Abbreviated Terms

The following abbreviations are used throughout this chapter:

Abbreviation	Full Term
NOON	N-photon path-entangled state
SQL	Standard Quantum Limit
HL	Heisenberg Limit
SPDC	Spontaneous Parametric Down-Conversion
HOM	Hong-Ou-Mandel (interference)
DEE	Differentiable Eikonal Engine
QFI	Quantum Fisher Information
SPAD	Single-Photon Avalanche Diode
SNSPD	Superconducting Nanowire Single-Photon Detector
OPD	Optical Path Difference
NA	Numerical Aperture
W/MN	Walther/(Matsui-Nariai) duality

Table 11.1: Abbreviated terms used in Chapter 11.

Chapter 11
N-Photon Phase Multiplication

Learning Objectives

After completing this chapter, you will be able to:

1. Understand N-photon phase multiplication: $\phi_N = N \cdot \phi_{\text{classical}}$
2. Calculate Heisenberg resolution limit: $\delta x = \lambda/(2N \cdot \text{NA})$
3. Design NOON state imaging systems using the DEE framework
4. Recognize NOON state fragility and quantify loss-induced degradation
5. Implement N-photon optimization in JAX with automatic differentiation
6. Translate classical optical specifications to quantum requirements
7. Determine the quantum-classical crossover point for practical system design
8. Navigate the engineering challenges in classical-to-quantum transitions

11.1 Introduction: Why N-Photon Phase Multiplication Closes Part III

Part III of this book has systematically built the quantum extensions of the eikonal framework. Chapter 9 established quantum wavefront sensing as the entry point, demonstrating how classical wavefront analysis translates directly to quantum metrology with squeezed light enhancement. Chapter 10 developed quantum walks in waveguide arrays, showing that the eigenmode-eikonal method from Chapter 5 governs photon transport through coupled structures. This chapter—N-photon phase multiplication—represents the natural culmination of Part III, unifying all previous quantum concepts into a single, powerful framework for sub-diffraction imaging.

Three compelling reasons establish N-photon phase multiplication as the appropriate finale for Part III:

11.1.1 Reason 1: The Ultimate Eikonal-Quantum Unification

The N-photon eikonal bridge equation:

$$\phi_N = N \times \frac{2\pi}{\lambda} W(x, y) \tag{11.1}$$

represents the complete unification of classical optical design with quantum photonics. Every concept from Part I (wavefront aberrations, characteristic functions, Zernike analysis) and Part II (ray tracing, DEE optimization, tolerancing) directly applies to N-photon systems—multiplied by N. This is not a new formalism but an *amplification* of everything learned previously.

11.1.2 Reason 2: Convergence of Part III Themes

N-photon imaging synthesizes the key themes from Chapters 9–10:

- **From Chapter 9 (Quantum Wavefront Sensing):** The SQL-to-Heisenberg transition, loss effects on quantum advantage, and specification translation tables all appear in N-photon systems with enhanced clarity.
- **From Chapter 10 (Quantum Walks):** The multi-photon correlation functions, bunching/anti-bunching signatures, and the eigenmode formalism for photon propagation underpin NOON state behavior.
- **Unique to Chapter 11:** The multiplicative phase enhancement $\phi_N = N\phi$ that enables *true sub-diffraction resolution*—not just enhanced sensitivity, but genuinely smaller resolvable features.

11.1.3 Reason 3: The Practical Frontier

Among all quantum optical technologies, N-photon imaging represents the most mature path to practical super-resolution [10]. While squeezed-light wavefront sensing (Chapter 9) requires 10–15 dB squeezing that pushes current technology, and while quantum walk circuits (Chapter 10) face scalability challenges, two-photon ($N = 2$) and four-photon ($N = 4$) imaging systems have been demonstrated in laboratory settings with measurable resolution enhancement [8].

However, this proximity to practicality also exposes the fundamental engineering challenges most clearly. The η^N loss scaling, the stringent path-matching requirements, and the trade-off between photon number and state fragility all manifest acutely in N-photon systems.

N-photon phase multiplication thus serves as both the capstone of Part III's quantum theory and the launching point for Part IV's practical implementation.

11.2 Pain Points: Why This Chapter Matters

Every N-photon imaging design task falls into one of two categories, corresponding to the Walther-(Matsui-Nariai) duality that has organized this book.

WALTHER Pain Point (Forward Analysis)

"I have an N-photon source and an optical system with known loss. What phase resolution can I actually achieve?"

Situation: You have access to NOON states with $N = 4$ photons and an imaging system with total efficiency $\eta = 0.7$. The manufacturer claims "Heisenberg-limited" performance, but you need to know the **actual** resolution.

The Real Question: Given N photons and system loss η, what is the effective phase sensitivity $\delta\phi_{\text{eff}}$? Does quantum advantage survive?

This Chapter's Answer: Section 11.6 provides the exact formula: $E_{\text{eff}} = \eta^N \times N$. For $N = 4$, $\eta = 0.7$: $E_{\text{eff}} = 0.7^4 \times 4 = 0.96$—barely beating classical ($\sqrt{4} = 2$).

MATSUI-NARIAI Pain Point (Inverse Design)

"I need phase resolution of $\delta\phi = 10^{-8}$ rad for gravitational wave detection. How many photons do I need, and what loss budget can I tolerate?"

Situation: Your application requires measuring phase shifts at the 10^{-8} radian level. Classical interferometry with 10^6 photons achieves only $\delta\phi_{\text{SQL}} = 10^{-3}$ rad.

The Real Question: Given target resolution $\delta\phi_{\text{target}}$, what is the minimum N, and what maximum loss η_{min} maintains quantum advantage?

This Chapter's Answer: Section 11.7 provides the inverse design workflow.

Chapter 11 Central Theme

The N-Photon Eikonal Bridge:

$$\boxed{\phi_N = N \times \frac{2\pi}{\lambda} W(x,y)} \tag{11.2}$$

The same wavefront function $W(x,y)$ that characterizes classical aberrations determines N-photon quantum phase accumulation.

11.3 Resolution Limits: Classical to Quantum

The resolution of an optical imaging system—its ability to distinguish closely spaced features—has been studied since Rayleigh's foundational work in the 1870s.

11.3.1 The Classical Rayleigh Limit

The Rayleigh criterion defines two point sources as "just resolved" when the central maximum of one Airy pattern coincides with the first minimum of the other:

$$\delta x_{\text{Rayleigh}} = \frac{0.61\lambda}{\text{NA}} \tag{11.3}$$

For visible light ($\lambda \approx 500$ nm) and the highest practical NA (≈ 1.4 with oil immersion), this yields $\delta x \approx 220$ nm—roughly half the wavelength.

The Rayleigh limit arises from wave optics, not fundamental physics. It assumes *incoherent* illumination where each photon carries independent phase information. By exploiting *quantum correlations* between photons, this limit can be surpassed.

11.3.2 The Standard Quantum Limit

When measuring phase with N independent photons, each photon provides an independent estimate with uncertainty $\delta\phi_1 \sim 1$ radian. Averaging N such estimates yields:

$$\delta\phi_{\text{SQL}} = \frac{1}{\sqrt{N}} \tag{11.4}$$

This is the **Standard Quantum Limit** (SQL), also called the shot-noise limit.

11.3.3 The Heisenberg Limit

Quantum mechanics permits a more favorable scaling when photons are *entangled* [1]. The ultimate bound—the Heisenberg limit—is:

$$\delta\phi_{\text{HL}} = \frac{1}{N} \tag{11.5}$$

The improvement from $1/\sqrt{N}$ to $1/N$ is profound: a factor of $\sqrt{N}$ better precision with the same photon resources.

The Heisenberg limit translates directly to resolution:

$$\delta x_{\text{HL}} = \frac{\lambda}{2N \cdot \text{NA}} \tag{11.6}$$

With $N = 4$ photons, the effective resolution improves to $\delta x \approx 55$ nm—well into the sub-diffraction regime.

