

Weight-adjusted DG methods for elastic wave propagation in arbitrary heterogeneous media

Jesse Chan

Department of Computational and Applied Math, Rice University

ICOSAHOM 2018
July 12, 2018

Collaborators and acknowledgements

- Prof. T. Warburton: Virginia Tech, Department of Mathematics
- Dr. Russell Hewett: TOTAL Research and Technology USA
- Prof. Maarten de Hoop: Rice University, Department of Computational Mathematics
- Khemraj Shukla: Oklahoma State University, Dept. of Geophysics.

High order DG methods for wave propagation

- Unstructured (tetrahedral) meshes for geometric flexibility.
- Low numerical dissipation and dispersion.
- High order approximations: more accurate per unknown.
- Explicit time stepping: high performance on many-core.

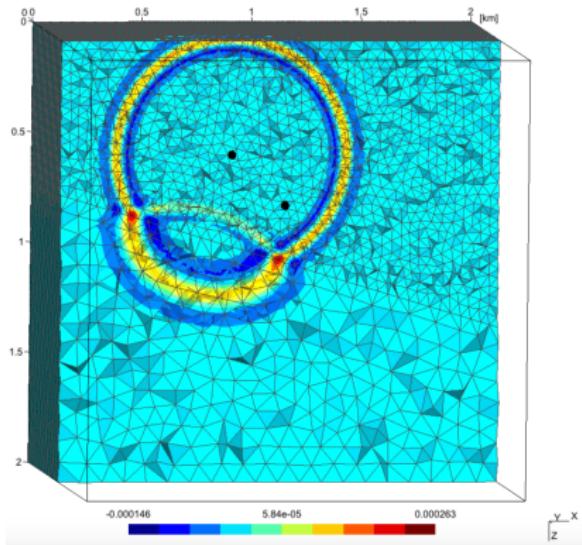
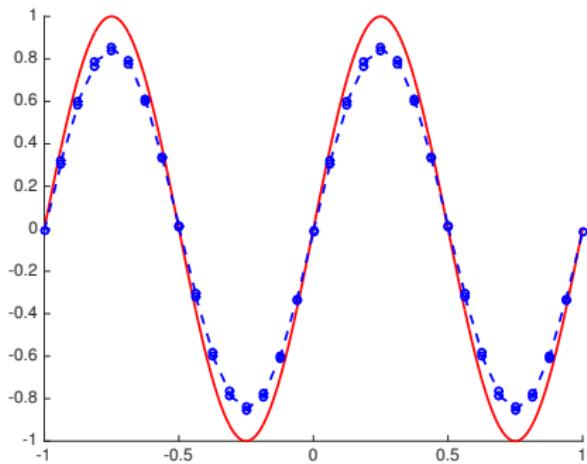


Figure courtesy of Axel Modave.

High order DG methods for wave propagation

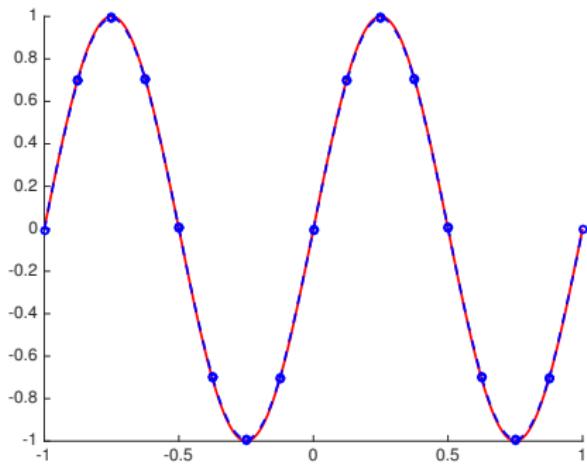
- Unstructured (tetrahedral) meshes for geometric flexibility.
- Low numerical dissipation and dispersion.
- High order approximations: more accurate per unknown.
- Explicit time stepping: high performance on many-core.



Fine linear approximation.

High order DG methods for wave propagation

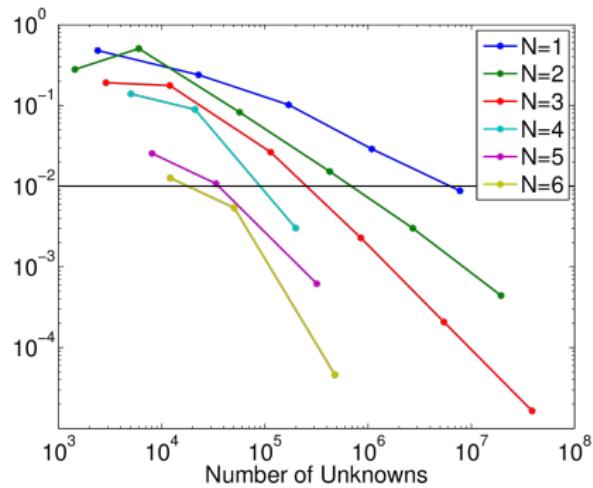
- Unstructured (tetrahedral) meshes for geometric flexibility.
- Low numerical dissipation and dispersion.
- High order approximations: more accurate per unknown.
- Explicit time stepping: high performance on many-core.



Coarse quadratic approximation.

High order DG methods for wave propagation

- Unstructured (tetrahedral) meshes for geometric flexibility.
- Low numerical dissipation and dispersion.
- High order approximations: more accurate per unknown.
- Explicit time stepping: high performance on many-core.



Max errors vs. dofs.

High order DG methods for wave propagation

- Unstructured (tetrahedral) meshes for geometric flexibility.
- Low numerical dissipation and dispersion.
- High order approximations: more accurate per unknown.
- Explicit time stepping: high performance on many-core.



Graphics processing units (GPU).

Outline

- 1 Weight-adjusted DG methods: acoustics
- 2 Extension to elastic wave propagation
- 3 Acoustic-elastic coupling, poroelasticity

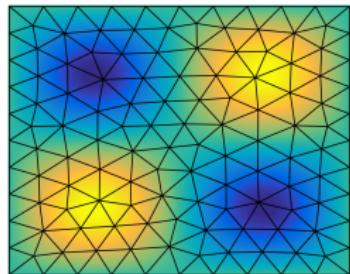
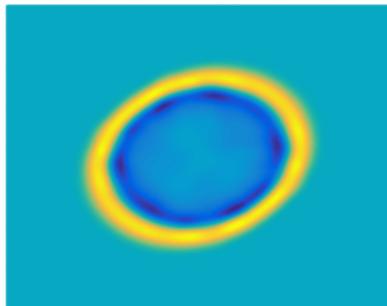
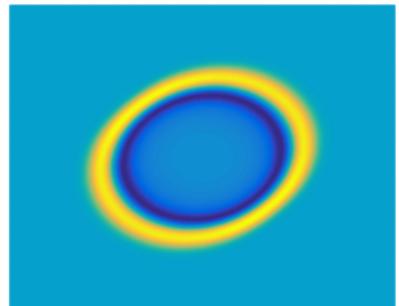
Outline

1 Weight-adjusted DG methods: acoustics

2 Extension to elastic wave propagation

3 Acoustic-elastic coupling, poroelasticity

High order methods and arbitrary heterogeneous media

(a) Mesh and exact c^2 (b) Piecewise const. c^2 (c) High order c^2

- Efficient implementations on **triangular** or **tetrahedral** meshes assume piecewise constant wavespeed c^2 .
- High order vs constant approx. of media: spurious reflections.
- This talk: generalized mass lumping for DG with provable stability, high order accuracy, simple numerical fluxes (central flux + penalty).

Energy stable discontinuous Galerkin formulations

- Model problem: acoustic wave equation (pressure-velocity system)

$$\frac{1}{c^2(\mathbf{x})} \frac{\partial p}{\partial t} = \nabla \cdot \mathbf{u}, \quad \frac{\partial \mathbf{u}}{\partial t} = \nabla p$$

- Local formulation on element D^k

$$\int_{D^k} \frac{1}{c^2(\mathbf{x})} \frac{\partial p}{\partial t} q = \int_{D^k} \nabla \cdot \mathbf{u} q + \frac{1}{2} \int_{\partial D^k} (\llbracket \mathbf{u} \rrbracket \cdot \mathbf{n} + \tau_p \llbracket p \rrbracket) q$$

$$\int_{D^k} \frac{\partial \mathbf{u}}{\partial t} \mathbf{v} = \int_{D^k} \nabla p \cdot \mathbf{v} + \frac{1}{2} \int_{\partial D^k} (\llbracket p \rrbracket + \tau_u \llbracket \mathbf{u} \rrbracket \cdot \mathbf{n}) \mathbf{v}$$

- Energy stability (penalty terms weakly enforce continuity conditions)

$$\frac{\partial}{\partial t} \left(\sum_k \int_{D^k} \frac{p^2}{c^2(\mathbf{x})} + |\mathbf{u}|^2 \right) = - \sum_k \int_{\partial D^k} \tau_p \llbracket p \rrbracket^2 + \tau_u \llbracket \mathbf{u} \cdot \mathbf{n} \rrbracket^2 \leq 0.$$

Semi-discrete formulation and weighted mass matrices

- Weighted mass matrix \mathbf{M}_{1/c^2} : SPD, induces weighted norm on \mathbf{p}

$$(\mathbf{M}_{1/c^2})_{ij} = \int_{D^k} \frac{1}{c^2(\mathbf{x})} \phi_j(\mathbf{x}) \phi_i(\mathbf{x}) d\mathbf{x}.$$

- Semi-discrete form: face mass and weak derivative matrices \mathbf{M}_f , \mathbf{S}_i .

$$\mathbf{M}_{1/c^2} \frac{d\mathbf{p}}{dt} = \sum_{i=1}^d \mathbf{S}_i \mathbf{u}_i + \sum_{\text{faces}} \mathbf{M}^f \mathbf{F}_p,$$

$$\mathbf{M} \frac{d\mathbf{u}_i}{dt} = \mathbf{S}_i \mathbf{p} + \sum_{\text{faces}} \mathbf{M}^f \mathbf{F}_u.$$

- Must build and factorize \mathbf{M}_{1/c^2} separately over each element (unless c^2 constant and $(\mathbf{M}_{1/c^2})^{-1} = c^2 \mathbf{M}^{-1}$).

