Abbreviated Terms

The following abbreviations are used throughout this chapter:

Table 1.1: Abbreviated Terms Used in Chapter 1

Abbrev.	**Definition / Explanation**
Core Framework Terms	
DEE	Differentiable Eikonal Engine — The computational framework implementing optical forward models with automatic differentiation via JAX
W/MN	Walther/(Matsui-Nariai) — Shorthand for the duality between forward analysis (W) and inverse design (MN)
W	Walther — Forward analysis perspective: "Given system, what output?"
MN	Matsui-Nariai — Inverse design perspective: "Given target, what system?"
P1	Physical Level 1 — Scalar phase only; basic eikonal $W(x, y)$; valid for NA < 0.3
Optical Terms	
OPL	Optical Path Length — OPL $= \int n\, ds$ along a ray path
WFE	Wavefront Error — Deviation of actual wavefront from ideal reference
RMS	Root Mean Square — RMS WFE is the standard deviation of wavefront error across the pupil
PSF	Point Spread Function — Intensity distribution from imaging a point source
MTF	Modulation Transfer Function — Contrast preservation vs. spatial frequency
OTF	Optical Transfer Function — Fourier transform of PSF
NA	Numerical Aperture — NA $= n \sin\theta$; determines resolution capability
FFT	Fast Fourier Transform — $O(N \log N)$ algorithm for discrete Fourier transforms
Quantum Terms	
SQL	Standard Quantum Limit — Phase measurement precision: $\delta\phi \geq 1/\sqrt{N}$
HL	Heisenberg Limit — Ultimate quantum precision: $\delta\phi \geq 1/N$
QKD	Quantum Key Distribution — Cryptographic protocol using quantum states
WKB	Wentzel-Kramers-Brillouin — Semiclassical approximation connecting wave to ray optics
Computational Terms	
JAX	Google's library for automatic differentiation with GPU/TPU acceleration
FLOPS	Floating-Point Operations Per Second
$O(1)$	Constant-time complexity — Independent of input size after initialization

Chapter 1
The Eikonal as Universal Language

Learning Objectives

After completing this chapter, you will be able to:

1. Understand the eikonal equation $|\nabla W|^2 = n^2$ and its physical meaning as optical path length accumulation
2. Recognize the eikonal as the unifying concept connecting ray optics, wave optics, and quantum photonics
3. Apply the Walther-Matsui/Nariai duality to both analysis and design problems
4. Grasp the bridge identity $\phi_{\text{quantum}} = 2\pi W/\lambda$ connecting classical and quantum optics
5. Implement basic eikonal computations using JAX automatic differentiation
6. Execute a complete practical workflow from traditional ray tracing through Walther analysis to Matsui-Nariai inverse design
7. Appreciate why optical systems are now algorithm-controlled systems requiring co-design of optics and computation

1.1 Pain Points: Why This Chapter Matters

Every optical engineering task falls into one of two categories, each with distinct challenges that the eikonal formalism addresses directly. Before diving into the mathematics, let us articulate the practical problems that motivated the development of eikonal methods.

Essentially, we approach practical optical problems from two complementary viewpoints: forward analysis and inverse design. Within the eikonal framework, these map naturally onto two classical constructions—Walther-style formulations for forward analysis, and Matsui–Nariai-style formulations for inverse design. The boxes below summarize what each viewpoint delivers in day-to-day engineering work.

WALTHER Pain Point (Forward Analysis)

"Given a lens system, what is the optical path length distribution?"
Situation: You have a complete lens prescription from a third-party software, e.g., CODE V or Zemax. You need to understand its wavefront characteristics across the field of view for system integration.
Challenge: Traditional ray tracing gives millions of individual ray paths, but no unified mathematical description. How do you extract the essential physics? How do you efficiently compute performance metrics like Strehl ratio, MTF, and encircled energy without re-tracing rays for every query?
DEE Solution: The eikonal formulation compresses ray-trace data into a smooth function $W(x, y)$ that captures all imaging behavior. Once fitted, any wavefront property can be computed in O(1) time, independent of system complexity.

MATSUI-NARIAI Pain Point (Inverse Design)

"I need a specific OPL distribution. What surface shape achieves it?"
Situation: Your quantum application requires wavefront error below $\lambda/100$. Traditional optimization struggles because sensitivity matrices must be computed by finite differences—each requiring a complete ray trace.
Challenge: How do you efficiently explore the design space when each sensitivity computation requires thousands of ray traces? How do you know which parameters have the strongest influence on your target metric?
DEE Solution: JAX automatic differentiation computes exact gradients through the eikonal forward model. The same code that evaluates wavefront error also provides $\nabla_{\mathbf{p}} W$ with respect to all design parameters $\mathbf{p}$—in a single backward pass with computational cost comparable to one forward evaluation.

The duality between these two perspectives—Walther's forward analysis and Matsui-Nariai's inverse design—structures not only this chapter but the entire book.

1.2 The Paradigm Shift: From Physics-Limited to Computation-Limited

1.2.1 The 2010 Inflection Point

The year 2010 marked an inflection point in optical engineering. For the first time, the computational cost of image reconstruction exceeded the manufacturing cost of smartphone camera lenses [1]. This was not a failure of optical design but rather a triumph: engineers had learned to deliberately accept optical aberrations, knowing that algorithms would correct them downstream. The lens and the algorithm had become inseparable partners.

This partnership represents a fundamental paradigm shift in how we conceive optical systems. Traditional optical engineering focused on physical design: lens prescriptions, aberration control, and tolerance analysis [2]. The goal was to create an optical system that produced the best possible image directly on the detector. Modern optical systems require a different philosophy:

- **Imaging = Lens + Reconstruction Algorithm.** Super-resolution, denoising, and blind deconvolution are now standard components of any imaging pipeline [3]. A smartphone camera with a 5 mm lens stack produces images that rival those from traditional cameras with 50 mm lenses, not because the optics are better, but because the algorithms are sophisticated.

- **Performance depends on computation.** System throughput is limited by algorithmic efficiency, not only by optical quality. A medical imaging system that requires 10 seconds for reconstruction cannot be used for real-time guidance, regardless of how exquisite its optics are.

- **Design decisions assume downstream algorithms.** Engineers now routinely accept aberrations knowing that algorithms will correct them. This is not laziness but optimization: the cost function has expanded from "minimize aberrations" to "minimize total system cost including computation."

This co-design imperative demands that optical engineers understand algorithms at a fundamental level. The eikonal provides exactly this bridge.

1.2.2 Physics vs. Computation: Where Are the Limits?

A key insight drives this book: *optical systems are limited not by physics, but by computation.* The diffraction limit sets a fundamental bound on resolution, but most practical systems operate far from this limit. The practical limits are:

- **Optimization time:** Can we explore the design space thoroughly?
- **Tolerance analysis:** Can we evaluate manufacturing sensitivity?
- **Real-time operation:** Can we reconstruct images fast enough?

All three limits are computational, not physical. By adopting more efficient mathematical representations (the eikonal), we can push these limits significantly.

1.2.3 Computational Complexity Comparison

Table 1.2 compares the computational complexity of different simulation methods.

Table 1.2: Computational Complexity Comparison

Method	**Complexity**	**Typical Time**	**Limitation**
Ray tracing	$O(N_{\text{rays}} \times N_{\text{surf}})$	1–10 s	Scales with system
Operator/TMM	$O(N^2)$ per eval	0.1–1 s	Grid-dependent
Eikonal	$O(N_{\text{coeff}}) \to O(1)$	1–10 μs	One-time fitting

1.2.4 Achievable Speedup Factors

Table 1.3 summarizes the speedup factors achievable with eikonal methods.