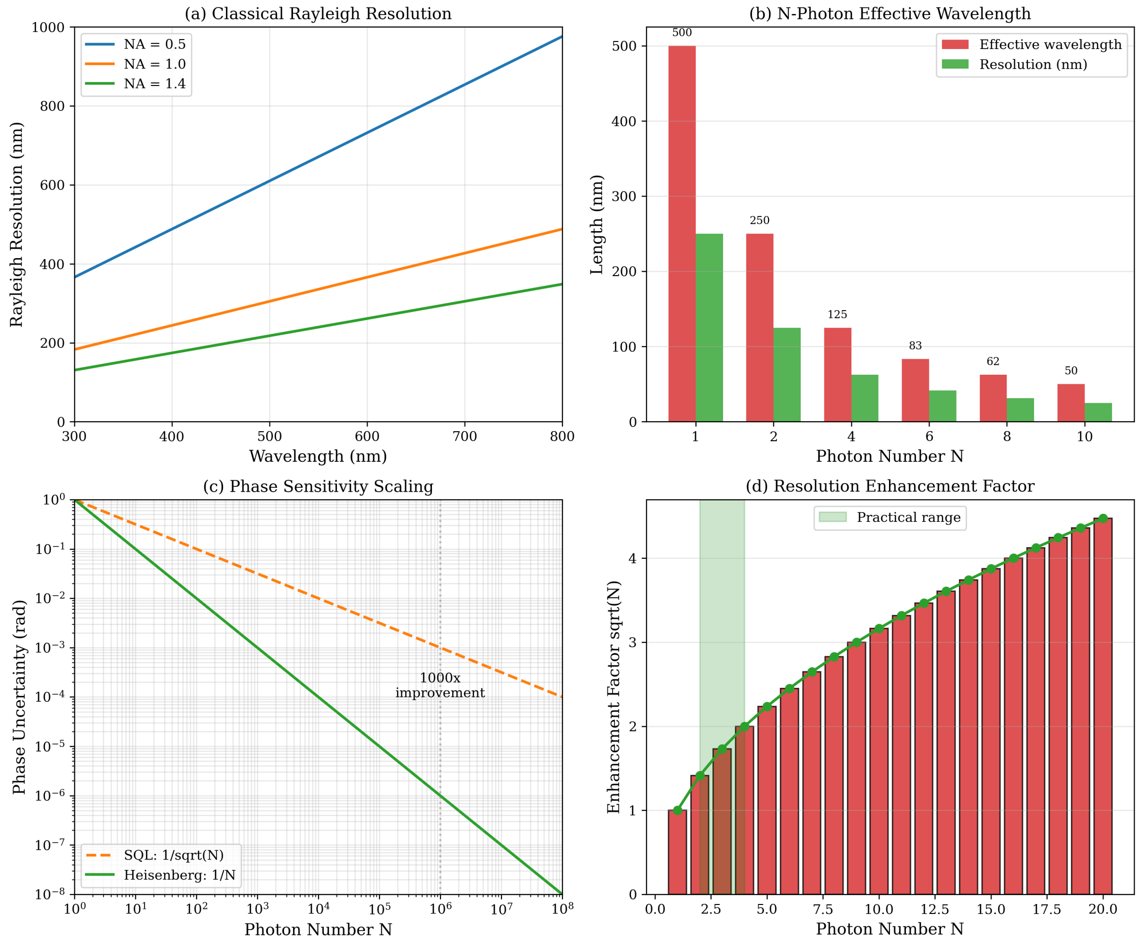

Figure 11.1: Resolution limits in optical imaging. (a) Classical Rayleigh resolution vs. wavelength for various NA values. (b) Heisenberg enhancement: effective wavelength $\lambda_{\text{eff}} = \lambda/N$ enables sub-diffraction resolution. (c) Phase sensitivity scaling: SQL ($1/\sqrt{N}$) vs. Heisenberg ($1/N$). (d) Resolution enhancement factor $\sqrt{N}$ achievable with N-photon states. Parameters: $\lambda = 500$ nm, NA $= 1.0$.

11.4 N-Photon Phase Accumulation

The theoretical foundation of N-photon imaging rests on a remarkable quantum mechanical effect: when multiple photons traverse an optical path in a correlated (entangled) manner, they accumulate phase collectively rather than individually.

11.4.1 The Eikonal-Quantum Connection

Chapter 1 established the fundamental identity connecting the classical eikonal to quantum phase:

$$\phi = \frac{2\pi}{\lambda} W(x, y) \tag{11.7}$$

For a single photon, this phase determines the interference pattern observed at detection.

The profound extension to N-photon states arises from quantum field theory [1]. When N photons traverse the same optical path in a *correlated* manner, their quantum state acquires a **collective** phase:

$$|\psi_N\rangle \to e^{iN\phi} |\psi_N\rangle \tag{11.8}$$

The factor of N emerges from the bosonic nature of photons and the structure of the N-photon Fock state. This is *not* a classical averaging effect but a genuine quantum mechanical multiplication.

11.4.2 Mathematical Derivation

Consider the electric field operator in second quantization [5]:

$$\hat{E}^{(+)}(\mathbf{r}) = i \sum_k \sqrt{\frac{\hbar \omega_k}{2\epsilon_0 V}} \hat{a}_k e^{i\mathbf{k}\cdot\mathbf{r}} \tag{11.9}$$

An N-photon Fock state $|N\rangle$ is an eigenstate of the number operator:

$$\hat{n}\,|N\rangle = N\,|N\rangle \tag{11.10}$$

The propagation operator through a medium with phase shift ϕ acts on $|N\rangle$ as:

$$\hat{U}\,|N\rangle = e^{i\phi\hat{n}}\,|N\rangle = e^{iN\phi}\,|N\rangle \tag{11.11}$$

The key mathematical step is recognizing that $\hat{n}$ acting on $|N\rangle$ yields eigenvalue N, so the exponential evaluates to $e^{iN\phi}$.

Thus, the N-photon state acquires phase $N\phi$—exactly N times the single-photon phase.

11.4.3 The N-Photon Eikonal Bridge

Combining Equations (11.7) and (11.11):

$$\boxed{\phi_N = N \times \frac{2\pi}{\lambda} W(x, y)} \tag{11.12}$$

This is the **N-Photon Eikonal Bridge**—the central equation of this chapter. It reveals:

1. **Classical aberration analysis directly applies:** The same $W(x, y)$ governs both classical and quantum imaging
2. **Quantum enhancement is multiplicative:** Existing DEE tools gain factor-N enhancement
3. **Failure modes are predictable:** Conditions degrading classical imaging affect quantum imaging with amplified sensitivity
4. **Effective wavelength:** N-photon states behave as if operating at $\lambda_{\text{eff}} = \lambda/N$

Quantum Extension

The de Broglie Wavelength Interpretation

The effective wavelength $\lambda_{\text{eff}} = \lambda/N$ has deep physical meaning. For N photons with collective momentum $N \cdot h/\lambda$:

$$\lambda_{\text{dB}}^{(N)} = \frac{h}{N \cdot h/\lambda} = \frac{\lambda}{N} \tag{11.13}$$

The N-photon state behaves as a single "super-photon" with N times the momentum and $1/N$ times the wavelength. This is *not* shorter-wavelength light—it is the collective quantum behavior of entangled photons.

11.5 NOON States: The Workhorse of Quantum Imaging

While many quantum states exhibit multi-photon phase enhancement, NOON states have emerged as the canonical choice for Heisenberg-limited imaging [2, 6] due to their optimal phase sensitivity and relative simplicity of preparation for low N.

