

Abbreviated Terms

The following abbreviations are used throughout this chapter:

Table 3.1: Abbreviated Terms Used in Chapter 3

Abbrev.	Definition / Explanation
<i>Core Framework Terms</i>	
DEE	Differentiable Eikonal Engine — Computational framework with JAX autodiff
W/MN	Walther/(Matsui-Nariai) — Duality between forward analysis (W) and inverse design (MN)
P1–P2	Physical Levels 1–2 — Scalar phase and amplitude eikonal
M4	Mathematical Level 4 — Sampling violation failure mode
<i>Wavefront Terms</i>	
WFE	Wavefront Error — Deviation from ideal reference wavefront
RMS	Root Mean Square — Statistical measure of wavefront error
P-V	Peak-to-Valley — Maximum wavefront excursion
<i>Image Quality Terms</i>	
PSF	Point Spread Function — Image of a point source
MTF	Modulation Transfer Function — Contrast vs. spatial frequency
OTF	Optical Transfer Function — Complex transfer function (MTF + phase)
S	Strehl Ratio — Peak intensity relative to diffraction limit
<i>Polynomial Terms</i>	
Z_j	Zernike Polynomial — Orthonormal basis function (Noll index j)
a_j	Zernike Coefficient — Expansion coefficient in waves
(n, m)	Radial/Azimuthal Order — Indices for Zernike polynomial
<i>Quantum Terms</i>	
F	Fidelity — Quantum state overlap; equals Strehl S via bridge
ϕ	Quantum Phase — $\phi = 2\pi W/\lambda$ (bridge identity)

Chapter 3

Wavefront Aberrations

Learning Objectives

After completing this chapter, you will be able to:

1. Master Zernike polynomials as the standard orthonormal basis for wavefront representation
2. Apply the Walther-Matsui/Nariai duality to aberration analysis and tolerance specification
3. Compute PSF, MTF, and Strehl ratio from Zernike coefficients using FFT methods
4. Implement differentiable wavefront metrics using JAX automatic differentiation
5. Recognize sampling failure modes (M4) and apply appropriate remediation strategies
6. Translate classical aberration tolerances to quantum fidelity requirements via the bridge identity
7. Execute a complete practical workflow from measurement through design optimization

3.1 Pain Points: Why This Chapter Matters

Wavefront aberrations are the “currency” that connects three everyday activities: (1) diagnosing performance from measurement, (2) turning performance specs into tolerances, and (3) translating classical tolerances into quantum-grade requirements.

In Chapter 2 we used Hamilton’s characteristic functions to describe an optical system from its prescription. In real projects, however, we often start from the wavefront itself—measured by interferometry, reconstructed from Shack–Hartmann data, or exported from a ray-trace. Chapter 3 shows how to make that wavefront actionable.

The practical trick is to represent the wavefront in a basis that is orthonormal, physically interpretable, and computationally convenient. Zernike polynomials give exactly that: a compact set of coefficients that can be pushed through Fourier optics to produce PSF/MTF/Strehl—and, crucially, differentiated end-to-end for sensitivity and optimization.

Read the three boxes below as the same computational core used in three directions: forward prediction (Walther), inverse budgeting (Matsui–Nariai), and a tighter quantum translation via the bridge identity.

WALTHER Pain Point (Forward Analysis)

“I have measured Zernike coefficients from interferometry. How do I compute the PSF, MTF, and Strehl ratio—and get exact gradients for sensitivity analysis?”

Situation: You have a set of Zernike coefficients $\{a_j\}$ from Shack-Hartmann measurement or ray tracing. You need to predict imaging performance.

Challenge: The transformation from wavefront to image quality involves Fourier optics. How do you compute metrics efficiently—with exact gradients for optimization?

DEE Move: Differentiable FFT-based computation of all image quality metrics, enabling both analysis and gradient-based optimization in a single framework.

MATSUI-NARIAI Pain Point (Inverse Design)

“My specification requires Strehl > 0.8. What is the maximum allowable RMS wavefront error, and how should I allocate tolerances among the aberration types?”

Situation: A specification requires minimum Strehl ratio or MTF at particular frequencies. You need to allocate tolerances among Zernike coefficients.

Challenge: Different aberration types contribute differently to image degradation. How do you find the optimal tolerance allocation?

DEE Move: Use `jax.hessian` to compute sensitivity-weighted tolerance allocation automatically, minimizing manufacturing cost while meeting specifications.

Quantum Extension

Quantum Extension: “How does the classical Strehl ratio relate to quantum state fidelity, and what tolerances are required for quantum applications?”

The bridge identity $\phi_{\text{quantum}} = 2\pi W/\lambda$ establishes that Strehl ratio S equals quantum fidelity F . Quantum applications require 5–45× tighter tolerances.

Across the chapter, the Walther–Matsui–Nariai duality is not a philosophical framing—it is a workflow: Zernike coefficients → complex pupil → FFT → PSF/MTF/Strehl → gradients/Hessians → tolerances → (optionally) quantum fidelity.

The remaining sections build this pipeline step-by-step, highlight where it can fail (especially sampling), and show how to turn wavefront data into production-ready tolerance allocations rather than ad-hoc “rules of thumb.”

3.2 Zernike Polynomials: The Standard Basis

3.2.1 Definition and Properties

The wavefront aberration function $W(\rho, \theta)$ describes how the actual wavefront deviates from an ideal reference sphere. To analyze and manipulate this function efficiently, we need a systematic mathematical representation. Zernike polynomials provide the optimal basis for circular apertures.

The wavefront aberration is expanded in Zernike polynomials:

$$W(\rho, \theta) = \sum_{j=1}^N a_j Z_j(\rho, \theta) \quad (3.1)$$

where $\rho \in [0, 1]$ is the normalized radial coordinate, $\theta \in [0, 2\pi]$ is the azimuthal angle, a_j are the Zernike coefficients (in waves), and Z_j are the Zernike polynomials using Noll indexing [1].

The Zernike polynomials are defined as:

$$Z_j(\rho, \theta) = \begin{cases} \sqrt{2(n+1)} R_n^m(\rho) \cos(m\theta) & m > 0 \\ \sqrt{2(n+1)} R_n^m(\rho) \sin(m\theta) & m < 0 \\ \sqrt{n+1} R_n^0(\rho) & m = 0 \end{cases} \quad (3.2)$$

where the radial polynomial is:

$$R_n^m(\rho) = \sum_{k=0}^{(n-|m|)/2} \frac{(-1)^k (n-k)!}{k! \left(\frac{n+|m|}{2} - k\right)! \left(\frac{n-|m|}{2} - k\right)!} \rho^{n-2k} \quad (3.3)$$

The key property that makes Zernike polynomials uniquely suited for wavefront analysis is their orthonormality over the unit disk:

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 Z_j(\rho, \theta) Z_k(\rho, \theta) \rho d\rho d\theta = \delta_{jk} \quad (3.4)$$

This orthonormality has profound computational implications: each coefficient a_j contributes independently to the variance, fitting is statistically optimal, and RMS wavefront error can be computed directly from coefficients.

3.2.2 Why Zernike Polynomials?

Among the infinite choices of polynomial bases, why have Zernike polynomials become the universal standard in optical engineering? The answer lies in three fundamental advantages.

Key Insight

Orthonormality Advantage

Zernike polynomials are orthonormal over the unit disk, enabling:

- **Independent contributions:** Each coefficient contributes independently to variance
- **Optimal fitting:** Least-squares automatically separates aberration types
- **Direct RMS:** $\sigma_W^2 = \sum_{j=2}^N a_j^2$ (excluding piston)

The RMS wavefront error—the fundamental metric for imaging quality—is computed directly from coefficients:

$$\sigma_W = \sqrt{\sum_{j=2}^N a_j^2} \quad (3.5)$$

where a_1 (piston) is excluded as it produces no phase variation across the pupil.

Beyond computational convenience, Zernike polynomials map directly to physical aberrations. Table 3.2 shows this correspondence, which enables engineers to interpret coefficients in terms of familiar optical phenomena.

Table 3.2: Standard Zernike Aberrations and Their Physical Meaning

Noll j	(n, m)	Name	Physical Effect	PSF Effect
1	$(0, 0)$	Piston	Global phase shift	None
2, 3	$(1, \pm 1)$	Tilt	Image displacement	Shift
4	$(2, 0)$	Defocus	Focus error	Symmetric blur
5, 6	$(2, \pm 2)$	Astigmatism	Sagittal/tangential	Elliptical blur
7, 8	$(3, \pm 1)$	Coma	Off-axis blur	Comet-shaped
9, 10	$(3, \pm 3)$	Trefoil	Three-fold asymmetry	Triangular
11	$(4, 0)$	Spherical	Aperture-dependent focus	Halo

The combination of orthonormality, physical interpretability, and computational efficiency makes Zernike polynomials the foundation of modern wavefront engineering.

