

# Vibrations, Structural & Computational Dynamics

October 20<sup>th</sup> 2020

$$\omega^2 Mx = Kx \Leftrightarrow (K - \lambda M)x = 0$$

ISAE/SUPAERO TOULOUSE  
SA403 - ADVANCED COMPUTATIONAL STRUCTURAL MECHANICS

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# Vibrations, Structural & Computational Dynamics

## Scope/Contents

Let's have fun & consider the next topics in more details in the next slides :

- Reminders
- s-dof system
- Industry Flavours of Dynamics Analysis
- Computational Approaches of Dynamics Analysis
- Reduction of Structures for Dynamics Analysis






## Dynamic Definition

5/67

# Reminder

## Dynamic Definition

Approach About dof	Keep it simple...but not simpler s-dof	... for fun... n-dof	... Reality continuum
User's Face			

**Table 1:** Classification of Dynamic Classical Models vs. Difficulty.

**Nota :** The art to switch from a dynamic analysis to a s-dof analysis does exist. For example [Miles'equation](#).

# Reminder

## Single dof System

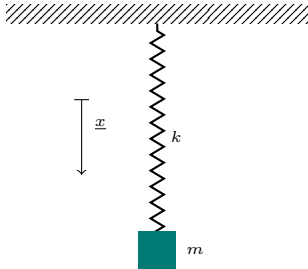


Figure 1: Single dof System without Damping device/sketch.

One has from LAGRANGE or D'ALEMBERT principle :

$$m\ddot{x} + c\dot{x} + kx = F \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\ddot{x} + 2\xi\omega_0\dot{x} + \omega_0^2 x = \frac{F}{m} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

with the damping factor

$$\xi = \frac{c}{2m\omega_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

And :

$$\omega_0 = \sqrt{\frac{k}{m}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Pulsation  
is a very important quantity in Dynamics of Structures.

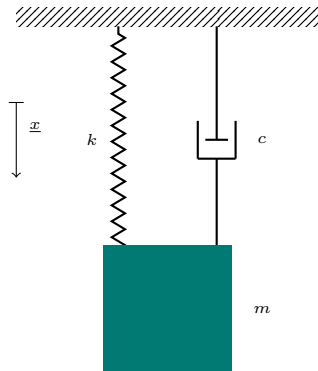


Figure 2: Single dof System with Damping device/sketch.

$$m\ddot{x} + c\dot{x} + kx = F \quad (7)$$

$$\ddot{x} + 2\xi\omega_0\dot{x} + \omega_0^2x = \frac{F}{m} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

with the damping factor

[illegible]

And :

$$\omega_0 = \sqrt{\frac{k}{m}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Pulsation  $\omega$  is a very important quantity in Dynamics of Structures.





Reminder      Single dof System

# Reminder

## Single dof System

[illegible]

[illegible]

The frequencies  $f_i$  are defined as :

[illegible]



# Reminder

## Natural vs. Forced

A dynamic system is either in :

### 1. free vibration

- No external excitations  $F(t) = 0$
- **Natural** Response/Vibration of the Structure

### 2. forced vibration

- External excitations  $F(t) \neq 0$ 
  - Either  $F(t)$  is periodic ( $\sim$  long timescales : many periods )
  - Or  $F(t)$  is transient ( $\sim$  short timescales)
- **Time** Response in Magnitude and Phase dependent upon the excitation (can be referred as the excitation Spectra)

# Reminder

## Damping definition

Planned for 2021.

# Reminder

## Modal basis definition

Planned for 2021.

# Reminder

## Time domain vs. frequency domain

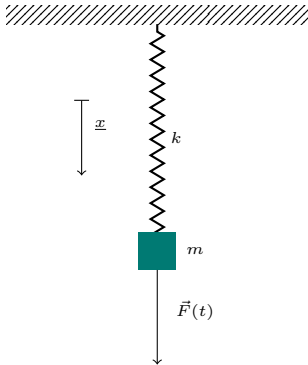
Planned for 2021.

## Rigid body vs. elastic body

The main business of complex dynamics phenomena is to bring the modelizaiton towards elastic (at least) solids.

