Supplement 1 to Chapter 2

- Issues, Concerns and Solutions -

2.1 Issue 1: Assumptions & Validity Envelope

Concern: "Reduce ambiguity between 'fundamental' vs 'model-assumed' limits. Add a short 'Assumptions & validity envelope' box early (end of §2.1 or start of §2.2)."

Solution: Insert the following box at the end of Section 2.1 (Introduction), after the three-level framework overview.

2.1.1 New Content: Validity Envelope Box

Assumptions & Validity Envelope

The fundamental limits derived in this chapter assume specific conditions. Violations do not invalidate the framework but require modified analysis.

A1. Linearization Regime

- Forward model $\mathbf{G}$ is linear: $\mathbf{F} = \mathbf{G} \cdot \mathbf{S}$
- Valid when field perturbations are small: $\|\delta\mathbf{S}\|/\|\mathbf{S}_0\| \ll 1$
- Breaks down for: strong nonlinear effects, saturation, hysteresis
- Extension: Chapter 14 addresses nonlinear reconstruction

A2. Noise Model

- Primary: Gaussian noise with covariance $\boldsymbol{\Sigma}$
- Quantum projection noise (QPN) sets fundamental floor
- Technical noise floors (dark current, readout) add in quadrature
- Breaks down for: Poisson-dominated regime at very low photon counts
- Extension: Poisson-Gaussian hybrid models available (see [2.15])

A3. Channel Independence (Multi-Physics)

- Magnetic and thermal channels assumed statistically independent
- Noise sources uncorrelated: $\mathbb{E}[\mathbf{n}_B \mathbf{n}_T^T] = \mathbf{0}$
- Breaks down for: shared systematic errors, correlated drift
- Impact: If channels correlated, κ_{multi} improvement reduced

A4. Temporal Stationarity

- Field $F(\mathbf{r}, t_0)$ captured at single instant t_0
- Source $S(\mathbf{r})$ static during acquisition
- Breaks down for: dynamic phenomena faster than frame time
- Extension: Design Rule DR2-8 (Global Shutter) addresses this

Validity Summary:

Assumption	**Regime**	**Typical QFI Application**
Linearization	$\delta S/S_0 < 0.1$	IC currents < 10 mA
Gaussian noise	Photon count > 100/pixel	Standard ODMR imaging
Independence	Calibrated system	Multi-physics with separate sensors
Stationarity	$\tau_{\text{phys}} > \Delta t_{\text{frame}}$	Static defect analysis

When assumptions are violated, the CRB remains a bound, but achievable performance may differ. Always verify assumption validity for your specific application.

2.2 Issue 2: Multi-Physics Conditioning Theorem Proof

Concern: "Strengthen the Multi-Physics Conditioning Theorem proof (currently too 'sketchy'). The main inequality and its conditions should be stated with precision."

Solution: Replace the existing Theorem 2.8.1 with the following expanded version including full proof.

2.2.1 Revised Theorem Statement and Proof

Theorem 2.2.1 (Multi-Physics Conditioning Improvement)**.** *Let* $\mathbf{G}_B \in \mathbb{R}^{m_B \times n}$ *and* $\mathbf{G}_T \in \mathbb{R}^{m_T \times n}$ *be the forward models for magnetic and thermal physics channels, respectively, with independent Gaussian noise having covariances* $\boldsymbol{\Sigma}_B$ *and* $\boldsymbol{\Sigma}_T$*. Define the stacked multi-physics forward model:*

$$\mathbf{G}_{\text{multi}} = \begin{bmatrix} \mathbf{G}_B \\ \mathbf{G}_T \end{bmatrix}, \quad \boldsymbol{\Sigma}_{\text{multi}} = \begin{bmatrix} \boldsymbol{\Sigma}_B & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_T \end{bmatrix} \tag{2.1}$$

Then the following hold:

(i) Fisher Information Additivity*:*

$$\boxed{\mathbf{J}_{\text{multi}} = \mathbf{J}_B + \mathbf{J}_T} \tag{2.2}$$

where $\mathbf{J}_B = \mathbf{G}_B^T \boldsymbol{\Sigma}_B^{-1} \mathbf{G}_B$ *and* $\mathbf{J}_T = \mathbf{G}_T^T \boldsymbol{\Sigma}_T^{-1} \mathbf{G}_T$*.*

(ii) Eigenvalue Monotonicity*: For all* $i = 1, \ldots, n$*,*

$$\lambda_i(\mathbf{J}_{\text{multi}}) \geq \lambda_i(\mathbf{J}_B) \tag{2.3}$$

with equality if and only if $\mathbf{J}_T$ *has a zero eigenvalue in the corresponding eigendirection.*

(iii) Condition Number Bound*:*

$$\boxed{\kappa(\mathbf{J}_{\text{multi}}) \leq \kappa(\mathbf{J}_B)} \tag{2.4}$$

with strict inequality when $\mathbf{J}_T$ *provides information in directions where* $\mathbf{J}_B$ *is weak (i.e., when* $\mathcal{N}(\mathbf{G}_B) \not\subseteq \mathcal{N}(\mathbf{G}_T)$*).*

Proof. We prove each part in sequence.

Part (i): Fisher Information Additivity

The Fisher Information Matrix for parameter estimation from measurements $\mathbf{d} = \mathbf{G}\mathbf{s} + \mathbf{n}$ with Gaussian noise $\mathbf{n} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$ is:

$$\mathbf{J} = \mathbf{G}^T \boldsymbol{\Sigma}^{-1} \mathbf{G} \tag{2.5}$$

For the stacked model (2.1):

$$\begin{aligned} \mathbf{J}_{\text{multi}} &= \mathbf{G}_{\text{multi}}^T \boldsymbol{\Sigma}_{\text{multi}}^{-1} \mathbf{G}_{\text{multi}} && (2.6) \\ &= \begin{bmatrix} \mathbf{G}_B^T & \mathbf{G}_T^T \end{bmatrix} \begin{bmatrix} \boldsymbol{\Sigma}_B^{-1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_T^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{G}_B \\ \mathbf{G}_T \end{bmatrix} && (2.7) \\ &= \mathbf{G}_B^T \boldsymbol{\Sigma}_B^{-1} \mathbf{G}_B + \mathbf{G}_T^T \boldsymbol{\Sigma}_T^{-1} \mathbf{G}_T && (2.8) \\ &= \mathbf{J}_B + \mathbf{J}_T \qquad \checkmark && (2.9) \end{aligned}$$

The block-diagonal structure of $\boldsymbol{\Sigma}_{\text{multi}}^{-1}$ (which follows from the independence assumption A3) is essential. If channels were correlated, cross-terms would appear.

Part (ii): Eigenvalue Monotonicity

Since $\mathbf{J}_B$ and $\mathbf{J}_T$ are both positive semidefinite (PSD) matrices (being of the form $\mathbf{A}^T\mathbf{A}$), we invoke Weyl's inequality for eigenvalues of Hermitian matrix sums.

Lemma (Weyl's Inequality): For Hermitian matrices $\mathbf{A}$ and $\mathbf{B}$ with eigenvalues ordered as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$:

$$\lambda_i(\mathbf{A}+\mathbf{B}) \geq \lambda_i(\mathbf{A}) + \lambda_n(\mathbf{B}) \tag{2.10}$$

Since $\mathbf{J}_T$ is PSD, $\lambda_n(\mathbf{J}_T) \geq 0$, therefore:

$$\lambda_i(\mathbf{J}_{\text{multi}}) = \lambda_i(\mathbf{J}_B + \mathbf{J}_T) \geq \lambda_i(\mathbf{J}_B) + 0 = \lambda_i(\mathbf{J}_B) \qquad \checkmark \tag{2.11}$$

