WEIGHT-ADJUSTED DISCONTINUOUS GALERKIN METHODS: MATRIX-VALUED WEIGHTS AND ELASTIC WAVE PROPAGATION IN HETEROGENEOUS MEDIA

JESSE CHAN

Abstract. Weight-adjusted inner products [1, 2] replace weighted L^2 inner products with easily invertible weight-adjusted approximations. When paired with discontinuous Galerkin (DG) methods, the result is a low storage, energy stable, and (for sufficiently regular weights) high order accurate method for wave propagation in the presence of general heterogeneous media and curvilinear meshes. In this work, we extend weight-adjusted DG (WADG) methods to weighted inner products with matrix-valued weights, focusing on the linear elastic wave equation as an application. We present a unified DG formulation for isotropic and anisotropic media based on the discretization of the symmetric form of the linear elasticity wave equation, with upwind-like dissipation incorporated through simple penalty terms in the numerical fluxes. This formulation is provably stable for arbitrary stiffness tensors and curvilinear meshes. A convergence analysis is given, and numerical results confirm the stability and high order accuracy of WADG for several problems in elastic wave propagation.

1. Introduction. Efficient and accurate simulations of elastic wave propagation form the foundation for various seismic applications, including imaging and inversion techniques, as well as rupture and earthquake simulations. The most common methods for the simulation of waves are finite difference methods [3]; however, finite element methods have garnered interest due to their low numerical dispersion and ability to accomodate geometrically flexible unstructured meshes.

Typical methods for time-domain wave propagation utilize explicit time stepping methods, since the hyperbolic partial differential equations (PDEs) governing wave propagation typically admit less restrictive stable timestep restrictions than parabolic equations. However, finite element methods require the inversion of a global mass matrix when paired with explicit time integrators. On hexahedral meshes, this can be avoided by using spectral element methods (SEM), which utilize mass lumping to produce a diagonal mass matrix [4]. This global inversion can also be avoided through the use of discontinuous Galerkin (DG) methods, which result in a locally invertible block diagonal mass matrix. Due to difficulties in extending mass-lumping techniques to tetrahedra, high order DG methods are often employed for seismic simulations which use simplicial meshes [5, 6, 7, 8, 9, 10]. Such methods also lend themselves well to efficient implementations using Graphics Processing Units (GPUs) [11, 12, 13, 14].

Most high order DG methods on simplicial meshes assume that material coefficients are constant over each element, which can result in spurious reflections and the loss of high order accuracy [15, 16, 17]. This limitation can be overcome for DG methods by incorporating sub-element variation in media through weighted mass matrices, which results in an energy stable method [16, 17]. For DG methods on hexahedral meshes, mass lumping reduces the construction of weighted mass matrices to entry-wise scalings of a diagonal mass matrix. However, on tetrahedral meshes, this approach necessitates the pre-computation and storage of factorizations for each local mass matrix, which greatly increases both storage costs and data transferred at high orders of approximation. These costs are especially problematic for accelerator architectures such as GPUs, which possess limited memory.

Storage costs associated with weighted mass matrices can be avoided by approximating weighted L^2 inner products using weight-adjusted inner products, which result in easily invertible approximations to weighted mass matrices [1, 2]. For sufficiently regular weights, high order accuracy is also retained. When paired with an energy stable DG formulation, weight-adjusted inner products result in weight-adjusted DG methods (WADG), which preserve energy stability and high order accuracy while retaining a low asymptotic storage cost. Additionally, unlike mass-lumping techniques, WADG does not rely on the use of inexact quadrature rules, and reduces to the exact inversion of mass matrices for constant weights.

Weight-adjusted DG methods have been applied to both acoustic wave propagation in heterogeneous media and on curvilinear meshes [1, 2]. Both cases involve scalar weighting functions. In this work, we extend weight-adjusted inner products to matrix-valued weights, and demonstrate the usefulness of this generalization to elastic wave propagation in arbitrary media. A new energy stable DG formulation is introduced based on the symmetric form of the elastic wave equations, with upwind-like numerical dissipation introduced through simple penalty fluxes [18, 10].

This work proceeds as follows. Sections 2 and 3 begin by presenting an energy stable DG formulation with simple penalty fluxes for the elastic wave equation in symmetric hyperbolic form, and discusses issues

related to storage and inversion of local mass matrices for material coefficients with sub-element variations. Section 4 extends weight-adjusted approximations to weighted L^2 inner products and mass matrices to the case of matrix-valued weights, and proves interpolation estimates which account for the regularity of the matrix weight. These results are incorporated into a weight-adjusted DG method for the linear elastic wave equations in Section 5. Finally, numerical results in Section 6 demonstrate the accuracy of this method for several analytic solutions in linear elasticity.

2. Symmetric form of the elastic wave equation . We begin with the linear elastic wave equation under small deformations in a domain $\Omega \in \mathbb{R}^d$. These equations can be written a first order velocity-stress system for velocity $\boldsymbol{v}$ and symmetric stress tensor $\boldsymbol{S}$

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} = \nabla \cdot \boldsymbol{S} + \boldsymbol{f}$$

$$\frac{\partial \boldsymbol{S}}{\partial t} = \frac{1}{2}\underline{\boldsymbol{C}}\left(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T\right),$$

where $\boldsymbol{f}$ is the body force per unit volume, ρ is density, and $\underline{\boldsymbol{C}}$ is the symmetric constitutive stiffness tensor relating stress and strain. We rewrite these equations as a symmetric hyperbolic system of PDEs [19]

(1)
$$\rho \frac{\partial \boldsymbol{v}}{\partial t} = \sum_{i=1}^{d} \boldsymbol{A}_i^T \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{x}_i} + \boldsymbol{f}$$

$$\boldsymbol{C}^{-1}\frac{\partial \boldsymbol{\sigma}}{\partial t} = \sum_{i=1}^{d} \boldsymbol{A}_i \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_i}.$$

where $\boldsymbol{C}$ is the symmetric matrix form of the constitutive tensor $\underline{\boldsymbol{C}}$ and σ is a vector of length $N_d = \frac{d(d+1)}{2}$, the number of unique entries of the stress tensor $\boldsymbol{S}$ in d dimensions. We note that the matrices $\boldsymbol{A}_i$ are constant, while ρ and $\boldsymbol{C}$ can vary spatially. Furthermore, we will assume that ρ and $\boldsymbol{C}$ are positive-definite and bounded pointwise such that

$$0 < \rho_{\min} \leq \rho(\boldsymbol{x}) \leq \rho_{\max} < \infty, \qquad \forall \boldsymbol{x} \in \mathbb{R}^d,$$

$$0 < c_{\min} \leq \boldsymbol{u}^T \boldsymbol{C}(\boldsymbol{x})\boldsymbol{u} \leq c_{\max} < \infty, \qquad \forall \boldsymbol{u} \in \mathbb{R}^{N_d}, \quad \forall \boldsymbol{x} \in \mathbb{R}^d,$$

which also implies that $\boldsymbol{C}^{-1}$ is pointwise positive-definite and bounded.

In two dimensions, $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2)^T$ and $\boldsymbol{\sigma} = (\sigma_{xx}, \sigma_{yy}, \sigma_{xy})^T$

$$\boldsymbol{S} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix},$$

while the matrices $\boldsymbol{A}_i$ are

$$\boldsymbol{A}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \boldsymbol{A}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In three dimensions, the velocity is $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)^T$, while $\boldsymbol{\sigma} = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{xy}, \sigma_{xz})^T$ denotes the unique entries of the stress tensor $\boldsymbol{S}$

$$\boldsymbol{S} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{pmatrix}.$$

The matrices $\boldsymbol{A}_i$ are then

$$\boldsymbol{A}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \boldsymbol{A}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{A}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In general anisotropic media, $\boldsymbol{C}$ is symmetric and positive-definite. For two-dimensional isotropic media, $\boldsymbol{C}$ is given as

$$\boldsymbol{C} = \begin{pmatrix} 2\mu + \lambda & \lambda & 0 \\ \lambda & 2\mu + \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix},$$

where λ, μ are Lame parameters. For three-dimensional isotropic media, $\boldsymbol{C}$ is given instead by

$$\begin{pmatrix} 2\mu + \lambda & \lambda & \lambda & \\ \lambda & 2\mu + \lambda & \lambda & \\ \lambda & \lambda & 2\mu + \lambda & \\ & & & \mu \boldsymbol{I}^{3\times 3} \end{pmatrix}.$$

We will consider both spatially varying isotropic and anisotropic media in this work.

3. An energy stable discontinuous Galerkin formulation for elastic wave propagation. The advantage of the symmetric first order formulation of the elastic wave equations (1) is that it is straight-forward to derive an energy stable discontinuous Galerkin formulation. We assume that the domain Ω is Lipschitz and exactly triangulated by a mesh Ω_h, which consists of elements D^k. We further assume that each element D^k is the image of a reference element $\widehat{D}$ under the local elemental mapping

$$\boldsymbol{x}^k = \boldsymbol{\Phi}^k \widehat{\boldsymbol{x}},$$

where $\boldsymbol{x}^k = \{x^k, y^k, z^k\}$ denote physical coordinates on D^k and $\widehat{\boldsymbol{x}} = \{\widehat{x}, \widehat{y}, \widehat{z}\}$ denote coordinates on the reference element. We denote the determinant of the Jacobian of $\boldsymbol{\Phi}^k$ as J, and refer to it as the Jacobian for the remainder of this work.

We will approximate components over each element D^k from an approximation space $V_h\left(D^k\right)$, which we define as the composition of the mapping $\boldsymbol{\Phi}^k$ and a reference approximation space $V_h\left(\widehat{D}\right)$

$$V_h\left(D^k\right) = \boldsymbol{\Phi}^k \circ V_h\left(\widehat{D}\right).$$

For the remainder of this work, we will take $V_h\left(\widehat{D}\right) = P^N\left(\widehat{D}\right)$, where $P^N\left(\widehat{D}\right)$ is the polynomial space of total degree N on the reference simplex. In two dimensions,

$$P^N\left(\widehat{D}\right) = \left\{\widehat{x}^i \widehat{y}^j, \quad 0 \leq i + j \leq N\right\},$$

and in three dimensions,

$$P^N\left(\widehat{D}\right) = \left\{\widehat{x}^i \widehat{y}^j \widehat{z}^k, \quad 0 \leq i + j + k \leq N\right\}.$$

We denote the L^2 inner product and norm over D^k by $(\cdot, \cdot)_{D^k}$, such that

$$(\boldsymbol{f}, \boldsymbol{g})_{D^k} = \int_{D^k} \boldsymbol{f} \cdot \boldsymbol{g}\, \mathrm{d}x = \int_{\widehat{D}} \boldsymbol{f} \cdot \boldsymbol{g} J, \qquad \|\boldsymbol{f}\|_{D^k}^2 = (\boldsymbol{f}, \boldsymbol{f})_{D^k},$$

where $\boldsymbol{f}, \boldsymbol{g}$ are vector-valued functions. Global L^2 inner products and norms are using local L^2 inner products and norms

$$(\boldsymbol{f}, \boldsymbol{g})_{\Omega_h} = \sum_{D^k \in \Omega_h} \int_{D^k} (\boldsymbol{f}, \boldsymbol{g})_{D^k}, \qquad \|\boldsymbol{f}\|_{\Omega_h}^2 = \sum_{D^k \in \Omega_h} \|\boldsymbol{f}\|_{D^k}^2.$$

We define also the L^2 inner product and norm over the boundary ∂D^k of an element

$$\langle u, v \rangle_{\partial D^k} = \int_{\partial D^k} uv\, \mathrm{d}x = \sum_{f \in \partial D^k} \int_{\widehat{f}} uv J^f\, \mathrm{d}\widehat{x}, \qquad \|u\|_{\partial D^k}^2 = \langle u, u \rangle,$$

where J^f is the Jacobian of the mapping from a reference face (triangle) $\widehat{f}$ to a physical face f of an element.

Let f be a face of an element D^k with neighboring element $D^{k,+}$ and unit outward normal $\boldsymbol{n}$. We define u^- and u^+ denote interior and exterior values of a discontinuous function u such that

$$u^- = u|_{f \cap D^k}, \qquad u^+ = u|_{f \cap \partial D^{k,+}}.$$

The jump and average of a scalar function $u \in V_h(\Omega_h)$ over f are then defined as

$$[\![u]\!] = u^+ - u^-, \qquad \{\!\{u\}\!\} = \frac{u^+ + u^-}{2}.$$

Jumps and averages of a vector-valued functions $\boldsymbol{u} \in \mathbb{R}^m$ and $\boldsymbol{S} \in \mathbb{R}^{m \times n}$ are then defined component-wise

$$([\![\boldsymbol{u}]\!])_i = [\![\boldsymbol{u}_i]\!], \qquad 1 \le i \le m, \qquad ([\![\boldsymbol{S}]\!])_{ij} = [\![\boldsymbol{S}_i]\!], \qquad 1 \le i \le m, \quad 1 \le j \le n.$$

We can now specify a unified DG formulation for the linear elastic wave equation (1) in either isotropic or anisotropic media. Symmetric hyperbolic systems readily admit a DG formulation based on penalty fluxes [20]. For the linear elastic wave equation in symmetric first order form, this formulation is given as

$$\sum_{D^k \in \Omega_h} \left(\rho \frac{\partial \boldsymbol{v}}{\partial t}, \boldsymbol{w} \right)_{D^k} = \sum_{D^k \in \Omega_h} \left(\left(\sum_{i=1}^d \boldsymbol{A}_i^T \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{x}_i} + \boldsymbol{f}, \boldsymbol{w} \right)_{D^k} + \left\langle \frac{1}{2} \boldsymbol{A}_n^T [\![\boldsymbol{\sigma}]\!] + \frac{\tau_v}{2} \boldsymbol{A}_n^T \boldsymbol{A}_n [\![\boldsymbol{v}]\!], \boldsymbol{w} \right\rangle_{\partial D^k} \right)$$

(2) $$\sum_{D^k \in \Omega_h} \left(\boldsymbol{C}^{-1} \frac{\partial \boldsymbol{\sigma}}{\partial t}, \boldsymbol{q} \right)_{D^k} = \sum_{D^k \in \Omega_h} \left(\left(\sum_{i=1}^d \boldsymbol{A}_i \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_i}, \boldsymbol{q} \right)_{D^k} + \left\langle \frac{1}{2} \boldsymbol{A}_n [\![\boldsymbol{v}]\!] + \frac{\tau_\sigma}{2} \boldsymbol{A}_n \boldsymbol{A}_n^T [\![\boldsymbol{\sigma}]\!], \boldsymbol{q} \right\rangle_{\partial D^k} \right),$$

for all $\boldsymbol{w}, \boldsymbol{q} \in V_h(\Omega_h)$. Here, $\tau_v, \tau_\sigma \ge 0$ are penalty parameters and $\boldsymbol{A}_n$ is the normal matrix defined on a face f as $\boldsymbol{A}_n = \sum_{i=1}^d \boldsymbol{n}_i \boldsymbol{A}_i$. In two dimensions, $\boldsymbol{A}_n$ is

$$\boldsymbol{A}_n = \begin{pmatrix} n_x & 0 \\ 0 & n_y \\ n_y & n_x \end{pmatrix}.$$

while in three dimensions, $\boldsymbol{A}_n$ is

$$\boldsymbol{A}_n = \begin{pmatrix} n_x & 0 & 0 \\ 0 & n_y & 0 \\ 0 & 0 & n_z \\ 0 & n_z & n_y \\ n_z & 0 & n_x \\ n_y & n_z & 0 \end{pmatrix}.$$

3.1. Boundary conditions. In this work, we assume boundary conditions on velocity and traction of the form

$$\boldsymbol{v} = \boldsymbol{v}_{\text{bc}}, \qquad \boldsymbol{S}\boldsymbol{n} = \boldsymbol{t}_{\text{bc}}$$

where $\boldsymbol{v}_{\text{bc}}$ and $\boldsymbol{t}_{\text{bc}}$ are given values. Traction boundary conditions where $\boldsymbol{t}_{\text{bc}} = 0$ are referred to as free-surface boundary conditions. We follow [21, 22] and impose boundary conditions on the DG formulation through exterior values and jumps of the solution. Boundary conditions on the normal component of the stress can be imposed by noting that the numerical flux contains the term $[\![\boldsymbol{A}_n^T \boldsymbol{\sigma}]\!] = \boldsymbol{S}\boldsymbol{n}$.