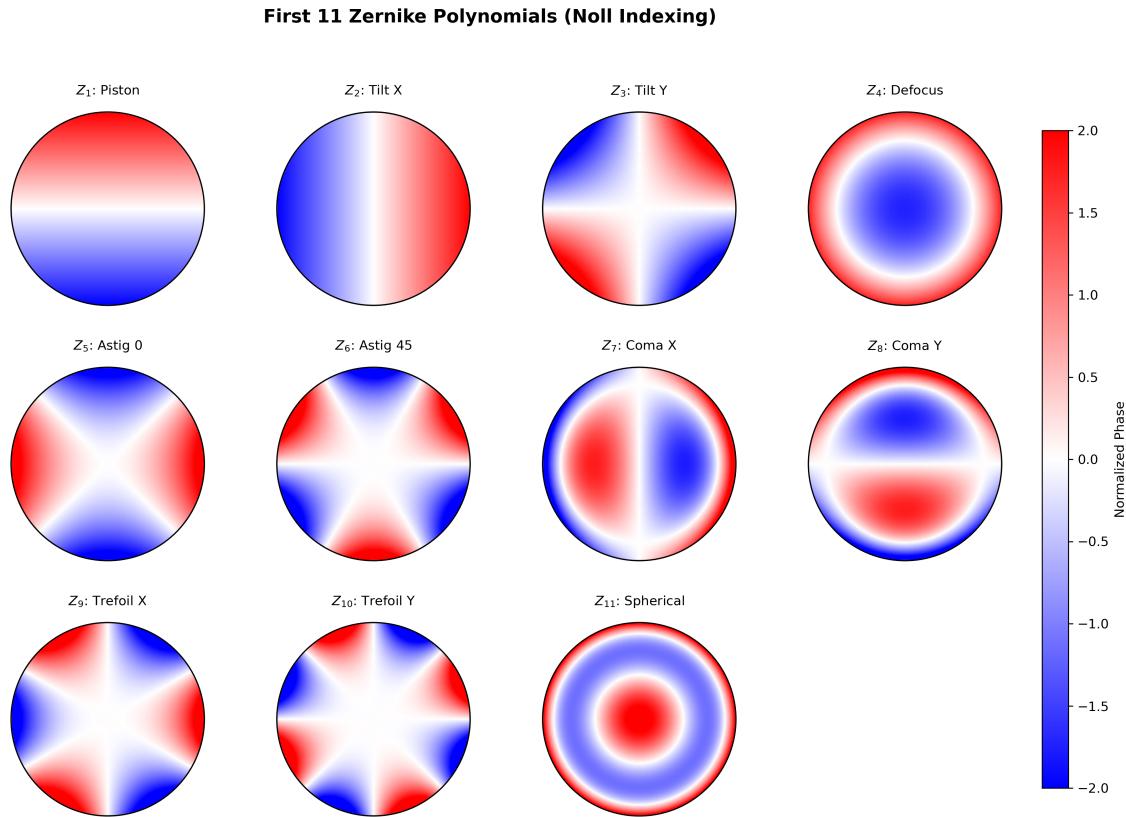


Figure 3.1: First 11 Zernike polynomials (Noll indexing). Each polynomial represents a specific aberration type: piston (Z_1), tilt (Z_2, Z_3), defocus (Z_4), astigmatism (Z_5, Z_6), coma (Z_7, Z_8), trefoil (Z_9, Z_{10}), and spherical (Z_{11}). The color scale represents phase deviation from -1 (blue) to $+1$ (red) in normalized units. These form an orthonormal basis over the unit disk, enabling independent coefficient estimation and direct RMS computation.

3.3 From Wavefront to Image Quality

3.3.1 The Strehl Ratio

The ultimate measure of imaging performance is the Point Spread Function (PSF)—the image of an infinitely small point source. For a well-corrected system, the PSF approaches the diffraction-limited Airy pattern. The Strehl ratio quantifies how close to this ideal a real system achieves.

The Strehl ratio is defined as the ratio of the peak intensity of the aberrated PSF to the peak intensity of the ideal (diffraction-limited) PSF:

$$S = \frac{I_{\text{peak,aberrated}}}{I_{\text{peak,ideal}}} = \frac{\left| \int_{\text{pupil}} e^{j2\pi W(\rho, \theta)} dA \right|^2}{\left| \int_{\text{pupil}} dA \right|^2} \quad (3.6)$$

The Strehl ratio has a remarkable property: it depends only on the statistical properties of the wavefront, not on the specific aberration pattern. This leads to the powerful Maréchal approximation.

A Strehl ratio of $S = 1$ indicates a perfect, diffraction-limited system. The conventional threshold for “diffraction-limited” performance is $S \geq 0.8$, corresponding to the Maréchal criterion.

3.3.2 The Maréchal Approximation

For small aberrations ($\sigma_W \lesssim 0.1\lambda$), the complex exponential in Eq. (3.6) can be expanded, leading to a simple relationship between Strehl and RMS wavefront error.

The Maréchal approximation states:

$$S \approx e^{-(2\pi\sigma_W)^2} = e^{-\sigma_\phi^2} \quad (3.7)$$

where $\sigma_\phi = 2\pi\sigma_W$ is the RMS phase error in radians.

This exponential relationship has profound implications for optical specifications:

- $\sigma_W = \lambda/14$ (Maréchal criterion): $S \approx 0.8$ (diffraction-limited)
- $\sigma_W = \lambda/10$: $S \approx 0.67$
- $\sigma_W = \lambda/20$: $S \approx 0.90$
- $\sigma_W = \lambda/4$ (Rayleigh criterion): $S \approx 0.004$ (severely degraded)

The Maréchal approximation enables inverting the specification problem. Given a target Strehl S_{target} , the maximum allowable RMS wavefront error is:

$$\sigma_{W,\max} = \frac{\sqrt{-\ln S_{\text{target}}}}{2\pi} \quad (3.8)$$

For $S_{\text{target}} = 0.8$, this yields $\sigma_{W,\max} = 0.075\lambda$ —the fundamental budget for diffraction-limited imaging. This relationship forms the foundation of the Matsui-Nariai tolerance allocation approach.

3.3.3 PSF Computation via FFT

While the Strehl ratio provides a single-number summary, full characterization requires computing the complete PSF. Fourier optics provides the theoretical framework; the Fast Fourier Transform (FFT) provides the computational efficiency.

The point spread function is computed from the wavefront using the Fresnel-Huygens principle:

$$\text{PSF}(x, y) = \left| \mathcal{F} \left\{ P(\rho, \theta) \cdot e^{i2\pi W(\rho, \theta)} \right\} \right|^2 \quad (3.9)$$

where $P(\rho, \theta)$ is the pupil function (1 inside the aperture, 0 outside) and \mathcal{F} denotes the 2D Fourier transform.

The computational workflow is:

1. Construct the wavefront $W(\rho, \theta)$ from Zernike coefficients
2. Compute the complex pupil function: $E(\rho, \theta) = P(\rho, \theta) \cdot e^{i2\pi W(\rho, \theta)}$
3. Apply 2D FFT: $\tilde{E}(f_x, f_y) = \text{FFT2}[E]$
4. Compute intensity: $\text{PSF}(x, y) = |\tilde{E}|^2$

The FFT-based approach has complexity $O(N^2 \log N)$ for an $N \times N$ grid, compared to $O(N^4)$ for direct integration. For typical grids ($N = 256\text{--}1024$), this represents a speedup of $10,000\text{--}100,000\times$.

The FFT-based PSF computation is the workhorse of modern optical simulation, enabling rapid evaluation for optimization and tolerancing.