Weight-adjusted DG: generalized mass lumping

- Weight-adjusted DG (WADG): $\mathbf{M} (\mathbf{M}_{1/w})^{-1} \mathbf{M}$ is SPD and an energy stable approximation of weighted mass matrix.

$$\mathbf{M}_w \frac{d\mathbf{U}}{dt} \approx \mathbf{M} (\mathbf{M}_{1/w})^{-1} \mathbf{M} \frac{d\mathbf{U}}{dt} = \text{right hand side.}$$

- Bypasses inverse of weighted matrix $(\mathbf{M}_w)^{-1}$

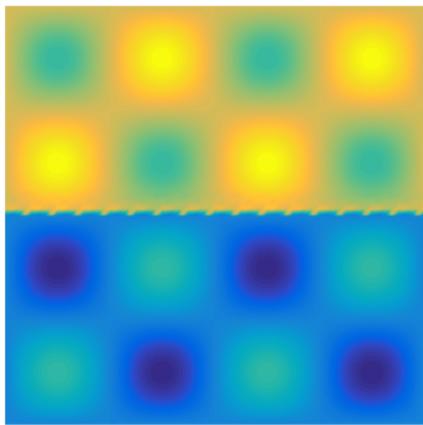
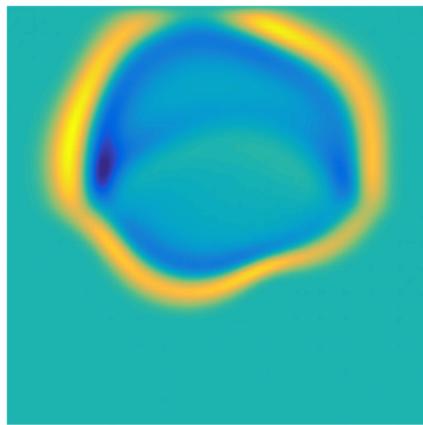
$$(\mathbf{M} (\mathbf{M}_{1/w})^{-1} \mathbf{M})^{-1} = \mathbf{M}^{-1} \mathbf{M}_{1/w} \mathbf{M}^{-1}.$$

- Low storage: matrix-free application of $\mathbf{M}^{-1} \mathbf{M}_{1/w}$ using quadrature.

$$\mathbf{M} = \mathbf{V}_q^T \mathbf{W} \mathbf{V}_q, \quad \mathbf{V}_q = \text{eval. at quad. points}, \quad \mathbf{W} = \text{quad. weights}$$

$$(\mathbf{M})^{-1} \mathbf{M}_{1/w} \text{RHS} = \underbrace{\widehat{\mathbf{M}}^{-1} \mathbf{V}_q^T \mathbf{W}}_{P_q} \text{diag}(1/w) \mathbf{V}_q \text{ (RHS).}$$

Acoustic wave equation: variable coefficients

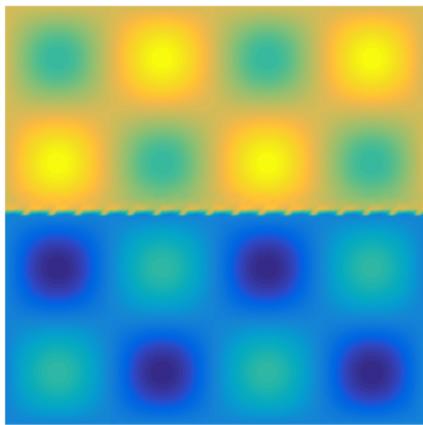
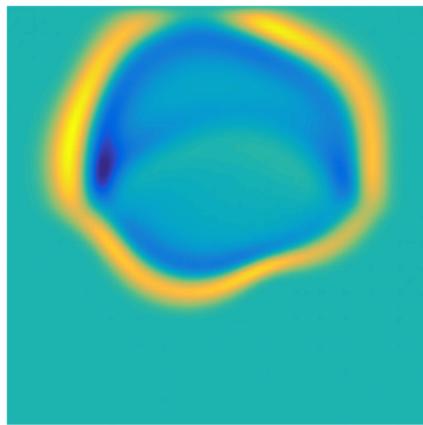
(a) Weight $w = c^2(x, y)$ 

(b) Standard DG

WADG retains high order accuracy: weighted projection error bound:

$$\|u - P_w u\|_{L^2} \leq Ch^{N+1} \|w\|_{W^{N+1,\infty}} \left\| \frac{\sqrt{J}}{w} \right\|_{L^\infty} \|u\|_{W^{N+1,2}}.$$

Acoustic wave equation: variable coefficients

(a) Weight $w = c^2(x, y)$ 

(b) Weighted-adjusted DG

WADG retains high order accuracy: weighted projection error bound:

$$\|u - P_w u\|_{L^2} \leq Ch^{N+1} \|w\|_{W^{N+1,\infty}} \left\| \frac{\sqrt{J}}{w} \right\|_{L^\infty} \|u\|_{W^{N+1,2}}.$$

Low storage WADG: efficiency on GPUs

	$N = 1$	$N = 2$	$N = 3$	$N = 4$	$N = 5$	$N = 6$	$N = 7$
\mathbf{M}_w^{-1}	.66	2.79	9.90	29.4	73.9	170.5	329.4
WADG	0.59	1.44	4.30	13.9	43.0	107.8	227.7
Speedup	1.11	1.94	2.30	2.16	1.72	1.58	1.45

Time (ns) per element: storing/applying $\mathbf{M}_{1/w}^{-1}$ vs WADG (deg. $2N$ quadrature).

- Efficiency on GPUs: reduce memory accesses and data movement.
- (Tuned) low storage WADG faster than storing and applying $\mathbf{M}_{1/w}^{-1}$!

Outline

- 1 Weight-adjusted DG methods: acoustics
- 2 Extension to elastic wave propagation
- 3 Acoustic-elastic coupling, poroelasticity

Matrix-valued weights and elastic wave propagation

- Symmetric hyperbolic system: velocity \mathbf{v} , stress $\boldsymbol{\sigma} = \text{vec}(\mathbf{S})$ with $\boldsymbol{\sigma} = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{xz}, \sigma_{xy})^T$.

$$\rho \frac{\partial \mathbf{v}}{\partial t} = \sum_{i=1}^d \mathbf{A}_i^T \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}_i}, \quad \mathbf{C}^{-1} \frac{\partial \boldsymbol{\sigma}}{\partial t} = \sum_{i=1}^d \mathbf{A}_i \frac{\partial \mathbf{v}}{\partial \mathbf{x}_i}.$$

- Factoring out the constitutive stiffness tensor \mathbf{C} results in simple and spatially constant matrices \mathbf{A}_i .

$$\mathbf{C} = \underbrace{\begin{pmatrix} 2\mu + \lambda & \lambda & \lambda \\ \lambda & 2\mu + \lambda & \lambda \\ \lambda & \lambda & 2\mu + \lambda \end{pmatrix}}_{\text{for isotropic media}} + \mu \mathbf{I}_{3 \times 3}, \quad \mathbf{A}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Hughes and Marsden 1978. Classical elastodynamics as a linear symmetric hyperbolic system.

DG formulation for elasticity: energy stability

- Analogous to acoustics: numerical fluxes independent of media!

$$\int_{D^k} \left(\mathbf{C}^{-1} \frac{\partial \boldsymbol{\sigma}}{\partial t} \right)^T \mathbf{q} = \int_{D^k} \sum_{i=1}^d \mathbf{A}_i \frac{\partial \mathbf{v}}{\partial \mathbf{x}_i} \mathbf{q} + \frac{1}{2} \int_{\partial D^k} \left(\mathbf{A}_n [\![\mathbf{v}]\!] + \tau_\sigma \mathbf{A}_n \mathbf{A}_n^T [\![\boldsymbol{\sigma}]\!] \right) \mathbf{q},$$

$$\int_{D^k} \left(\rho \frac{\partial \mathbf{v}}{\partial t} \right)^T \mathbf{w} = \int_{D^k} \sum_{i=1}^d \mathbf{A}_i^T \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}_i} \mathbf{w} + \frac{1}{2} \int_{\partial D^k} \left(\mathbf{A}_n^T [\![\boldsymbol{\sigma}]\!] + \tau_v \mathbf{A}_n^T \mathbf{A}_n [\![\mathbf{v}]\!] \right) \mathbf{w}.$$

- Energy method: take $\mathbf{q} = \boldsymbol{\sigma}$, $\mathbf{w} = \mathbf{v}$

$$\int_{D^k} \left(\mathbf{C}^{-1} \frac{\partial \boldsymbol{\sigma}}{\partial t} \right)^T \boldsymbol{\sigma} = \frac{\partial}{\partial t} \int_{D^k} \left(\boldsymbol{\sigma}(\mathbf{x})^T \mathbf{C}^{-1}(\mathbf{x}) \boldsymbol{\sigma}(\mathbf{x}) \right)$$

$$\int_{D^k} \left(\rho \frac{\partial \mathbf{v}}{\partial t} \right)^T \mathbf{v} = \frac{\partial}{\partial t} \int_{D^k} \rho(\mathbf{x}) |\mathbf{v}|^2.$$

DG formulation for elasticity: energy stability, cont.

