 projection + IBP.
- Jump stabilization $s(u, v)$ gives entropy inequality.

Chan (2017). *On discretely entropy conservative and entropy stable discontinuous Galerkin methods.*

Entropy stable high order DG: implementation

- Efficient reformulation (Hadamard product: low-memory evaluation)

$$P_N\left(\left.\frac{\partial P_N \boldsymbol{f}_S(\boldsymbol{u}_x,\boldsymbol{u}_y)}{\partial x}\right|_{y=x}\right)$$
$$=\boldsymbol{P}_q \operatorname{diag}\left(\boldsymbol{V}_q\boldsymbol{D}\boldsymbol{P}_q\boldsymbol{F}_S\right)=\boldsymbol{P}_q\left(\left(\boldsymbol{V}_q\boldsymbol{D}\boldsymbol{P}_q\circ\boldsymbol{F}_S\right)\boldsymbol{1}\right).$$

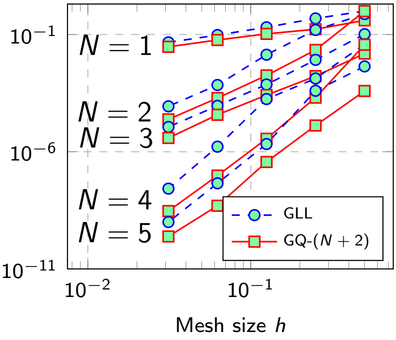
- Explicit time-stepping right hand side evaluation:
  1. Compute $P_N\left(\boldsymbol{v}(\boldsymbol{u})\right)$.
  2. Evaluate $\boldsymbol{u}=\boldsymbol{u}(P_N\left(\boldsymbol{v}(\boldsymbol{u})\right))$ at volume, face quadratures.
  3. Compute $\boldsymbol{RHS}(\boldsymbol{u})=2\left(\boldsymbol{D}_h\circ\boldsymbol{F}_S\left(\boldsymbol{u}_x,\boldsymbol{u}_y\right)\right)\boldsymbol{1}$

- Simplifications for diag-norm SBP (nodal collocation): avoid computing projections, combine volume + surface operations.

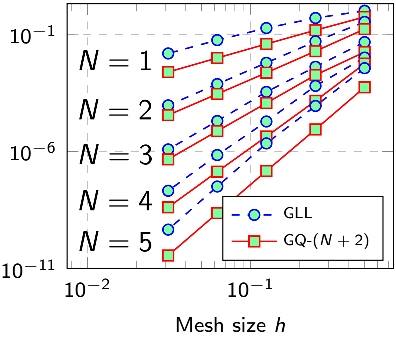
Talk outline

Numerical experiments: compressible Euler equations

- Entropy conservative (EC) and Lax-Friedrichs (LF) fluxes.
- No additional stabilization, filtering, or limiting.
- L^2 rates: odd/even decoupling for EC, $O(h^{N+1})$ for LF.



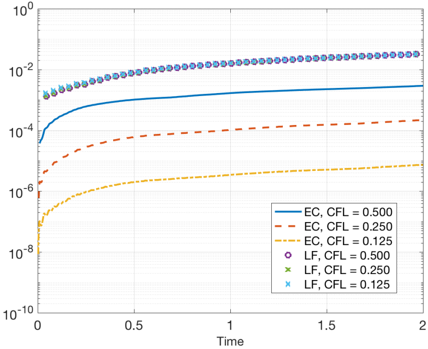
(c) Entropy conservative flux



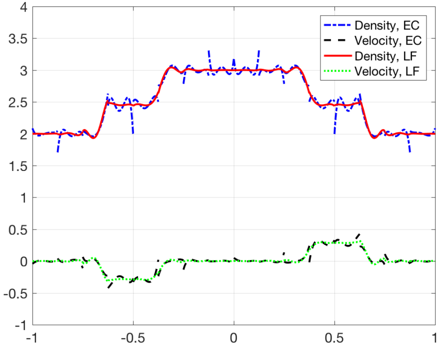
(d) With Lax-Friedrichs penalization

Numerical experiments: entropy conservation

- Entropy conservation: *semi-discrete*, not fully discrete.
- $\Delta S(\boldsymbol{u}) = |S(\boldsymbol{u}(x,t)) - S(\boldsymbol{u}(x,0))| \to 0$ as as $\Delta t \to 0$.