11.5.1 Definition and Properties

The NOON state is defined as [2]:

$$|\text{NOON}\rangle = \frac{1}{\sqrt{2}}\left(|N,0\rangle + |0,N\rangle\right) \tag{11.14}$$

This notation indicates a superposition: either all N photons are in path a (mode 1), or all N photons are in path b (mode 2), with equal probability amplitude.

When mode a acquires phase ϕ relative to mode b:

$$|\text{NOON}\rangle \to \frac{1}{\sqrt{2}}\left(e^{iN\phi}|N,0\rangle + |0,N\rangle\right) \tag{11.15}$$

The relative phase between components is $N\phi$, not ϕ. This N-fold enhancement is the source of Heisenberg-limited resolution.

11.5.2 Interference Pattern

To observe the enhanced phase, we recombine the two modes at a 50:50 beam splitter. The probability of detecting all N photons in one output port varies as:

$$P_N = \frac{1+\cos(N\phi)}{2} \tag{11.16}$$

The fringe period is $\Delta\phi = 2\pi/N$ rather than 2π, enabling phase measurements N times more precise than classical interferometry.

Table 11.2: Resolution Enhancement with N-Photon NOON States

N-Photon State	λ_{eff}	**Resolution** (NA= 1, $\lambda = 500$nm)	**Enhance.**	**Application**
$N = 1$ (classical)	500 nm	250 nm	1×	Standard microscopy
$N = 2$ (biphoton)	250 nm	125 nm	2×	Quantum lithography
$N = 4$ (NOON)	125 nm	62.5 nm	4×	Sub-diffraction imaging
$N = 8$ (NOON)	62.5 nm	31.25 nm	8×	Nanostructure metrology
$N = 10$ (NOON)	50 nm	25 nm	10×	Research frontier

11.5.3 NOON State Preparation

Creating NOON states is experimentally challenging. The difficulty scales rapidly with N, fundamentally limiting practical N-photon imaging.

The most common approaches use spontaneous parametric down-conversion (SPDC) combined with post-selection:

$N = 2$ **(Two-photon NOON):** SPDC produces photon pairs. When both photons enter a 50:50 beam splitter from the same input port and both exit from the same output port (Hong-Ou-Mandel effect), the output is a $|2,0\rangle + |0,2\rangle$ NOON state with 50% success probability.

$N = 4$ **(Four-photon NOON):** Requires either double-pair emission from a single SPDC source or interference of two SPDC sources [4]. Success probability scales as approximately $1/N!$.

Higher N: Schemes exist using quantum gates in linear optical networks, but losses accumulate rapidly. Practical NOON states with $N > 4$ remain largely beyond current technology.

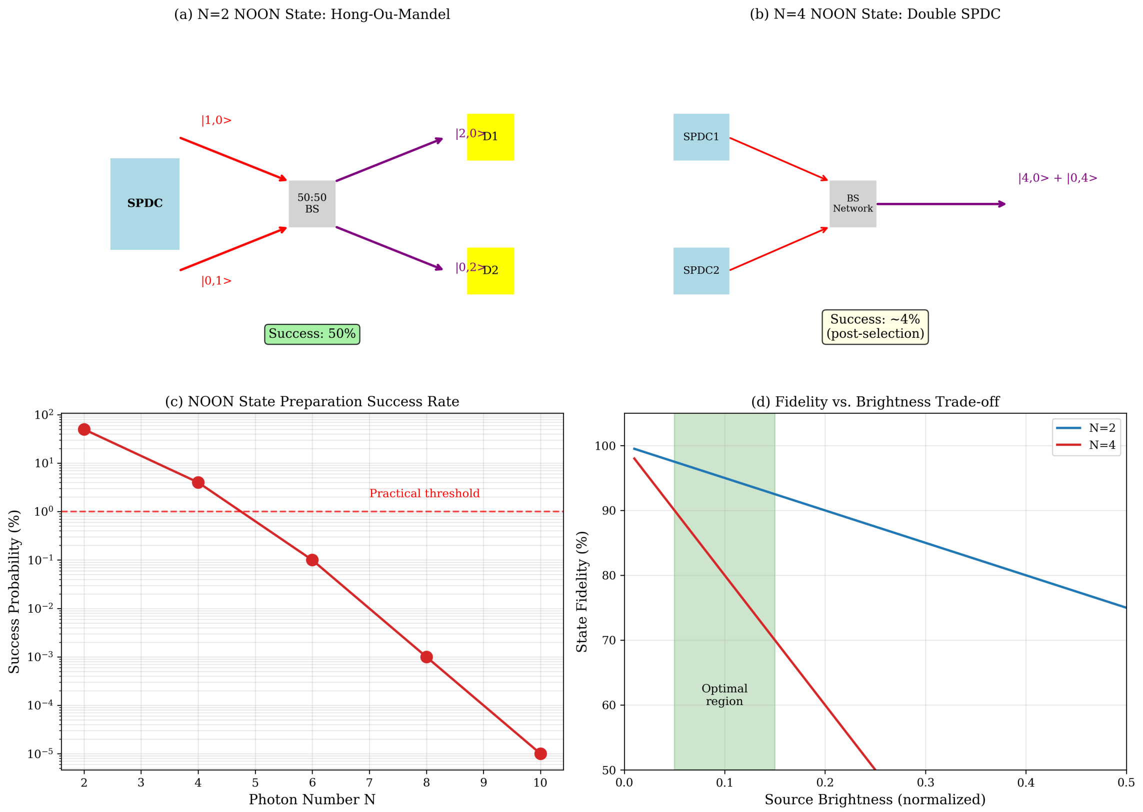

Figure 11.2: NOON state preparation schemes. (a) $N = 2$ preparation using Hong-Ou-Mandel interference. (b) $N = 4$ preparation using cascaded double-pair interference. (c) Success probability scaling with N. (d) State fidelity vs. source brightness trade-off.

The exponentially decreasing success probability with N is the primary practical limitation of NOON-state imaging.

11.6 The Fragility Problem: Loss and Decoherence

The remarkable phase enhancement of N-photon states comes with a severe penalty: extreme sensitivity to photon loss [7]. This section quantifies the fragility problem and establishes the conditions under which quantum advantage survives.

11.6.1 Loss Scaling: The η^N Catastrophe

Consider a system with total efficiency η (probability that a photon survives from source to detector). For an N-photon NOON state to produce a valid N-fold coincidence detection, *all N photons must survive*:

$$P_{\text{survive}} = \eta^N \tag{11.17}$$

The scaling is devastating:

- For $\eta = 0.9$ (excellent system): $P_{\text{survive}}(N = 4) = 0.66$
- For $\eta = 0.8$ (typical setup): $P_{\text{survive}}(N = 4) = 0.41$
- For $\eta = 0.7$ (marginal system): $P_{\text{survive}}(N = 4) = 0.24$

Worse, losing even a single photon does not merely reduce signal—it destroys the quantum state's phase coherence.