For a face which lies on a boundary, velocity boundary conditions are imposed by setting

$$[\![\boldsymbol{v}]\!] = 2 \left(\boldsymbol{v}_{\text{bc}} - \boldsymbol{v}^- \right), \qquad [\![\boldsymbol{A}_n^T \boldsymbol{\sigma}]\!] = [\![\boldsymbol{S}\boldsymbol{n}]\!] = 0,$$

while traction boundary conditions are imposed by

$$[\![\boldsymbol{A}_n^T \boldsymbol{\sigma}]\!] = [\![\boldsymbol{S}\boldsymbol{n}]\!] = 2 \left(\boldsymbol{t}_{\text{bc}} - \boldsymbol{S}^- \boldsymbol{n} \right) = 2 \left(\boldsymbol{t}_{\text{bc}} - \boldsymbol{A}_n^T \boldsymbol{\sigma}^- \right), \qquad [\![\boldsymbol{v}]\!] = 0.$$

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For problems which involve the truncation of infinite or large domains, absorbing boundary conditions are required. For such cases, we impose simple non-reflective boundary conditions [21] through jumps

$$[\![\boldsymbol{A}_n^T\boldsymbol{\sigma}]\!] = [\![\boldsymbol{S}\boldsymbol{n}]\!] = -\boldsymbol{S}^-\boldsymbol{n} = -\boldsymbol{A}_n^T\boldsymbol{\sigma}^-, \qquad [\![\boldsymbol{v}]\!] = -\boldsymbol{v}^-.$$

We note that more accurate absorbing conditions can be imposed using, for example, perfectly matched layers [23] or high order absorbing boundary conditions [24, 25].

In all cases, boundary conditions are imposed by computing numerical fluxes using these modified jumps. This imposition guarantees energy stability for all three sets of boundary conditions.

3.2. Energy stability. It is straightforward to show that the formulation (2) is energy stable for zero velocity and traction boundary conditions, as well as non-reflective boundary conditions. Integrating by parts the velocity equations gives

$$\left(\rho\frac{\partial\boldsymbol{v}}{\partial t},\boldsymbol{w}\right)_{D^k} = -\left(\sum_{i=1}^d\boldsymbol{\sigma},\boldsymbol{A}_i\frac{\partial\boldsymbol{w}}{\partial\boldsymbol{x}_i}\right)_{D^k} + \left\langle\boldsymbol{A}_n^T\left\{\!\!\left\{\boldsymbol{\sigma}\right\}\!\!\right\} + \frac{\tau_v}{2}\boldsymbol{A}_n^T\boldsymbol{A}_n[\![\boldsymbol{v}]\!],\boldsymbol{w}\right\rangle_{\partial D^k}$$

$$\left(\boldsymbol{C}^{-1}\frac{\partial\boldsymbol{\sigma}}{\partial t},\boldsymbol{q}\right)_{D^k} = \left(\sum_{i=1}^d\boldsymbol{A}_i\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{x}_i},\boldsymbol{q}\right)_{D^k} + \left\langle\frac{1}{2}\boldsymbol{A}_n[\![\boldsymbol{v}]\!] + \frac{\tau_\sigma}{2}\boldsymbol{A}_n\boldsymbol{A}_n^T[\![\boldsymbol{\sigma}]\!],\boldsymbol{q}\right\rangle_{\partial D^k}$$

Taking $(\boldsymbol{w},\boldsymbol{q}) = (\boldsymbol{v},\boldsymbol{\sigma})$ and adding both equations together yields

$$\sum_{D^k\in\Omega_h}\frac{1}{2}\frac{\partial}{\partial t}\left((\rho\boldsymbol{v},\boldsymbol{v})_{D^k} + \left(\boldsymbol{C}^{-1}\boldsymbol{\sigma},\boldsymbol{\sigma}\right)_{D^k}\right) =$$

$$\sum_{D^k\in\Omega_h}\left\langle\boldsymbol{A}_n^T\left\{\!\!\left\{\boldsymbol{\sigma}\right\}\!\!\right\} + \frac{\tau_v}{2}\boldsymbol{A}_n^T\boldsymbol{A}_n[\![\boldsymbol{v}]\!],\boldsymbol{v}\right\rangle_{\partial D^k} + \left\langle\frac{1}{2}\boldsymbol{A}_n[\![\boldsymbol{v}]\!] + \frac{\tau_\sigma}{2}\boldsymbol{A}_n\boldsymbol{A}_n^T[\![\boldsymbol{\sigma}]\!],\boldsymbol{\sigma}\right\rangle_{\partial D^k}$$

$$\sum_{D^k\in\Omega_h}\sum_{f\in\partial D^k}\int_f\boldsymbol{A}_n^T\left\{\!\!\left\{\boldsymbol{\sigma}\right\}\!\!\right\}\boldsymbol{v} + \frac{\tau_v}{2}\boldsymbol{A}_n^T\boldsymbol{A}_n[\![\boldsymbol{v}]\!]\boldsymbol{v} + \frac{1}{2}\boldsymbol{A}_n[\![\boldsymbol{v}]\!]\boldsymbol{\sigma} + \frac{\tau_\sigma}{2}\boldsymbol{A}_n\boldsymbol{A}_n^T[\![\boldsymbol{\sigma}]\!]\boldsymbol{\sigma}.$$

Let Γ_h denote the set of unique faces in Ω_h, and let $\Gamma_v, \Gamma_\sigma, \Gamma_{\text{abc}}$ denote the parts of the boundary where velocity, traction, and non-reflective boundary conditions are imposed, respectively. We separate surface terms into contributions from interior shared faces and from boundary faces. On an interior shared face, we sum contributions from the two adjacent elements to yield

$$\sum_{f\in\Gamma_h\backslash\partial\Omega}\int_f\boldsymbol{A}_n^T\left\{\!\!\left\{\boldsymbol{\sigma}\right\}\!\!\right\}\boldsymbol{v} + \frac{\tau_v}{2}\boldsymbol{A}_n^T\boldsymbol{A}_n[\![\boldsymbol{v}]\!]\boldsymbol{v} + \frac{1}{2}\boldsymbol{A}_n[\![\boldsymbol{v}]\!]\boldsymbol{\sigma} + \frac{\tau_\sigma}{2}\boldsymbol{A}_n\boldsymbol{A}_n^T[\![\boldsymbol{\sigma}]\!]\boldsymbol{\sigma}$$

$$= -\sum_{f\in\Gamma_h\backslash\partial\Omega}\int_f\frac{\tau_v}{2}\left|\boldsymbol{A}_n[\![\boldsymbol{v}]\!]\right|^2 + \frac{\tau_\sigma}{2}\left|\boldsymbol{A}_n^T[\![\boldsymbol{\sigma}]\!]\right|^2.$$

For faces which lie on the boundary Γ_v where velocity boundary conditions are imposed, $[\![\boldsymbol{v}]\!] = -2\boldsymbol{v}^-$, $[\![\boldsymbol{A}_n^T\boldsymbol{\sigma}]\!] = 0$, and $\boldsymbol{A}_n^T\left\{\!\!\left\{\boldsymbol{\sigma}\right\}\!\!\right\} = \boldsymbol{S}\boldsymbol{n}^-$, implying that

$$\sum_{f\in\Gamma_v}\int_f\boldsymbol{A}_n^T\left\{\!\!\left\{\boldsymbol{\sigma}\right\}\!\!\right\}\boldsymbol{v} + \frac{\tau_v}{2}\boldsymbol{A}_n^T\boldsymbol{A}_n[\![\boldsymbol{v}]\!]\boldsymbol{v} + \frac{1}{2}\boldsymbol{A}_n[\![\boldsymbol{v}]\!]\boldsymbol{\sigma} + \frac{\tau_\sigma}{2}\boldsymbol{A}_n\boldsymbol{A}_n^T[\![\boldsymbol{\sigma}]\!]\boldsymbol{\sigma}$$

$$= \sum_{f\in\Gamma_v}\int_f\boldsymbol{A}_n^T\boldsymbol{\sigma}^-\boldsymbol{v}^- - \boldsymbol{A}_n\boldsymbol{v}^-\boldsymbol{\sigma}^- - \tau_v\boldsymbol{A}_n^T\boldsymbol{A}_n\left|\boldsymbol{v}^-\right|^2 = -\sum_{f\in\Gamma_v}\int_f\tau_v\boldsymbol{A}_n^T\boldsymbol{A}_n\left|\boldsymbol{v}^-\right|^2.$$

For faces which lie on Γ_σ, $\boldsymbol{A}_n^T[\![\boldsymbol{\sigma}]\!] = -2\boldsymbol{A}_n^T\boldsymbol{\sigma}^-$, $\boldsymbol{A}_n^T\left\{\!\!\left\{\boldsymbol{\sigma}\right\}\!\!\right\} = 0$, and $[\![\boldsymbol{v}]\!] = 0$, yielding a similar contribution

$$\sum_{f\in\Gamma_\sigma}\int_f\boldsymbol{A}_n^T\left\{\!\!\left\{\boldsymbol{\sigma}\right\}\!\!\right\}\boldsymbol{v} + \frac{\tau_v}{2}\boldsymbol{A}_n^T\boldsymbol{A}_n[\![\boldsymbol{v}]\!]\boldsymbol{v} + \frac{1}{2}\boldsymbol{A}_n[\![\boldsymbol{v}]\!]\boldsymbol{\sigma} + \frac{\tau_\sigma}{2}\boldsymbol{A}_n\boldsymbol{A}_n^T[\![\boldsymbol{\sigma}]\!]\boldsymbol{\sigma} = -\sum_{f\in\Gamma_\sigma}\int_f\tau_\sigma\boldsymbol{A}_n\boldsymbol{A}_n^T\left|\boldsymbol{\sigma}^-\right|^2.$$

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Finally, for faces in Γ_{abc} we have $\boldsymbol{A}_n^T \{\!\{\boldsymbol{\sigma}\}\!\} = \frac{1}{2}\boldsymbol{A}_n^T\boldsymbol{\sigma}^-$, $\boldsymbol{A}_n^T [\![\boldsymbol{\sigma}]\!] = -\boldsymbol{A}_n^T\boldsymbol{\sigma}^-$, and $[\![\boldsymbol{v}]\!] = -\boldsymbol{v}^-$, yielding

$$\sum_{f \in \Gamma_{\mathrm{abc}}} \int_f \boldsymbol{A}_n^T \{\!\{\boldsymbol{\sigma}\}\!\}\, \boldsymbol{v} + \frac{\tau_v}{2}\boldsymbol{A}_n^T\boldsymbol{A}_n [\![\boldsymbol{v}]\!]\boldsymbol{v} + \frac{1}{2}\boldsymbol{A}_n[\![\boldsymbol{v}]\!]\boldsymbol{\sigma} + \frac{\tau_\sigma}{2}\boldsymbol{A}_n\boldsymbol{A}_n^T[\![\boldsymbol{\sigma}]\!]\boldsymbol{\sigma} =$$

$$- \sum_{f \in \Gamma_{\mathrm{abc}}} \int_f \frac{\tau_v}{2}\boldsymbol{A}_n^T\boldsymbol{A}_n \left|\boldsymbol{v}^-\right|^2 + \frac{\tau_\sigma}{2}\boldsymbol{A}_n\boldsymbol{A}_n^T\left|\boldsymbol{\sigma}^-\right|^2.$$

Combining all face contributions together gives the following result:

THEOREM 1. *The DG formulation* (2) *is energy stable for* $\tau_v, \tau_\sigma \geq 0$, *in the sense that*

$$\sum_{D^k \in \Omega_h} \frac{1}{2}\frac{\partial}{\partial t}\left((\rho\boldsymbol{v}, \boldsymbol{v})_{D^k} + (\boldsymbol{C}^{-1}\boldsymbol{\sigma}, \boldsymbol{\sigma})_{D^k}\right) = -\sum_{f \in \Gamma_h \backslash \partial\Omega} \int_f \frac{\tau_v}{2}\left|\boldsymbol{A}_n[\![\boldsymbol{v}]\!]\right|^2 + \frac{\tau_\sigma}{2}\left|\boldsymbol{A}_n^T[\![\boldsymbol{\sigma}]\!]\right|^2$$

(3)
$$- \sum_{f \in \Gamma_v} \int_f \tau_v\left|\boldsymbol{A}_n\boldsymbol{v}^-\right|^2 - \sum_{f \in \Gamma_\sigma} \int_f \tau_\sigma\left|\boldsymbol{A}_n^T\boldsymbol{\sigma}^-\right|^2 - \sum_{f \in \Gamma_{\mathrm{abc}}} \int_f \frac{\tau_v}{2}\left|\boldsymbol{A}_n\boldsymbol{v}^-\right|^2 + \frac{\tau_\sigma}{2}\left|\boldsymbol{A}_n^T\boldsymbol{\sigma}^-\right|^2 \leq 0.$$

Since ρ and $\boldsymbol{C}^{-1}$ are positive definite, the left hand side of (3) is an L^2-equivalent norm on $(\boldsymbol{u}, \boldsymbol{\sigma})$, and Theorem 3 implies that the magnitude of the DG solution $(\boldsymbol{u}, \boldsymbol{\sigma})$ is non-increasing in time. This also shows that dissipation present for positive penalization constants τ_v, τ_σ acts on non-conforming components with non-zero jumps $\boldsymbol{A}_n^T[\![\boldsymbol{v}]\!]$ and $\boldsymbol{A}_n[\![\boldsymbol{\sigma}]\!]$. In fact, it was shown in [20] that, in the limit as $\tau_v, \tau_\sigma \to \infty$, the eigenspaces of DG discretizations split into a conforming part consisting of $\boldsymbol{u}, \boldsymbol{\sigma}$ which satisfy

$$\boldsymbol{A}_n[\![\boldsymbol{u}]\!] = 0, \qquad \boldsymbol{A}_n^T[\![\boldsymbol{\sigma}]\!] = 0$$

and an non-conforming part (defined through the L^2 orthogonal complement) corresponding to eigenvalues contain real parts which approach $-\infty$. For the linear elastic wave equations, these conditions are equivalent to the requirement of C^0 continuity for $\boldsymbol{u}$ and normal continuity of the stress tensor $[\![\boldsymbol{S}]\!]\boldsymbol{n} = 0$.