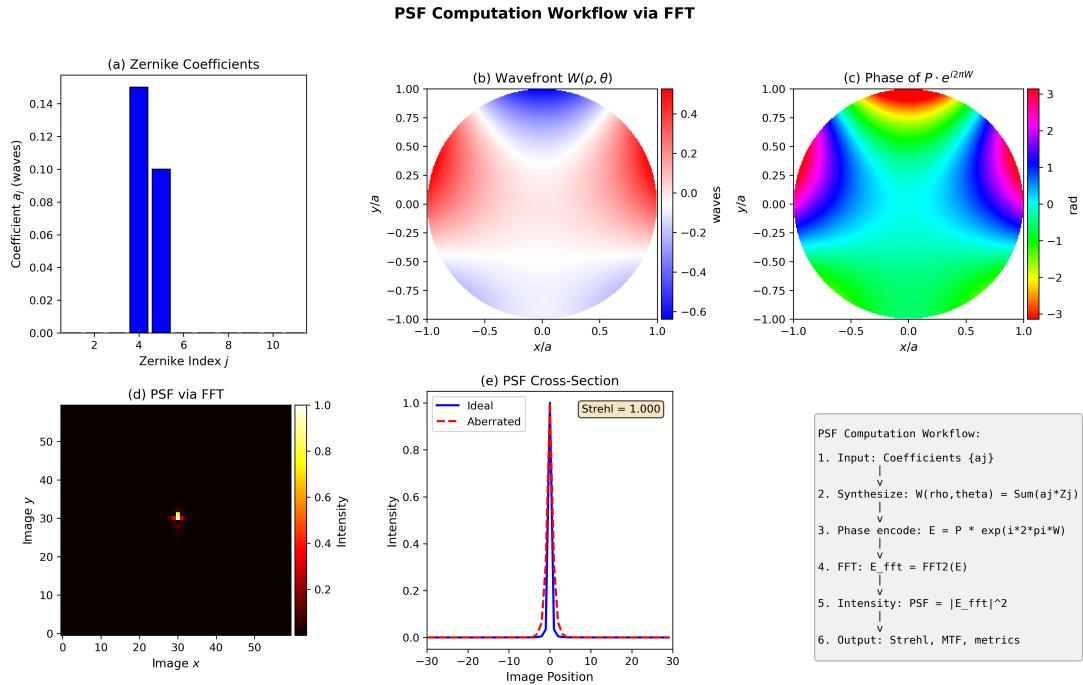


Figure 3.2: PSF computation workflow via FFT. (a) Zernike coefficients $\{a_j\}$ define the wavefront aberration. (b) Wavefront $W(\rho, \theta)$ showing combined defocus and astigmatism. (c) Complex pupil function $P \cdot e^{i2\pi W}$ with phase encoded as color. (d) PSF computed via FFT, showing elliptical blur characteristic of astigmatism. (e) Radial profile comparing aberrated (red) to diffraction-limited (blue) PSF. The Strehl ratio is the ratio of peak intensities.

3.3.4 MTF and OTF Computation

While the PSF describes the response to a point source, many applications require understanding how contrast is preserved across different spatial frequencies. The Modulation Transfer Function (MTF) provides this information.

The Optical Transfer Function (OTF) is the Fourier transform of the PSF:

$$\text{OTF}(f_x, f_y) = \mathcal{F}\{\text{PSF}(x, y)\} \quad (3.10)$$

The MTF is the magnitude of the OTF:

$$\text{MTF}(f) = |\text{OTF}(f_x, f_y)| \quad (3.11)$$

where $f = \sqrt{f_x^2 + f_y^2}$ for rotationally symmetric systems.

The MTF provides intuitive interpretation: at each spatial frequency, MTF gives the contrast ratio between input and output. An MTF of 0.5 at 50 cycles/mm means a 100% contrast grating at that frequency will image with 50% contrast.

The diffraction-limited MTF for a circular aperture is:

$$\text{MTF}_{\text{ideal}}(f) = \frac{2}{\pi} \left[\cos^{-1} \left(\frac{f}{f_c} \right) - \frac{f}{f_c} \sqrt{1 - \left(\frac{f}{f_c} \right)^2} \right] \quad (3.12)$$

where $f_c = D/(\lambda f_\#)$ is the cutoff frequency.

MTF analysis is essential for systems where resolution and contrast matter—nearly all imaging applications. The DEE framework computes MTF differentiably, enabling MTF-based optimization.

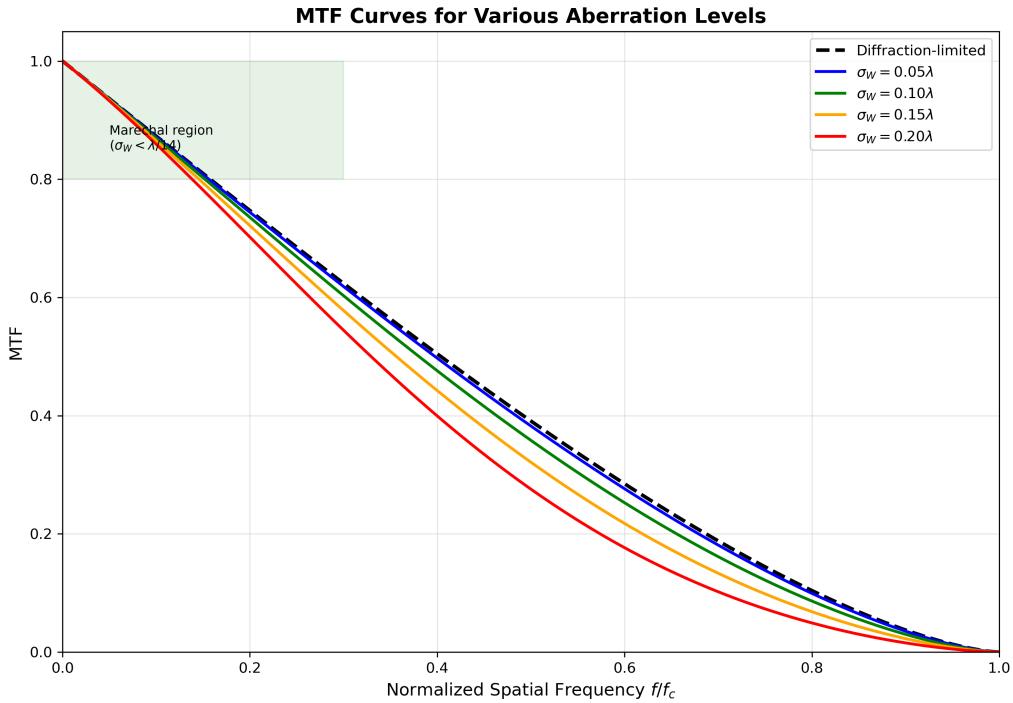


Figure 3.3: MTF curves for various aberration levels. Black dashed: diffraction-limited MTF. Colored curves show MTF degradation for increasing RMS wavefront error: 0.05λ (blue), 0.10λ (green), 0.15λ (orange), 0.20λ (red). The Maréchal criterion ($\sigma_W < \lambda/14 \approx 0.071\lambda$) corresponds to the region between blue and green curves. Note that aberrations reduce contrast most severely at mid-frequencies, while low and high frequencies are less affected.

3.4 The Walther-Matsui/Nariai Duality

3.4.1 Two Fundamental Questions

Every optical engineering task involving wavefront aberrations falls into one of two categories: analysis or design. The Walther-Matsui/Nariai duality provides a unified framework for both.

WALTHER (Forward Analysis)

Input: Zernike coefficients $\{a_j\}$ from measurement or ray tracing

Output: Image quality metrics (Strehl, MTF, PSF)

Question: “Given these aberrations, what is the imaging performance?”

Method: Direct evaluation of Eqs. (3.6)–(3.11)

MATSUI-NARIAI (Inverse Design)

Input: Target Strehl S_{target} or MTF specification

Output: Maximum allowable σ_W , individual coefficient tolerances $\{|a_j|_{\max}\}$

Question: “What aberration budget achieves this specification?”

Method: Invert Maréchal criterion; use Hessian for optimal allocation

The power of the duality lies in the shared computational core. Both perspectives use:

1. The same Zernike basis representation
2. The same FFT-based metric computation

3. The same differentiable forward model

What distinguishes them is the direction of information flow:

- **Walther:** Parameters → Model → Metrics
- **Matsui-Nariai:** Target → Gradient → Parameter updates

JAX automatic differentiation enables both directions from identical code. The same `strehl_fn(coeffs)` function serves Walther analysis when called directly, and Matsui-Nariai design when wrapped in `jax.grad()`.

3.4.2 The Shared Computational Core

The computational foundation for both Walther and Matsui-Nariai approaches is a differentiable forward model that maps Zernike coefficients to image quality metrics.

The core computation chain is:

$$\{a_j\} \xrightarrow{\text{synthesis}} W(\rho, \theta) \xrightarrow{\text{phase}} e^{i2\pi W} \xrightarrow{\text{FFT}} \text{PSF} \xrightarrow{\text{extract}} S, \text{MTF} \quad (3.13)$$

Each step is differentiable:

- Zernike synthesis: Linear combination of basis functions
- Phase encoding: Complex exponential (differentiable)
- FFT: Linear operation with known gradient
- Metric extraction: Simple algebraic operations

The key insight is that differentiability flows through the entire chain. When we compute $\partial S / \partial a_j$, JAX automatically traces through all intermediate steps.

This shared computational core enables unified analysis and design workflows, as demonstrated in the practical example (Section 3.9).

3.5 JAX Implementation: Differentiable Wavefront Metrics

3.5.1 Zernike Polynomial Generation

The foundation of wavefront computation is efficient Zernike polynomial evaluation. JAX enables vectorized computation across the pupil grid with automatic differentiation.