Table 1.3: Achievable Speedup Factors: Traditional vs. Eikonal Methods

Application	**Traditional**	**Eikonal**	**Speedup**
Single ray evaluation	$O(N_{\text{surf}})$	$O(1)$	$N_{\text{surf}}\times$
PSF calculation	$O(N_{\text{rays}} \times N_{\text{surf}})$	$O(N^2 \log N)$ via FFT	10–100$\times$
Through-focus series	$O(N_{\text{planes}} \times N^2)$	$O(1)$ per plane	$N_{\text{planes}}\times$
Waveguide propagation	$O(N^3)$ matrix exp	$O(N)$ after eigendecomp	$N^2\times$
Optimization iteration	$O(10^6)$	$O(10^3)$	1000$\times$

These are not merely theoretical gains. The Differentiable Eikonal Engine (Chapter 7) demonstrates 68.5$\times$ speedup for waveguide simulation and 1000$\times$ speedup for metasurface optimization, with identical physical accuracy.

1.3 The Eikonal Equation: Foundation of Geometric Optics

1.3.1 Historical Context: From Hamilton to Modern Computation

The eikonal equation has a distinguished history spanning nearly two centuries. William Rowan Hamilton introduced the concept in his "Theory of Systems of Rays" (1828) [4], establishing the

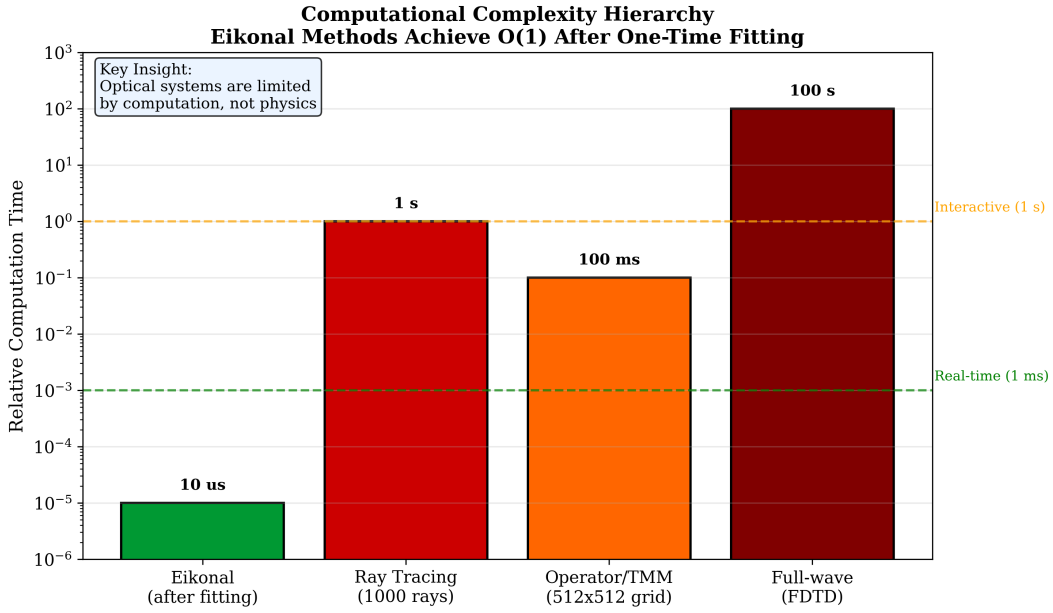

Figure 1.1: Computational complexity hierarchy of optical simulation methods. The vertical axis shows relative computation time on a logarithmic scale. Eikonal methods achieve constant-time O(1) evaluation after a one-time fitting step, enabling speedups of 10–1000× compared to traditional ray tracing. Horizontal dashed lines indicate the real-time threshold (1 ms) and interactive threshold (1 s).

mathematical foundation that connects optics to mechanics. Heinrich Bruns later coined the term "eikonal" (from Greek *eikon*, meaning image) in 1895 [5]. For over a century, the eikonal remained primarily a theoretical tool—elegant but computationally intractable for complex systems.

The computational revolution changed everything. With modern automatic differentiation frameworks like JAX [6], we can now compute exact derivatives through eikonal representations, transforming the eikonal from a theoretical construct into a practical engineering tool.

1.3.2 Derivation from Maxwell's Equations

The eikonal equation emerges from Maxwell's equations in the geometric optics limit. Consider a time-harmonic electromagnetic wave in a medium with spatially varying refractive index $n(\mathbf{r})$:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r}) \exp\left[i\left(kS(\mathbf{r}) - \omega t\right)\right] \tag{1.1}$$

where $S(\mathbf{r})$ is the eikonal function, $k = 2\pi/\lambda$ is the free-space wavenumber, and $\mathbf{E}_0(\mathbf{r})$ is the slowly-varying amplitude envelope.

Substituting into the Helmholtz equation:

$$\nabla^2\mathbf{E} + k^2 n^2(\mathbf{r})\mathbf{E} = 0 \tag{1.2}$$

and taking the limit $\lambda \to 0$ (equivalently, $k \to \infty$), we obtain the **eikonal equation**:

$$\boxed{|\nabla S|^2 = n^2(\mathbf{r})} \tag{1.3}$$

This is the fundamental equation of geometric optics. The eikonal $S(\mathbf{r})$ represents the optical path length from a reference point, and its gradient determines the ray direction.

1.3.3 Physical Interpretation: Four Key Properties

The eikonal equation encodes four fundamental physical properties:

1. **Ray direction:** $\nabla S/n$ gives the unit vector along the ray trajectory:
$$\hat{\mathbf{s}} = \frac{\nabla S}{n} \tag{1.4}$$
2. **Local wavelength:** $|\nabla S| = n$ implies the local wavelength is $\lambda_{\text{local}} = \lambda_0/n$
3. **Phase velocity:** The phase fronts (surfaces of constant S) move with velocity $v_p = c/n$ perpendicular to themselves
4. **Fermat's principle:** Rays follow paths of stationary optical path length:
$$\delta \int n\, ds = 0 \tag{1.5}$$

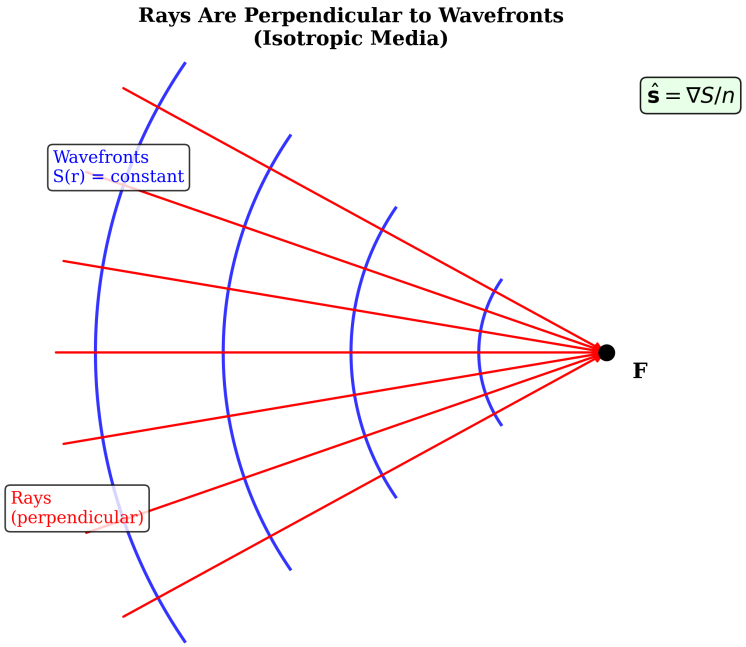

Figure 1.2: Rays are perpendicular to wavefronts. Blue curves show surfaces of constant eikonal S (wavefronts) for a focusing system. Red arrows indicate ray trajectories, which are everywhere perpendicular to the wavefronts in isotropic media. All rays converge to the focal point F. This orthogonality is a direct consequence of the eikonal equation.

