Abbreviated Terms

The following abbreviations are used throughout this chapter:

Abbreviation	Full Term
QWS	Quantum Wavefront Sensing
SQL	Standard Quantum Limit
HL	Heisenberg Limit
OPA	Optical Parametric Amplifier
MPLC	Multi-Plane Light Converter
QFI	Quantum Fisher Information
WFE	Wavefront Error
RMS	Root Mean Square
PSF	Point Spread Function
DEE	Differentiable Eikonal Engine
W	Walther (forward analysis)
MN	Matsui-Nariai (inverse design)
LO	Local Oscillator
PLL	Phase Lock Loop
AO	Adaptive Optics
LIGO	Laser Interferometer Gravitational-Wave Observatory

Table 9.1: Chapter 9 Notation Summary

Symbol	Meaning	Units/Range
$W(\rho, \theta)$	Classical wavefront	[waves]
ϕ	Quantum phase	[rad]
a_n	Zernike coefficient	[waves]
$Z_n(\rho, \theta)$	Zernike polynomial	dimensionless
N	Photon number	$\mathbb{Z}^+$
$\delta\phi$	Phase uncertainty	[rad]
r	Squeezing parameter	dimensionless
η	System efficiency	$[0, 1]$
F_Q	Quantum Fisher information	[rad^{-2}]
$\mathbf{H}$	Classical Hessian	[waves^{-2}]
Q	Quantum advantage factor	dimensionless

Chapter 9
Quantum Wavefront Sensing

Learning Objectives

After completing this chapter, you will be able to:

1. Recognize Zernike coefficients as quantum observables—no reformulation needed
2. Understand and apply SQL ($1/\sqrt{N}$) vs Heisenberg limit ($1/N$) scaling
3. Design squeezed-light enhanced wavefront sensors with quantitative specifications
4. Compute quantum advantage factors including realistic loss effects
5. Apply the bridge identity $\phi = 2\pi W/\lambda$ to transfer classical design skills
6. Implement quantum wavefront sensing algorithms in JAX with automatic differentiation
7. Translate classical wavefront sensor specifications to quantum requirements
8. Identify failure modes and their mitigation strategies
9. Execute complete design workflows for gravitational wave optics and adaptive optics

9.1 Introduction: Why Quantum Wavefront Sensing Opens Part III

Part III of this book transitions from classical optical design to quantum photonic applications. Of all possible quantum optical systems, why does *quantum wavefront sensing* serve as our entry point? The answer lies in both pedagogical and practical considerations that make QWS the ideal bridge between the classical foundations of Parts I–II and the advanced quantum applications that follow.

Three compelling reasons establish QWS as the natural starting point for quantum extensions:

9.1.1 Reason 1: Direct Knowledge Transfer

Wavefront sensing is arguably the most mature subfield of optical engineering. Every optical engineer has direct experience with Shack-Hartmann sensors, interferometers, and Zernike polynomial analysis. The profound insight of this chapter is that *this knowledge transfers directly to quantum systems*:

Key Insight

Zernike coefficients ARE quantum observables. The classical wavefront expansion $W(\rho, \theta) = \sum_n a_n Z_n(\rho, \theta)$ maps *exactly* to quantum measurement operators. No reformulation is needed—your classical intuition transfers directly. The bridge identity $\phi = 2\pi W/\lambda$ is not an analogy; it is a mathematical identity.

This direct mapping means that classical tolerance analysis, aberration budgets, and merit function optimization apply unchanged to quantum systems—with one critical modification: the sensitivity requirements tighten by factors of 10–1000×.

9.1.2 Reason 2: Experimental Maturity

Unlike many quantum technologies that remain laboratory curiosities, quantum-enhanced wavefront sensing has achieved practical deployment. The LIGO gravitational wave detectors have used squeezed light since 2019 [4, 5], demonstrating that quantum enhancement works in billion-dollar instruments requiring extreme reliability. Table 9.2 summarizes the technology readiness.

Table 9.2: Quantum Wavefront Sensing Technology Maturity

Technology	**Current State**	**TRL**
Squeezed light sources (OPA)	Commercial availability	9
Balanced homodyne detection	Routine laboratory use	8
Mode sorters (MPLC)	Research demonstrations	5–6
Integrated quantum sensors	Early prototypes	3–4

9.1.3 Reason 3: Foundation for Advanced Chapters

The concepts introduced here—quantum Fisher information, Heisenberg scaling, loss budgets, and the specification tightening framework—recur throughout Chapters 10–12. Mastering QWS provides the vocabulary and analytical tools for understanding quantum walks (Chapter 10), N-photon imaging (Chapter 11), and quantum-to-production workflows (Chapter 12).

The structure of this chapter follows the established W/MN duality framework. We begin with forward analysis (Walther: given a quantum sensor, what is its performance?) and proceed to inverse design (Matsui-Nariai: given a sensitivity target, how do we build the sensor?). Two comprehensive practical examples demonstrate the complete workflow.

By the end of this chapter, you will possess the tools to design quantum-enhanced wavefront sensors that achieve precision beyond classical limits, with full understanding of when quantum enhancement is beneficial and when classical approaches suffice.

Quantum Caveat: State of the Art

The quantum enhancement claims in this chapter are grounded in established physics (Caves 1981 [1], LIGO deployment 2019 [5]). However, readers should note:

1. **Idealized conditions:** Theoretical limits assume perfect mode matching and coherence—practical systems achieve 50–80% of theoretical advantage
2. **Loss sensitivity:** Quantum advantage degrades rapidly with optical loss; systems with $\eta < 50\%$ show no practical benefit
3. **No experimental validation in this book:** The predictions herein are theoretical; validation would require dedicated experiments
4. **State of the art:** As of 2025, 15 dB squeezing has been demonstrated [6]; the examples use conservative 10–13 dB values

Let us back to highlight the pain points in the view of optical engineers:

Optical Engineer's Pain Points

WALTHER (Forward Analysis): "I have a wavefront sensor with known specifications (lenslet count, detector noise, integration time). What is my fundamental phase sensitivity limit, and is it quantum-limited?"

MATSUI-NARIAI (Inverse Design): "I need $\delta\phi < 10^{-6}$ rad for my gravitational wave detector optics. What photon flux, squeezing level, and system efficiency do I need to achieve this?"

The Gap: Classical optical engineers have decades of wavefront sensing experience but lack the quantum framework to understand ultimate limits or leverage quantum enhancement. Quantum physicists understand squeezed states but cannot translate this to practical sensor specifications. **This chapter bridges that gap.**

9.2 The Classical-Quantum Bridge for Wavefront Sensing

The power of the eikonal framework lies in its universality. The same mathematical structure that describes classical wavefront aberrations also describes quantum phase—not by analogy, but by identity. This section establishes the precise mathematical correspondence that enables classical optical engineers to leverage their existing expertise for quantum applications.

9.2.1 The Central Identity

The bridge identity from Chapter 1 applies directly to wavefront sensing:

$$\boxed{\phi_{\text{quantum}} = \frac{2\pi}{\lambda} W_{\text{eikonal}}} \tag{9.1}$$

This is *not* an analogy—it is an exact mathematical identity. The classical wavefront aberration $W(x, y)$ *is* the quantum phase $\phi(x, y)$ scaled by $2\pi/\lambda$. The implications are profound:

1. **Same infrastructure:** Code that computes wavefront aberrations directly computes quantum phases
2. **Same observables:** Zernike coefficients measured classically are identical to quantum expectation values
3. **Same optimization:** Minimizing RMS WFE maximizes quantum state fidelity

The conversion is bidirectional. Given a classical wavefront specification (e.g., RMS WFE $< \lambda/20$), we immediately obtain the quantum phase requirement:

$$\delta\phi_{\max} = \frac{2\pi}{\lambda} \cdot \frac{\lambda}{20} = \frac{\pi}{10} \approx 0.31 \text{ rad} \tag{9.2}$$

For quantum applications requiring fidelity $F > 0.99$, the phase tolerance is:

$$\delta\phi < \sqrt{2(1-F)} = \sqrt{2(0.01)} = 0.14 \text{ rad} \Rightarrow \delta W < \frac{\lambda}{45} \tag{9.3}$$

The bridge identity enables direct translation of classical specifications to quantum requirements and vice versa, without any intermediate reformulation.

