1. Properties of GD functions

Interior GD functions are defined as scalings and translations of a "canonical" GD function $\phi(x)$. Assuming that the polynomial degree p is odd, let $\ell_j(x)$ be the $(p+1)$ Lagrange functions defined using equispaced nodal points x_j such that

$$\{x_0, \dots, x_p\} = \{-(a-1), \dots, a\}, \qquad a = \frac{p+1}{2}.$$

This choice of nodal points ensures that the interval $[0,1]$ lies is the middle of $[x_0, x_p]$. We define the canonical GD function $\phi(x)$ piecewise on the intervals $I_0, \dots, I_{2p-2}$, where $I_j = [-(p-1)+j, -p+j]$

$$\phi(x) = \begin{cases} \ell_j(x-j), & x \in I_j \\ 0 & \text{otherwise} \end{cases}.$$

Symbolic computations then yield the following lemma:

Lemma 1. *Let ϕ be the canonical GD function. For $p = 1, 3, \dots, 75$,*

(1) $\int_{-\infty}^{\infty} \phi^2 \leq \int_{-\infty}^{\infty} \phi = 1$

(2) *The GD basis functions satisfy*

$$\sum_{j=-p}^{p} \left| \int \phi_0(x)\phi_0(x-j) \right| \leq 2.$$

(3) *For any degree p polynomial $u(x)$,*

$$\int_{-\infty}^{\infty} u\phi_i = u(0).$$

Proof. We use symbolic software to verify the first part (1) and (2). The equality in (1) can be shown directly. Because $\operatorname{supp}(\phi) = [-(p-1), p-1]$, $\int_{-\infty}^{\infty} \phi(x) = \int_{-(p-1)}^{p-1} \phi(x)$. By the definition of $\phi(x)$,

$$\int_{-(p-1)}^{(p-1)} \phi(x) = \sum_{j=0}^{p} \int_{I_j} \ell_j(x - x_j) = \int_0^1 \sum_{j=0}^{p} \ell_j(x) = \int_0^1 1 = 1.$$

To prove (3), we first show that $\int_{-(p-1)}^{p-1} x^k \phi = 0$ for $0 < k \leq p$. For $k > 0$ odd, this holds by the symmetry of $\phi(x)$ across 0. For $0 < k \leq p$ even, this condition is equivalent to

$$\int_{-(p-1)}^{p-1} x^k \phi_i(x) = 2\int_0^{p-1} x^k \phi_i(x) = \sum_{j=0}^{a} \int_0^1 \ell_j(x)(x-j)^k = 0.$$

which we verify for $p = 1, \dots, 75$ using symbolic software. Then, (3) follows from (1) and a Taylor representation of $u(x)$ around $x = 0$.

$$u(x) = u(0) + u'(0)x + u''(0)x^2 + \ldots + u^{(p)}(0)x^p.$$

□

We conjecture that Lemma 1 holds for all $p > 0$ odd. A translation and scaling of ϕ then provides the following corollary:

Corollary 1. *Let x_i be equispaced points with spacing h, and let $\phi_i(x) = \phi\left((x - x_j)/h\right)$ be the GD function at x_i. For any degree p polynomial $u(x)$,*

$$\frac{1}{h}\int_{-\infty}^{\infty} u\phi_i = u(x_i).$$

2. Accuracy and energy stability of the lumped GD mass matrix

Using Lemma 1, we can show that the lumped GD mass matrix is high order accurate in the following sense:

Lemma 2. *Let $\widetilde{\boldsymbol{M}}$ denote the non-symmetric lumped GD mass matrix. Then, $\widetilde{\boldsymbol{M}}^{-1}\boldsymbol{Q}$ is a $(p+1)$ order accurate approximation to the first derivative.*

Proof. Let $u(x)$ be a degree p polynomial with GD coefficients $\boldsymbol{u}_i = u(x_i)$. Since the GD basis can reproduce polynomials of degree p, $\boldsymbol{\delta u} = \boldsymbol{M}^{-1}\boldsymbol{Q}\boldsymbol{u}$ are the GD coefficients of the exact derivative $\left.\frac{\partial u}{\partial x}\right|_{x_i}$.

The lumped GD mass matrix $\widetilde{\boldsymbol{M}}$ is $(p+1)$ order accurate if $\widetilde{\boldsymbol{M}}\boldsymbol{\delta u} = \boldsymbol{Q}\boldsymbol{u}$ as well. Since $\boldsymbol{\delta u}$ is again polynomial, high order accuracy is ensured if $\boldsymbol{M}\boldsymbol{u} = \widetilde{\boldsymbol{M}}\boldsymbol{u}$ for all coefficients $\boldsymbol{u}$ which correspond to polynomials of degree p. Since the boundary rows of $\boldsymbol{M}$ and $\widetilde{\boldsymbol{M}}$ are identical, $(\boldsymbol{M}\boldsymbol{u})_i = (\widetilde{\boldsymbol{M}}\boldsymbol{u})_i$ for all indices i corresponding to boundary GD functions. For $\boldsymbol{u}$ polynomial and i corresponding to interior GD functions, Corollary 1 guarantees $(\boldsymbol{M}\boldsymbol{u})_i = (\widetilde{\boldsymbol{M}}\boldsymbol{u})_i$. □

We now show that, for linear symmetric hyperbolic PDEs, the lumped GD mass matrix preserves semi-discrete energy stability. We assume that there are sufficiently many elements relative to the order p. Then, under a reordering of degrees of freedom, the GD mass matrix $\boldsymbol{M}$ is

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^T & \boldsymbol{C} \end{bmatrix},$$

where $\boldsymbol{A}$ is the sub-matrix consisting of rows and columns of $\boldsymbol{M}$ corresponding to boundary GD functions, and $\boldsymbol{C}$ is the sub-matrix corresponding to interior GD functions. Since $\boldsymbol{M}$ is SPD, the matrices $\boldsymbol{A}, \boldsymbol{C}$ are also SPD. We assume for simplicity that $h = 1$, such that the lumped mass matrix $\widetilde{\boldsymbol{M}}$ is

$$\widetilde{\boldsymbol{M}} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}.$$

We require that $\boldsymbol{x}^T\widetilde{\boldsymbol{M}}\boldsymbol{x}$ induces a norm on $\boldsymbol{x}$, which holds if $\widetilde{\boldsymbol{M}}$ is positive definite.