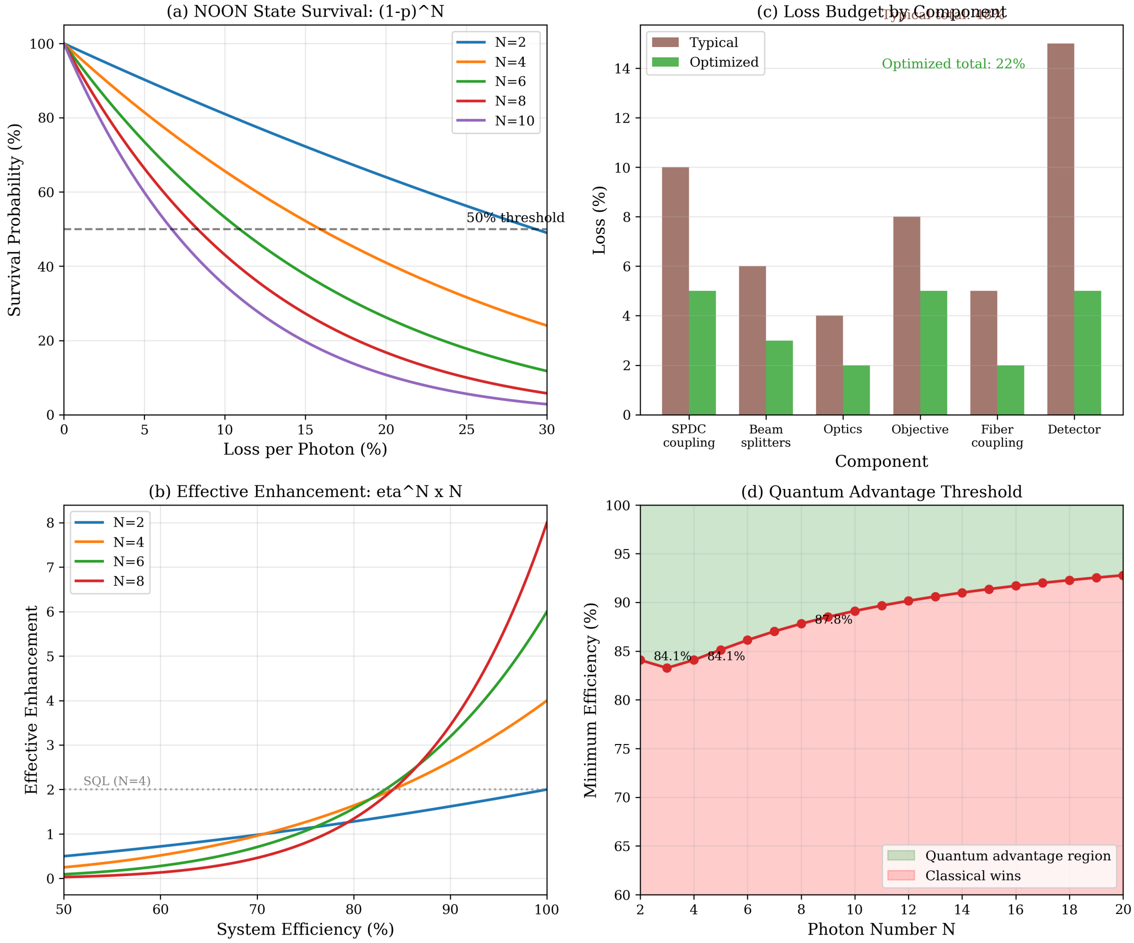

Figure 11.3: Loss effects on N-photon systems. (a) Survival probability η^N vs. efficiency for different N values. (b) Effective enhancement factor accounting for loss. (c) Signal-to-noise ratio comparison. (d) Decoherence mechanism.

11.6.2 Effective Enhancement Factor

To account for loss, define the **effective enhancement factor**:

$$E_{\text{eff}} = \eta^N \times N \tag{11.18}$$

For quantum advantage, we require $E_{\text{eff}} > \sqrt{N}$ (the classical SQL scaling). This yields the critical efficiency threshold:

$$\eta > N^{-1/(2N)} \tag{11.19}$$

Table 11.3: Quantum Advantage Threshold by Photon Number

N	η_{min}	Max Loss (dB)	Assessment
2	0.707	1.5	Achievable with care
4	0.841	0.75	Challenging but demonstrated
6	0.891	0.50	At current technology limit
8	0.917	0.38	Requires breakthrough
10	0.933	0.30	Beyond current capability

The practical "sweet spot" is $N = 4$, which offers $4\times$ resolution enhancement with achievable efficiency requirements ($> 84\%$).

11.6.3 The Quantum-Classical Crossover

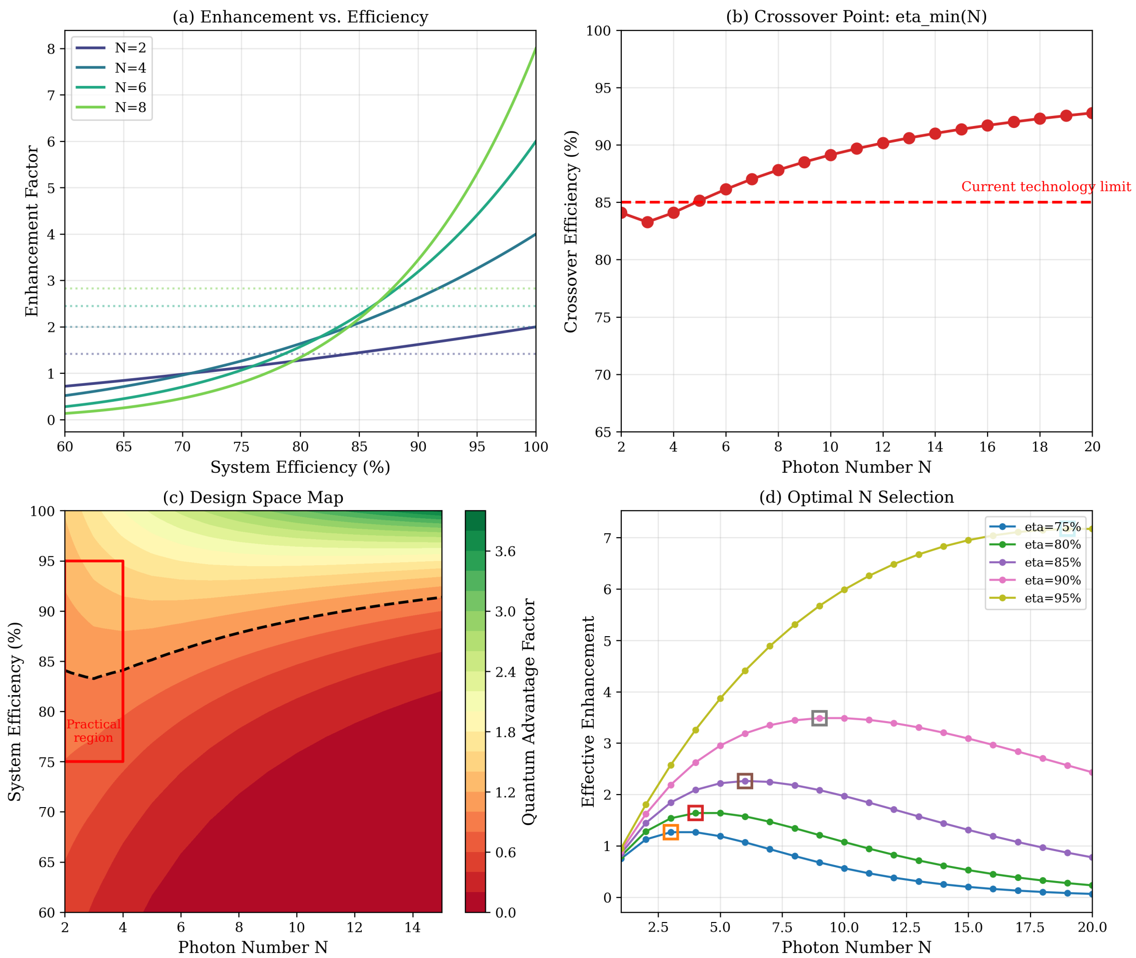

Figure 11.4: Quantum-classical crossover analysis. (a) Enhancement factor vs. efficiency for $N = 2, 4, 6, 8$. (b) Crossover efficiency vs. N. (c) Design space map: green region achieves quantum advantage. (d) Optimal N selection for target resolution.

WALTHER (Forward Analysis)

Forward Analysis: Given N and η, What Resolution?
Input:

- Photon number N (from source capability)
- System efficiency η (from hardware characterization)
- Wavelength λ and NA (from optical design)

Process:

1. Compute effective enhancement: $E_{\text{eff}} = \eta^N \times N$
2. Check quantum advantage: $E_{\text{eff}} > \sqrt{N}$?