9.2.2 Zernike Coefficients as Quantum Observables

The classical wavefront expansion takes on new meaning when we recognize that optical phase is a quantum mechanical observable.

The classical expansion:

$$W(\rho,\theta) = \sum_n a_n Z_n(\rho,\theta) \tag{9.4}$$

corresponds to a quantum measurement in the Zernike mode basis. Each coefficient a_n is the expectation value of a projection operator onto the n-th Zernike mode [7]:

$$a_n = \langle \hat{Z}_n \rangle = \mathrm{Tr}[\rho \hat{Z}_n] \tag{9.5}$$

where $\hat{Z}_n$ is the Zernike mode projector and ρ is the quantum state of the optical field.

This correspondence has a remarkable consequence: the mode sorter technology developed for classical wavefront sensing (Multi-Plane Light Converters, MPLCs) directly implements quantum mode-selective measurement. No new measurement technology is required—the same hardware measures both classical and quantum wavefronts.

Classical wavefront sensing infrastructure serves quantum metrology with only source modifications (adding squeezing) and detection upgrades (balanced homodyne instead of intensity detection).

9.3 Quantum Limits: SQL and Heisenberg Bound

Every measurement is limited by fundamental quantum fluctuations. Understanding these limits is essential for determining when quantum enhancement provides real benefit and when classical methods are sufficient.

9.3.1 The Standard Quantum Limit (SQL)

When measuring optical phase with coherent light (the output of a standard laser), the phase uncertainty is bounded by shot noise:

$$\boxed{\delta\phi_{\mathrm{SQL}} = \frac{1}{\sqrt{N}}} \tag{9.6}$$

where N is the total number of detected photons. This is the **Standard Quantum Limit**—the best achievable sensitivity with classical light sources.

For a Shack-Hartmann sensor with N_{lenslet} lenslets, each receiving n photons:

$$\delta\phi_{\text{per lenslet}} = \frac{1}{\sqrt{n}}, \quad \delta\phi_{\mathrm{global}} = \frac{1}{\sqrt{N_{\mathrm{lenslet}} \cdot n}} = \frac{1}{\sqrt{N_{\mathrm{total}}}} \tag{9.7}$$

Numerical example: A typical wavefront sensor receiving 10^6 photons achieves:

$$\delta\phi_{\mathrm{SQL}} = \frac{1}{\sqrt{10^6}} = 10^{-3} \text{ rad} = 0.057 \Rightarrow \delta W = \frac{\lambda}{6300} \tag{9.8}$$

The SQL appears to be the ultimate classical limit. However, the "standard" in SQL refers to *standard quantum states* (coherent states), not the ultimate quantum limit.

9.3.2 The Heisenberg Limit

Quantum mechanics permits better scaling using entangled or squeezed states:

$$\boxed{\delta\phi_{\mathrm{HL}} = \frac{1}{N}} \tag{9.9}$$

This is the **Heisenberg Limit**—the ultimate bound on phase sensitivity allowed by quantum mechanics. The improvement over SQL is dramatic:

$$\frac{\delta\phi_{\mathrm{SQL}}}{\delta\phi_{\mathrm{HL}}} = \sqrt{N} \tag{9.10}$$

For $N = 10^6$ photons, this represents a 1000× improvement in sensitivity.

Achieving the Heisenberg limit requires exotic quantum states (NOON states, spin-squeezed states) that are extremely fragile. Practical systems use *squeezed states* that achieve intermediate scaling between SQL and Heisenberg.

The practical quantum advantage lies in squeezed-state enhancement, which provides 4–20× improvement over SQL with currently available technology, rather than attempting to reach the fragile Heisenberg limit.

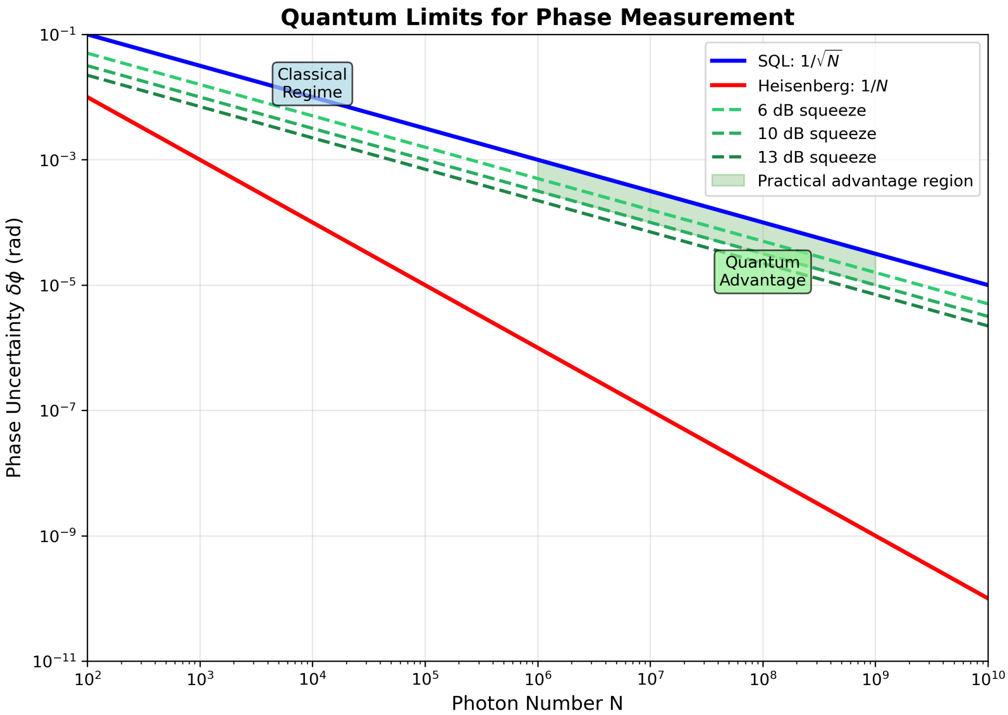

Figure 9.1: Quantum limits for phase measurement. The Standard Quantum Limit (SQL, blue) scales as $1/\sqrt{N}$, representing the best achievable with coherent light. The Heisenberg Limit (red) scales as $1/N$, representing the ultimate quantum bound. Squeezed states (green curves) achieve intermediate performance, with improvement factor depending on squeezing level r. Practical systems with 10–13 dB squeezing operate in the green region.

9.4 Squeezed Light Enhancement

Squeezed states of light redistribute quantum uncertainty between conjugate variables (amplitude and phase), enabling sub-SQL phase sensitivity at the cost of increased amplitude noise. This section develops the theory and practical implementation of squeezed-light wavefront sensing.