3.3. The semi-discrete matrix system for DG. The solution to (2) can be approximated by discretizing in space and using an explicit time integrator, which requires only evaluations of local contributions over D^k to the DG formulation

$$\left(\rho\frac{\partial\boldsymbol{v}}{\partial t}, \boldsymbol{w}\right)_{D^k} = \left(\sum_{i=1}^d \boldsymbol{A}_i^T\frac{\partial\boldsymbol{\sigma}}{\partial\boldsymbol{x}_i}, \boldsymbol{w}\right)_{D^k} + \left\langle \frac{1}{2}\boldsymbol{A}_n^T[\![\boldsymbol{\sigma}]\!] + \frac{\tau_v}{2}\boldsymbol{A}_n^T\boldsymbol{A}_n[\![\boldsymbol{v}]\!], \boldsymbol{w}\right\rangle_{\partial D^k}$$

(4)
$$\left(\boldsymbol{C}^{-1}\frac{\partial\boldsymbol{\sigma}}{\partial t}, \boldsymbol{q}\right)_{D^k} = \left(\sum_{i=1}^d \boldsymbol{A}_i\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{x}_i}, \boldsymbol{q}\right)_{D^k} + \left\langle \frac{1}{2}\boldsymbol{A}_n[\![\boldsymbol{v}]\!] + \frac{\tau_\sigma}{2}\boldsymbol{A}_n\boldsymbol{A}_n^T[\![\boldsymbol{\sigma}]\!], \boldsymbol{q}\right\rangle_{\partial D^k}.$$

Let $\{\phi_i\}_{i=1}^{N_p}$ be a basis for $P^N\left(\widehat{D}\right)$. We define the reference mass matrix $\widehat{\boldsymbol{M}}$ and the physical mass matrix $\boldsymbol{M}$ for an element D^k as

$$\left(\widehat{\boldsymbol{M}}\right)_{ij} = \int_{\widehat{D}} \phi_j\phi_i\, \mathrm{d}\widehat{\boldsymbol{x}}, \qquad (\boldsymbol{M})_{ij} = \int_{D^k} \phi_j\phi_i\, \mathrm{d}\boldsymbol{x} = \int_{\widehat{D}} \phi_j\phi_i J\, \mathrm{d}\widehat{\boldsymbol{x}}.$$

For affine mappings, J is constant and $\boldsymbol{M} = J\widehat{\boldsymbol{M}}$. We also define weak differentiation matrices $\boldsymbol{S}_k$ and face mass matrix $\boldsymbol{M}_f$ such that

$$(\boldsymbol{S}_k)_{ij} = \int_{D^k} \frac{\partial\phi_j}{\partial\boldsymbol{x}_k}\phi_i\, \mathrm{d}\boldsymbol{x}, \qquad (\boldsymbol{M}_f)_{ij} = \int_f \phi_j\phi_i\, \mathrm{d}\boldsymbol{x} = \int_{\widehat{f}} \phi_j\phi_i J^f\, \mathrm{d}\widehat{\boldsymbol{x}},$$

where J^f is the Jacobian of the mapping from a reference face $\widehat{f}$ to f. For affinely mapped simplices, J^f is also constant and $\boldsymbol{M}_f = J^f\widehat{\boldsymbol{M}}_f$, where the definition of the reference face mass matrix $\widehat{\boldsymbol{M}}_f$ is analogous to the definition of the reference mass matrix $\widehat{\boldsymbol{M}}$.

Finally, we define weighted mass matrices. Let $w(\boldsymbol{x}) \in \mathbb{R}$ and $\boldsymbol{W}(\boldsymbol{x}) \in \mathbb{R}^{m \times n}$. Then, scalar and matrix-weighted mass matrices $\boldsymbol{M}_w$ and $\boldsymbol{M}_{\boldsymbol{W}}$ are defined through

$$
(\boldsymbol{M}_w)_{ij} = \int_{D^k} w(\boldsymbol{x}) \phi_j(\boldsymbol{x}) \phi_i(\boldsymbol{x}) \, \mathrm{d}x, \qquad \boldsymbol{M}_{\boldsymbol{W}} = \begin{pmatrix} \boldsymbol{M}_{\boldsymbol{W}_{1,1}} & \cdots & \boldsymbol{M}_{\boldsymbol{W}_{1,n}} \\ \vdots & \ddots & \vdots \\ \boldsymbol{M}_{\boldsymbol{W}_{m,1}} & \cdots & \boldsymbol{M}_{\boldsymbol{W}_{m,n}} \end{pmatrix},
$$

where $\boldsymbol{M}_{\boldsymbol{W}_{i,j}}$ is the scalar weighted mass matrix weighted by the (i, j) entry of $\boldsymbol{W}$.

Local contributions to the DG variational form may then be evaluated in a quadrature-free manner using these matrices. Let $\boldsymbol{\Sigma}_i, \boldsymbol{V}_i$ denote vectors containing degrees of freedom for solution components $\boldsymbol{\sigma}_i, \boldsymbol{v}_i$ such that

$$
\boldsymbol{\sigma}_i(\boldsymbol{x}, t) = \sum_{j=1}^{N_p} \boldsymbol{\Sigma}_i(t) \phi_j(\boldsymbol{x}), \qquad 1 \le i \le N_d
$$

$$
\boldsymbol{v}_i(\boldsymbol{x}, t) = \sum_{j=1}^{N_p} \boldsymbol{V}_i(t) \phi_j(\boldsymbol{x}), \qquad 1 \le i \le d.
$$

Then, the local DG formulation can be written as a block system of ordinary differential equations (ODEs) by concatenating $\boldsymbol{\Sigma}_i, \boldsymbol{V}_i$ into single vectors $\boldsymbol{\Sigma}, \boldsymbol{V}$ and using the Kronecker product $\otimes$

$$
\boldsymbol{M}_{\rho \boldsymbol{I}} \frac{\partial \boldsymbol{V}}{\partial t} = \sum_{i=1}^{d} \left(\boldsymbol{A}_i^T \otimes \boldsymbol{S}_i \right) \boldsymbol{\Sigma} + \sum_{f \in \partial D^k} \left(\boldsymbol{I} \otimes \boldsymbol{M}_f \right) \boldsymbol{F}_v
$$

(5)
$$
\boldsymbol{M}_{\boldsymbol{C}^{-1}} \frac{\partial \boldsymbol{\Sigma}}{\partial t} = \sum_{i=1}^{d} \left(\boldsymbol{A}_i \otimes \boldsymbol{S}_i \right) \boldsymbol{V} + \sum_{f \in \partial D^k} \left(\boldsymbol{I} \otimes \boldsymbol{M}_f \right) \boldsymbol{F}_\sigma.
$$

where $\boldsymbol{M}_f \boldsymbol{F}_v$ and $\boldsymbol{M}_f \boldsymbol{F}_\sigma$ represent contributions to the velocity and stress equations from surface integrals.

In order to apply standard time integration methods, we must invert $\boldsymbol{M}_{\rho \boldsymbol{I}}$ and $\boldsymbol{M}_{\boldsymbol{C}^{-1}}$ to isolate $\frac{\partial \boldsymbol{v}}{\partial t}$ and $\frac{\partial \boldsymbol{\sigma}}{\partial t}$ on the left hand side. While the inversion of $\boldsymbol{M}_{\rho \boldsymbol{I}}$ and $\boldsymbol{M}_{\boldsymbol{C}^{-1}}$ can be parallelized from element to element, doing so typically requires either the precomputation and storage of large dense matrix inverses or the on-the-fly construction and solution of a large dense matrix at every timestep. The former option requires a large amount of storage, while the latter option is computationally expensive and difficult to parallelize. This cost can be overcome for $\rho, \boldsymbol{C}$ which are constant over an element D^k, in which case $\boldsymbol{M}_{\rho \boldsymbol{I}}$ is block diagonal with identical blocks $\boldsymbol{M}_\rho = \rho \boldsymbol{M}$, while $\boldsymbol{M}_{\boldsymbol{C}^{-1}}$ reduces to

$$
\boldsymbol{M}_{\boldsymbol{C}^{-1}} = \begin{pmatrix} \boldsymbol{C}_{1,1}^{-1} \boldsymbol{M} & \cdots & \boldsymbol{C}_{1,N_d}^{-1} \boldsymbol{M} \\ \vdots & \ddots & \vdots \\ \boldsymbol{C}_{N_d,1}^{-1} \boldsymbol{M} & \cdots & \boldsymbol{C}_{N_d,N_d}^{-1} \boldsymbol{M} \end{pmatrix} = \left(\boldsymbol{C}^{-1} \otimes \boldsymbol{M} \right).
$$

Then, $\boldsymbol{M}_\rho^{-1} = \frac{1}{\rho} \boldsymbol{M}^{-1} = \frac{1}{J\rho} \widehat{\boldsymbol{M}}^{-1}$, and $\boldsymbol{M}_{\boldsymbol{C}^{-1}}^{-1} = \boldsymbol{C} \otimes \boldsymbol{M}^{-1} = \boldsymbol{C} \otimes \left(\frac{1}{J} \widehat{\boldsymbol{M}}^{-1} \right)$, and each matrix inverse can be applied using the inverse of the reference mass matrix $\widehat{\boldsymbol{M}}^{-1}$ and the values of $\rho, \boldsymbol{C}$, and J on each element. Applying this observation to 5 then yields the following local system of ODEs

$$
\frac{\partial \boldsymbol{V}}{\partial t} = \sum_{i=1}^{d} \left(\frac{1}{\rho} \boldsymbol{A}_i^T \otimes \boldsymbol{D}_i \right) \boldsymbol{\Sigma} + \sum_{f \in \partial D^k} \left(\frac{1}{\rho} \boldsymbol{I} \otimes \boldsymbol{L}_f \right) \boldsymbol{F}_v
$$

$$
\frac{\partial \boldsymbol{\Sigma}}{\partial t} = \sum_{i=1}^{d} \left(\boldsymbol{C} \boldsymbol{A}_i \otimes \boldsymbol{D}_i \right) \boldsymbol{V} + \sum_{f \in \partial D^k} \left(\boldsymbol{C} \otimes \boldsymbol{L}_f \right) \boldsymbol{F}_\sigma,
$$

where we have introduced the differentiation matrix $\boldsymbol{D}_i = \boldsymbol{M}^{-1}\boldsymbol{S}_i$ and lift matrix $\boldsymbol{L}_f = \boldsymbol{M}^{-1}\boldsymbol{M}_f$. For affine elements, both derivative and lift matrices may be applied using only geometric factors and reference derivative and lift matrices.

Unfortunately, if ρ and $\boldsymbol{C}$ vary spatially within an element, the above approach can no longer be used to invert $\boldsymbol{M}_\rho$ and $\boldsymbol{M}_{\boldsymbol{C}^{-1}}$ in an efficient and low-storage manner. For isotropic media, one way to address sub-element variations in material parameters is to diagonalize the matrix $\boldsymbol{C}$ through a change of variables [26]. This results in a local system of ODEs with only scalar weighted mass matrices [16], which can be treated using scalar weight-adjusted approximations. We take a different approach in addressing these issues by approximating the matrix-weighted mass matrix $\boldsymbol{M}_{\boldsymbol{C}^{-1}}$ and corresponding matrix-weighted L^2 inner product with a weight-adjusted approximation which is low storage, simple to invert, energy stable, and provably high order accurate for spatially varying weights $\rho, \boldsymbol{C}$ with sufficiently regularity.

4. Weight-adjusted inner products for matrix-valued weights. Weight-adjusted inner products are high order accurate approximations of weighted L^2 inner products. These can be interpreted as generalizations of mass lumping techniques, reducing to mass lumping when integrals are evaluated with appropriate quadrature rules. These weight-adjusted inner products result in weight-adjusted mass matrices, whose inverses approximate the inverse of a weighted L^2 mass matrix.

We wish to apply weight-adjusted approximations to avoid the inversion of $\boldsymbol{M}_\rho$ and $\boldsymbol{M}_{\boldsymbol{C}^{-1}}$. Approximating the inverse of $\boldsymbol{M}_\rho$ can be done using weight-adjusted approximations for scalar weights [1, 2], which we review in Section 4.1. We then extend scalar weight-adjusted approximations to matrix-valued weights in Section 4 to approximate the inverse of $\boldsymbol{M}_{\boldsymbol{C}^{-1}}$.

4.1. Scalar weight adjusted inner products. We introduce standard Lebesgue L^p norms and their associated L^p spaces over a general domain Ω

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega u^p\right)^{1/p}, \qquad L^p(\Omega) = \left\{u : \Omega \to \mathbb{R}, \quad \|u\|_{L^p(\Omega)} < \infty\right\}$$

for $1 \le p < \infty$. For $p = \infty$, these are defined as

$$\|u\|_{L^\infty(\Omega)} = \inf\left\{C \ge 0 : |u(\boldsymbol{x})| \le C \quad \forall \boldsymbol{x} \in \Omega\right\}, \qquad L^\infty(\Omega) = \left\{u : \Omega \to \mathbb{R}, \quad \|u\|_{L^\infty(\Omega)} < \infty\right\}.$$

These induce L^p Sobolev seminorms and norms of degree s

$$|u|_{W^{s,p}(\Omega)} = \left(\sum_{|\alpha|=s} \|D^\alpha u\|_{L^p(\Omega)}^p\right)^{1/p}, \qquad |u|_{W^{s,\infty}(\Omega)} = \max_{|\alpha|=s} \|D^\alpha u\|_{L^\infty(\Omega)}$$

$$\|u\|_{W^{s,p}(\Omega)} = \left(\sum_{|\alpha|\le s} \|D^\alpha u\|_{L^p(\Omega)}^p\right)^{1/p}, \qquad \|u\|_{W^{s,\infty}(\Omega)} = \max_{|\alpha|\le s} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

where $\alpha = \{\alpha_1, \alpha_2, \alpha_3\}$ is a multi-index such that

$$D^\alpha u = \frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial y^{\alpha_2}} \frac{\partial^{\alpha_3}}{\partial z^{\alpha_3}} u,$$

We define two operators $T_w : L^2(D^k) \to P^N(D^k)$ and $T_w^{-1} : L^2(D^k) \to P^N(D^k)$ such that

$$T_w u = \Pi_N(wu), \qquad \left(wT_w^{-1}u, v\right)_{D^k} = (u, v)_{D^k}.$$

A weighted L^2 inner product $(wu, v)_{D^k}$ can be approximated by a weight-adjusted inner product

$$(wu, v)_{D^k} = (T_w u, v)_{D^k} \approx \left(T_{1/w}^{-1}u, v\right)_{D^k}.$$

based on the observation that $T_{1/w}^{-1}u \approx uw$. This approximation is made precise by the following estimates for approximations of the product uw and weighted moments on affinely mapped elements:

THEOREM 2 (Theorem 5 in [1]). *Let D^k be quasi-regular with representative size $h = \mathrm{diam}\left(D^k\right)$. For $N \geq 0$, $w \in W^{N+1,\infty}\left(D^k\right)$, and $u \in W^{N+1,2}\left(D^k\right)$,*

$$\|uw - T_w u\|_{L^2(D^k)} \leq C_w h^{N+1} \|u\|_{W^{N+1,2}(D^k)}, \tag{6}$$

$$\left\|uw - T_{1/w}^{-1} u\right\|_{L^2(D^k)} \leq C_w h^{N+1} \|u\|_{W^{N+1,2}(D^k)}. \tag{7}$$

where C_w depends on w as follows:

$$C_w = C \|w\|_{L^\infty(D^k)} \left\|\frac{1}{w}\right\|_{L^\infty(D^k)} \|w\|_{W^{N+1,\infty}(D^k)}.$$

THEOREM 3 (Theorem 6 in [1]). *Let $u \in W^{N+1,2}\left(D^k\right)$, $w \in W^{N+1,\infty}\left(D^k\right)$, and $v \in P^M\left(D^k\right)$ for $0 \leq M \leq N$; then*

$$\left|(wu, v)_{L^2(D^k)} - \left(T_{1/w}^{-1} u, v\right)\right|$$
$$\leq C h^{2N+2-M} \|w\|_{L^\infty(D^k)} \left\|\frac{1}{w}\right\|_{L^\infty(D^k)}^2 \|w\|_{W^{N+1,\infty}(D^k)}^2 \|u\|_{W^{N+1,2}(D^k)} \|v\|_{L^\infty(D^k)}.$$

Both theorems rely on a scalar weighted interpolation estimate derived in [18, 1] for a general non-affine element D^k.

