Analysis of Noisy, Deterministic Arnoldi's Method of Minimized Iterations

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Abstract

In this paper, I will show that a bias in the eigenvalues appears when the matrix-vector multiplication operation contains noise. (This isn't worded very well.)

1 Perfect Arithmetic

2 Noisy Matrix-Vector Product

For this analysis, the matrix I will use is

$$A \equiv \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} \tag{1}$$

with eigenvalaues, $\sigma_A = [\alpha, \beta, \gamma]$.

The matrix-vector product is noisy for this analysis. I will represent the noise in this operation by adding a noise vector

$$\xi = \begin{bmatrix} \xi_{(i,1)} \\ \xi_{(i,2)} \\ \xi_{(i,3)} \end{bmatrix} \tag{2}$$

where i is the noise from the i-th Arnoldi iteration. I have outlined the Arnoldi method in Algorithm **??** for reference.

I will begin with the vector

$$q_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} . \tag{3}$$

With perfect arithmetic, Arnoldi's method will break down due to invariance of the subspace after one iteration. With any starting vector, Arnoldi's method will break down due to invariance after 3 iterations, but since I have chosen the first eigenvector as a starting eigenvector, it will break down after one iteration.

Algorithm 1: Arnoldi Process (freely borrowed from Watkins(2002))

$q_1 = q / \|q\|_2$
for $k = 1, \ldots, m-1$ **do**
  $\tilde{q}_{k+1} \leftarrow A q_k$
  for $j = 1, \ldots, k$ **do** ▷ `Orthogonalize`
    $h_{jk} \leftarrow \langle q_j, q_{k+1} \rangle$
    $q_{k+1} \leftarrow q_{k+1} - q_j h_{jk}$
  $h_{k+1,k} \leftarrow \|q_{k+1}\|_2$
  if $h_{k+1,k} = 0$ **then** ▷ `Span` $\{q_1, \ldots, q_k\}$ `is invariant under` A
    quit.
  $q_{k+1} \leftarrow q_{k+1} / h_{k+1,k}$

2.1 Iteration 1

We begin with the matrix-vector product Aq_1.

$$\hat{q}_2 = Aq_1 = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \xi_{(1,1)} \\ \xi_{(1,2)} \\ \xi_{(1,3)} \end{bmatrix} \tag{4}$$

We next orthogonalize $\hat{q}_2$.

$$\hat{q}_2 = \hat{q}_2 - h_{1,1} q_1, \tag{5}$$

where

$$h_{1,1} = \langle q_1, \hat{q}_2 \rangle = \alpha + \xi_{(1,1)}. \tag{6}$$

So

$$\hat{q}_2 = \begin{bmatrix} \alpha + \xi_{(1,1)} \\ \xi_{(1,2)} \\ \xi_{(1,3)} \end{bmatrix} - \left(\alpha + \xi_{(1,1)}\right) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \xi_{(1,2)} \\ \xi_{(1,3)} \end{bmatrix} = \hat{q}_2. \tag{7}$$

Finally, we normalize $\hat{q}_2$.

$$h_{2,1} = \|\hat{q}_2\|_2 = \left(\xi_{(1,2)}^2 + \xi_{(1,3)}^2\right)^{(1/2)} \tag{8}$$

$$q_2 = \frac{\hat{q}_2}{h_{2,1}} = \left(\xi_{(1,2)}^2 + \xi_{(1,3)}^2\right)^{(-1/2)} \begin{bmatrix} 0 \\ \xi_{(1,2)} \\ \xi_{(1,3)} \end{bmatrix} \tag{9}$$

With perfect arithmetic the vector q_2 would be [0] and Arnoldi's method would break down, but with the addition of noise in the matrix-vector product, the vector is non-zero and Arnoldi's method continues. At this point, the upper Hessenberg matrix looks like

$$H = \begin{bmatrix} \alpha + \xi_{(1,1)} \\ \left(\xi_{(1,2)}^2 + \xi_{(1,3)}^2\right)^{(1/2)} \end{bmatrix}. \tag{10}$$

We use H—without the bottom row—to find the eigenvalues of A. The eigenvalue of H' is

$$\sigma'_A = \alpha + \xi_{(1,1)} \tag{11}$$

the estimate of the eigenvalue is off by the amount of the noise in the matrix-vector product.

