Teste de Cálculo de Várias Variáveis-EMAp-30/10/2020

Questão 1) Deseja-se construir uma peça de zinco que tem a forma da superfície do cilindro $x^2 + y^2 = 4$, compreendida entre os planos $z = 0$ e $x + y + z = 2$, $z \geq 0$.
Se o metro quadrado do zinco custa ***A*** reais calcule o preço total da peça.

Questão 2) Calcule $\int_\sigma \ ydx + (3y^3 - x)dy + zdz$, para cada um dos caminhos

$\sigma(t) = (t, t^n, 0)$, $0 \leq t \leq 1$, $n = 1, 2, 3, \dots \dots$

Questão 3) Calcular $\iint_S \ xy \, dS$ onde S é a superfície do tetraedro com lados

$z = 0$, $y = 0$, $x + z = 1$ e $x = y$.

Questão 4) A temperature em $\mathbb{R}^3$ é dada por $T(x, y, z) = 3x^2 + 3z^2$. Calcular o fluxo do calor através da superfície $x^2 + z^2 = 2$, $0 \leq y \leq 2$ e $k = 1$

(1) $r = (2\cos\theta, 2\,\text{sen}\,\theta, z)$

$z \geq 0 \Rightarrow 2 - x - y \geq 0 \Rightarrow 2 - 2\,\text{sen}\,\theta - 2\cos\theta \geq 0$

$\Rightarrow \text{sen}\,\theta + \cos\theta \leq 1 \Rightarrow \frac{\pi}{2} \leq \theta \leq 2\pi.$

$$\begin{cases} \theta \in \left[\frac{\pi}{2}, 2\pi\right] \\ z \in [0, 2 - 2\cos\theta - 2\,\text{sen}\,\theta] \end{cases}$$

$r_\theta = (-2\,\text{sen}\,\theta, 2\cos\theta, 0)$

$r_z = (0,0,1)$

$$\Rightarrow r_\theta \times r_z = \begin{vmatrix} i & j & k \\ -2\,\text{sen}\,\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} =$$

$= (2\cos\theta, 2\,\text{sen}\,\theta, 0) \Rightarrow \|N\| = \sqrt{4\cos^2\theta + 4\,\text{sen}^2\theta} = 2$

$$\Rightarrow \iint_S \|N\|\, dS = \int_{\frac{\pi}{2}}^{2\pi} \int_0^{2-2\cos\theta-2\,\text{sen}\,\theta} \|N\|\, dz\, d\theta =$$

$$= 2\int_{\frac{\pi}{2}}^{2\pi} (2 - 2\cos\theta - 2\,\mathrm{sen}\,\theta)\, d\theta = 4\left[\theta - \mathrm{sen}\,\theta + \cos\theta\right]_{\frac{\pi}{2}}^{2\pi}$$

$$= 4\left[2\pi + 1 - \frac{\pi}{2} + 1\right] = 4\left[\frac{3\pi}{2} + 2\right] = 6\pi + 8$$

$\therefore$ O preço é $(6\pi + 8)A$.

(2) $\sigma(t) = (t, t^n, 0) \Rightarrow \sigma'(t) = (1, nt^{n-1}, 0) \Rightarrow$

$$\Rightarrow \frac{dx}{dt} = 1; \quad \frac{dy}{dt} = nt^{n-1}; \quad \frac{dz}{dt} = 0 \Rightarrow$$

$$\Rightarrow \int_\sigma y\,dx + (3y^3 - x)\,dy + z\,dz =$$

$$= \int_0^1 \left(y \cdot \frac{dx}{dt} + (3y^3 - x) \cdot \frac{dy}{dt} + z \cdot \frac{dz}{dt} \right) dt =$$

$$= \int_0^1 \left(t^n \cdot 1 + (3t^{3n} - t)\, n t^{n-1} + 0 \right) dt =$$

$$= \int_0^1 \left(t^m + 3m t^{4m-1} - m t^m\right) dt =$$

$$= \left[\frac{t^{m+1}}{m+1} + \frac{3t^{4m}}{4} - \frac{m t^{m+1}}{m+1} \right]_0^1 =$$