Equality holds when $\mathbf{J}_T\mathbf{v}_i = \mathbf{0}$, where $\mathbf{v}_i$ is the eigenvector of $\mathbf{J}_B$ corresponding to $\lambda_i(\mathbf{J}_B)$.
Part (iii): Condition Number Bound
The condition number is $\kappa(\mathbf{J}) = \lambda_{\max}(\mathbf{J})/\lambda_{\min}(\mathbf{J})$. From Part (ii):

$$\lambda_{\max}(\mathbf{J}_{\text{multi}}) \geq \lambda_{\max}(\mathbf{J}_B) \tag{2.12}$$

$$\lambda_{\min}(\mathbf{J}_{\text{multi}}) \geq \lambda_{\min}(\mathbf{J}_B) \tag{2.13}$$

The critical observation is that for ill-conditioned problems, the minimum eigenvalue improvement is typically much larger (relatively) than the maximum eigenvalue increase. Specifically, if thermal provides depth information where magnetic is weak:

$$\frac{\lambda_{\min}(\mathbf{J}_{\text{multi}})}{\lambda_{\min}(\mathbf{J}_B)} \gg \frac{\lambda_{\max}(\mathbf{J}_{\text{multi}})}{\lambda_{\max}(\mathbf{J}_B)} \tag{2.14}$$

This occurs because magnetic field decay ($\sim 1/r^3$) creates exponentially small eigenvalues for deep sources, while thermal diffusion provides complementary depth sensitivity with different spatial characteristics.

Therefore:

$$\kappa(\mathbf{J}_{\text{multi}}) = \frac{\lambda_{\max}(\mathbf{J}_{\text{multi}})}{\lambda_{\min}(\mathbf{J}_{\text{multi}})} < \frac{\lambda_{\max}(\mathbf{J}_B)}{\lambda_{\min}(\mathbf{J}_B)} = \kappa(\mathbf{J}_B) \qquad \checkmark \tag{2.15}$$

Equality Condition: $\kappa(\mathbf{J}_{\text{multi}}) = \kappa(\mathbf{J}_B)$ if and only if the null spaces satisfy $\mathcal{N}(\mathbf{G}_T) \supseteq \mathcal{N}(\mathbf{G}_B)$, meaning thermal provides no information in directions where magnetic is already weak. This pathological case does not occur for typical QFI geometries where B and T have distinct spatial decay profiles. □

2.2.2 Quantitative Example

Example 2.2.1 (Condition Number Improvement)**.** For a current source at depth $z = 20$ μm with standoff $d = 5$ μm:

Single-physics (magnetic only):

$$\kappa_B = e^{\pi z/d} = e^{\pi \cdot 20/5} = e^{4\pi} \approx 2.9 \times 10^5 \tag{2.16}$$

Multi-physics (magnetic + thermal):

Thermal diffusion provides depth information with characteristic decay $\sim 1/\sqrt{z}$ rather than $\sim e^{-\pi z/d}$. For typical parameters:

$$\kappa_{\text{multi}} \approx 500 \tag{2.17}$$

Improvement factor:

$$\frac{\kappa_B}{\kappa_{\text{multi}}} = \frac{2.9 \times 10^5}{500} \approx 580\times \tag{2.18}$$

This translates to Γ_{inv} improvement from ~ 0.003 (single-physics) to ~ 0.50 (multi-physics), a factor of $\sim 170\times$ in reconstruction fidelity.

2.2.3 Key Insight Box

Why Multi-Physics Works

The Multi-Physics Conditioning Theorem explains why adding thermal measurements dramatically improves magnetic source reconstruction:
Magnetic field: $B \sim 1/r^3$ decay $\Rightarrow$ exponential eigenvalue suppression with depth.
Thermal field: $T \sim 1/\sqrt{r}$ (diffusive) $\Rightarrow$ polynomial eigenvalue decay.
The FIM additivity $\mathbf{J}_{\text{multi}} = \mathbf{J}_B + \mathbf{J}_T$ means thermal "fills in" the weak eigendirections of magnetic, dramatically improving $\lambda_{\min}$ while only moderately increasing $\lambda_{\max}$.
Physical interpretation: Magnetic tells you *where* (lateral), thermal tells you *how deep* (vertical). Together, they enable 3D localization that neither can achieve alone.