For $h = 1$, we have the following property of $\boldsymbol{C}$:

Lemma 3. *The maximum eigenvalue of $|\boldsymbol{C}|$ is bounded by 2.*

Proof. The proof follows directly from bounding the Rayleigh quotient of $|\boldsymbol{C}|$ and using the banded Toeplitz nature of the matrix. Expanding $\boldsymbol{u}^T\left|\boldsymbol{C}\right|\boldsymbol{u}$ out and using symmetry gives

$$\begin{aligned}
\boldsymbol{u}^T \left|\boldsymbol{C}\right| \boldsymbol{u} &\leq \sum_{i=0}^{K} \sum_{j=\max(0,i-p)}^{\min(K,i+p)} \left|\boldsymbol{u}_i \boldsymbol{u}_j\right| \left|\int \phi_i \phi_j\right| \\
&\leq \sum_{i=0}^{K} \left(\boldsymbol{u}_i^2 \int \phi_i^2 + 2 \sum_{j=i+1}^{\min(K,i+p)} \left|\boldsymbol{u}_i \boldsymbol{u}_j\right| \left|\int \phi_i \phi_j\right| \right).
\end{aligned}$$

Applying Young's inequality bounds this sum from above by

$$\begin{aligned}
&\leq \sum_{i=0}^{K} \left(\boldsymbol{u}_i^2 \int \phi_i^2 + \sum_{j=i+1}^{\min(K,i+p)} \left(\boldsymbol{u}_i^2 + \boldsymbol{u}_j^2\right) \left|\int \phi_i \phi_j\right| \right) \\
&= \sum_{i=0}^{K} \boldsymbol{u}_i^2 \left(\sum_{j=i}^{\min(K,i+p)} \left|\int \phi_i \phi_j\right| \right) + \sum_{j=i+1}^{i+p} \boldsymbol{u}_j^2 \left|\int \phi_i \phi_j\right|. \qquad (1)
\end{aligned}$$

Distributing the terms of the latter sum in (1) yields that

$$\begin{aligned}
\boldsymbol{u}^T \left|\boldsymbol{C}\right| \boldsymbol{u} &\leq \sum_{i=0}^{K} \boldsymbol{u}_i^2 \left(\sum_{j=\max(i-p,0)}^{\min(K,i+p)} \left|\int \phi_i \phi_j\right| \right) \\
&\leq \sum_{i=0}^{K} \boldsymbol{u}_i^2 \left(\sum_{j=-p}^{p} \left|\int \phi_0 \phi_j\right| \right) \leq 2 \sum_{i=0}^{K} \boldsymbol{u}_i^2 = 2\boldsymbol{u}^T \boldsymbol{u}
\end{aligned}$$

where we have used translation invariance of the interior GD basis functions and property (2) of Lemma 1. □

We can then show the following:

Lemma 4. *The lumped GD mass matrix $\widetilde{\boldsymbol{M}}$ is positive definite in the sense that*

$$0 < \boldsymbol{x}^T \widetilde{\boldsymbol{M}} \boldsymbol{x}, \qquad \forall \boldsymbol{x} \in \mathbb{R}^n, \quad \boldsymbol{x} \neq 0.$$

Proof. Let $\boldsymbol{x} = [\boldsymbol{u}, \boldsymbol{v}]^T$. Using that $\boldsymbol{u}^T \boldsymbol{C} \boldsymbol{u} \leq \boldsymbol{u}^T \left|\boldsymbol{C}\right| \boldsymbol{u}$ and Lemma 4

$$\begin{aligned}
0 < \boldsymbol{x}^T \boldsymbol{M} \boldsymbol{x} &\leq \boldsymbol{u}^T \boldsymbol{A} \boldsymbol{u} + 2\boldsymbol{u}^T \boldsymbol{B} \boldsymbol{v} + \boldsymbol{v}^T \left|\boldsymbol{C}\right| \boldsymbol{v} \\
&\leq \boldsymbol{u}^T \boldsymbol{A} \boldsymbol{u} + 2\boldsymbol{u}^T \boldsymbol{B} \boldsymbol{v} + 2\boldsymbol{v}^T \boldsymbol{v} \\
&\leq 2\left(\boldsymbol{u}^T \boldsymbol{A} \boldsymbol{u} + \boldsymbol{u}^T \boldsymbol{B} \boldsymbol{v} + \boldsymbol{v}^T \boldsymbol{v}\right) = 2\boldsymbol{x}^T \widetilde{\boldsymbol{M}} \boldsymbol{x}
\end{aligned}$$

□

Remark. *The bound in Lemma 4 is sufficient to show positive definiteness of the non-symmetric lumped GD mass matrix. However, numerical experiments indicate that the maximum eigenvalue $\lambda_{\max}$ of $\boldsymbol{C}$ achieves a tighter bound $\lambda_{\max} \leq 1$ for all p and K tested.*

Ideally, the norms induced by $\boldsymbol{M}$ and $\widetilde{\boldsymbol{M}}$ should also be equivalent and induce equivalent measures of energy. Numerical experiments that this is indeed the case.