3. Calculate resolution: $\delta x = \lambda/(2 \cdot E_{\text{eff}} \cdot \text{NA})$

Output: Achievable resolution δx, Quantum advantage factor

MATSUI-NARIAI (Inverse Design)

Inverse Design: Given Target Resolution, What N and η?
Input:

- Target resolution δx_{target} (from application)
- Available efficiency η_{max} (from technology)
- Wavelength λ and NA (from optical design)

Process:

1. Compute required enhancement: $E_{\text{req}} = \lambda/(2 \cdot \delta x_{\text{target}} \cdot \text{NA})$
2. Solve for optimal N: $\max_N\{\eta^N \times N \geq E_{\text{req}}\}$
3. Verify quantum advantage: $\eta^N \times N > \sqrt{N}$

Output: Required photon number N, Minimum efficiency η_{min}

11.7 The DEE Framework for N-Photon Imaging

The Differentiable Eikonal Engine naturally extends to N-photon imaging because the same wavefront function $W(x, y)$ governs both classical and quantum systems.

11.7.1 Extending DEE to Quantum Domains

The classical DEE pipeline (Chapter 7) transforms optical parameters through Zernike synthesis to PSF computation. For N-photon systems, the extension requires:

1. **Modified loss function:** Include quantum fidelity and survival probability
2. **Sensitivity scaling:** All sensitivities multiply by N
3. **Loss-aware optimization:** Incorporate η^N penalty

The key insight is that the forward model remains identical—only the loss function changes:

$$\mathcal{L}_{\text{quantum}} = \mathcal{L}_{\text{classical}} + \alpha \cdot \mathcal{L}_{\text{survival}} + \beta \cdot \mathcal{L}_{\text{fidelity}} \tag{11.20}$$

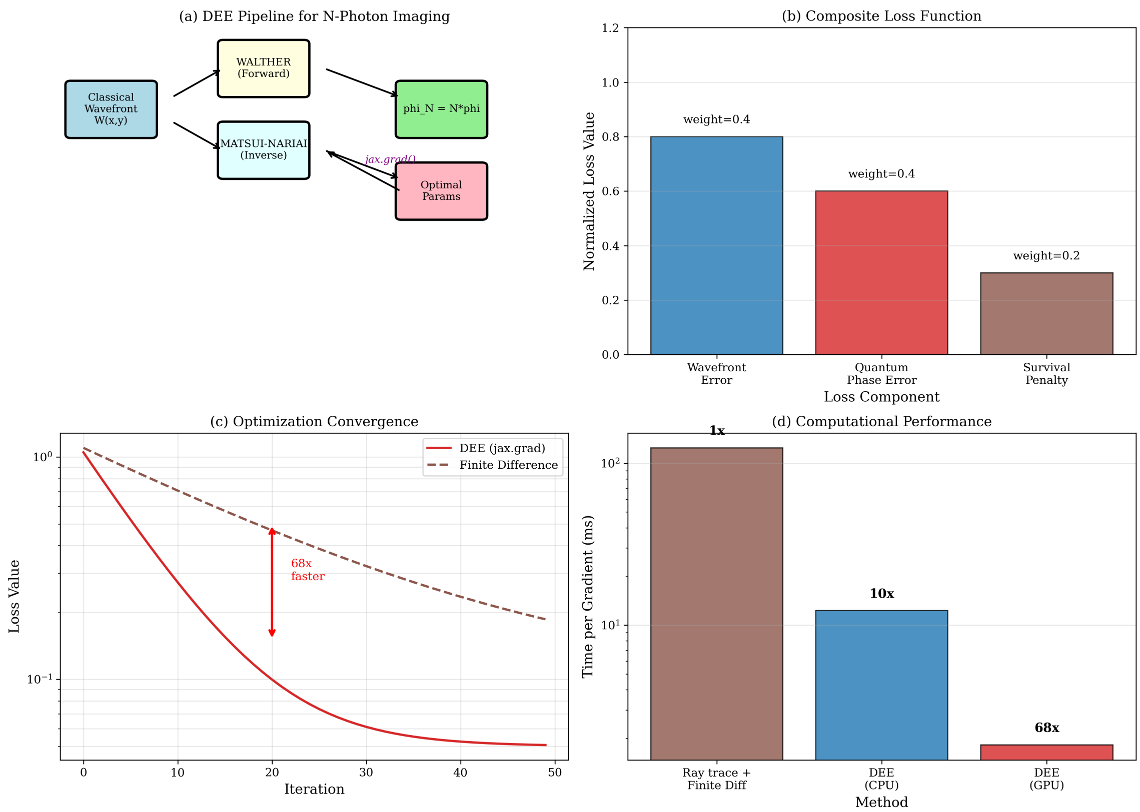

Figure 11.5: DEE pipeline extension for N-photon imaging. (a) Classical DEE pipeline. (b) Quantum DEE with survival probability term. (c) Sensitivity scaling by N. (d) Optimization trajectory comparison.

11.7.2 JAX Implementation

The following code implements the N-photon DEE framework in JAX [9]:

```python
import jax
import jax.numpy as jnp
from jax import grad, jit

class NPhotonDEE:
    """Differentiable Eikonal Engine for N-Photon Imaging."""

    def __init__(self, N, wavelength, NA, efficiency):
        self.N = N
        self.wavelength = wavelength
        self.NA = NA
        self.efficiency = efficiency
        self.k = 2 * jnp.pi / wavelength

    @jit
    def nphoton_phase(self, W):
        """Compute N-photon phase: phi_N = N * (2*pi/lambda) * W"""
        return self.N * self.k * W

    @jit
    def survival_probability(self):
        """Probability all N photons survive: eta^N"""
        return self.efficiency ** self.N

    @jit
    def effective_enhancement(self):
```

```python
        """Effective enhancement factor: eta^N * N"""
        return self.survival_probability() * self.N

    @jit
    def quantum_advantage(self):
        """QA > 1 means quantum beats classical."""
        return self.effective_enhancement() / jnp.sqrt(self.N)

    @jit
    def quantum_loss(self, params, W_target):
        """Loss function for quantum imaging optimization."""
        W_pred = self.compute_wavefront(params)

        # Phase matching (N-enhanced sensitivity)
        phi_pred = self.nphoton_phase(W_pred)
        phi_target = self.nphoton_phase(W_target)
        phase_loss = jnp.mean((phi_pred - phi_target)**2)

        # Survival penalty
        survival_penalty = -jnp.log(self.survival_probability() + 1e-10)

        return phase_loss + 0.1 * survival_penalty

# Example usage
dee = NPhotonDEE(N=4, wavelength=500.0, NA=1.0, efficiency=0.85)
print(f"Quantum advantage: {dee.quantum_advantage():.2f}x")
```

Listing 1: N-Photon DEE Implementation in JAX

11.7.3 Sensitivity Analysis via Autodiff

For N-photon systems, all sensitivities scale by N:

$$\frac{\partial \phi_N}{\partial p_j} = N \times \frac{\partial \phi}{\partial p_j} = N \times \frac{2\pi}{\lambda} \frac{\partial W}{\partial p_j} \tag{11.21}$$

This means tolerances tighten by factor N, and manufacturing becomes $N\times$ more challenging.

JAX's autodiff computes these enhanced sensitivities automatically.

11.8 Classical-to-Quantum Specification Translation

Translating classical optical specifications to quantum requirements is one of the most practically valuable skills for engineers entering quantum photonics.

