On the penalty parameter in first order DG formulations

Jesse Chan

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1 Introduction

2 DG numerical fluxes

We consider a first order system of hyperbolic equations

$$\boldsymbol{A}_0 \frac{\partial \boldsymbol{U}}{\partial t} + \sum_{i=1}^{d} \frac{\partial \boldsymbol{F}_i(\boldsymbol{U})}{\partial \boldsymbol{x}_i} = 0,$$

which may alternatively be written as

$$\boldsymbol{A}_0 \frac{\partial \boldsymbol{U}}{\partial t} + \sum_{i=1}^{d} \frac{\partial \left(\boldsymbol{A}_i U\right)}{\partial \boldsymbol{x}_i}, \qquad \boldsymbol{A}_i = \frac{\partial \boldsymbol{F}_i(\boldsymbol{U})}{\partial \boldsymbol{U}}$$

where $\boldsymbol{A}_i$ are symmetric matrices. The semi-discrete DG formulation for such systems may be written as

$$\sum_{D^k \in \Omega_h} \left(\boldsymbol{A}_0 \frac{\partial \boldsymbol{U}}{\partial t}, \boldsymbol{V}\right)_{L^2(D^k)} = \sum_{D^k \in \Omega_h} \left(\left(\boldsymbol{F}_i(\boldsymbol{U}), \boldsymbol{V}_{,i}\right)_{L^2(D^k)} - \langle \boldsymbol{F}_i^*(\boldsymbol{U}) \boldsymbol{n}_i, \boldsymbol{V} \rangle_{\partial D^k} \right)$$

where $\boldsymbol{n}$ is the outward normal on a face f of D^k, and $\boldsymbol{F}^*$ is a numerical flux depending defined on shared faces between two elements.

For convergence, $\boldsymbol{F}^* = \boldsymbol{F}^*(\boldsymbol{U})$ must be consistent such that, for exact solutions $\boldsymbol{U}$,

$$\boldsymbol{F}^* = \boldsymbol{F}(\boldsymbol{U}).$$

Let f be a shared face between two elements $D^{k,+}$ and $D^{k,-}$, and let $\boldsymbol{F}^+, \boldsymbol{F}^-$ be evaluations of $\boldsymbol{F}(\boldsymbol{U})$ restricted to $D^{k,+}$ and $D^{k,-}$, respectively. Typical DG fluxes are defined as the sum of a consistent averaging of $\boldsymbol{F}^+$ and $\boldsymbol{F}^-$ and a penalization term

$$\boldsymbol{F}^* = \{\{\boldsymbol{F}_n(\boldsymbol{U})\}\} - \boldsymbol{W} [\![\boldsymbol{U}]\!], \tag{1}$$

where $\boldsymbol{W}$ is some positive-definite matrix, which is required for energy stability.

The upwind numerical flux is a well-known flux of the form (1). For some normal vector $\boldsymbol{n}$, let $[\![\boldsymbol{A}_n]\!] = \sum_{i=1}^{d} [\![\boldsymbol{A}_i \boldsymbol{n}_i]\!]$. By add citation, $[\![\boldsymbol{A}_n]\!]$ contains real eigenvalues, and admits an eigenvalue decomposition

$$[\![\boldsymbol{A}_n]\!] = \boldsymbol{V} \boldsymbol{\Lambda} \boldsymbol{V}^{-1}, \qquad \boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{pmatrix}.$$

For problems with continuous coefficients, the upwind numerical flux can be defined as

$$\boldsymbol{F}^* = \boldsymbol{A}^+ \boldsymbol{U}^- + \boldsymbol{A}^- \boldsymbol{U}^+,$$

where the matrices $\boldsymbol{A}^+, \boldsymbol{A}^-$ are constructed from the positive and negative eigenvalues

$$\begin{aligned} \boldsymbol{A}^+ &= \frac{1}{2} \boldsymbol{V} \left(\boldsymbol{\Lambda} + |\boldsymbol{\Lambda}| \right) \boldsymbol{V}^{-1} \\ \boldsymbol{A}^- &= \frac{1}{2} \boldsymbol{V} \left(\boldsymbol{\Lambda} - |\boldsymbol{\Lambda}| \right) \boldsymbol{V}^{-1}, \end{aligned}$$

and $|\Lambda|$ is the diagonal matrix whose entries consist of the absolute values of the eigenvalues $|\lambda_i|$. This can be rewritten as

$$\boldsymbol{F}^* = \{\{\boldsymbol{A}\boldsymbol{U}\}\} - \boldsymbol{V} \left| \boldsymbol{\Lambda} \right| \boldsymbol{V}^{-1} [\![\boldsymbol{U}]\!].$$

An alternative to upwind fluxes are penalty fluxes, which penalize appropriately defined jumps of the solution. One such penalty flux is given as

$$\boldsymbol{F}^* = \{\{\boldsymbol{A}\boldsymbol{U}\}\} - \boldsymbol{A}^T \boldsymbol{W} [\![\boldsymbol{A}\boldsymbol{U}]\!].$$

where $\boldsymbol{W}$ is some positive-definite weighting matrix. In contrast to Lax-Friedrichs fluxes, the penalty flux enforce a weaker continuity.

