# Entropy stable Gauss collocation discontinuous Galerkin methods

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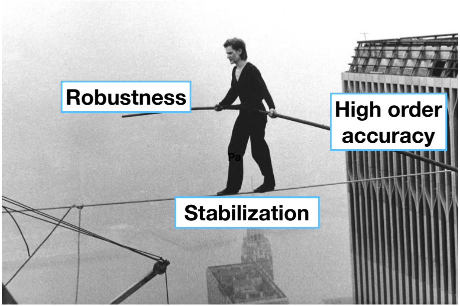
# Discretely entropy stable schemes

Goal: address the stability and robustness of high order methods for nonlinear conservation laws.




- Aim for stability independently of artificial viscosity, limiters, discretization errors.
- *Mechanical* approach to stability based on algebraic properties of a discretization.
- Semi-discrete stability property for high order DG methods.

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Finite volume methods: Tadmor, Chandrashekar, Ray, Svard, Fjordholm, Mishra, LeFloch, Rohde, . . .
High order tensor product elements: Fisher, Carpenter, Gassner, Winters, Kopriva, Hindenlang, Persson, Pazner, . . .
High order general elements: Chen and Shu, Crean, Hicken, DCDR Fernandez, Zingg, . . .

# Entropy stability for nonlinear problems

- Generalizes energy stability to nonlinear systems of conservation laws (Burgers', shallow water, compressible Euler, MHD).

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial x} = 0.$$

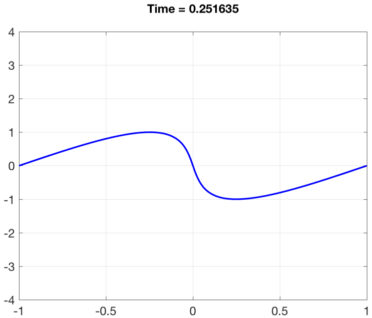
- Continuous entropy inequality: given a convex entropy function $S(\boldsymbol{u})$ and "entropy potential" $\psi(\boldsymbol{u})$,

$$\int_\Omega \boldsymbol{v}^T \left(\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{f}(\boldsymbol{u})}{\partial x}\right) = 0, \qquad \boxed{\boldsymbol{v} = \frac{\partial S}{\partial \boldsymbol{u}}}$$

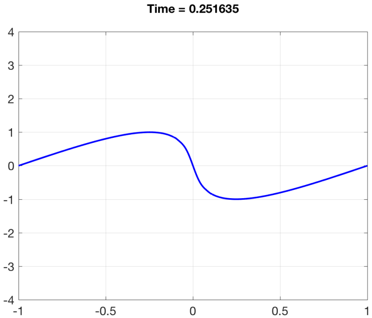
$$\Longrightarrow \int_\Omega \frac{\partial S(\boldsymbol{u})}{\partial t} + \left(\boldsymbol{v}^T \boldsymbol{f}(\boldsymbol{u}) - \psi(\boldsymbol{u})\right)\Big|_{-1}^{1} \leq 0.$$

- Proof of entropy inequality relies on chain rule, integration by parts.

# High order methods typically unstable for nonlinear PDEs




(a) Exact solution




(b) 8th order DG

- High order methods tend to blow up for under-resolved solutions (shocks, turbulence), sensitive to discretization.
- Instability tied to loss of the chain rule.

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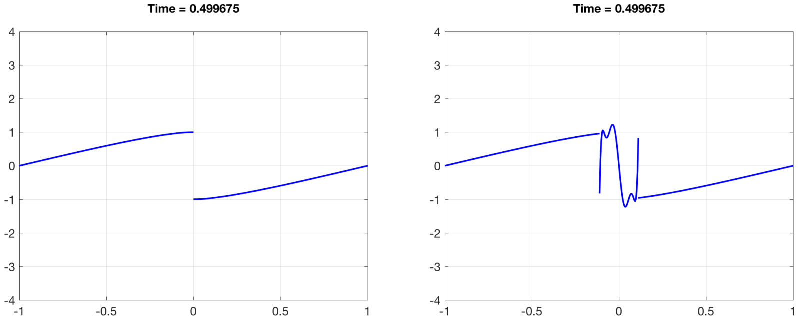


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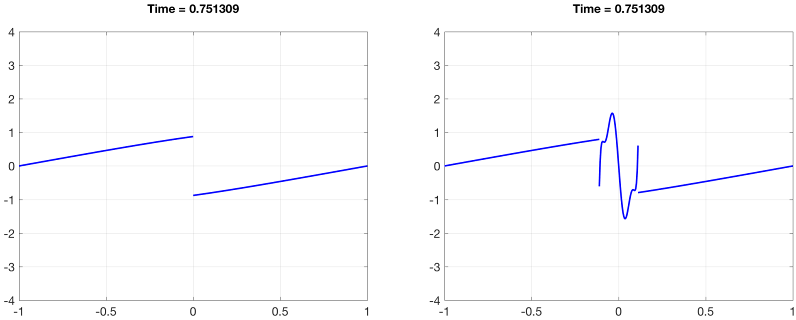


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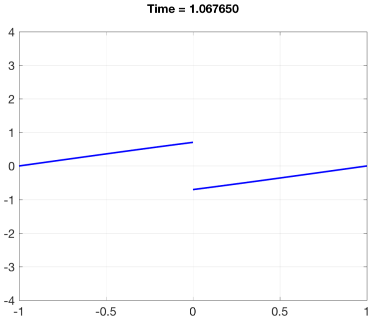


