FLEXIBLE SURFACES IN $\mathbb{C}P^2$ AND $S^2 \times S^2$

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ABSTRACT. A surface Σ in a 4-manifold M is called flexible if any mapping class of the surface arises as the restriction of a diffeomorphism $(M, \Sigma) \to (M, \Sigma)$. We construct flexible surfaces in $\mathbb{C}P^2$ and $S^2 \times S^2$ within any prescribed non-characteristic homology class. Within characteristic homology classes there is a spin structure obstructing flexibility and we construct so-called spin-flexible representatives.

1. INTRODUCTION

Let $\Sigma^2 \subset M^4$ be a closed oriented surface in a closed oriented 4-manifold M. Associated to the pair (M, Σ) is the *relative mapping class group* $\mathrm{Mod}(M, \Sigma) = \pi_0(\mathrm{Diff}^+(M, \Sigma))$, where $\mathrm{Diff}^+(M, \Sigma)$ consists of orientation preserving diffeomorphisms of M fixing Σ *setwise*. Restriction determines a homomorphism $\mathcal{R} : \mathrm{Mod}(M, \Sigma) \to \mathrm{Mod}^{\pm}(\Sigma) = \pi_0(\mathrm{Diff}(\Sigma))$. Following Hirose [Hir02], the *extendable subgroup* $\mathcal{E}(M, \Sigma)$ of Σ in M is the image of $\mathcal{R}$ in $\mathrm{Mod}(\Sigma) = \pi_0(\mathrm{Diff}^+(\Sigma))$. The surface Σ is called *flexible* in M if $\mathcal{E}(M, \Sigma) = \mathrm{Mod}(\Sigma)$. If Σ is *characteristic*[1] in M and $H_1(M; \mathbb{Z}) = 0$ there is an obstruction to flexibility, namely *Rochlin's quadratic form* $q_\Sigma : H_1(\Sigma; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ (Proposition 2.5). The mapping class group $\mathrm{Mod}(\Sigma)$ acts on the set of quadratic forms on $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ by pullback. By definition of q_Σ, $\mathcal{E}(M, \Sigma) \leq \mathrm{Mod}(\Sigma)[q_\Sigma]$, where $\mathrm{Mod}(\Sigma)[q_\Sigma]$ denotes the stabilizer of q_Σ. We say Σ is *spin-flexible* if $\mathcal{E}(M, \Sigma) = \mathrm{Mod}(\Sigma)[q_\Sigma]$. A guiding question of Hirose, appearing as Problem 6.3 in [Hir15], is as follows.

Question 1.1. *Assume that M is simply connected. Let $x \in H_2(M; \mathbb{Z})$ be an element which is not characteristic. Does x admit a flexible representative?*

We resolve this question for $\mathbb{C}P^2$ and $S^2 \times S^2$. In particular we prove

Theorem A. *Let $M = \mathbb{C}P^2$ or $S^2 \times S^2$. Fix $x \in H_2(M; \mathbb{Z})$. If x is characteristic, there exists a spin-flexible representative for x, otherwise there is a flexible representative for x.*

We may further ask for flexible representatives which minimize genus in their homology class. Recall that smooth degree d plane curves in $\mathbb{C}P^2$ represent $d[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2; \mathbb{Z})$, minimize genus in their homology class (Thom Conjecture, [KM94]), and are characteristic if and only if d is odd. In this direction we establish that a single internal *stabilization* suffices for (spin) flexibility.

Theorem B. *Let $X_d \subset \mathbb{C}P^2$ be a smooth plane curve of degree d. Let $T \subset S^4$ be the boundary of a solid torus in S^4. Let Z_d be the stabilization $X_d \# T \subset \mathbb{C}P^2 \# S^4 \cong \mathbb{C}P^2$. If d is even, Z_d is flexible. If d is odd, Z_d is spin-flexible.*

This result may be viewed as a generalization of [Hir05, Theorem 5.1] to all $d \neq 3$.