2.1 Example: acoustic wave equation

Consider the isotropic acoustic wave equation in pressure-velocity form

$$\begin{aligned} \frac{\partial p}{\partial t} &= \nabla \cdot \boldsymbol{u} \\ \frac{\partial \boldsymbol{u}}{\partial t} &= \nabla p. \end{aligned}$$

Let $\boldsymbol{U}$ denote the group variable $\boldsymbol{U} = (p, u, v)$, where u and v are the x and y components of velocity. Then, in two dimensions, the isotropic wave equation is given as

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{A}_x \boldsymbol{U}}{\partial x} + \frac{\partial \boldsymbol{A}_y \boldsymbol{U}}{\partial y} = 0, \qquad \boldsymbol{A}_x = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{A}_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The normal flux matrix $\boldsymbol{A}_n$ is then

$$\boldsymbol{A}_n = \begin{pmatrix} 0 & \boldsymbol{n}_x & \boldsymbol{n}_y \\ \boldsymbol{n}_x & 0 & 0 \\ \boldsymbol{n}_y & 0 & 0 \end{pmatrix}$$

implying that the penalty fluxes (with $\boldsymbol{W} = \tau \boldsymbol{I}$) are

$$\{\{\boldsymbol{A}_n \boldsymbol{U}\}\} - \boldsymbol{A}_n^T [\![\boldsymbol{A}_n \boldsymbol{U}]\!] = \begin{pmatrix} \{\{\boldsymbol{u}_n\}\} \\ \{\{p \boldsymbol{n}_x\}\} \\ \{\{p \boldsymbol{n}_y\}\} \end{pmatrix} - \tau \begin{pmatrix} [\![p]\!] \\ [\![\boldsymbol{u}_{\boldsymbol{n}}]\!] \boldsymbol{n}_x \\ [\![\boldsymbol{u}_{\boldsymbol{n}}]\!] \boldsymbol{n}_y \end{pmatrix}$$

For $\tau = 1$, these fluxes coincide with the upwind fluxes.

3 Dependence of spectra on the penalty parameter

3.1 Eigenmodes

Reproduce Gershgorin proof here.

3.2 Eigenvalues

Let $\boldsymbol{A} = \boldsymbol{B} + \boldsymbol{C}$ be the DG discretization matrix, which is assumed to be the sum of a skew-symmetric matrix $\boldsymbol{B}$ and symmetric, negative-definite matrix $\boldsymbol{C}$.

Let $\lambda, \boldsymbol{u}$ be an eigenpair of $\boldsymbol{A}$. Let $\lambda = \alpha + i\beta$ and $\boldsymbol{u} = \boldsymbol{v} + i\boldsymbol{w}$, where α, β and $\boldsymbol{v}, \boldsymbol{w}$ are the real and imaginary parts of λ and $\boldsymbol{u}$, respectively. Then, expanding and grouping terms in

$$\boldsymbol{A}(\boldsymbol{v} + i\boldsymbol{w}) = (\alpha + i\beta)(\boldsymbol{v} + i\boldsymbol{w})$$

we have that

$$\begin{aligned}\boldsymbol{A}\boldsymbol{v} &= \alpha\boldsymbol{v} - \beta\boldsymbol{w} \\ \boldsymbol{A}\boldsymbol{w} &= \beta\boldsymbol{v} + \alpha\boldsymbol{w}.\end{aligned}$$

Assuming that $\boldsymbol{u}^*\boldsymbol{u} = \|\boldsymbol{v}\|^2 + \|\boldsymbol{w}\|^2 = 1$, multiplying both sides by $\boldsymbol{v}^T, \boldsymbol{w}^T$ and straightforward manipulations using the skew-symmetry of $\boldsymbol{B}$ and symmetry of $\boldsymbol{C}$ yields

$$\begin{aligned}\tau\left(\boldsymbol{v}^T\boldsymbol{C}\boldsymbol{v} + \boldsymbol{w}^T\boldsymbol{C}\boldsymbol{w}\right) &= \alpha \\ \boldsymbol{v}^T\boldsymbol{B}\boldsymbol{w} - \boldsymbol{w}^T\boldsymbol{B}\boldsymbol{w} &= \beta.\end{aligned}$$

Since $\boldsymbol{C}$ is symmetric, negative-definite, and independent of τ, the quantity

$$\left|\boldsymbol{v}^T\boldsymbol{C}\boldsymbol{v} + \boldsymbol{w}^T\boldsymbol{C}\boldsymbol{w}\right| \leq \|\boldsymbol{C}\|.$$

Since this is bounded independently of τ, we conclude that $|\alpha| = O(\tau)$.

Use Gershgorin result to show that α diverges for eigenmodes the non-conforming subspace.

4 Numerical experiments