Discretely entropy stable discontinuous Galerkin methods

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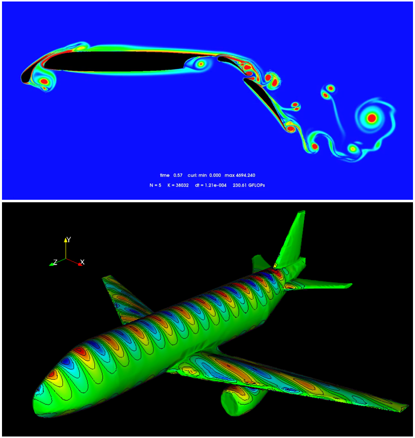
High order methods for hyperbolic PDEs

- Time-dependent solutions of wave and fluid PDEs.
- Low numerical dissipation and dispersion (waves and vortices).
- High order approximations: more accurate per unknown.
- Many-core architectures (matrix free explicit time-stepping).



Figures courtesy of T. Warburton.

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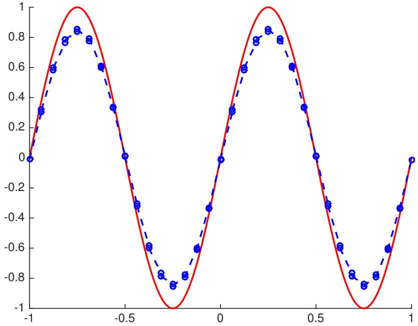

Fine linear approximation.

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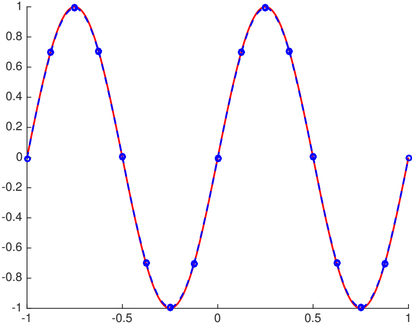

Coarse quadratic approximation.

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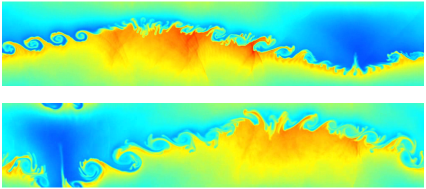

Figure courtesy of Per-Olof Persson.

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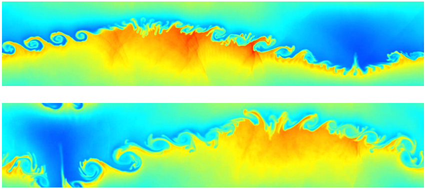

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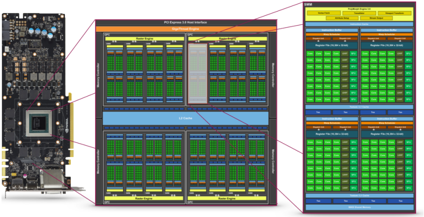

A graphics processing unit (GPU).

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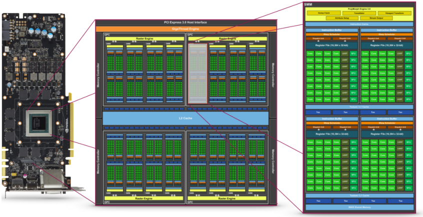


A graphics processing unit (GPU).

Goal: address inherent instability of high order methods!

Talk outline

1. High order DG methods: linear problems
2. Entropy stability for nonlinear conservation laws
3. Entropy stable formulations
4. Numerical experiments, higher dimensions, curved meshes

Talk outline

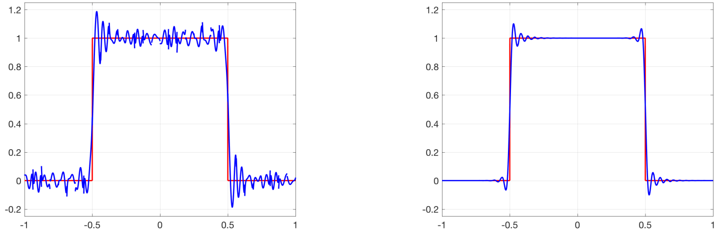
High order DG methods for linear problems

- Constant linear advection on $\Omega = [-1,1]$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \qquad u(-1) = u(1), \qquad \Longrightarrow \frac{\partial}{\partial t}\|u\|^2_{L^2(\Omega)} = 0.$$

- Local DG formulation on element D^k: let $[\![u]\!] = u^+ - u$ and $\tau \geq 0$

$$\left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}, v\right)_{D^k} + \left\langle \frac{n_x - \tau\,|n_x|}{2}[\![u]\!], v\right\rangle_{\partial D^k} = 0, \qquad \forall v \in P^N(D^k).$$



(a) Central flux ($\tau = 0$)

(b) Upwind flux ($\tau = 1$)

DG energy estimates for linear advection

- Let $V_h = \bigoplus_k P^N(D^k)$, define global DG derivative $D_h^x : V_h \to V_h$:

$$\left(D_h^x u, v\right)_{\Omega} = \sum_k \left(\frac{\partial u}{\partial x}, v\right)_{D^k} + \frac{1}{2}\left\langle [\![u]\!], v n_x\right\rangle_{\partial D^k}, \qquad v \in V_h,$$

- Advection formulation: positive semi-definite penalization $s_\tau(u, v)$

$$\left(\frac{\partial u}{\partial t} + D_h^x u, v\right)_{\Omega} + \underbrace{\sum_k \left\langle -\tau \frac{|n_x|}{2} [\![u]\!], v\right\rangle_{\partial D^k}}_{s_\tau(u,v)} = 0.$$