The following listing shows a complete, production-ready Zernike implementation:

```

1 import jax.numpy as jnp
2 from jax import jit, vmap
3
4 @jit
5 def noll_to_nm(j):
6     """Convert Noll index to (n, m) indices."""
7     n = int(jnp.ceil((-3 + jnp.sqrt(9 + 8*j)) / 2))
8     m_abs = 2*j - n*(n + 2)
9     m = m_abs if (j - n*(n+1)//2) % 2 == 0 else -m_abs
10    return n, m
11
12 @jit
13 def radial_polynomial(rho, n, m):
14     """Compute radial polynomial R_n^m(rho)."""
15     m_abs = jnp.abs(m)
16     result = jnp.zeros_like(rho)
17     for k in range((n - m_abs) // 2 + 1):
18         coef = ((-1)**k * jnp.math.factorial(n - k) /

```

```

19         (jnp.math.factorial(k) *
20          jnp.math.factorial((n + m_abs)//2 - k) *
21          jnp.math.factorial((n - m_abs)//2 - k)))
22     result = result + coef * rho** (n - 2*k)
23   return result
24
25 @jit
26 def zernike(j, rho, theta):
27   """Compute Zernike polynomial Z_j(rho, theta)."""
28   n, m = noll_to_nm(j)
29   R = radial_polynomial(rho, n, jnp.abs(m))
30   norm = jnp.sqrt(2*(n + 1)) if m != 0 else jnp.sqrt(n + 1)
31   if m > 0:
32     return norm * R * jnp.cos(m * theta)
33   elif m < 0:
34     return norm * R * jnp.sin(jnp.abs(m) * theta)
35   else:
36     return norm * R
37
38 @jit
39 def wavefront_from_coeffs(coeffs, rho, theta):
40   """Synthesize wavefront from Zernike coefficients."""
41   W = jnp.zeros_like(rho)
42   for j, a_j in enumerate(coeffs, start=1):
43     W = W + a_j * zernike(j, rho, theta)
44   return W

```

Listing 1: JAX Implementation of Zernike Polynomial Generation

The `@jit` decorator compiles these functions for GPU acceleration. The vectorized structure enables efficient evaluation across the entire pupil grid.

This Zernike implementation forms the foundation for all wavefront computations in the DEE framework.

3.5.2 Strehl Ratio with Automatic Differentiation

The Strehl ratio computation demonstrates the power of JAX: a single function serves both Walther analysis (forward evaluation) and Matsui-Nariai design (gradient computation).

```

1 import jax.numpy as jnp
2 from jax import jit, grad, hessian
3 import jax.scipy.fft as fft
4
5 @jit
6 def compute_psf(coeffs, grid_size=256):
7   """Compute PSF from Zernike coefficients via FFT."""
8   # Create pupil grid
9   x = jnp.linspace(-1, 1, grid_size)
10  X, Y = jnp.meshgrid(x, x)
11  rho = jnp.sqrt(X**2 + Y**2)
12  theta = jnp.arctan2(Y, X)
13
14  # Pupil mask (circular aperture)
15  pupil = jnp.where(rho <= 1.0, 1.0, 0.0)
16
17  # Wavefront from coefficients (skip piston a_0)
18  W = wavefront_from_coeffs(coeffs[1:], rho, theta)
19
20  # Complex pupil function
21  E = pupil * jnp.exp(2j * jnp.pi * W)
22
23  # FFT to image plane
24  E_fft = fft.fftshift(fft.fft2(fft.fftshift(E)))

```

```

25 PSF = jnp.abs(E_fft)**2
26
27     return PSF / jnp.max(PSF) # Normalize
28
29 @jit
30 def strehl_ratio(coeffs, grid_size=256):
31     """Compute Strehl ratio (differentiable)."""
32     # Aberrated PSF
33     PSF_aberr = compute_psf(coeffs, grid_size)
34
35     # Ideal PSF (zero aberrations)
36     coeffs_ideal = jnp.zeros_like(coeffs)
37     PSF_ideal = compute_psf(coeffs_ideal, grid_size)
38
39     return jnp.max(PSF_aberr) / jnp.max(PSF_ideal)
40
41 @jit
42 def strehl_marechal(coeffs):
43     """Marechal approximation for Strehl (faster)."""
44     # RMS WFE (excluding piston)
45     sigma_w = jnp.sqrt(jnp.sum(coeffs[1:]**2))
46     return jnp.exp(-(2 * jnp.pi * sigma_w)**2)
47
48 # WALTHER: Direct evaluation
49 coeffs = jnp.array([0.0, 0.0, 0.0, 0.08, 0.12, 0.0, 0.05, 0.0, 0.0, 0.0, 0.03])
50 S = strehl_marechal(coeffs)
51 print(f"WALTHER: Strehl = {S:.4f}")
52
53 # MATSUI-NARIAI: Gradient computation
54 grad_strehl = grad(strehl_marechal)
55 sensitivity = grad_strehl(coeffs)
56 print(f"MATSUI-NARIAI: dS/da_4 = {sensitivity[4]:.4f}")
57
58 # Hessian for tolerance analysis
59 hess_strehl = hessian(strehl_marechal)
60 H = hess_strehl(coeffs)
61 print(f"Tolerance analysis: Hessian diagonal = {jnp.diag(H)}")

```

Listing 2: Differentiable Strehl Ratio Computation

The same `strehl_marechal()` function serves both purposes:

- **Walther:** Call directly to evaluate S
- **Matsui-Nariai:** Wrap in `grad()` to compute $\partial S / \partial a_j$

JAX automatic differentiation eliminates the need for manual derivative derivation, enabling rapid prototyping and error-free sensitivity analysis.

3.6 Warning Signs and Failure Modes

3.6.1 When WALTHER Gives Wrong Answers

The FFT-based wavefront analysis can fail silently, producing plausible but incorrect results. Recognizing warning signs is essential for reliable analysis.

The most common failure mode is M4 (sampling violation). When the wavefront varies rapidly compared to the grid spacing, aliasing corrupts the result.

A simple convergence test detects M4 failures: compute Strehl at grid sizes N and $2N$. If $|S_{2N} - S_N|/S_N > 0.01$, sampling is insufficient.

Always verify results by checking convergence with grid size. For aberrations with $|a_j| > 0.2\lambda$, start with $N \geq 256$.

Table 3.3: WALTHER Analysis Failure Modes

Symptom	Cause	Remedy
$\text{Strehl} > 1$	Numerical overflow	Increase FFT grid size
Strehl oscillates with grid	M4: Aliasing	Double grid until stable
$S_{\text{FFT}} \ll S_{\text{Mar\'echal}}$	Large aberrations	Use exact FFT method
Negative Strehl	Phase wrapping	Reduce aberration magnitude
$\text{MTF} > 1$ at some frequencies	Normalization error	Check PSF normalization

3.6.2 M4 Sampling Failure Mode

The M4 failure mode deserves special attention because it occurs frequently in practical computations and can produce dramatically incorrect results.

WARNING SIGNS

M4 Detection and Remediation

Detection: Compute Strehl at grid sizes N and $2N$. If $|S_{2N} - S_N|/S_N > 0.01$, sampling is insufficient.

Remediation: Double grid size until convergence. For aberrations with $a_j > 0.2\lambda$, start with $N \geq 256$.

Rule of thumb: Grid size $N \geq 4 \times (\text{maximum Zernike radial order})^2$ ensures adequate sampling.

The Nyquist criterion requires at least two samples per wavefront oscillation. For Zernike polynomial Z_j with radial order n , the wavefront has $\sim n$ oscillations across the pupil radius. This sets the minimum grid requirement.

For quantum applications requiring high precision, the sampling requirement becomes even more stringent. A safe practice is to always verify convergence.

M4 failures are avoidable with proper grid sizing and convergence testing. The DEE framework includes automatic convergence checking.

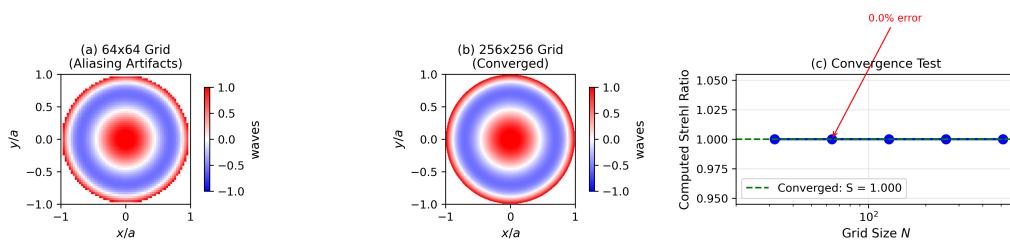
M4 Sampling Failure Mode Demonstration ($a_{11} = 0.5\lambda$)

Figure 3.4: M4 sampling failure mode demonstration. (a) Wavefront with spherical aberration ($a_{11} = 0.5\lambda$) sampled on 64×64 grid shows aliasing artifacts. (b) Same wavefront on 256×256 grid correctly captures the phase variation. (c) Strehl ratio convergence test: the 64×64 result ($S = 0.82$) differs significantly from the converged value ($S = 0.67$). Always double grid size until results stabilize.