$$= \frac{1}{m+1} + \frac{3}{4} - \frac{m}{m+1} = \frac{7-m}{4m+4}.$$

③ Vértices:

$$\begin{cases} z = 0 \\ y = 0 \\ x + z = 1 \end{cases} \Rightarrow (1,0,0)$$

$$\begin{cases} z = 0 \\ y = 0 \\ x = y \end{cases} \Rightarrow (0,0,0)$$

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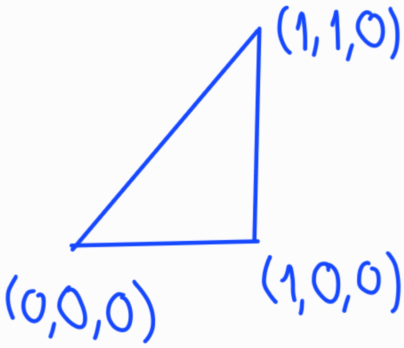
$$\begin{cases} y = 0 \\ x + z = 1 \\ x = y \end{cases} \Rightarrow (0,0,1)$$

Temos 4 faces a analisar separadamente:

$S_1 =$



$\Rightarrow r_1 = (x,y,0)$; $x \in [0,1]$, $y \in [x,1]$

$$r_{1_x} = (1,0,0)$$

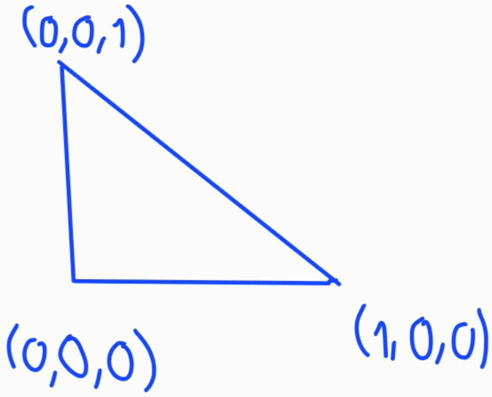
$$r_{1_y} = (0,1,0)$$

$$\Rightarrow r_{1_y} \times r_{1_x} = (0,0,-1) \Rightarrow \|N_1\| = 1$$

$$f(r_1(x,y)) = xy \Rightarrow \iint_{S_1} xy \, dS_1 = \int_0^1 \int_x^1 xy \cdot 1 \, dy\, dx =$$

$$= \int_0^1 \left[\frac{xy^2}{2}\right]_x^1 dx = \int_0^1 \left(\frac{x}{2} - \frac{x^3}{2}\right) dx = \left[\frac{x^2}{4} - \frac{x^4}{8}\right]_0^1 =$$