- \mathbf{A}_i constant, integration by parts if quadrature exact for degree $(2N-1)$

$$\int_{D^k} \sum_{i=1}^d \mathbf{A}_i \frac{\partial \mathbf{v}}{\partial \mathbf{x}_i} \cdot \mathbf{q} + \frac{1}{2} \int_{\partial D^k} \left(\mathbf{A}_n [\![\mathbf{v}]\!] + \tau_\sigma \mathbf{A}_n \mathbf{A}_n^T [\![\boldsymbol{\sigma}]\!] \right) \cdot \mathbf{q},$$

$$\int_{D^k} \sum_{i=1}^d \mathbf{A}_i^T \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{x}_i} \cdot \mathbf{w} + \frac{1}{2} \int_{\partial D^k} \left(\mathbf{A}_n^T [\![\boldsymbol{\sigma}]\!] + \tau_v \mathbf{A}_n^T \mathbf{A}_n [\![\mathbf{v}]\!] \right) \cdot \mathbf{w}$$

- Energy stability (penalty weakly enforces continuity conditions):

$$\sum_{D^k} \frac{1}{2} \frac{\partial}{\partial t} \int_{D^k} \rho |\mathbf{v}|^2 + \boldsymbol{\sigma}^T \mathbf{C}^{-1} \boldsymbol{\sigma} =$$

$$- \sum_{\text{faces}} \int_f \frac{\tau_v}{2} |\mathbf{A}_n [\![\mathbf{v}]\!]|^2 + \frac{\tau_\sigma}{2} \left| \mathbf{A}_n^T [\![\boldsymbol{\sigma}]\!] \right|^2 \leq 0.$$

- $\tau_v, \tau_\sigma > 0$ penalizes normal jumps $[\![\mathbf{v}]\!] \cdot \mathbf{n} \approx 0, [\![\mathbf{S}]\!] \cdot \mathbf{n} \approx 0$.

DG formulation for elasticity: energy stability, cont.

- \mathbf{A}_i constant, integration by parts if quadrature exact for degree $(2N-1)$

$$\int_{D^k} \sum_{i=1}^d -\mathbf{v} \cdot \mathbf{A}_i^T \frac{\partial \mathbf{q}}{\partial \mathbf{x}_i} + \int_{\partial D^k} \left(\mathbf{A}_n \{ \{ \mathbf{v} \} \} + \frac{\tau_\sigma}{2} \mathbf{A}_n \mathbf{A}_n^T [\sigma] \right) \cdot \mathbf{q},$$

$$\int_{D^k} \sum_{i=1}^d \mathbf{A}_i^T \frac{\partial \sigma}{\partial \mathbf{x}_i} \cdot \mathbf{w} + \frac{1}{2} \int_{\partial D^k} \left(\mathbf{A}_n^T [\sigma] + \tau_v \mathbf{A}_n^T \mathbf{A}_n [\mathbf{v}] \right) \cdot \mathbf{w}$$

- Energy stability (penalty weakly enforces continuity conditions):

$$\sum_{D^k} \frac{1}{2} \frac{\partial}{\partial t} \int_{D^k} \rho |\mathbf{v}|^2 + \boldsymbol{\sigma}^T \mathbf{C}^{-1} \boldsymbol{\sigma} =$$

$$- \sum_{\text{faces}} \int_f \frac{\tau_v}{2} |\mathbf{A}_n [\mathbf{v}]|^2 + \frac{\tau_\sigma}{2} \left| \mathbf{A}_n^T [\sigma] \right|^2 \leq 0.$$

- $\tau_v, \tau_\sigma > 0$ penalizes normal jumps $[\mathbf{v}] \cdot \mathbf{n} \approx 0$, $[\mathbf{S}] \cdot \mathbf{n} \approx 0$.

DG formulation for elasticity: energy stability, cont.

- \mathbf{A}_i constant, integration by parts if quadrature exact for degree $(2N-1)$

$$\int_{D^k} \sum_{i=1}^d -\mathbf{v} \cdot \mathbf{A}_i^T \frac{\partial \sigma}{\partial \mathbf{x}_i} + \int_{\partial D^k} \left(\mathbf{A}_n \{ \{ \mathbf{v} \} \} + \frac{\tau_\sigma}{2} \mathbf{A}_n \mathbf{A}_n^T [\sigma] \right) \cdot \sigma,$$

$$\int_{D^k} \sum_{i=1}^d \mathbf{A}_i^T \frac{\partial \sigma}{\partial \mathbf{x}_i} \cdot \mathbf{v} + \frac{1}{2} \int_{\partial D^k} \left(\mathbf{A}_n^T [\sigma] + \tau_v \mathbf{A}_n^T \mathbf{A}_n [\mathbf{v}] \right) \cdot \mathbf{v}$$

- Energy stability (penalty weakly enforces continuity conditions):

$$\sum_{D^k} \frac{1}{2} \frac{\partial}{\partial t} \int_{D^k} \rho |\mathbf{v}|^2 + \boldsymbol{\sigma}^T \mathbf{C}^{-1} \boldsymbol{\sigma} =$$

$$- \sum_{\text{faces}} \int_f \frac{\tau_v}{2} |\mathbf{A}_n [\mathbf{v}]|^2 + \frac{\tau_\sigma}{2} \left| \mathbf{A}_n^T [\sigma] \right|^2 \leq 0.$$

- $\tau_v, \tau_\sigma > 0$ penalizes normal jumps $[\mathbf{v}] \cdot \mathbf{n} \approx 0$, $[\mathbf{S}] \cdot \mathbf{n} \approx 0$.

DG formulation for elasticity: energy stability, cont.

- \mathbf{A} ; constant, integration by parts if quadrature exact for degree $(2N-1)$

Sum volume terms, which cancel to zero +

$$\int_{\partial D^k} \left(\mathbf{A}_n \{\{\mathbf{v}\}\} + \frac{\tau_\sigma}{2} \mathbf{A}_n \mathbf{A}_n^T [\![\sigma]\!] \right) \cdot \boldsymbol{\sigma} + \frac{1}{2} \left(\mathbf{A}_n^T [\![\sigma]\!] + \tau_v \mathbf{A}_n^T \mathbf{A}_n [\![\mathbf{v}]\!] \right) \cdot \mathbf{v}$$

- Energy stability (penalty weakly enforces continuity conditions):

$$\begin{aligned} \sum_{D^k} \frac{1}{2} \frac{\partial}{\partial t} \int_{D^k} \rho |\mathbf{v}|^2 + \boldsymbol{\sigma}^T \mathbf{C}^{-1} \boldsymbol{\sigma} = \\ - \sum_{\text{faces}} \int_f \frac{\tau_v}{2} |\mathbf{A}_n [\![\mathbf{v}]\!]|^2 + \frac{\tau_\sigma}{2} \left| \mathbf{A}_n^T [\![\boldsymbol{\sigma}]\!] \right|^2 \leq 0. \end{aligned}$$

- $\tau_v, \tau_\sigma > 0$ penalizes normal jumps $[\![\mathbf{v}]\!] \cdot \mathbf{n} \approx 0, [\![\mathbf{S}]\!] \cdot \mathbf{n} \approx 0$.

DG formulation for elasticity: energy stability, cont.

- \mathbf{A}_n constant, integration by parts if quadrature exact for degree $(2N-1)$

Sum volume terms, which cancel to zero +

$$\int_{\partial D^k} \left(\mathbf{A}_n \{\!\{ \mathbf{v} \}\!} + \frac{\tau_\sigma}{2} \mathbf{A}_n \mathbf{A}_n^T [\![\boldsymbol{\sigma}]\!] \right) \cdot \boldsymbol{\sigma} + \frac{1}{2} \left(\mathbf{A}_n^T [\![\boldsymbol{\sigma}]\!] + \tau_v \mathbf{A}_n^T \mathbf{A}_n [\![\mathbf{v}]\!] \right) \cdot \mathbf{v}$$

- Energy stability (penalty weakly enforces continuity conditions):

$$\begin{aligned} \sum_{D^k} \frac{1}{2} \frac{\partial}{\partial t} \int_{D^k} \rho |\mathbf{v}|^2 + \boldsymbol{\sigma}^T \mathbf{C}^{-1} \boldsymbol{\sigma} = \\ - \sum_{\text{faces}} \int_f \frac{\tau_v}{2} |\mathbf{A}_n [\![\mathbf{v}]\!]|^2 + \frac{\tau_\sigma}{2} \left| \mathbf{A}_n^T [\![\boldsymbol{\sigma}]\!] \right|^2 \leq 0. \end{aligned}$$

- $\tau_v, \tau_\sigma > 0$ penalizes normal jumps $[\![\mathbf{v}]\!] \cdot \mathbf{n} \approx 0, [\![\mathbf{S}]\!] \cdot \mathbf{n} \approx 0$.

DG formulation for elasticity: energy stability, cont.

- \mathbf{A} ; constant, integration by parts if quadrature exact for degree $(2N-1)$

Sum over all elements: central flux terms cancel +

$$\sum_k \int_{\partial D^k} \frac{\tau_\sigma}{2} \mathbf{A}_n \mathbf{A}_n^T [\sigma] \cdot \sigma + \frac{\tau_v}{2} \mathbf{A}_n^T \mathbf{A}_n [v] \cdot v$$

- Energy stability (penalty weakly enforces continuity conditions):

$$\begin{aligned} \sum_{D^k} \frac{1}{2} \frac{\partial}{\partial t} \int_{D^k} \rho |v|^2 + \sigma^T \mathbf{C}^{-1} \sigma = \\ - \sum_{\text{faces}} \int_f \frac{\tau_v}{2} |\mathbf{A}_n [v]|^2 + \frac{\tau_\sigma}{2} |\mathbf{A}_n^T [\sigma]|^2 \leq 0. \end{aligned}$$

- $\tau_v, \tau_\sigma > 0$ penalizes normal jumps $[v] \cdot n \approx 0, [S] \cdot n \approx 0$.