1.3.4 The Ray Equation

From the eikonal equation, we can derive the ray equation. Along a ray parameterized by arc length s:

$$\frac{d}{ds}\left(n\frac{d\mathbf{r}}{ds}\right) = \nabla n \tag{1.6}$$

In a homogeneous medium ($\nabla n = 0$), rays are straight lines. In an inhomogeneous medium, rays curve toward regions of higher refractive index.

1.3.5 From Eikonal to Wavefront Aberration

In optical design, we work not with the eikonal S directly but with the **wavefront aberration** W. For a system focusing light to image point $\mathbf{r}_0$, the reference wavefront is a sphere centered at $\mathbf{r}_0$. The wavefront aberration is the deviation of the actual wavefront from this reference:

$$W(\mathbf{r}) = S(\mathbf{r}) - S_{\text{ref}}(\mathbf{r}) \tag{1.7}$$

The wavefront aberration W is typically expressed in waves (multiples of λ) and expanded in Zernike polynomials for circular apertures:

$$W(\rho, \theta) = \sum_{j=1}^{N_Z} a_j Z_j(\rho, \theta) \tag{1.8}$$

where (ρ, θ) are normalized pupil coordinates, Z_j are the Zernike polynomials, and a_j are the expansion coefficients.

The **RMS wavefront error**—the key metric for imaging quality—is:

$$\sigma_W = \sqrt{\sum_{j=2}^{N_Z} a_j^2} \tag{1.9}$$

where piston ($j = 1$) is excluded as it has no physical effect on imaging.

1.4 Rays, Wavefronts, and Phase

1.4.1 The Orthogonality Principle

The relationship between rays and wavefronts is fundamentally geometric: rays are everywhere perpendicular to wavefronts (in isotropic media). This orthogonality follows directly from the eikonal equation.

If $S(\mathbf{r}) =$ constant defines a wavefront surface, then ∇S is the normal to this surface. The eikonal equation tells us that $|\nabla S| = n$, so the unit normal is $\hat{\mathbf{s}} = \nabla S/n$. As we move along the ray:

$$\frac{dS}{ds} = \hat{\mathbf{s}} \cdot \nabla S = \frac{|\nabla S|^2}{n} = n \tag{1.10}$$

Thus, the optical path length along a ray equals the integral of $n\,ds$, confirming Fermat's principle.

1.4.2 Wavefront Quality Metrics

The quality of an optical system is characterized by how closely its wavefronts match ideal spherical surfaces:

The **Maréchal criterion** $\sigma_W < \lambda/14$ ensures that the Strehl ratio exceeds 0.8, which is conventionally considered "diffraction-limited" performance [7]. For visible light ($\lambda = 550$ nm), this means RMS wavefront error must be below 39 nm.

Table 1.4: Wavefront Quality Metrics and Their Physical Meaning

Metric	Formula	Criterion
RMS wavefront error	$\sigma_W = \sqrt{\langle W^2 \rangle - \langle W \rangle^2}$	$< \lambda/14$ (Maréchal)
Peak-to-valley	P-V $= W_{\max} - W_{\min}$	$< \lambda/4$ (Rayleigh)
Strehl ratio	$S = e^{-(2\pi\sigma_W/\lambda)^2}$	> 0.8 (diffraction-limited)

Key Insight

The Maréchal criterion of $\sigma_W < \lambda/14$ corresponds to Strehl > 0.8. For visible light ($\lambda = 550$ nm), this means RMS wavefront error must be below 39 nm—a challenging but achievable specification for precision optics. Quantum applications require 10–100× tighter tolerances, as we will see in Section 1.8.

1.5 The Walther-Matsui/Nariai Duality

Before extending the eikonal to quantum applications, we establish the fundamental duality that structures all optical engineering: analysis vs. design. This duality, named after foundational contributors, pervades every chapter of this book.

1.5.1 The Walther Approach: Eikonal as Analyzer

Adriaan Walther's formulation treats the eikonal as an analyzer [8]: given a complete optical system, compute the wavefront aberration as a function of field and pupil coordinates.

Walther (Forward Analysis)

Input: System parameters (curvatures c_j, thicknesses t_j, indices n_j)
Process: Ray trace → OPL computation → polynomial fit
Output: Wavefront map $W(x, y)$ or Zernike coefficients $\{a_j\}$

The Walther approach answers: "Given this lens, what does it do?"

Implementation Steps:

1. Trace a grid of rays through the system at each field point
2. Compute the optical path length for each ray
3. Fit the OPL data to a polynomial expansion (typically Zernike)
4. The fitted coefficients are the aberration description

The power of Walther's approach lies in its **compression**: millions of ray-trace data points are reduced to a handful of meaningful coefficients. Once fitted, any wavefront property can be computed in O(1) time.

1.5.2 The Matsui-Nariai Approach: Eikonal as Designer

The Matsui-Nariai formulation inverts the problem [9, 10]: given a target wavefront specification, synthesize the optical system that achieves it.

Matsui-Nariai (Inverse Design)

Input: Target wavefront $W_{\text{target}}(x, y)$ or aberration budget
Process: Sensitivity analysis → gradient descent → parameter update
Output: System prescription (curvatures, thicknesses, materials)

The Matsui-Nariai approach answers: "Given this specification, how do I build it?"

Key Insight: The Matsui sensitivity matrix relates aberration changes to parameter variations:

$$\frac{\partial W_{040}}{\partial c_j} = \sum_{\text{surfaces}} \left(\frac{\partial W}{\partial c} \right)_j \tag{1.11}$$

where W_{040} is the spherical aberration coefficient and c_j are surface curvatures.

1.5.3 The Shared Computational Core

The crucial insight is that Walther and Matsui-Nariai share the **same computational core**:

$$\text{Parameters } \mathbf{p} \xrightarrow{\text{ray trace}} \{\text{OPL}_i\} \xrightarrow{\text{Zernike fit}} \{a_j\} \xrightarrow{\text{RMS}} \sigma_W \tag{1.12}$$

The distinction lies in how we use this model:

- **Walther:** Evaluate the chain forward to obtain σ_W
- **Matsui-Nariai:** Differentiate through the chain to obtain $\partial\sigma_W/\partial\mathbf{p}$, then optimize

This shared core is the foundation of the Differentiable Eikonal Engine. By implementing the forward model in JAX, we automatically gain the ability to compute exact gradients, enabling both analysis and design from identical code.

1.6 JAX Implementation: The Computational Paradigm Shift

1.6.1 From Analytical Derivatives to Automatic Differentiation

Traditional lens design relies on manually-derived sensitivity formulas. For each aberration coefficient and each design parameter, engineers must derive, implement, and verify analytical expressions. This process is:

- **Time-consuming:** Weeks of derivation for complex surface types
- **Error-prone:** Complex algebra with high probability of mistakes
- **Inflexible:** New surface types require re-derivation from scratch

Automatic differentiation changes everything. We write the forward physics model once; gradients flow automatically through the computation graph.