# Single dof System

## Forced Harmonic State - General



One has from LAGRANGE or D'ALEMBERT principle :

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t) \quad . \quad . \quad . \quad . \quad (13)$$

and

$$F(t) = F_0 \cos \Omega t = F_0 \mathcal{R} [\exp i\Omega t] \quad . \quad . \quad . \quad . \quad (14)$$

The response is the sum of a transient response  $x_t(t)$  and a constant response  $x_p(t)$  :

- $x_t(t)$  is the natural response
- $x_p(t)$  is the response after the transient response has vanished : with a pulsation of same  $\Omega$  than the excitation and with a magnitude and a phase to be derived

Figure 3: Single dof System without Damping device/sketch.



# Single dof System

## Forced Harmonic State - Undamped

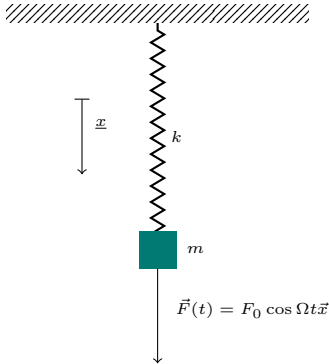


Figure 4: Single dof System.

One has from LAGRANGE or D'ALEMBERT principle :

$$m\ddot{x}(t) + kx(t) = F_0 \cos \Omega t \quad . \quad . \quad . \quad . \quad . \quad (15)$$

One has :

$$x_t(t) = X_t \cos(\omega_0 t - \varphi_t) \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and :

$$x_p(t) = X_p \cos \Omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

inserted in Equation 15 one has the magnitude

$$X_p = \frac{F_0}{k - m\Omega^2} = \frac{1}{\omega_0^2 - \Omega^2} \cdot \frac{F_0}{m} \quad . \quad . \quad . \quad . \quad (18)$$

and of course full solution

$$x(t) = x_t(t) + x_p(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

**Nota :** Without damping the excitation and the time response are in phase.

# Single dof System

## Forced Harmonic State - Damped

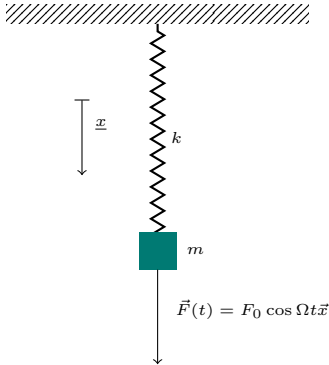


Figure 5: Single dof System.

One has from LAGRANGE or D'ALEMBERT principle :

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F_0 \cos \Omega t \quad . \quad . \quad . \quad (20)$$

One has :

$$x_t(t) = X_t \exp(-\xi \omega_0 t) \cos(\omega_0 t - \varphi_t) \quad . \quad . \quad (21)$$

and :

$$x_p(t) = X_p \cos(\Omega t - \varphi_p) \quad . \quad . \quad . \quad . \quad (22)$$

inserted in Equation 15 one has the magnitude

$$X_p = \frac{F_0}{m} \cdot \frac{1}{\sqrt{(\omega_0^2 - \Omega^2)^2 + 4(\xi \Omega \omega_0)^2}} \quad (23)$$

$$\varphi_p = \tan^{-1} \frac{2\xi \omega_0 \Omega}{\omega_0^2 - \Omega^2} \quad . \quad . \quad . \quad . \quad (24)$$

and of course full solution

$$x(t) = x_t(t) + x_p(t) \quad . \quad . \quad . \quad . \quad . \quad (25)$$

**Nota :** amplification of the applied force is maximum when  $\Omega \rightarrow \omega_0$ .

## Forced Harmonic State - Damped

- [illegible]

- 

# Some classical real world examples

Taipei 101 Tower

Picture from peellden [2005]



# Some classical real world examples

Tapei 101 Tower

Picture from du Plessis [2010]

# Some classical real world examples

## Tapei 101 Tower

*Situé à Taïwan, la tour Taipei 101 mesure plus de 500 mètres de haut pour une masse totale de près de 700 000 tonnes. Construite en 2003, elle est restée le plus haut gratte-ciel du monde jusqu'en 2007 avec l'inauguration du Burj Dubaï (828 m).*

*Son TMD est constitué d'une boule d'acier de 660 tonnes pour un rayon de 2,7 m suspendue entre le 92e et le 87e étage. Elle pendule grâce à 4 câbles d'acier de 11,5 m et est amortie par 8 vérins hydrauliques. Sous l'action des typhons, le déplacement des étages les plus hauts peut être de 3 m.*

*Ainsi, il est prédit par les constructeurs que les oscillations de la tour peuvent être atténuées de 30 % à 40 %. Le système est étudié pour résister à un tremblement de terre de magnitude 7 sur l'échelle de Richter. Son efficacité a été vérifiée lors du séisme Sichuan qui a frappé Taiwan en 2008. De plus, le TMD ne représente que 0.2 % du coût total de construction du bâtiment.*

wikipedia — 2020.