(e) $\Delta S(\boldsymbol{u})$ for various Δt

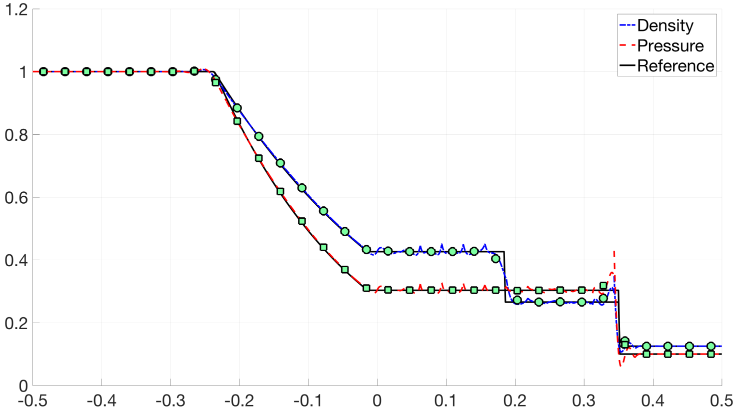


(f) $\rho(x), u(x)$ $(N = 4, K = 16)$

$\Delta S(\boldsymbol{u})$ and solution for entropy conservative (EC) and Lax-Friedrichs (LF) fluxes.

Numerical experiments: Sod shock tube

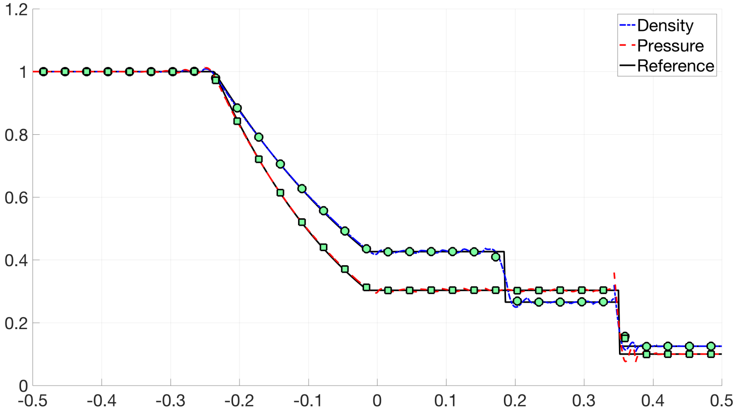
- Cell averages overlaid as circles.
- CFL of .125 used for both GLL-$(N+1)$and GQ-$(N+2)$.



$N = 4, K = 32$, $(N+1)$ point Gauss-Lobatto-Legendre quadrature.

Numerical experiments: Sod shock tube

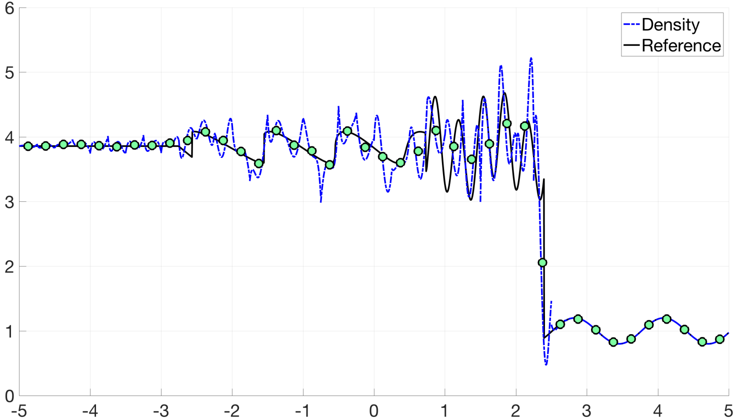
- Cell averages overlaid as circles.
- CFL of .125 used for both GLL-$(N+1)$and GQ-$(N+2)$.



$N = 4$, $K = 32$, $(N+2)$ point Gaussian quadrature.