2.2 Iteration 2

The second iteration proceeds in the same way, matrix-vector product, orthogonalization, and normalization, but the terms become more complicated.

$$\begin{aligned}
\hat{q}_3 = Aq_2 &= \begin{bmatrix} 0 \\ \beta\xi_{(1,2)}\left(1/h_{2,1}\right) \\ \gamma\xi_{(1,3)}\left(1/h_{2,1}\right) \end{bmatrix} + \begin{bmatrix} \xi_{(2,1)} \\ \xi_{(2,2)} \\ \xi_{(2,3)} \end{bmatrix} && (12) \\
&= \begin{bmatrix} \xi_{(2,1)} \\ \beta\xi_{(1,2)}\left(1/h_{2,1}\right) + \xi_{(2,2)} \\ \gamma\xi_{(1,3)}\left(1/h_{2,1}\right) + \xi_{(2,3)} \end{bmatrix} && (13)
\end{aligned}$$

Now we orthogonalize. We begin by orthogonalizing against q_1.

$$\hat{q}_3 = \hat{q}_3 - h_{1,2}q_1 \tag{14}$$

$$h_{1,2} = \langle \hat{q}_3, q_1 \rangle = \xi_{(2,1)} \tag{15}$$

$$\hat{q}_3 = \begin{bmatrix} \xi_{(2,1)} \\ \beta\xi_{(1,2)}\left(1/h_{2,1}\right) + \xi_{(2,2)} \\ \gamma\xi_{(1,3)}\left(1/h_{2,1}\right) + \xi_{(2,3)} \end{bmatrix} - \xi_{(2,1)} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{16a}$$

$$\hat{q}_3 = \begin{bmatrix} 0 \\ \beta\xi_{(1,2)}\left(1/h_{2,1}\right) + \xi_{(2,2)} \\ \gamma\xi_{(1,3)}\left(1/h_{2,1}\right) + \xi_{(2,3)} \end{bmatrix} . \tag{16b}$$

Now we continue by orthogonalizing q_2.

$$\hat{q_3} = \hat{q_3} - h_{2,2}q_2 \tag{17}$$

$$\begin{aligned} h_{2,2} &= \langle \hat{q_3}, q_2 \rangle && (18a) \\ &= \frac{\beta \xi_{(1,2)} \xi_{(1,2)}}{\left(h_{2,1}\right)^2} + \frac{\xi_{(2,2)} \xi_{(1,2)}}{h_{2,1}} && (18b) \\ &= \frac{\beta \xi_{(1,2)}^2 + \xi_{(2,2)} \xi_{(1,2)} h_{2,1}}{\left(\xi_{(1,2)}^2 + \xi_{(1,3)}^2\right)} && (18c) \end{aligned}$$

$$\begin{aligned} \hat{q_3} = &\begin{bmatrix} 0 \\ \beta \xi_{(1,2)} \left(1/h_{2,1}\right) + \xi_{(2,2)} \\ \gamma \xi_{(1,3)} \left(1/h_{2,1}\right) + \xi_{(2,3)} \end{bmatrix} && (19a) \\ &- \left(\frac{\beta \xi_{(1,2)}^2 + \xi_{(2,2)} \xi_{(1,2)} h_{2,1}}{\left(\xi_{(1,2)}^2 + \xi_{(1,3)}^2\right)} \right) \left(1/h_{2,1}\right) \begin{bmatrix} 0 \\ \xi_{(1,2)} \\ \xi_{(1,3)} \end{bmatrix} \\ \hat{q_3} = &\begin{bmatrix} 0 \\ \beta \xi_{(1,2)} \left(1/h_{2,1}\right) + \xi_{(2,2)} \\ \gamma \xi_{(1,3)} \left(1/h_{2,1}\right) + \xi_{(2,3)} \end{bmatrix} && (19b) \\ &- \left(\frac{\beta \xi_{(1,2)}^2 + \xi_{(2,2)} \xi_{(1,2)} h_{2,1}}{\left(\xi_{(1,2)}^2 + \xi_{(1,3)}^2\right)^{3/2}} \right) \begin{bmatrix} 0 \\ \xi_{(1,2)} \\ \xi_{(1,3)} \end{bmatrix} \\ & && (19c) \end{aligned}$$

H looks like

$$H = \begin{bmatrix} \alpha + \xi_{(1,1)} & \xi_{(2,1)} \\ \sqrt{\xi_{(1,2)}^2 + \xi_{(1,3)}^2} & \frac{\beta \xi_{(1,2)}^2 + \xi_{(2,2)} \xi_{(1,2)} \left(\xi_{(1,2)}^2 + \xi_{(1,3)}^2\right)^{1/2}}{\left(\xi_{(1,2)}^2 + \xi_{(1,3)}^2\right)} \end{bmatrix} \tag{20}$$