- Energy method (periodic domains): integrate by parts, take $v = u$.

$$\left(D_h^x u, v\right)_{\Omega} = \langle u, v\rangle_{\partial\Omega} - \left(u, D_h^x v\right)_{\Omega}, \qquad \text{(integration-by-parts)}$$

$$\Longrightarrow \frac{1}{2}\frac{\partial}{\partial t}\|u\|^2_{L^2(\Omega)} = -s_\tau(u, u) \leq 0.$$

Talk outline

Entropy stability for nonlinear conservation laws

- System of nonlinear conservation laws, convex entropy $S(\boldsymbol{u})$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial x} = 0.$$

- Nonlinear entropy inequality: chain rule + entropy potential ψ

$$\boldsymbol{v}^T \left(\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial x} \right) = 0, \qquad \boldsymbol{v} = \frac{\partial S}{\partial \boldsymbol{u}}$$

$$\Longrightarrow \int_\Omega \frac{\partial S(\boldsymbol{u})}{\partial t} + \left(\boldsymbol{v}^T \boldsymbol{f}(\boldsymbol{u}) - \psi(\boldsymbol{u}) \right) \Big|_{-1}^{1} \leq 0.$$

- Periodic linear advection: square entropy, $\boldsymbol{v} = \boldsymbol{u}$

$$\int_\Omega \frac{\partial S(u)}{\partial t} \leq 0, \qquad S(u) = \frac{u^2}{2}.$$

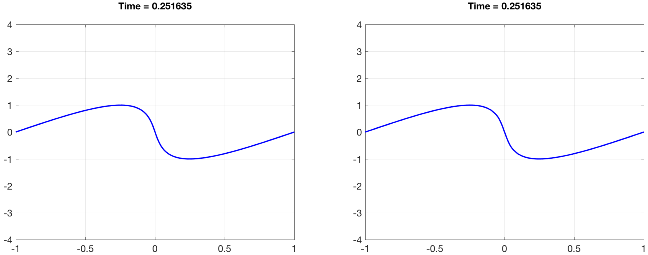
Discrete instability for nonlinear conservation laws

- How to differentiate $f(u)$? Burgers' equation: $f(u) = u^2/2$

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0, \qquad u \in P^N(D^k), \quad u^2 \notin P^N(D^k).$$

- Loss of chain rule with L^2 projection P_N and inexact quadrature.

$$\left(\frac{\partial u}{\partial t} + \frac{1}{2} D_h^x P_N u^2, v\right)_\Omega = 0, \qquad \frac{1}{2}\frac{\partial P_N u^2}{\partial x} \neq P_N\left(u\frac{\partial u}{\partial x}\right)$$



(a) $N = 7, K = 8$ (b) $N = 7, K = 9$

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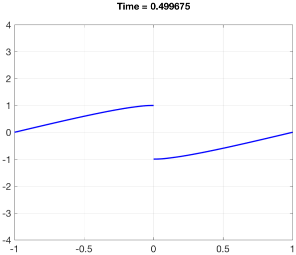

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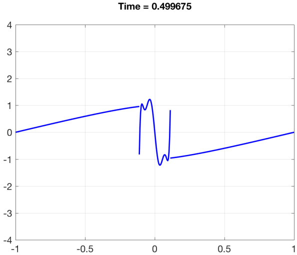

(b) $N = 7, K = 9$

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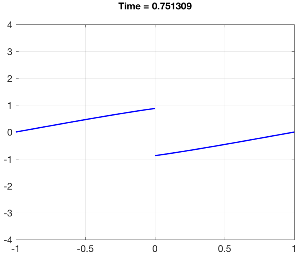

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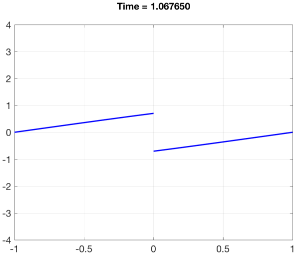

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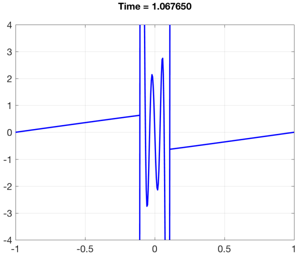


(b) $N = 7, K = 9$

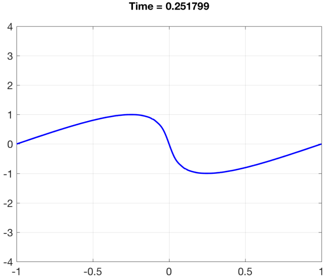
Talk outline

Burgers' equation: energy stable formulations

- Split formulation (replace $\frac{\partial}{\partial x}$ with $D_h^x P_N$ + stabilization).

$$\frac{\partial u}{\partial t} + \frac{1}{3}\left(\frac{\partial u^2}{\partial x} + u\frac{\partial u}{\partial x}\right) = 0$$

$$\Longrightarrow \left(\frac{\partial u}{\partial t} + \frac{1}{3}\left(D_h^x P_N u^2 + u D_h^x u\right), v\right) + s_\tau(u,v) = 0.$$



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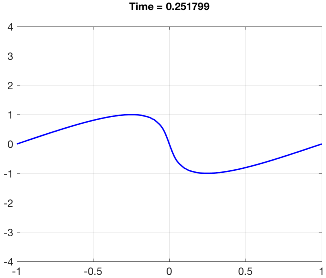