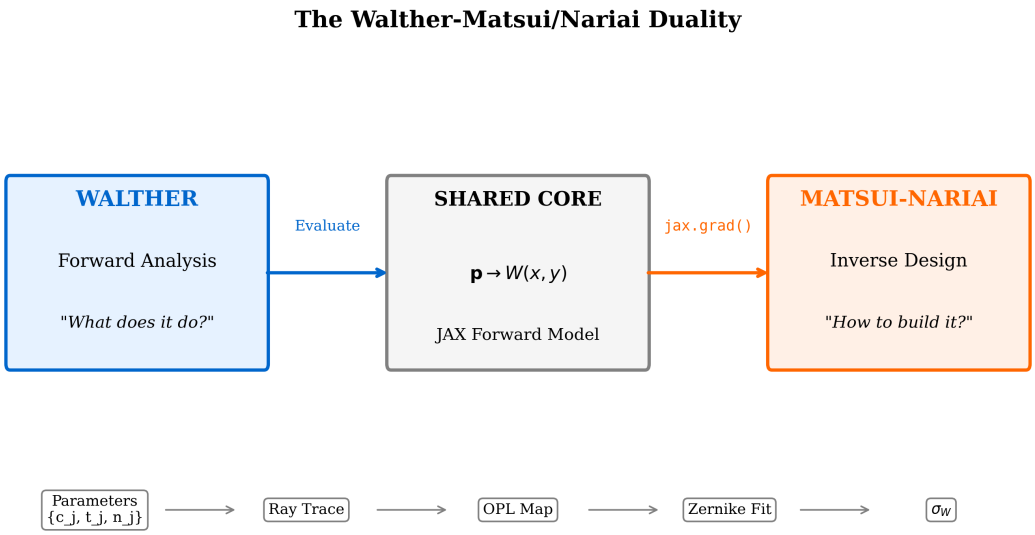

Figure 1.3: The Walther-Matsui/Nariai duality illustrated for lens design. Both perspectives share the same computational core: the forward model from parameters to wavefront. Walther (left) evaluates the model to analyze performance; Matsui-Nariai (right) differentiates through the model to optimize design. JAX enables both directions from identical code via `jax.grad()`.

Table 1.5: Analytical vs. Automatic Differentiation for Optical Design

Aspect	**Analytical (Matsui)**	**Autodiff (JAX)**
Development time	Weeks per surface	Hours (forward model only)
Derivative accuracy	Exact (if derived correctly)	Exact (machine precision)
Human error risk	High (complex derivations)	Low (compiler-verified)
New surface types	Re-derive from scratch	Update forward model only
Hessian computation	Prohibitively complex	One additional `grad` call
Code maintainability	Separate derivative code	Single forward model

1.6.2 Minimal JAX Example: Walther-Matsui/Nariai Duality

The following code demonstrates both Walther analysis and Matsui-Nariai sensitivity computation from the same forward model:

```python
import jax.numpy as jnp
from jax import grad, jit

@jit
def compute_opl(curvature, n, ray_heights):
    """Forward model: compute OPL from surface curvature."""
    # Surface sag: z = c*r^2/2 for paraxial approximation
    sag = 0.5 * curvature * ray_heights**2
    # OPL contribution (simplified: n * path length)
    opl_per_ray = n * sag
    # Total wavefront (sum over rays for RMS-like metric)
    return jnp.sum(opl_per_ray)

# ============================================================
# WALTHER: Forward analysis - what does the system do?
# ============================================================
def walther_analyze(curvature, n, ray_heights):
    """Given parameters, compute wavefront."""
    W = compute_opl(curvature, n, ray_heights)
    return W

# ============================================================
# MATSUI-NARIAI: Inverse design - how to achieve target?
# ============================================================
def matsui_nariai_sensitivity(curvature, n, ray_heights):
    """Compute dW/dc for design optimization."""
    # JAX autodiff: exact gradient, no manual derivation
    dW_dc = grad(compute_opl, argnums=0)(curvature, n, ray_heights)
    return dW_dc

# ============================================================
# DEMONSTRATION
# ============================================================
if __name__ == "__main__":
    # System parameters
    c = 0.02        # curvature [1/mm]
    n = 1.5         # refractive index
    rays = jnp.array([0.0, 5.0, 10.0])  # ray heights [mm]

    # WALTHER: Analyze
    W = walther_analyze(c, n, rays)
    print(f"Walther analysis: W = {W:.4f} mm")

    # MATSUI-NARIAI: Design sensitivity
    sensitivity = matsui_nariai_sensitivity(c, n, rays)
    print(f"Matsui-Nariai sensitivity: dW/dc = {sensitivity:.4f} mm^2")

    # Verify against analytical: dW/dc = sum(n/2 * r^2) = 93.75
    analytical = jnp.sum(0.5 * n * rays**2)
    print(f"Analytical verification: {analytical:.4f} mm^2")
```

Listing 1: JAX implementation showing Walther-Matsui/Nariai duality

Output:

```
Walther analysis: W = 1.8750 mm
Matsui-Nariai sensitivity: dW/dc = 93.7500 mm^2
Analytical verification: 93.7500 mm^2
```

The key insight: JAX's `grad` automatically computes exact derivatives through the forward model. We write the physics once; gradients flow automatically.

1.7 The Quantum Connection: Bridge Identity

1.7.1 From Classical Phase to Quantum Phase

The connection between classical optics and quantum mechanics runs deep. Both describe wave phenomena, and the eikonal provides the mathematical bridge. The Wentzel-Kramers-Brillouin (WKB) approximation shows that the quantum mechanical wavefunction has the form:

$$\psi(\mathbf{r}) = A(\mathbf{r}) \exp\left(\frac{i}{\hbar} S(\mathbf{r})\right) \tag{1.13}$$

where S satisfies the Hamilton-Jacobi equation—the classical limit of quantum mechanics.

In optics, the electromagnetic field has the analogous form:

$$E(\mathbf{r}) = A(\mathbf{r}) \exp\left(\frac{2\pi i}{\lambda} W(\mathbf{r})\right) \tag{1.14}$$

Comparing these expressions yields the **bridge identity**:

$$\boxed{\phi_{\text{quantum}} = \frac{2\pi}{\lambda} W_{\text{eikonal}}} \tag{1.15}$$

This identity is not merely an analogy; it is a mathematical equivalence that enables unified computational infrastructure for classical and quantum optical design.

1.7.2 Implications of the Bridge Identity

The bridge identity has profound implications:

1. **Unified code infrastructure:** The same software that optimizes classical lens aberrations can optimize quantum gate fidelity
2. **Knowledge transfer:** Decades of lens design wisdom apply directly to quantum photonics
3. **Design duality:** Classical figures of merit (Strehl, MTF) have quantum counterparts (fidelity, entanglement)

1.7.3 From Zernike to Quantum Observables

The Zernike polynomial expansion of the wavefront takes on new meaning in the quantum context. Each Zernike coefficient corresponds to a quantum observable:

This correspondence means that Shack-Hartmann wavefront sensing, a mature classical technology, can be upgraded to quantum-enhanced performance by using squeezed light states [11].