Context and Related Work. Recall that a surface in a 4-manifold M is *unknotted* if it is the boundary of a handlebody[2] in M. Montesinos established [Mon83, Theorem 5.4] that an unknotted torus in S^4 is spin-flexible and Hirose [Hir02, Theorem 1.1] generalized this to unknots of all higher genera. On the other hand, there are examples (non-trivial spun tori) of *knotted surfaces* in S^4 whose extendable subgroup is much *smaller* (namely of infinite index, see [Hir93] for precise details). In $\mathbb{C}P^2$ unknots are no longer characteristic and are *flexible* surfaces [Hir05, Theorem 3.1]. A broader version of Question 1.1 asks only for the *existence* of flexible surfaces in a simply connected 4-manifold. In this regard, Hirose and Yasuhara construct flexible surfaces of

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[1]Recall that a class in $H^2(M; \mathbb{Z})$ is *characteristic* if it is an integral lift of the second Stiefel-Whitney class of M. A closed surface $\Sigma \subset M$ is characteristic if the Poincaré dual of $[\Sigma] \in H_2(M; \mathbb{Z})$ is.

[2]In this paper we exclusively work with orientable surfaces.

any genus in $\mathbb{C}P^2, S^2 \times S^2, \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ or any elliptic surface [HY08, Theorem 3.1]. Yet another version allows one to stabilize the ambient 4-manifold with an $S^2 \times S^2$ summand. Hirose and Yasuhara prove [HY08, Theorem 4.1] that for any $\Sigma^2 \subset M^4$ with M simply connected, the surface $\Sigma \# S^2 \times \{*\} \subset M \# S^2 \times S^2$ is flexible. In particular, if one is allowed to change *both* the ambient manifold and homology class, the best result is possible.

Let $X_d \subset \mathbb{C}P^2$ be a smooth degree d plane curve. Hirose proved that X_3 and X_4 are flexible surfaces in $\mathbb{C}P^2$ [Hir05, Theorem 4.2]. Moreover, Hirose proved that, for an unknot $\Sigma_g \subset S^4$ of genus g, the surface $X_3 \# \Sigma_g$ is spin-flexible [Hir05, Theorem 5.1]. A corollary of work of Salter is that $X_5 \subset \mathbb{C}P^2$ is spin-flexible [Sal25, Theorem A]. The situation for plane curves of higher degree is, to the best of the authors knowledge, unknown. However, $\mathcal{E}(\mathbb{C}P^2, X_d)$ is a finite index subgroup of $\mathrm{Mod}(X_d)$ for $d \geq 6$. This follows from work of Salter [Sal19, Theorem 1.3].

Precisely, let $\mathcal{P}_d \subset |\mathcal{O}(d)|$ denote the locus of smooth degree d plane curves. The tautological family over $\mathcal{P}_d$ determines a monodromy representation $\rho_d : \pi_1(\mathcal{P}_d, X_d) \to \mathrm{Mod}(X_d)$. By the isotopy extension theorem, the homomorphism ρ_d factors through an *embedded monodromy representation* $\widetilde{\rho}_d : \pi_1(\mathcal{P}_d, X_d) \to \mathrm{Mod}(\mathbb{C}P^2, X_d)$. In particular, the image of ρ_d is a lower bound for $\mathcal{E}(\mathbb{C}P^2, X_d)$. Extending work in [Sal19], Calderon-Salter [CS21] establish that the image of ρ_d in $\mathrm{Mod}(X_d)$ coincides with the *stabilizer* $\mathrm{Mod}(X_d)[\phi_d]$ of a $(d-3)$-spin structure ϕ_d.[3] In particular,

$$\mathrm{Mod}(X_d)[\phi_d] \leq \mathcal{E}(\mathbb{C}P^2, X_d).$$

The higher spin structure ϕ_d is an instance of a more general phenomena used in proving both Theorems A and B - it arises from a specific *framing*[4] of the surface X_d in an affine piece $\mathbb{C}^2 \subset \mathbb{C}P^2$. Let $B^4_\varepsilon(0) \subset \mathbb{C}^2$ be a ball of radius $\varepsilon > 0$ about the origin $0 \in \mathbb{C}^2$. Up to isotopy, $X_d \cap B^4_\varepsilon(0) \subset \mathbb{C}^2$ is a Seifert surface for the closure of a positive braid. Ferretti [Fer25], using work of Stallings [Sta78], associates to the closure of any positive braid β a *framed* Seifert surface $F_\beta \subset S^3$. Our representatives are obtained by capping off such Seifert surfaces. In fixing only the ambient homology class we may stabilize with an unknotted surface. Our techniques rest on the contrast between the geometric rigidity of the framed surfaces and the methods of Hirose established for unknots in S^4 [Hir02].