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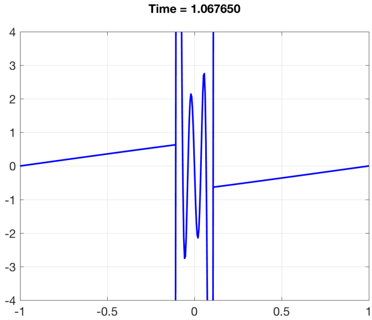


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1. Summation-by-parts and high order DG
2. "Decoupled" block SBP operators
3. Numerical experiments
4. A mortar approach to non-conforming interfaces

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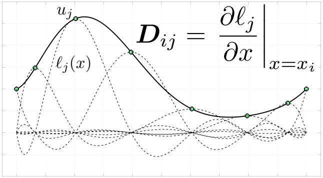
# Nodal DG, summation-by-parts (SBP), flux differencing




$$\boldsymbol{B} = \begin{bmatrix} -1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

- Mimic integration by parts algebraically

$$\boldsymbol{Q} = \boldsymbol{B} - \boldsymbol{Q}^T, \qquad \boldsymbol{Q} = \boldsymbol{M}\boldsymbol{D}, \qquad \boldsymbol{M} \text{ diagonal mass matrix.}$$

- Typical collocation scheme:

$$\boldsymbol{M}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + \boldsymbol{Q}\boldsymbol{f}(\boldsymbol{u}) = 0 \quad \Longrightarrow \quad \boldsymbol{M}_{ii}\frac{\mathrm{d}\boldsymbol{u}_i}{\mathrm{d}t} + \sum_j \boldsymbol{Q}_{ij}\boldsymbol{f}(\boldsymbol{u}_j) = 0.$$

- Flux differencing: recover standard form if $\boldsymbol{f}_S(\boldsymbol{u}_i, \boldsymbol{u}_j) = \frac{1}{2}(\boldsymbol{u}_i + \boldsymbol{u}_j)$.

$$\boldsymbol{M}_{ii}\frac{\mathrm{d}\boldsymbol{u}_i}{\mathrm{d}t} + \sum_j \boldsymbol{Q}_{ij} 2\boldsymbol{f}_S(\boldsymbol{u}_i, \boldsymbol{u}_j) = 0 \quad \Longrightarrow \quad \boxed{\boldsymbol{M}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + 2(\boldsymbol{Q} \circ \boldsymbol{F}_S)\boldsymbol{1} = 0}.$$

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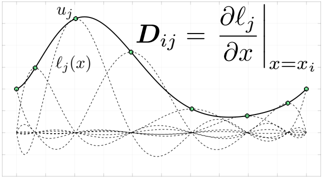


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$$Q = B - Q^T, \qquad Q = MD, \qquad M \text{ diagonal mass matrix.}$$

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$$M\frac{\mathrm{d}u}{\mathrm{d}t} + Qf(u) = 0 \quad \Longrightarrow \quad M_{ii}\frac{\mathrm{d}u_i}{\mathrm{d}t} + \sum_j Q_{ij} f(u_j) = 0.$$

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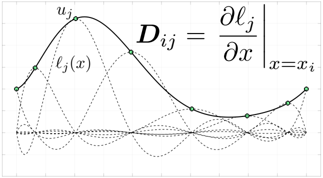


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# Entropy stable schemes: a brief derivation

- Skew-symmetric formulation on each element:

$$\boldsymbol{M}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{dt}} + 2\left(\boldsymbol{Q} \circ \boldsymbol{F}_S\right)\mathbf{1} = 0$$

- Trick: use Tadmor's entropy conservative numerical flux for $\boldsymbol{f}_S, \boldsymbol{f}^*$

$$\begin{aligned} \boldsymbol{f}_S(\boldsymbol{u},\boldsymbol{u}) &= \boldsymbol{f}(\boldsymbol{u}), && \text{(consistency)}\\ \boldsymbol{f}_S(\boldsymbol{u},\boldsymbol{v}) &= \boldsymbol{f}_S(\boldsymbol{v},\boldsymbol{u}), && \text{(symmetry)}\\ \left(\boldsymbol{v}_L - \boldsymbol{v}_R\right)^T \boldsymbol{f}_S\left(\boldsymbol{u}_L,\boldsymbol{u}_R\right) &= \psi_L - \psi_R, && \text{(conservation).} \end{aligned}$$

- Proof of entropy conservation: multiply by $\boldsymbol{v}^T$

$$\boldsymbol{v}^T\boldsymbol{M}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{dt}} + \boldsymbol{v}^T\left(\left(\boldsymbol{Q} - \boldsymbol{Q}^T\right) \circ \boldsymbol{F}_S\right)\mathbf{1} + \boldsymbol{v}^T\boldsymbol{B}\boldsymbol{f}^* = 0.$$

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$$\boldsymbol{M}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{dt}} + \underbrace{\left(\left(\boldsymbol{Q} - \boldsymbol{Q}^T\right) \circ \boldsymbol{F}_S\right)\boldsymbol{1}}_{\text{SBP property}} + \left(\boldsymbol{B} \circ \boldsymbol{F}_S\right)\boldsymbol{1} = 0$$

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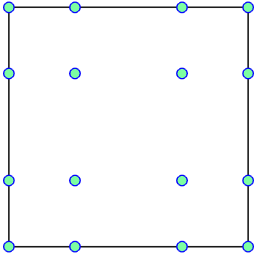

(a) Lobatto nodes

- Existing high order entropy stable schemes based on Lobatto nodes.
- Gauss nodes: advantages in increasing quadrature accuracy?
- Challenge in extending to Gauss nodes: inter-element coupling terms.

Fisher, Carpenter, et al. (2013, 2014), Gassner, Winters, Hindenlang, Kopriva (2016, 2018), . . .