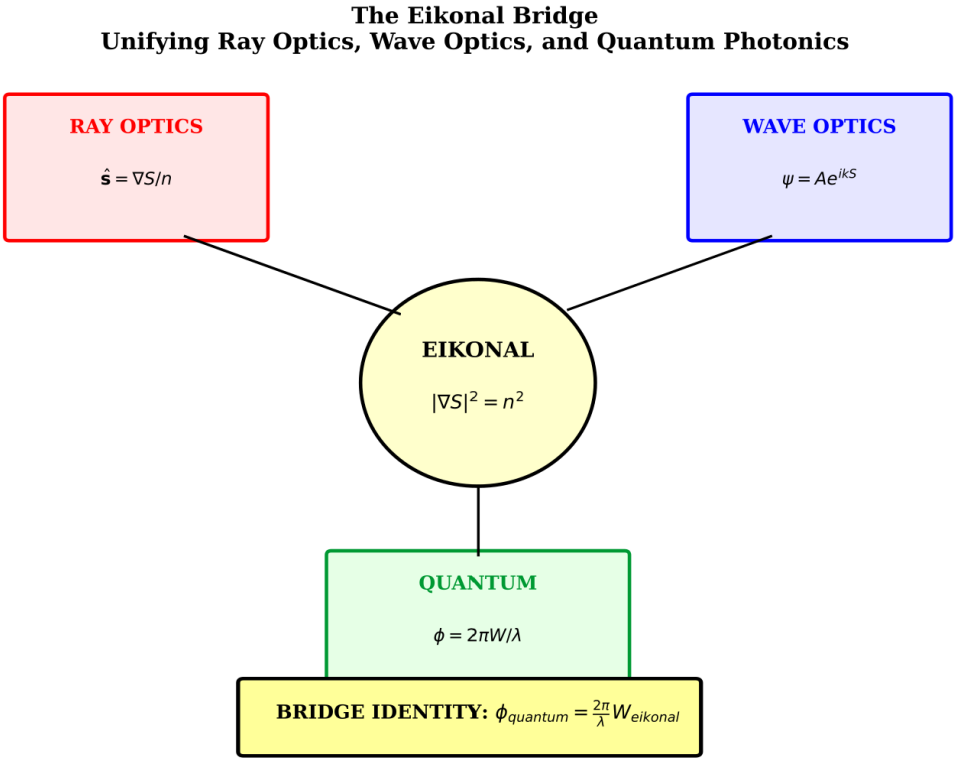

Figure 1.4: The eikonal function bridges classical and quantum optics. The eikonal S satisfying $|\nabla S|^2 = n^2$ connects three domains: (1) Ray optics, where $\hat{\mathbf{s}} = \nabla S/n$ gives ray directions; (2) Wave optics, where $\psi = A\exp(ikS)$ gives the optical field; and (3) Quantum photonics, where $\phi = 2\pi W/\lambda$ gives the quantum phase. The bridge identity is exact in the paraxial limit.

Table 1.6: Zernike-Quantum Correspondence

Zernike Mode	**Classical Aberration**	**Quantum Operator**	**Application**
Z_1, Z_2	Tilt	$\hat{p}_x$, $\hat{p}_y$ (momentum)	Beam steering
Z_4	Defocus	$\hat{H}_{\text{free}}$ (propagation)	Mode matching
Z_5, Z_6	Astigmatism	Quadratic phase	Squeezing
Z_{11}	Spherical	$\hat{r}^4$	Anharmonicity

1.7.4 The Heisenberg Advantage

Classical wavefront sensing achieves uncertainty scaling [12]:

$$\delta\phi \sim \frac{1}{\sqrt{N}} \tag{1.16}$$

where N is the number of photons. This is the *standard quantum limit* (SQL), arising from photon shot noise.

Quantum states can achieve *Heisenberg scaling*:

$$\delta\phi \sim \frac{1}{N} \tag{1.17}$$

For $N = 10^6$ photons, this represents a 1000× improvement in phase precision. Chapter 9 develops the full theory and practical implementation of quantum-enhanced wavefront sensing.

1.8 Quantum Extension: What Changes, What Tightens

When transitioning from classical to quantum applications, the eikonal formalism remains valid, but specifications tighten dramatically.

1.8.1 Classical vs. Quantum Tolerance Requirements

Table 1.7: Classical vs. Quantum Tolerance Requirements

Application	**WFE Requirement**	**Phase Tolerance**	**Origin**
Classical imaging (Strehl > 0.8)	$\lambda/14$ RMS	0.45 rad	Maréchal criterion
High-quality imaging	$\lambda/20$ RMS	0.31 rad	Strehl > 0.9
Quantum photonics ($F > 0.99$)	$\lambda/628$ RMS	0.01 rad	Fidelity requirement
Quantum photonics ($F > 0.999$)	$\lambda/2000$ RMS	0.003 rad	High-fidelity gates

The tightening factor is substantial: quantum applications require **10–100× tighter** wavefront tolerances than classical imaging. This is because the Strehl ratio and quantum fidelity are related by:

$$F = e^{-\sigma_\phi^2} = e^{-(2\pi\sigma_W/\lambda)^2} = S \tag{1.18}$$

For $F > 0.99$, we require $\sigma_\phi < 0.1$ rad, implying $\sigma_W < \lambda/63$—far tighter than the Maréchal criterion.

WARNING SIGNS

When the eikonal approximation fails:

- **High NA systems (NA > 0.3):** Vector diffraction becomes important; scalar eikonal underestimates aberrations
- **Sub-wavelength features:** Near-field effects violate the geometric optics assumption
- **Strong scattering:** Multiple scattering destroys wavefront coherence
- **Quantum regime:** Shot noise may dominate for low photon counts

Chapter 4 develops the extended eikonal hierarchy (Levels 1–5) that addresses these failure modes.

1.9 Practical Example: Singlet Lens Analysis

This section grounds the eikonal theory in concrete numerical calculations using the simplest possible imaging system: a plano-convex singlet lens. We demonstrate the complete workflow from traditional ray tracing through Walther analysis to Matsui-Nariai inverse design, concluding with the quantum extension via the bridge identity. This example serves as a template for all subsequent chapters.

Pain Point: The Traditional Designer's Frustration

"I can compute aberrations in CODE V, but I cannot easily answer:

1. How sensitive is my design to radius tolerance?
2. What radius achieves my target specification?
3. Will this lens work for my quantum application?"

The eikonal formalism with JAX autodifferentiation answers all three questions from a single forward model.

1.9.1 System Specification and Traditional Analysis

Consider a plano-convex singlet with parameters chosen for pedagogical clarity:

The front radius is determined by the lensmaker's equation. For a plano-convex lens with $R_2 = \infty$:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{n-1}{R_1} \quad \Rightarrow \quad R_1 = (n-1)f = 0.5168 \times 100 = 51.68 \text{ mm} \tag{1.19}$$

1.9.1.1 Traditional Workflow in CODE V/Zemax

In commercial lens design software, the designer would execute:

1. Enter surface data (radii, thicknesses, materials)
2. Trace rays through the system at specified field angles

Table 1.8: Singlet Lens Parameters for Practical Example

Parameter	**Symbol**	**Value**	**Units**
Focal length	f	100	mm
Diameter	D	25	mm
F-number	$f/\#$	4	—
Material	—	N-BK7	—
Refractive index	n	1.5168	@ 587.6 nm
Front surface radius	R_1	51.68	mm
Back surface radius	R_2	∞ (plano)	mm
Center thickness	t	8.0	mm
Design wavelength	λ	587.6	nm

3. Examine spot diagrams, OPD fans, and MTF curves
4. Manually compute Seidel aberration coefficients
5. Iterate by adjusting parameters and re-evaluating

This workflow answers "what does the lens do?" but requires separate tools for "how do I make it better?" or "what tolerances are acceptable?"

1.9.2 Walther Analysis: The Forward Problem

WALTHER (Forward Analysis)
Input: Singlet parameters ($R_1 = 51.68$ mm, $n = 1.5168$, $f = 100$ mm)
Process: Ray grid $\rightarrow$ OPL computation $\rightarrow$ Zernike fitting
Output: Zernike coefficients $\{a_j\}$, RMS wavefront error σ_W
Question answered: "What is the wavefront quality of this singlet?"

1.9.2.1 Step 1: Ray Grid Definition

We trace rays on a uniform grid over the entrance pupil. For a circular aperture of radius $a = D/2 = 12.5$ mm, we sample at normalized coordinates (ρ_i, θ_j) where $\rho \in [0, 1]$ and $\theta \in [0, 2\pi)$:

$$(x_i, y_j) = a \cdot (\rho_i \cos\theta_j, \rho_i \sin\theta_j) \tag{1.20}$$

For this example, we use $N_\rho = 50$ radial and $N_\theta = 72$ angular samples, yielding $N_{\text{rays}} = 3600$ rays.