Remark 1.2. (Higher spin structures and Rochlin's form) When d is odd, this story is not disjoint from the topological one. In this case, $d - 3$ is even and ϕ_d has a mod 2 reduction $\overline{\phi_d}$, which *coincides* with the 2-spin structure corresponding to Rochlin's quadratic form (see [Lib88] for details) and we have

$$\mathrm{Mod}(X_d)[\phi_d] \leq \mathcal{E}(\mathbb{C}P^2, X_d) \leq \mathrm{Mod}(X_d)[\overline{\phi_d}].$$

General overview. Consider $M = \mathbb{C}P^2$ or $S^2 \times S^2$, equipped with its standard handlebody decomposition. Recall that any homology class $x \in H_2(M; \mathbb{Z})$ admits a representative $\Sigma \subset M$ which intersects the 0-handle $B^4 \subset M$ in a Seifert surface $F_\Sigma = \Sigma \cap B^4 = \Sigma \cap \partial B^4$ where $\Sigma \backslash \mathrm{int}(F_\Sigma) = D_\Sigma$ is a collection of closed 2-disks (Subsection 4.2). Let $\theta_{F_\Sigma} : H_1(F_\Sigma; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ be the Seifert form and $i_\Sigma : F_\Sigma \to \Sigma$ inclusion. We begin by assuming that the surface Σ is characteristic.

Let $q_\Sigma : H_1(\Sigma; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ denote Rochlin's quadratic form. Then $i_\Sigma^* q_\Sigma = \theta_{F_\Sigma}$ (Proposition 2.9). Note that θ_{F_Σ} is always defined. In general, extending θ_{F_Σ} over $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ is a geometric witness for the failure of Σ to be characteristic. This is the origin of flexibility (we refer the reader to Figure 6 which constructs a flexible representative for $2[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2; \mathbb{Z})$).

According to Calderon-Salter [CS21, Proposition 5.1], if the genus of Σ is at least 3, the stabilizer

$$\mathrm{Mod}(\Sigma)[q_\Sigma] = \mathcal{T}_{q_\Sigma} := \langle T_\gamma \,|\, \gamma \subset \Sigma \text{ nonseparating simple closed curve}, q_\Sigma(\gamma) = 1 \rangle.$$

To prove Theorem A we construct a representative Σ for which $\mathcal{T}_{q_\Sigma} \leq \mathcal{E}(M, \Sigma)$. A consequence of this approach is that one must deal with *arbitrary* curves. This is done in two steps.

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[3]An r-spin structure on a surface Σ is a cohomology class ξ on the unit tangent bundle $\xi \in H^1(UT\Sigma; \mathbb{Z}/r\mathbb{Z})$ which assigns 1 to a positively oriented fiber.

[4]Trivialization of the tangent bundle.

First we establish a lower bound for $\mathcal{E}(M, \Sigma)$ which supports a *change-of-coordinates* principle[5]. It is the image of a *framed* mapping class group (Definition 3.3). Second, we stabilize with an unknotted surface in order to manufacture more extendable mapping classes.