**Nota :** Further materials... about 2008 Tuned Mass Damper V&V Oberkampff and Roy [2010].



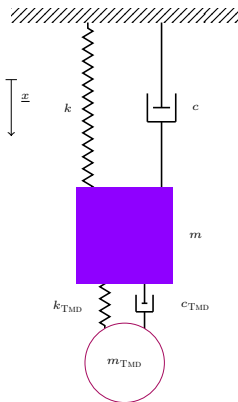
Constructing Taipei 101  
Real World Project



Constructing Taipei 101  
Real World Project

# Some classical real world examples

## Tapei 101 Tower



**Figure 6:** Single dof System with Damping & Tuned Mass Damper devices. The tuned mass damper basically permits the system of mass  $m$  to dissipate energy.

# Some classical real world examples

## Chinook Helicopter Ground Test

An Helicopter Ground Test that went particularly well.



Copyright © 2008-2009



# Some classical real world examples

## A/C Take-off

Main loads-drivers for Engine Attachments under a take-off loadcase are :

- Engine Thrust & Torque
- Air Intake  $F_y$  &  $F_z$  aero resultants @A1



- Inertial  $n_z \sim 1.2$
- Overall Aeroloads acting on Nacelle :  $F_y$ ,  $F_z$  and  $M_x$ ,  $M_y$  and  $M_z$
- No sensitivity to A/C precession rate ( $\dot{\alpha} \cdot \sum I \cdot \omega$  low means no gyroscopic effect)

# Some classical real world examples

## A/C Take-off

The F/T A/C parameters allow to find easily the date  $t$  for which  $\max n_z$  is reached. FT11  $n_z=1.2$  is reached for take-off. Numerical values for illustration purposes.

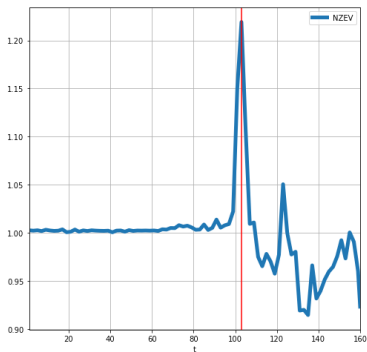


Figure 7: A330neo FT11 -  $n_z$  vs.  $t$ .

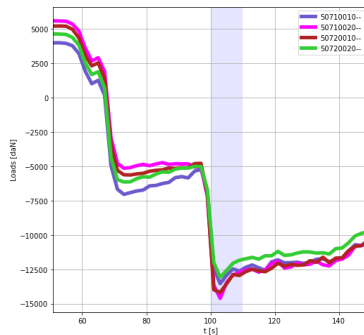


Figure 8: A330neo FT11 - E/M FM [daN] vs.  $t$ .

# Some classical real world examples

## A/C Engine Failure Loads

Dynamics phenomena involved e.g. within :

- Noise / Pax
- Engine
- Rotorburst
- Imbalance
- Fluid/Structure interaction e.g. Fuel
- and of course many more...

For an A/C an infamous dynamic event is loss of part of an engine as turbomachinery may lose parts to due their wear or whatever... The structure is regulated by two Technical Papers

- CS25.362 (Engine Failure Loads) from CS-25 [EASA \[2018\]](#)
- AMC 25-24 (Sustained Engine Imbalance)

# Some classical real world examples

## A/C Engine Failure Loads

### CS 25.362 Engine failure loads

ED Decision 2009/017/R

- (a) For engine mounts, pylons and adjacent supporting airframe structure, an ultimate loading condition must be considered that combines 1g flight loads with the most critical transient dynamic loads and vibrations, as determined by dynamic analysis, resulting from failure of a blade, shaft, bearing or bearing support, or bird strike event. Any permanent deformation from these ultimate load conditions should not prevent continued safe flight and landing.
- (b) The ultimate loads developed from the conditions specified in paragraph (a) are to be:
  - (1) multiplied by a factor of 1.0 when applied to engine mounts and pylons; and
  - (2) multiplied by a factor of 1.25 when applied to adjacent supporting airframe structure.