Numerical experiments: sine-shock interaction

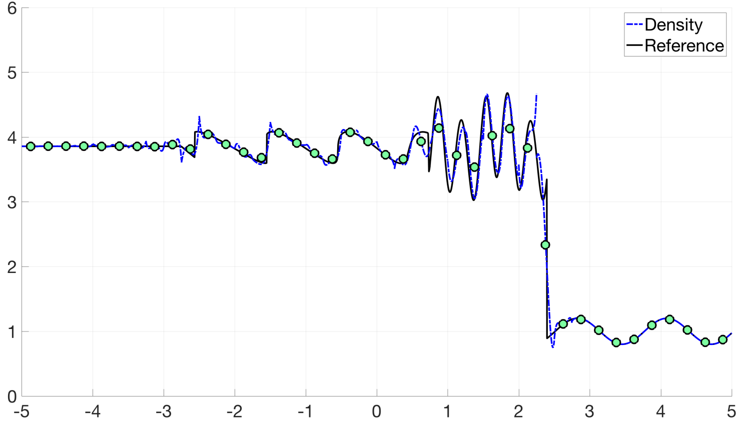
- Reference solution, smaller CFL (.05 vs .125) for GQ-$(N+2)$.



$N = 4$, $K = 40$, $(N + 1)$ point Gauss-Lobatto-Legendre quadrature.

Numerical experiments: sine-shock interaction

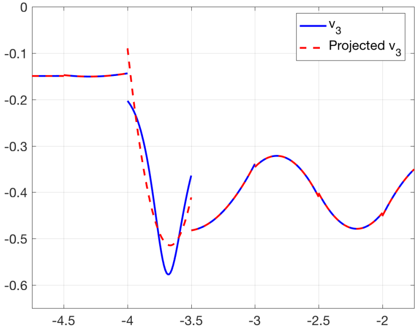
- Reference solution, smaller CFL (.05 vs .125) for GQ-$(N+2)$.



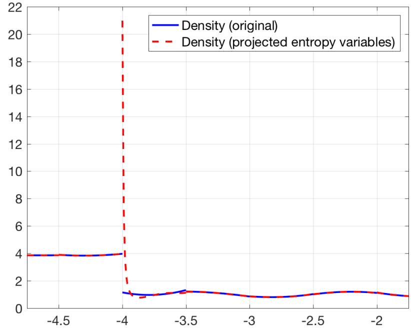
$N = 4, K = 40$, $(N+2)$ point Gaussian quadrature.

Numerical experiments: CFL restrictions

- For GLL-$(N+1)$ quadrature, $\boldsymbol{u}\left(P_N \boldsymbol{v}\right) = \boldsymbol{u}$ at GLL points.
- For GQ-$(N+2)$, discrepancy between L^2 projection and interpolation.
- Still need positivity of thermodynamic quantities!



(g) $v_3(x), \left(P_N v_3\right)(x)$



(h) $\rho(x), \rho\left(\left(P_N \boldsymbol{v}\right)(x)\right)$

Extension to higher dimensions

- Define global gradient, divergence, e.g.

$$\left(\nabla_h \cdot \boldsymbol{u}, v\right)_\Omega = \sum_k \left(\nabla \cdot \boldsymbol{u}, v\right)_{D^k} + \left\langle \frac{1}{2} [\![\boldsymbol{u}]\!] \cdot \boldsymbol{n}, v \right\rangle_{\partial D^k}$$

$$\left(\nabla_h u, \boldsymbol{v}\right)_\Omega = \sum_k \left(\nabla u, \boldsymbol{v}\right)_{D^k} + \left\langle \frac{1}{2} [\![u]\!] \boldsymbol{n}, \boldsymbol{v} \right\rangle_{\partial D^k}$$

- Flux differencing: let $\boldsymbol{u}_x = \boldsymbol{u}\left(P_N \boldsymbol{v}(\boldsymbol{x})\right), \boldsymbol{u}_y = \boldsymbol{u}\left(P_N \boldsymbol{v}(\boldsymbol{y})\right)$

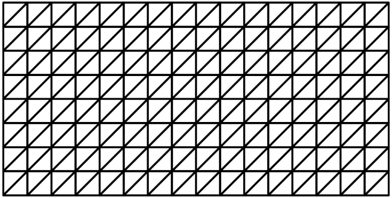
$$\left(\frac{\partial \boldsymbol{u}}{\partial t} + \left(2\nabla_h \cdot \boldsymbol{f}_S(\boldsymbol{u}_x, \boldsymbol{u}_y)\right)\big|_{\boldsymbol{y}=\boldsymbol{x}}, \boldsymbol{w}\right)_\Omega = 0, \qquad \forall \boldsymbol{w} \in V_h.$$