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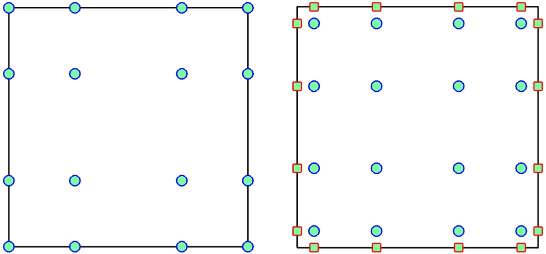

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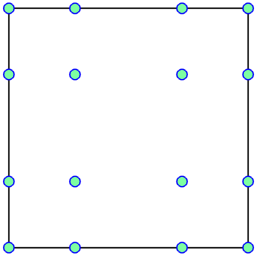

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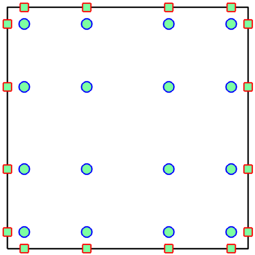

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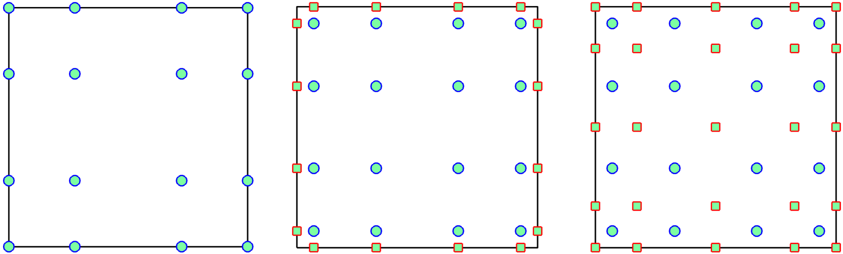

(a) Lobatto nodes (b) Gauss nodes (c) Staggered grid

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# Generalized SBP operators

- Differentiation matrix $\boldsymbol{D}$: evaluates derivatives at Gauss points.
- Define boundary interpolation matrix $\boldsymbol{E}$

$$\boldsymbol{E} = \begin{bmatrix} \ell_1(-1), & \dots & \ell_{N+1}(-1) \\ \ell_1(1), & \dots & \ell_{N+1}(1) \end{bmatrix}$$

- Generalized SBP property: let $\boldsymbol{Q} = \boldsymbol{M}\boldsymbol{D}$,

$$\boldsymbol{Q} + \boldsymbol{Q}^T = \boldsymbol{E}^T\boldsymbol{B}\boldsymbol{E}, \qquad \boldsymbol{B} = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}$$

$$\Longrightarrow \int_{-1}^{1} \frac{\partial I_N u}{\partial x} I_N v + \int_{-1}^{1} I_N u \frac{\partial I_N v}{\partial x} = \left(I_N u\right)\left(I_N v\right)\Big|_{-1}^{1}.$$

where $I_N$ is the degree $N$ polynomial interpolation operator

Fernández, Hicken, Zingg (2014), Fernández, Boom, Zingg (2014)

# Entropy stability on multiple elements

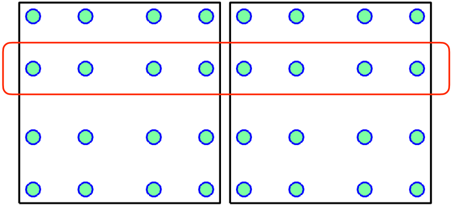

Figure: Coupling between Gauss nodes on neighboring elements.

- Re-deriving the entropy stable formulation with GSBP operators:

$$\boldsymbol{M}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{dt}} + 2\left(\boldsymbol{Q} \circ \boldsymbol{F}_S\right)\mathbf{1} = 0.$$

- The presence of the interpolation matrix $\boldsymbol{E}$ increases inter-element coupling, complicates BC imposition.

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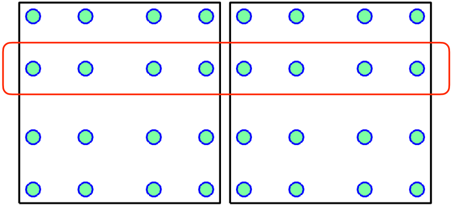

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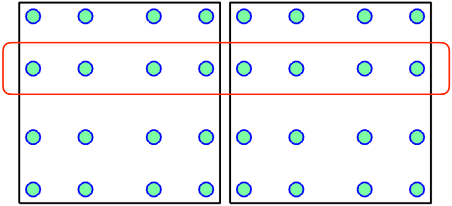

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- The presence of the interpolation matrix $\boldsymbol{E}$ increases inter-element coupling, complicates BC imposition.

# Talk outline

# A "decoupled" SBP operator

- Goal: SBP property without the boundary interpolation operator $\boldsymbol{E}$

$$\boldsymbol{Q}_N = \begin{bmatrix} \boldsymbol{Q} - \frac{1}{2}\boldsymbol{E}^T\boldsymbol{B}\boldsymbol{E} & \frac{1}{2}\boldsymbol{E}^T\boldsymbol{B} \\ -\frac{1}{2}\boldsymbol{B}\boldsymbol{E} & \frac{1}{2}\boldsymbol{B} \end{bmatrix},$$

- If $\boldsymbol{Q} + \boldsymbol{Q}^T = \boldsymbol{E}^T\boldsymbol{B}\boldsymbol{E}$, then $\boldsymbol{Q}_N$ satisfies the SBP property