[Amdt 25/8]

Figure 9: CS25.362 Flavour.

# Some classical real world examples

## A/C Engine Failure Loads

### 4. BACKGROUND.

- a. Requirements. [CS 25.362](#) (“Engine failure loads”) requires that the engine mounts, pylons, and adjacent supporting airframe structure be designed to withstand 1g flight loads combined with the transient dynamic loads resulting from each engine structural failure condition. The aim being to ensure that the aeroplane is capable of continued safe flight and landing after sudden engine stoppage or engine structural failure, including ensuing damage to other parts of the engine.
- b. Engine failure loads. Turbine engines have experienced failure conditions that have resulted in sudden engine deceleration and, in some cases, seizures. These failure conditions are usually caused by internal structural failures or ingestion of foreign objects, such as birds or ice. Whatever the source, these conditions may produce significant structural loads on the engine, engine mounts, pylon, and adjacent supporting airframe structure. With the development of larger high-bypass ratio turbine engines, it became apparent that engine seizure torque loads alone did not adequately define the full loading imposed on the engine mounts, pylons, and adjacent supporting airframe structure. The progression to high-bypass ratio turbine engines of larger diameter and fewer blades with larger chords has increased the magnitude of the transient loads that can be produced during and following engine failures. Consequently, it is considered necessary that the applicant performs a dynamic analysis to ensure that representative loads are determined during and immediately following an engine failure event.

A dynamic model of the aircraft and engine configuration should be sufficiently detailed to characterise the transient loads for the engine mounts, pylons, and adjacent supporting airframe structure during the failure event and subsequent run down.

Figure 10: CS25.362 AMC Flavour - Background.

# Some classical real world examples

## A/C Engine Failure Loads

### 5. EVALUATION OF TRANSIENT FAILURE CONDITIONS

- a. Evaluation. The applicant's evaluation should show that, from the moment of engine structural failure and during spool-down to the time of windmilling engine rotational speed, the engine-induced loads and vibrations will not cause failure of the engine mounts, pylon, and adjacent supporting airframe structure. (Note: The effects of continued rotation (windmilling) are described in [AMC 25-24](#)).

Major engine structural failure events are considered as ultimate load conditions, since they occur at a sufficiently infrequent rate. For design of the engine mounts and pylon, the ultimate loads may be taken without any additional multiplying factors. At the same time, protection of the basic airframe is assured by using a multiplying factor of 1.25 on those ultimate loads for the design of the adjacent supporting airframe structure.

- b. Blade loss condition. The loads on the engine mounts, pylon, and adjacent supporting airframe structure should be determined by dynamic analysis. The analysis should take into account all significant structural degrees of freedom. The transient engine loads should be determined for the blade failure condition and rotor speed approved per CS-E, and over the full range of blade release angles to allow determination of the critical loads for all affected components.

The loads to be applied to the pylon and airframe are normally determined by the applicant based on the integrated model, which includes the validated engine model supplied by the engine manufacturer.

The calculation of transient dynamic loads should consider:

- the effects of the engine mounting station on the aeroplane (i.e., right side, left side, inboard position, etc.); and
- the most critical aeroplane mass distribution (i.e., fuel loading for wing-mounted engines and payload distribution for fuselage-mounted engines).

For calculation of the combined ultimate airframe loads, the 1g component should be associated with typical flight conditions.

Figure 11: CS25.362 AMC Flavour - Transient Loads.

## A/C Engine Failure Loads

CS-25.362 AMC §6 Analysis Methodology – 2018.

# Some classical real world examples

## A/C Engine Failure Loads

### 7. MATHEMATICAL MODELLING AND VALIDATION

- a. Components of the integrated dynamics model. The applicant should calculate airframe dynamic responses with an integrated model of the engine, engine mounts, pylon, and adjacent supporting airframe structure. The model should provide representative connections at the engine-to-pylon interfaces, as well as all interfaces between components (e.g., inlet-to-engine and engine-to-thrust reverser). The integrated dynamic model used for engine structural failure analyses should be representative of the aeroplane to the highest frequency needed to accurately represent the transient response. The integrated dynamic model consists of the following components that must be validated:
  - Airframe structural model.
  - Propulsion structural model (including the engine model representing the engine type-design).