9.4.1 Squeezing Parameter and Phase Sensitivity

A squeezed state is characterized by the squeezing parameter r, which determines the redistribution of quantum uncertainty. The phase uncertainty becomes:

$$\boxed{\delta\phi_{\text{squeezed}} = \frac{e^{-r}}{\sqrt{N}}} \tag{9.11}$$

The improvement over SQL is the quantum advantage factor:

$$Q = \frac{\delta\phi_{\text{SQL}}}{\delta\phi_{\text{squeezed}}} = e^r \tag{9.12}$$

dB Convention: Squeezing is commonly expressed in decibels:

$$r_{\text{dB}} = 20\log_{10}(e^r)\cdot r = 8.686\cdot r \tag{9.13}$$

Table 9.3: Squeezing Level and Quantum Advantage

Squeezing (dB)	**Parameter r**	**Advantage $Q = e^r$**	**Achieved**
3	0.35	1.4×	Routine
6	0.69	2.0×	Commercial
10	1.15	3.2×	LIGO (2019)
13	1.50	4.5×	Research
15	1.73	5.6×	State-of-art

These theoretical improvements assume perfect systems with no optical loss. Real systems suffer significant degradation, as addressed in Section 9.5.

With 10–13 dB of squeezing (readily achievable with current technology), wavefront sensors can improve their sensitivity by a factor of 3–5× over the SQL baseline.

9.5 Loss Effects and Efficiency Budget

Optical loss is the primary enemy of quantum enhancement. Unlike classical systems where loss simply reduces signal strength, quantum systems suffer fundamental degradation of their quantum properties. Understanding and minimizing loss is critical for achieving practical quantum advantage.

9.5.1 Loss Degradation Formula

When squeezed light passes through a lossy channel with efficiency η (transmission), the effective squeezing parameter degrades according to:

$$\boxed{r_{\text{eff}} = r - \frac{1}{2}\ln\left(\frac{1}{\eta}\right)} \tag{9.14}$$

This formula reveals the devastating effect of loss:

- At $\eta = 90\%$ (10% loss): $r_{\text{eff}} = r - 0.053$ (0.46 dB degradation)
- At $\eta = 70\%$ (30% loss): $r_{\text{eff}} = r - 0.178$ (1.55 dB degradation)
- At $\eta = 50\%$ (50% loss): $r_{\text{eff}} = r - 0.347$ (3.01 dB degradation)

The quantum advantage factor after loss becomes:

$$Q_{\text{eff}} = e^{r_{\text{eff}}} = e^{r} \cdot \eta^{1/2} \tag{9.15}$$

For a system starting with 13 dB squeezing and 70% efficiency:

$$Q_{\text{eff}} = 4.5 \times 0.84 = 3.8\times \tag{9.16}$$

Still significant, but reduced from the ideal 4.5×.

9.5.2 Complete Loss Budget Example

Consider a quantum wavefront sensor with the following component chain:

Table 9.4: Detailed Loss Budget for Quantum Wavefront Sensor

Component	Transmission	Loss (dB)
OPA output coupling	98%	0.09
Collimating optics (2 surfaces)	99% each	0.09
Test optic (4 surfaces)	99.5% each	0.09
Mode sorter	92%	0.36
Relay to detector	98%	0.09
Detector quantum efficiency	95%	0.22
Total	**82.7%**	**0.83**

With 13 dB input squeezing and 82.7% efficiency:

$$r_{\text{eff}} = 1.50 - 0.5\ln(1/0.827) = 1.50 - 0.095 = 1.41 \tag{9.17}$$

$$\text{Effective squeezing} = 8.69 \times 1.41 = 12.2\text{ dB} \tag{9.18}$$

$$Q_{\text{eff}} = e^{1.41} = 4.1\times \tag{9.19}$$

Achieving high quantum advantage requires aggressive loss minimization throughout the optical train. Each component must be optimized for maximum transmission.

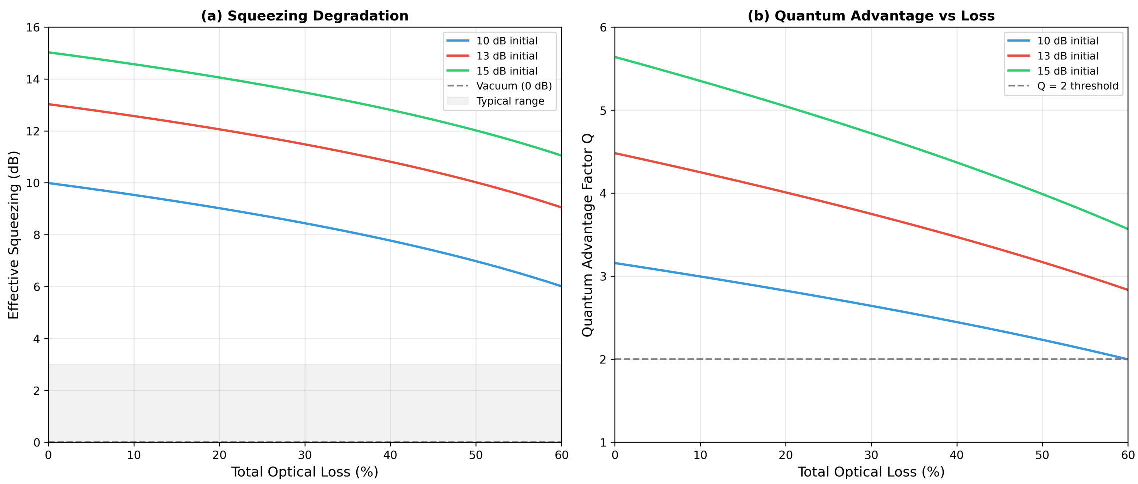

Figure 9.2: Squeezing degradation vs. optical loss. (a) Effective squeezing (dB) as a function of total optical loss for initial squeezing levels of 10, 13, and 15 dB. (b) Quantum advantage factor $Q = \delta\phi_{\text{SQL}}/\delta\phi_{\text{squeezed}}$ vs. loss. The dashed line marks $Q = 2$ (minimum useful advantage).

9.6 Quantum Wavefront Sensor Architecture

A complete quantum wavefront sensor combines several subsystems: squeezed light generation, mode-selective optics, and balanced homodyne detection. This section presents the hardware architecture and critical specifications.

9.6.1 Hardware Chain

A complete quantum wavefront sensor comprises five subsystems:

1. **Squeezed light source:** OPA with PPKTP or PPLN crystal, producing >10 dB squeezing
2. **Test optic interface:** Mode-matched illumination and collection with high transmission
3. **Mode sorter:** Multi-plane light converter (MPLC) for Zernike decomposition
4. **Balanced homodyne detector array:** One detector pair per Zernike mode
5. **Bayesian estimator:** Real-time coefficient extraction with uncertainty quantification

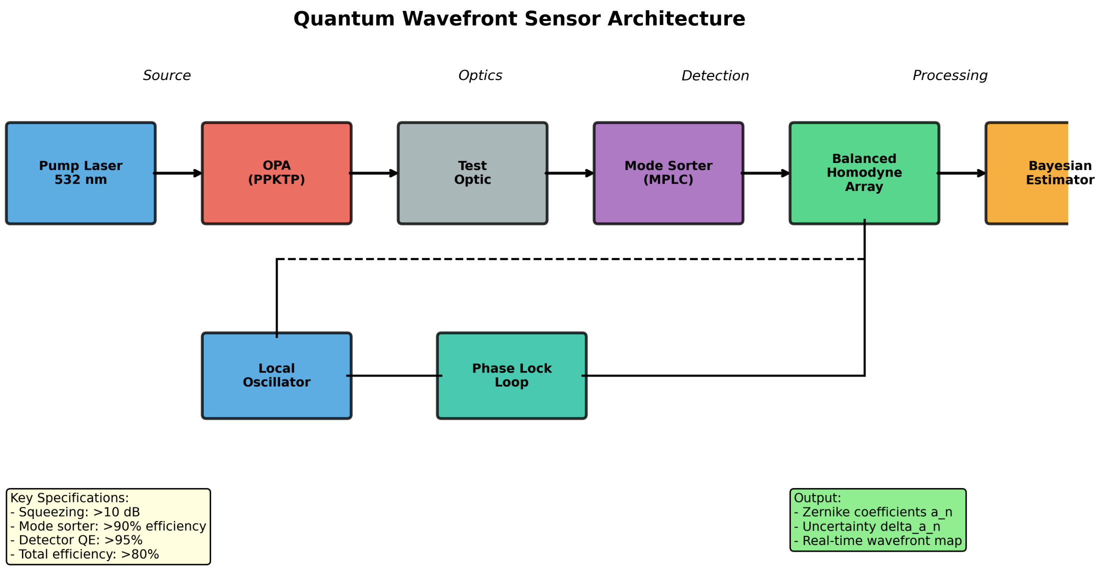

Figure 9.3: Quantum wavefront sensor architecture. Top: hardware chain from OPA source through mode sorter to balanced homodyne array. Bottom: signal flow and processing. The phase-lock loop (PLL) maintains coherence between signal and local oscillator. Key specifications: squeezing >10 dB, mode sorter efficiency >90%, detector QE >95%.