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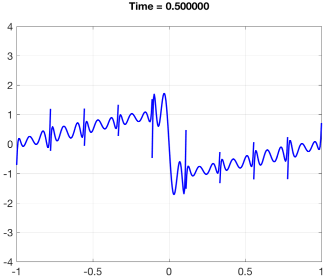

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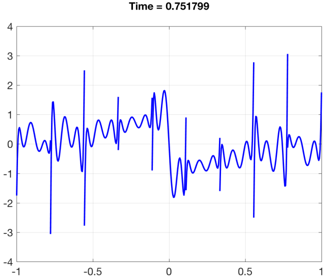

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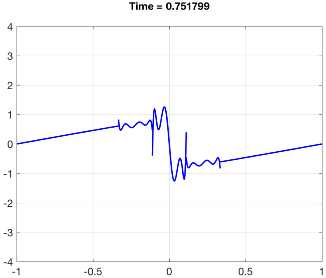

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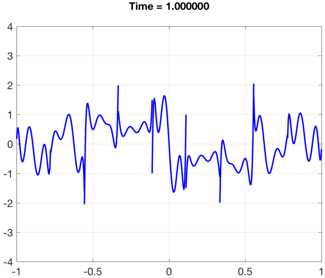

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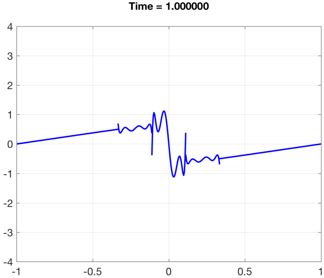

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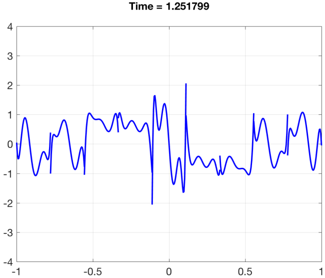

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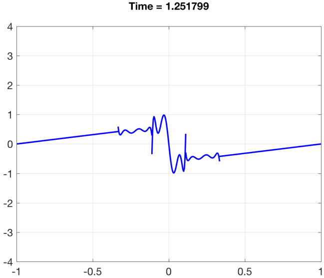

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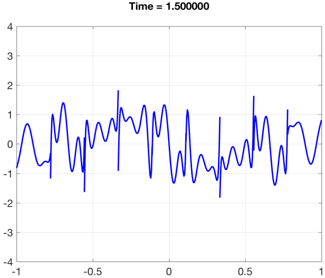

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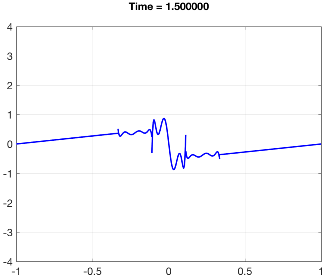

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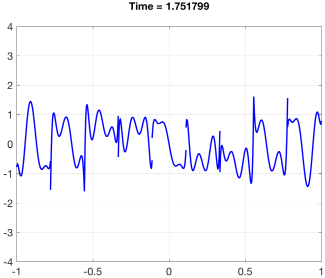

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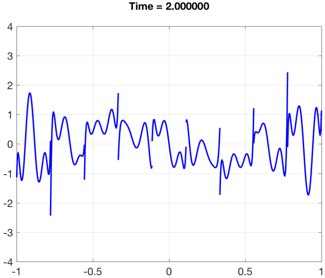

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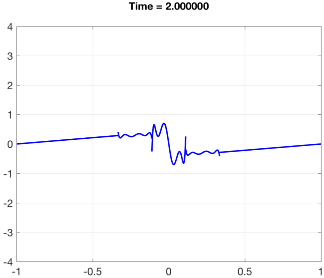

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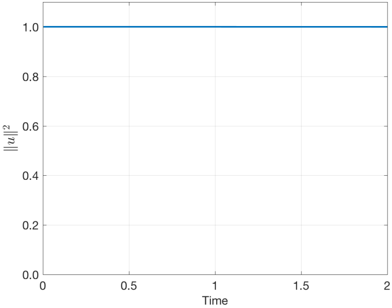
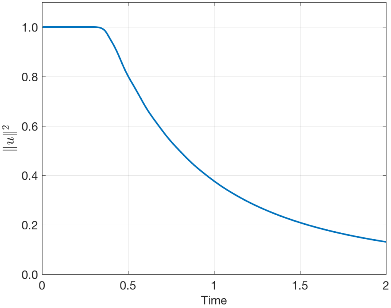


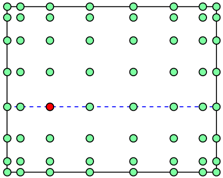
(a) Energy conservative ($\tau = 0$)



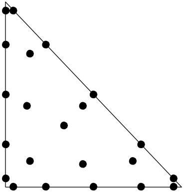
(b) Energy stable ($\tau = 1$)

Entropy conservative summation-by-parts (SBP) methods

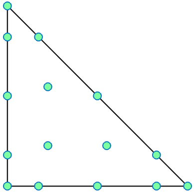
- Pioneering work relies on mass lumping, nodal collocation + SBP.
- Need at least $2N-1$ accurate volume and surface quadrature!
- Difficult to generalize beyond polynomials (splines, pyramids, etc).



