$$f_n(x) = [g(x)]^n = \underbrace{g(x) \cdot g(x) \cdot \ldots \cdot g(x)}_{n \text{ fatores}}$$

$$f_n'(x) = g'(x) \cdot [g(x)]^{n-1} + g(x) \cdot f_{n-1}'(x)$$

$$f_n'(x) = g'(x) \cdot f_{n-1}(x) + g(x) \cdot f_{n-1}'(x)$$

$$\underbrace{\frac{f_n'(x)}{[g(x)]^n}}_{F(n)} = \frac{g'(x) \cdot f_{n-1}(x)}{[g(x)]^n} + \underbrace{\frac{f_{n-1}'(x)}{[g(x)]^{n-1}}}_{F(n-1)}$$

$$F(n) = \frac{g'(x) \cdot f_{n-1}(x)}{[g(x)]^n} + \cancel{F(n-1)}$$

$$\cancel{F(n-1)} = \frac{g'(x) \cdot f_{n-2}(x)}{[g(x)]^{n-1}} + \cancel{F(n-2)}$$

$$\vdots$$

$$\cancel{F(2)} = \frac{g'(x) \cdot f_1(x)}{[g(x)]^2} + F(1)$$

$$+$$

$$F(n) = F(1) + \sum_{k=2}^{n} \frac{g'(x) \cdot f_{k-1}(x)}{[g(x)]^k} = \frac{f_1'(x)}{g(x)} + g'(x) \cdot \sum_{k=2}^{n} \frac{[g(x)]^{k-1}}{[g(x)]^k}$$

$$= \frac{g'(x)}{g(x)} + g'(x) \sum_{k=2}^{m} \frac{1}{g(x)} = \frac{g'(x)}{g(x)} + g'(x) \cdot (m-1) \cdot \frac{1}{g(x)} =$$

$$= \frac{g'(x)}{g(x)} + m \cdot \frac{g'(x)}{g(x)} - \frac{g'(x)}{g(x)} = m \cdot \frac{g'(x)}{g(x)}$$

$$\Rightarrow \frac{f_m'(x)}{[g(x)]^m} = m \cdot \frac{g'(x)}{g(x)} \Rightarrow f_m'(x) = m \cdot [g(x)]^{m-1} \cdot g'(x)$$

$$\therefore \frac{d}{dx}\left[(g(x))^m\right] = m \cdot [g(x)]^{m-1} \cdot g'(x)$$

Prova por indução:

$$m = 1 \Rightarrow \frac{d}{dx}(g(x)) = 1 \cdot [g(x)]^0 \cdot g'(x) = g'(x) \quad \text{(OK!)}$$

Suponha que $\frac{d}{dx}\left[(g(x))^k\right] = k \cdot [g(x)]^{k-1} \cdot g'(x)$ para algum $k \in \mathbb{N}$. Temos:

$$\frac{d}{dx}\left[(g(x))^{k+1}\right] = \frac{d}{dx}\left[(g(x))^k \cdot g(x)\right] =$$

$$= \frac{d}{dx}\left[(g(x))^k\right] \cdot g(x) + (g(x))^k \cdot g'(x) =$$

$$= k \cdot [g(x)]^{k-1} \cdot g'(x) \cdot g(x) + [g(x)]^k \cdot g'(x) =$$

$$= k \cdot [g(x)]^k \cdot g'(x) + [g(x)]^k \cdot g'(x) =$$

$$= (k+1) \cdot [g(x)]^k \cdot g'(x) \quad \text{(OK!)} \quad \blacksquare$$

A1 :