9.6.2 Mode Sorting Technology

The mode sorter is the critical component enabling parallel Zernike measurement. Multi-plane light converters (MPLCs) perform the unitary transformation [8]:

$$|Z_n\rangle \rightarrow |\text{pixel}_n\rangle \tag{9.20}$$

converting each Zernike mode to a spatially separated output. State-of-the-art MPLCs achieve:

- Mode count: 15–45 Zernike modes
- Efficiency: 85–95% per mode
- Crosstalk: <1% between adjacent modes

The mode sorter efficiency directly impacts quantum advantage. A 90% efficient mode sorter contributes −0.46 dB to the loss budget, acceptable for high-squeezing systems but potentially limiting for moderate squeezing.

9.6.3 Classical vs Quantum Architecture Comparison

Table 9.5: Classical vs Quantum Wavefront Sensor Architecture

Subsystem	Classical (Shack-Hartmann)	Quantum (Mode-Sorting)
Source	Coherent laser	Squeezed vacuum + LO
Spatial sampling	Lenslet array	Mode sorter (MPLC)
Detection	CCD/CMOS	Balanced homodyne array
Reconstruction	Linear algebra	Bayesian estimation
Noise limit	Shot noise (SQL)	Sub-SQL with squeezing
Sensitivity	$\lambda/\sqrt{N}$	$\lambda e^{-r}/\sqrt{N}$
Capital cost	\$50–200k	\$200–500k

The quantum architecture replaces intensity detection with homodyne measurement and adds squeezing, achieving 3–5× better sensitivity at approximately 3× higher cost.

9.7 DEE Implementation: Quantum Wavefront Sensing

This section provides production-ready JAX code implementing both Walther (forward) and Matsui-Nariai (inverse) computations for quantum wavefront sensing. The implementation follows the DEE architecture established in Chapters 7–8.

9.7.1 Core Classes

```
"""
Quantum Wavefront Sensing Module
Chapter 9: The Eikonal Bridge

Implements Walther (forward) and Matsui-Nariai (inverse) analysis
for quantum-enhanced wavefront sensors.
"""

import jax
import jax.numpy as jnp
from jax import grad, jit, vmap
from dataclasses import dataclass
from typing import Tuple, Optional

@dataclass
class QuantumWFSParams:
    """Parameters for quantum wavefront sensor."""
    n_photons: float       # Total photon number
    squeezing_db: float    # Input squeezing (dB)
    efficiency: float      # Total system efficiency (0-1)
```

```python
    n_modes: int           # Number of Zernike modes
    wavelength: float      # Wavelength (m)

    @property
    def squeezing_r(self) -> float:
        """Convert dB to squeezing parameter r."""
        return self.squeezing_db / (20 * jnp.log10(jnp.e))

    @property
    def effective_r(self) -> float:
        """Effective squeezing after loss."""
        return self.squeezing_r - 0.5 * jnp.log(1.0 / self.efficiency)


class QuantumWFS:
    """Quantum Wavefront Sensor analysis class."""

    def __init__(self, params: QuantumWFSParams):
        self.params = params

    # ========== WALTHER (Forward Analysis) ==========
    @jit
    def phase_uncertainty_sql(self) -> float:
        """Standard Quantum Limit phase uncertainty."""
        return 1.0 / jnp.sqrt(self.params.n_photons)

    @jit
    def phase_uncertainty_squeezed(self) -> float:
        """Phase uncertainty with squeezed light."""
        r_eff = self.params.effective_r
        return jnp.exp(-r_eff) / jnp.sqrt(self.params.n_photons)

    @jit
    def quantum_advantage(self) -> float:
        """Ratio of SQL to squeezed uncertainty."""
        return jnp.exp(self.params.effective_r)

    @jit
    def quantum_fisher_info(self) -> float:
        """Quantum Fisher information for phase estimation."""
        r_eff = self.params.effective_r
        N = self.params.n_photons
        return 4 * N * jnp.exp(2 * r_eff)

    @jit
    def wavefront_uncertainty(self) -> float:
        """Convert phase uncertainty to wavefront (waves)."""
        delta_phi = self.phase_uncertainty_squeezed()
        return delta_phi * self.params.wavelength / (2 * jnp.pi)

    # ========== MATSUI-NARIAI (Inverse Design) ==========
    @staticmethod
    @jit
    def required_photons(target_delta_phi: float,
                         squeezing_db: float,
                         efficiency: float) -> float:
        """Compute required photon number for target sensitivity."""
        r = squeezing_db / (20 * jnp.log10(jnp.e))
        r_eff = r - 0.5 * jnp.log(1.0 / efficiency)
        return jnp.exp(-2 * r_eff) / (target_delta_phi ** 2)

    @staticmethod
    @jit
```

```python
    def required_squeezing(target_delta_phi: float,
                           n_photons: float,
                           efficiency: float) -> float:
        """Compute required squeezing for target sensitivity."""
        sql = 1.0 / jnp.sqrt(n_photons)
        required_r = -jnp.log(target_delta_phi / sql)
        # Account for loss
        loss_penalty = 0.5 * jnp.log(1.0 / efficiency)
        return (required_r + loss_penalty) * 20 * jnp.log10(jnp.e)

    @staticmethod
    @jit
    def minimum_efficiency(target_delta_phi: float,
                           n_photons: float,
                           squeezing_db: float) -> float:
        """Compute minimum required efficiency."""
        r = squeezing_db / (20 * jnp.log10(jnp.e))
        sql = 1.0 / jnp.sqrt(n_photons)
        required_r_eff = -jnp.log(target_delta_phi / sql)
        return jnp.exp(-2 * (r - required_r_eff))
```