(c) Tensor product SBP nodes (GLL quadrature)



(d) Diag-norm SBP nodes ($N = 4$, 22 pts)



(e) Non-SBP nodes ($N = 4$, 15 pts)

Fisher and Carpenter (2013). *High-order ES finite difference schemes for nonlinear conservation laws: Finite domains.*

Gassner, Winters, and Kopriva (2016). *Split form nodal DG schemes with SBP property for the comp. Euler equations.*

Chen and Shu (2017). *ES high order DG methods with suitable quadrature rules for hyperbolic conservation laws.*

Flux differencing: split formulations and beyond

- Tadmor's entropy conservative finite volume flux

$$\begin{aligned} \boldsymbol{f}_S(\boldsymbol{u},\boldsymbol{u}) &= \boldsymbol{f}(\boldsymbol{u}), && \text{(consistency)}\\ \boldsymbol{f}_S(\boldsymbol{u},\boldsymbol{v}) &= \boldsymbol{f}_S(\boldsymbol{v},\boldsymbol{u}), && \text{(symmetry)}\\ \left(\boldsymbol{v}_L-\boldsymbol{v}_R\right)^T \boldsymbol{f}\left(\boldsymbol{u}_L,\boldsymbol{u}_R\right) &= \psi_L-\psi_R, && \text{(conservation).} \end{aligned}$$

- Easy example: Burgers' equation, let $u_L = u(x), u_R = u(y)$

$$f_S(u_L,u_R) = \frac{1}{6}\left(u_L^2 + u_L u_R + u_R^2\right),$$

$$\frac{\partial f(u)}{\partial x} \Longrightarrow 2\frac{\partial f_S\left(u(x),u(y)\right)}{\partial x}\bigg|_{y=x} = \frac{1}{3}\frac{\partial u^2}{\partial x} + \frac{1}{3}u\frac{\partial u}{\partial x} + \frac{1}{3}u^2\frac{\partial \cancel{1}}{\cancel{\partial} x}.$$

- Harder example: compressible Euler (entropy conservative mass flux)

$$f_S^{\rho}(\boldsymbol{u}_L,\boldsymbol{u}_R) = \{\{\rho\}\}^{\log}\{\{u\}\}, \qquad \{\{u\}\}^{\log} = \frac{u_L-u_R}{\log u_L - \log u_R}.$$

Tadmor, Eitan (1987). *The numerical viscosity of entropy stable schemes for systems of conservation laws. I.*

Chandrashekar (2013). *Kinetic energy preserving and entropy stable FV schemes for compressible Euler and NS equations.*

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(\boldsymbol{v}_L - \boldsymbol{v}_R)^T \boldsymbol{f}(\boldsymbol{u}_L, \boldsymbol{u}_R) &= \psi_L - \psi_R, && \text{(conservation).}
\end{aligned}
$$

- Easy example: Burgers' equation, let $u_L = u(x), u_R = u(y)$

$$
f_S(u_L, u_R) = \frac{1}{6}\left(u_L^2 + u_L u_R + u_R^2\right),
$$

$$
\frac{\partial f(u)}{\partial x} \Longrightarrow 2\left.\frac{\partial f_S(u(x), u(y))}{\partial x}\right|_{y=x} = \frac{1}{3}\frac{\partial u^2}{\partial x} + \frac{1}{3}u\frac{\partial u}{\partial x} + \frac{1}{3}u^2\frac{\cancel{\partial 1}}{\cancel{\partial x}}.
$$

- Harder example: compressible Euler (entropy conservative mass flux)

$$
f_S^\rho(\boldsymbol{u}_L, \boldsymbol{u}_R) = \{\{\rho\}\}^{\log}\{\{u\}\}, \qquad \{\{u\}\}^{\log} = \frac{u_L - u_R}{\log u_L - \log u_R}.
$$

Tadmor, Eitan (1987). *The numerical viscosity of entropy stable schemes for systems of conservation laws. I.*

Chandrashekar (2013). *Kinetic energy preserving and entropy stable FV schemes for compressible Euler and NS equations.*

Flux differencing: split formulations and beyond

- Tadmor's entropy conservative finite volume flux

$$\begin{aligned} \boldsymbol{f}_S(\boldsymbol{u},\boldsymbol{u}) &= \boldsymbol{f}(\boldsymbol{u}), && \text{(consistency)}\\ \boldsymbol{f}_S(\boldsymbol{u},\boldsymbol{v}) &= \boldsymbol{f}_S(\boldsymbol{v},\boldsymbol{u}), && \text{(symmetry)}\\ (\boldsymbol{v}_L-\boldsymbol{v}_R)^T\,\boldsymbol{f}\,(\boldsymbol{u}_L,\boldsymbol{u}_R) &= \psi_L-\psi_R, && \text{(conservation).} \end{aligned}$$

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Chandrashekar (2013). *Kinetic energy preserving and entropy stable FV schemes for compressible Euler and NS equations.*

Discrete entropy conservation: a continuous formulation

Theorem (Chan 2017)

Let $\boldsymbol{u}_x = \boldsymbol{u}\left((P_N \boldsymbol{v})(x)\right), \boldsymbol{u}_y = \boldsymbol{u}\left((P_N \boldsymbol{v})(y)\right)$*, and let* $\boldsymbol{u}$ *solve*

$$\left(\frac{\partial \boldsymbol{u}}{\partial t} + \left.\left(2D_h^x \boldsymbol{f}_S(\boldsymbol{u}_x, \boldsymbol{u}_y)\right)\right|_{y=x}, \boldsymbol{w}\right)_\Omega = 0, \qquad \forall \boldsymbol{w} \in V_h.$$