Precisely, let $S = F_\Sigma \# \Sigma_g \subset S^3$, where $\Sigma_g \subset S^3$ is an unknotted surface of genus g. Capping S off with D_Σ yields a representative $Z = S \cup D_\Sigma$. We prove that for $g \geq 5$ and $\gamma \subset S$ a nonseparating simple closed curve with

$$\theta_S(\gamma) = \mathrm{lk}(\gamma, \gamma^+) = 1 \mod 2,$$

one has $T_\gamma \in \mathcal{E}(M, Z)$ (Lemma 4.6). As $Z \backslash S = D_\Sigma$ is a collection of disks and $i_Z^* q_Z = \theta_S$, it follows that

$$\mathcal{T}_{q_Z} \leq \mathcal{E}(M, Z).$$

Assume now that x is not a characteristic element. Lemma 4.6 still applies, needing only the pair (B^4, S). To establish flexibility, it is enough to prove that for any nonseparating simple closed curve $\delta \subset S$ with

$$\theta_S(\delta) = \mathrm{lk}(\delta, \delta^+) = 0 \mod 2$$

one has $T_\delta \in \mathcal{E}(M, Z)$. We show that isotoping δ across a disk in D_Σ yields a curve $\widetilde{\delta}$ on S with $\theta_S(\widetilde{\delta}) = 1 \mod 2$ (Proposition 4.8). By Lemma 4.6, $T_{\widetilde{\delta}} \in \mathcal{E}(M, Z)$. On Z, the curves δ and $\widetilde{\delta}$ are isotopic and so $T_\delta \in \mathcal{E}(M, Z)$. As $\mathrm{Mod}(Z)$ is generated by Dehn twists on nonseparating simple closed curves, Z is flexible.

Outline of the paper. In Section 2 we recall foundational material regarding extendable mapping classes. We do so in detail for we conclude the section by showing that the methods of Hirose established in [Hir02] prove spin-flexibility for properly embedded unknots in the 4-ball. In Section 3 we recall relevant background regarding *framed* and *spin* mapping class groups. In Section 4 we prove Theorem A. Starting with a family of framed Seifert surfaces for homogeneous braid closures studied in [Fer25], we use techniques from Section 3 to establish, after stabilizing with an unknot of sufficiently large genus, that the Dehn twist on any nonseparating simple closed curve with odd self-linking number is an extendable mapping class (Lemma 4.6). Thereafter we readily construct (spin) flexible representatives in $\mathbb{C}P^2$ and $S^2 \times S^2$. Flexibility for non-characteristic classes is a consequence of Proposition 4.8. In Section 5 we prove Theorem B. Methods from Section 4 are adapted to the case of a *single* stabilization, using the *geometric monodromy* associated to a positive braid identified in [Fer25].

Conventions and notation. We work in the smooth category throughout. All manifolds are tacitly oriented, and all submanifolds are embedded. For a simple closed curve γ on a surface Σ, denote the left handed Dehn twist along γ by $T_\gamma : \Sigma \to \Sigma$. We conflate curves on surfaces, as well as diffeomorphisms, with their associated isotopy classes. For a pair of curves $\alpha, \beta \subset \Sigma$, we denote their geometric intersection number by $i(\alpha, \beta)$. Given a manifold M and subspace $A \subset M$, let $\mathrm{Diff}_\partial(M, A)$ denote diffeomorphisms of M fixing A *setwise*, and restricting to the identity on ∂M. For a submanifold $S \subset M$, we denote the normal bundle of S in M by $\nu_{S \subset M}$, implicitly identified with an open neighborhood of S in M.

2. Extension Methods and Rochlin's Quadratic Form

We recall the principal techniques for extending mapping classes of surfaces in 4-manifolds and the obstructions that arise in doing so. We conclude by noting that properly embedded unknots in the 4-ball are spin-flexible rel boundary (Proposition 2.15).

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[5]Change-of-coordinates refers to a family of results allowing one to exchange configurations of curves on a surface through diffeomorphisms of the surface provided the configurations have the same 'combinatorics'.

Extendable subgroups. Let $(\Sigma^2, \partial\Sigma) \subset (M^4, \partial M)$ be a properly embedded oriented compact surface in a oriented 4-manifold M. Let $\mathrm{Mod}(M, \Sigma) = \pi_0(\mathrm{Diff}_\partial^+(M, \Sigma))$ be the smooth *relative mapping class group* of the pair (M, Σ). Restriction determines a homomorphism

$$\mathcal{R} : \mathrm{Mod}(M, \Sigma) \to \mathrm{Mod}^\pm(\Sigma) = \pi_0(\mathrm{Diff}_\partial(\Sigma, \partial\Sigma)).$$

Definition 2.1. Define $\mathcal{E}_\partial(M, \Sigma)$ to be the image of $\mathcal{R}$ in $\mathrm{Mod}(\Sigma) = \pi_0(\mathrm{Diff}_\partial^+(\Sigma, \partial\Sigma)) \leq \mathrm{Mod}^\pm(\Sigma)$.