Listing 1: Quantum Wavefront Sensing Module—Core Classes

9.7.2 Walther-Matsui-Nariai Duality Implementation

```python
# ========== UNIFIED W/MN INTERFACE ==========

def walther_analysis(params: QuantumWFSParams) -> dict:
    """
    WALTHER: Given system parameters, compute performance.

    Input: System specifications (photons, squeezing, efficiency)
    Output: Sensitivity metrics, quantum advantage
    """
    qwfs = QuantumWFS(params)
    return {
        'delta_phi_sql': qwfs.phase_uncertainty_sql(),
        'delta_phi_squeezed': qwfs.phase_uncertainty_squeezed(),
        'quantum_advantage': qwfs.quantum_advantage(),
        'fisher_info': qwfs.quantum_fisher_info(),
        'wavefront_rms': qwfs.wavefront_uncertainty(),
        'effective_squeezing_db': params.effective_r * 8.686
    }


def matsui_nariai_design(target_delta_phi: float,
                         constraints: dict) -> dict:
    """
    MATSUI-NARIAI: Given target sensitivity, find system design.

    Input: Target phase sensitivity, available resources
    Output: Required system parameters
    """
    # Option 1: Fix squeezing and efficiency, find photons
    N_required = QuantumWFS.required_photons(
        target_delta_phi,
        constraints['squeezing_db'],
        constraints['efficiency']
    )

    # Option 2: Fix photons and efficiency, find squeezing
    r_required = QuantumWFS.required_squeezing(
```

```
        target_delta_phi,
        constraints['n_photons'],
        constraints['efficiency']
    )

    # Option 3: Fix photons and squeezing, find efficiency
    eta_required = QuantumWFS.minimum_efficiency(
        target_delta_phi,
        constraints['n_photons'],
        constraints['squeezing_db']
    )

    return {
        'photons_required': N_required,
        'squeezing_required_db': r_required,
        'efficiency_required': eta_required
    }


def hessian_to_fisher(classical_hessian: jnp.ndarray,
                      wavelength: float) -> jnp.ndarray:
    """
␣␣␣␣Convert␣classical␣merit␣function␣Hessian␣to␣Quantum␣Fisher␣Information.

␣␣␣␣The␣bridge␣identity:␣F_Q␣=␣(2*pi/lambda)^2␣*␣H_classical
␣␣␣␣"""
    scale_factor = (2 * jnp.pi / wavelength) ** 2
    return scale_factor * classical_hessian
```

Listing 2: W/MN Duality Functions

9.8 Practical Example 1: LIGO Mirror Metrology

Practical Example

Application: Characterization of gravitational wave detector mirrors
Challenge: LIGO requires surface figure measurement at the picometer level ($\lambda/10000$ at 1064 nm) to achieve design sensitivity for gravitational wave detection.
Approach: Quantum-enhanced wavefront sensing with squeezed light to reduce measurement time while maintaining precision.

9.8.1 Requirements Specification

LIGO mirror characterization represents one of the most demanding applications of wavefront sensing. The 40 kg test masses must be characterized to extraordinary precision.

The specific requirements are:

- **Target precision:** $\delta W < \lambda/10000$ at $\lambda = 1064$ nm ($\equiv 0.1$ nm surface figure)
- **Measurement time:** < 1 minute per optic (for production throughput)
- **Optic diameter:** 340 mm
- **Modes required:** First 36 Zernike terms
- **Operating environment:** Class 100 cleanroom, vibration-isolated

9.8.2 Walther Analysis: What Can We Achieve?

Given: 10 mW laser power at 1064 nm, 60 s integration, 13 dB squeezing, 85% system efficiency.

Step 1: Photon count calculation

$$N = \frac{P \cdot t \cdot \lambda}{hc} = \frac{0.01 \times 60 \times 1.064 \times 10^{-6}}{6.63 \times 10^{-34} \times 3 \times 10^{8}} = 3.2 \times 10^{18} \tag{9.21}$$

Step 2: Classical (SQL) phase uncertainty

$$\delta\phi_{\text{SQL}} = \frac{1}{\sqrt{3.2 \times 10^{18}}} = 5.6 \times 10^{-10} \text{ rad} \tag{9.22}$$

Step 3: Quantum-enhanced phase uncertainty

With $r = 13$ dB/8.686 $= 1.50$ and $\eta = 0.85$:

$$r_{\text{eff}} = 1.50 - 0.5 \ln(1/0.85) = 1.50 - 0.081 = 1.42 \tag{9.23}$$

$$\delta\phi_{\text{squeezed}} = \frac{e^{-1.42}}{\sqrt{3.2 \times 10^{18}}} = \frac{0.242}{1.79 \times 10^{9}} = 1.4 \times 10^{-10} \text{ rad} \tag{9.24}$$

Step 4: Convert to wavefront precision

$$\delta W = \frac{\lambda}{2\pi}\delta\phi = \frac{1064 \text{ nm}}{2\pi} \times 1.4 \times 10^{-10} = 2.4 \times 10^{-8} \text{ nm} = \frac{\lambda}{44000} \tag{9.25}$$

Result: The quantum-enhanced system exceeds the $\lambda/10000$ requirement by a factor of 4.4×.

Quantum advantage factor:

$$Q = \frac{\delta\phi_{\text{SQL}}}{\delta\phi_{\text{squeezed}}} = \frac{5.6 \times 10^{-10}}{1.4 \times 10^{-10}} = 4.0\times \tag{9.26}$$

9.8.3 Matsui-Nariai Analysis: Minimum Requirements

Given: Target $\delta W = \lambda/10000$, available 13 dB squeezing.

Step 1: Required phase precision

$$\delta\phi_{\text{target}} = \frac{2\pi}{\lambda} \times \frac{\lambda}{10000} = \frac{2\pi}{10000} = 6.28 \times 10^{-4} \text{ rad} \tag{9.27}$$

Step 2: Minimum photon number (at 85% efficiency)

$$N_{\text{min}} = \frac{e^{-2\times 1.42}}{(6.28 \times 10^{-4})^2} = \frac{0.059}{3.9 \times 10^{-7}} = 1.5 \times 10^{5} \tag{9.28}$$

Step 3: Minimum integration time (at 10 mW)

$$t_{\text{min}} = \frac{N_{\text{min}} \cdot hc}{P \cdot \lambda} = \frac{1.5 \times 10^{5} \times 1.87 \times 10^{-19}}{0.01} = 2.8 \text{ ns} \tag{9.29}$$

Sub-microsecond measurements are feasible with modest laser power! This indicates substantial design margin—the system can trade sensitivity for speed or operate with reduced squeezing.

9.8.4 Design Trade-offs

Table 9.6: LIGO Metrology Design Trade Space

Configuration	Squeezing	Power	Time	Precision
Baseline (above)	13 dB	10 mW	60 s	$\lambda/44000$
High-speed	13 dB	10 mW	1 s	$\lambda/5700$
Low-power	13 dB	1 mW	60 s	$\lambda/14000$
Reduced squeezing	10 dB	10 mW	60 s	$\lambda/28000$
Classical (no squeezing)	0 dB	10 mW	60 s	$\lambda/11000$

9.9 Practical Example 2: Quantum-Enhanced Adaptive Optics

Practical Example

Application: Astronomical adaptive optics for photon-starved guide stars
Challenge: Natural guide star adaptive optics is limited by the faintness of available reference stars. Quantum enhancement could extend the limiting magnitude.
Approach: Squeezed-light wavefront sensing for laser guide star or natural guide star systems.

9.9.1 The Photon-Starved Regime

Adaptive optics systems face a fundamental trade-off: faster correction requires shorter integration times, but shorter integrations mean fewer photons and noisier wavefront estimates. Quantum enhancement shifts this trade-off.