Then $\boldsymbol{u}$ *satisfies* $\int_\Omega \frac{\partial S(\boldsymbol{u})}{\partial t} + \int_{\partial\Omega} \left(P_N \boldsymbol{v}^T \boldsymbol{f}(\boldsymbol{u}_x) - \psi(\boldsymbol{u}_x)\right) n_x = 0.$

Chan (2017). *On discretely entropy conservative and entropy stable discontinuous Galerkin methods.*

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Sketch of proof

Step 1 (time term): take $\boldsymbol{w} = P_N \boldsymbol{v}(\boldsymbol{u})$*. Assume continuity in time and method of lines s.t.* $\frac{\partial \boldsymbol{u}}{\partial t} \in V_h$*.*

$$\left(\frac{\partial \boldsymbol{u}}{\partial t}, P_N \boldsymbol{v}\right)_\Omega = \left(\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v}\right)_\Omega = \left(\frac{\partial S}{\partial \boldsymbol{u}} \frac{\partial \boldsymbol{u}}{\partial t}, 1\right)_\Omega = \left(\frac{\partial S(\boldsymbol{u})}{\partial t}, 1\right)_\Omega$$

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Discrete entropy conservation: a continuous formulation

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Let $\boldsymbol{u}_x = \boldsymbol{u}\left((P_N \boldsymbol{v})(x)\right), \boldsymbol{u}_y = \boldsymbol{u}\left((P_N \boldsymbol{v})(y)\right)$*, and let* $\boldsymbol{u}$ *solve*

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Sketch of proof

Step 2 (spatial term): integrate by parts.

$$\left(\left.\left(D_h^x \boldsymbol{f}_S(\boldsymbol{u}_x, \boldsymbol{u}_y)\right)\right|_{y=x}, P_N \boldsymbol{v}\right)_\Omega$$
$$+ \left\langle \boldsymbol{f}_S\left(\boldsymbol{u}_x, \boldsymbol{u}_x\right), (P_N \boldsymbol{v}) n_x \right\rangle_{\partial\Omega} - \left(\left. D_h^x \left(\boldsymbol{f}_S(\boldsymbol{u}_x, \boldsymbol{u}_y)\left(P_N \boldsymbol{v}\right)(x)\right)\right|_{y=x}, 1\right)_\Omega .$$

Chan (2017). *On discretely entropy conservative and entropy stable discontinuous Galerkin methods.*

Discrete entropy conservation: a continuous formulation

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Let $\boldsymbol{u}_x = \boldsymbol{u}\left((P_N\boldsymbol{v})(x)\right), \boldsymbol{u}_y = \boldsymbol{u}\left((P_N\boldsymbol{v})(y)\right)$, *and let* $\boldsymbol{u}$ *solve*

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Sketch of proof

Step 2 (spatial term): gather volume terms, use conservation and IBP.

$$\left(D_h^x \left.\left(\boldsymbol{f}_S(\boldsymbol{u}_x, \boldsymbol{u}_y)\left((P_N\boldsymbol{v})(x) - (P_N\boldsymbol{v})(y)\right)\right)\right|_{y=x}, 1\right)_\Omega$$
$$= \left(D_h^x \left.\left(\psi(\boldsymbol{u}_x) - \psi(\boldsymbol{u}_y)\right)\right|_{y=x}, 1\right)_\Omega = \langle \psi(\boldsymbol{u}_x), 1 n_x\rangle_{\partial\Omega}.$$

Chan (2017). *On discretely entropy conservative and entropy stable discontinuous Galerkin methods.*

Discrete entropy conservation: a continuous formulation

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Let $\boldsymbol{u}_x = \boldsymbol{u}\left((P_N \boldsymbol{v})(x)\right), \boldsymbol{u}_y = \boldsymbol{u}\left((P_N \boldsymbol{v})(y)\right)$*, and let* $\boldsymbol{u}$ *solve*

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Then $\boldsymbol{u}$ *satisfies* $\int_\Omega \frac{\partial S(\boldsymbol{u})}{\partial t} + \int_{\partial\Omega} \left(P_N \boldsymbol{v}^T \boldsymbol{f}(\boldsymbol{u}_x) - \psi(\boldsymbol{u}_x)\right) n_x = 0.$

- Difficulty: $\boldsymbol{u} \in V_h$, but $\boldsymbol{v}(\boldsymbol{u}) \notin V_h$! Need $\boldsymbol{u} = \boldsymbol{u}\left(P_N \boldsymbol{v}\right)$ for

$$\left(P_N \boldsymbol{v}_L - P_N \boldsymbol{v}_R\right)^T \boldsymbol{f}\left(\boldsymbol{u}_L, \boldsymbol{u}_R\right) = \psi_L - \psi_R.$$

- Proof requires only (inexact) quadrature-based L^2 projection + IBP.
- Jump stabilization $s_\tau(u, v)$ (e.g. Lax-Friedrichs) for entropy inequality.