8) a) $f(x) = (x^2 - 7x + 1)^{20} (x^3 - 7)^{15}$

$$f'(x) = \frac{d}{dx}\left[(x^2 - 7x + 1)^{20}\right] \cdot (x^3 - 7)^{15} +$$

$$+ (x^2 - 7x + 1)^{20} \cdot \frac{d}{dx}\left[(x^3 - 7)^{15}\right] =$$

$$= 20 (x^2 - 7x + 1)^{19} \cdot (2x - 7) \cdot (x^3 - 7)^{15} +$$

$$+ (x^2 - 7x + 1)^{20} \cdot 15 \cdot (x^3 - 7)^{14} \cdot 3x^2 =$$

$$= (x^2 - 7x + 1)^{19} (x^3 - 7)^{14} \left[20(2x - 7)(x^3 - 7) +\right.$$

$$\left. + (x^2 - 7x + 1) \cdot 45 \cdot x^2\right]$$

b) $f(x) = \dfrac{x - 7}{x^3 + 3x^2 + 2x + 1}$

$$f'(x) = \frac{(x^3 + 3x^2 + 2x + 1) \cdot 1 - (x - 7) \cdot (3x^2 + 6x + 2)}{(x^3 + 3x^2 + 2x + 1)^2}$$

$$= \frac{\cancel{x^3} + \cancel{3x^2} + \cancel{2x} + 1 - \cancel{3x^3} - \cancel{6x^2} - \cancel{2x} + \cancel{21}x^2 + 42\cancel{x} + 14}{(x^3 + 3x^2 + 2x + 1)^2}$$

$$= \frac{-2x^3 + 18x^2 + 42x + 15}{(x^3 + 3x^2 + 2x + 1)^2}$$

c) $f(x) = \sqrt[5]{x^4}\left(\frac{x}{7} - \frac{1}{4}\right) = x^{4/5}\left(\frac{x}{7} - \frac{1}{4}\right)$

$$f'(x) = \frac{4}{5}x^{1/5}\left(\frac{x}{7} - \frac{1}{4}\right) + x^{4/5} \cdot \frac{1}{7}$$

d) $f(x) = \left(\frac{e^x + 7}{7 + e^{-x}}\right)^{100}$

$$f'(x) = 100\left(\frac{e^x + 7}{7 + e^{-x}}\right)^{99} \cdot \frac{(7 + e^{-x}) \cdot e^x - (e^x + 7) \cdot (-e^{-x})}{(7 + e^{-x})^2} =$$

$$= 100\left(\frac{e^x + 7}{7 + e^{-x}}\right)^{99} \cdot \frac{7e^x + \overbrace{1 + 1}^{2} + 7e^{-x}}{(7 + e^{-x})^2}$$

2 a) $\lim_{x \to 0} \frac{x^3 + 5x^2 - 3x}{2 - \sqrt{x^2 + 4}} \cdot \frac{2 + \sqrt{x^2 + 4}}{2 + \sqrt{x^2 + 4}} =$

$$= \lim_{x \to 0} \frac{(x^3 + 5x^2 - 3x)(2 + \sqrt{x^2 + 4})}{-x^2} =$$

$$= \lim_{x \to 0} -\frac{(x^2 + 5x - 3)(2 + \sqrt{x^2 + 4})}{x} =$$

$$= \lim_{x \to 0} -\left(x + 5 - \frac{3}{x}\right)\left(2 + \sqrt{x^2 + 4}\right) = 4 \lim_{x \to 0}\left(\frac{3}{x} - x - 5\right)$$

$$\left.\begin{array}{l}\lim\limits_{x\to 0^-}\left(\dfrac{3}{x}-x-5\right)=-\infty \\ \lim\limits_{x\to 0^+}\left(\dfrac{3}{x}-x-5\right)=\infty\end{array}\right\}\nexists \lim .$$

b) $$\lim_{x\to 1}\frac{x^3-1}{3(x-1)^2}=\lim_{x\to 1}\frac{(x-1)(x^2+x+1)}{3(x-1)^2}=$$

$$=\lim_{x\to 1}\frac{x^2+x+1}{3(x-1)} ;$$

$$\left.\begin{array}{l}\lim\limits_{x\to 1^-}\dfrac{x^2+x+1}{3(x-1)}=\lim\limits_{x\to 1^-}\dfrac{1}{(x-1)}=-\infty \\ \lim\limits_{x\to 1^+}\dfrac{x^2+x+1}{3(x-1)}=\lim\limits_{x\to 1^+}\dfrac{1}{(x-1)}=+\infty\end{array}\right\}\nexists \lim .$$

c) $$\lim_{x\to 0}\frac{\operatorname{sen} x}{x}=1$$

$$\lim_{x\to 0}\frac{\operatorname{tg} x}{x}=\lim_{x\to 0}\frac{\operatorname{sen} x}{x}\cdot\frac{1}{\cos x}=\lim_{x\to 0}\frac{\operatorname{sen} x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x}=$$

$$=1\cdot\frac{1}{1}=1 \quad ; \quad x\to 0 \Rightarrow \operatorname{tg} x\cong \operatorname{sen} x\cong x$$

