Problem 2. $\forall i((0 \le i < N) \to (p_i \wedge q_i)) \to \forall i((0 \le i \le 2N) \to (x_0 \le x_i))$ holds for every $N \in \mathbb{N}$

Proof: We prove this statement using induction on N.

Basis: For the basis of induction, we establish that $\forall i((0 \le i < 0) \to (p_i \wedge q_i)) \to \forall i((0 \le i \le 0) \to (x_0 \le x_i))$. The antecedent of this conditional is vacuously true, since $0 \le i < 0$ is false for any $i \in \mathbb{N}$. Therefore, it will suffice to prove that the consequent, $\forall i((0 \le i \le 0) \to (x_0 \le x_i))$, is true. Suppose that $0 \le m \le 0$. Then $m = 0$. Consequently, $x_0 \le x_m$ holds trivially, because $x_0 \le x_0$. Thus, the theorem holds when $N = 0$.

IH: For the induction hypothesis, suppose that $\forall i((0 \le i < N) \to (p_i \wedge q_i)) \to \forall i((0 \le i \le 2N) \to (x_0 \le x_i))$.

IS: Suppose that $\forall i((0 \le i < N+1) \to p_i \wedge q_i))$. Furthermore, suppose that $0 \le m \le 2(N+1)$. Either $m < N+1$ or $m \ge N+1$. We consider each case in detail:

i) If $m < N+1$, then, since $\forall i((0 \le i < N+1) \to p_i \wedge q_i))$ holds by assumption, we must have $\forall i((0 \le i < N) \to p_i \wedge q_i))$ trivially. Since $\forall i((0 \le i < N) \to p_i \wedge q_i))$, $\forall i((0 \le i \le 2N) \to (x_0 \le x_i))$ holds by the *IH*. Since $0 \le m < N+1$, we must have $0 \le m \le 2N$. Therefore, because $0 \le m \le 2N$ and $\forall i((0 \le i \le 2N) \to (x_0 \le x_i))$, we have $x_0 \le x_m$.

ii) If $m \ge N+1$, then it is easy to check that either $m = 2i+1$ or $m = 2i+2$ for some $i < N+1$. Since $i < N+1$, $p_i \wedge q_i$ by our initial supposition. If $m = 2i+1$, then, since p_i holds, $x_{2i} \le x_m$. If $m = 2i+2$, then, since q_i holds, $x_{2i} \le x_m$ as well. Thus, in either of our two cases, $x_{2i} \le x_m$. Since $i < N+1$, $2i \le 2N$ trivially, which means that $x_0 \le x_{2i}$ by the *IH*. Since $x_0 \le x_{2i}$ and $x_{2i} \le x_m$, we must have $x_0 \le x_m$.

In both of the two cases, (i) and (ii), we see that $x_0 \le x_m$, which means that $\forall i((0 \le i \le 2(N+1)) \to (x_0 \le x_i))$. $\square$

Problem 3. $\forall i((i \ge 0) \to (p_i \wedge q_i))$ is inductive.

Proof: We have already shown that $\forall i((i \ge 0) \to p_i)$ is inductive, so we must prove the same thing for the predicate q.

Suppose that in a certain state, we satisfy $\forall i((i \ge 0) \to (p_i \wedge q_i))$. Suppose that $i \ge 0$. Then, q_i holds by supposition, which means that $x_{2i} \le x_{2i+2}$. Now, suppose that we transition to a new state. We want to show that in this new state, $x'_{2i} \le x'_{2i+2}$. This transition was either a push or a pop. We consider each case:

i) If the transition was a push, $x'_{2i} = min(x_{2i}, x_{2i-1})$ and $x'_{2i+2} = x'_{2(i+1)} = min(x_{2i+2}, x_{2i+1})$. So, we want to show that $min(x_{2i}, x_{2i-1}) \le min(x_{2i+2}, x_{2i+1})$. This gives us four more sub-cases:

a) First of all, $x_{2i} \le x_{2i+2}$ by the fact that q_i holds. Therefore, if $min(x_{2i}, x_{2i-1}) = x_{2i}$ and $min(x_{2i+2}, x_{2i+1}) = x_{2i+2}$, then $min(x_{2i}, x_{2i-1}) \le min(x_{2i+2}, x_{2i+1})$.
b) Now, if $min(x_{2i}, x_{2i-1}) = x_{2i-1}$ and $min(x_{2i+2}, x_{2i+1}) = x_{2i+2}$, then $x_{2i-1} \le x_{2i}$ trivially. We have already seen from case (a) that $x_{2i} \le x_{2i+2}$. Since $x_{2i} \le x_{2i+2}$ and $x_{2i-1} \le x_{2i}$, we must have $x_{2i-1} \le x_{2i+2}$,

which means $min(x_{2i}, x_{2i-1}) \leq min(x_{2i+2}, x_{2i+1})$.
c) Since p is an inductive property, we must have $x_{2i} \leq x_{2i+1}$. Therefore, if $min(x_{2i}, x_{2i-1}) = x_{2i}$ and $min(x_{2i+2}, x_{2i+1}) = x_{2i+1}$, then $min(x_{2i}, x_{2i-1}) \leq min(x_{2i+2}, x_{2i+1})$.
d) Finally, $x_{2i-1} \leq x_{2i+1}$, since q holds by supposition. Therefore, if $min(x_{2i}, x_{2i-1}) = x_{2i-1}$ and $min(x_{2i+2}, x_{2i+1}) = x_{2i+1}$, then $min(x_{2i}, x_{2i-1}) \leq min(x_{2i+2}, x_{2i+1})$.

In all four cases, we showed that $x'_{2i} \leq x'_{2i+2}$, which means that q_i is an inductive property across all push operations.