Chan (2017). *On discretely entropy conservative and entropy stable discontinuous Galerkin methods.*

Entropy stable high order DG: implementation

- Explicit time-stepping right hand side evaluation:
  1. Compute $P_N(\boldsymbol{v}(\boldsymbol{u}))$.
  2. Evaluate $\boldsymbol{u} = \boldsymbol{u}(P_N(\boldsymbol{v}(\boldsymbol{u})))$ at volume, face quadratures.
  3. Compute $\boldsymbol{RHS}(\boldsymbol{u}) = 2\left(\boldsymbol{D}_h^x \circ \boldsymbol{f}_S(\boldsymbol{u}_x, \boldsymbol{u}_y)\right)\mathbf{1}$

- Efficient reformulation (Hadamard product: low-memory evaluation)

$$P_N\left(\left.\frac{\partial P_N \boldsymbol{f}_S(\boldsymbol{u}_x, \boldsymbol{u}_y)}{\partial x}\right|_{y=x}\right)$$
$$= \boldsymbol{P}_q \operatorname{diag}\left(\boldsymbol{V}_q \boldsymbol{D} \boldsymbol{P}_q \boldsymbol{F}_S\right) = \boldsymbol{P}_q\left(\left(\left(\boldsymbol{V}_q \boldsymbol{D} \boldsymbol{P}_q\right) \circ \boldsymbol{F}_S\right)\mathbf{1}\right).$$

- Simplifications for diag-norm SBP (nodal collocation): avoid computing projections, combine volume + surface operations.

Talk outline

1 High order DG methods: linear problems

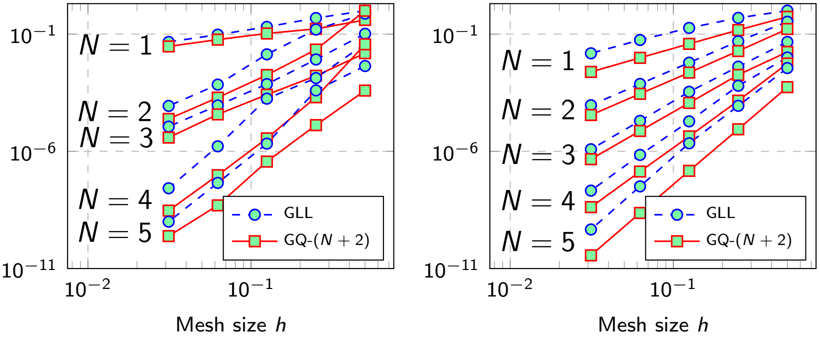
2 Entropy stability for nonlinear conservation laws

3 Entropy stable formulations

4 Numerical experiments, higher dimensions, curved meshes

Numerical experiments: compressible Euler equations

- Gauss-Legendre-Lobatto (GLL) vs Gauss (GQ) quadratures.
- Entropy conservative (EC) and Lax-Friedrichs (LF) fluxes.
- No additional stabilization, filtering, or limiting.

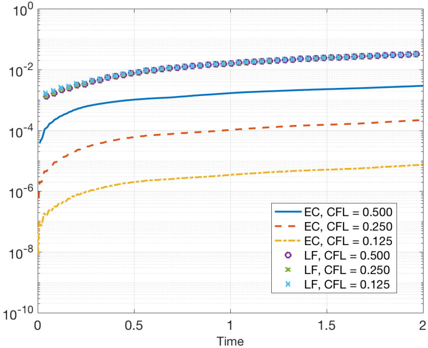


(f) Entropy conservative flux

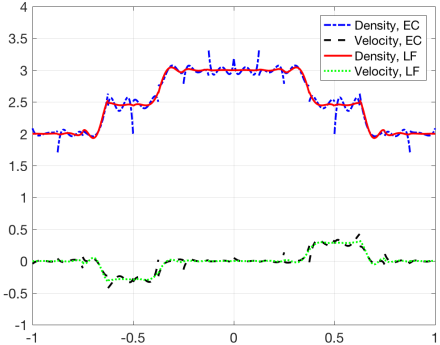
(g) With Lax-Friedrichs penalization

Numerical experiments: entropy conservation

- Entropy conservation: *semi-discrete*, not fully discrete.
- $\Delta S(\boldsymbol{u}) = |S(\boldsymbol{u}(x,t)) - S(\boldsymbol{u}(x,0))| \to 0$ as as $\Delta t \to 0$.



(h) $\Delta S(\boldsymbol{u})$ for various Δt

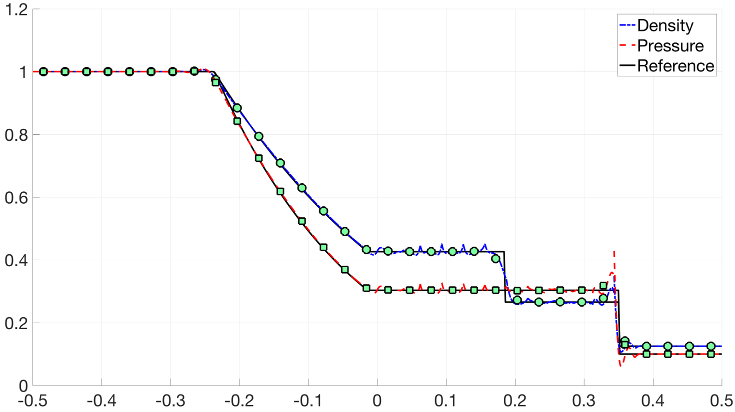


(i) $\rho(x), u(x)$ $(N = 4, K = 16)$

$\Delta S(\boldsymbol{u})$ and solution for entropy conservative (EC) and Lax-Friedrichs (LF) fluxes.

Numerical experiments: Sod shock tube

- Cell averages overlaid as circles.
- CFL of .125 used for both GLL-($N+1$)and GQ-($N+2$).



$N = 4$, $K = 32$, $(N+1)$ point Gauss-Lobatto-Legendre quadrature.

