⑥ 16.13

$A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ $A^{-1} = A$ $A^{-1}A = AA = I_n$

$I_n = \begin{bmatrix} a^2 + b^2 \\ ab - ab \end{bmatrix} = \begin{bmatrix} a^2 + b^2 \\ ab - ab \end{bmatrix}$

$T(e_1)$

$T(e_2) = \begin{bmatrix} ab - ab \\ b^2 + a^2 \end{bmatrix} = \begin{bmatrix} ab - ab \\ b^2 + a^2 \end{bmatrix} \rightarrow \begin{bmatrix} a^2 + b^2 & ab - ab \\ ab - ab & b^2 + a^2 \end{bmatrix} = I_n$

$\therefore a^2 + b^2 = 1$ & $ab - ab = 0$.

⑦ 16.16

a) $A^{-1}(\lambda x) = x$
$\lambda A^{-1}(x) = x$
$A^{-1}(x) = x/\lambda$

b) $A^{-1}(v) = u$, $A^{-1}(2v) = 2A^{-1}(v)$
$\therefore 2A^{-1}(v) = 2u$
$A^{-1}(2v) = 2u$.

c) $A^{-1}(2v_1) = u_1$ $A^{-1}(3v_2) = u_2$ $A^{-1}(3v_1 + 2v_2)$
$3/2(A^{-1}(2v_1)) = u_1 \cdot 3/2$ $2/3(A^{-1}(3v_2)) = u_2 \cdot 2/3$

$= \frac{3}{2}u_1 + \frac{2}{3}u_2$

(14) a) $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

b) $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ 4×3 $D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ 3×4

$CD = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ 4×3 $F = $ 3×4

$EF = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

This is not possible, since no possible combination of $F(e_n) \cdot E$ would produce $EF(e_4)$

(15) $M = \begin{array}{ccc} & j & 7000 \\ i & \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 \ldots & \\ & & & 0 \end{bmatrix} & \\ 7000 & & \end{array}$

Every ij entry in a M^2 matrix is given by the dot product of the i row with the j column of the M matrix, therefore, every ij entry is the number of mutual friends i and j have.

(16) a) $A = \begin{array}{cc} & j \\ & \begin{bmatrix} a_1 & & \\ & \ldots & \\ & & \\ a_{15} & & a_{55} \end{bmatrix} \\ i & \end{array}$ $\vec{V} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$

If $A\vec{V}$ is the number of birds that move from island to island after one year, then $A(A\vec{V})$ or $A^2\vec{V}$ is the number of birds that moved from island to island after two years.

b) Every ij entry in A^2 is the dot product of the i row and j column of A, and therefore the fraction of birds that moved over two years from island i to island j.

c) iA produces the cumulative fraction of birds that travel from every island j.

(17) a) $Q_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} ax + by \\ bx + cy \end{bmatrix}$

$= \begin{bmatrix} x(ax+by) + y(bx+cy) \end{bmatrix}$

$= ax^2 + byx + byx + cy^2$

$= ax^2 + 2byx + cy^2$

b) $f(x,y)$ has five terms, however a 2×2 matrix can produce at most 4 terms when multiplied by a vector in $\mathbb{R}^2$ $\therefore$ no, there does not exist such a matrix.

c) $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x^2+3y^2-5z^2+6yz+xy$

$= x(xa+by+cz) + y(xb+dy+ez) + z(xc+ey+fz)$

$= \underbrace{x^2a}_{a=1} + \underbrace{bxy + bxy}_{b=\frac{1}{2}} + \underbrace{czx + czx}_{c=0} + \underbrace{ezy + ezy}_{e=3} + \underbrace{dy^2}_{d=3} + \underbrace{fz^2}_{f=-5}$

$A = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 3 & 3 \\ 0 & 3 & -5 \end{bmatrix}$