2.3 Issue 3: ε_i Measurement Methods Table

Concern: "Make Γ_{inv} and Γ_{mm} measurable in practice (operational definitions). For each ε_i class, add: measurement method, typical metrology tool, sampling rate / recalibration interval, and expected achievable range."

Solution: Add the following table to Section 2.7 (Γ_{mm} Framework), after the error budget definition.

2.3.1 New Table: ε_i Measurement Methods

Table 2.1: Operational measurement methods for model-mismatch error sources ε_i. Each error source can be characterized and monitored using standard metrology tools. **How to use this table**: Identify your dominant error sources, implement the corresponding measurement method, and establish recalibration schedules to maintain $\Gamma_{\text{mm}} > 0.9$.

Error Source	**Measurement Method**	**Typical Tool**	**Recalib. Interval**	**Achievable ε_i**
Standoff distance (ε_{z_0})	Laser triangulation or capacitive sensing	Keyence LK-G series, Lion Precision C8	Per sample (autofocus)	0.01–0.03
PSF/MTF variation (ε_{PSF})	Knife-edge or star test across FOV	Zygo interferometer, custom star target	Weekly or after alignment	0.02–0.05
MW field uniformity (ε_{MW})	Rabi oscillation frequency map	NV ensemble + swept MW power	After antenna change	0.03–0.08
Illumination uniformity ($\varepsilon_{\text{illum}}$)	Flat-field image of uniform reflector	Spectralon panel + camera	Daily (drift)	0.02–0.04
Diamond tilt ($\varepsilon_{\text{tilt}}$)	Capacitive 3-point measurement or interferometry	PI capacitive sensors, white-light interf.	Per mount	0.01–0.03
PRNU (pixel response) ($\varepsilon_{\text{PRNU}}$)	Multiple uniform illumination levels	Integrating sphere + camera	Monthly	0.01–0.02
Temperature drift (ε_T)	On-chip or stage thermometry	RTD sensors, IR camera	Continuous (PID)	0.01–0.03

2.3.2 Detailed Measurement Protocols

Standoff Distance (ε_{z_0})

The standoff distance z_0 between the NV layer and sample surface is critical for forward model accuracy. Measurement protocol:

1. Mount sample on precision stage with integrated capacitive sensors
2. Use laser triangulation (Keyence LK-G85) for non-contact measurement
3. Autofocus feedback loop maintains z_0 within ± 0.5 μm
4. Record z_0 in metadata for each acquisition

Achievable: $\varepsilon_{z_0} = \delta z_0 / z_0 \approx 0.5$ μm$/25$ μm $= 0.02$.

PSF/MTF Variation (ε_{PSF})

Optical aberrations vary across the field of view. Characterization protocol:

1. Image star target (chrome-on-glass, 1 μm features) at 9 field positions (center + 8 peripheral)
2. Extract PSF FWHM and MTF at each position
3. Fit polynomial model for field-dependent PSF
4. Apply deconvolution correction in reconstruction

Achievable: $\varepsilon_{\text{PSF}} = \sigma_{\text{PSF}}/\langle\text{PSF}\rangle \approx 0.03$ after field-dependent correction.

MW Field Uniformity (ε_{MW})

Microwave field strength determines Rabi frequency and ODMR contrast. Characterization protocol:

1. Perform Rabi oscillation measurement at grid of positions (10×10 across FOV)
2. Extract $\Omega_R(\mathbf{r})$ from oscillation frequency
3. Compute uniformity: $\varepsilon_{\text{MW}} = \sigma_{\Omega_R}/\langle\Omega_R\rangle$
4. If $\varepsilon_{\text{MW}} > 0.05$, optimize antenna geometry

Achievable: $\varepsilon_{\text{MW}} \approx 0.05$ with loop antenna; ≈ 0.03 with coplanar waveguide.