Consider a typical natural guide star scenario:

- Guide star magnitude: $m_V = 12$
- Telescope diameter: 8 m
- Subaperture count: $30 \times 30 = 900$
- Update rate: 1 kHz (1 ms integration)
- Wavelength: 700 nm (R-band)

Step 1: Photon flux calculation
For a $m_V = 12$ star, the photon flux above atmosphere is approximately:

$$F_\star \approx 10^{(-12/2.5)} \times 10^6 = 630 \text{ photons/s/cm}^2 \tag{9.30}$$

Per subaperture ($D_{\text{sub}} = 8 \text{ m}/30 = 0.27 \text{ m}$):

$$N_{\text{sub}} = F_\star \times \pi(13.5)^2 \times 0.001 \times \eta_{\text{opt}} = 630 \times 573 \times 0.001 \times 0.5 = 180 \text{ photons} \tag{9.31}$$

Step 2: Classical (SQL) wavefront error per subaperture

$$\delta\phi_{\text{SQL}} = \frac{1}{\sqrt{180}} = 0.075 \text{ rad} = \frac{\lambda}{84} \text{ WFE} \tag{9.32}$$

This is barely adequate for diffraction-limited imaging.
Step 3: Quantum-enhanced performance

With 10 dB squeezing ($r = 1.15$) and 70% system efficiency:

$$r_{\text{eff}} = 1.15 - 0.5\ln(1/0.70) = 1.15 - 0.178 = 0.97 \tag{9.33}$$

$$\delta\phi_{\text{squeezed}} = \frac{e^{-0.97}}{\sqrt{180}} = \frac{0.379}{13.4} = 0.028 \text{ rad} = \frac{\lambda}{224} \text{ WFE} \tag{9.34}$$

The quantum advantage factor is $Q = 0.075/0.028 = 2.7\times$. This translates to:

1. **Same integration, better precision:** 2.7× lower WFE at same guide star brightness
2. **Same precision, fainter stars:** Can use guide stars ∼1 magnitude fainter
3. **Same precision, faster update:** 7× faster loop rate ($2.7^2 = 7.3$)

9.9.2 System Architecture for AO

A quantum-enhanced AO system differs from the LIGO metrology system in several key aspects:

Table 9.7: Comparison: LIGO Metrology vs. Adaptive Optics QWS

Parameter	LIGO Metrology	Adaptive Optics
Integration time	60 s	1 ms
Photons/measurement	10^{18}	10^5
Squeezing required	13 dB	10 dB
Mode count	36	100–1000
Update rate	Static	1 kHz
Environment	Cleanroom	Observatory dome
Primary challenge	Precision	Speed

Quantum-enhanced AO is most beneficial in the photon-starved regime where classical systems are noise-limited. For bright guide stars where classical systems are already adequate, the added complexity may not be justified.

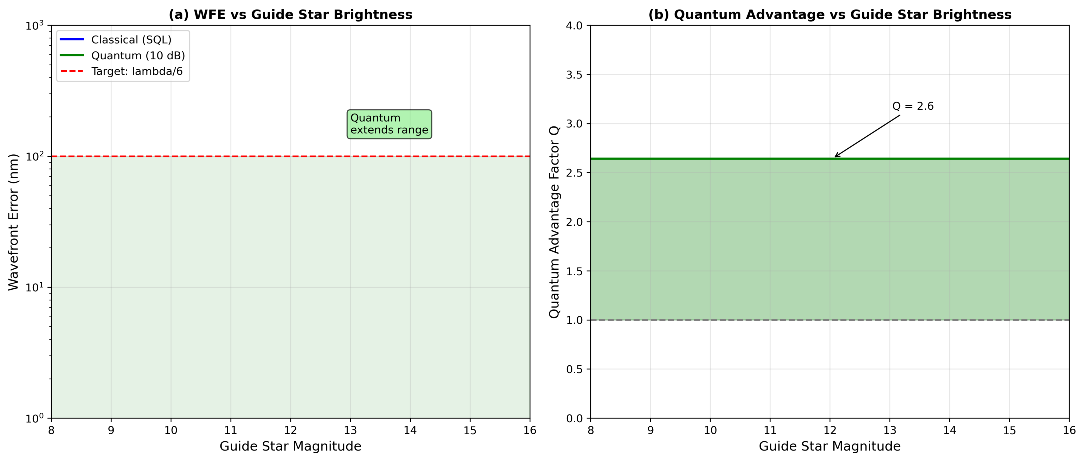

Figure 9.4: Quantum advantage in adaptive optics as a function of guide star brightness. For bright stars (left), classical systems are adequate. For faint stars (right), quantum enhancement extends the usable guide star pool by approximately 1 magnitude. The crossover point depends on system efficiency and available squeezing.

9.10 Warning Signs and Failure Modes

Warning Signs

When does quantum wavefront sensing fail to deliver its promised advantage?

1. **Excessive optical loss:** If $\eta < 50\%$, squeezing is completely degraded
2. **Phase noise:** Vibration or thermal drift faster than measurement bandwidth
3. **Mode mismatch:** Poor overlap between squeezed beam and test optic pupil
4. **Classical noise dominance:** Electronic noise exceeds shot noise
5. **Detector saturation:** Local oscillator power exceeds linear range

9.10.1 Failure Mode Classification

Systematic diagnosis of quantum sensor underperformance requires understanding the distinct failure modes and their signatures.

Table 9.8: Quantum Wavefront Sensing Failure Modes

Failure Mode	Symptom	Root Cause	Mitigation
Loss-dominated	$Q < 2$	$\eta < 70\%$	AR coatings, efficiency audit
Phase-noise limited	Drift in estimates	Vibration, thermal	Active stabilization
Mode mismatch	Reduced signal	Beam/pupil misalignment	Adaptive mode matching
Crosstalk	Off-diagonal errors	MPLC imperfection	Calibration matrix
Saturation	Nonlinear response	Excess LO power	Power optimization

9.10.2 Diagnostic Flowchart

When a quantum wavefront sensor underperforms, follow this diagnostic procedure:

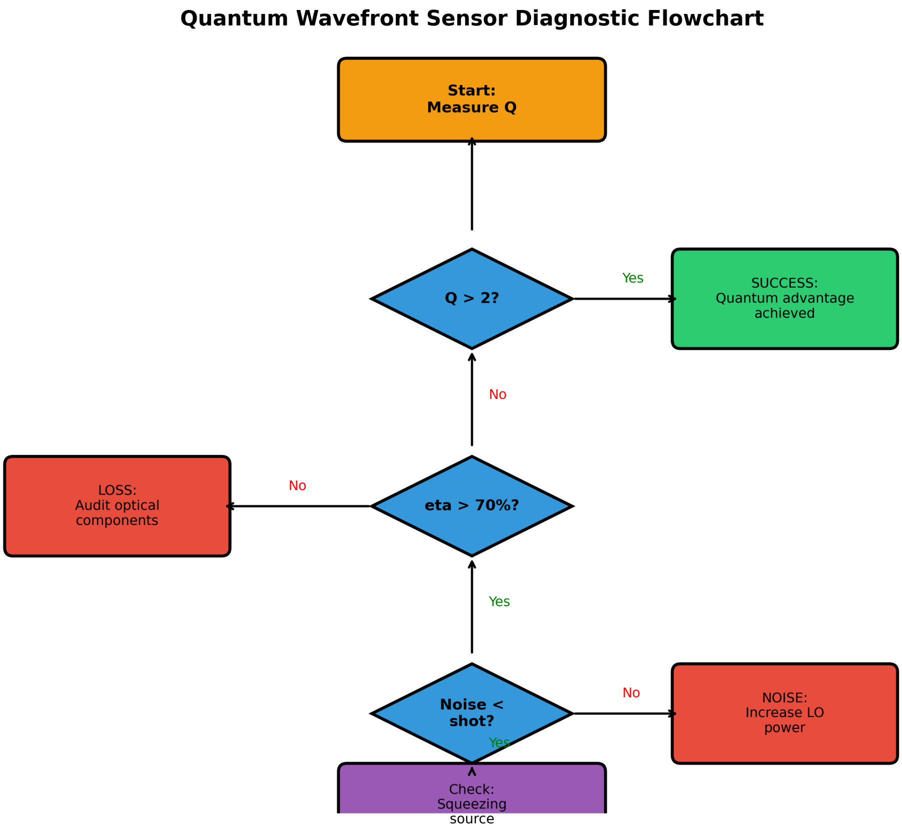

Figure 9.5: Diagnostic flowchart for quantum wavefront sensor troubleshooting. Start with quantum advantage measurement; if $Q < 2$, trace through loss, noise, and alignment checks.