Remark 2.2. Several calculations of $\mathcal{E}(M, \Sigma)$ in the literature ([Hir05], [Hir02]) construct extensions of mapping classes by isotopies of Σ in M. In particular they actually compute the *smooth monodromy group* of Σ in M. The same is done throughout this paper. In the presence of boundary all isotopies are rel boundary.

Remark 2.3. That the target of restriction $\mathcal{R} : \mathrm{Mod}(M, \Sigma) \to \mathrm{Mod}^\pm(\Sigma)$ is the *extended* mapping class group is a feature. For example, complex conjugation on $\mathbb{C}P^2$ yields an orientation reversing diffeomorphism on a smooth plane curve (defined by a homogeneous polynomial with real coefficients). Of course, complex conjugation is still an orientation preserving diffeomorphism of $\mathbb{C}P^2$. On the other hand, any isotopy of a surface in M must (after isotopy extension) preserve orientation and act trivially on the homology of M.

2.1. **Rochlin's quadratic form.** A *membrane* for the pair (M, Σ) is an embedded oriented surface $F \subset \mathrm{int}(M)$, with $\partial F \subset \Sigma$ a simple closed curve and interior transverse to Σ. If $H_1(M; \mathbb{Z}) = 0$ any curve on Σ has a membrane. In extracting obstructions to extending mapping classes, a guiding principle is that invariants associated to membranes must be preserved by $\mathcal{E}_\partial(M, \Sigma)$ (this avenue is pursued in [LNSW13], for example, by recording the minimum genus of amongst membranes along a simple closed curve).

Given a membrane F for the pair (M, Σ), the *relative Euler number* $e_\Sigma(F) \in \mathbb{Z}$ is defined as follows. Up to isotopy, we may assume that F meets Σ normally along ∂F. Let $\nu_F \to F$ denote the normal bundle of F in M. As F is homotopy equivalent to a wedge of circles and ν_F is an orientable 2-plane bundle, there exists a trivialization $\nu_F \cong F \times \mathbb{R}^2$. The image of a nowhere vanishing section of the normal bundle $\nu_{\partial F \subset \Sigma}$ sits inside ν_F and the trivialization then determines a smooth map $\partial F \to \mathbb{R}^2 - \{0\}$. The degree of this map is the relative Euler number $e_\Sigma(F) \in \mathbb{Z}$. We remind the reader that the relative Euler number itself is not an invariant of ∂F by the *spinning* construction, which changes $e_\Sigma(F)$ at the expense of creating intersections between $\mathrm{int}(F)$ and Σ (see [FK78]).

We record the presence of Rochlin's form in the *rel boundary* setting. Recall that a *quadratic form* $q : H_1(\Sigma; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ is a function satisfying

$$q(x + y) = q(x) + q(y) + \langle x, y \rangle$$

for all $x, y \in H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$. Here $\langle -, - \rangle$ denotes the mod 2 intersection pairing, arising from the homomorphism $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z}) \to H_1(\Sigma, \partial\Sigma; \mathbb{Z}/2\mathbb{Z})$ induced by inclusion and Poincaré duality. The mapping class group $\mathrm{Mod}(\Sigma)$ acts on the set of quadratic forms on $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ by pullback $(f \cdot q)(x) := q(f_*^{-1}(x))$.