Figure 12: CS25.362 AMC Flavour - Mathematical Modelling.



# Some classical real world examples

## A/C Engine Failure Loads



Picture from [Rolls-Royce \[2014\]](#)



Copyright © 2014 Rolls-Royce

# Some classical real world examples

## A/C Engine Failure Loads

Picture from **Rolls-Royce** [2020]



# Some classical real world examples

## A/C Engine Failure Loads



Picture from **Rolls-Royce** [2020]

# Some classical real world examples

## A/C Engine Failure Loads



Downloaded by F. Orange Nguyen  
Nasa X-45 Project

# Some classical (future) real world examples

A/C Engine Failure Loads



Picture from AIRBUS.



# Some classical (future) real world examples

## A/C Engine Failure Loads



Picture from AIRBUS.

# Some classical (future) real world examples

## A/C Engine Failure Loads

The propeller can be set into equations this way...



# Some classical real world examples

## A/C Engine Failure Loads

Engines can lead to e.g. two phenomena linked with flow instabilities :

- Surge
- Stall

Have a look to  
*Dixon and Hall [2013]*  
for more insights.

Picture from Airbus.



# Some classical real world examples

## A/C Engine Failure Loads

If one engine is shut down the Aircraft has to withstand Windmilling Loads.

Picture from Airbus (Same credits as Slide 41/67).



# Some classical real world examples

## Rotordynamics

References : Lalanne and Ferraris [1998], Muszynska [2005].

# Some classical real world examples

## Aeroelasticity

Planned for 2021.

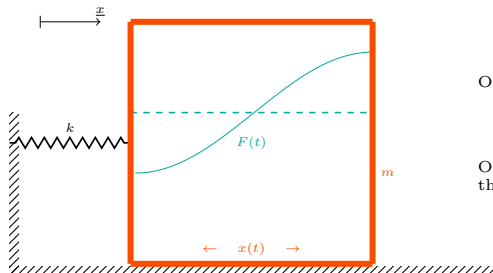
# Some classical real world examples

## Flight Dynamics

Planned for 2021.

## Fluid Structure Interacton

A flavour of Fluid Structure Interaction (e.g. de Langre [2001]).



One can assume e.g. :

$$\ddot{x} + x = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

One can derive the coupling equations that govern the two parameters  $(x, F)$ .

$$M\ddot{F} + KF = M_{SF}\ddot{x} \quad . \quad . \quad . \quad . \quad (28)$$

$$\ddot{x} + x = -m_{FS}\ddot{F} - m_F\ddot{x} \quad . \quad (29)$$

**Figure 13:** A Fluid Structure Interaction Simple Configuration. One can easily think e.g. to a bio-fuel tank embedded in a flying machine. Under manoeuvres the bio-fuel is swung.

Inertial coupling does exist for the two parameters  $(x, F)$ . One can limit the participation of the fluid to the dynamics of the structure installing anti-swung walls within the tank.

# Some classical real world examples

## Impact

Planned for 2021.

# Transition Slides

So touchy industrial dynamics phenomena could not be solved without massive usage of solvers involving state of the art numerical analysis algorithm.



## The FEA Formulation

## Basics

One solves POISSON's equation :

[illegible]

It is considered  $f \in L^2(\Omega)$ . It is assumed  $u \in H^2(\Omega)$ . Thus by GREEN formula

[illegible]

If it is chosen  $v \in H_0^1(\Omega)$  one has

[illegible]

The equation (32) is called the variational formulation or the weak form of the differential equation (30).

# The FEA Formulation

## Basics

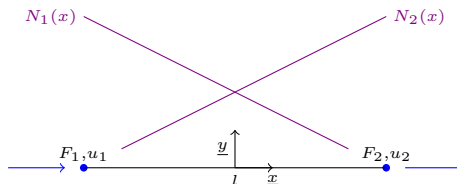


Figure 14: Simple axial element assumption sketch

Linear form function

$$\begin{cases} N_1(x) = \frac{1}{l} \left( \frac{l}{2} - x \right) \\ N_2(x) = \frac{1}{l} \left( x + \frac{l}{2} \right) \end{cases} \quad \dots \dots \dots (33)$$

Strain worth

$$\epsilon = B \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{l} & \frac{1}{l} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \dots \dots \dots (34)$$