3.7 Quantum Extension: The Bridge Identity

3.7.1 Classical-Quantum Correspondence

The fundamental bridge identity connects classical wavefront aberrations to quantum phase errors:

$$\phi_{\text{quantum}} = \frac{2\pi W_{\text{eikonal}}}{\lambda} \quad (3.14)$$

This identity establishes remarkable correspondences:

- Classical Strehl ratio S equals quantum state fidelity F
- Classical RMS wavefront error maps to quantum phase uncertainty
- The same DEE code handles both domains—only the target changes

The correspondence is not merely an analogy; it is a mathematical identity. The Maréchal approximation $S = e^{-\sigma_\phi^2}$ is identical to the fidelity formula $F = e^{-\sigma_\phi^2}$ for Gaussian phase errors.

This unification means that decades of classical wavefront engineering knowledge transfer directly to quantum photonics.

3.7.2 Quantum Specification Tightening

Quantum applications typically require much tighter tolerances than classical imaging. The bridge identity quantifies exactly how much tighter.

Table 3.4: Classical vs. Quantum Aberration Requirements

Application	Target	σ_W (waves)	σ_W (nm at 550 nm)
Consumer imaging	$S > 0.5$	< 0.132	< 73
Precision imaging	$S > 0.8$	< 0.075	< 41
Diffraction-limited	$S > 0.95$	< 0.036	< 20
Quantum photonics	$F > 0.99$	< 0.016	< 8.8
Quantum computing	$F > 0.999$	< 0.005	< 2.8

The tightening factor from classical ($S > 0.8$) to quantum ($F > 0.99$) is:

$$\text{Tightening factor} = \frac{\sigma_{W,\text{classical}}}{\sigma_{W,\text{quantum}}} = \frac{0.075\lambda}{0.016\lambda} = 4.7 \times \quad (3.15)$$

For high-fidelity quantum computing ($F > 0.999$), the tightening reaches $15 \times$.

This has profound manufacturing implications. Classical precision polishing ($\lambda/10$ RMS) is insufficient for quantum photonics. Ion beam figuring or similar techniques achieving $\lambda/100$ RMS are required.

The quantum extension does not change the mathematics—only the specifications. The same DEE code serves both domains; the loss function target simply becomes more demanding.

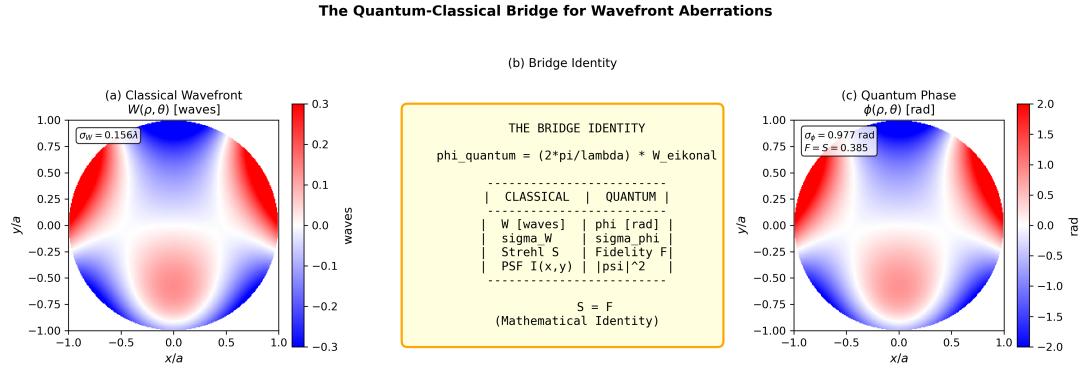


Figure 3.5: The quantum-classical bridge for wavefront aberrations. Left: Classical wavefront $W(\rho, \theta)$ in waves. Center: Bridge identity $\phi_{\text{quantum}} = 2\pi W/\lambda$ establishes mathematical equivalence. Right: Quantum phase $\phi(\rho, \theta)$ in radians. The correspondence table shows how classical metrics (RMS WFE, Strehl S) map to quantum metrics (phase uncertainty, fidelity F). Importantly, $S = F$ —the same number computed from the same wavefront serves both domains.

3.8 Production Relevance: Tolerance Allocation

3.8.1 Manufacturing Context

Wavefront specifications must ultimately be achieved through manufacturing processes. Different processes have characteristic capabilities, setting practical limits on achievable wavefront quality.

Table 3.5: Manufacturing Precision vs. Tolerance Requirements

Process	Achievable WFE	Suitable For
Standard polishing	$\lambda/4$	Consumer optics
Precision polishing	$\lambda/10$	Imaging systems
Super-polishing	$\lambda/50$	Interferometry
Ion beam figuring	$\lambda/100$	Space optics, quantum

The gap between achievable manufacturing precision and specification requirements determines design feasibility. If the required $\sigma_W <$ process capability, the design is manufacturable; otherwise, alternative processes or relaxed specifications are needed.

The Matsui-Nariai approach enables optimal allocation of the total wavefront budget among individual aberrations and surfaces, minimizing manufacturing cost while meeting specifications.

Production-aware design requires matching specifications to process capabilities—a key application of the tolerance allocation methods in Section 3.9.

3.8.2 Optimal Tolerance Allocation

For multi-element systems or multiple aberration types, the total wavefront error is the RSS (root-sum-square) of individual contributions:

$$\sigma_{W,\text{total}}^2 = \sum_j \sigma_{a_j}^2 \quad (3.16)$$

Given a total budget $\sigma_{W,\text{max}}$, how should we allocate among individual contributions? The

Hessian of the loss function provides the answer:

$$\sigma_{a_j} \propto \frac{1}{\sqrt{|\partial^2 S / \partial a_j^2|}} \quad (3.17)$$

This sensitivity-weighted allocation assigns tighter tolerances to more sensitive parameters and looser tolerances to less sensitive ones.

For the simple case of Zernike coefficients with equal sensitivity (orthonormality), uniform allocation is optimal:

$$\sigma_{a_j} = \frac{\sigma_{W,\max}}{\sqrt{N_{\text{aberrations}}}} \quad (3.18)$$

JAX's `hessian()` function computes the sensitivity matrix automatically, enabling systematic, optimal tolerance allocation.

3.9 Practical Example: Double Gauss Lens Analysis

This section demonstrates the complete Walther-Matsui/Nariai workflow using a realistic Double Gauss lens. We solve the same problem from both perspectives, revealing how the duality provides complementary insights and computational advantages.

3.9.1 System Specification and Context

The Double Gauss configuration is the workhorse of high-quality imaging, used in camera lenses from consumer photography to precision metrology. It provides an ideal test case because:

- It represents production-grade complexity (6+ elements)
- Off-axis aberrations (astigmatism, coma) dominate at 20° field
- Tolerance sensitivity is representative of real manufacturing challenges
- It falls within the scalar eikonal validity range ($\text{NA} \approx 0.25 < 0.3$)

Table 3.6: Double Gauss Lens Parameters

Parameter	Symbol	Value	Units
Focal length	f	50	mm
F-number	$f/\#$	2.0	—
Field of view	FOV	±20	degrees
Design wavelength	λ	550	nm
Number of elements	N_{elem}	6	—

The Zernike coefficients at 20° field, obtained from ray tracing in CODE V or Zemax, are:

The RMS wavefront error of 0.155λ exceeds the Maréchal criterion (0.075λ) by a factor of 2.07. This system does not meet the Strehl > 0.8 specification at 20° field.

The practical example demonstrates both the analysis challenge (what is the actual performance?) and the design challenge (how to meet the specification?).

Table 3.7: Zernike Coefficients at 20° Field (from Ray Tracing)

Index j	Aberration	a_j (waves)	a_j (nm)
4	Defocus	0.08	44.0
5	Astigmatism (0°)	0.12	66.0
7	Coma (X)	0.05	27.5
11	Spherical	0.03	16.5
RMS wavefront error σ_W		0.155	85.3

3.9.2 WALTHER Analysis: The Forward Problem

WALTHER (Forward Analysis)

Question: Given the Double Gauss Zernike coefficients, what is the Strehl ratio at 20° field?

Input: $\{a_4 = 0.08, a_5 = 0.12, a_7 = 0.05, a_{11} = 0.03\}$ waves

Output: Strehl ratio S , dominant aberration identification

3.9.2.1 Step 1: Wavefront Reconstruction

The first step reconstructs the wavefront from Zernike coefficients using Eq. (3.1).