(18) a) $T = \frac{PV}{k}$ $\left(\frac{\partial T}{\partial V}\right)_P = \frac{Pk - 0}{k^2} = \frac{P}{k}$

b) $V = \frac{kT}{P}$ $\left(\frac{\partial V}{\partial P}\right)_T = \frac{0 - 1(kT)}{P^2} = \frac{-kT}{P^2}$

c) $P = \frac{kT}{V}$ $\left(\frac{\partial P}{\partial T}\right)_V = \frac{kV - 0}{V^2} = \frac{k}{V}$

d) $\frac{P}{k} \cdot \frac{-kT}{P^2} \cdot \frac{k}{V} = \frac{P}{k} \cdot -\left(\frac{PV}{1}\right)\left(\frac{1}{P}\right) \cdot \frac{k}{V} = -1$

(19) $G\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \frac{3}{7}$ $G(x) = Af\left(Bf\left(\begin{bmatrix}0\\3\end{bmatrix}\right)\right)$ $C\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix}$

a) $G\left(\begin{bmatrix}C_{11}-0\\C_{21}-3\end{bmatrix}\right) = G\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + DG\begin{bmatrix}C_{11}-0\\C_{21}-3\end{bmatrix}$

$DG = A \cdot Df\Big|_{Bf\left(\left[\begin{smallmatrix}0\\3\end{smallmatrix}\right]\right)} \cdot B \cdot Df\Big|_{\left[\begin{smallmatrix}0\\3\end{smallmatrix}\right]}$

$Df = \begin{bmatrix}\partial f_1/\partial x & \partial f_1/\partial y\\ \partial f_2/\partial x & \partial f_2/\partial y\end{bmatrix} = \begin{bmatrix}1/(1+x)^2 & 0\\ 0 & 1/(1+y)^2\end{bmatrix}$

$\star\ Bf\left(\begin{bmatrix}0\\3\end{bmatrix}\right) = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0\\3/4\end{bmatrix} = \begin{bmatrix}0\\3/4\end{bmatrix}$

$E = B \cdot Df\Big|_{\left[\begin{smallmatrix}0\\3\end{smallmatrix}\right]} = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1/16\end{bmatrix} = \begin{bmatrix}1 & 1\\1/16 & 0\end{bmatrix}$

$M = A \cdot Df\Big|_{Bf\left(\left[\begin{smallmatrix}0\\3\end{smallmatrix}\right]\right) = \left[\begin{smallmatrix}0\\3/4\end{smallmatrix}\right]} = \begin{bmatrix}1 & 1\end{bmatrix} \cdot \begin{bmatrix}1 & 0\\0 & 16/49\end{bmatrix} = \begin{bmatrix}1 & 16/49\end{bmatrix}$

$DG = M \cdot E = \begin{bmatrix}1 & 16/49\end{bmatrix}\begin{bmatrix}1 & 1\\1/16 & 0\end{bmatrix} = \begin{bmatrix}50/49 & 1\end{bmatrix}$

$= 3/7 + \begin{bmatrix}50/49 & 1\end{bmatrix}\begin{bmatrix}C_{11}-0\\C_{21}-3\end{bmatrix}$

$= 3/7 + 50/49(C_{11}-0) + C_{21} - 3$

$\therefore$ since a change in $50/49\, C_{11} > C_{21}$,

C_{11} should be changed by $+0.01$ for $G\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$ to increase as much as possible.

b) $G\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \frac{61}{65}$ $\quad G(x) = Af\left(Bf\left(\begin{bmatrix}2\\1\end{bmatrix}\right)\right)$

$$G\left(\begin{bmatrix}c_{12}-2\\c_{22}-1\end{bmatrix}\right) = G\left(\begin{bmatrix}0\\1\end{bmatrix}\right) + DG\begin{bmatrix}c_{12}-2\\c_{22}-1\end{bmatrix}$$

$$DG = A \cdot Df\Big|_{Bf\left(\begin{bmatrix}2\\1\end{bmatrix}\right)} \cdot B \cdot Df\Big|_{\begin{bmatrix}2\\1\end{bmatrix}}$$

$$E = B \cdot Df\Big|_{\begin{bmatrix}2\\1\end{bmatrix}} = \begin{bmatrix}1/9 & 0\\0 & 1/4\end{bmatrix} \rightarrow \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1/9 & 0\\0 & 1/4\end{bmatrix} = \begin{bmatrix}1/9 & 1/4\\1/9 & 0\end{bmatrix}$$