THEOREM 4 (Theorem 3.1 in [1]). *Let D^k be a quasi-regular element with representative size $h = \mathrm{diam}\left(D^k\right)$. For $N \geq 0$, $w \in W^{N+1,\infty}$, and $u \in W^{N+1,2}$,*

$$\left\|u - \frac{1}{w}\Pi_N(wu)\right\|_{D^k} \leq C_N h^{N+1} \left\|\sqrt{J}\right\|_{L^\infty(D^k)} \left\|\frac{1}{\sqrt{J}}\right\|_{L^\infty(D^k)} \left\|\frac{1}{w}\right\|_{L^\infty} \|w\|_{W^{N+1,\infty}} \|u\|_{W^{N+1,2}}.$$

4.2. Extension to matrix weights. We now aim to generalize weight-adjusted inner products to the case of matrix-valued weights. We first define appropriate generalizations of norms used in Section 4.1 to vector-valued functions. Let $D^{\boldsymbol{\alpha}}\boldsymbol{v}$ denote component-wise differentiation of $\boldsymbol{v}$ with respect to a d-dimensional multi-index $\boldsymbol{\alpha}$. Then, vector L^p Sobolev norms for $\boldsymbol{v}(\boldsymbol{x}) \in \mathbb{R}^m$ can be defined as

$$\|\boldsymbol{v}\|_{W^{k,p}}^p = \sum_{i=1}^m \|\boldsymbol{v}_i\|_{W^{k,2}}^p = \sum_{|\alpha| \leq k} \|D^\alpha \boldsymbol{v}\|_{L^p}^p, \qquad 1 \leq p < \infty,$$

$$\|\boldsymbol{v}\|_{W^{k,\infty}} = \max_i \|\boldsymbol{v}_i\|_{W^{k,\infty}}.$$

The corresponding Sobolev spaces $W^{k,p}$ and $W^{k,\infty}$ are defined similar to the scalar case.

Let $\boldsymbol{W}(\boldsymbol{x})$ be a matrix-valued weight function which is pointwise symmetric positive-definite matrix

$$0 < w_{\min} \leq \|\boldsymbol{W}(\boldsymbol{x})\|_2 \leq w_{\max} < \infty, \qquad 0 < \tilde{w}_{\min} \leq \left\|\boldsymbol{W}^{-1}(\boldsymbol{x})\right\|_2 \leq \tilde{w}_{\max} < \infty, \qquad \forall \boldsymbol{x} \in \Omega.$$

We define a kth order Sobolev norm for $\boldsymbol{W}(\boldsymbol{x})$ in terms of the induced p-norm

$$\|\boldsymbol{W}(\boldsymbol{x})\|_{k,p,\infty}^p = \sum_{|\boldsymbol{\alpha}| \leq k} \sup_{\boldsymbol{x}} \|D^{\boldsymbol{\alpha}} \boldsymbol{W}(\boldsymbol{x})\|_p^p$$

While this norm is not sub-multiplicative, the following bound holds

$$\|\boldsymbol{W}\boldsymbol{v}\|_{W^{k,p}}^p = \sum_{|\alpha| \leq k} \|D^\alpha (\boldsymbol{W}\boldsymbol{v})\|_{L^p}^p \leq C_N \int \sum_{|\alpha| \leq k} \sum_{|\beta| \leq |\alpha|} \left\|\left(D^\beta \boldsymbol{W}\right)\left(D^{\alpha-\beta}\boldsymbol{v}\right)\right\|_p^p$$
$$\leq C_N \int \left(\sum_{|\alpha| \leq k} \|(D^\alpha \boldsymbol{W})\|\right)^p \left(\sum_{|\alpha| \leq k} \|D^\alpha \boldsymbol{v}\|_p\right)^p$$
$$\leq C_N \|\boldsymbol{W}\|_{k,p,\infty}^p \|\boldsymbol{v}\|_{k,p}^p,$$

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where we have used Leibniz's rule, Cauchy-Schwarz, and the arithmetic-geometric mean inequality.

The following theorem extends Theorem 4 to matrix weights by computing weighted interpolation estimates for the quantity $\boldsymbol{W}^{-1}\Pi_N\left(\boldsymbol{W}\boldsymbol{v}\right)$.

THEOREM 5. *Let D^k be a quasi-regular element with representative size $h = \mathrm{diam}\left(D^k\right)$. For $N \geq 0$, $\boldsymbol{W} \in \left(W^{N+1,\infty}\right)^{d \times d}$, and $\boldsymbol{v} \in \left(W^{N+1,2}\right)^d$,*

$$\left\|\boldsymbol{v} - \boldsymbol{W}^{-1}\Pi_N\left(\boldsymbol{W}\boldsymbol{v}\right)\right\|_{L^2} \leq Ch^{N+1}\left\|\sqrt{J}\right\|_{L^\infty(D^k)}\left\|\frac{1}{\sqrt{J}}\right\|_{L^\infty(D^k)}\tilde{w}_{\max}\left\|\boldsymbol{W}\right\|_{N+1,2,\infty}\left\|\boldsymbol{v}\right\|_{W^{N+1,2}}$$

Proof. The proof is the similar to the scalar case. Using vector-valued versions of Bramble-Hilbert and a scaling argument for quasi-regular elements yields

$$\begin{aligned}
\left\|\boldsymbol{v} - \boldsymbol{W}^{-1}\Pi_N\left(\boldsymbol{W}\boldsymbol{v}\right)\right\|_{L^2} &\leq C_1\left\|\sqrt{J}\right\|_{L^\infty(D^k)}\sup_{\boldsymbol{x}}\left\|\boldsymbol{W}^{-1}\right\|_2\left\|\boldsymbol{W}\boldsymbol{v} - \Pi_N\left(\boldsymbol{W}\boldsymbol{v}\right)\right\|_{L^2} \\
&\leq C_1\left\|\sqrt{J}\right\|_{L^\infty(D^k)}\sup_{\boldsymbol{x}}\left\|\boldsymbol{W}^{-1}\right\|_2\left|\boldsymbol{W}\boldsymbol{v}\right|_{W^{N+1,2}\left(\widehat{D}\right)} \\
&\leq C_2h^{N+1}\left\|\sqrt{J}\right\|_{L^\infty(D^k)}\left\|\frac{1}{\sqrt{J}}\right\|_{L^\infty(D^k)}\sup_{\boldsymbol{x}}\left\|\boldsymbol{W}^{-1}\right\|_2\left\|\boldsymbol{W}\boldsymbol{v}\right\|_{W^{N+1,2}(D^k)} \\
&\leq C_3h^{N+1}\left\|\sqrt{J}\right\|_{L^\infty(D^k)}\left\|\frac{1}{\sqrt{J}}\right\|_{L^\infty(D^k)}\tilde{w}_{\max}\left\|\boldsymbol{W}\right\|_{N+1,2,\infty}\left\|\boldsymbol{v}\right\|_{W^{N+1,2}(D^k)}. \quad \square
\end{aligned}$$

4.2.1. Weight-adjusted approximations with matrix weights. Let $\Pi_N\boldsymbol{u}$ be defined as the L^2 projection applied to each component of the vector-valued function $\boldsymbol{u}$. We then define the operator $T_{\boldsymbol{W}}$ analogously to the scalar case

$$T_{\boldsymbol{W}}\boldsymbol{v} = \Pi_N\left(\boldsymbol{W}\boldsymbol{v}\right).$$

The inverse operator $T_{\boldsymbol{W}}^{-1}$ is defined implicitly via

$$\left(\boldsymbol{W}T_{\boldsymbol{W}}^{-1}\boldsymbol{v}, \boldsymbol{w}\right) = \left(\boldsymbol{v}, \boldsymbol{w}\right).$$

This definition is a straightforward generalization of T_w^{-1} to matrix-valued weights $\boldsymbol{W}$. Conveniently, all properties of T_w, T_w^{-1} for scalar $w(\boldsymbol{x})$ carry over to matrix weights $\boldsymbol{W}(\boldsymbol{x})$ as well [1].

LEMMA 6. *Let Π_N denote the component-wise L^2 projection, and let $\boldsymbol{W} \in \left(L^\infty\right)^m$. Then, $T_{\boldsymbol{W}}$ satisfies the following properties:*

1. $T_{\boldsymbol{W}}^{-1}T_{\boldsymbol{W}} = \Pi_N$
2. $\Pi_N T_{\boldsymbol{W}}^{-1} = T_{\boldsymbol{W}}^{-1}\Pi_N = T_{\boldsymbol{W}}^{-1}$
3. $\left\|T_{\boldsymbol{W}}^{-1}\right\|_{D^k} \leq \tilde{w}_{\max}.$
4. $\left(T_{\boldsymbol{W}}^{-1}\boldsymbol{v}, \boldsymbol{w}\right)$ *forms an inner product on $\left(P^N\right)^m \times \left(P^N\right)^m$, which is equivalent to the L^2 inner product with equivalence constants $C_1 = \tilde{w}_{\min}, C_2 = \tilde{w}_{\max}$.*

Proof. The proofs of properties 1 and 2 are consequences of the definition of $T_{\boldsymbol{W}}$, and are identical to proofs for the scalar case. Property 3 is a straightforward extension from the scalar case. Let $\boldsymbol{v} \in \left(P^N\right)^m$. Then,

$$\left(T_{\boldsymbol{W}}^{-1}\boldsymbol{v}, \boldsymbol{v}\right) = \left(\boldsymbol{W}^{-1}\boldsymbol{W}T_{\boldsymbol{W}}^{-1}\boldsymbol{v}, \boldsymbol{v}\right) \leq \sup_{\boldsymbol{x}}\left\|\boldsymbol{W}^{-1}(\boldsymbol{x})\right\|\left(\boldsymbol{W}T_{\boldsymbol{W}}^{-1}\boldsymbol{v}, \boldsymbol{v}\right) = \tilde{w}_{\max}\left\|\boldsymbol{v}\right\|_{L^2}^2.$$

Property 4 then simply requires the lower bound

$$\left(T_{\boldsymbol{W}}^{-1}\boldsymbol{v}, \boldsymbol{v}\right) = \left(\boldsymbol{W}^{-1}\boldsymbol{W}T_{\boldsymbol{W}}^{-1}\boldsymbol{v}, \boldsymbol{v}\right) \geq \inf_{\boldsymbol{x}}\left\|\boldsymbol{W}^{-1}(\boldsymbol{x})\right\|\left\|\boldsymbol{v}\right\|_{L^2}^2 = \tilde{w}_{\min}\left\|\boldsymbol{v}\right\|_{L^2}^2. \quad \square$$

Using Theorem 5, we may also show that the matrix-valued weight-adjusted inner product is also high order accurate for sufficiently regular $\boldsymbol{W}$.

THEOREM 7. *Let D^k be a quasi-regular element with representative size $h = \operatorname{diam}\left(D^k\right)$. For $N > 0$*

$$\left\|\boldsymbol{W}\boldsymbol{v} - T_{\boldsymbol{W}^{-1}}^{-1}\boldsymbol{v}\right\|_{D^k} \leq C_{\boldsymbol{W}} h^{N+1} \left\|\boldsymbol{v}\right\|_{W^{N+1,2}(D^k)}$$

with constant $C_{\boldsymbol{W}}$ depending on $\boldsymbol{W}$ and N

$$C_{\boldsymbol{W}} = C_N \left\|\sqrt{J}\right\|_{L^\infty(D^k)} \left\|\frac{1}{\sqrt{J}}\right\|_{L^\infty(D^k)} \tilde{w}_{\max} w_{\max} \left\|\boldsymbol{W}\right\|_{N+1,2,\infty}.$$

Proof. The proof is again similar to the scalar case. The triangle inequality gives

$$\left\|\boldsymbol{W}\boldsymbol{v} - T_{\boldsymbol{W}^{-1}}^{-1}\boldsymbol{v}\right\|_{D^k} \leq \left\|\boldsymbol{W}\boldsymbol{v} - \Pi_N(\boldsymbol{W}\boldsymbol{v})\right\|_{D^k} + \left\|\Pi_N(\boldsymbol{W}\boldsymbol{v}) - T_{\boldsymbol{W}^{-1}}^{-1}\boldsymbol{v}\right\|_{D^k}.$$

The former term is bounded by interpolation estimates and by arguments used in the proof of Theorem 5. The latter term is bounded

$$\begin{aligned}
\left\|\Pi_N(\boldsymbol{W}\boldsymbol{v}) - T_{\boldsymbol{W}^{-1}}^{-1}\boldsymbol{v}\right\|_{D^k} &= \left\|T_{\boldsymbol{W}^{-1}}^{-1} T_{\boldsymbol{W}^{-1}} \Pi_N(\boldsymbol{W}\boldsymbol{v}) - T_{\boldsymbol{W}^{-1}}^{-1} \Pi_N \boldsymbol{v}\right\|_{D^k} \\
&= \left\|T_{\boldsymbol{W}^{-1}}^{-1} \Pi_N \left(T_{\boldsymbol{W}^{-1}} \Pi_N(\boldsymbol{W}\boldsymbol{v})\right) - T_{\boldsymbol{W}^{-1}}^{-1} \Pi_N \boldsymbol{v}\right\|_{D^k} \\
&\leq C_N \left\|T_{\boldsymbol{W}^{-1}}^{-1}\right\| \left\|T_{\boldsymbol{W}^{-1}} \Pi_N(\boldsymbol{W}\boldsymbol{v}) - \Pi_N \boldsymbol{v}\right\|_{D^k} \\
&\leq C_N \left\|T_{\boldsymbol{W}^{-1}}^{-1}\right\| \left\|\Pi_N\left(\boldsymbol{W}^{-1} \Pi_N(\boldsymbol{W}\boldsymbol{v})\right) - \Pi_N \boldsymbol{v}\right\|_{D^k} \\
&\leq C_N w_{\max} \left\|\boldsymbol{W}^{-1} \Pi_N(\boldsymbol{W}\boldsymbol{v}) - \boldsymbol{v}\right\|_{D^k}.
\end{aligned}$$

where we have used $\left\|T_{\boldsymbol{W}^{-1}}^{-1}\right\| \leq w_{\max}$ (Lemma 6) and $\left\|\Pi_N\right\|_{D^k} = 1$. An application of Theorem 5 to bound $\left\|\boldsymbol{W}^{-1}\Pi_N(\boldsymbol{W}\boldsymbol{v}) - \boldsymbol{v}\right\|_{D^k}$ completes the proof. □

We note that these estimates are tight, in the sense that we recover estimates for scalar weight-adjusted inner products when the matrix weight is taken to be $\boldsymbol{W} = w\boldsymbol{I}$.