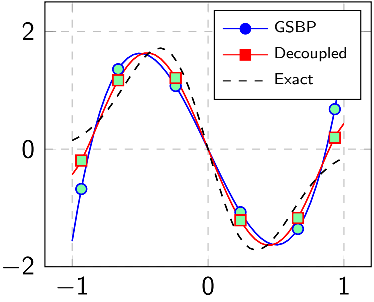
$$\boxed{\boldsymbol{Q}_N + \boldsymbol{Q}_N{}^T = \begin{bmatrix} \boldsymbol{0} & \\ & \boldsymbol{B} \end{bmatrix}} \sim \boxed{\int_{-1}^{1} \frac{\partial I_N u}{\partial x} v + u \frac{\partial I_N v}{\partial x} = uv\Big|_{-1}^{1}.}$$

- $\boldsymbol{Q}_N$ approximates $f\frac{\partial g}{\partial x}$ by $\boldsymbol{u}$ using data at $\boldsymbol{x} = [\boldsymbol{x}_{\rm vol}, \boldsymbol{x}_{\rm face}]$

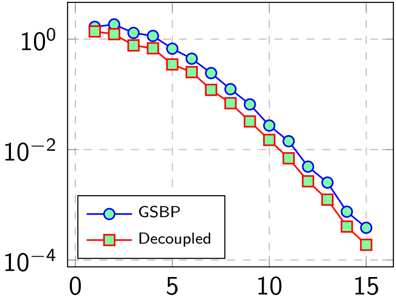
$$\boldsymbol{M}\boldsymbol{u} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{E} \end{bmatrix}^T \operatorname{diag}(\boldsymbol{f})\, \boldsymbol{Q}_N \boldsymbol{g}, \qquad \boldsymbol{f}_i, \boldsymbol{g}_i = f(\boldsymbol{x}_i), g(\boldsymbol{x}_i).$$

- Reduces to traditional SBP operator under appropriate quadrature.

# Decoupled SBP operators add boundary corrections




(a) Derivative approximations




(b) $L^2$ error w.r.t. degree $N$

- Equivalent to a variational problem for a polynomial $u(\boldsymbol{x}) \approx f\frac{\partial g}{\partial x}$.

$$\int_{-1}^{1} u(\boldsymbol{x})v(\boldsymbol{x}) = \int_{-1}^{1} f\frac{\partial I_N g}{\partial x}v + (g - I_N g)\frac{(fv + I_N(fv))}{2}\bigg|_{-1}^{1}.$$

# Entropy stable schemes using decoupled SBP operators

- Replace SBP operator with decoupled SBP operator

$$\boldsymbol{M}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} + \left(\left(\boldsymbol{Q} - \boldsymbol{Q}^T\right) \circ \boldsymbol{F}_S\right)\boldsymbol{1} + \boldsymbol{B}\boldsymbol{f}^* = 0, \qquad \text{(standard SBP)}$$

- $\boldsymbol{F}_S$ is the matrix of flux evaluations between solution values at *both* volume and face nodes

$$\left(\boldsymbol{F}_S\right)_{ij} = \boldsymbol{f}_S\left(\widetilde{\boldsymbol{u}}_i, \widetilde{\boldsymbol{u}}_j\right), \qquad \widetilde{\boldsymbol{u}} = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{u}_f \end{bmatrix}$$

- Face values of $\boldsymbol{u}_f$: interpolate entropy variables (entropy projection). Resulting scheme is entropy stable.

---

Chan (2018). *On discretely entropy conservative and entropy stable discontinuous Galerkin methods.*

Parsani et al. (2016), *Entropy Stable Staggered Grid Discontinuous Spectral Collocation Methods*

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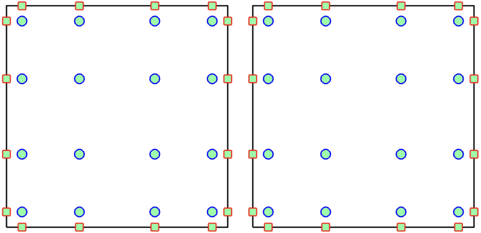
# Higher dimensions and curved meshes



$$\left( \underbrace{\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ -\boldsymbol{B}^T & \boldsymbol{C} \end{bmatrix}}_{\boldsymbol{Q}_N^i - \left(\boldsymbol{Q}_N^i\right)^T} \circ \underbrace{\begin{bmatrix} \boldsymbol{F}_S^{vv} & \boldsymbol{F}_S^{vf} \\ \boldsymbol{F}_S^{fv} & \boldsymbol{F}_S^{ff} \end{bmatrix}}_{\boldsymbol{F}_S} \right) \mathbf{1}$$

- Multi-dimensional operators defined using Kronecker products

$$\boldsymbol{Q}^1 = \boldsymbol{Q}_{\mathrm{1D}} \otimes \boldsymbol{M}_{\mathrm{1D}}, \qquad \boldsymbol{E} = \begin{bmatrix} \boldsymbol{E}_{\mathrm{1D}} \otimes \boldsymbol{I}_2 \\ \boldsymbol{I}_2 \otimes \boldsymbol{E}_{\mathrm{1D}} \end{bmatrix}, \quad \boldsymbol{B}^1 = \begin{bmatrix} \boldsymbol{B}_{\mathrm{1D}} \otimes \boldsymbol{I}_2 & \\ & \mathbf{0} \end{bmatrix}.$$

- Curved meshes: discretize geometric terms in split form, ensure GCL.
- Compute geometric terms at Lobatto points, interpolate to Gauss.

Thomas and Lombard (1979), Kopriva (2006).