$$M = A \cdot Df\Big|_{Bf\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}4/3\\2/3\end{bmatrix}} \begin{bmatrix}9/49 & 0\\0 & 9/25\end{bmatrix} \rightarrow \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}9/49 & 0\\0 & 9/25\end{bmatrix} = \begin{bmatrix}\frac{9}{49} & \frac{9}{25}\end{bmatrix}$$

$$M \cdot E = \begin{bmatrix}\frac{9}{49} & \frac{9}{25}\end{bmatrix}\begin{bmatrix}1/9 & 1/4\\1/9 & 0\end{bmatrix} = \begin{bmatrix}\frac{74}{1225} & \frac{9}{196}\end{bmatrix}$$

$$G\left(\begin{bmatrix}c_{12}-2\\c_{22}-1\end{bmatrix}\right) = \frac{61}{65} + \begin{bmatrix}74/1225 & 9/196\end{bmatrix}\begin{bmatrix}c_{12}-2\\c_{22}-1\end{bmatrix}$$

$$= 61/65 + 74/1225(c_{12}-2) + 9/196(c_{22}-1)$$

$\therefore$ since a change in $74/1225\, c_{12} > 9/196\, c_{22}$, c_{12} should be changed by -0.01 for $G\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$ to decrease as much as possible.

(13) a) $\frac{\partial}{\partial x}(F(x,y)) = \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x}$

$= \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} = -\frac{\partial F}{\partial x}$

$= \frac{\partial y}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial y} = -\frac{F_x(x,y)}{F_y(x,y)}$

b) $= \frac{3x^2}{2y} = -\frac{F_x(x,y)}{F_y(x,y)}$ $\quad y = x^{3/2}$

$x^{3/2} = y \quad \frac{\partial y}{\partial x} = \frac{3}{2}x^{\frac{1}{2}} \cdot \frac{x^{3/2}}{x^{3/2}} = \frac{3x^2}{2x^{3/2}} = \frac{3x^2}{2y}$

(1) d) N/A

e)

$$\begin{array}{ccc} T(e_1) & T(e_2) & T(e_3) \\ \begin{bmatrix} 10 + 44 \\ 12 + 52 \end{bmatrix} & \begin{bmatrix} 20 + 55 \\ 24 + 65 \end{bmatrix} & \begin{bmatrix} 30 + 66 \\ 30 + 78 \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} 54 & 75 & 96 \\ 64 & 89 & 114 \end{bmatrix}$$

g) 14

h) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

(20) I spent 14 hours outside of class working on Math stuff.

(2) $T_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$T_2(e_1) = \dfrac{\cos\theta}{\cos^2\theta + \sin^2\theta} = \begin{bmatrix} \cos^2\theta \\ \sin\theta\cos\theta \end{bmatrix}$

$T_2(e_2) = \dfrac{\sin\theta}{\cos^2\theta + \sin^2\theta} = \begin{bmatrix} \sin\theta\cos\theta \\ \sin^2\theta \end{bmatrix}$

$T_2 = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

a) $= \begin{bmatrix} \sin\theta\cos\theta & -\cos^2\theta \\ \sin^2\theta & -\cos\theta\sin\theta \end{bmatrix} \times \dfrac{\cos\theta}{\sin\theta}$ $\quad \frac{\pi}{2}, \pi/0$

$= \begin{bmatrix} \sin\theta\cos\theta & -\cos^2\theta \\ \sin\theta\cos\theta & -\cos^2\theta \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\theta & -\cos^2\theta \\ 0 & 0 \end{bmatrix}$

b) $\sin\theta\cos\theta\, x_1 - \cos^2\theta\, x_2 = 0$

$x_1 = \dfrac{\cos^2\theta}{\sin\theta\cos\theta} x_2 \qquad N(A) = \begin{bmatrix} \dfrac{\cos\theta}{\sin\theta} \\ 1 \end{bmatrix} = \text{span}\left(\begin{bmatrix} \dfrac{\cos\theta}{\sin\theta} \\ 1 \end{bmatrix}\right)$