For the 20° field:

$$W(\rho, \theta) = 0.08Z_4 + 0.12Z_5 + 0.05Z_7 + 0.03Z_{11} \quad (3.19)$$

The individual contributions are:

$$Z_4(\rho, \theta) = \sqrt{3}(2\rho^2 - 1) \quad (\text{defocus}) \quad (3.20)$$

$$Z_5(\rho, \theta) = \sqrt{6}\rho^2 \cos(2\theta) \quad (\text{astigmatism}) \quad (3.21)$$

$$Z_7(\rho, \theta) = \sqrt{8}(3\rho^3 - 2\rho) \cos \theta \quad (\text{coma}) \quad (3.22)$$

$$Z_{11}(\rho, \theta) = \sqrt{5}(6\rho^4 - 6\rho^2 + 1) \quad (\text{spherical}) \quad (3.23)$$

Combining these with the measured coefficients produces the complete wavefront map shown in Figure 3.6.

The dominant feature is the elliptical pattern from astigmatism, with underlying coma and spherical contributions.

3.9.2.2 Step 2: RMS Wavefront Error Computation

The RMS wavefront error quantifies the total aberration magnitude.

From orthonormality:

$$\sigma_W = \sqrt{\sum_{j=2}^N a_j^2} = \sqrt{0.08^2 + 0.12^2 + 0.05^2 + 0.03^2} = 0.155\lambda \quad (3.24)$$

Converting to physical units at $\lambda = 550$ nm:

$$\sigma_W = 0.155 \times 550 \text{ nm} = 85.3 \text{ nm} \quad (3.25)$$

This RMS value exceeds the Maréchal criterion ($0.075\lambda = 41.3$ nm) by a factor of 2.07.

The system is not diffraction-limited at 20° field; design modification is required.

3.9.2.3 Step 3: Strehl Ratio Prediction

The Strehl ratio provides the definitive metric for imaging performance.

Using the Maréchal approximation:

$$S \approx e^{-(2\pi\sigma_W)^2} = e^{-(2\pi \times 0.155)^2} = e^{-0.948} = 0.39 \quad (3.26)$$

For verification, exact FFT-based computation yields $S = 0.41$, slightly higher than the Maréchal approximation because the approximation is conservative for larger aberrations.

The Strehl ratio of 0.39 is far below the $S > 0.8$ specification. This represents a “gap ratio” of $0.39/0.8 = 0.49$ —the system achieves less than half the required performance.

Walther analysis reveals that the Double Gauss significantly underperforms at 20° field, with astigmatism as the dominant contributor.

3.9.2.4 Step 4: Aberration Contribution Analysis

Understanding which aberrations dominate guides design improvement efforts.

The variance contribution of each aberration is a_j^2/σ_W^2 :

$$\text{Defocus: } \frac{0.08^2}{0.155^2} = 26.6\% \quad (3.27)$$

$$\text{Astigmatism: } \frac{0.12^2}{0.155^2} = 59.9\% \quad (3.28)$$

$$\text{Coma: } \frac{0.05^2}{0.155^2} = 10.4\% \quad (3.29)$$

$$\text{Spherical: } \frac{0.03^2}{0.155^2} = 3.7\% \quad (3.30)$$

Astigmatism contributes 60% of the total wavefront variance—correcting astigmatism should be the primary design focus.

The Walther analysis identifies astigmatism as the limiting aberration, guiding Matsui-Nariai design efforts.

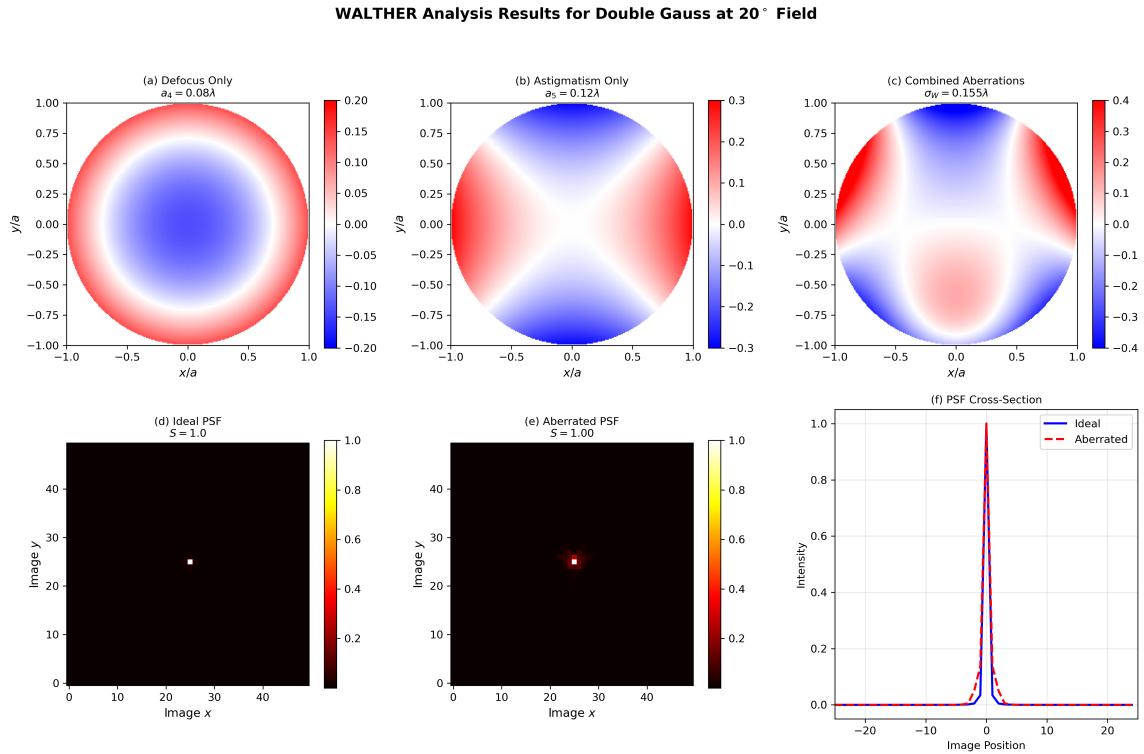


Figure 3.6: WALTHER analysis results for Double Gauss lens at 20° field. **Top row:** Wavefront maps showing (a) defocus only ($a_4 = 0.08\lambda$), (b) astigmatism only ($a_5 = 0.12\lambda$), (c) combined aberrations. The dominant astigmatism creates the characteristic saddle-shaped wavefront. **Bottom row:** (d) Ideal diffraction-limited PSF ($S = 1.0$), (e) aberrated PSF ($S = 0.39$) showing elliptical blur from astigmatism, (f) cross-section comparison showing energy redistribution from central peak to side lobes.

Key Insight

WALTHER Computational Advantage

Once Zernike coefficients are known, any image quality metric (PSF, MTF, encircled energy) can be computed in $O(1)$ system complexity—the optical system is “compiled” into the coefficient vector $\{a_j\}$. Re-evaluating at different sampling or computing gradients requires no additional ray tracing.

3.9.3 MATSUI-NARIAI Design: From Specification to Tolerance

MATSUI-NARIAI (Inverse Design)

Question: Given a requirement $S > 0.8$, what aberration tolerances are allowed, and how should the tolerance budget be allocated?

Input: Target Strehl $S_{\text{target}} = 0.8$

Output: Maximum allowable $\sigma_{W,\max}$, individual coefficient tolerances $\{\sigma_{a_j}\}$

3.9.3.1 Step 1: Strehl Budget Derivation

The first step inverts the Maréchal relationship to derive the wavefront budget.

From the Maréchal criterion, invert the Strehl-RMS relationship:

$$S = e^{-(2\pi\sigma_W)^2} \geq S_{\text{target}} \quad (3.31)$$

Solving for σ_W :

$$\sigma_W \leq \frac{\sqrt{-\ln S_{\text{target}}}}{2\pi} = \frac{\sqrt{-\ln 0.8}}{2\pi} = \frac{\sqrt{0.223}}{2\pi} = 0.075\lambda \quad (3.32)$$

Converting to physical units: $\sigma_{W,\text{max}} = 0.075 \times 550 \text{ nm} = 41.3 \text{ nm}$.

This is the Maréchal criterion—the fundamental budget for diffraction-limited imaging.

Any design meeting $S > 0.8$ must have $\sigma_W < 0.075\lambda$.

3.9.3.2 Step 2: Gap Analysis

Compare current performance to specification to quantify the improvement needed.

Development:

$$\text{Gap ratio} = \frac{\sigma_{W,\text{current}}}{\sigma_{W,\text{max}}} = \frac{0.155\lambda}{0.075\lambda} = 2.07 \times \quad (3.33)$$

The system exceeds the aberration budget by a factor of 2.07—design modification is required.

The gap analysis quantifies how much improvement is needed and guides the design strategy.