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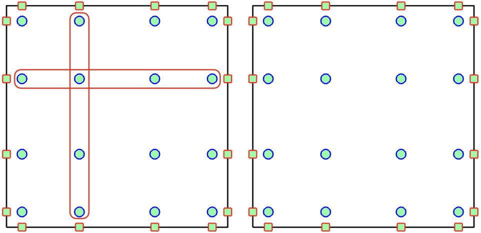

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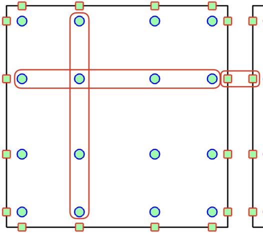

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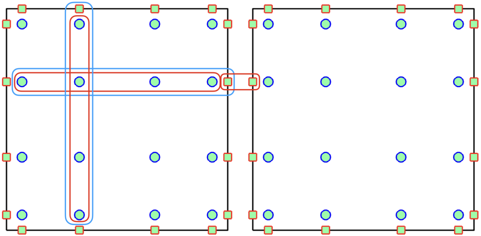

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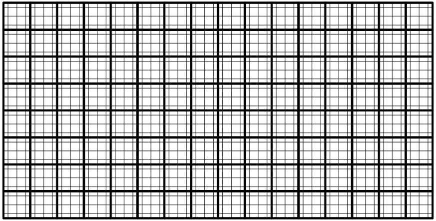
# Talk outline

# Accurate quadrature can reduce error on curved meshes



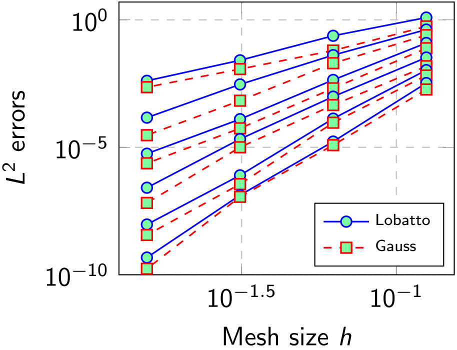



Figure: $L^2$ errors for the 2D isentropic vortex at time $T = 5$ for degree $N = 2, \ldots, 7$ Lobatto and Gauss collocation schemes (similar behavior in 3D).

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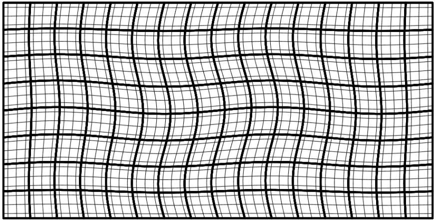

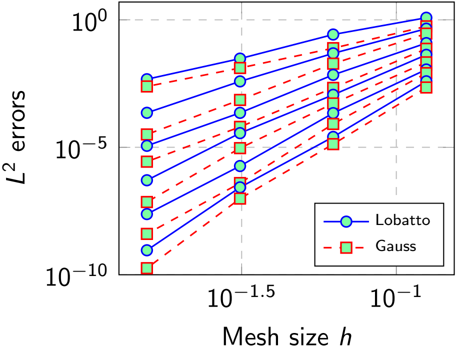


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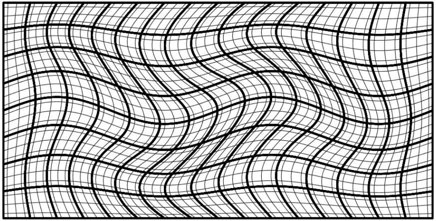

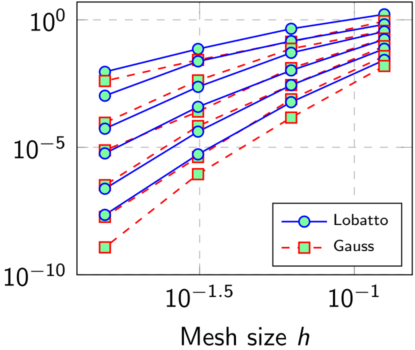


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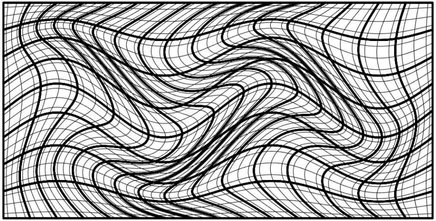

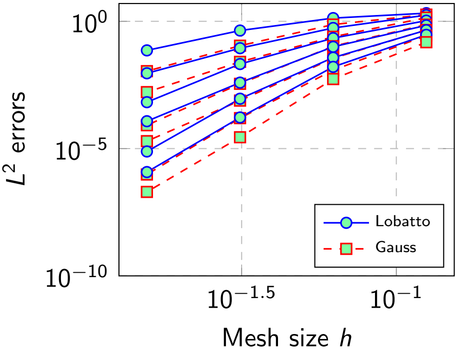


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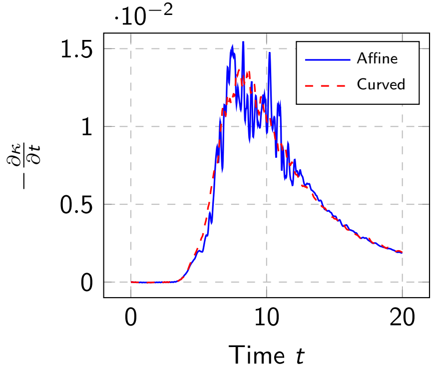
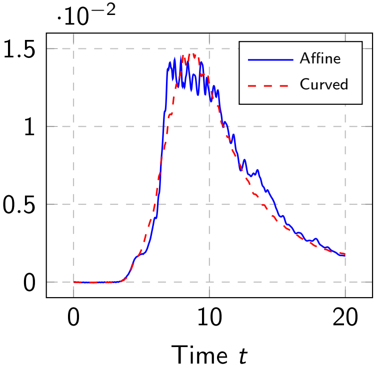
# 3D inviscid Taylor-Green vortex