$x_2 = 1$

c) $\text{image} = C(A) = \begin{bmatrix} \sin\theta\cos\theta \\ \sin^2\theta \end{bmatrix} = \text{span}\left(\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\right)$

(3) $T_1 = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$

$T_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$T_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$

a) $= \begin{bmatrix} -\cos\theta\sin\theta & -\sin^2\theta \\ \cos^2\theta & \sin\theta\cos\theta \end{bmatrix} \times \frac{\sin\theta}{\cos\theta}$

$= \begin{bmatrix} -\cos\theta\sin\theta & -\sin^2\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} = \begin{bmatrix} -\cos\theta\sin\theta & -\sin^2\theta \\ 0 & 0 \end{bmatrix}$

$-\cos\theta\sin\theta\, x_1 - \sin^2\theta\, x_2 = 0$

$x_1 = \frac{\sin^{\cancel{2}}\theta \cdot x_2}{-\cos\theta\cancel{\sin\theta}} = \begin{bmatrix} \sin\theta \\ -\cos\theta \\ 1 \end{bmatrix}$

$x_2 = x_2$

b) $N(A) = \text{span}\left(\begin{bmatrix} \sin\theta \\ -\cos\theta \\ 1 \end{bmatrix}\right)$

c) $C(A) = \text{span}\left(\begin{bmatrix} -\cos\cancel{\theta}\sin\theta \\ \cos^2\cancel{\theta} \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}\right)$

8 a) i) $$2\begin{bmatrix}3\\1\end{bmatrix}-1\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}6\\2\end{bmatrix}-\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}5\\0\end{bmatrix}$$

ii) $$1\begin{bmatrix}3\\1\end{bmatrix}+2\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix}+\begin{bmatrix}2\\4\end{bmatrix}=\begin{bmatrix}5\\5\end{bmatrix}$$

b) i) $$a\begin{bmatrix}3\\1\end{bmatrix}+b\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix} \qquad V_\beta=\begin{bmatrix}1\\-1\end{bmatrix}$$

$a=1$, $b=-1$

ii) $$a\begin{bmatrix}3\\1\end{bmatrix}+b\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}9\\8\end{bmatrix} \qquad v_\beta=\begin{bmatrix}2\\3\end{bmatrix}$$

$a=2$, $b=3$

9 a) i) $$0\begin{bmatrix}1\\0\\-1\end{bmatrix}+1\begin{bmatrix}0\\1\\1\end{bmatrix}+1\begin{bmatrix}1\\1\\-1\end{bmatrix}=\begin{bmatrix}1\\2\\0\end{bmatrix}$$

ii) $$2\begin{bmatrix}1\\0\\-1\end{bmatrix}+3\begin{bmatrix}0\\1\\1\end{bmatrix}-1\begin{bmatrix}1\\1\\-1\end{bmatrix}=\begin{bmatrix}2\\0\\-2\end{bmatrix}+\begin{bmatrix}0\\3\\3\end{bmatrix}+\begin{bmatrix}-1\\-1\\1\end{bmatrix}=\begin{bmatrix}1\\2\\2\end{bmatrix}$$

b) i) $$a\begin{bmatrix}1\\0\\-1\end{bmatrix}+b\begin{bmatrix}0\\1\\1\end{bmatrix}+c\begin{bmatrix}1\\1\\-1\end{bmatrix}=\begin{bmatrix}0\\1\\1\end{bmatrix} \qquad v_\beta=\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$b=1$, $a=0$, $c=0$

4

-2 -4

ii) $$a\begin{bmatrix}1\\0\\-1\end{bmatrix}+b\begin{bmatrix}0\\1\\1\end{bmatrix}+c\begin{bmatrix}1\\1\\-4\end{bmatrix}=\begin{bmatrix}2\\2\\-1\end{bmatrix} \qquad v_\beta=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$a=1$, $b=1$, $c=1$

(4) $\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 2 & -1 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 2 & 1 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{array}\right]$

$= \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

(5) $\begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$ Rank $< m$ so not invertible.