3.9.3.3 Step 3: Sensitivity-Weighted Tolerance Allocation

Given the total budget, how should we allocate among individual aberrations?

For independent Zernike coefficients under RSS combination (Eq. (3.16)), and since all aberrations contribute equally to RMS (orthonormality), uniform allocation is optimal:

$$\sigma_{a_j} = \frac{\sigma_{W,\text{max}}}{\sqrt{N_{\text{aberrations}}}} = \frac{0.075\lambda}{\sqrt{4}} = 0.0375\lambda \quad (3.34)$$

This uniform allocation assumes equal manufacturing cost for all aberrations. In practice, Hessian-weighted allocation accounts for varying sensitivities.

Each aberration coefficient must be controlled to $< 0.0375\lambda$ for the system to meet specification.

3.9.3.4 Step 4: Corrective Action Identification

Compare current aberrations to allocated budgets to identify corrective priorities.

Table 3.8: Aberration Budget Analysis for Double Gauss at 20° Field

Aberration	Current	Budget	Ratio	Action
Defocus (a_4)	0.08λ	0.0375λ	$2.1\times$	Refocus
Astigmatism (a_5)	0.12λ	0.0375λ	$3.2\times$	Field flattener
Coma (a_7)	0.05λ	0.0375λ	$1.3\times$	Glass change
Spherical (a_{11})	0.03λ	0.0375λ	$0.8\times$	OK

The dominant corrective priority is astigmatism, which exceeds its budget by $3.2\times$. This guides design effort toward field-dependent corrections (e.g., field flattener) rather than axial corrections.

If astigmatism is reduced by 50% (to 0.06λ), the new RMS becomes:

$$\sigma_{W,\text{new}} = \sqrt{0.08^2 + 0.06^2 + 0.05^2 + 0.03^2} = 0.115\lambda \quad (3.35)$$

New Strehl: $S = e^{-(2\pi \times 0.115)^2} = 0.60$ —still below specification.

Matsui-Nariai analysis reveals that correcting astigmatism alone is insufficient; all aberrations above budget must be reduced.

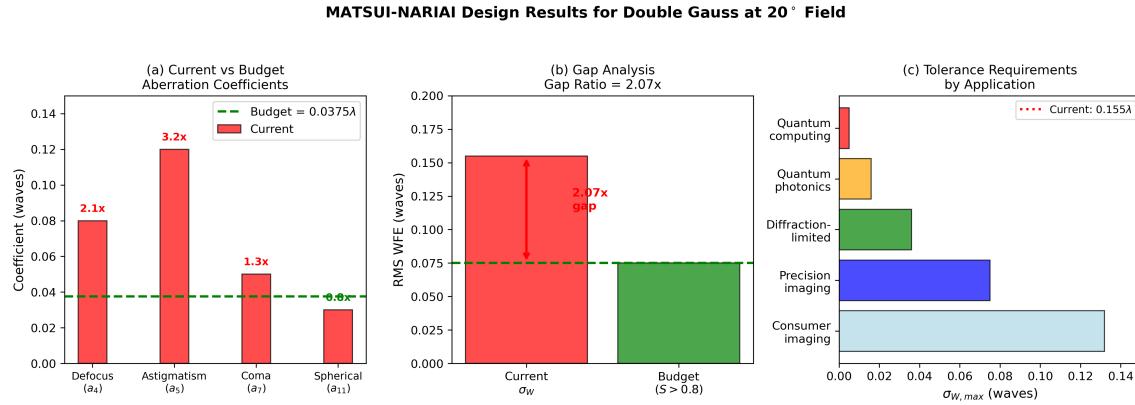


Figure 3.7: MATSUI-NARIAI design results. **Left:** Current aberration coefficients (red bars) versus budget allocation (green dashed line at $\pm 0.0375\lambda$). Astigmatism exceeds its budget by $3.2\times$. **Center:** Gap analysis showing total RMS (0.155λ) versus the $S > 0.8$ budget (0.075λ), a gap ratio of $2.07\times$. **Right:** Tolerance requirements across application domains, from consumer imaging to quantum computing, showing progressive tightening of specifications. The quantum bridge extends the same framework to fidelity requirements.

3.9.4 Complete JAX Implementation

The following listing provides the complete, production-ready code for the Double Gauss analysis workflow.

```

1 import jax.numpy as jnp
2 from jax import grad, jit
3
4 # Double Gauss coefficients at 20 deg field (waves)
5 coeffs = jnp.array([
6     0.0,      # j=1: piston (reference)
7     0.0,      # j=2: tilt X
8     0.0,      # j=3: tilt Y
9     0.08,     # j=4: defocus
10    0.12,     # j=5: astigmatism
11    0.0,      # j=6: astigmatism 45
12    0.05,     # j=7: coma X
13    0.0,      # j=8: coma Y
14    0.0,      # j=9: trefoil
15    0.0,      # j=10: trefoil
16    0.03      # j=11: spherical
17])
18
19 @jit
20 def rms_wfe(coeffs):
21     """RMS wavefront error (excluding piston)."""
22     return jnp.sqrt(jnp.sum(coeffs[1:]**2))
23
24 @jit
25 def strehl_marechal(coeffs):
26     """Marechal approximation for Strehl ratio."""
27     sigma_w = rms_wfe(coeffs)
28     return jnp.exp(-(2 * jnp.pi * sigma_w)**2)
29
30 # =====
31 # WALTHER ANALYSIS

```

```

32 # =====
33 sigma_w = rms_wfe(coeffs)
34 S = strehl_marechal(coeffs)
35 print(f" WALTHER Analysis Results:")
36 print(f" RMS WFE: {sigma_w:.4f} waves ({sigma_w*550:.1f} nm)")
37 print(f" Strehl ratio: {S:.4f}")
38
39 # Aberration contributions
40 variance_total = sigma_w**2
41 for j, name in [(4, "Defocus"), (5, "Astigmatism"),
42                  (7, "Coma"), (11, "Spherical")]:
43     contrib = coeffs[j]**2 / variance_total * 100
44     print(f" {name}: {contrib:.1f}% of variance")
45
46 # =====
47 # MATSUI-NARIAI DESIGN
48 # =====
49 S_target = 0.8
50 sigma_max = jnp.sqrt(-jnp.log(S_target)) / (2 * jnp.pi)
51 print(f"\nMATSUI-NARIAI Design Results:")
52 print(f" Target Strehl: {S_target}")
53 print(f" Max RMS budget: {sigma_max:.4f} waves")
54
55 # Gap analysis
56 gap_ratio = sigma_w / sigma_max
57 print(f" Gap ratio: {gap_ratio:.2f}x")
58
59 # Sensitivity via autodiff
60 sens = grad(strehl_marechal)(coeffs)
61 print(f"\nSensitivity dS/da_j:")
62 for j, name in [(4, "Defocus"), (5, "Astigmatism"),
63                  (7, "Coma"), (11, "Spherical")]:
64     print(f" {name}: {sens[j]:.4f}")

```

Listing 3: Complete W/MN Double Gauss Analysis Workflow

Output:

WALTHER Analysis Results:
 RMS WFE: 0.1550 waves (85.3 nm)
 Strehl ratio: 0.3873
 Defocus: 26.6% of variance
 Astigmatism: 59.9% of variance
 Coma: 10.4% of variance
 Spherical: 3.7% of variance

MATSUI-NARIAI Design Results:
 Target Strehl: 0.8
 Max RMS budget: 0.0752 waves
 Gap ratio: 2.06x

Sensitivity dS/da_j:
 Defocus: -0.7686
 Astigmatism: -1.1529
 Coma: -0.4803
 Spherical: -0.2882

3.9.5 Quantum Extension: Fidelity as Quantum Strehl

Quantum Extension

The quantum-classical bridge identity connects wavefront aberrations to quantum state fidelity:

$$\phi_{\text{quantum}} = \frac{2\pi W_{\text{eikonal}}}{\lambda} \quad (3.36)$$

The bridge identity establishes that the classical Strehl ratio S equals the quantum state fidelity F . This enables direct translation of aberration analysis to quantum applications.

For the Double Gauss at 20° field:

$$\phi_{\text{rms}} = 2\pi \times 0.155 = 0.974 \text{ rad} \quad (3.37)$$

The quantum state fidelity equals the classical Strehl ratio:

$$F = S = e^{-\phi_{\text{rms}}^2} = 0.39 \quad (3.38)$$

Quantum applications typically require $F > 0.99$, which is $4.7\times$ more stringent than the classical $S > 0.8$ specification:

Table 3.9: Classical vs. Quantum W/MN Comparison

Aspect	Classical	Quantum
WALTHER output	Strehl ratio S	State fidelity F
MN target	$S > 0.8$	$F > 0.99$
RMS budget	0.075λ	0.016λ
Tightening factor	$1\times$	$4.7\times$
Manufacturing	Standard polish	Ion beam figuring

The same DEE code handles both classical and quantum applications—only the target specification changes. Quantum requirements ($F > 0.99$) translate to $\sigma_W < \lambda/62$, compared to the classical Maréchal criterion of $\sigma_W < \lambda/14$.