11.8.1 Core Tightening Factors

The fundamental principle is that quantum systems require specifications N times tighter than classical:

Table 11.4: Classical to Quantum Specification Translation ($N = 4$)

Parameter	**Classical**	**Quantum**	**Rule**
RMS wavefront error	$\lambda/14$	$\lambda/56$	$\div N$
Surface figure (P-V)	$\lambda/4$	$\lambda/16$	$\div N$
AR coating reflectance	0.5%	0.125%	$\div N$
Beam splitter ratio	$50\% \pm 2\%$	$50\% \pm 0.5\%$	$\div N$
Path length match	1 μm	250 nm	$\div N$
Angular alignment	10 arcsec	2.5 arcsec	$\div N$
Temperature stability	± 1 K	± 0.25 K	$\div N$

The cost implications are severe. Quantum-grade components typically cost 3–100× more than classical equivalents.

11.8.2 Loss Budget Allocation

For N-photon systems, total efficiency η is the product of component efficiencies:

$$\eta = \eta_{\text{source}} \times \eta_{\text{coupling}} \times \eta_{\text{optics}} \times \eta_{\text{detector}} \tag{11.22}$$

Table 11.5: Loss Budget Allocation for $N = 4$ System

Component	**Efficiency**	**Loss (dB)**	**Technology**
SPDC source coupling	92%	0.36	PPKTP, fiber-coupled
Optical path (6 surfaces)	97%	0.13	Super-polish
Beam splitters	99%	0.04	Precision 50:50
Detector efficiency	95%	0.22	SNSPD at 2.5 K
Total	**85%**	**0.71**	Above 84% ✓

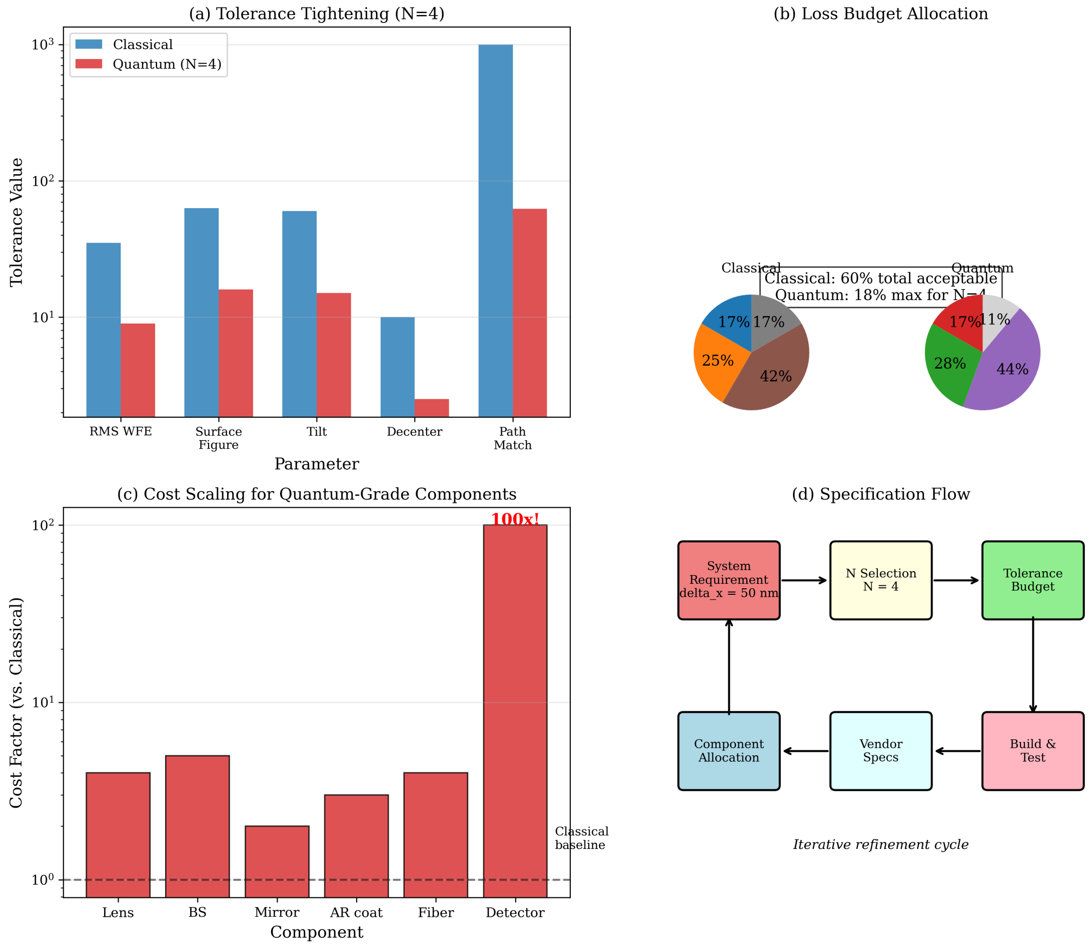

Figure 11.6: Specification translation from classical to quantum domains. (a) Tolerance scaling. (b) Loss budget comparison. (c) Cost scaling. (d) Specification flow.

Successful N-photon system design requires early loss budget planning with margin for integration losses.

11.9 Practical Engineering Considerations

This section addresses the hardware, workflow, and decision-making aspects of N-photon system implementation.

11.9.1 Hardware Chain

A complete N-photon imaging system requires:

1. **Pump laser:** High-stability, single-frequency source
2. **SPDC crystal:** BBO, PPKTP, or PPLN for photon pair generation
3. **State preparation:** Beam splitters, delay lines for NOON state creation
4. **Sample interface:** Quantum-grade objective with matched arms
5. **Detection:** SPADs or SNSPDs
6. **Coincidence electronics:** Sub-ns timing resolution

Table 11.6: NOON State Generation Rates and Integration Times

N	**Scheme**	**Rate**	**Time for 10^6 counts**
2	Single SPDC + HOM	10^4 Hz	100 seconds
4	Double SPDC	10 Hz	28 hours
6	Triple SPDC cascade	10^{-2} Hz	3.2 years
8	Cascaded selection	10^{-3} Hz	32 years

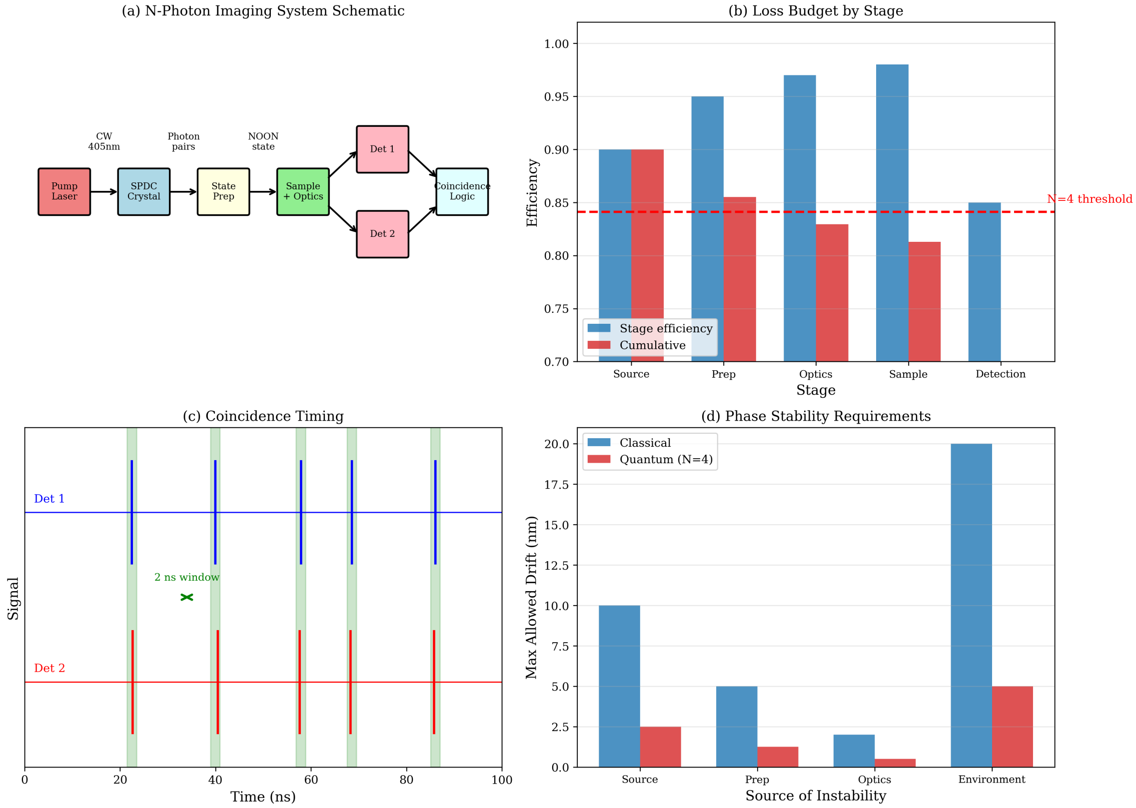

Figure 11.7: Hardware chain for N-photon quantum imaging. (a) System schematic. (b) Loss budget allocation. (c) Timing diagram. (d) Phase stability requirements.