Definition 2.4. Let $q : H_1(\Sigma; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ be a quadratic form. The stabilizer of q, denoted $\mathrm{Mod}(\Sigma)[q]$, is

$$\mathrm{Mod}(\Sigma)[q] = \{f \in \mathrm{Mod}(\Sigma) \,|\, (f \cdot q) = q\}.$$
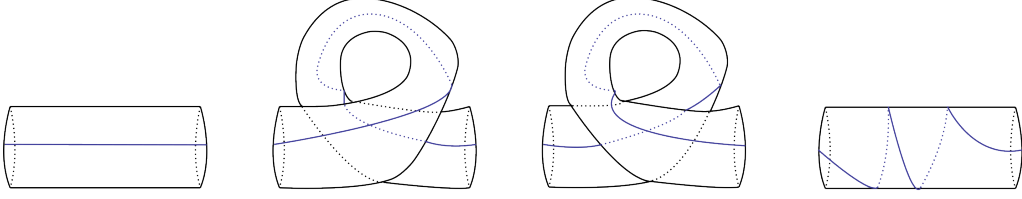
Proposition 2.5. *Suppose that $H_1(M; \mathbb{Z}) = 0$ and $\Sigma \subset M$ is characteristic, that is, $[\Sigma, \partial\Sigma] \in H_2(M, \partial M; \mathbb{Z}/2\mathbb{Z})$ is dual to $w_2(M) \in H^2(M; \mathbb{Z}/2\mathbb{Z})$ under Lefschetz duality. Then Rochlin's quadratic form $q_\Sigma : H_1(\Sigma; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ is defined and*

$$\mathcal{E}(M, \Sigma) \leq \mathrm{Mod}(\Sigma)[q_\Sigma].$$

Proof. We briefly recall some of the details, referring the reader to [Klu21, Lemma 2]. For a simple closed curve $\gamma \subset \Sigma$, pick a membrane F with $\partial F = \gamma$. Then

$$q_\Sigma(\gamma) = |\mathrm{int}(F) \cap \Sigma| + e_\Sigma(F) \mod 2.$$

The condition that Σ be characteristic in M ensures this is independent of the choice of membrane for γ. Transporting membranes with a diffeomorphism $(M, \Sigma) \to (M, \Sigma)$ shows that $\mathcal{E}(M, \Sigma) \leq \mathrm{Mod}(\Sigma)[q_\Sigma]$. $\square$

Figure 1. A well-known isotopy of a standard annulus in $S^3 \times I$ rel boundary inducing the square of a (right handed) twist along a core curve.

Definition 2.6. We say that $\Sigma \subset M$ is *flexible rel boundary* if $\mathcal{E}_\partial(M, \Sigma) = \text{Mod}(\Sigma)$. A characteristic surface $\Sigma \subset M$ is *spin flexible rel boundary* if $\mathcal{E}_\partial(M, \Sigma) = \text{Mod}(\Sigma)[q_\Sigma]$.

The Arf invariant. Provided the intersection pairing on $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ is nondegenerate, quadratic forms on $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ are in the same orbit under $\text{Mod}(\Sigma)$ if and only if they have the same *Arf invariant*, a $\mathbb{Z}/2\mathbb{Z}$-valued invariant of a quadratic form (see [RW14] for details). Quadratic forms with Arf invariant one (resp. zero) are called *odd* (resp. *even*). In the closed setting the Arf invariant captures ambient topology as follows.

Theorem 2.7 (Freedman-Kirby [FK78]). *Assume that M is closed, simply connected and that $\Sigma \subset M$ is characteristic. Then*

$$\text{Arf}(q_\Sigma) = \frac{1}{8}(\sigma(M) - \Sigma \cdot \Sigma) \mod 2,$$

where $\sigma(M)$ denotes the signature of M, and $\Sigma \cdot \Sigma$ the self-intersection of Σ in M.

Remark 2.8. Whilst we work exclusively in the smooth category, we remark that Rochlin's form can be defined in the topologically locally flat category (see [Klu21] for discussion on this matter).

Rochlin's form in $3\frac{1}{2}$ dimensions. We record a well known perspective on Rochlin's form for surfaces which interpolate between 3 and 4 dimensions. Recall that, for a Seifert surface $S \subset S^3 = \partial B^4$, there is the self-linking form $\theta_S : H_1(S; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$, defined on primitive elements by

$$\theta_S(\gamma) = \text{lk}(\gamma, \gamma^+) \mod 2$$

where γ^+ denotes a positive push-off of a curve $\gamma \subset S$, and lk denotes the linking number of the 1-cycles in S^3.