Numerical experiments: Sod shock tube

- Cell averages overlaid as circles.
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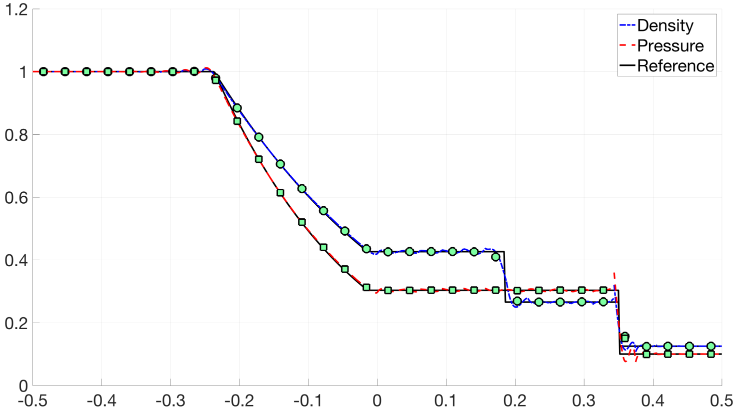
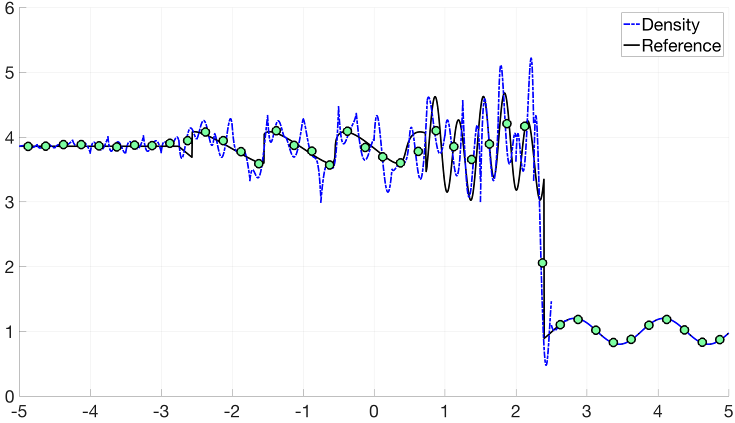


$N = 4, K = 32$, $(N+2)$ point Gauss quadrature.

Numerical experiments: sine-shock interaction

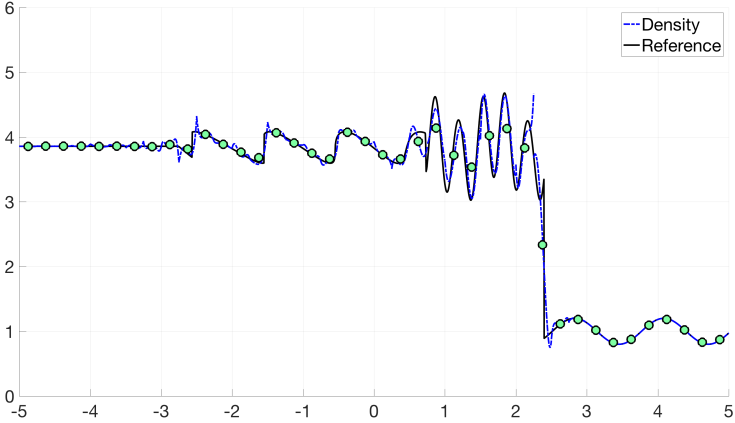
- GQ-$(N+2)$ needs smaller CFL (.05 vs .125) for stability.



$N = 4$, $K = 40$, $CFL = .05$, $(N+1)$ point Gauss-Lobatto-Legendre quadrature.

Numerical experiments: sine-shock interaction

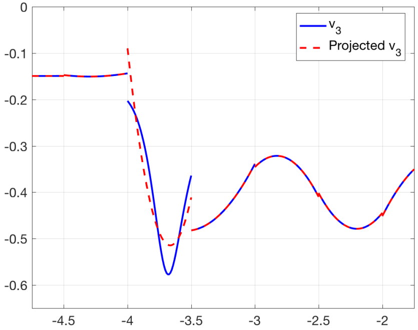
- GQ-($N+2$) needs smaller CFL (.05 vs .125) for stability.



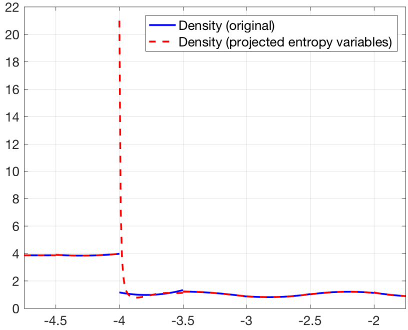
$N = 4$, $K = 40$, $CFL = .05$, $(N+2)$ point Gauss quadrature.

Numerical experiments: CFL restrictions

- For GLL-$(N+1)$ quadrature, $\boldsymbol{u}\left(P_N \boldsymbol{v}\right) = \boldsymbol{u}$ at GLL points.
- For GQ-$(N+2)$, discrepancy between L^2 projection and interpolation.
- Still need positivity of thermodynamic quantities!