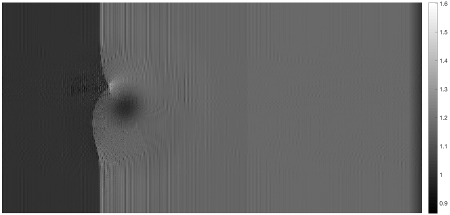

(a) GLL collocation



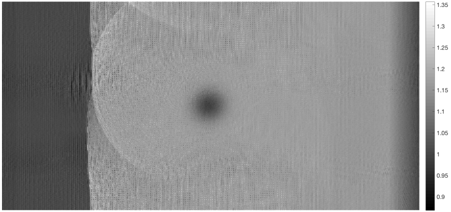

(b) Gauss collocation

Figure: Kinetic energy dissipation rate for entropy stable GLL and Gauss collocation schemes with $N = 7$ and $h = \pi/8$.

# Shock vortex interaction




(a) Entropy conservative flux, $T = .3$




(b) Entropy conservative flux, $T = .7$

Figure: Shock vortex interaction problem using high order entropy stable Gauss collocation schemes with $N = 4, h = 1/100$.

Jiang, Shu (1998). *Efficient Implementation of Weighted ENO Schemes.*

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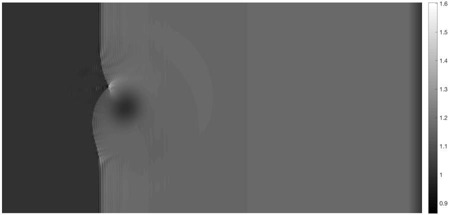
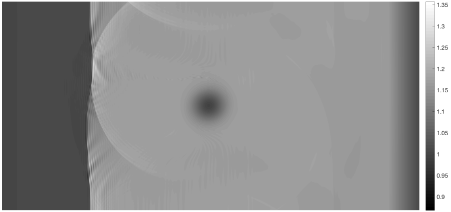



(a) Lax-Friedrichs flux, $T = .3$

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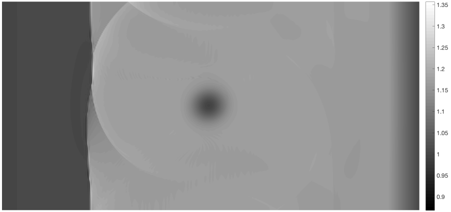
# Shock vortex interaction


(a) Matrix dissipation flux, $T = .3$

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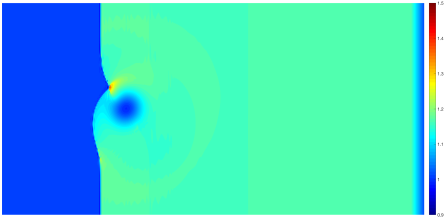

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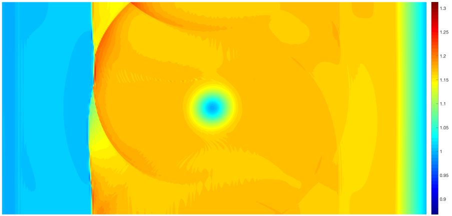

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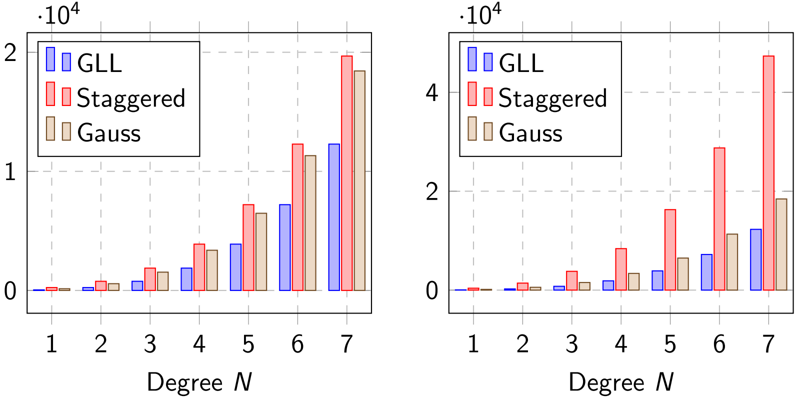
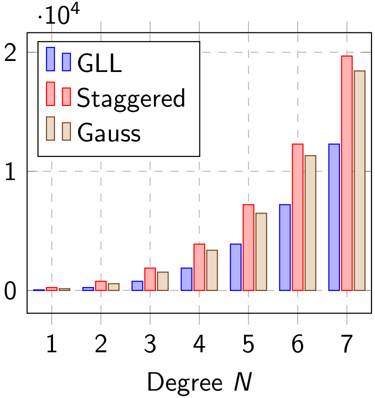
# Cost estimates: flux evaluations and matrix operations




(a) Two-point flux evaluations (b) Est. matrix op. FLOPS

*Note: does not significantly reduce total flux evaluations.*

# Cost estimates: flux evaluations and matrix operations

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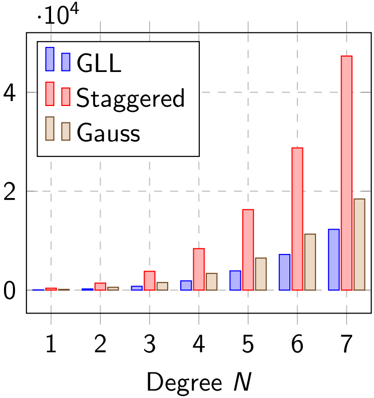