DG formulation for elasticity: energy stability, cont.

- \mathbf{A} ; constant, integration by parts if quadrature exact for degree $(2N-1)$

Sum over all elements: combine penalty flux terms+

$$\sum_k \int_{\partial D^k} \frac{\tau_\sigma}{2} \mathbf{A}_n^T [\![\boldsymbol{\sigma}]\!] \cdot \mathbf{A}_n^T \boldsymbol{\sigma} + \frac{\tau_v}{2} \mathbf{A}_n [\![\mathbf{v}]\!] \cdot \mathbf{A}_n \mathbf{v}$$

- Energy stability (penalty weakly enforces continuity conditions):

$$\begin{aligned} \sum_{D^k} \frac{1}{2} \frac{\partial}{\partial t} \int_{D^k} \rho |\mathbf{v}|^2 + \boldsymbol{\sigma}^T \mathbf{C}^{-1} \boldsymbol{\sigma} = \\ - \sum_{\text{faces}} \int_f \frac{\tau_v}{2} |\mathbf{A}_n [\![\mathbf{v}]\!]|^2 + \frac{\tau_\sigma}{2} \left| \mathbf{A}_n^T [\![\boldsymbol{\sigma}]\!] \right|^2 \leq 0. \end{aligned}$$

- $\tau_v, \tau_\sigma > 0$ penalizes normal jumps $[\![\mathbf{v}]\!] \cdot \mathbf{n} \approx 0, [\![\mathbf{S}]\!] \cdot \mathbf{n} \approx 0$.

DG formulation for elasticity: energy stability, cont.

- \mathbf{A} ; constant, integration by parts if quadrature exact for degree $(2N-1)$

Sum over all elements: combine penalty flux terms+

$$\sum_k \int_{\partial D^k} \frac{\tau_\sigma}{2} \mathbf{A}_n^T [\sigma] \cdot \mathbf{A}_n^T [\sigma] + \frac{\tau_v}{2} \mathbf{A}_n [\mathbf{v}] \cdot \mathbf{A}_n [\mathbf{v}]$$

- Energy stability (penalty weakly enforces continuity conditions):

$$\begin{aligned} \sum_{D^k} \frac{1}{2} \frac{\partial}{\partial t} \int_{D^k} \rho |\mathbf{v}|^2 + \boldsymbol{\sigma}^T \mathbf{C}^{-1} \boldsymbol{\sigma} = \\ - \sum_{\text{faces}} \int_f \frac{\tau_v}{2} |\mathbf{A}_n [\mathbf{v}]|^2 + \frac{\tau_\sigma}{2} \left| \mathbf{A}_n^T [\sigma] \right|^2 \leq 0. \end{aligned}$$

- $\tau_v, \tau_\sigma > 0$ penalizes normal jumps $[\mathbf{v}] \cdot \mathbf{n} \approx 0, [\mathbf{S}] \cdot \mathbf{n} \approx 0$.

DG formulation for elasticity: energy stability, cont.

- \mathbf{A} ; constant, integration by parts if quadrature exact for degree $(2N-1)$

Sum over all elements: combine penalty flux terms+

$$\sum_k \int_{\partial D^k} \frac{\tau_\sigma}{2} \mathbf{A}_n^T [\sigma] \cdot \mathbf{A}_n^T [\sigma] + \frac{\tau_v}{2} \mathbf{A}_n [\mathbf{v}] \cdot \mathbf{A}_n [\mathbf{v}]$$

... and done!

- Energy stability (penalty weakly enforces continuity conditions):

$$\begin{aligned} \sum_{D^k} \frac{1}{2} \frac{\partial}{\partial t} \int_{D^k} \rho |\mathbf{v}|^2 + \boldsymbol{\sigma}^T \mathbf{C}^{-1} \boldsymbol{\sigma} = \\ - \sum_{\text{faces}} \int_f \frac{\tau_v}{2} |\mathbf{A}_n [\mathbf{v}]|^2 + \frac{\tau_\sigma}{2} \left| \mathbf{A}_n^T [\sigma] \right|^2 \leq 0. \end{aligned}$$

- $\tau_v, \tau_\sigma > 0$ penalizes normal jumps $[\mathbf{v}] \cdot \mathbf{n} \approx 0, [\mathbf{S}] \cdot \mathbf{n} \approx 0$.

Semi-discrete system: matrix-valued weighted

- Matrix-weighted mass matrix: let $\mathbf{W} \in \mathbb{R}^{d \times d}$ be SPD with entries w_{ij}

$$\mathbf{M}_W = \begin{pmatrix} \mathbf{M}_{w_{11}} & \dots & \mathbf{M}_{w_{1d}} \\ \vdots & \ddots & \vdots \\ \mathbf{M}_{w_{d1}} & \dots & \mathbf{M}_{w_{dd}} \end{pmatrix}$$

- Semi-discrete DG formulation involves \mathbf{C}^{-1} -weighted mass matrix

$$\mathbf{M}_{\mathbf{C}^{-1}} \frac{\partial \Sigma}{\partial t} = \sum_{i=1}^d (\mathbf{A}_i \otimes \mathbf{S}_i) \mathbf{V} + \sum_{\text{faces}} (\mathbf{I} \otimes \mathbf{M}^f) \mathbf{F}_\sigma,$$

$$\mathbf{M}_{\rho I} \frac{\partial \mathbf{V}}{\partial t} = \sum_{i=1}^d (\mathbf{A}_i^T \otimes \mathbf{S}_i) \Sigma + \sum_{\text{faces}} (\mathbf{I} \otimes \mathbf{M}^f) \mathbf{F}_v.$$

Weight-adjusted DG: matrix-valued weights

- Matrix-weighted mass matrix large, hard to invert

$$\mathbf{M}_{\mathbf{C}^{-1}} = \begin{pmatrix} \mathbf{M}_{\mathbf{C}_{11}^{-1}} & \dots & \mathbf{M}_{\mathbf{C}_{1d}^{-1}} \\ \vdots & \ddots & \vdots \\ \mathbf{M}_{\mathbf{C}_{d1}^{-1}} & \dots & \mathbf{M}_{\mathbf{C}_{dd}^{-1}} \end{pmatrix}$$

- Weight-adjusted approximation for \mathbf{C}^{-1} decouples each component

$$\mathbf{M}_{\mathbf{C}^{-1}}^{-1} \approx (\mathbf{I} \otimes \mathbf{M}^{-1}) \mathbf{M}_{\mathbf{C}} (\mathbf{I} \otimes \mathbf{M}^{-1}).$$

- Evaluate RHS components at quadrature points, apply $\mathbf{C}(\mathbf{x}_i)$ to component vectors at quadrature points, project back to polynomials.
- Recovers Kronecker product $\mathbf{M}_{\mathbf{C}^{-1}}^{-1} = \mathbf{C} \otimes \mathbf{M}^{-1}$ for constant \mathbf{C}^{-1} .

Energy stability and DG spectra

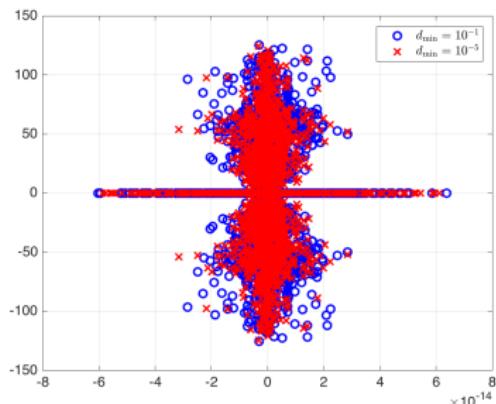
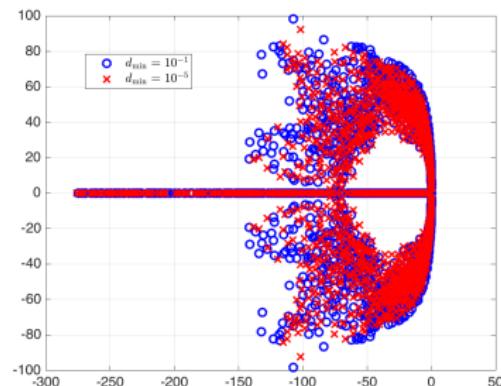
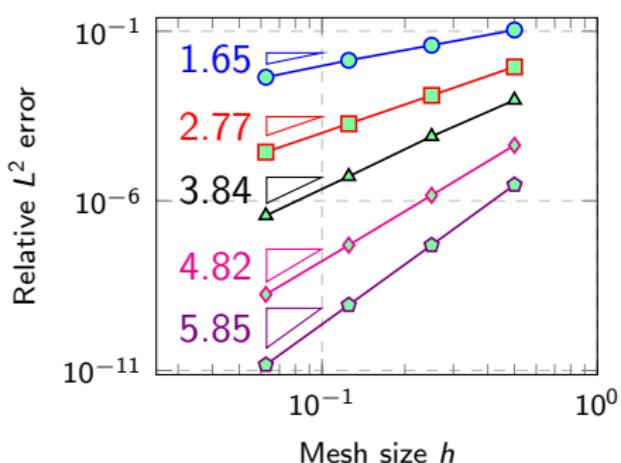
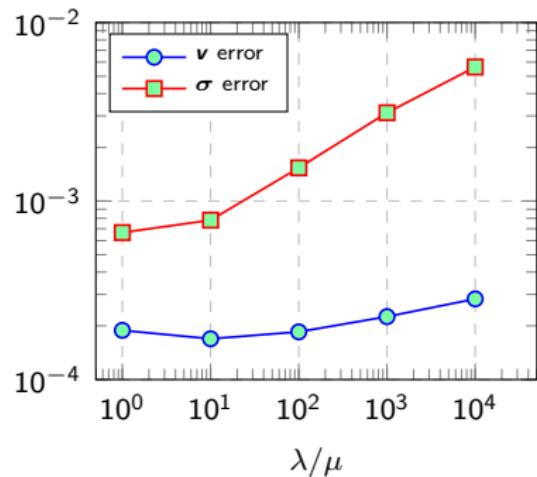
(a) Central flux ($\tau_v, \tau_\sigma = 0$)(b) Penalty flux ($\tau_v, \tau_\sigma = 1$)

Figure: DG spectra with random heterogeneities at each quadrature point.

