MULTI-PATCH TIME-DOMAIN DISCONTINUOUS GALERKIN METHODS WITH IMPROVED CFL CONDITIONS

JESSE CHAN, JOHN EVANS

Abstract.
Achieve extra h-resolution for the same CFL condition by using high order B-splines as a local approximation space on tensor product elements. Alternatively, achieve an $O(h_N/N)$ CFL with respect to the fine-grid mesh size $h_N = h/N$.
Achieve a spectral element-type implementation using quadrature nodes for B-splines.

1. Introduction. Original IGA paper [1]

The "value of continuity" for direct solvers [2] is to decrease the number of degrees of freedom while maintaining a comparable resolution.

DG on tets using exact geometry [3]

Multi-patch IGA [4, 5].

Can improve work estimates [6], but not CFL.

Improving CFL: Dual-grid filter [7]. Adding conservation constraints [8, 9], increasing coupling between elements. Central DG [10].

Hermite methods [11, 12].

The work here more closely resembles the recently introduced Galerkin difference methods [13], which constructs extended-support basis functions from .

2. Stable timestep restrictions for time-domain DG methods. Under the method of lines, a PDE is transformed to a system of ordinary differential equations (ODEs)

$$\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{dt}} = \boldsymbol{A}_h \boldsymbol{U}$$

by discretizing in space. This system may then be solved using an explicit time stepping, which avoids the inversion of a global matrix. Since such a scheme requires only matrix-vector products, the storage of a global matrix may also be avoided through a matrix-free implementation. Under explicit time stepping, the eigenvalues of $\boldsymbol{A}_h$ must lie within the stability region of the chosen scheme.

Bounds on real and imaginary parts of eigenvalues can be estimated using theorems of Bendixson and Hirsch [14], as is done in [15].

We illustrate this using the advection equation.

$$\begin{aligned}
V^T A_h U &= \sum_k \left(\beta \cdot \nabla u, v\right)_{L^2(D^k)} + \left\langle f^*(u), v\right\rangle_{L^2(\partial D^k)} \\
&\leq \|\beta\|_{L^\infty} \sum_k \|\nabla u\| \|v\| + \|f^*(u)\|_{L^2(\partial D^k)} \|v\|_{L^2(\partial D^k)} \\
&\lesssim \|\beta\|_{L^\infty} \sum_k \left(C_I \|J\boldsymbol{G}\|_{L^\infty(\widehat{D})} + C_N^2 \left\|J^f\right\|_{L^\infty(\widehat{D})}\right) \|1/J\|_{L^\infty(\widehat{D})} \|u\|_{L^2(D^k)} \|v\|_{L^2(D^k)}.
\end{aligned}$$

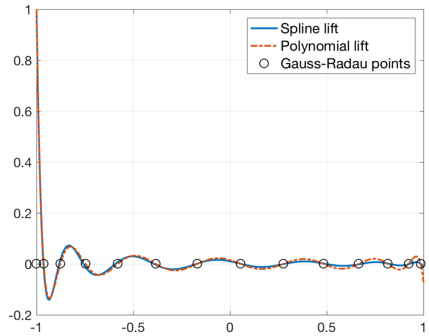
The CFL condition is then determined by mesh-dependent geometric factors of $O(h)$ and order-dependent constants in the inverse and trace inequalities C_I, C_N.

For shape regular meshes, the ratio between volume and surface area is proportional to h, so

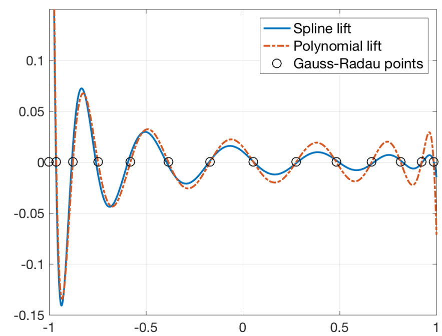
$$\left\|J^f\right\|_{L^\infty(D^k)} \left\|J^{-1}\right\|_{L^\infty(D^k)} = O(h).$$

For polynomial approximation spaces, it can be shown that C_I, C_N both depend quadratically on the degree N. Thus, the spectral radius can be bounded by

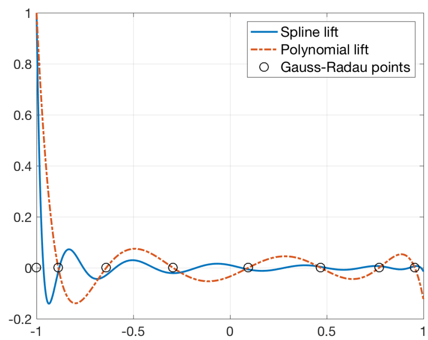
$$\rho\left(\boldsymbol{A}_h\right) = O\left(\frac{h}{\|\beta\|_{L^\infty} N^2}\right).$$