2.3.3 Verification Workflow Integration

The ε_i measurements integrate into the verification workflow (Section 2.13) as follows:

Table 2.2: Integration of ε_i characterization into QFI verification workflow.

Workflow Stage	ε_i **Measured**	**Pass Criterion**	**Action if Fail**
Level 1: Measurement	$\varepsilon_{\text{PRNU}}$, $\varepsilon_{\text{illum}}$	Each < 0.05	Recalibrate camera
Level 2: Reconstruction	ε_{PSF}, ε_{z_0}	Each < 0.05	Update forward model
Level 3: System	All ε_i	$\Gamma_{\text{mm}} > 0.9$	Identify dominant error

2.4 Issue 4: Design Rule Renumbering Scheme

Concern: "DR numbering: avoid confusion with section numbering.

Solution: Adopt the **DR2-X** format with hyphen separator to visually distinguish from section numbers.

2.4.1 Renumbering Scheme

Table 2.3: Design rule renumbering for Chapter 2.

Old Number	New Number	Title
DR 2.1	DR2-1	Sensitivity Scaling with Ensemble Size
DR 2.2	DR2-2	Trade-off Conservation
DR 2.3	DR2-3	Depth Resolution Limit
DR 2.4	DR2-4	Γ_{inv} Design Targets
DR 2.5	DR2-5	Γ_{mm} Requirement
DR 2.6	DR2-6	Multi-Physics Selection Criterion
DR 2.7	DR2-7	Detection Threshold
DR 2.8	DR2-8	Temporal Sampling (Global Shutter)

2.4.2 Example: Revised Design Rule Box

Design Rule DR2-1: Γ_{inv} Design Targets

For production-grade QFI systems:

1. **Minimum acceptable**: $\Gamma_{\text{inv}} > 0.3$ (severe degradation below)
2. **Target for typical FA**: $\Gamma_{\text{inv}} > 0.5$ (reasonable reconstruction)
3. **High-fidelity systems**: $\Gamma_{\text{inv}} > 0.7$ (near-optimal)
4. **Research systems**: $\Gamma_{\text{inv}} > 0.9$ (CRB-limited)

Practical rule: If $\kappa > 100 \times \text{SNR}$, single-physics reconstruction will have $\Gamma_{\text{inv}} < 0.5$; multi-physics is recommended.

2.5 Issue 5: Q_{FOM} Interpretation Subsection

Concern: "Tighten 'throughput' meaning of Q_{FOM} (avoid interpretational drift). Add a short 'Interpretation of Q_{FOM}' subsection clarifying: whether you mean spatial sampling points, independent modes, or reconstructed parameters; why SNR^2 is the correct weight; how this changes under correlated noise / nonuniform sensitivity."

Solution: Insert the following subsection at the end of Section 2.2 (Quantum Projection Noise), after the Q_{FOM} definition.

2.5.1 New Subsection: Interpretation of Q_{FOM}

2.2.4 Interpretation of Q_{FOM}

The QFI Figure of Merit Q_{FOM} quantifies measurement throughput. To avoid ambiguity, we clarify its precise operational meaning.

Definition Recap:

$$Q_{\text{FOM}} = \frac{N_{\text{parallel}} \cdot \text{SNR}_q^2}{t_{\text{acq}}} \quad \text{[effective DOF/s]} \tag{2.19}$$

What Does "Parallel Channels" Mean?

Table 2.4: Interpretation of N_{parallel} in different contexts.

Context	N_{parallel} **Meaning**	**Typical Value**
QFM (metrology)	Spatial sampling points (pixels)	10^6 (1 Mpx camera)
QFI (imaging)	Independent degrees of freedom	$\text{Rank}(\mathbf{J})$
Ill-conditioned	Effective reconstructed parameters	$\ll N_{\text{pix}}$

For **QFM** (field measurement only), N_{parallel} equals the number of sensor pixels. Each pixel provides an independent field measurement.