4.2.2. Approximation of weighted mass matrix inverses. The advantage of using weight-adjusted inner products is that the corresponding weight-adjusted mass matrices are straightforward to invert. For scalar weights, the weight-adjusted mass matrix approximates the weighted L^2 mass matrix and its inverse by

$$\boldsymbol{M}_w \approx \boldsymbol{M}\boldsymbol{M}_{1/w}^{-1}\boldsymbol{M}, \qquad \boldsymbol{M}_w^{-1} \approx \boldsymbol{M}^{-1}\boldsymbol{M}_{1/w}\boldsymbol{M}^{-1}.$$

By evaluating $\boldsymbol{M}_{1/w}$ in a matrix-free fashion (for example, using a sufficiently accurate quadrature rule), the inverse of the weight-adjusted mass matrix $\boldsymbol{M}^{-1}\boldsymbol{M}_{1/w}\boldsymbol{M}^{-1}$ can be applied in a low storage fashion.

For weight-adjusted inner products with matrix-valued weights, the corresponding weight-adjusted mass matrices approximate weighted L^2 mass matrices and inverses in a similar fashion

$$\begin{aligned}
\boldsymbol{M}_{\boldsymbol{W}} &\approx (\boldsymbol{I} \otimes \boldsymbol{M}) \boldsymbol{M}_{\boldsymbol{W}^{-1}}^{-1} (\boldsymbol{I} \otimes \boldsymbol{M}) \\
\boldsymbol{M}_{\boldsymbol{W}}^{-1} &\approx \left(\boldsymbol{I} \otimes \boldsymbol{M}^{-1}\right) \boldsymbol{M}_{\boldsymbol{W}^{-1}} \left(\boldsymbol{I} \otimes \boldsymbol{M}^{-1}\right).
\end{aligned}$$

(8)

We note that, when $\boldsymbol{W}$ is constant over D^k, $\boldsymbol{M}_{\boldsymbol{W}}$ reduces to the Kronecker product of the inverse stiffness tensor and the local mass matrix

$$\boldsymbol{M}_{\boldsymbol{W}} = \boldsymbol{W} \otimes \boldsymbol{M}, \qquad \boldsymbol{M}_{\boldsymbol{W}}^{-1} = \boldsymbol{W}^{-1} \otimes \boldsymbol{M}^{-1}.$$

In this case, we also have $\boldsymbol{M}_{\boldsymbol{W}^{-1}} = \boldsymbol{W}^{-1} \otimes \boldsymbol{M}$, and substituting this explicit inverse into the weight adjusted mass matrix inverse $\left(\boldsymbol{I} \otimes \boldsymbol{M}^{-1}\right) \boldsymbol{M}_{\boldsymbol{W}^{-1}} \left(\boldsymbol{I} \otimes \boldsymbol{M}^{-1}\right)$ in (8) recovers the exact inversion of $\boldsymbol{M}_{\boldsymbol{W}}$.

5. An energy stable weight-adjusted discontinuous Galerkin formulation for elastic wave propagation. We construct a weight-adjusted DG method by simply replacing the weighted L^2 inner products appearing in the left hand side of the local DG formulation (4) with weight-adjusted approximations

$$\left(T_{1/\rho}^{-1} \frac{\partial \boldsymbol{v}}{\partial t}, \boldsymbol{w}\right)_{D^k} = \left(\sum_{i=1}^{d} \boldsymbol{A}_i^T \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{x}_i}, \boldsymbol{w}\right)_{D^k} + \left\langle \frac{1}{2}\boldsymbol{A}_n^T [\![\boldsymbol{\sigma}]\!] + \frac{\tau_v}{2}\boldsymbol{A}_n \boldsymbol{A}_n^T [\![\boldsymbol{v}]\!], \boldsymbol{w}\right\rangle_{\partial D^k}$$

$$\left(T_{\boldsymbol{C}}^{-1} \frac{\partial \boldsymbol{\sigma}}{\partial t}, \boldsymbol{q}\right)_{D^k} = \left(\sum_{i=1}^{d} \boldsymbol{A}_i \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_i}, \boldsymbol{q}\right)_{D^k} + \left\langle \frac{1}{2}\boldsymbol{A}_n [\![\boldsymbol{v}]\!] + \frac{\tau_\sigma}{2}\boldsymbol{A}_n^T \boldsymbol{A}_n [\![\boldsymbol{\sigma}]\!], \boldsymbol{q}\right\rangle_{\partial D^k}.$$

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Since the right hand side of the WADG formulation is identical to the right hand side of the DG formulation (2), WADG preserves a variant of the energy stability in Theorem 3

$$\sum_{D^k \in \Omega_h} \frac{1}{2} \frac{\partial}{\partial t} \left(\left(T_{1/\rho}^{-1} \boldsymbol{v}, \boldsymbol{v} \right)_{D^k} + \left(T_{\boldsymbol{C}}^{-1} \boldsymbol{\sigma}, \boldsymbol{\sigma} \right)_{D^k} \right) \leq - \sum_{f \in \Gamma_h \setminus \partial \Omega} \int_f \frac{\tau_v}{2} \left| \boldsymbol{A}_n [\![\boldsymbol{v}]\!] \right|^2 + \frac{\tau_\sigma}{2} \left| \boldsymbol{A}_n^T [\![\boldsymbol{\sigma}]\!] \right|^2$$

$$- \sum_{f \in \Gamma_v} \int_f \tau_v \left| \boldsymbol{A}_n \boldsymbol{v}^- \right|^2 - \sum_{f \in \Gamma_\sigma} \int_f \tau_\sigma \left| \boldsymbol{A}_n^T \boldsymbol{\sigma}^- \right|^2 - \sum_{f \in \Gamma_{\mathrm{abc}}} \int_f \frac{\tau_v}{2} \left| \boldsymbol{A}_n \boldsymbol{v}^- \right|^2 + \frac{\tau_\sigma}{2} \left| \boldsymbol{A}_n^T \boldsymbol{\sigma}^- \right|^2 \leq 0.$$

The substitution of weight-adjusted inner products for weighted L^2 inner products replaces the weighted L^2 mass matrices in (5) by their weight-adjusted approximations. Inverting these weight-adjusted mass matrices yields the following local system of ODEs for $\boldsymbol{V}, \boldsymbol{\Sigma}$

$$\frac{\partial \boldsymbol{V}}{\partial t} = \left(\boldsymbol{I} \otimes \boldsymbol{M}^{-1} \right) \boldsymbol{M}_{(\rho^{-1} \boldsymbol{I})} \left(\sum_{i=1}^{d} \left(\boldsymbol{A}_i^T \otimes \boldsymbol{D}_i \right) \boldsymbol{\Sigma} + \sum_{f \in \partial D^k} \left(\boldsymbol{I} \otimes \boldsymbol{L}_f \right) \boldsymbol{F}_v \right)$$

$$\frac{\partial \boldsymbol{\Sigma}}{\partial t} = \left(\boldsymbol{I} \otimes \boldsymbol{M}^{-1} \right) \boldsymbol{M}_{\boldsymbol{C}} \left(\sum_{i=1}^{d} \left(\boldsymbol{A}_i \otimes \boldsymbol{D}_i \right) \boldsymbol{V} + \sum_{f \in \partial D^k} \left(\boldsymbol{I} \otimes \boldsymbol{L}_f \right) \boldsymbol{F}_\sigma \right).$$

In practice, the matrices $\left(\boldsymbol{I} \otimes \boldsymbol{M}^{-1} \right) \boldsymbol{M}_{(\rho^{-1} \boldsymbol{I})}$ and $\left(\boldsymbol{I} \otimes \boldsymbol{M}^{-1} \right) \boldsymbol{M}_{\boldsymbol{C}}$ are applied in a matrix-free fashion using reference element matrices and values of $\rho, \boldsymbol{C}$ at quadrature points. After fusing operations together, this procedure can be boiled down to multiplication by two rectangular matrices [2].

One drawback of the analysis presented in this work is that accuracy is not guaranteed in the isotropic incompressible limit $\mu/\lambda \to 0$. In this case, the stiffness matrix $\boldsymbol{C}$ becomes singular, and the constant $\tilde{w}_{\mathrm{max}}$ in the upper bound on $\left\| \boldsymbol{C}^{-1} \right\|$ blows up. However, numerical experiments in Section 6.2.3 suggest that, while taking $\mu/\lambda \approx 0$ (or $\boldsymbol{C}$ nearly singular) results in larger relative errors for $\boldsymbol{\sigma}$, the accuracy of the WADG solution for $\boldsymbol{v}$ does not degrade significantly for near-incompressible materials.

5.1. Energy stability on curvilinear meshes. We have shown the energy stability of the DG formulation (2) on meshes of affine elements by assuming exact integration of all terms. However, energy stability can still be guaranteed if integrals are evaluated inexactly using quadrature. Instead of discretizing the "strong" DG formulation (2), we discretize the "skew-symmetric" formulation, where the right hand side of the velocity equations is integrated by parts,[1] resulting in the formulation

$$\int_{\widehat{D}} T_{(\rho J)^{-1}}^{-1} \frac{\partial \boldsymbol{v}}{\partial t} \boldsymbol{w} = - \int_{\widehat{D}} \left(\sum_{i=1}^{d} \boldsymbol{\sigma}, \boldsymbol{A}_i \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{x}_i} \right) J + \sum_{f \in \partial D^k} \int_{\widehat{f}} \left(\boldsymbol{A}_n^T \{\!\{\boldsymbol{\sigma}\}\!\} + \frac{\tau_v}{2} \boldsymbol{A}_n^T \boldsymbol{A}_n [\![\boldsymbol{v}]\!] \right) \boldsymbol{w} J^f$$

$$\int_{\widehat{D}} T_{(J^{-1} \boldsymbol{C})}^{-1} \frac{\partial \boldsymbol{\sigma}}{\partial t} \boldsymbol{q} = \int_{\widehat{D}} \left(\sum_{i=1}^{d} \boldsymbol{A}_i \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_i}, \boldsymbol{q} \right) J + \sum_{f \in \partial D^k} \int_{\widehat{f}} \left(\frac{1}{2} \boldsymbol{A}_n [\![\boldsymbol{v}]\!] + \frac{\tau_\sigma}{2} \boldsymbol{A}_n \boldsymbol{A}_n^T [\![\boldsymbol{\sigma}]\!] \right) \boldsymbol{q} J^f$$

where we have incorporated spatial variations of J into the definition of the weights on the left hand side.

Since the proof of stability in Theorem 3 requires only algebraic manipulations of this formulation, the same energy stability result holds for this formulation. More importantly, a discrete version of energy stability is still guaranteed in the presence of inexact quadrature, where integrals in Theorem 3 are replaced with quadrature approximations. This is especially important for discretizations on curvilinear meshes, where the exact integration of spatially varying geometric factors and Jacobians can be either inhibitively expensive for high order curvilinear mappings or impossible for rational mappings [27, 28].

We note that, to ensure energy stability on curved and non-affine elements, the "skew-symmetric" formulation must be evaluated explicitly using quadrature, which is typically more expensive than quadrature-free evaluations used to evaluate the "strong" DG formulation (2). These costs can be slightly reduced for curvilinear meshes by evaluating the DG formulation using the "skew-symmetric" formulation on curvilinear elements and the more efficient "strong" DG formulation on affine elements [2].

---

[1]The choice of which equation to integrate by parts is arbitrary, since integrating the stress equations by parts also results in an energy stable formulation.

5.2. Convergence analysis. Using estimates from Section 4.2.1, we can extend the semi-discrete convergence analysis in [29, 18, 1] to linear elastic wave propagation on meshes of affine elements. Techniques in [18] can be used to extend this analysis to curvilinear elements.

In terms of group variables $\boldsymbol{U} = (\boldsymbol{v}, \boldsymbol{\sigma})$ and $\boldsymbol{V} = (\boldsymbol{w}, \boldsymbol{q}) \in (V_h(\Omega_h))^d \times (V_h(\Omega_h))^{N_d}$, the variational formulation can be written as

$$\left(T_{\boldsymbol{A}_0^{-1}}^{-1} \frac{\partial \boldsymbol{U}}{\partial t}, \boldsymbol{V} \right)_\Omega + a(\boldsymbol{U}, \boldsymbol{V}) + b(\boldsymbol{U}, \boldsymbol{V}) = (\boldsymbol{f}, \boldsymbol{V})$$

$$a(\boldsymbol{U}, \boldsymbol{V}) = \sum_{D^k \in \Omega_h} \left(-\left(\sum_{i=1}^d \boldsymbol{\sigma}, \boldsymbol{A}_i \frac{\partial \boldsymbol{w}}{\partial \boldsymbol{x}_i} \right)_{D^k} + \left(\sum_{i=1}^d \boldsymbol{A}_i \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_i}, \boldsymbol{q} \right)_{D^k} \right)$$

$$b(\boldsymbol{U}, \boldsymbol{V}) = \sum_{D^k \in \Omega_h} \left(\left\langle \boldsymbol{A}_n^T \{\!\{\boldsymbol{\sigma}\}\!\} + \frac{\tau_v}{2} \boldsymbol{A}_n^T \boldsymbol{A}_n [\![\boldsymbol{v}]\!], \boldsymbol{w} \right\rangle_{\partial D^k} + \left\langle \frac{1}{2} \boldsymbol{A}_n [\![\boldsymbol{v}]\!] + \frac{\tau_\sigma}{2} \boldsymbol{A}_n \boldsymbol{A}_n^T [\![\boldsymbol{\sigma}]\!], \boldsymbol{q} \right\rangle_{\partial D^k} \right),$$

where $\boldsymbol{A}_0 = \mathrm{diag}\left(\rho \boldsymbol{I}, \boldsymbol{C}^{-1} \right)$. Energy stability implies that

$$b(\boldsymbol{U}, \boldsymbol{U}) = \frac{1}{2} \sum_{D^k \in \Omega_h} \left(\tau_v \|[\![\boldsymbol{A}_n \boldsymbol{v}]\!]\|_{\partial D^k}^2 + \tau_\sigma \|[\![\boldsymbol{A}_n^T \boldsymbol{\sigma}]\!]\|_{\partial D^k}^2 \right)$$

$$\left(T_{\boldsymbol{A}_0^{-1}}^{-1} \frac{\partial \boldsymbol{U}}{\partial t}, \boldsymbol{U} \right)_\Omega + b(\boldsymbol{U}, \boldsymbol{U}) = (\boldsymbol{f}, \boldsymbol{U}).$$

Let $\boldsymbol{U}, \boldsymbol{U}_h$ denote the exact and discrete WADG solutions, respectively. Since the DG formulation (2) is consistent, these solutions satisfy

$$\left(\boldsymbol{A}_0 \frac{\partial \boldsymbol{U}}{\partial t}, \boldsymbol{V} \right) + a(\boldsymbol{U}, \boldsymbol{V}) + b(\boldsymbol{U}, \boldsymbol{V}) = (\boldsymbol{f}, \boldsymbol{V})$$

(9)

$$\left(T_{\boldsymbol{A}_0^{-1}}^{-1} \frac{\partial \boldsymbol{U}_h}{\partial t}, \boldsymbol{V} \right) + a(\boldsymbol{U}_h, \boldsymbol{V}) + b(\boldsymbol{U}_h, \boldsymbol{V}) = (\boldsymbol{f}, \boldsymbol{V})$$

for all $\boldsymbol{V} \in (V_h(\Omega_h))^d \times (V_h(\Omega_h))^{N_d}$. We decompose the error $\boldsymbol{U} - \boldsymbol{U}_h$ into a projection error $\boldsymbol{\epsilon}$ and discretization error $\boldsymbol{\eta}$.