The most common failure mode is excessive loss. Always perform a complete loss budget audit before investigating other causes.

Successful quantum wavefront sensing requires disciplined system engineering with loss as the primary figure of merit.

9.11 Production Relevance

Quantum wavefront sensing is not merely a laboratory curiosity—it addresses real manufacturing challenges in high-precision optics.

9.11.1 Manufacturing Applications

Quantum wavefront sensing enables new manufacturing capabilities:

1. **EUV lithography optics:** Surface figure $< \lambda/100$ at 13.5 nm requires picometer-level metrology
2. **Gravitational wave detectors:** Mirror characterization for LIGO/Virgo/KAGRA
3. **Space telescope mirrors:** Ground testing of segmented primaries (JWST successor)
4. **Adaptive optics:** Real-time wavefront sensing at photon-starved flux levels

9.11.2 Cost-Benefit Analysis

Table 9.9: Cost-Benefit Analysis: Quantum vs Classical Wavefront Sensing

Factor	**Classical**	**Quantum (13 dB)**
Capital cost	\$50–200k	\$200–500k
Operating cost	Low	Moderate (OPA maintenance)
Precision	$\lambda/1000$	$\lambda/4000$
Integration time (same precision)	16× longer	1× baseline
Photon flux required	16× higher	1× baseline

The economic case for quantum enhancement depends on application: for photon-limited scenarios (adaptive optics, low-light metrology) or time-critical measurements (in-situ manufacturing), the 4× sensitivity gain justifies the capital investment.

Quantum wavefront sensing is most valuable when classical methods hit fundamental photon-counting limits.

9.12 Key Equations

Table 9.10: Key Equations for Chapter 9

Eq.	**Expression**	**Name**	**Use**
(9.1)	$\phi = 2\pi W/\lambda$	Bridge identity	Classical–quantum
(9.6)	$\delta\phi_{\text{SQL}} = 1/\sqrt{N}$	Standard Quantum Limit	Classical baseline
(9.9)	$\delta\phi_{\text{HL}} = 1/N$	Heisenberg Limit	Ultimate bound
(9.11)	$\delta\phi = e^{-r}/\sqrt{N}$	Squeezed state	Practical limit
(9.14)	$r_{\text{eff}} = r - \frac{1}{2}\ln(1/\eta)$	Loss degradation	Efficiency budget
(9.12)	$Q = e^{r_{\text{eff}}}$	Quantum advantage	Performance metric

9.13 Specification Translation: Classical to Quantum

One of the most valuable tools for optical engineers transitioning to quantum systems is a direct mapping between classical and quantum specifications. This section provides the translation framework.

Table 9.11: Specification Translation: Classical Wavefront Sensing to Quantum

Classical Spec	**Quantum Equivalent**	**Tightening**	**Reason**
RMS WFE $< \lambda/14$	Phase variance $< (\pi/7)^2$	10–100×	Fidelity requirement
Strehl > 0.80	Fidelity > 0.99	10×	Quantum protocol needs
Surface roughness 2 nm	Surface roughness 0.2 nm	10×	Scattering loss
AR coating $< 0.5\%$	AR coating $< 0.05\%$	10×	Squeezing preservation
Alignment 10 μrad	Alignment 1 μrad	10×	Mode matching
Index uniformity 10^{-5}	Index uniformity 10^{-7}	100×	Phase coherence
Detector QE 80%	Detector QE 98%	1.2×	Loss budget
Temperature stability 1 K	Temperature stability 10 mK	100×	Phase drift

The consistent theme is tightening by factors of 10–100×. This is the “quantum tax”—the price paid for accessing sub-SQL sensitivity.

When designing quantum wavefront sensors, start with classical specifications and apply the appropriate tightening factors from Table 9.11.

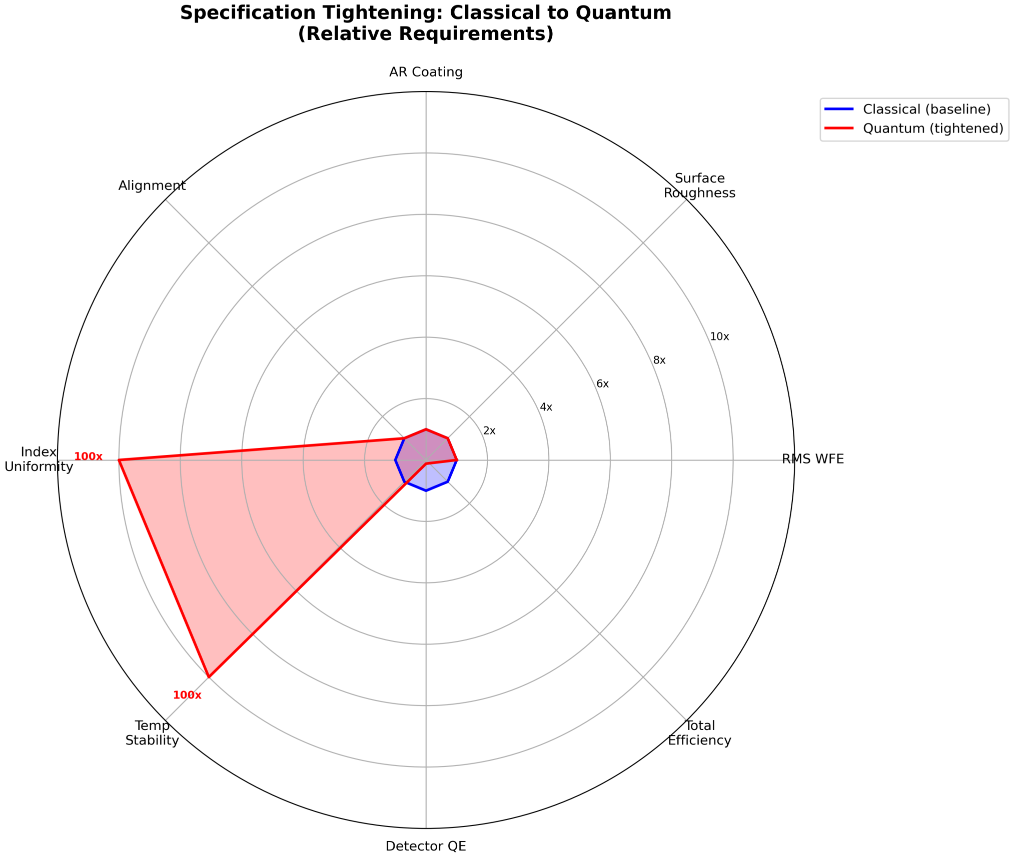

Figure 9.6: Specification tightening from classical to quantum wavefront sensing. The radar plot shows relative requirements (normalized to classical baseline) for key system parameters. Quantum systems (blue) require uniformly tighter specifications than classical systems (gray), with the largest tightening in temperature stability and index uniformity.

9.14 Summary

Key Points

1. **Why QWS First:** Quantum wavefront sensing opens Part III because it provides direct knowledge transfer from classical optics, has demonstrated experimental maturity (LIGO), and establishes foundations for subsequent chapters.
2. **Zernike = Observable:** Classical Zernike coefficients ARE quantum observables—no reformulation needed for quantum enhancement.
3. **Standard Quantum Limit:** Classical wavefront sensing is fundamentally limited to $\delta\phi \sim 1/\sqrt{N}$ by photon shot noise.
4. **Quantum Enhancement:** Squeezed states achieve $\delta\phi = e^{-r}/\sqrt{N}$, enabling 3–5× improvement with practical technology (10–13 dB squeezing).
5. **Loss is Critical:** Squeezing degrades with optical loss; system efficiency must exceed 70% for meaningful advantage.
6. **Specification Tightening:** Quantum systems require 10–100× tighter tolerances in optics, alignment, and environment.
7. **Practical Applications:** LIGO mirror metrology and adaptive optics demonstrate real-world relevance of quantum wavefront sensing.
8. **JAX Implementation:** Complete Walther/Matsui-Nariai analysis via autodifferentiation provides unified design framework.