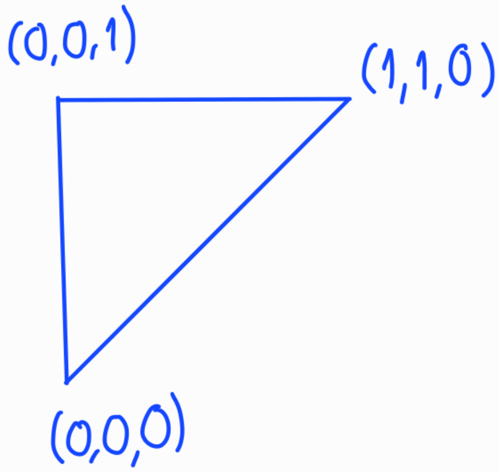
$$= \frac{1}{8};$$

$S_2 =$ 

$\Rightarrow \pi_2 = (x,0,z) \; ; \quad x \in [0,1]$

$\qquad z \in [0, 1-x]$

$$f(\pi_2(x,z)) = 0 \Rightarrow \iint_{S_2} xy \, dS_2 = 0 \; ;$$

$S_3 =$ 

$\Rightarrow \pi_3 = (x,x,z) \; ; \quad x \in [0,1]$

$\qquad z \in [0,1-x]$

$$\pi_{3_x} = (1,1,0)$$

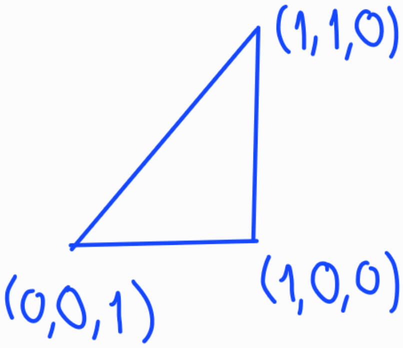
$$\pi_{3_z} = (0,0,1)$$

$$\Rightarrow \pi_{3_x} \times \pi_{3_z} = (1,-1,0) \Rightarrow \|N_3\| = \sqrt{2}$$

$$f(\pi_3(x,z)) = x^2 \Rightarrow \iint_{S_3} xy \, dS_3 = \int_0^1 \int_0^{1-x} x^2 \sqrt{2} \, dz \, dx =$$

$$= \sqrt{2} \int_0^1 x^2 (1-x) \, dx = \sqrt{2} \int_0^1 (x^2 - x^3) \, dx = \sqrt{2} \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 =$$

$$= \frac{\sqrt{2}}{3} - \frac{\sqrt{2}}{4} \; ;$$

$S_4 =$ 

$$\Rightarrow \; r_4 = (x, y, 1-x); \quad x \in [0,1], \quad y \in [x,1]$$

$$r_{4_x} = (1, 0, -1)$$

$$r_{4_y} = (0, 1, 0)$$

$$\Rightarrow \; r_{4_x} \times r_{4_y} = (1, 0, 1) \; \Rightarrow \; \|N_4\| = \sqrt{2}$$

$$f\left(r_4(x,y)\right) = xy \; \Rightarrow \; \iint_{S_4} xy \, dS_4 = \int_0^1 \int_x^1 xy \cdot \sqrt{2} \, dy \, dx =$$

$$= \sqrt{2} \int_0^1 \left[\frac{xy^2}{2} \right]_x^1 dx = \sqrt{2} \int_0^1 \left(\frac{x}{2} - \frac{x^3}{2} \right) dx = \sqrt{2} \left[\frac{x^2}{4} - \frac{x^4}{8} \right]_0^1 =$$

$$= \frac{\sqrt{2}}{8} .$$

$$\therefore \iint_S xy \, dS = \frac{1}{8} + 0 + \frac{\sqrt{2}}{3} - \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{8} = \frac{1}{8} + \frac{5\sqrt{2}}{24} .$$

④ $F = -K\nabla T = -1\left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right) = -(6x, 0, 6z)$

$= (-6x, 0, -6z)$;

$r = (\sqrt{2}\cos\theta, y, \sqrt{2}\,\text{sen}\,\theta)$, $\begin{array}{l} y \in [0,2] \\ \theta \in [0, 2\pi] \end{array} \Rightarrow$

$r_y = (0,1,0)$

$r_\theta = (-\sqrt{2}\,\text{sen}\,\theta, 0, \sqrt{2}\cos\theta)$ $\Rightarrow r_y \times r_\theta = (\sqrt{2}\cos\theta, 0, \sqrt{2}\,\text{sen}\,\theta)$

$F(r(\theta, y)) = (-6\sqrt{2}\cos\theta, 0, -6\sqrt{2}\,\text{sen}\,\theta)$

$$\therefore \iint_S F \cdot dS = \int_0^2 \int_0^{2\pi} (-12\cos^2\theta - 12\,\text{sen}^2\theta)\, d\theta\, dy =$$

$$= 2(-12)\int_0^{2\pi} (\text{sen}^2\theta + \cos^2\theta)\, d\theta = -24 \int_0^{2\pi} d\theta = -48\pi.$$