## Basics

Stiffness matrix is derived as

$$K = \int_{\Omega} B^T E B d\Omega = S \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{E}{l^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} dx . . . . . \quad (35)$$

The simple bar element works merely as<sup>†</sup>

[illegible]

with stiffness matrix

[illegible]

**Nota :** Mass matrix is derived as

[illegible]

Finite Elements Solver User's user can usually output  $K$ ,  $M$  or any matrix in an file for double check  $\mathcal{E}$

~~analysis.~~

... a  $F = k u$  spring ☺

# The FEA Formulation

## Basics

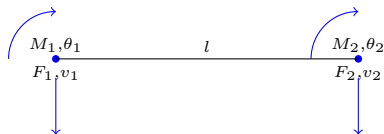


Figure 15: Simple bending element assumption sketch

For a simple bending element one has the relationship

$$\begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix} = \frac{2EI}{l^3} \begin{bmatrix} 6 & -3l & -6 & -3l \\ -3l & 2l^2 & 3l & l^2 \\ -6 & 3l & 6 & 3l \\ -3l & l^2 & 3l & 2l^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \quad (39)$$

# The FEA Formulation

## Advanced

Planned for 2021.

# The FEA Formulation

## Dynamic Reduction

Among the most interesting & funny concept within Dynamics FEA is the Model Reduction. On one hand it can be considered as classical business for Static Analysis with the associated GUYAN approach. On the other hand the approach for dynamic analysis is not numerically exact because upon eigenvalues desired dependent. Nevertheless it remains usual business within large companies dynamics analysis. Mainly to ease the exchange of FEM between e.g. stakeholders of a project. I follow Komzsik [2005] notations.

## The FEA Formulation

## Dynamic Reduction

For GUYAN Static Reduction the framework is linear elasticity. One solves

[illegible]

with o-set interior degrees of freedom and a-set exterior degrees of freedom one can split the equation (40)

[illegible]

whose first line of (41) means

$$\begin{aligned}
& K_{oo}u_o + K_{oa}u_a = F_o \\
& \Leftrightarrow K_{oo}^{-1}(K_{oo}u_o + K_{oa}u_a) = K_{oo}^{-1}F_o \\
& \Leftrightarrow u_o = K_{oo}^{-1}F_o - K_{oo}^{-1}K_{oa}u_a \\
& \Leftrightarrow u_o = \underbrace{K_{oo}^{-1}F_o}_{u_o^0} - \underbrace{K_{oo}^{-1}K_{oa}}_{-G_{oa}}u_a
\end{aligned} \quad (42)$$

or

[illegible]

with  $G_{oa}$  boundary transformation and  $u_o^0$  fixed boundary displacement.

# The FEA Formulation

## Dynamic Reduction

Thus the second line of (41) means with the definition (43) of  $u_o$

$$\begin{aligned}
 & K_{oa}^T u_o + K_{aa} u_a = F_a \\
 \Leftrightarrow & K_{oa}^T \left( u_o^0 + G_{oa} u_a \right) + K_{aa} u_a = F_a \\
 \Leftrightarrow & \left( \underbrace{K_{oa}^T G_{oa} + K_{aa}}_{\overline{K_{aa}}} \right) u_a = F_a - \underbrace{K_{oa}^T u_o^0}_{\overline{F_a}} \quad \dots \dots \dots (44)
 \end{aligned}$$

with  $\overline{K_{aa}}$  boundary stiffness and  $\overline{F_a}$  boundary load. Typically with a classical NASTRAN run one has  $\overline{K_{aa}} \equiv$  KAAX and  $\overline{F_a} \equiv$  PAX. Note that (44) defines the boundary displacement field associated to the ASET (NASTRAN language). For Dynamic framework :



# The FEA Formulation

## Dynamic Reduction

$$\begin{bmatrix} K_{oo} - \lambda M_{oo} & K_{ot} - \lambda M_{ot} \\ K_{ot}^T - \lambda M_{ot}^T & K_{aa} - \lambda M_{aa} \end{bmatrix} \begin{Bmatrix} u_o \\ u_t \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (45)$$

One considers first the eigenvalue problem :

$$[K_{oo} - \lambda M_{oo}] \Phi_o = 0 \quad (46)$$

One is lead to assume a diagonal matrix  $\Lambda_{q_o q_o}$  with  $q_o$  the eigenvalues and another matrix  $\Phi_{o q_o}$  with classical :

$$\Phi_{o q_o}^T M_{oo} \Phi_{o q_o} = I_{q_o q_o} \quad (47)$$

And for the stiffness :

$$\Phi_{o q_o}^T K_{oo} \Phi_{o q_o} = \Lambda_{q_o q_o} \quad (48)$$

The installed  $q_o$  is not today a computational issue for low frequency range of interest for e.g. an Aircraft. It is a gloabl rule of thumb to compute a higher number than in the end the eigenvalues of interest.