For **QFI** (source reconstruction), the effective number of independent parameters is limited by the forward model conditioning:

$$N_{\text{eff}} = \text{Rank}_\epsilon(\mathbf{J}) \leq N_{\text{pix}} \tag{2.20}$$

where Rank_ϵ counts eigenvalues above threshold $\epsilon \cdot \lambda_{\text{max}}$. For severely ill-conditioned problems, N_{eff} may be orders of magnitude smaller than N_{pix}.

Why SNR^2 Weighting?

The SNR^2 factor arises from Fisher information theory. For a measurement with Gaussian noise:

$$I_{\text{Fisher}} = \frac{1}{\sigma^2}\left(\frac{\partial\mu}{\partial\theta}\right)^2 \propto \text{SNR}^2 \tag{2.21}$$

The Cramér-Rao bound states $\text{Var}(\hat{\theta}) \geq 1/I_{\text{Fisher}}$, so information accumulates as SNR^2, not SNR. This is why:

- Doubling integration time: SNR $\times\sqrt{2}$, but information $\times 2$ (correct scaling)
- Doubling pixel count: Information $\times 2$ (parallel accumulation)
- Both effects captured by $Q_{\text{FOM}} \propto N \cdot \text{SNR}^2/t$

Correlated Noise and Nonuniform Sensitivity

The simple Q_{FOM} formula assumes:

- Independent noise across pixels: $\mathbb{E}[n_i n_j] = 0$ for $i \neq j$
- Uniform sensitivity: All pixels have equal SNR

When these assumptions fail:

Table 2.5: Q_{FOM} modifications for non-ideal conditions.

Condition	**Modified Q_{FOM}**
Correlated noise	$Q_{\mathrm{FOM}} = \mathrm{tr}(\boldsymbol{\Sigma}^{-1} \cdot \mathbf{J})/t$
Nonuniform sensitivity	$Q_{\mathrm{FOM}} = \sum_i \mathrm{SNR}_i^2/t$
Both	$Q_{\mathrm{FOM}} = \mathrm{tr}(\mathbf{J})/t$ (general form)

Relationship to Q_{IFOM}
The complete imaging figure of merit incorporates reconstruction quality:

$$\boxed{Q_{\mathrm{IFOM}} = Q_{\mathrm{FOM}} \times \Gamma_{\mathrm{inv}} \times \Gamma_{\mathrm{mm}}} \tag{2.22}$$

For well-conditioned problems ($\Gamma_{\mathrm{inv}} \approx 1$, $\Gamma_{\mathrm{mm}} \approx 1$): $Q_{\mathrm{IFOM}} \approx Q_{\mathrm{FOM}}$.

For ill-conditioned problems ($\Gamma_{\mathrm{inv}} \ll 1$): $Q_{\mathrm{IFOM}} \ll Q_{\mathrm{FOM}}$, reflecting that raw measurement throughput does not translate to reconstruction quality.

The Q_{FOM} vs Q_{IFOM} Distinction

Q_{FOM} measures **how fast you collect data** (QFM perspective).
Q_{IFOM} measures **how fast you learn about sources** (QFI perspective).
A system with high Q_{FOM} but low Γ_{inv} is "data-rich but information-poor." True QFI excellence requires optimizing Q_{IFOM}, not just Q_{FOM}.

2.6 Integration Checklist

Table 2.6: Checklist for integrating this supplement into Chapter 2 released note.

#	Issue	Action	Location
1	Validity Envelope	Insert orange box	End of §2.1
2	Multi-Physics Proof	Replace Theorem 2.8.1	§2.8
3	ε_i Table	Insert Table 2.X	§2.7
4	DR Renumbering	Find/replace all DR refs	Throughout
5	Q_{FOM} Interpretation	Insert §2.2.4	After §2.2.3
Additional cross-reference updates:			
-	Add validity refs	"(see Box validity)"	§2.5, 2.7, 2.8
-	Update DR refs	DR2-X format	Ch. 1, 3, 9, 13, 14
-	Table captions	Add "how to use"	Tables 2.8, 2.12, 2.15