Proposition 2.9 (Page 117 of [Sav12]). *Let $\Sigma^2 \subset M^4$ be a characteristic surface, $H_1(M; \mathbb{Z}) = 0$ and $B^4 \subset M$ an embedded 4-ball. Assume that $\Sigma \cap B^4 = \Sigma \cap \partial B^4 = S$ a Seifert surface for a link. Let $i : S \to \Sigma$ be inclusion and θ_S the linking form on S. Then $i^* q_\Sigma = \theta_S$ where $q_\Sigma : H_1(\Sigma; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ denotes Rochlin's form.*
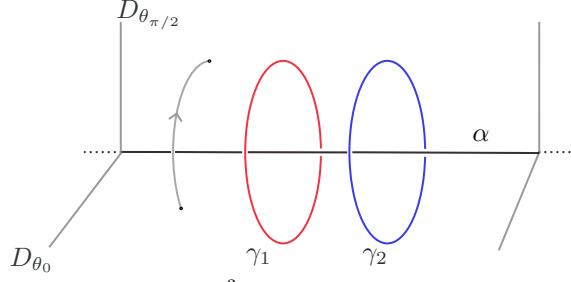
The above is a statement of equality on the nose. We record an instance in which the linking form and Rochlin's form, whilst not defined on the same surface, may still be identified.

Proposition 2.10. *For $S \subset S^3 = \partial B^4$ a Seifert surface, let S' denote the result of pushing the interior of S into the interior of B^4. The surface $(S', \partial S') \subset (B^4, S^3)$ is characteristic and Rochlin's form $q_{S'}$ is defined on S'. The isotopy used to push the surface into B^4 yields an identification $S' \cong S$, and the pullback of θ_S to S' coincides with Rochlin's form $q_{S'}$.*

Proof. The argument is identical to the proof of Proposition 2.9 in [Sav12, Page 117]. $\square$

2.2. **Extension methods.** We now turn to examples of elements in $\mathcal{E}_\partial(M, \Sigma)$ for pairs (M, Σ).

Tube isotopy. We begin with a quintessential and well-known isotopy of a surface rendering a non-trivial extendable mapping class ([Hir93]). Consider a standard annulus F in $S^3 \times \{1/2\} \subset S^3 \times [0, 1]$ with core curve c. Depicted in Figure 1 is an isotopy of F in $S^3 \times [0, 1]$ (rel boundary) which induces the square of a (right-handed) Dehn twist along c.

Figure 2. Depicted is an axis $\alpha \subset S^3$ for a 2-component unlink $\gamma_1 \cup \gamma_2$. The unlink is transverse to the pages of the open book decomposition $\pi : S^3 \backslash \alpha \to S^1$.

Hopf band curves and fibered subsurfaces. Let B^4 denote the closed unit ball in $\mathbb{R}^4$, $S^3 = \partial B^4$. Let A be an unknotted annulus in S^3 with one full positive or negative twist, so A is a positive or negative *Hopf band*. By pushing the interior of A into B^4, we obtain a properly embedded surface $A' \subset B^4$. Let a denote the core curve of the annulus A'. A key technical result, establish by Hirose in [Hir05, Proposition 2.1], is

Proposition 2.11. *We have $T_a \in \mathcal{E}_\partial(B^4, A')$. In fact, T_a is a smooth monodromy element for the pair (B^4, A'), that is, exists a diffeomorphism $T \in \mathrm{Diff}^+_\partial(B^4)$, smoothly isotopic to the identity rel boundary for which $T|_{A'} = T_a$.*

We record the prototypical application of Proposition 2.11. Let $S \subset S^3$ be a Seifert surface for a link.

Definition 2.12. We call a curve $\gamma \subset S$ a *Hopf band curve* if $\overline{\nu_{\gamma \subset S}}$ is a Hopf band in S^3. In particular, γ is an unknot in S^3 and, after orienting γ, $\mathrm{lk}_S(\gamma, \gamma^+) = \pm 1$ where γ^+ denotes a positive push-off of $\gamma \subset S$.