$$\boldsymbol{U} - \boldsymbol{U}_h = (\Pi_N \boldsymbol{U} - \boldsymbol{U}_h) - (\Pi_N \boldsymbol{U} - \boldsymbol{U}) = \boldsymbol{\eta} - \boldsymbol{\epsilon}.$$

We also introduce a consistency error $\boldsymbol{\delta} = \boldsymbol{A}_0 \boldsymbol{U} - T_{\boldsymbol{A}_0^{-1}}^{-1} \boldsymbol{U}$ resulting from the approximation of $\boldsymbol{A}_0 \boldsymbol{U}$ by a weight-adjusted inner product

$$\boldsymbol{A}_0 \frac{\partial \boldsymbol{U}}{\partial t} - T_{\boldsymbol{A}_0^{-1}}^{-1} \frac{\partial \boldsymbol{U}_h}{\partial t} = \frac{\partial}{\partial t} \left(\boldsymbol{A}_0 \boldsymbol{U} - T_{\boldsymbol{A}_0^{-1}}^{-1} \boldsymbol{U} \right) + \frac{\partial}{\partial t} T_{\boldsymbol{A}_0^{-1}}^{-1} (\Pi_N \boldsymbol{U} - \boldsymbol{U}_h) = \frac{\partial \boldsymbol{\delta}}{\partial t} + \frac{\partial}{\partial t} \left(T_{\boldsymbol{A}_0^{-1}}^{-1} \boldsymbol{\eta} \right)$$

where we have used that $\boldsymbol{T}_{\boldsymbol{A}_0^{-1}}^{-1} = \boldsymbol{T}_{\boldsymbol{A}_0^{-1}}^{-1} \Pi_N$. Subtracting the DG and WADG formulations in (9) and setting $\boldsymbol{V} = \boldsymbol{\eta}$ then yields

(10)
$$\frac{1}{2} \frac{\partial}{\partial t} \left(T_{\boldsymbol{A}_0^{-1}}^{-1} \boldsymbol{\eta}, \boldsymbol{\eta} \right)_\Omega + b(\boldsymbol{\eta}, \boldsymbol{\eta}) = \left(\frac{\partial \boldsymbol{\delta}}{\partial t}, \boldsymbol{\eta} \right) + a(\boldsymbol{\epsilon}, \boldsymbol{\eta}) + b(\boldsymbol{\epsilon}, \boldsymbol{\eta}),$$

where we have used that $a(\boldsymbol{\eta}, \boldsymbol{\eta}) = 0$ by skew symmetry.

We bound $a(\boldsymbol{\epsilon}, \boldsymbol{\eta}) + b(\boldsymbol{\epsilon}, \boldsymbol{\eta})$ on the right hand side by integrating by parts the stress equation and using the component-wise L^2 orthogonality of $\boldsymbol{\epsilon}$ to derivatives of $\boldsymbol{\eta}$. This reduces the term $a(\boldsymbol{\epsilon}, \boldsymbol{\eta}) + b(\boldsymbol{v}, \boldsymbol{\eta})$ to surface

contributions over each element, which can be combined with contributions from neighboring elements

$$\sum_{D^k \in \Omega_h} \left(\left\langle \boldsymbol{A}_n^T \left\{\!\left\{\boldsymbol{\epsilon}_\sigma\right\}\!\right\} + \frac{\tau_v}{2} \boldsymbol{A}_n^T \boldsymbol{A}_n [\![\boldsymbol{\epsilon}_v]\!], \boldsymbol{\eta}_v \right\rangle_{\partial D^k} + \left\langle \boldsymbol{A}_n \left\{\!\left\{\boldsymbol{\epsilon}_v\right\}\!\right\} + \frac{\tau_\sigma}{2} \boldsymbol{A}_n \boldsymbol{A}_n^T [\![\boldsymbol{\epsilon}_\sigma]\!], \boldsymbol{\eta}_\sigma \right\rangle_{\partial D^k} \right)$$

$$= \sum_{D^k \in \Omega_h} \left(\left\langle \left\{\!\left\{\boldsymbol{\epsilon}_\sigma\right\}\!\right\} - \frac{\tau_v}{2} \boldsymbol{A}_n [\![\boldsymbol{\epsilon}_v]\!], \boldsymbol{A}_n [\![\boldsymbol{\eta}_v]\!] \right\rangle_{\partial D^k} + \left\langle \left\{\!\left\{\boldsymbol{\epsilon}_v\right\}\!\right\} - \frac{\tau_\sigma}{2} \boldsymbol{A}_n^T [\![\boldsymbol{\epsilon}_\sigma]\!], \boldsymbol{A}_n^T [\![\boldsymbol{\eta}_\sigma]\!] \right\rangle_{\partial D^k} \right)$$

$$\leq \sum_{D^k \in \Omega_h} \left\| \left\{\!\left\{\boldsymbol{\epsilon}_\sigma\right\}\!\right\} - \frac{\tau_v}{2} \boldsymbol{A}_n [\![\boldsymbol{\epsilon}_v]\!] \right\|_{\partial D^k} \|\boldsymbol{A}_n [\![\boldsymbol{\eta}_v]\!]\|_{\partial D^k} + \left\| \left\{\!\left\{\boldsymbol{\epsilon}_v\right\}\!\right\} - \frac{\tau_\sigma}{2} \boldsymbol{A}_n^T [\![\boldsymbol{\epsilon}_\sigma]\!] \right\|_{\partial D^k} \|\boldsymbol{A}_n^T [\![\boldsymbol{\eta}_\sigma]\!]\|_{\partial D^k}$$

$$\leq C_\tau \sum_{D^k \in \Omega_h} \|\boldsymbol{\epsilon}\|_{\partial D^k} \frac{1}{2} \left(\tau_v \|\boldsymbol{A}_n [\![\boldsymbol{\eta}_v]\!]\|_{\partial D^k}^2 + \tau_\sigma \|\boldsymbol{A}_n^T [\![\boldsymbol{\eta}_\sigma]\!]\|_{\partial D^k}^2 \right)^{1/2}$$

where C_τ is proportional to $\max(\tau_v, \tau_\sigma)$. Using Young's inequality with $\alpha \geq 1/\max(\tau_v, \tau_\sigma, 1)$ sufficiently large yields the following bound

$$|a(\boldsymbol{\epsilon}, \boldsymbol{\eta}) + b(\boldsymbol{\epsilon}, \boldsymbol{\eta})| \leq \frac{C}{4\alpha} b(\boldsymbol{\eta}, \boldsymbol{\eta}) + \frac{C\alpha}{4} \sum_{D^k \in \Omega_h} \|\boldsymbol{\epsilon}\|_{\partial D^k}^2 .$$

This can be used to bound (10). Using Young's inequality on the remaining terms and absorbing terms involving $\boldsymbol{\eta}$ into the left hand side yields

$$\frac{1}{2} \frac{\partial}{\partial t} \left(T_{\boldsymbol{A}_0^{-1}}^{-1} \boldsymbol{\eta}, \boldsymbol{\eta} \right)_\Omega + \left(1 - \frac{C}{4\alpha} \right) b(\boldsymbol{\eta}, \boldsymbol{\eta}) \leq \left\| \frac{\partial \boldsymbol{\delta}}{\partial t} \right\|_{D^k} + \sum_{D^k \in \Omega_h} \frac{C\alpha}{4} \|\boldsymbol{\epsilon}\|_{\partial D^k}^2 .$$

Integrating from $[0, T]$, and using the arithmetic-geometric mean inequality yields

$$\frac{1}{2} \left(T_{\boldsymbol{A}_0^{-1}}^{-1} \boldsymbol{\eta}, \boldsymbol{\eta} \right)_\Omega + \left(1 - \frac{C}{4\alpha} \right) \int_0^T b(\boldsymbol{\eta}, \boldsymbol{\eta}) \leq \int_0^T \left\| \frac{\partial \boldsymbol{\delta}}{\partial t} \right\|_{D^k} + \int_0^T \sum_{D^k \in \Omega_h} C \|\boldsymbol{\epsilon}\|_{\partial D^k} .$$

Using Theorem 7 and a standard hp trace inequality $\|u\|_{\partial D^k} \leq C h^{-1/2} \|u\|_{L^2(\Omega)}$ [14] then yields

$$\|\boldsymbol{\eta}\| \leq C_{\boldsymbol{W}} T \sup_{t \in [0,T]} \left(\left\| \frac{\partial \boldsymbol{\delta}}{\partial t} \right\| + \|\boldsymbol{\epsilon}\| h^{-1/2} \right) .$$

Bounding each term on the right using interpolation estimates for sufficiently regular $\boldsymbol{U}$ yields the final result

$$\|\boldsymbol{U} - \boldsymbol{U}_h\| \leq (C_1 + C_2 T) h^{N+1/2} \left(\|\boldsymbol{U}\|_{W^{N+1,2}} + C_3 h^{1/2} \left\| \frac{\partial \boldsymbol{U}}{\partial t} \right\|_{W^{N+1,2}} \right) .$$

where C_2 depends on N and C_3 depends on $\|\boldsymbol{W}\|_{N+1,2,\infty}$. Todo - finish

6. Numerical experiments. The following sections present several numerical experiments validating the stability and accuracy of the proposed method in two and three dimensions. The energy stability of the method is also confirmed for examples with sub-cell variations in heterogeneous media and curvilinear meshes. The convergence of the new DG formulation in piecewise constant isotropic media is confirmed using analytic solutions, while the convergence of the method for high order approximations of heterogeneous media is confirmed using a fine grid reference solution. Finally, the method is applied to problems with anisotropy and stiffness matrices $\boldsymbol{C}$ with sub-element variations.

In all experiments, we follow [1] and compute application of weight-adjusted mass matrices using a quadrature exact for polynomials of degree $(2N + 1)$. Time integration is performed using the low-storage 4th order five-stage Runge-Kutta scheme of Carpenter and Kennedy [30], and the timestep is chosen based on the global restriction

$$dt = \min_f \frac{C_{\mathrm{CFL}}}{\|\boldsymbol{C}(\boldsymbol{x})\|_{2,\infty,\Omega} C_N \|J^f\|_{L^\infty(f)} \|J^{-1}\|_{L^\infty(f)}}$$

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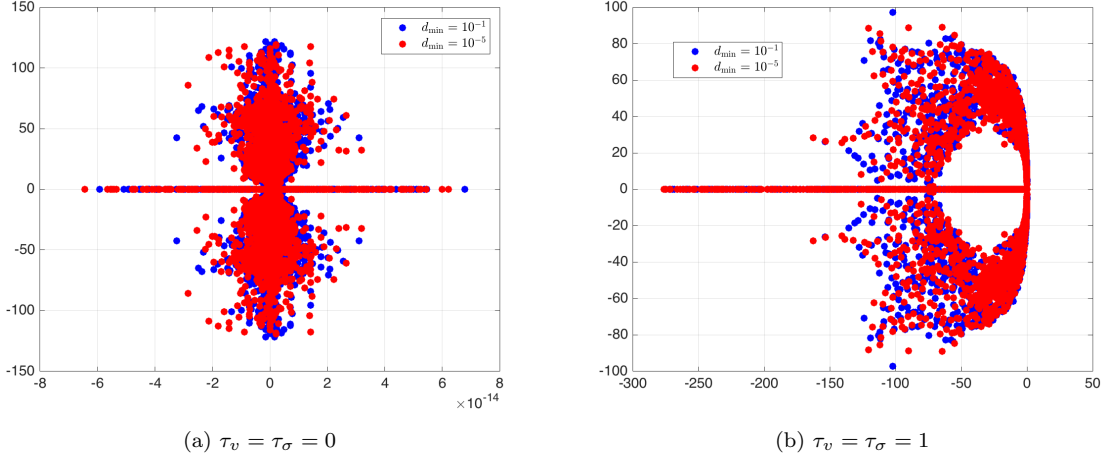(a) $\tau_v = \tau_\sigma = 0$ (b) $\tau_v = \tau_\sigma = 1$

Fig. 1: Spectra for $N = 4$ and $h = 1/4$, using randomly chosen $\boldsymbol{C}(\boldsymbol{x})$ with eigenvalues between $[d_{\min}, 1]$ at each quadrature point. For both $\tau_v = \tau_\sigma = 0$ and $\tau_v = \tau_\sigma = 1$, the largest real part of the spectra is $O(10^{-14})$.

where $C_N = O(N^2)$ is the order-dependent constant in the surface polynomial trace inequality [14], C_{CFL} is a tunable global CFL constant, and $\|\boldsymbol{C}\|_{2,\infty,\Omega}$ denotes the spatial supremum of the 2-norm of the stiffness matrix $\boldsymbol{C}(\boldsymbol{x})$.

$$\|\boldsymbol{C}\|_{2,\infty,\Omega} = \sup_{\boldsymbol{x} \in \Omega} \|\boldsymbol{C}(\boldsymbol{x})\|_2 \, .$$

6.1. Spectra and choice of penalty parameter. We first verify the energy stability of the WADG method for arbitrary heterogeneous media. We use a stiffness matrix constructed using random numbers and similarity transforms, such that at every quadrature point, $\boldsymbol{C}(\boldsymbol{x}) = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^T$, where $\boldsymbol{D}$ is a diagonal matrix with positive entries $d_{\min} \leq \boldsymbol{D}_{ii} \leq d_{\max}$ and $\boldsymbol{U}$ is a random unitary matrix. Let $\boldsymbol{A}_h$ denote the matrix induced by the global semi-discrete DG formulation, such that the time evolution of the solution $\boldsymbol{v}, \boldsymbol{\sigma}$ is governed by

$$\frac{\partial \boldsymbol{Q}}{\partial t} = \boldsymbol{A}_h \boldsymbol{Q},$$

with $\boldsymbol{Q}$ denotes a vector of degrees of freedom for $(\boldsymbol{v}, \boldsymbol{\sigma})$. Figure 1 shows computed eigenvalues of $\boldsymbol{A}_h$ for $[d_{\min}, d_{\max}] = [1/10, 1]$ and $[10^{-4}, 1]$, $\tau_v = \tau_\sigma = 0$ and $\tau_v = \tau_\sigma = 1$ under discretization parameters $N = 4$ and $h = 1/4$.

We note that, for $\tau_v = \tau_\sigma = 1$, $\|\boldsymbol{A}_h\|$ is approximately twice as large compared to when $\tau_v = \tau_\sigma = 0$. This can result in a more restrictive stability condition on the timestep. We observe that the growth in $\|\boldsymbol{A}_h\|$ for $\tau_v, \tau_\sigma = 1$ is due to the presence of eigenvalues with large negative real parts, which implies that larger timesteps may be used under time integrators with extended stability regions along the real axis [31].