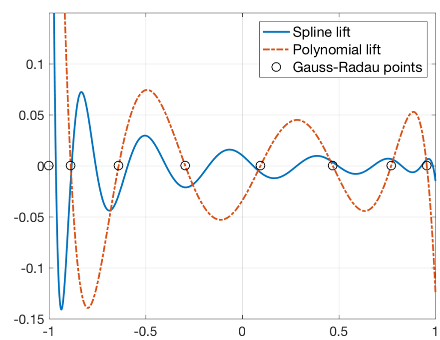
(a) $N_P = N + K_{\text{sub}}$



(b) Zoom in



(c) $N_P = N$



(d) Zoom in

Fig. 1: Comparison of lift functions for a degree N_P polynomial space and spline space with degree $N = 7$ and $K_{\text{sub}} = N$ elements. Degree N_P Gauss-Radau quadrature points are also shown for comparison.

2.1. Explicit constants in trace and inverse inequalities for polynomial and B-spline approximations. For polynomials, $C_I, C_N = O(N^2)$ on the reference element (see, for example [16, 17, 15]).

For B-splines with $K_{\text{sub}} = N$, $C_N = O(N)$ instead of $O(N^2)$.

For sufficiently regular solutions, convergence is in L^2 as $C(h/N)^{N+1}$ while still maintaining an $O(h/N^2)$ CFL. Or, if we measure by $h_N = h/N$, CFL is $O(h_N/N)$. This results from interpolation estimates for one-dimensional B-splines [18]

Show plots of lift functions for polynomials vs for B-spline bases - less oscillation for B-splines.

3. DG formulation. Skew-symmetric form is energy stable for acoustic, electromagnetic, and elastic wave propagation. Disadvantage of IGA-DG vs SEM is a lack of equivalence with strong and weak formulations [19]

Penalty fluxes [20] for dissipation.

Avoid mass matrix inversion using weight-adjusted DG [21, 22]. Especially important for IGA-DG since it decouples the inversion of the mass matrix into a Kronecker structure.

4. Quadrature . Reduce dimension of the search space by treating the weights as dependent variables and assuming symmetry of the rule. Initialize guess for quadrature rule using the previous order's rule to generate a polynomial interpolant from equispaced nodes to quadrature nodes [23, 24].

We note that these rules are generated by choosing the number of points N_q such that $\dim(S_{\text{target}}) \leq 2N_q$, and may not be optimal. The problem is that, to exactly integrate the target space, $N_q = O(N^2)$. Reduced quadratures are thus necessary.

4.1. Reduced quadrature rules. The derivative of a spline does not map into the same spline space; instead, it decreases both the order and continuity. The derivative operator then satisfies $\frac{\partial}{\partial x} : S^N_{N-1} \rightarrow S^{N-1}_{N-2}$. To define a derivative operator which maps from S^N_{N-1} to itself, we use an L^2 projection.

$$\int_{D^k} a(x) \frac{\partial u}{\partial x} v = \int_{\widehat{D}} a(x) \frac{\partial r}{\partial x} J \frac{\partial u}{\partial r} v = \int_{\widehat{D}} a(x) \left(\frac{\partial r}{\partial x} J \right) \Pi_N \left(\frac{\partial u}{\partial r} \right) v.$$

Can cast into collocation style using skew-symmetric formulation by using a degree $N + K_{\rm sub} - 1$ Gauss-Legendre or Gauss-Lobatto rule.

Problems here - high order accuracy is preserved if the mass matrix is under-integrated, but not if an isoparametric mapping is used.

5. Future work. GLL-based spectral elements have other advantages, notably a connection to summation by parts (SBP) operators. Wave propagation problems yield variational formulations which are a-priori stable under inexact quadrature. However, SBP properties are crucial to guaranteeing energy stability for more complex physics, such as nonlinear hyperbolic conservation laws.

REFERENCES

[1] Thomas JR Hughes, John A Cottrell, and Yuri Bazilevs. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. *Computer methods in applied mechanics and engineering*, 194(39):4135–4195, 2005.

[2] Daniel Garcia, David Pardo, Lisandro Dalcin, Maciej Paszyński, Nathan Collier, and Victor M Calo. The value of continuity: Refined isogeometric analysis and fast direct solvers. *Computer Methods in Applied Mechanics and Engineering*, 2016.