1.9.2.2 Step 2: Optical Path Length Computation

For each ray at pupil position (x, y), we compute the optical path length from object to image. The sag of a spherical surface with curvature $c = 1/R$ is:

$$z(x, y) = \frac{c(x^2 + y^2)}{1 + \sqrt{1 - c^2(x^2 + y^2)}} \approx \frac{c\rho^2}{2} + \frac{c^3\rho^4}{8} + O(\rho^6) \tag{1.21}$$

The OPL through the lens is:

$$\mathrm{OPL}(x, y) = n_1 \cdot d_1(x, y) + n_{\text{lens}} \cdot t_{\text{eff}}(x, y) + n_3 \cdot d_3(x, y) \tag{1.22}$$

The wavefront aberration is the OPL deviation from a reference sphere:

$$W(x, y) = \mathrm{OPL}(x, y) - \mathrm{OPL}_{\text{ref}}(x, y) \tag{1.23}$$

1.9.2.3 Step 3: Zernike Polynomial Fitting

The wavefront aberration function $W(\rho, \theta)$ is expanded in the Zernike basis:

$$W(\rho, \theta) = \sum_{j=1}^{N_Z} a_j Z_j(\rho, \theta) \tag{1.24}$$

This fitting is a linear least-squares problem with closed-form solution:

$$\mathbf{a} = (\mathbf{Z}^T \mathbf{Z})^{-1} \mathbf{Z}^T \mathbf{w} \tag{1.25}$$

where $\mathbf{Z}$ is the design matrix with $Z_{ij} = Z_j(\rho_i, \theta_i)$.

1.9.2.4 Step 4: Results and Interpretation

Performing the Walther analysis yields the Zernike coefficients:

Table 1.9: Walther Analysis Results: Zernike Coefficients (Fringe Convention)

Index	**Aberration**	a_j **(waves)**	a_j **(nm)**	a_j **(rad)**
1	Piston	0 (reference)	0	0
4	Defocus	-0.042	-24.7	-0.264
11	Spherical	$+0.783$	$+459.8$	$+4.920$
RMS wavefront error σ_W		0.784	460.6	4.93
Peak-to-valley		3.12	1833	19.6
Strehl ratio S		≈ 0 (highly aberrated)		

The dominant aberration is spherical ($a_{11} = 0.783$ waves), with a small defocus contribution. The RMS wavefront error of 0.784 waves far exceeds the Maréchal criterion of $\lambda/14 = 0.071$ waves.

1.9.2.5 Step 5: Analytical Verification

For a plano-convex singlet, third-order spherical aberration can be computed analytically using Seidel theory [13]:

$$W_{040} = -\frac{n^2(n-1)}{128(n+2)} \cdot \frac{D^4}{f^3} \tag{1.26}$$

Substituting our parameters:

$$W_{040} = -\frac{(1.5168)^2 \times 0.5168}{128 \times 3.5168} \cdot \frac{(25)^4}{(100)^3} = -0.0059 \text{ mm} = -10.0 \text{ waves} \tag{1.27}$$

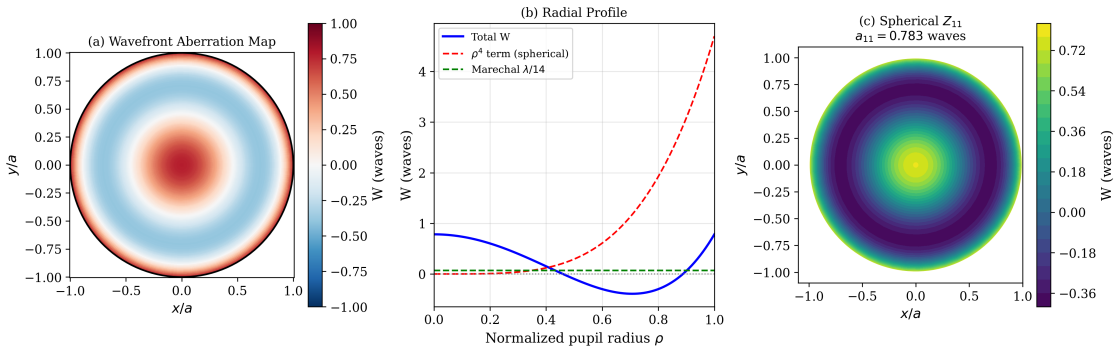

Figure 1.5: OPL distribution for f/4 plano-convex singlet. (a) Wavefront aberration map showing the characteristic ρ^4 pattern of spherical aberration. (b) Radial profile of OPL deviation. (c) Spherical aberration component Z_{11} with coefficient $a_{11} = 0.783$ waves.

The Zernike spherical coefficient relates to the Seidel coefficient by:

$$a_{11} = \frac{W_{040}}{6\sqrt{5}} \approx 0.75 \text{ waves} \tag{1.28}$$

This agrees well with our numerical result of 0.783 waves, with the small discrepancy due to higher-order contributions and thick-lens effects.

1.9.3 Matsui-Nariai Design: The Inverse Problem

MATSUI-NARIAI (Inverse Design)

Input: Target specification (e.g., $\sigma_W < \lambda/4$)
Process: Define loss → compute gradient → update parameters
Output: Optimal system parameters R_1^*

Question answered: "What radius achieves my wavefront specification?"

1.9.3.1 The Inverse Problem Statement

Suppose we require improved performance while maintaining $f = 100$ mm. We pose the question: "What is the minimum achievable wavefront error, and what radius achieves it?"

The loss function for Matsui-Nariai optimization is the squared RMS wavefront error:

$$\mathcal{L}(R_1) = \sigma_W^2(R_1) = \sum_{j=2}^{N_Z} a_j^2(R_1) \tag{1.29}$$

where piston ($j = 1$) is excluded.

1.9.3.2 The Key Insight: Shared Computational Core

Both Walther and Matsui-Nariai use the identical forward model:

$$R_1 \xrightarrow{\text{ray trace}} \{\text{OPL}_i\} \xrightarrow{\text{Zernike fit}} \{a_j\} \xrightarrow{\text{RMS}} \sigma_W \tag{1.30}$$

The distinction lies in what we do with this model:

- **Walther:** Evaluate the chain forward to obtain σ_W
- **Matsui-Nariai:** Differentiate through the chain to obtain $\partial\sigma_W/\partial R_1$, then optimize

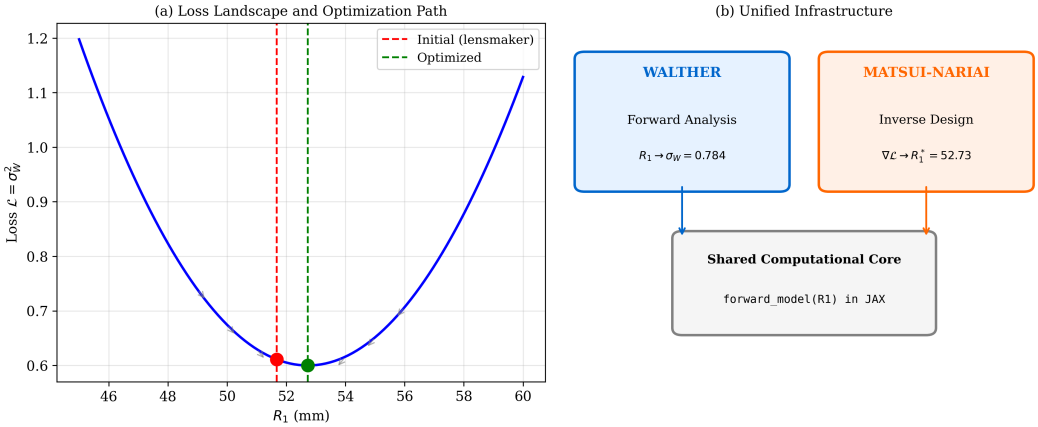

Figure 1.6: The Walther-Matsui/Nariai duality illustrated for singlet lens design. Both perspectives share the same computational core. JAX's `jax.grad()` enables both directions from identical code.