Alternatively, we can avoid growth in $\|\boldsymbol{A}_h\|$ by scaling τ. To avoid artificially increasing the stiffness of (5), it was argued in [20] that the magnitude of the penalty parameter should depend on the largest singular value of $\boldsymbol{A}_n$, which is $O(1)$ for elasticity in two and three dimensions. However, this does not factor in the effect of ρ and $\boldsymbol{C}^{-1}$ appearing in the left hand side. These coefficients can be taken into account by scaling only the stress penalization parameter with the average of local max norms over a face f

$$\tau_{\sigma,f} \approx \frac{2}{\|\boldsymbol{C}^+(\boldsymbol{x})\|_{2,\infty,f} + \|\boldsymbol{C}^-(\boldsymbol{x})\|_{2,\infty,f}},$$

where $\boldsymbol{C}^+, \boldsymbol{C}^-$ denote values of the stiffness matrix on elements which share the face f. It may also be advantageous to rescale τ_σ for each individual component of stress based on the rows of $\boldsymbol{C}$. Under this
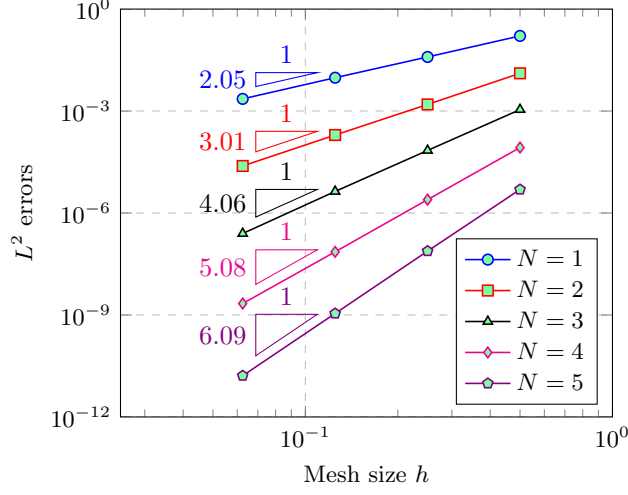
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Fig. 2: Convergence of L^2 errors for harmonic oscillation.

rescaling of the stress penalty parameter, we observe numerically that the norm of $\boldsymbol{A}_h$ remains roughly the same magnitude as for $\tau_p = \tau_\sigma = 0$.

In the following experiments, we use $\tau_v = \tau_\sigma = 1$. However, we do not observe significant changes in computed L^2 errors under the aforementioned rescaling of τ_σ.

6.2. Analytic solutions. Next, we study the accuracy and convergence of weight-adjusted DG method for several analytical solutions in linear elasticity. In all cases, the solution in expressed terms of the displacement vector $\boldsymbol{u}(\boldsymbol{x}, t) = (u_1, \ldots, u_d)$. Initial conditions for velocity and stress are computed through

$$\boldsymbol{v}(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{u}}{\partial t}, \qquad \boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{E} = \boldsymbol{C}\frac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right).$$

Unless otherwise stated, we report relative L^2 errors for all components of the solution $\boldsymbol{U} = (\boldsymbol{v}, \boldsymbol{\sigma})$

$$\frac{\|\boldsymbol{U} - \boldsymbol{U}_h\|_{L^2(\Omega)}}{\|\boldsymbol{U}\|_{L^2(\Omega)}} = \frac{\left(\sum_{i=1}^m \|\boldsymbol{U}_i - \boldsymbol{U}_{i,h}\|_{L^2(\Omega)}^2\right)^{1/2}}{\left(\sum_{i=1}^m \|\boldsymbol{U}_i\|_{L^2(\Omega)}^2\right)^{1/2}}.$$

6.2.1. Harmonic oscillation of a square. We first examine convergence on a unit square domain with $\lambda = \mu = \rho = 1$. The components of the displacement vector is given by

$$u_1(x, y, t) = \cos(\omega\pi t)\cos(\pi x)\sin(\pi y)$$
$$u_2(x, y, t) = -\cos(\omega\pi t)\sin(\pi x)\cos(\pi y),$$

where $\omega = \sqrt{2\mu}$. Figure 2 shows L^2 errors computed at time $T = 1$, using uniform triangular meshes constructed by bisecting a uniform mesh of quadrilaterals along the diagonal. For $N = 1, \ldots, 5$, optimal $O(h^{N+1})$ rates of convergence are observed.

6.2.2. Rayleigh and Lamb waves. Next, we examine the convergence of WADG for Rayleigh and Lamb waves, both of which test the imposition of traction-free boundary conditions.

Rayleigh waves are elastic surface waves which decay exponentially away from the surface. These waves

are given by the displacement vector

$$\boldsymbol{u}\left(x,y,t\right) = e^{-\omega x\sqrt{1-\xi^2}} \left(\begin{array}{c} \cos(\omega(y + c_r t)) \\ \sqrt{1-\xi^2}\sin(\omega(y + c_r t)) \end{array} \right)$$
$$+ \left(\frac{\xi^2}{2} - 1\right) e^{-\omega x\sqrt{1-\frac{\xi^2\mu}{2\mu+\lambda}}} \left(\begin{array}{c} \cos(\omega(y + c_r t)) \\ \sin(\omega(y + c_r t))/\sqrt{1-\frac{\xi^2\mu}{2\mu+\lambda}} \end{array} \right),$$

where ω is the wavespeed, c_r is the Rayleigh phase velocity $c_r = \xi\sqrt{\mu}$, and ξ satisfies

$$\sqrt{1-\xi^2}\sqrt{1-\frac{\xi^2\mu}{2\mu+\lambda}} - \left(\frac{\xi^2}{2} - 1\right)^2 = 0.$$

In our computations, we use $\rho = \mu = \lambda = 1$, $\xi = 0.949554083888034$, and $\omega = 2\pi$. We solve on the domain $[0,2] \times [0,1]$ using a sequence of uniform triangular meshes, and enforce traction-free boundary conditions at $x = 0$ and exact Dirichlet boundary conditions at $x = 2$. Periodic boundary conditions are applied at $y = 0$ and $y = 1$.

Lamb waves are supported by elastic waveguides with traction-free (free surface) boundary conditions at the top and bottom of the domain. The displacement of these waves is given by

$$u_1\left(x,y,t\right) = (-kB_1\cos(py) - qB_2\cos(qy))\sin(kx - \omega t)$$
$$u_2\left(x,y,t\right) = (-pB_1\sin(py) + kB_2\sin(qy))\cos(kx - \omega t)$$

where k is the wavenumber and ω is the frequency, and the constants p and q are defined as

$$p^2 = \frac{\omega^2}{2\mu+\lambda} - k^2, \qquad q^2 = \frac{\omega^2}{\mu} - k^2.$$

The wavenumber k and frequency ω are related through a dispersion relation. The ratio of the amplitudes B_1/B_2 can be determined using other parameters, implying that B_1, B_2 are unique up to a scaling constant. In our experiments, we use $\rho = \mu = 1$, $\lambda = 2$, $k = 2\pi$. For these values, $\omega = 13.137063197233$, $B_1 = 126.1992721468$ and $B_2 = 53.88807700007$. We solve on the domain $[-1,1] \times [-1/2, 1/2]$, with traction-free boundary conditions at $y = \pm 1/2$ and periodic boundary conditions at $x = \pm 1$.

Figure 3b shows L^2 errors for both waves at time $T = 1$. Computed convergence rates fall between the optimal rate of $O(h^{N+1})$ and theoretical rate of $O(h^{N+1/2})$ [32].

6.2.3. Rayleigh waves in near-incompressible materials. As noted in Section 5, error estimates for isotropic elasticity no longer hold in the incompressible limit with $\mu/\lambda \to 0$ limit due to the fact that $\boldsymbol{C}$ becomes singular. We use the propagation of Rayleigh waves to examine the behavior of WADG for near-incompressible materials. We follow [33, 34] and fix $\lambda = 1$ and set $\mu = 1, .1, .01, .001, .0001$. Since the Rayleigh wave propagates with speed proportional to $\sqrt{\mu}$, we compute L^2 errors at the final time $1/(4\sqrt{\mu})$ to ensure a fair comparison between solutions at different values of μ.

Table 1 shows relative errors for $\boldsymbol{v}, \boldsymbol{\sigma}$ at different orders and mesh sizes. The relative errors for $\boldsymbol{\sigma}$ grow as $\mu/\lambda \to 0$. This is not surprising, as the constant in the error estimates of Section 4.2.1 blows up in the incompressible limit. However, because the magnitude of $\boldsymbol{\sigma}$ also decreases as $\mu/\lambda \to 0$, relative errors for $\boldsymbol{v}$ remain roughly the same magnitude as the material approaches incompressibility. Decreasing μ by four orders of magnitude results in up to a ten-fold increase in relative error for $\boldsymbol{\sigma}$, but less than a two-fold increase in error for $\boldsymbol{v}$.

6.2.4. Stoneley waves. A Stoneley wave is supported along the interface between two solids [35]. Like Rayleigh waves, Stoneley waves decay exponentially away from the interface, and test the effectiveness of numerical fluxes across interfaces. We follow [22, 34] and use discontinuous media defined by

$$(\rho, \lambda, \mu) = \begin{cases} (10, 3, 3) & y > 0 \\ (1, 1, 1) & y < 0. \end{cases}$$
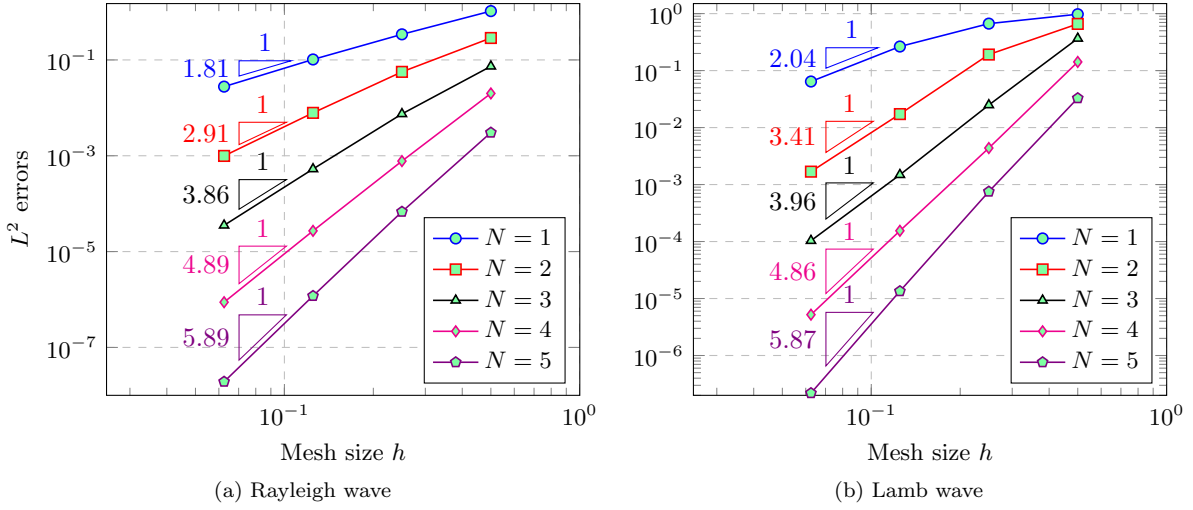
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(a) Rayleigh wave

(b) Lamb wave

Fig. 3: Convergence of L^2 errors for Rayleigh and Lamb wave solutions.

	$\mu = 1$	$\mu = .1$	$\mu = .01$	$\mu = .001$	$\mu = .0001$
$N = 2$, $h = 1/4$	3.1063e-02	2.6972e-02	3.0858e-02	4.0103e-02	5.6835e-02
$N = 3$, $h = 1/4$	3.1677e-03	2.6848e-03	2.9854e-03	3.6960e-03	4.9362e-03
$N = 4$, $h = 1/4$	2.8726e-04	2.4990e-04	2.9142e-04	3.9774e-04	5.2469e-04
$N = 2$, $h = 1/8$	3.1819e-03	2.5476e-03	2.8877e-03	3.5608e-03	4.7010e-03
$N = 3$, $h = 1/8$	1.8867e-04	1.6925e-04	1.8509e-04	2.2520e-04	2.8301e-04
$N = 4$, $h = 1/8$	8.4999e-06	7.5750e-06	8.1094e-06	1.0760e-05	1.4886e-05

(a) Relative error in $\boldsymbol{v}$

	$\mu = 1$	$\mu = .1$	$\mu = .01$	$\mu = .001$	$\mu = .0001$
$N = 2$, $h = 1/4$	6.8250e-02	7.4130e-02	1.2104e-01	2.1333e-01	4.1451e-01
$N = 3$, $h = 1/4$	9.2980e-03	1.0685e-02	1.8109e-02	3.2046e-02	5.7612e-02
$N = 4$, $h = 1/4$	9.7251e-04	1.1687e-03	2.1941e-03	4.1378e-03	6.8404e-03
$N = 2$, $h = 1/8$	9.8138e-03	1.1429e-02	2.0331e-02	3.7889e-02	7.6516e-02
$N = 3$, $h = 1/8$	6.6596e-04	7.8157e-04	1.5353e-03	3.1239e-03	5.6341e-03
$N = 4$, $h = 1/8$	3.4272e-05	4.2639e-05	8.8393e-05	1.9764e-04	3.6052e-04

(b) Relative error in $\boldsymbol{\sigma}$

Table 1: Behavior of WADG for linear elastic wave propagation in the incompressible limit as $\mu/\lambda \to 0$. Errors are shown for various orders and mesh resolutions.

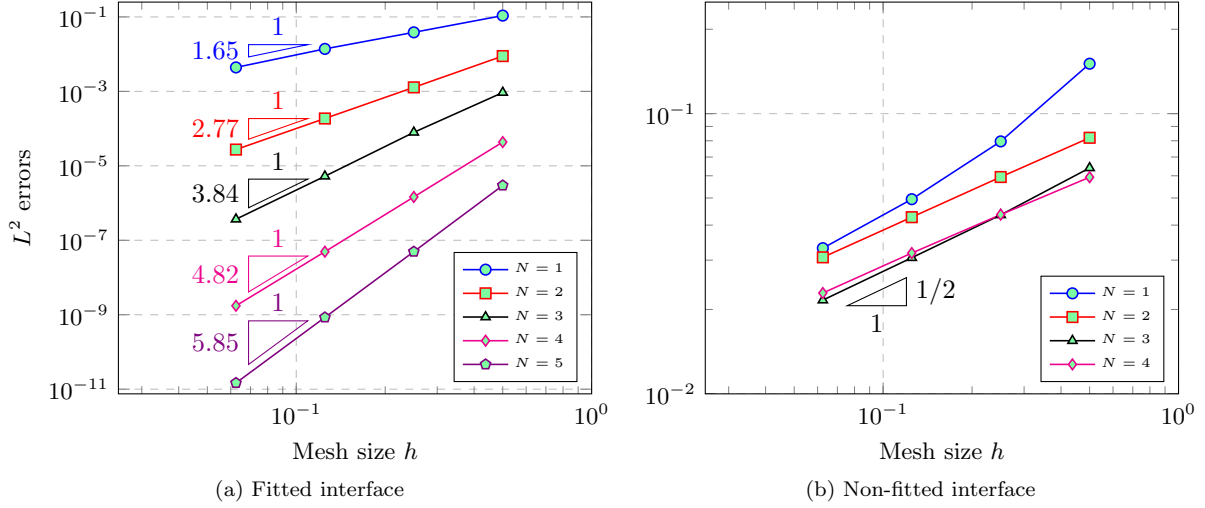(a) Fitted interface   (b) Non-fitted interface

Fig. 4: Convergence of WADG for a Stoneley wave using a fitted mesh aligned with the interface and a non-fitted mesh where the interface does not lie exactly on an element boundary.