## The FEA Formulation

## Dynamic Reduction

For the boundary

[illegible]

with the coupled boundary  $\Phi_{tq_o}$ . The dynamics reduction leads to the introduction of the matrix with  $q_t + q_o$  columns and with  $t + o$  rows.

$$S = \begin{bmatrix} \Phi_{oq_o} & & \\ & \Phi_{tq_o} & \\ & & \ddots \end{bmatrix} \quad (50)$$

with  $Su_d = u_a$  one has

[illegible]

Thus

[illegible]

Then the problem for the residual structure is :

$$\begin{bmatrix} \Lambda_{q_o q_o} - \lambda I_{q_o q_o} & K_{q_o q_t} - \lambda M_{q_o q_t} \\ K_{q_o q_t}^T - \lambda M_{q_o q_t}^T & \Lambda_{q_t q_t} - \lambda I_{q_t q_t} \end{bmatrix} \begin{Bmatrix} u_{q_o} \\ u_{q_t} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (53)$$

## The FEA Formulation

## Dynamic Reduction

with the coupling matrices  $M_{q_o q_t}$  and  $K_{q_o q_t}$  uilt as previously.

[illegible]

[illegible]

One ends up with a  $q_o + q_t$  problem. All the matrices are not square. The nature of the choice of the  $q_o$  leads thus to the introduction of a numerical error. This kind of CRAIG-BAMPTON Reduction is not numerically exact for the modal analysis.

### The NEWMARK family integration scheme

$$M \ddot{u} + C \dot{u} + Ku = f_{ext} . . . . . (56)$$

$$\text{or} \quad M\ddot{u} + C\dot{u} + f_{int} = f_{ext} . . . . . (57)$$

[illegible][illegible][illegible]

### The NEWMARK family integration scheme

[illegible][illegible]



### The NEWMARK family integration scheme

[illegible]

# Optimization under Dynamics Constraints

## Two Springs/Three Masses Case Study

One considers the next Two Springs/Three Masses system. One is interested in the frictionless harmonic answer of the system sketched in Figure 16.

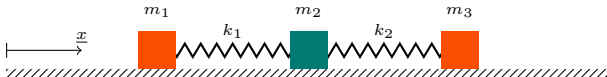


Figure 16: Two Springs/Three Masses system.

**Aim of Case study** (extracted from Yang [2010]): Choose masses and spring stiffnesses to satisfy next optimization (min-max Danskin [2012]) problem with  $S$  and  $k_0$  constants:

$$\text{Problem} \left\{ \begin{array}{l} \min_{p \in \{m_1, m_2, m_3, k_1, k_2\}} \omega = \max \omega_i \\ S = m_1 + m_2 + m_3 \\ m_1 = m_3 \\ k_1 = k_2 \geq k_0 \end{array} \right.$$



## Two Springs/Three Masses Case Study

**Answer :** One can reach the minimum-max  $\omega_i$

$$\min \max_i \omega_i = \left\{ \frac{8k_0}{S} \right\}^{\frac{1}{2}} . . . . . (67)$$

with the *ad hoc* mass split (cf. complete demonstration in Yang [2010]).

- Dynamics & Instability
- Component Modal Analysis
- ...

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- my nowadays regularly Dynamics Engineers focal points within Engine Manufacturer Companies
- my nowadays Aircraft Dynamics Specialists colleagues @ AIRBUS/Toulouse
- my nowadays Powerplant Integrated FEM Dynamics Specialists colleagues @ AIRBUS/Toulouse

# References

I did not copy/paste them only to get a nice Bibliography. I have been lead to open most of them either as a student or in my professional life. Some are really interesting to dive into more insight Structural & Computational Dynamics of Structures and I admit some content chapters, sections & every technical ... To complete the documentation of some finite elements software can be of great help (NASTRAN as Sitton [1997], Herling [1997], ABAQUS, ANSYS, MATLAB, ...)

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