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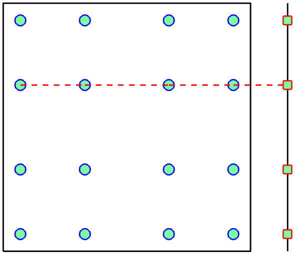
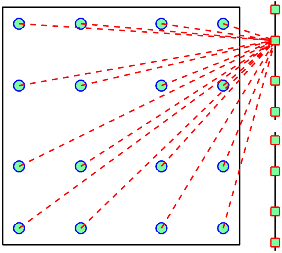
# Talk outline

# Non-conforming interfaces (with DCDR Fernandez)



(a) Conforming surface quadrature nodes



(b) Non-conforming surface nodes

- Volume/surface nodes interact through $\boldsymbol{f}_S(\boldsymbol{u}_i, \boldsymbol{u}_j)$ and interpolation.
- Fix: weakly couple conforming+non-conforming faces using a mortar.

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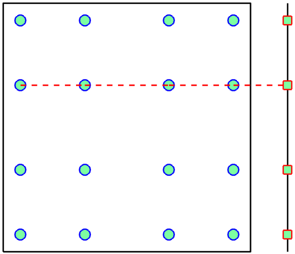

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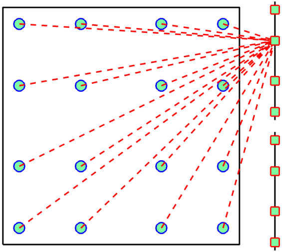

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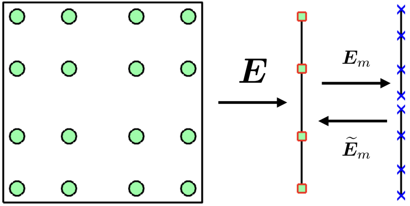
# A decoupled SBP approach to mortars

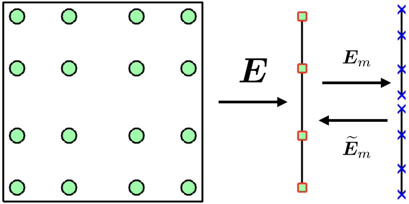



$$\underbrace{\begin{bmatrix} \boldsymbol{Q}_i - \boldsymbol{Q}_i^T & \boldsymbol{E}^T\boldsymbol{B}_i \\ -\boldsymbol{B}_i\boldsymbol{E} & \end{bmatrix}}_{\text{skew operator } \boldsymbol{Q}_N^i - \left(\boldsymbol{Q}_N^i\right)^T}$$

- Define appropriate interpolation operators $\boldsymbol{E}_m, \widetilde{\boldsymbol{E}}_m$ between conforming and non-conforming (mortar) nodes.
- Modify the skew-symmetric formulation as follows:

$$\boldsymbol{M}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{dt}} + \sum_{i=1}^{d} \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{E} \end{bmatrix}^T \begin{bmatrix} \boldsymbol{Q}_i - \boldsymbol{Q}_i^T & \boldsymbol{E}^T\boldsymbol{B}_i \\ -\boldsymbol{B}_i\boldsymbol{E} & \end{bmatrix} + \boldsymbol{E}^T\boldsymbol{B}_i\boldsymbol{f}_i^* = 0.$$

# A decoupled SBP approach to mortars

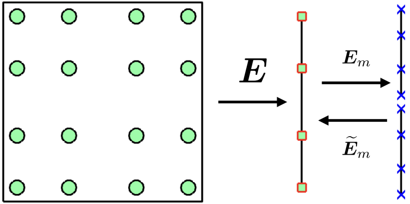



$$\underbrace{\begin{bmatrix} \boldsymbol{Q}_i - \boldsymbol{Q}_i^T & \boldsymbol{E}^T\boldsymbol{B}_i & \\ -\boldsymbol{B}_i\boldsymbol{E} & & \boldsymbol{B}_i\widetilde{\boldsymbol{E}}_m \\ & -\widetilde{\boldsymbol{B}}_i\boldsymbol{E}_m & \end{bmatrix}}_{\text{modified skew operator } \boldsymbol{Q}_N^i - \left(\boldsymbol{Q}_N^i\right)^T}$$

- Define appropriate interpolation operators $\boldsymbol{E}_m, \widetilde{\boldsymbol{E}}_m$ between conforming and non-conforming (mortar) nodes.
- Modify the skew-symmetric formulation as follows:

$$\boldsymbol{M}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{dt}} + \sum_{i=1}^{d} \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{E} \end{bmatrix}^T \begin{bmatrix} \boldsymbol{Q}_i - \boldsymbol{Q}_i^T & \boldsymbol{E}^T\boldsymbol{B}_i \\ -\boldsymbol{B}_i\boldsymbol{E} & \end{bmatrix} + \boldsymbol{E}^T\boldsymbol{B}_i\boldsymbol{f}_i^* = 0.$$

# A decoupled SBP approach to mortars

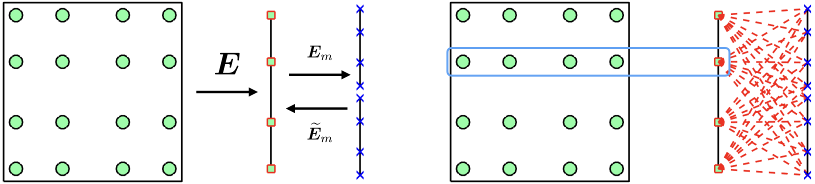



$$\underbrace{\begin{bmatrix} Q_i - Q_i^T & E^T B_i & \\ -B_i E & & B_i \widetilde{E}_m \\ & -\widetilde{B}_i E_m & \end{bmatrix}}_{\text{modified skew operator } Q_N^i - \left(Q_N^i\right)^T}$$