The displacement vector for a Stoneley wave is then given by

$$u_1(x,y,t) = \begin{cases} \mathrm{Re}\left(\left(ikB_1 e^{-kb_{1p}y} + kb_{1s}B_2 e^{-kb_{1s}y} \right) e^{i(ky-\omega t)} \right), & y > 0 \\ \mathrm{Re}\left(\left(-kb_{1p}B_1 e^{-kb_{1p}y} + ikB_2 e^{-kb_{1s}y} \right) e^{i(kx-\omega t)} \right), & y < 0 \end{cases}$$

$$u_2(x,y,t) = \begin{cases} \mathrm{Re}\left(\left(ikB_3 e^{kb_{2p}y} - kb_{2s}B_4 e^{kb_{2s}y} \right) e^{i(kx-\omega t)} \right), & y > 0 \\ \mathrm{Re}\left(\left(kb_{2p}B_3 e^{kb_{2p}y} + ikB_4 e^{kb_{2s}y} \right) e^{i(kx-\omega t)} \right), & y < 0 \end{cases},$$

where c_{st} is the Stoneley wave speed, and

$$k = \omega/c_{st}, \qquad b_{jp} = \sqrt{1 - \frac{c_{st}^2}{(2\mu_j + \lambda_j)/\rho_j}}, \qquad b_{js} = \sqrt{1 - \frac{c_{st}^2}{(\mu_j)/\rho_j}}, \qquad j = 1,2.$$

The Stoneley wave speed c_{st} can be determined based material parameters and interface conditions, and the amplitudes B_1, B_2, B_3, B_4 are determined from c_{st} up to scaling by a constant. For the parameters used in this study, we take $c_{st} = 0.546981324213884$, $B_1 = i0.2952173626624$, $B_2 = -0.6798795208473$, $B_3 = i0.5220044931212$, and $B_4 = -0.9339639688697$. We assume $k = 1$, which gives $\omega = c_{st}$.

We solve on the domain $[-1,1] \times [-5,5]$, and enforce Dirichlet boundary conditions at all boundaries using the exact solution. Figure 4 shows L^2 errors for two uniform meshes of triangles constructed by bisecting a quadrilateral mesh of $K_{1D} \times 5K_{1D}$ elements. Figure 4a shows errors when K_{1D} is even and the mesh is fitted to the interface at $y = 0$, while Figure 4b shows errors when K_{1D} is odd and the interface cuts through element interiors. When the mesh is fitted to the interface, computed convergence rates are between $O(h^{N+1/2})$ and $O(h^{N+1})$.

When the mesh is not fitted to the interface exactly, we compute the application of the weight-adjusted mass matrix inverse using a quadrature rule from Xiao and Gimbutas [36] which is exact for degree $2N + 1$ polynomials. Since the values of ρ, μ, and λ are positive at all quadrature points, the method is energy stable. However, since the exact solution is discontinuous, the error in elements cut by the interface is $O(1)$, resulting in L^2 errors which converge at rate $O(h^{1/2})$. We note that, when using piecewise constant approximations of μ and λ, we observe the same $O(h^{1/2})$ convergence rate, though errors are roughly twice as large in magnitude.

6.2.5. Convergence to a reference solution. Finally, we examine the accuracy of the WADG method for smoothly varying heterogeneous media by comparing against a reference spectral method so-
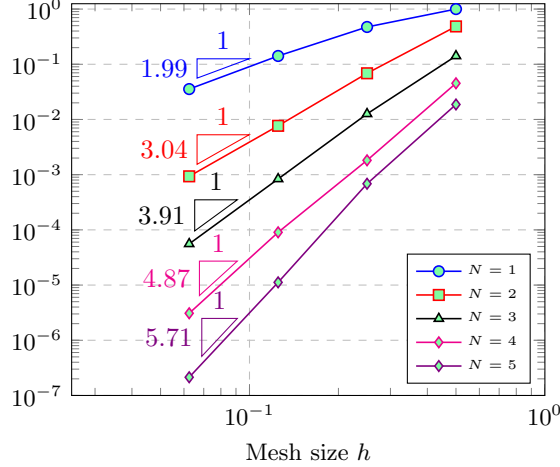
Fig. 5: Convergence of WADG under mesh refinement to a reference $N = 50$ spectral method solution with smoothly varying heterogeneous media for $N = 1, \ldots, 5$.

lution with order $N = 50$ on a unit square $[-1, 1]^2$. Boundary conditions are enforced weakly through DG-like fluxes [22]. We use a heterogeneous isotropic medium with ρ, λ, μ set to

$$\rho(\boldsymbol{x}) = 1, \qquad \lambda(\boldsymbol{x}) = 1 + .25 \sin(\pi x) \sin(\pi y), \qquad \mu(\boldsymbol{x}) = 1 + .25 \cos(\pi x) \cos(\pi y).$$

Initial stresses are set to zero, while the initial velocity is set to

$$v_1(\boldsymbol{x}) = \cos(\pi x) \sin(\pi y), \qquad v_2(\boldsymbol{x}) = -\sin(\pi x) \cos(\pi y).$$

Zero traction boundary conditions are enforced on all domain boundaries. Figure 5 shows L^2 errors with respect to the reference solution at time $T = 1/2$ for different mesh sizes and orders of approximation. Computed convergence rates fall between the optimal $O(h^{N+1})$ and predicted $O(h^{N+1/2})$.

6.3. Application examples. Next, we investigate the behavior of WADG for linear elasticity for several application examples.

6.3.1. Stiff inclusion. A common test of methods for linear elastic wave propagation is the stiff inclusion problem [21, 5, 34]. We solve on a domain $[-1, 1] \times [-.5, .5]$ with a rectangular inclusion located at $[-.5, .5] \times [-.1, .1]$. Outside of the inclusion, material parameters are taken to be

$$\rho = 1, \qquad \mu = 1, \qquad \lambda = 2.$$

Within the inclusion, material parameters are taken to be

$$\rho = 1, \qquad \mu = 100, \qquad \lambda = 200,$$

such that the wave speed in the rectangular inclusion is ten times that of the wave speed outside. A pulse is generated through velocity boundary conditions at $x = -1$

$$v_1(x, y, t) = \begin{cases} \sin(\pi t/t_0), & t < t_0 \\ 0, & t \geq t_0 \end{cases}, \qquad v_2(x, y, t) = 0.$$

In our experiments, we take $t_0 = .025$. Traction free boundary conditions are enforced at all other domain boundaries. A wave is generated when the pulse hits the inclusion, bouncing back and forth and producing reflected waves which propagate outward from the stiff region.
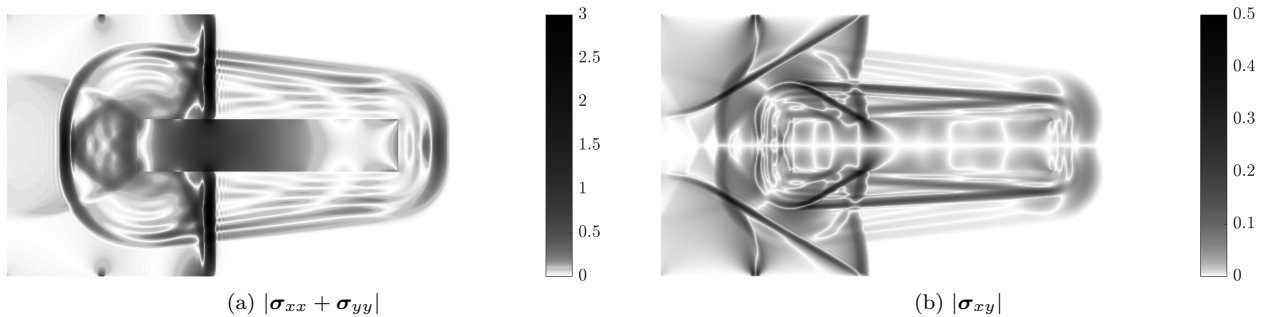
20

(a) $|\boldsymbol{\sigma}_{xx} + \boldsymbol{\sigma}_{yy}|$ (b) $|\boldsymbol{\sigma}_{xy}|$

Fig. 6: Values of different solution fields at $T = .4$ for the stiff inclusion problem of Leveque [21] with order of approximation $N = 5$.

We construct a mesh by dividing each element of a quadrilateral meshes along the diagonal to produce triangular meshes, using 100×50 quadrilateral elements in the x and y coordinates, respectively. Figure 6 shows values of $|\boldsymbol{\sigma}_{xx} + \boldsymbol{\sigma}_{yy}|$ at final time $T = .4$. Following [21], a nonlinear color scale is used in order to distinguish small-amplitude waves. The results for the interface-aligned mesh show qualitatively good agreement with results in the literature [21, 5, 34].

6.3.2. Heterogeneous anisotropic material. We next examine a model wave propagation problem in heterogeneous anisotropy media [37, 38, 7]. The density $\rho = 7100$ is constant over the domain, while the entries of the stiffness matrix $\boldsymbol{C}$ are taken to be

$$\boldsymbol{C}_{11} = .165, \quad \boldsymbol{C}_{12} = .05, \quad \boldsymbol{C}_{22} = .062, \quad \boldsymbol{C}_{33} = .0396, \qquad x < 0$$
$$\boldsymbol{C}_{11} = .165, \quad \boldsymbol{C}_{12} = .0858, \quad \boldsymbol{C}_{22} = .165, \quad \boldsymbol{C}_{33} = .0396, \qquad x > 0,$$

with the remaining entries determined by symmetry or set to zero if unspecified. For $x < 0$, this corresponds to an anisotropic material, while for $x > 0$, this corresponds to an isotropic material with $\mu = .0396, \lambda = .0858$.

The computational domain is taken to be $[-.16, .16]^2$, and following [38, 7], we use $N = 5$ and a triangular mesh constructed by subdividing a grid of 32×32 uniform quadrilaterals. This mesh corresponds to an element size of $.005$, which was used in [38, 7]. Forcing is applied to the y-component of the velocity by a Ricker pulse point source

$$f(\boldsymbol{x}, t) = \left(1 - 2(\pi f_0(t - t_0))^2\right) e^{-(\pi f_0(t-t_0))^2} \delta(x - x_0),$$

where $x_0 = -.02$, $f_0 = .17$, and $t_0 = 1/f_0$.[2]

6.4. A three-dimensional example. Finally, we present a three-dimensional example of elastic wave propagation in heterogeneous media with sub-element variations, anisotropy, and a discontinuity across an interface. We consider elastic wave propagation on the bi-unit cube $[-1, 1]^3$, with isotropic media for $z < 0$ and transversely isotropic media for $z > 0$. For isotropic media, we take $\mu(\boldsymbol{x}) = \lambda(\boldsymbol{x}) = w(\boldsymbol{x})$, where

$$w(\boldsymbol{x}) = 1 + \cos(k\pi x)\cos(k\pi y)\cos(k\pi z).$$

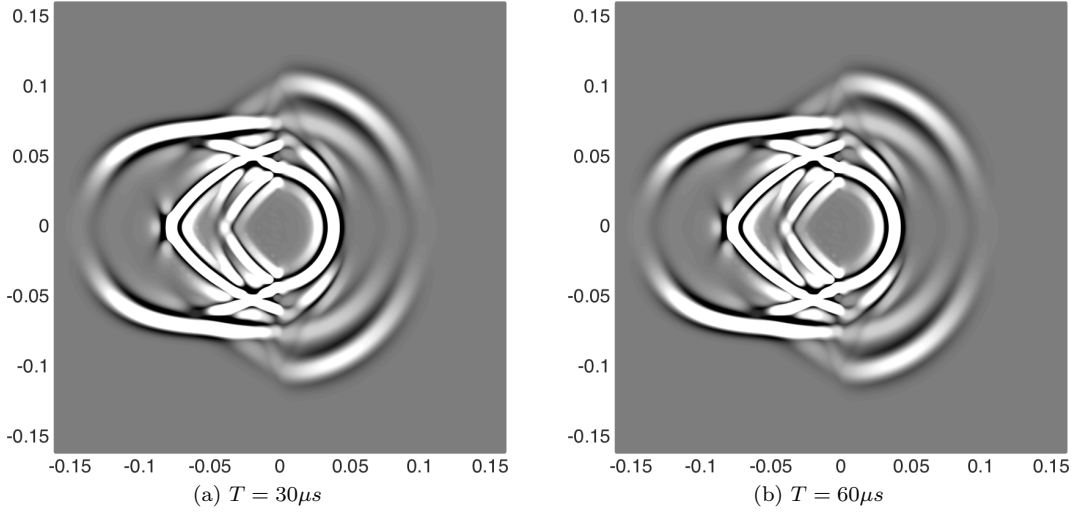[2] All values and units are adapted from [38, 7], and correspond to units of meters, kg, and microseconds.

(a) $T = 30\mu s$ (b) $T = 60\mu s$

Fig. 7: Example of wave propagation in heterogeneous anisotropic media. The vertical component of the velocity $\boldsymbol{v}_2$ is shown at $T = 30$ and $T = 60$ microseconds.

For the transversely isotropic media, we take the entries of the stiffness matrix $\boldsymbol{C}$ to be

$$\boldsymbol{C}_{11} = \frac{2\mu + \lambda}{4}$$
$$\boldsymbol{C}_{12} = \frac{2\mu + \lambda}{4}$$
$$\boldsymbol{C}_{13} = \frac{2\mu + \lambda}{4}$$
$$\boldsymbol{C}_{22} = \frac{2\mu + \lambda}{4}$$
$$\boldsymbol{C}_{23} = \frac{2\mu + \lambda}{4}$$
$$\boldsymbol{C}_{33} = \frac{2\mu + \lambda}{4}$$

We follow the implementation of GPU-accelerated DG methods outlined in [11], which breaks the computational work in each timestep *volume* and *surface* kernels for the evaluation of the DG formulation and an *update* step to apply a time integration method. For this work, apply the weight-adjusted mass matrix inverse within the update kernel as well. Strategies for volume and surface kernels follow [11], while computational approaches for WADG are outlined in [2]. More detailed computational performance analyses will be performed in future work. All three-dimensional computations are performed on an AMD Tahiti GPU.

7. Conclusions. This work presents a weight-adjusted discontinuous Galerkin (WADG) method for the linear elastic wave equations with arbitrary heterogeneous media. The method is energy stable and high order accurate for arbitrary stiffness matrices, and a slight modification results in an energy stable method for curvilinear meshes as well. The penalty numerical fluxes for this formulation are simple to derive and implement, and their lack of dependence on the stiffness matrix allows for a unified treatment of isotropic and anisotropic media. Numerical examples confirm the accuracy of this method for analytic solutions in of the elastic wave equations, as well as its high order accuracy with respect to a reference solution for smoothly varying heterogeneous media. The behavior of this method for several application examples in also shows good agreement with existing results in the literature for both isotropic and anisotropic heterogeneous media.

We note that the implementation of this method reduces to the application of the weight-adjusted mass matrix inverse and the evaluation of constant-coefficient terms in the DG formulation. Fast methods based on Bernstein-Bezier bases for the application of derivative and lift matrices for constant-coefficient terms can reduce this latter cost, especially at high orders of approximation [39]. Finally, we note that the approach presented here is more generally applicable to problems involving a weighted mass matrix with a matrix-valued weight, which we will explore in the future. These include hyperbolic systems of PDEs with a right hand side of the form $\boldsymbol{A}_0(\boldsymbol{x})\frac{\partial \boldsymbol{U}}{\partial t}$, such as Maxwell's equations for cloaking problems [40].

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