(10) a) $\frac{dBMI}{dt} = \frac{\partial BMI}{\partial w}\frac{dw}{dt} + \frac{\partial BMI}{\partial h}\frac{dh}{dt}$

$= \frac{10\,000}{h^2}\frac{dw}{dt} - \frac{20\,000}{h^3}\frac{dh}{dt}$

$= \frac{10000}{(140)^2}(0.4) - \frac{20\,000}{(140)^3}(0.6)$

$= 0.059$ points / month

b) Given that his current BMI is 16.8, a rate of 0.059 will only be worrying if prolonged for an extended period.

(11) a) $f\circ g = (\sin t\, e^{t2}, \sin^2 t\, e^t, \sin^3 t + e^{3t})$

$$D f\circ g = \begin{bmatrix} e^{2t}\cos t + 2e^{2t}\sin t \\ 2e^t \sin t \cos t + e^t \sin^2 t \\ 3\sin^2 t\cos t + 3e^3 t \end{bmatrix}$$

b) $$Df = \begin{bmatrix} y^2 & 2xy \\ 2xy & x^2 \\ 3x^2 & 3y^2 \end{bmatrix} = \begin{bmatrix} e^{t2} & 2\sin t e^t \\ 2\sin t e^t & \sin^2 t \\ 3\sin^2 t & 3e^{2t} \end{bmatrix}$$

$$Dg = \begin{bmatrix} \cos t \\ e^t \end{bmatrix}$$

$$DfDg = \begin{bmatrix} e^{t2} & 2\sin t\, e^t \\ 2\sin t\, e^t & \sin^2 t \\ 3\sin^2 t & 3e^2 t \end{bmatrix} \begin{bmatrix} \cos t \\ e^t \end{bmatrix}$$

$$= \begin{bmatrix} \cos t\, e^{t2} + 2\sin t\, e^{2t} \\ \cos t\, 2\sin t\, e^t + \sin^2 t\, e^t \\ \cos t\, 3\sin^2 t + 3e^3 t \end{bmatrix}$$

12 $Df(g(1,-1,3)) = \begin{bmatrix} 2y & 2x \\ 3 & -1 \end{bmatrix}_{(2,5)} = \begin{bmatrix} 10 & 4 \\ 3 & -1 \end{bmatrix}$

$Df(g(x)) \cdot Dg = D f \circ g$

$\begin{bmatrix} 10 & 4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 4 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 26 & -10 & 28 \\ -1 & -3 & -7 \end{bmatrix} = D f \circ g(1,-1,3)$

(2) $T_1 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$T_2(e_1) = \dfrac{\cos\theta}{\cos^2\theta + \sin^2\theta} = \begin{bmatrix} \cos^2\theta \\ \sin\theta\cos\theta \end{bmatrix}$

$T_2(e_2) = \dfrac{\sin\theta}{\cos^2\theta + \sin^2\theta} = \begin{bmatrix} \sin\theta\cos\theta \\ \sin^2\theta \end{bmatrix}$

$T_{2_1} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

a) $= \begin{bmatrix} \sin\theta\cos\theta & -\cos^2\theta \\ \sin^2\theta & -\cos\theta\sin\theta \end{bmatrix} \quad x \quad \dfrac{\cos\theta}{\sin\theta}$

$= \begin{bmatrix} \sin\theta\cos\theta & -\cos^2\theta \\ \sin\theta\cos\theta & -\cos^2\theta \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\theta & -\cos^2\theta \\ 0 & 0 \end{bmatrix}$

b) $\sin\theta\cos\theta\, x_1 - \cos^2\theta\, x_2 = 0$

$x_1 = \dfrac{\cos^2\theta\, x_2}{\sin\theta\cancel{\cos\theta}}$ $\quad N(A) = \begin{bmatrix} \dfrac{\cos\theta}{\sin\theta} \\ 1 \end{bmatrix} = \text{span}\left(\begin{bmatrix} \dfrac{\cos\theta}{\sin\theta} \\ 1 \end{bmatrix}\right)$

$x_2 = 1$

c) image $= C(A) = \begin{bmatrix} \sin\theta\cos\theta \\ \sin^2\theta \end{bmatrix} = \text{span}\left(\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\right)$