- Define appropriate interpolation operators $E_m, \widetilde{E}_m$ between conforming and non-conforming (mortar) nodes.
- Modify the skew-symmetric formulation as follows:

$$M \frac{\mathrm{d}u}{\mathrm{dt}} + \sum_{i=1}^{d} \begin{bmatrix} I \\ E \\ E_m E \end{bmatrix}^T \begin{bmatrix} Q_i - Q_i^T & E^T B_i & \\ -B_i E & & B_i \widetilde{E}_m \\ & -\widetilde{B}_i E_m & \end{bmatrix} + E^T E_m^T \widetilde{B}_i f_i^* = 0.$$

# Mortar implementation



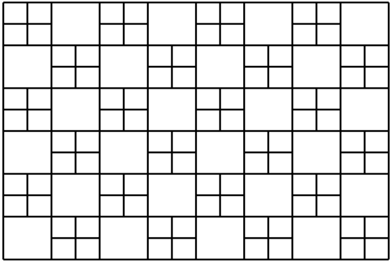

(a) Mortar operators (b) Mortar coupling

$$\boldsymbol{M}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{dt}} + \sum_{i=1}^{d} \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{E} \end{bmatrix}^T \left( \begin{bmatrix} \boldsymbol{Q}_i - \boldsymbol{Q}_i^T & \boldsymbol{E}^T\boldsymbol{B}_i \\ -\boldsymbol{B}_i\boldsymbol{E} & \end{bmatrix} \circ \boldsymbol{F}_S \right) \boldsymbol{1} + \boldsymbol{E}^T\boldsymbol{B}_i \widetilde{\boldsymbol{f}}_i^* = 0$$

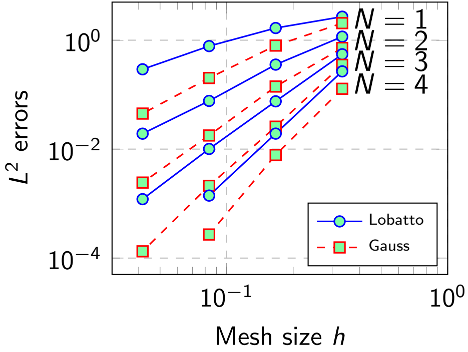
$$\widetilde{\boldsymbol{f}}_i^* = \widetilde{\boldsymbol{E}}_m \boldsymbol{f}_i^* + \left( \widetilde{\boldsymbol{E}}_m \circ \boldsymbol{F}_S^{sm} \right) \boldsymbol{1} - \widetilde{\boldsymbol{E}}_m \left( \boldsymbol{E}_m \circ \boldsymbol{F}_S^{ms} \right) \boldsymbol{1}$$

Can reformulate as an entropy stable correction to the numerical flux.

# Numerical results: non-conforming meshes



(a) Coarse non-conforming mesh




(b) Sub-optimal rates if under-integrated

The skew-symmetric formulation guarantees entropy stability for both Lobatto and Gauss quadratures, but Gauss is more accurate.

Chan (2019). *Skew-symmetric entropy stable modal discontinuous Galerkin formulations.*

# Summary and future work

- Gauss collocation DG can improve accuracy on curved meshes.
- Can treat non-conforming interfaces using a mortar-based coupling.
- This work is supported by DMS-1719818 and DMS-1712639.

Thank you! Questions?

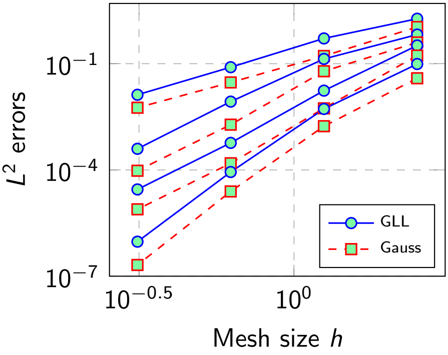
Chan (2019). *Skew-symmetric entropy stable modal discontinuous Galerkin formulations*.

Chan, Del Rey Fernandez, Carpenter (2018). *Efficient entropy stable Gauss collocation methods*.

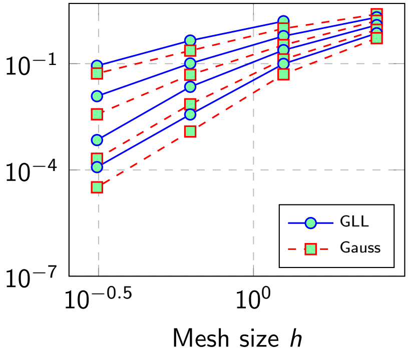
Chan, Wilcox (2018). *On discretely entropy stable weight-adjusted DG methods: curvilinear meshes*.

Chan (2018). *On discretely entropy conservative and entropy stable discontinuous Galerkin methods*.

# 3D isentropic vortex



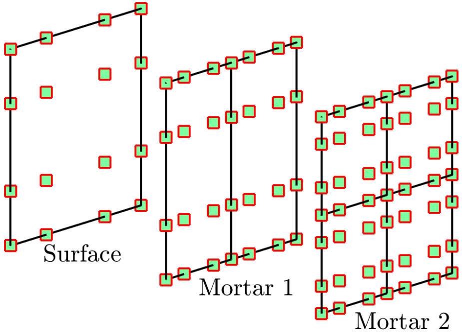

(a) Affine mesh




(b) Curved mesh

Figure: $L^2$ errors for the 3D isentropic vortex for $N = 2, \ldots, 5$.

# Entropy stable mortars in 3D




The mortar-based approach can reduce the number of flux evaluations for 3D non-conforming meshes.