- Guaranteed energy stability for both energy conservative and energy dissipative numerical interface fluxes.
- CFL can be improved by setting $\tau_\sigma \approx 1/\|\mathbf{C}\|$, $\tau_v \approx \|\rho\|$

Elastic wave propagation: convergence

- Convergence for harmonic oscillation, Rayleigh, Lamb, and Stoneley waves: between $O(h^{N+1})$ and $O(h^{N+1/2})$.
- σ error grows as $\|\mathbf{C}^{-1}\| \rightarrow \infty$ (e.g. incompressible limit $\lambda/\mu \rightarrow \infty$).

(a) L^2 errors (Stoneley wave)(b) $\|\mathbf{C}^{-1}\| \rightarrow \infty, N = 3, h = 1/8.$

Elastic wave propagation: stiff inclusion

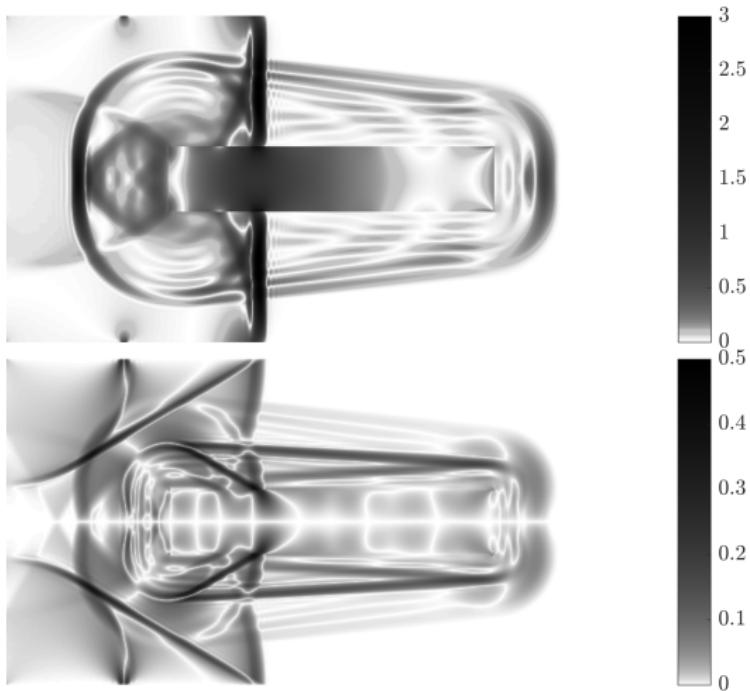
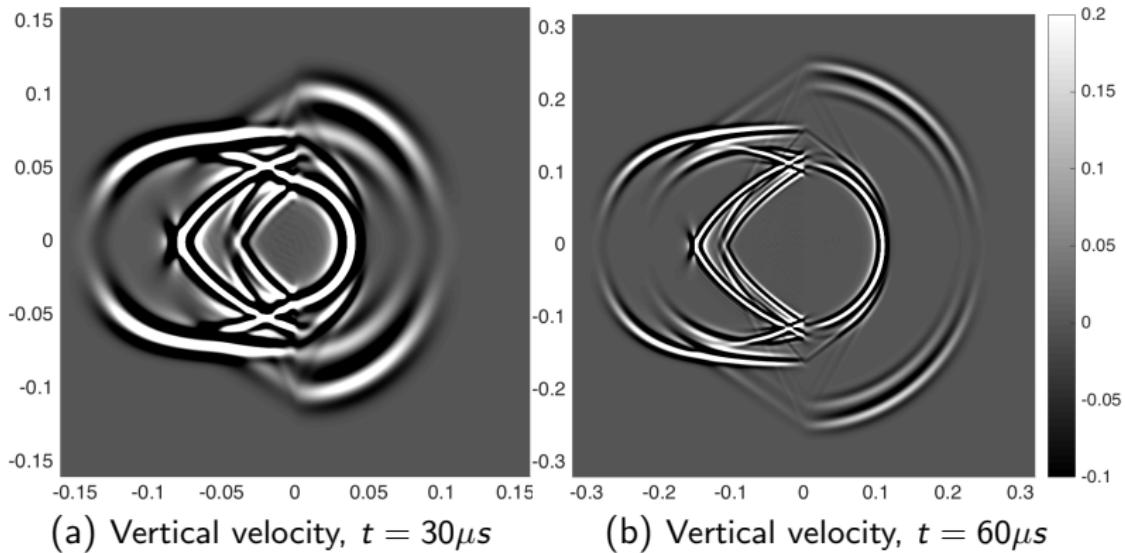


Figure: $\text{tr}(\sigma)$ and σ_{xy} for stiff inclusion with $N = 5$, $h \approx 1/50$.

Elastic wave propagation: anisotropy

Simple implementation for anisotropy - fluxes independent of \mathbf{C} .



Anisotropic heterogeneous media: transverse isotropy ($x < 0$), isotropy ($x > 0$).

Curved meshes and heterogeneous media

- Map curved elements D^k to reference element \hat{D} , introduce determinant of Jacobian of mapping J (varies over \hat{D}).
- RHS terms: discretize skew-symmetric form

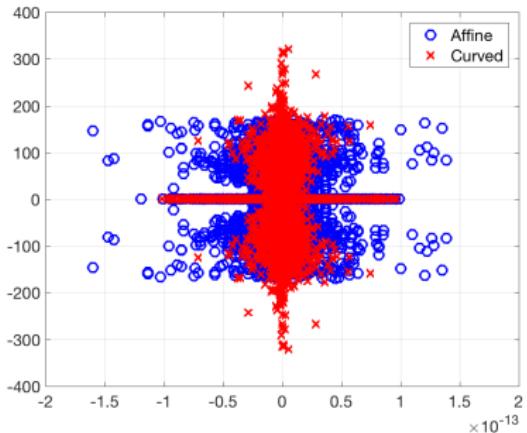
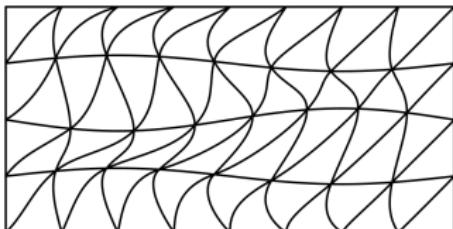
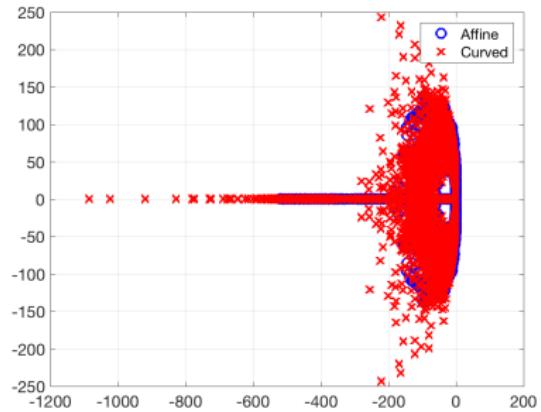
$$\int_{\hat{D}} \sum_{i=1}^d -\mathbf{v} \cdot \mathbf{A}_i^T \frac{\partial \mathbf{q}}{\partial \mathbf{x}_i} J + \int_{\partial D^k} \left(\mathbf{A}_n \{\{\mathbf{v}\}\} + \frac{\tau_\sigma}{2} \mathbf{A}_n \mathbf{A}_n^T [\![\sigma]\!] \right) \mathbf{q},$$

$$\int_{\hat{D}} \sum_{i=1}^d \mathbf{A}_i^T \frac{\partial \sigma}{\partial \mathbf{x}_i} \cdot \mathbf{w} J + \frac{1}{2} \int_{\partial D^k} \left(\mathbf{A}_n^T [\![\sigma]\!] + \tau_v \mathbf{A}_n^T \mathbf{A}_n [\![\mathbf{v}]\!] \right) \mathbf{w}.$$

- Time-derivative terms: J incorporated into matrix weighting.