[3] C. Michoski, J. Chan, L. Engvall, and J.A. Evans. Foundations of the blended isogeometric discontinuous Galerkin (BIDG) method. *Computer Methods in Applied Mechanics and Engineering*, 305:658 – 681, 2016.

[4] Ulrich Langer, Angelos Mantzaflaris, Stephen E Moore, and Ioannis Toulopoulos. Multipatch discontinuous Galerkin isogeometric analysis. In *Isogeometric Analysis and Applications 2014*, pages 1–32. Springer, 2015.

[5] Ulrich Langer and Stephen E Moore. Discontinuous Galerkin isogeometric analysis of elliptic PDEs on surfaces. In *Domain Decomposition Methods in Science and Engineering XXII*, pages 319–326. Springer, 2016.

[6] Jesse Chan and T Warburton. GPU-accelerated Bernstein-Bezier discontinuous Galerkin methods for wave problems. *arXiv preprint arXiv:1512.06025*, 2015.

[7] Timothy Warburton and Thomas Hagstrom. Taming the CFL number for discontinuous Galerkin methods on structured meshes. *SIAM Journal on Numerical Analysis*, 46(6):3151–3180, 2008.

[8] Zhiliang Xu and Yingjie Liu. A conservation constrained Runge-Kutta discontinuous Galerkin method with the improved CFL condition for conservation laws. 2010.

[9] Zhiliang Xu, Xu-Yan Chen, and Yingjie Liu. A new Runge-Kutta discontinuous Galerkin method with conservation constraint to improve CFL condition for solving conservation laws. *Journal of computational physics*, 278:348–377, 2014.

[10] Matthew A Reyna and Fengyan Li. Operator bounds and time step conditions for the DG and central DG methods. *Journal of Scientific Computing*, 62(2):532–554.

[11] John Goodrich, Thomas Hagstrom, and Jens Lorenz. Hermite methods for hyperbolic initial-boundary value problems. *Mathematics of computation*, 75(254):595–630, 2006.

[12] Arturo Vargas, Jesse Chan, Thomas Hagstrom, and Tim Warburton. Variations on hermite methods for wave propagation. *arXiv preprint arXiv:1509.08012*, 2015.

[13] Jeffrey W Banks and Thomas Hagstrom. On galerkin difference methods. *Journal of Computational Physics*, 313:310–327, 2016.

[14] Marvin Marcus and Henryk Minc. *A survey of matrix theory and matrix inequalities*, volume 14. Courier Corporation, 1992.

[15] Jesse Chan, Zheng Wang, Axel Modave, Jean-Francois Remacle, and T Warburton. GPU-accelerated discontinuous Galerkin methods on hybrid meshes. *Journal of Computational Physics*, 318:142–168, 2016.

[16] T Warburton and Jan S Hesthaven. On the constants in *hp*-finite element trace inverse inequalities. *Computer methods in applied mechanics and engineering*, 192(25):2765–2773, 2003.

[17] Sevtap Ozisik, Beatrice Riviere, and Tim Warburton. On the constants in inverse inequalities in l2. Technical Report CAAM TR10-19, Rice University, 2010.

[18] Yuri Bazilevs, L Beirao da Veiga, J Austin Cottrell, Thomas JR Hughes, and Giancarlo Sangalli. Isogeometric analysis: approximation, stability and error estimates for *h*-refined meshes. *Mathematical Models and Methods in Applied Sciences*, 16(07):1031–1090, 2006.

[19] David A Kopriva and Gregor Gassner. On the quadrature and weak form choices in collocation type discontinuous galerkin spectral element methods. *Journal of Scientific Computing*, 44(2):136–155, 2010.

[20] Jesse Chan and T Warburton. A short note on the penalty flux parameter for first order discontinuous galerkin formulations. *arXiv preprint arXiv:1611.00102*, 2016.

[21] Jesse Chan, Russell J Hewett, and T Warburton. Weight-adjusted discontinuous Galerkin methods: wave propagation in heterogeneous media. *arXiv preprint arXiv:1608.01944*, 2016.

[22] Jesse Chan, Russell J Hewett, and T Warburton. Weight-adjusted discontinuous Galerkin methods: curvilinear meshes. *arXiv preprint arXiv:1608.03836*, 2016.

[23] T Warburton. An explicit construction of interpolation nodes on the simplex. *Journal of engineering mathematics*, 56(3):247–262, 2006.

[24] Jesse Chan and T Warburton. A comparison of high order interpolation nodes for the pyramid. *SIAM Journal on Scientific Computing*, 37(5):A2151–A2170, 2015.