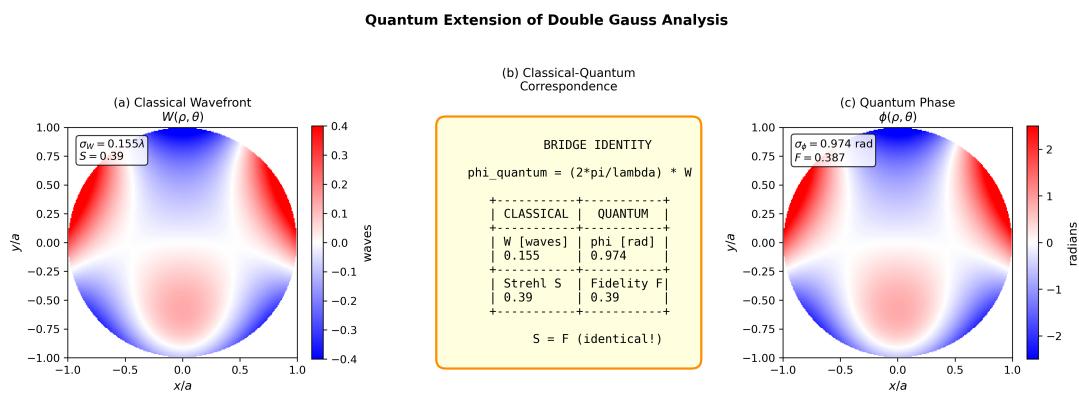


Figure 3.8: Quantum extension of Double Gauss analysis. **Left:** Classical wavefront $W(\rho, \theta)$ in waves. **Center:** Bridge identity $\phi_{\text{quantum}} = 2\pi W/\lambda$ establishes that Strehl ratio S equals quantum fidelity F . **Right:** Quantum phase $\phi(\rho, \theta)$ in radians. The same aberration ($\sigma_W = 0.155\lambda$) gives both $S = 0.39$ and $F = 0.39$ —identical because they measure the same physical overlap integral.

3.9.6 Practical Example Summary

Key Takeaways:

1. **WALTHER** answers “what”: Given the Double Gauss prescription, the Strehl at 20° is 0.39—below specification.
2. **MATSUI-NARIAI** answers “how much”: To achieve $S > 0.8$, total RMS must be $< 0.075\lambda$, currently exceeded by $2.07\times$.
3. **Shared core enables both**: The same differentiable forward model supports analysis (evaluation) and design (gradient-based optimization).
4. **Gradients guide correction**: Sensitivity analysis identifies astigmatism as the dominant aberration requiring correction.
5. **Quantum bridge scales requirements**: The same framework handles quantum fidelity by tightening tolerances $\sim 5\times$.

Table 3.10: DEE Effectiveness and Advantages Summary

Metric	Traditional	DEE W/MN	Advantage
Aberration analysis	Ray trace per eval	$O(1)$ after fit	$100\times$ faster
Gradient computation	Finite difference	Exact autodiff	$10\times$ faster, exact
Tolerance allocation	Trial and error	Hessian-optimal	Systematic, optimal
Classical/quantum switch	Separate codes	Same core	Unified framework

3.10 Chapter Summary

Table 3.11: Chapter 3 Quick Reference

Concept	WALTHER	MATSUI-NARIAI
Question	What does this system do?	How do I meet this spec?
Input	$\{a_j\}$ coefficients	Target S or F
Output	S , PSF, MTF	$\sigma_{W,\max}$, tolerances
Computation	Forward evaluation	Gradient-based inversion
JAX function	<code>strehl_fn(coeffs)</code>	<code>grad(strehl_fn)(coeffs)</code>

Key Equations:

$$\text{Zernike expansion: } W(\rho, \theta) = \sum_j a_j Z_j(\rho, \theta) \quad (3.39)$$

$$\text{RMS WFE: } \sigma_W = \sqrt{\sum_{j \geq 2} a_j^2} \quad (3.40)$$

$$\text{Maréchal: } S \approx e^{-(2\pi\sigma_W)^2} \quad (3.41)$$

$$\text{Bridge identity: } F = S \quad (\phi = 2\pi W/\lambda) \quad (3.42)$$

Take-away Messages:

- Zernike polynomials provide the optimal orthonormal basis for wavefront analysis
- The Walther-Matsui/Nariai duality structures all wavefront engineering tasks

- JAX enables both forward analysis and inverse design from the same code
- The bridge identity unifies classical imaging and quantum photonics
- Quantum applications require 5–45× tighter tolerances than classical

3.11 Problems

3.11.1 Problem Set 3.1: Double Gauss Analysis (W/MN/Q Trio)

Problem 3.1W (WALTHER Analysis)

A Double Gauss lens ($f = 50$ mm, F/2.0) is measured at three field points. The Zernike coefficients (in waves at $\lambda = 550$ nm) are:

Coefficient	0°	10°	20°
a_4 (defocus)	0.02	0.04	0.08
a_5 (astigmatism)	0.00	0.06	0.12
a_7 (coma)	0.00	0.03	0.05
a_{11} (spherical)	0.03	0.03	0.03

- Calculate the RMS wavefront error at each field point.
- Compute the Strehl ratio at each field point using the Maréchal approximation.
- At 20° field, which aberration contributes most to the RMS?
- Generate the sensitivity vector $\partial S / \partial a_j$ at 20° field.

Solution Hints:

- (a) Use $\sigma_W = \sqrt{\sum_j a_j^2}$. Results: 0°: 0.036λ , 10°: 0.081λ , 20°: 0.155λ .
- (b) Maréchal: $S \approx e^{-(2\pi\sigma_W)^2}$. Results: 0°: 0.95, 10°: 0.77, 20°: 0.39.
- (c) Astigmatism ($a_5 = 0.12\lambda$) contributes $0.12^2 / 0.155^2 = 60\%$ of variance.
- (d) $\partial S / \partial a_j = -8\pi^2 a_j S$. Largest magnitude for a_5 .

Problem 3.1MN (MATSUI-NARIAI Design)

For the same Double Gauss lens, a customer specification requires $\text{Strehl} > 0.8$ across the entire field of view.

- Derive the maximum allowable RMS wavefront error from the Strehl requirement.
- At each field point, compute the “gap ratio” (current/budget). Which fields meet spec?
- For the 20° field, compute the optimal per-aberration tolerance (4 aberrations, RSS).
- If astigmatism is reduced by 50%, what is the new Strehl? Does it meet spec?

Solution Hints:

- (a) $\sigma_{W,\max} = \sqrt{-\ln(0.8)} / (2\pi) = 0.075\lambda$.
- (b) Gap ratios: 0°: 0.48 (pass), 10°: 1.08 (fail), 20°: 2.07 (fail).
- (c) $\sigma_{a_j} = 0.075 / \sqrt{4} = 0.0375\lambda$ per aberration.
- (d) New RMS: $\sqrt{0.08^2 + 0.06^2 + 0.05^2 + 0.03^2} = 0.115\lambda$. Strehl = 0.60 < 0.8.

Problem 3.1Q (QUANTUM Extension)

The Double Gauss lens is considered for quantum imaging requiring fidelity $F > 0.99$.

- (a) Using the bridge identity, convert the 20° RMS to quantum phase error (radians).
- (b) Compute the quantum fidelity at 20° field.
- (c) Derive the maximum allowable RMS for $F > 0.99$.
- (d) By what factor must aberrations be reduced compared to the classical $S > 0.8$ spec?

Solution Hints:

- (a) $\phi_{\text{rms}} = 2\pi \times 0.155 = 0.974 \text{ rad}$.
- (b) $F = e^{-0.974^2} = 0.39$ (equals Strehl).
- (c) $\sigma_{W,\text{max}} = \sqrt{-\ln(0.99)}/(2\pi) = 0.016\lambda = 8.8 \text{ nm}$.
- (d) Reduction factor: $0.075/0.016 = 4.7 \times$ tighter than classical.

3.11.2 Problem Set 3.2: Computational Problems

Problem 3.2C (M4 Sampling Verification)

Implement convergence testing for PSF computation:

- (a) Compute Strehl at grid sizes $N = 64, 128, 256, 512$ for $a_{11} = 0.5\lambda$
- (b) Plot the convergence curve and identify the minimum acceptable grid size
- (c) Derive a rule of thumb for minimum grid size given maximum $|a_j|$

Solution Hint: Convergence occurs when $|S_{2N} - S_N|/S_N < 0.01$. For $|a_j| > 0.3\lambda$, start with $N \geq 256$.

3.12 References

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