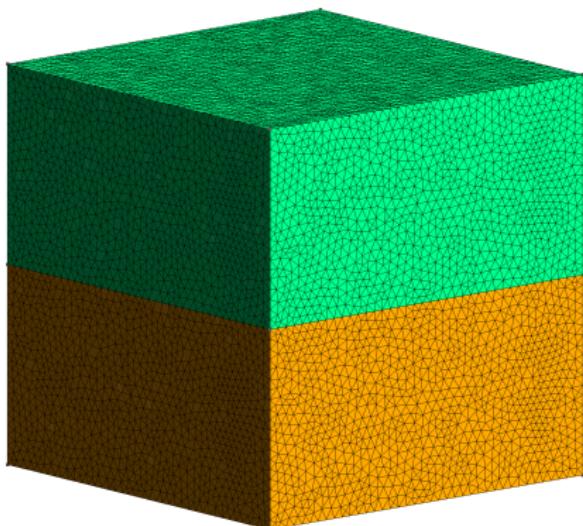
$$\int_{D^k} \left(\mathbf{C}^{-1} \frac{\partial \sigma}{\partial t} \right)^T \mathbf{q} = \int_{\hat{D}} \left(\mathbf{J} \mathbf{C}^{-1} \frac{\partial \sigma}{\partial t} \right)^T \mathbf{q}.$$

Curved meshes and random heterogeneous media

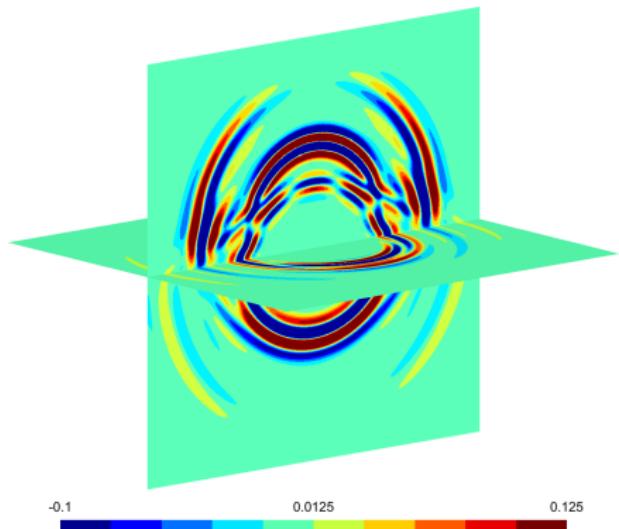
(b) Central flux ($\tau_v, \tau_\sigma = 0$)(c) Central flux ($\tau_v, \tau_\sigma = 1$)

A skew-symmetric discretization guarantees discrete energy stability.

Elastic wave propagation: 3D isotropic media



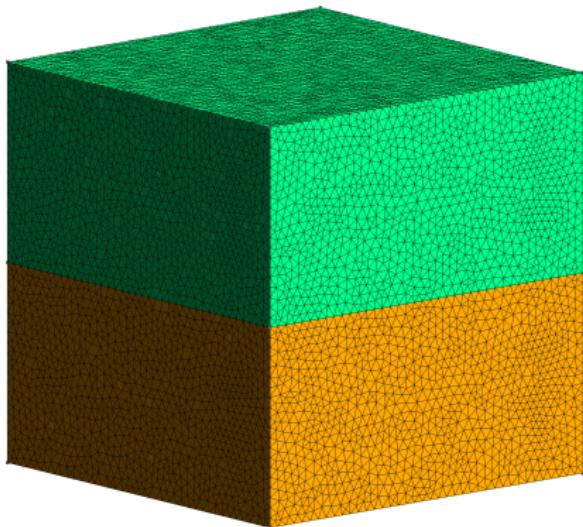
(a) Computational mesh



(b) Homogeneous isotropic media

Figure: $\text{tr}(\sigma)$ with $\mu(x) = 1 + H(y) + \frac{1}{2} \cos(3\pi x) \cos(3\pi y) \cos(3\pi z)$, $N = 5$.

Elastic wave propagation: 3D isotropic media



(a) Computational mesh

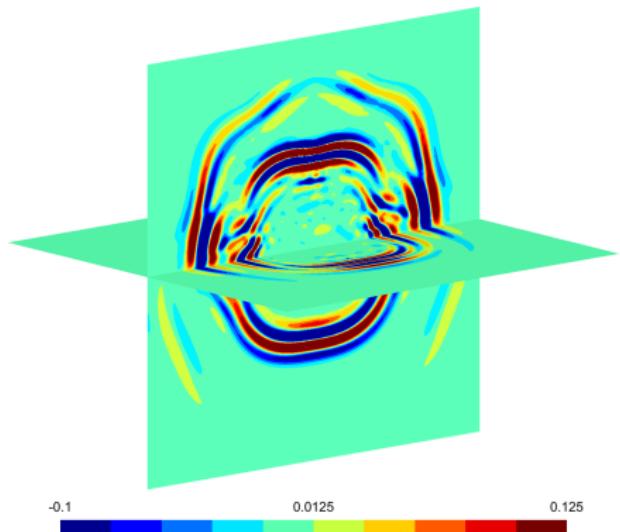
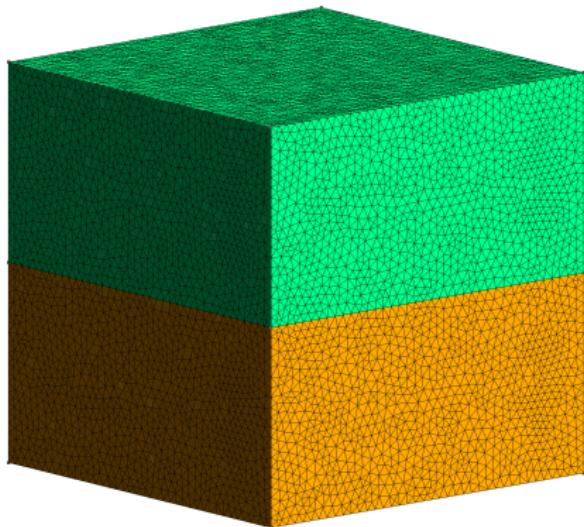
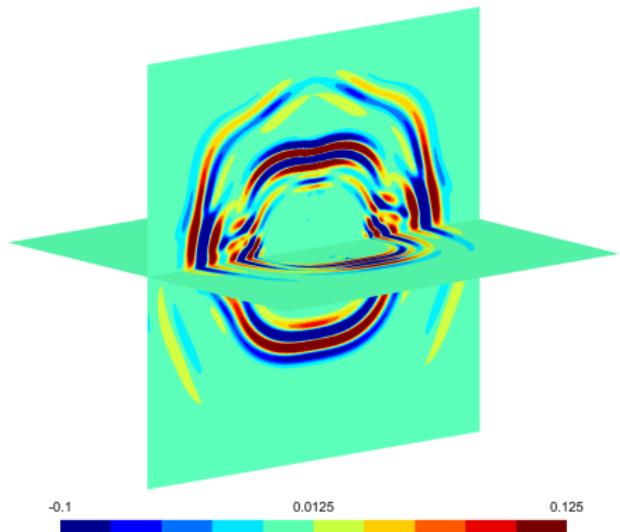
(b) Piecewise constant $C(x)$

Figure: $\text{tr}(\sigma)$ with $\mu(x) = 1 + H(y) + \frac{1}{2} \cos(3\pi x) \cos(3\pi y) \cos(3\pi z)$, $N = 5$.

Elastic wave propagation: 3D isotropic media



(a) Computational mesh



(b) High order $C(x)$

Figure: $\text{tr}(\sigma)$ with $\mu(x) = 1 + H(y) + \frac{1}{2} \cos(3\pi x) \cos(3\pi y) \cos(3\pi z)$, $N = 5$.

Outline

- 1 Weight-adjusted DG methods: acoustics
- 2 Extension to elastic wave propagation
- 3 Acoustic-elastic coupling, poroelasticity

Acoustic-elastic coupling

- Coupling conditions for fluid (acoustic) and solid (elastic) media:

$$\mathbf{S} \cdot \mathbf{n} = p\mathbf{n}, \quad \mathbf{v} \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n}.$$

- Replace $\llbracket p \rrbracket$ and $\llbracket \sigma \rrbracket$ with residuals of coupling conditions

$$\llbracket p \rrbracket = \mathbf{n} \cdot (\mathbf{S}^+ \cdot \mathbf{n}) - p, \quad \text{(acoustic side)}$$

$$\mathbf{A}_n^T \llbracket \sigma \rrbracket = \llbracket \mathbf{S} \rrbracket \cdot \mathbf{n} = p^+ \mathbf{n} - \mathbf{S} \cdot \mathbf{n}, \quad \text{(elastic side)}.$$

- Energy stable for arbitrary acoustic and elastic media.

Acoustic-elastic coupling: arbitrary heterogeneous media

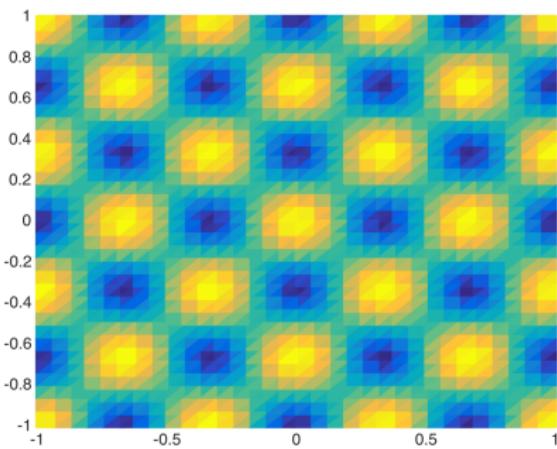
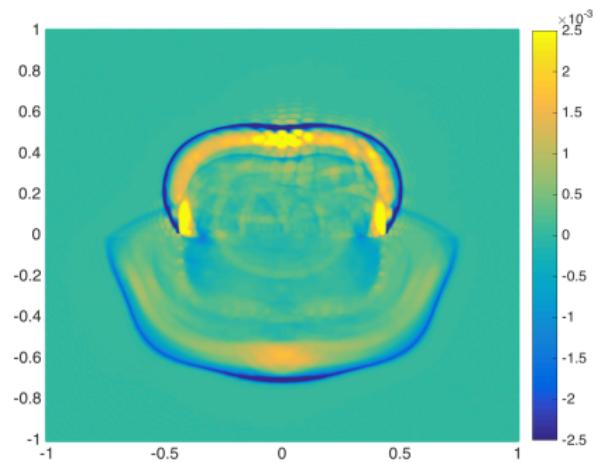
(a) Low order $c^2(x), \mu(x)$ (b) $\text{tr}(\sigma)$

Figure: Acoustic-elastic waves from a Ricker pulse ($N = 10, h = 1/16$).