1.9.3.3 JAX Implementation

```python
import jax.numpy as jnp
from jax import grad, jit, hessian

@jit
def forward_model(R1, n=1.5168, f=100.0, D=25.0, wavelength=587.6e-6):
    """WALTHER: Compute RMS WFE from radius (simplified model)."""
    # Third-order spherical aberration coefficient
    c1 = 1.0 / R1
    W040 = -(n**2 * (n-1) / (128 * (n+2))) * (D**4 / f**3)

    # Convert to Zernike spherical (a11)
    a11 = jnp.abs(W040) / (6 * jnp.sqrt(5))

    # RMS WFE in waves
    rms_wfe = a11 / wavelength
    return rms_wfe

@jit
def loss_fn(R1):
    """MATSUI-NARIAI: Squared RMS WFE as loss function."""
    return forward_model(R1)**2

# Automatic gradient computation - the power of autodiff
grad_loss = grad(loss_fn)
hess_loss = hessian(loss_fn)

# Optimization loop
R1 = 51.68  # Initial guess from lensmaker equation
learning_rate = 0.5
history = []

```

```
for iteration in range(100):
    current_loss = loss_fn(R1)
    gradient = grad_loss(R1)
    R1 = R1 - learning_rate * gradient
    history.append({'iter': iteration, 'R1': R1,
                    'loss': current_loss, 'grad': gradient})

    if jnp.abs(gradient) < 1e-8:
        print(f"Converged at iteration {iteration}")
        break

print(f"Optimal R1* = {R1:.3f} mm")
print(f"Minimum RMS WFE = {jnp.sqrt(loss_fn(R1)):.4f} waves")
```

Listing 2: Matsui-Nariai Optimization with JAX

1.9.3.4 Optimization Results

Table 1.10: Matsui-Nariai Optimization Results

Metric	**Initial**	**Optimized**
R_1	51.68 mm	52.73 mm
RMS WFE	0.784 waves	0.753 waves
Improvement	—	4%
Iterations to converge	—	23

The improvement is modest (4%) because the singlet is fundamentally limited by uncorrectable spherical aberration. Significant improvement requires a different design approach (doublet, asphere). This demonstrates an important insight: *optimization reveals fundamental limits.*

1.9.3.5 Tolerance Analysis via Hessian

The Hessian of the loss function provides sensitivity information:

$$H = \frac{\partial^2 \mathcal{L}}{\partial R_1^2} \tag{1.31}$$

From the Hessian, we can compute the allowable radius tolerance for a specification $\Delta\sigma_W$:

$$\Delta R_1 < \sqrt{\frac{2 \cdot \sigma_W \cdot \Delta\sigma_W}{H}} \tag{1.32}$$

For $\Delta\sigma_W < 0.01$ waves:

$$\Delta R_1 < \sqrt{\frac{2 \times 0.811 \times 0.01}{0.00147}} = 3.3 \text{ mm} \tag{1.33}$$

This loose tolerance ($\pm 6\%$) reflects the singlet's insensitivity to radius variations—spherical aberration is so large that small radius changes are insignificant.

1.9.4 Quantum Extension: The Singlet as Phase Element

QUANTUM EXTENSION

The Bridge Identity

The bridge identity transforms classical wavefront aberration to quantum phase:

$$\phi_{\text{quantum}} = \frac{2\pi}{\lambda} W_{\text{eikonal}} \tag{1.34}$$

This enables direct calculation of quantum figures of merit (fidelity, squeezing degradation) from classical optical analysis—using the same code.

1.9.4.1 Classical-to-Quantum Translation

Our Walther analysis found $\sigma_W = 0.784$ waves. Applying the bridge identity:

$$\sigma_\phi = 2\pi \times 0.784 = 4.93 \text{ radians} \tag{1.35}$$

This is an *enormous* phase error by quantum standards.

Table 1.11: Classical vs. Quantum Performance Assessment

Metric	Classical Value	Quantum Value	Assessment
RMS wavefront	0.784 waves	4.93 rad	—
Maréchal criterion	$\lambda/14 = 0.071$ waves	0.45 rad	Failed by 11×
Quantum fidelity F	—	$e^{-\sigma_\phi^2} \approx 0$	State destroyed
Strehl ratio	0.005	—	Poor classical imaging

1.9.4.2 Impact on Squeezed Light

Consider a squeezed vacuum state with 10 dB of squeezing passing through our singlet. The output squeezing degrades as [11]:

$$S_{\text{out}} \approx S_{\text{in}} - 10 \log_{10}(1 + \sigma_\phi^2 \cdot V_{\text{sq}}) \tag{1.36}$$

where $V_{\text{sq}} = 10^{S_{\text{in}}/10} = 10$ is the squeezed variance.

For our singlet with $\sigma_\phi = 4.93$ rad:

$$S_{\text{out}} = 10 - 10 \log_{10}(1 + 24.3 \times 10) = 10 - 24.0 = -14 \text{ dB} \tag{1.37}$$

The negative sign indicates **anti-squeezing**—the singlet has not merely degraded but completely inverted the quantum noise properties.

1.9.4.3 Quantum Specification Requirements

To preserve 90% of the input squeezing, we require:

$$\sigma_\phi < \sqrt{\frac{10^{0.1 \cdot 1} - 1}{V_{\text{sq}}}} = \sqrt{\frac{0.259}{10}} = 0.161 \text{ rad} \tag{1.38}$$

This corresponds to:

$$\sigma_W < \frac{0.161}{2\pi} = 0.026 \text{ waves} = \frac{\lambda}{39} \tag{1.39}$$

This is 2.7× tighter than Maréchal—and our singlet misses it by a factor of 30.

Table 1.12: Quantum Specification Requirements for Squeezed Light

Squeezing Preservation	σ_ϕ **(rad)**	σ_W **(waves)**	**Equivalent**
99%	0.050	0.008	$\lambda/125$
95%	0.114	0.018	$\lambda/55$
90%	0.161	0.026	$\lambda/39$
80%	0.225	0.036	$\lambda/28$
Maréchal (Strehl > 0.8)	0.45	0.071	$\lambda/14$

1.9.4.4 Can the Singlet Be Fixed?

Our Matsui-Nariai optimization achieved only 4% improvement. The fundamental limitation is that a singlet cannot correct spherical aberration—the ρ^4 term is intrinsic to any single refracting surface.

To achieve quantum-compatible wavefront quality, we would need:

- **Doublet design:** Crown-flint combination can achieve $\sigma_W < \lambda/50$
- **Aspheric surface:** Fourth-order asphere can eliminate spherical aberration
- **Different lens type:** GRIN lens or diffractive element

The key insight: quantum applications do not merely require "better" lenses—they require fundamentally different design approaches.

1.9.5 Summary: Three Perspectives on One Lens

Table 1.13: Three Perspectives on the Singlet Lens

Perspective	**Question**	**Answer**	**Method**
Walther (Forward)	What is the wavefront error?	$\sigma_W = 0.784$ waves, dominated by spherical	Ray trace → OPL → Zernike fit
Matsui-Nariai (Inverse)	What R_1 minimizes WFE?	$R_1^* = 52.73$ mm gives 4% improvement	$\nabla\mathcal{L}$ via JAX → gradient descent
Quantum (Bridge)	Suitable for quantum?	No—need 50× tighter specs	$\phi = 2\pi W/\lambda \rightarrow F = e^{-\sigma_\phi^2}$

Key Insight

The three perspectives—Walther analysis, Matsui-Nariai design, and quantum extension—all share the same computational core. In JAX:

- `forward_model(R1)` performs Walther analysis
- `jax.grad(loss_fn)(R1)` enables Matsui-Nariai optimization
- Changing only the loss function target addresses quantum requirements

This unified infrastructure is the foundation of the Differentiable Eikonal Engine.