11.9.2 When to Use Quantum vs. Classical

Table 11.7: Decision Matrix: Quantum vs. Classical Imaging

Factor	**Favor Classical**	**Favor Quantum**
Resolution requirement	> 100 nm	< 50 nm
Sample damage threshold	Low	High
Integration time budget	Minutes	Hours–Days
System efficiency	$< 80\%$	$> 85\%$
Cost sensitivity	High	Low
Application type	Production	Research

Quantum imaging is most appropriate for research applications requiring sub-50 nm resolution.

11.10 Comprehensive Practical Example: Two-Photon Lithography

Practical Example

Design Problem Statement
Requirement: Design a lithography system capable of writing 50 nm features in photoresist.
Constraints:

- Available wavelength: $\lambda = 405$ nm (UV diode laser)
- Maximum NA: 1.35 (oil immersion)
- Budget: Research-grade ($500k)

Questions:

1. Is classical lithography sufficient?
2. If quantum needed, what photon number N?
3. What component specifications required?
4. What is expected write time?

11.10.1 Step 1: Classical Feasibility Assessment

The classical (Rayleigh) resolution limit is:

$$\delta x_{\text{classical}} = \frac{0.61 \times 405 \text{ nm}}{1.35} = 183 \text{ nm} \tag{11.23}$$

Even with the highest practical NA, classical lithography achieves only 183 nm resolution—far from the 50 nm target.

Assessment: Classical approach **cannot meet requirement**. Quantum enhancement required.

11.10.2 Step 2: Quantum Design Selection

The required enhancement factor is:

$$E_{\text{required}} = \frac{\delta x_{\text{classical}}}{\delta x_{\text{target}}} = \frac{183 \text{ nm}}{50 \text{ nm}} = 3.66 \tag{11.24}$$

For Heisenberg-limited operation: $E_{\text{ideal}} = N$, therefore $N \geq 4$ is required ideally.
With realistic efficiency $\eta = 0.85$:

Table 11.8: Enhancement Factor vs. Photon Number at $\eta = 0.85$

N	η^N	E_{eff}	Resolution (nm)	Meets Target?
2	0.72	1.45	126	No
4	0.52	2.09	88	No
6	0.38	2.25	81	No

Revised Design: Use $N = 2$ with $\lambda = 266$ nm (frequency-doubled):

$$\delta x = \frac{0.61 \times 266 \text{ nm}}{1.35 \times E_{\text{eff}}} = \frac{162 \text{ nm}}{1.35 \times 1.45} = 48 \text{ nm} \checkmark \tag{11.25}$$

$N = 2$ at $\lambda = 266$ nm achieves 48 nm resolution.

11.10.3 Step 3: Component Specification

Loss budget allocation for $\eta \geq 0.707$:

Table 11.9: Component Specifications for $N = 2$, $\lambda = 266$ nm System

Component	**Spec**	**Efficiency**	**Notes**
SPDC source (BBO)	Type-I, 5 mm	90%	Phase-matched
Collection optics	AR < 0.2%	98%	UV fused silica
50:50 beam splitter	50% ± 0.5%	99.5%	Cube type
Objective lens	NA 1.35, UV	92%	$\lambda/28$ WFE
SNSPD detectors	266 nm QE	95%	Cooled to 2.5 K
Total		**79%**	Above 70.7% ✓

11.10.4 Step 4: Path Matching Requirements

The path matching requirement for $N = 2$:

$$\Delta L < \frac{\lambda}{N} = \frac{266 \text{ nm}}{2} = 133 \text{ nm} \tag{11.26}$$

Aberration matching requirement:

$$\Delta W < \frac{\lambda}{4N} = \frac{266 \text{ nm}}{8} = 33 \text{ nm RMS} \tag{11.27}$$

11.10.5 Step 5: Write Time Estimation

For 1 mm^2 area at 50 nm resolution:

$$\text{Pixels} = \left(\frac{1 \text{ mm}}{50 \text{ nm}}\right)^2 = 4 \times 10^8 \text{ pixels} \tag{11.28}$$

With 5×10^4 NOON states/second:

$$\text{Total time} = 4 \times 10^8 \times 20 \text{ ms} \approx 93 \text{ days} \tag{11.29}$$

Two-photon lithography at 50 nm is feasible but suitable only for small-area research patterns.