4 b) $\lim\limits_{n\to\infty} x_n=\infty$; $y_n=f(x_n)=\dfrac{3}{x_n}-\dfrac{1}{x_n^2} \Rightarrow$

$$\Rightarrow \lim_{n \to \infty} y_n = \lim_{n\to\infty}\left(\frac{3}{x_n} - \frac{1}{x_n^2}\right) = 3\lim_{n\to\infty}\frac{1}{x_n} -$$

$$- \lim_{n\to\infty}\frac{1}{x_n^2} = 3.0 - 0 = 0.$$

5 i) $|f(x)| \leq 3x^2 \Rightarrow -3x^2 \leq f(x) \leq 3x^2$

$\lim_{x\to 0}(-3x^2) = \lim_{x\to 0}(3x^2) = 0 \Rightarrow$ Teo. Confronto:

$\lim_{x\to 0} f(x) = 0$; $|f(0)| \leq 0 \Rightarrow |f(0)| = 0 \Rightarrow f(0) = 0$

$\lim_{x\to 0} f(x) = f(0) \Rightarrow f$ é contínua em 0.

ii) $$f'_-(x) = \lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-}\frac{f(0+h)}{h} = \lim_{h\to 0^-}\frac{f(h)}{h}$$

$$= 0 \quad ; \quad -3h^2 \leq f(h) \leq 3h^2 \Rightarrow -3h \geq \frac{f(h)}{h} \geq 3h \Rightarrow$$

$$\Rightarrow \lim_{h\to 0^-}\frac{f(h)}{h} = 0 ;$$

$$f'_+(x) = \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{f(0+h)}{h} = \lim_{h\to 0^+} \frac{f(h)}{h}$$

$= 0$; $-3h^2 \leq f(h) \leq 3h^2 \Rightarrow -3h \leq \frac{f(h)}{h} \leq 3h \Rightarrow$

$\Rightarrow \lim_{h\to 0^+} \frac{f(h)}{h} = 0$;

$f'_-(0) = f'_+(0) \Rightarrow f$ é dif. em 0.

iii) $f'(0) = 0$.

$\boxed{C}$,,.

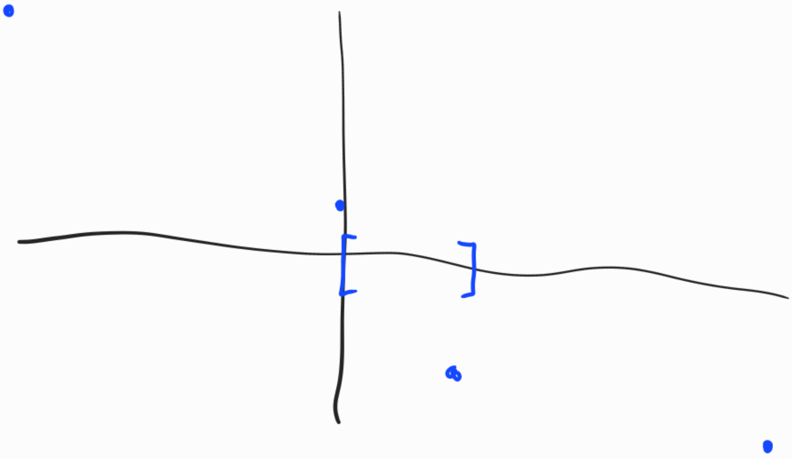
7 $\cos x = x \Rightarrow g(x) = \cos x - x \Rightarrow g$ é contínua.

$g(0) = 1 - 0 = 1 > 0$

$g(1) < 0$

$\lim_{x\to -\infty} g(x) = \infty$

$\lim_{x\to \infty} g(x) = -\infty$



$\exists c \in [0,1]$ tq. $g(c) = \cos c - c = 0 \Rightarrow c = \cos c$. $\boxed{B}$,,.