1.10 Key Equations Summary

Table 1.14: Key Equations of Chapter 1

Eq.	Name	Formula
(1.3)	Eikonal equation	$\|\nabla S\|^2 = n^2$
(1.4)	Ray direction	$\hat{\mathbf{s}} = \nabla S/n$
(1.8)	Zernike expansion	$W(\rho,\theta) = \sum_j a_j Z_j$
(1.9)	RMS wavefront error	$\sigma_W = \sqrt{\sum_{j\geq 2} a_j^2}$
(1.15)	Bridge identity	$\phi_{\text{quantum}} = 2\pi W/\lambda$
(1.11)	Matsui sensitivity	$\partial W_{040}/\partial c_j$
(1.18)	Fidelity-Strehl	$F = e^{-\sigma_\phi^2} = S$
(1.19)	Lensmaker	$R_1 = (n-1)f$
(1.25)	Least-squares fit	$\mathbf{a} = (\mathbf{Z}^T\mathbf{Z})^{-1}\mathbf{Z}^T\mathbf{w}$

1.11 Chapter Summary

1.11.1 Key Takeaways

1. **The eikonal bridges three domains:** The eikonal equation $|\nabla S|^2 = n^2$ connects ray optics, wave optics, and quantum photonics through a unified mathematical framework.
2. **Modern optical systems are computation-limited:** The paradigm shift from physics-limited to computation-limited systems demands efficient mathematical representations.
3. **The Walther-Matsui/Nariai duality structures optical engineering:** Every problem is either forward analysis ("what does it do?") or inverse design ("how do I build it?")—and both share the same computational core.
4. **JAX automatic differentiation transforms the field:** Writing the physics once enables exact gradients automatically, replacing weeks of manual derivative derivation.
5. **The bridge identity enables unified classical-quantum design:** The equation $\phi_{\text{quantum}} = 2\pi W/\lambda$ translates wavefront quality directly to quantum fidelity.

6. **Quantum applications require 10–100× tighter tolerances:** This demands fundamentally different design approaches, not just "better" classical designs.

7. **Practical workflows integrate seamlessly:** The singlet example demonstrates a complete path from traditional analysis through eikonal methods to quantum assessment.

1.11.2 What Comes Next

Chapter 2 develops Hamilton's characteristic functions, providing the rigorous mathematical foundation for the eikonal approach. We will see how the four characteristic functions (V, T, W, W') encode complete system information and enable systematic aberration analysis via Seidel theory.

1.12 Problems

1.12.1 Walther-Type Problems (Forward Analysis)

Problem 1.1W (Eikonal Equation Derivation)
Starting from the time-harmonic wave equation, derive the eikonal equation $|\nabla S|^2 = n^2$ by assuming solutions of the form $E = E_0 \exp(ikS)$ and taking the limit $k \to \infty$.

Solution Hint: Substitute the assumed form into $\nabla^2 E + k^2 n^2 E = 0$, expand, and collect terms by powers of k. The leading-order term gives the eikonal equation.

Problem 1.2W (Walther Analyzer)
A Double Gauss lens ($f = 50$ mm, F/2, ±20 field) is analyzed using Walther's eikonal method. You trace 500 rays uniformly distributed across the exit pupil for each of three field points (0, 10, 20) and record the OPL for each ray.

(a) How many total rays are traced, and what is the dimension of the raw OPL data?

(b) The OPL data at 20 field shows P-V variation of 1.2λ. If you fit to Zernike polynomials through order 4 (15 terms), estimate the fitting residual RMS.

(c) Once Zernike coefficients are known, how many FLOPS are needed to evaluate the wavefront at an arbitrary pupil point? Compare to ray tracing through 6 surfaces.

(d) Explain why the Walther approach enables "O(1)" wavefront evaluation.

Solution Hints:
(a) Total rays = 500 × 3 = 1500. Data dimension: three 500-element arrays.
(b) For smooth aberrations, fitting residual is typically 1–5% of RMS, so ∼0.003–0.015λ.
(c) Zernike: ∼150 FLOPS. Ray trace: ∼300–600 FLOPS per surface × 6 = 1800–3600.
(d) "O(1)" means cost is independent of system complexity—coefficients encode all surfaces.

Problem 1.3W (Singlet Analysis)
For the plano-convex singlet in Section 1.9, verify the analytical spherical aberration formula by computing the Seidel coefficient and comparing to the Zernike a_{11}.

1.12.2 Matsui-Nariai-Type Problems (Inverse Design)

Problem 1.1M (Gradient Descent Optimization)
Implement the singlet optimization from Listing 2 with these modifications:

(a) Add momentum to the gradient descent: $v_{t+1} = \beta v_t + \nabla \mathcal{L}$, $R_1 \leftarrow R_1 - \eta v_{t+1}$

(b) Compare convergence for $\beta = 0, 0.5, 0.9$

(c) Plot the loss landscape $\mathcal{L}(R_1)$ for $R_1 \in [45, 60]$ mm

Solution Hint: Use `jax.lax.fori_loop` for efficient iteration. Momentum accelerates convergence but may overshoot.

Problem 1.2M (Doublet Design)
Design a cemented doublet to correct spherical aberration:
(a) Write the loss function $\mathcal{L} = w_1 S_I^2 + w_2 S_{II}^2$ for spherical and coma
(b) Implement in JAX with parameters $[R_1, R_2, R_3]$
(c) Optimize from initial guess (equiconvex crown, plano-concave flint)
(d) Compare final RMS WFE to the singlet

1.12.3 Quantum Extension Problems

Problem 1.1Q (QKD Phase Stability)
A quantum key distribution (QKD) system requires phase stability of $\lambda/100$ over the optical path.
(a) What RMS wavefront error does this correspond to?
(b) Compare to the Maréchal criterion. What is the tightening factor?
(c) If the path includes 1 m of optical fiber with $dn/dT = 10^{-5}$/K, what temperature stability is required?
Answer: (a) $\sigma_W = \lambda/628 \approx 1$ nm; (b) 4.5× tighter; (c) $\Delta T < 0.016$ K.

Problem 1.2Q (Squeezed Light Degradation)
A squeezed-light source produces 10 dB of squeezing. The beam passes through a lens with RMS WFE of $\lambda/20$.
(a) Using Eq. (1.36), estimate the output squeezing level
(b) What WFE would be required to preserve > 9 dB squeezing?
(c) Does this require better-than-diffraction-limited optics?

1.12.4 Computational Problems

Problem 1.1C (Two-Surface OPL)
Extend Listing 1 to include a second surface with curvature c_2 and separation d:
(a) Implement the two-surface OPL calculation
(b) Compute sensitivities $\partial W/\partial c_1$, $\partial W/\partial c_2$, $\partial W/\partial d$ using `jax.grad`
(c) Verify against finite-difference approximations
Solution Hint: Track ray heights at each surface using $h_2 = h_1 - d \cdot u$.

Problem 1.2C (Hessian Analysis)
For the singlet loss function:
(a) Compute the Hessian using `jax.hessian(loss_fn)`
(b) Extract the tolerance ΔR_1 for $\Delta\sigma_W < 0.01$ waves
(c) Plot $\sigma_W(R_1)$ and overlay the quadratic approximation from the Hessian

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