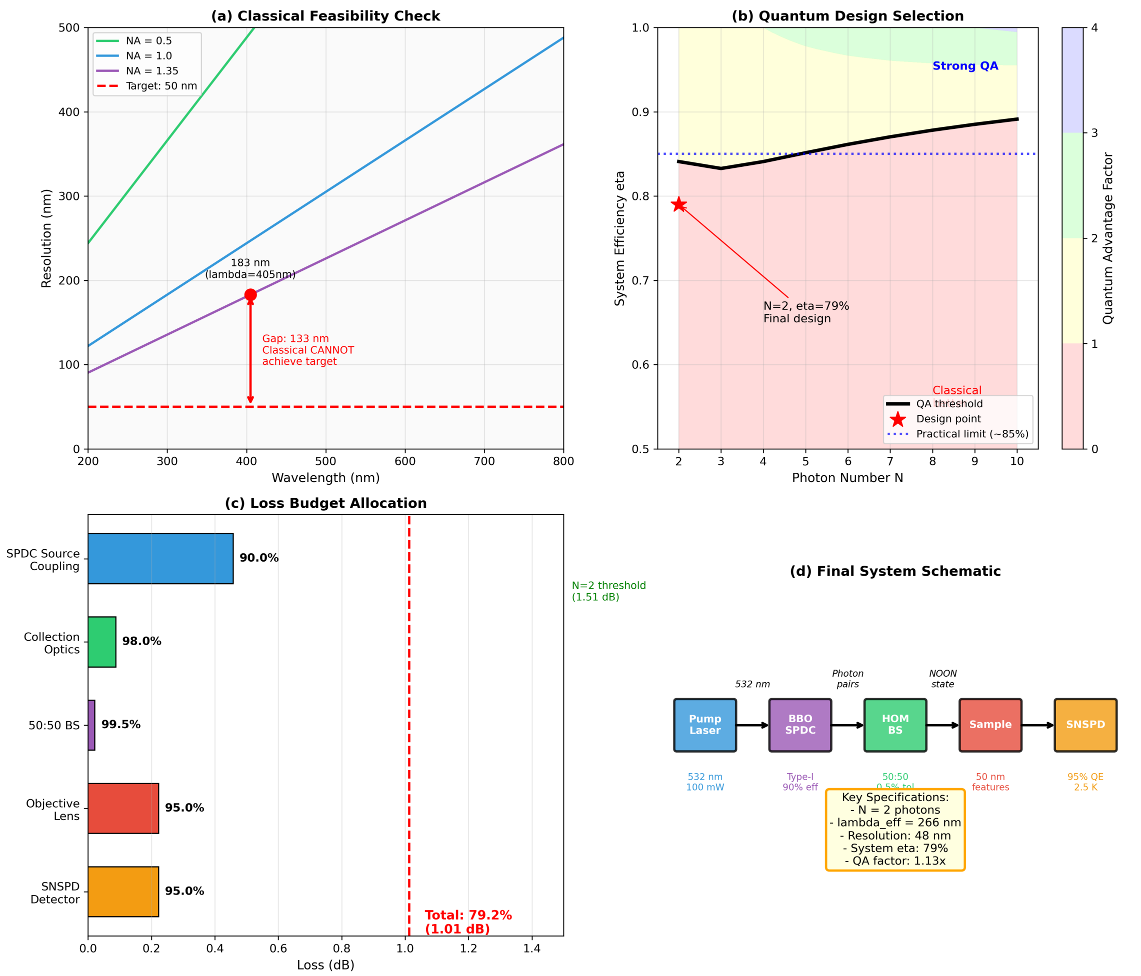

Figure 11.8: Practical workflow for the two-photon lithography design. (a) Classical feasibility check. (b) Quantum design selection. (c) Loss budget allocation. (d) Final system schematic.

11.11 Key Equations

Table 11.10: Key Equations for Chapter 11

Eq.	Expression	Name
(11.3)	$\delta x_{\text{Rayleigh}} = 0.61\lambda/\text{NA}$	Rayleigh limit
(11.4)	$\delta\phi_{\text{SQL}} = 1/\sqrt{N}$	Standard Quantum Limit
(11.5)	$\delta\phi_{\text{HL}} = 1/N$	Heisenberg Limit
(11.12)	$\phi_N = N \times (2\pi/\lambda)W(x, y)$	N-Photon Eikonal Bridge
(11.14)	$\lvert\text{NOON}\rangle = (\lvert N, 0\rangle + \lvert 0, N\rangle)/\sqrt{2}$	NOON state
(11.16)	$P_N = (1 + \cos N\phi)/2$	NOON interference
(11.17)	$P_{\text{survive}} = \eta^N$	Survival probability
(11.18)	$E_{\text{eff}} = \eta^N \times N$	Effective enhancement
(11.19)	$\eta > N^{-1/(2N)}$	QA threshold

Chapter Summary

Chapter 11 Key Points

1. **N-photon phase multiplication:** Entangled N-photon states acquire collective phase $\phi_N = N \cdot \phi_{\text{classical}}$, experiencing effective wavelength λ/N [1].
2. **Heisenberg resolution limit:** $\delta x = \lambda/(2N \cdot \text{NA})$ represents the ultimate resolution with N photons, offering $\sqrt{N}$ improvement over SQL.
3. **NOON states:** The canonical states $|\text{NOON}\rangle = (|N, 0\rangle + |0, N\rangle)/\sqrt{2}$ for Heisenberg-limited imaging.
4. **State fragility:** Coherence destroyed by any photon loss; system efficiency $> 80\%$ required for practical quantum advantage.
5. **Crossover condition:** Quantum beats classical when $\eta^N \times N > \sqrt{N}$, i.e., $\eta > N^{-1/(2N)}$.
6. **DEE extension:** Same JAX autodiff pipeline optimizes both classical and quantum systems—only the loss function changes.
7. **Specification translation:** Classical tolerances tighten by factor N for quantum operation.
8. **Practical sweet spot:** $N = 4$ offers best balance of enhancement ($4\times$) and achievable efficiency (84%).

Problems

Problem 11.1: Walther: Forward Analysis — Basic

A $N = 2$ photon imaging system operates at $\lambda = 800$ nm with a 0.5 NA objective. The system has total efficiency $\eta = 0.85$.
(a) Calculate the classical Rayleigh resolution.
(b) Calculate the two-photon Heisenberg resolution (ideal, no loss).
(c) Calculate the effective resolution accounting for loss.
(d) Determine the quantum advantage factor.
(e) Is quantum imaging beneficial for this system?

Solution Hints

(a) $\delta x_{\text{classical}} = 0.61 \times 800/0.5 = 976$ nm
(b) $\delta x_{\text{HL}} = 800/(2 \times 2 \times 0.5) = 400$ nm
(c) $E_{\text{eff}} = 0.85^2 \times 2 = 1.445$; $\delta x_{\text{eff}} = 800/(2 \times 1.445 \times 0.5) = 553$ nm
(d) $\text{QA} = 1.445/\sqrt{2} = 1.02$
(e) Marginally beneficial (QA > 1), but improvement is minimal.

Problem 11.2: Matsui-Nariai: Inverse Design — Lithography

Design an N-photon lithography system for writing 50 nm features.
(a) Specify the required N and NA combination (assume $\lambda = 500$ nm).
(b) Calculate the maximum tolerable optical loss per component (assume 6 surfaces).

(c) Determine the aberration matching requirement between arms.
(d) Estimate the exposure time for a 1 mm^2 area at 10^5 pair/s rate.

Solution Hints

(a) Classical: $\delta x = 0.61 \times 500/1.4 = 218$ nm. Need $218/50 = 4.36\times$ enhancement. Min $N = 5$.
(b) For $N = 4$, threshold $\eta > 0.84$. Per-surface: $(0.84)^{1/6} = 0.97$, i.e., 3% loss max.
(c) Path matching: $\Delta L < \lambda/N = 125$ nm. Aberration: $\Delta W < \lambda/4N = 31$ nm RMS.
(d) Pixels $= (10^6/50)^2 = 4 \times 10^8$. Time per pixel ≈ 10 ms. Total ≈ 46 days.

Problem 11.3: Walther: Crossover Analysis

A research lab has an imaging system with efficiency $\eta = 0.90$ and can produce NOON states with $N = 2, 4, 6$.
(a) Calculate the effective enhancement for each N value.
(b) Determine which values of N achieve quantum advantage.
(c) Identify the optimal choice of N.
(d) If efficiency drops to $\eta = 0.80$, which N values still work?

Solution Hints

(a) $N = 2$: $E = 0.81 \times 2 = 1.62$. $N = 4$: $E = 0.656 \times 4 = 2.62$. $N = 6$: $E = 0.531 \times 6 = 3.19$.
(b) Threshold: $E > \sqrt{N}$. $N = 2$: $1.62 > 1.41$ ✓. $N = 4$: $2.62 > 2$ ✓. $N = 6$: $3.19 > 2.45$ ✓. All achieve QA.
(c) $N = 6$ has highest E_{eff}, but $N = 4$ is practical sweet spot.
(d) At $\eta = 0.80$: $N = 2$: $1.28 > 1.41$? No. $N = 4$: $1.64 > 2$? No. None achieve QA at 80%.

Problem 11.4: JAX Implementation

Implement the N-photon DEE optimizer in JAX for a simple lens system.
(a) Write the forward model computing N-photon phase from wavefront.
(b) Implement the quantum loss function including survival probability.
(c) Use `jax.grad` to compute sensitivities.
(d) Verify that sensitivities scale by factor N.

Solution Hints

Key code structure:

```
@jit
def nphoton_phase(W, N, wavelength):
    k = 2 * jnp.pi / wavelength
    return N * k * W

@jit
def quantum_loss(params, N, eta, target):
    W = compute_wavefront(params)
    phi = nphoton_phase(W, N, wavelength)
    survival = eta**N
    phase_error = jnp.mean((phi - target)**2)
    return phase_error - 0.1*jnp.log(survival)
```

```
grad_fn = jax.grad(quantum_loss)
```

Verify: $\partial\phi_N/\partial p = N \times \partial\phi/\partial p$.

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