Acoustic-elastic coupling: arbitrary heterogeneous media

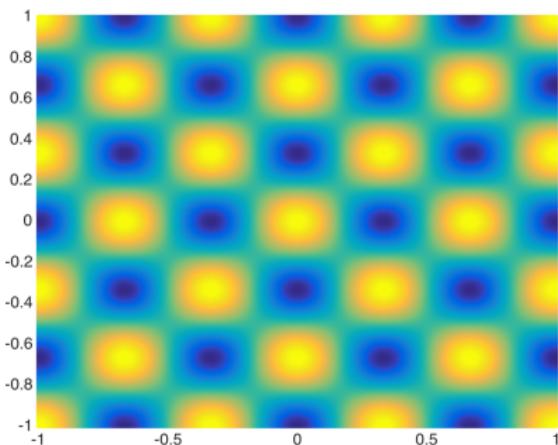
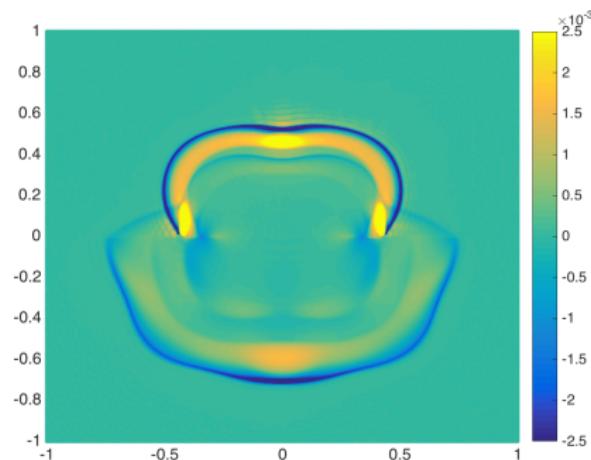
(a) High order $c^2(x), \mu(x)$ (b) $\text{tr}(\sigma)$

Figure: Acoustic-elastic waves from a Ricker pulse ($N = 10, h = 1/16$).

Poroelasticity: low frequency Biot's system

- Biot's system adds pore pressure p and relative fluid velocity \mathbf{q} .
- Symmetric first order system with augmented stress, velocity.

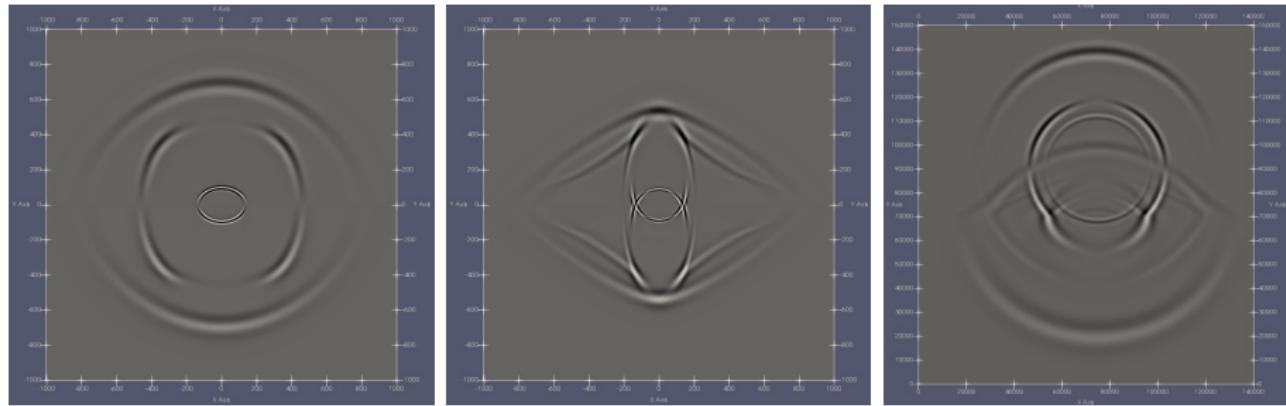
$$\tilde{\boldsymbol{\sigma}} = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{xz}, \sigma_{xy}, p)^T, \quad \tilde{\mathbf{v}} = (v_x, v_y, v_z, q_x, q_y, q_z)^T$$

$$\mathbf{E}_v \frac{\partial \tilde{\mathbf{v}}}{\partial t} = \sum_{i=1}^d \mathbf{A}_i^T \frac{\partial \tilde{\boldsymbol{\sigma}}}{\partial \mathbf{x}_i} - \mathbf{D} \tilde{\mathbf{v}}, \quad \mathbf{E}_\sigma \frac{\partial \tilde{\boldsymbol{\sigma}}}{\partial t} = \sum_{i=1}^d \mathbf{A}_i \frac{\partial \tilde{\mathbf{v}}}{\partial \mathbf{x}_i}.$$

- $\mathbf{E}_s, \mathbf{E}_v$ SPD, \mathbf{A}_i spatially const. + sparse (a single ± 1 per row).

$$\mathbf{E}_v = \begin{pmatrix} \rho & & \rho_f & & \\ & \ddots & & & \ddots \\ \rho_f & & m_1 & & \\ & \ddots & & & \ddots \end{pmatrix}, \quad \mathbf{E}_s = \begin{pmatrix} \mathbf{C}^{-1} & \mathbf{C}^{-1}\boldsymbol{\alpha} \\ \boldsymbol{\alpha}^T \mathbf{C}^{-1} & \frac{1}{M} + \boldsymbol{\alpha}^T \mathbf{C}^{-1} \boldsymbol{\alpha} \end{pmatrix},$$

Numerical results for Biot



(a) Orthotropic media, 1.56 ms

(b) Epoxy-glass, 1.8 ms

(c) Isotropic interface, 230 ms

Inviscid Biot solution (mass particle velocity) showing fast P, slow S, and slow P waves. Simulation utilizes $N = 3$ and a uniform mesh of 128 elements per side.

Systematic derivation of WADG formulations: can symmetrize a system by defining an appropriate convex entropy (energy function).

Carcione 1996. Wave propagation in anisotropic, saturated porous media: plane wave theory and numerical simulation.

Lemoine, Ou, LeVeque 2013. High-resolution finite volume modeling of wave propagation in orthotropic poroelastic media.

Summary and acknowledgements

- Weight-adjusted DG (WADG) for acoustic and elastic wave propagation in heterogeneous media and on curved meshes.
- Generalized mass lumping: energy stability and high order accuracy.
- Significantly simpler formulations using symmetric forms of PDEs.
- This work is supported by DMS-1719818 and DMS-1712639.

Thank you! Questions?



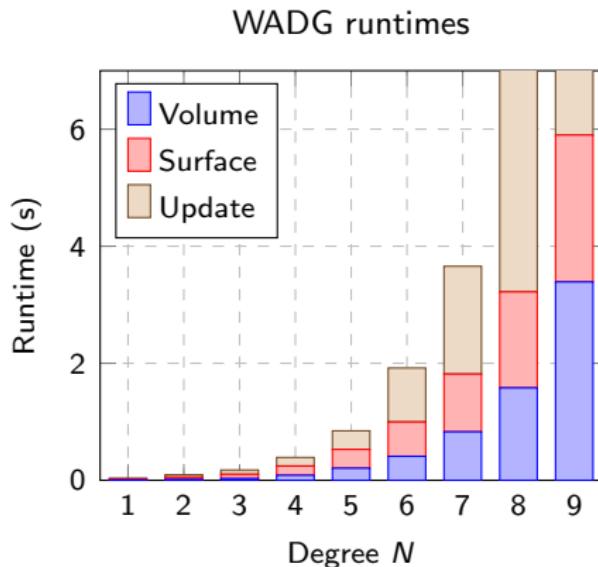
Chan, Hewett, Warburton. 2017. Weight-adjusted DG methods: wave propagation in heterogeneous media (SISC).

Chan 2018. Weight-adjusted DG methods: matrix-valued weights and elastic wave prop. in heterogeneous media (IJNME).

Additional slides

Computational costs at high orders of approximation

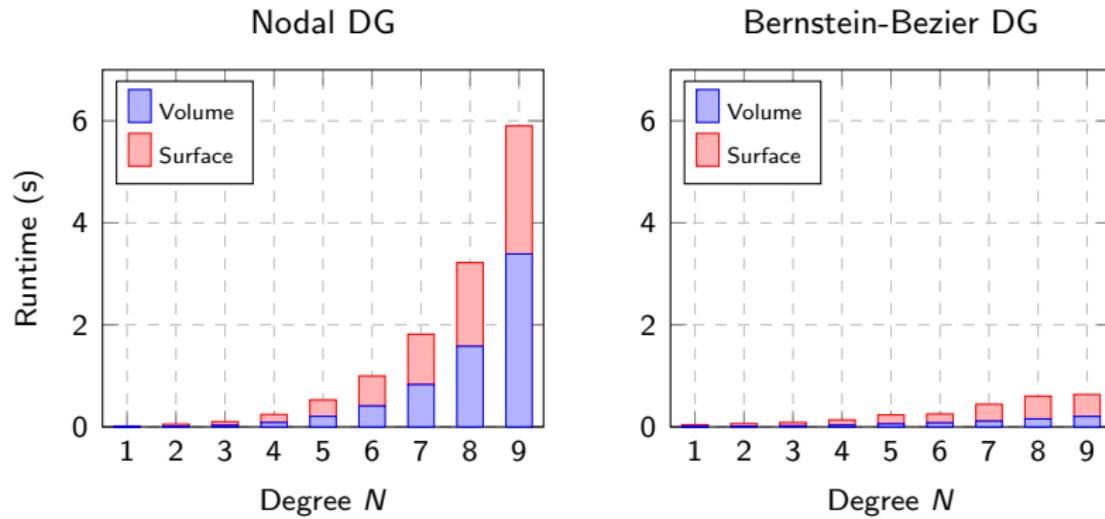
Note: WADG at high orders becomes **expensive!**



- Large **dense** matrices: $O(N^6)$ work per tet.
- High orders usually use tensor-product elements: $O(N^4)$ vs $O(N^6)$ cost, but less geometric flexibility.
- Idea: choose basis such that matrices are **sparse**.