(j) $v_3(x), \left(P_N v_3\right)(x)$



(k) $\rho(x), \rho\left(\left(P_N \boldsymbol{v}\right)(x)\right)$

Extension to higher dimensions

- Define global gradient, divergence, e.g.

$$\left(\nabla_h \cdot \boldsymbol{u}, v\right)_\Omega = \sum_k \left(\nabla \cdot \boldsymbol{u}, v\right)_{D^k} + \left\langle \frac{1}{2} [\![\boldsymbol{u}]\!] \cdot \boldsymbol{n}, v \right\rangle_{\partial D^k}$$

$$\left(\nabla_h u, \boldsymbol{v}\right)_\Omega = \sum_k \left(\nabla u, \boldsymbol{v}\right)_{D^k} + \left\langle \frac{1}{2} [\![u]\!] \, \boldsymbol{n}, \boldsymbol{v} \right\rangle_{\partial D^k}$$

- Flux differencing: let $\boldsymbol{u}_x = \boldsymbol{u}\left(P_N \boldsymbol{v}(\boldsymbol{x})\right), \boldsymbol{u}_y = \boldsymbol{u}\left(P_N \boldsymbol{v}(\boldsymbol{y})\right)$

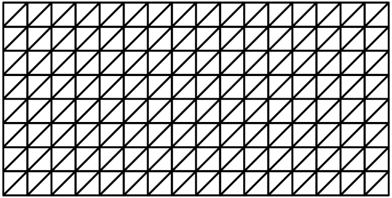
$$\left(\frac{\partial \boldsymbol{u}}{\partial t} + \left(2\nabla_h \cdot \boldsymbol{f}_S(\boldsymbol{u}_x, \boldsymbol{u}_y)\right)\big|_{\boldsymbol{y}=\boldsymbol{x}}, \boldsymbol{w}\right)_\Omega = 0, \qquad \forall \boldsymbol{w} \in V_h.$$

- Entropy stability on curved meshes: modify flux using $\boldsymbol{G}_{ij} = \frac{\partial \boldsymbol{x}_i}{\partial \widehat{\boldsymbol{x}}_j}$

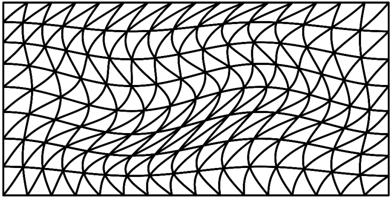
$$\tilde{\boldsymbol{f}}_S(\boldsymbol{u}_L, \boldsymbol{u}_R) = \{\{J\boldsymbol{G}\}\} \, \boldsymbol{f}_S(\boldsymbol{u}_L, \boldsymbol{u}_R).$$

Numerical results: two-dimensional curvilinear meshes

- Vortex problem at $T = 5$, $\Omega = [0, 20] \times [-5, 5]$, CFL $= .25$.
- Avoid weighted mass inverse using weight-adjusted approximation.



(a) Uniform mesh

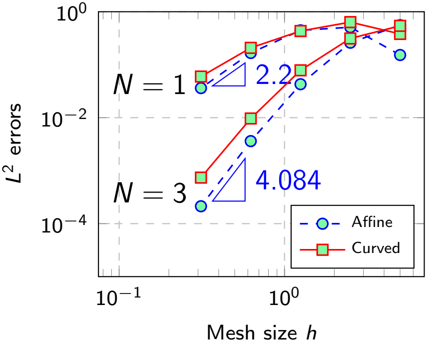


(b) Warped mesh

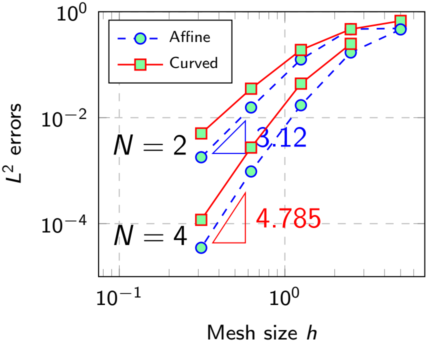
Chan, Hewett, and Warburton (2016). *Weight-adjusted discontinuous Galerkin methods: curvilinear meshes.*

Numerical results: two-dimensional curvilinear meshes

L^2 error converges at $O(h^{N+1})$ up to time-stepper accuracy (LSERK-45).



(a) Degree $N = 1, N = 3$



(b) Degree $N = 2, N = 4$

Summary and acknowledgements

- Derived discretely entropy stable high order discontinuous Galerkin methods using a continuous formulation.
- Future work: regularization, multi-GPU (with Lucas Wilcox).
- This research is supported by the National Science Foundation under awards DMS-1712639 and DMS-1719818.

Thank you! Questions?

Chan, Hewett, and Warburton (2016). *Weight-adjusted discontinuous Galerkin methods: curvilinear meshes.*

Chan (2017). *On discretely entropy conservative and entropy stable discontinuous Galerkin methods.*

Additional slides

Global DG differentiation operator

- Let $v \in V_h$, $u, w \notin V_h$ with u, w bounded; modified D_h^x

$$
\begin{aligned}
\left(D_h^x u, vw\right)_\Omega = &\sum_k \left(\frac{\partial P_N u}{\partial x}, vw\right)_{D^k} \\
&+ \frac{1}{2}\left\langle u^+ - P_N u, vwn_x\right\rangle_{\partial D^k} \\
&+ \frac{1}{2}\left\langle u - P_N u, P_N\left(vw\right) n_x\right\rangle_{\partial D^k}.
\end{aligned}
$$

- Integration-by-parts property

$$
\left(D_h^x u, vw\right)_\Omega = \left\langle u, vw\right\rangle_{\partial\Omega} - \left(u, D_h^x\left(vw\right)\right)_\Omega.
$$

- Coupling only through surface values (in contrast to D_h^x with $[\![P_N u]\!]$).

Chen and Shu (2017). *ES high order DG methods with suitable quadrature rules for hyperbolic conservation laws.*