- Entropy stability on curved meshes: modify flux using $\boldsymbol{G}_{ij} = \frac{\partial \boldsymbol{x}_i}{\partial \widehat{\boldsymbol{x}}_j}$

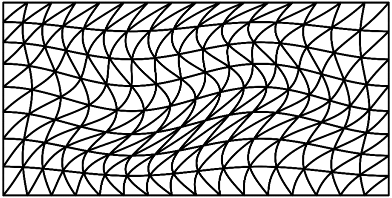
$$\tilde{\boldsymbol{f}}_S(\boldsymbol{u}_L, \boldsymbol{u}_R) = \{\{J\boldsymbol{G}\}\}\, \boldsymbol{f}_S(\boldsymbol{u}_L, \boldsymbol{u}_R).$$

Numerical results: two-dimensional curvilinear meshes

- Vortex problem at $T = 5$, $\Omega = [0, 20] \times [-5, 5]$, CFL $= .25$.
- Avoid weighted mass inverse using weight-adjusted approximation.



(a) Uniform mesh

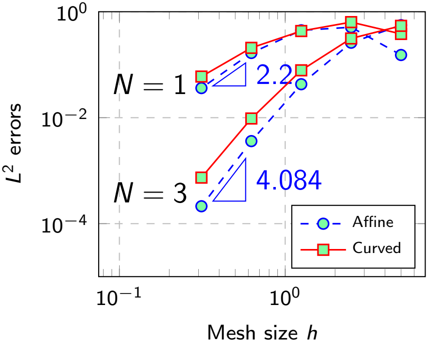


(b) Warped mesh

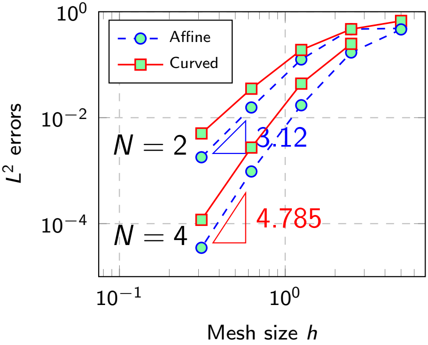
Chan, Hewett, and Warburton (2016). *Weight-adjusted discontinuous Galerkin methods: curvilinear meshes.*

Numerical results: two-dimensional curvilinear meshes

L^2 error converges at $O(h^{N+1})$ up to time-stepper accuracy (LSERK-45).



(a) Degree $N = 1, N = 3$



(b) Degree $N = 2, N = 4$

Summary and acknowledgements

- Derived discretely entropy stable high order discontinuous Galerkin methods using a continuous formulation.
- Future work: regularization, multi-GPU (with Lucas Wilcox).
- This research is supported by the National Science Foundation under awards DMS-1712639 and DMS-1719818.

Thank you! Questions?

Chan, Hewett, and Warburton (2016). *Weight-adjusted discontinuous Galerkin methods: curvilinear meshes.*

Chan (2017). *On discretely entropy conservative and entropy stable discontinuous Galerkin methods.*

Additional slides

Global DG differentiation operator

- Let $v \in V_h$, $u, w \notin V_h$ with u, w bounded; modified D_h^x

$$\left(D_h^x u, vw\right)_\Omega = \sum_k \left(\frac{\partial P_N u}{\partial x}, vw\right)_{D^k} + \frac{1}{2}\left\langle u^+ - P_N u, vw\boldsymbol{n}_x\right\rangle_{\partial D^k} + \frac{1}{2}\left\langle u - P_N u, P_N\left(vw\right)\boldsymbol{n}_x\right\rangle_{\partial D^k}.$$

- Integration-by-parts property

$$\left(D_h^x u, vw\right)_\Omega = \left\langle u, vw\right\rangle_{\partial\Omega} - \left(u, D_h^x\left(vw\right)\right)_\Omega.$$

- Coupling only through surface values (in contrast to D_h^x with $[\![P_N u]\!]$).

Chen and Shu (2017). *ES high order DG methods with suitable quadrature rules for hyperbolic conservation laws.*