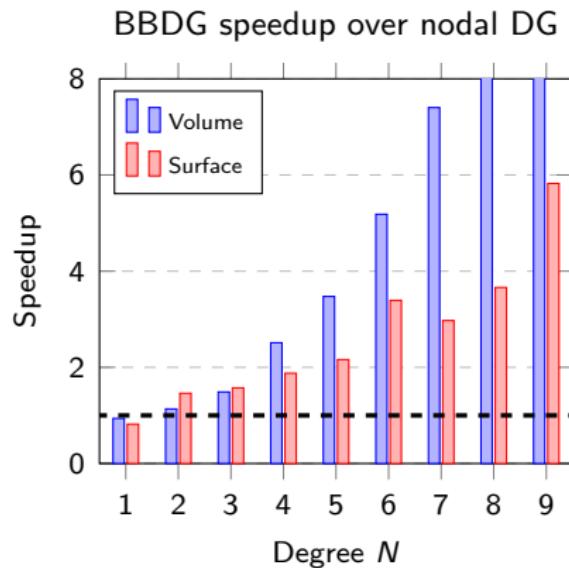
WADG runtimes for 50 timesteps, 98304 elements.

BBDG: efficient volume, surface kernels



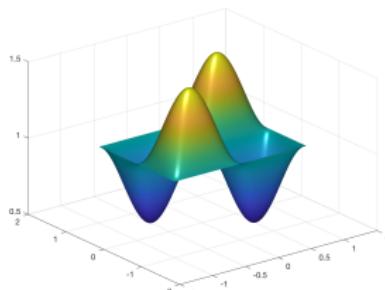
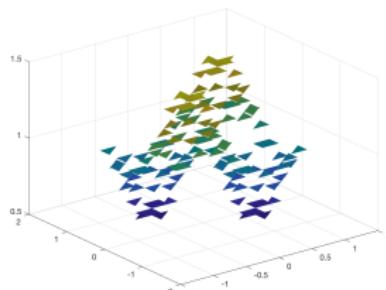
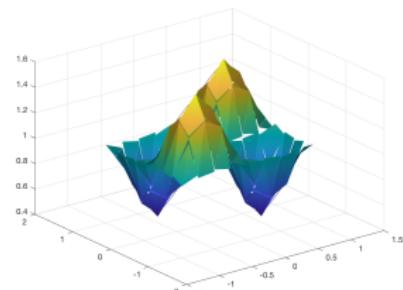
Bernstein-Bezier speedup requires constant coefficient RHS evaluation!

BBDG: efficient volume, surface kernels

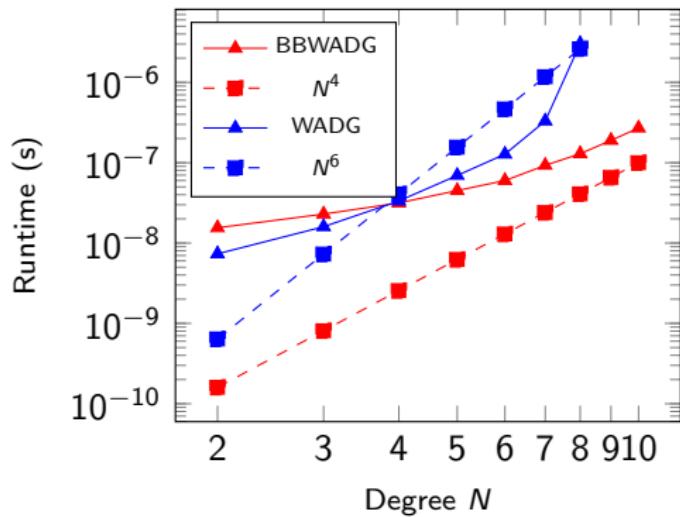


Bernstein-Bezier speedup requires constant coefficient RHS evaluation!

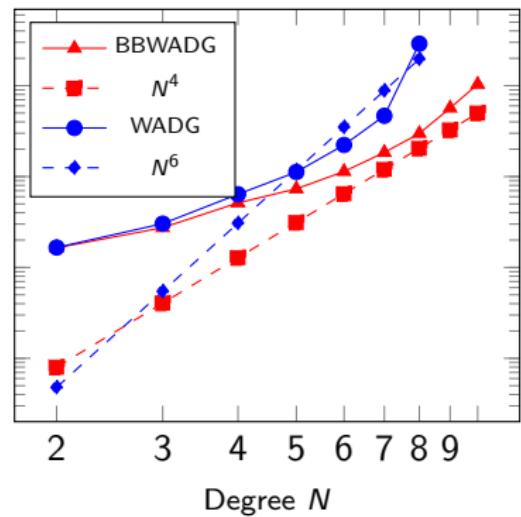
BBWADG: polynomial multiplication and projection

(a) Exact c^2 (b) $M = 0$ approximation(c) $M = 1$ approximation

- WADG reuses fast Bernstein volume and surface kernels.
- Fast $O(N^3)$ Bernstein algorithms for polynomial multiplication: represent $c^2 \in P^M$, $p(\mathbf{x}) \in P^N$, and construct $c^2(\mathbf{x})p(\mathbf{x}) \in P^{M+N}$.
- Fast $O(N^4)$ polynomial L^2 projection $\tilde{\mathbf{P}}_N : P^{M+N} \rightarrow P^N$.

BBWADG: computational runtime for $M = 1$ 

(a)

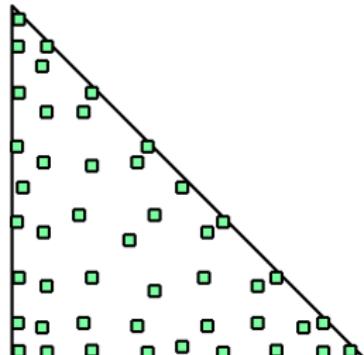
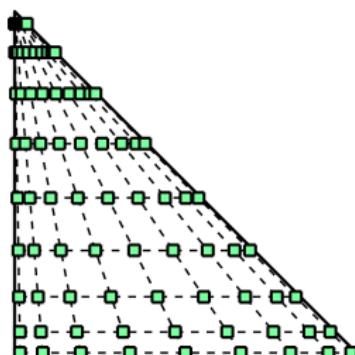


(b)

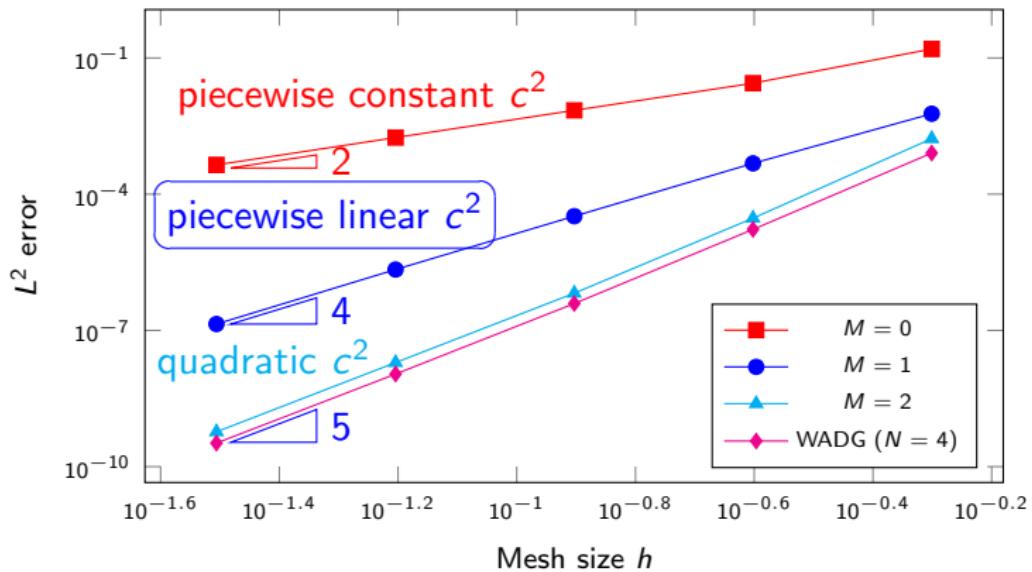
BBWADG: update kernel speedup over WADG (acoustics)

	$N = 3$	$N = 4$	$N = 5$	$N = 6$	$N = 7$	$N = 8$
WADG	1.60e-8	3.34e-8	6.94e-8	1.28e-7	3.31e-7	3.03e-6
BBWADG	2.20e-8	3.30e-8	4.42e-8	6.01e-8	9.46e-8	1.31e-7
Speedup	0.7260	1.0127	1.5706	2.1258	3.4938	23.1591

For $N \geq 8$, quadrature (and WADG) becomes much more expensive.

(a) $N = 7$ quadrature(b) $N = 8$ quadrature

BBWADG: approximating c^2 and accuracy



Approximating smooth $c^2(x)$ using L^2 projection:
 $O(h^2)$ for $M = 0$, $O(h^4)$ for $M = 1$, $O(h^{M+3})$ for $0 < M \leq N - 2$.