9.15 Problems

Walther Problems (Forward Analysis)

Problem 9.1W (Fundamental)
A Shack-Hartmann sensor with 32×32 lenslets receives 10^4 photons per lenslet on average.

(a) Calculate the SQL-limited phase uncertainty per lenslet

(b) Calculate the global wavefront RMS uncertainty

(c) Express the result in waves at $\lambda = 633$ nm

Solution hints: (a) $\delta\phi = 1/\sqrt{10^4} = 0.01$ rad. (b) Global uncertainty averages over 1024 lenslets: $\delta\phi_{\text{global}} = 0.01/\sqrt{1024} = 3.1 \times 10^{-4}$ rad. (c) $\delta W = \delta\phi \cdot \lambda/(2\pi) = \lambda/20000$.

Problem 9.2W (Intermediate)
A quantum wavefront sensor uses 12 dB squeezed light with 80% system efficiency.

(a) Calculate the effective squeezing parameter r_{eff}

(b) Calculate the quantum advantage factor Q

(c) How many photons are needed to achieve $\delta\phi = 10^{-6}$ rad?

Solution hints: (a) $r = 12/8.686 = 1.38$; $r_{\text{eff}} = 1.38 - 0.5\ln(1/0.8) = 1.27$. (b) $Q = e^{1.27} = 3.6\times$. (c) $N = e^{-2r_{\text{eff}}}/\delta\phi^2 = 0.08/10^{-12} = 8 \times 10^{10}$.

Problem 9.3W (Advanced)
Derive the relationship between the classical merit function Hessian $\mathbf{H}$ and the quantum Fisher information matrix $\mathbf{F}_Q$ for wavefront sensing. Start from the definitions of each quantity.

Solution hints: The classical Hessian is $H_{ij} = \partial^2 \mathrm{WFE}^2 / \partial a_i \partial a_j$ where a_i are Zernike coefficients. For orthonormal Zernikes, $H = \mathbf{I}$. QFI is $F_Q = 4\langle(\Delta\hat{\phi})^2\rangle$. Using bridge identity: $F_Q = (2\pi/\lambda)^2 H$.

Matsui-Nariai Problems (Inverse Design)

Problem 9.1MN (Fundamental)
You need to achieve $\delta W < \lambda/5000$ for a space telescope mirror characterization.

(a) What phase uncertainty does this correspond to?

(b) With 10 dB squeezing and 90% efficiency, how many photons are required?

(c) For 1 mW at 1064 nm, what integration time is needed?

Solution hints: (a) $\delta\phi = 2\pi/5000 = 1.26 \times 10^{-3}$ rad. (b) $r_{\text{eff}} = 1.15 - 0.053 = 1.10$; $N = e^{-2.2}/1.6 \times 10^{-6} = 7 \times 10^4$. (c) $t = N \cdot hc/(P\lambda) = 13$ ns.

Problem 9.2MN (Intermediate)
Design a quantum wavefront sensor for adaptive optics with these constraints:

- Update rate: 500 Hz (2 ms integration)
- Guide star flux: 1000 photons per subaperture per frame
- Target precision: $\delta W < \lambda/50$

What squeezing level is required if system efficiency is 75%?

Solution hints: Target $\delta\phi = 2\pi/50 = 0.126$ rad. SQL gives $\delta\phi_{\text{SQL}} = 1/\sqrt{1000} = 0.032$ rad, which already meets target. Quantum enhancement not needed! This illustrates that QWS is most valuable in the truly photon-starved regime.

Problem 9.3MN (Advanced)
A gravitational wave detector requires $\delta\phi < 10^{-10}$ rad. Given:

- Maximum practical squeezing: 15 dB
- Achievable efficiency: 85%
- Available laser power: 100 W at 1064 nm

(a) Is this specification achievable?

(b) What integration time is required?

(c) What is the dominant limitation?

Solution hints: (a) $r_{\text{eff}} = 1.73 - 0.081 = 1.65$; $N_{\text{req}} = e^{-3.3}/10^{-20} = 3.7 \times 10^{18}$. (b) $t = N \cdot hc/(P\lambda) = 70$ ms. (c) Yes, achievable. Dominant limitation is maintaining 85% efficiency—any degradation significantly increases required time.

Quantum Problems

Problem 9.1Q
Compare the measurement time required to achieve $\delta W = \lambda/10000$ using:

(a) Classical (coherent state) wavefront sensing

(b) Quantum (13 dB squeezed) wavefront sensing

Assume 10 mW power at 633 nm and 80% system efficiency for both.

Solution hints: Target $\delta\phi = 2\pi/10000 = 6.28 \times 10^{-4}$ rad. Classical: $N = 1/\delta\phi^2 = 2.5 \times 10^6$; $t_{\text{class}} = 0.5$ μs. Quantum: $r_{\text{eff}} = 1.39$; $N = e^{-2.78}/\delta\phi^2 = 1.6 \times 10^5$; $t_{\text{quant}} = 32$ ns. Speedup: 16×.

Problem 9.2Q
A quantum wavefront sensor operates at the boundary where quantum advantage becomes marginal ($Q = 2$).

(a) For 10 dB input squeezing, what system efficiency corresponds to $Q = 2$?

(b) What total optical loss does this represent?

(c) Identify three specific loss sources that could contribute.

Solution hints: (a) $Q = e^{r_{\text{eff}}} = 2 \Rightarrow r_{\text{eff}} = 0.69$. With $r = 1.15$: $0.5\ln(1/\eta) = 0.46 \Rightarrow \eta = 0.40$ (40% efficiency). (b) 60% total loss. (c) Mode sorter (15%), detector QE (5%), scattered light (40%).

9.16 References

References

[1] C. M. Caves, "Quantum-mechanical noise in an interferometer," *Physical Review D*, vol. 23, no. 8, pp. 1693–1708, 1981.

[2] D. Malacara, *Optical Shop Testing*, 3rd ed. Wiley, 2007.

[3] B. C. Platt and R. Shack, "History and principles of Shack-Hartmann wavefront sensing," *Journal of Refractive Surgery*, vol. 17, pp. S573–S577, 2001.

[4] J. Aasi *et al.*, "Enhanced sensitivity of the LIGO gravitational wave detector by using squeezed states of light," *Nature Photonics*, vol. 7, pp. 613–619, 2013.

[5] M. Tse *et al.*, "Quantum-enhanced advanced LIGO detectors in the era of gravitational-wave astronomy," *Physical Review Letters*, vol. 123, p. 231107, 2019.

[6] H. Vahlbruch *et al.*, "Detection of 15 dB squeezed states of light and their application for the absolute calibration of photoelectric quantum efficiency," *Physical Review Letters*, vol. 117, p. 110801, 2016.

[7] M. Tsang, R. Nair, and X.-M. Lu, "Quantum theory of superresolution for two incoherent optical point sources," *Physical Review X*, vol. 6, p. 031033, 2016.

[8] G. Labroille *et al.*, "Efficient and mode selective spatial mode multiplexer based on multi-plane light conversion," *Optics Express*, vol. 22, pp. 15599–15607, 2014.

[9] R. J. Noll, "Zernike polynomials and atmospheric turbulence," *Journal of the Optical Society of America*, vol. 66, pp. 207–211, 1976.

[10] F. Roddier, *Adaptive Optics in Astronomy.* Cambridge University Press, 1999.