Push the interior of S into $\mathrm{int}(B^4)$ to obtain a properly embedded surface S'. Explicitly, using a collar $S^3 \times [0,1]$ about $\partial B^4 = S^3 \times \{0\}$, let

$$S' = \partial S \times [0, 1/2] \cup \mathrm{int}(S) \times \{1/2\}$$

and $\gamma \subset \mathrm{int}(S) \subset S^3 \times \{1/2\}$ be a simple closed curve. Applying Proposition 2.11 yields

Lemma 2.13. *Suppose that the annulus $A = \overline{\nu_{\gamma \subset S}}$ is a Hopf band in $S^3 \times \{1/2\}$. Let ψ be an isotopy of B^4 which pushes the interior of A futher into $\mathrm{int}(B^4)$. There exists a diffeomorphism $T : B^4 \to B^4$ for which $\psi^{-1} \circ T \circ \psi : B^4 \to B^4$ is the identity on the boundary $S^3 \times \{0\}$, $T(S') = S'$ and induces the Dehn twist along a core curve of the annulus A. In particular, $T_\gamma \in \mathcal{E}_\partial(B^4, S')$.*

A straightforward generalization of these methods shows that the monodromy of any *fibered subsurface* can be realized as an extendable mapping class.

Twists along on separating unlinks. In the sequel we make use of an argument in [Hir02, Lemma 2.2]. For the benefit of the reader we include a reformulation (see also [LS25] formulating this in the more general setting of arbitrary open book decompositions).
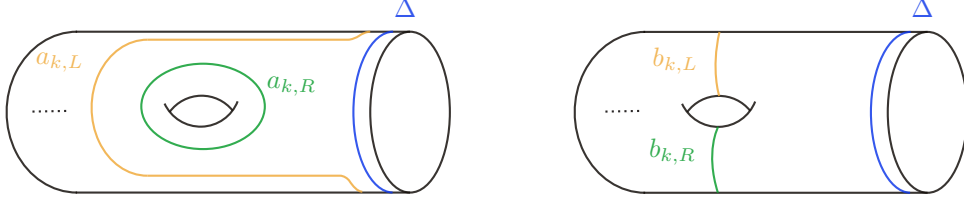
Proposition 2.14 (cf. Lemma 2.2 in [Hir02])**.** *If a surface $\Sigma \subset M^4$ intersects an $S^3 \times [0,1] \subset \mathrm{int}(M^4)$ in the cylinder on an n component unlink $\gamma_1, \ldots, \gamma_n \subset S^3$, the multitwist $T_{\gamma_1} \ldots T_{\gamma_n} \in \mathcal{E}_\partial(M, \Sigma)$.*

Proof. Let $\alpha \subset S^3$ be an axis for the unlink $\gamma_1 \cup \cdots \cup \gamma_n$ (see Figure 2). Consider the standard open book decomposition $\pi : S^3 \backslash \alpha \cong \mathrm{int}(D^2) \times S^1 \to S^1$. Pick n distinct points $\{x_1, \ldots, x_n\} \subset \mathrm{int}(D^2)$. We may assume the unlink $\gamma_1 \cup \cdots \cup \gamma_n$ is of the form $\{x_1, \ldots, x_n\} \times S^1$. By isotopy extension, the monodromy of π yields an isotopy

$$\varphi : S^3 \times [0,1] \to S^3$$

with $\varphi|_{S^3 \times \{0\}} = \varphi|_{S^3 \times \{1\}} = \mathrm{Id}_{S^3}$. The isotopy φ spins the points $\{x_1, \ldots, x_n\}$ around the axis α whilst fixing the unlink $\gamma_1 \cup \cdots \cup \gamma_n$ setwise. Define $\Phi : S^3 \times [0,1] \to S^3 \times [0,1]$ by $\Phi(x,t) = (\varphi(x,t), t)$. The map Φ induces a Dehn twist on each of the annuli $\gamma_1 \times [0,1] \cup \cdots \cup \gamma_n \times [0,1]$. Extending Φ by the identity over M shows $T_{\gamma_1} \cdots T_{\gamma_n} \in \mathcal{E}_\partial(M, \Sigma)$. $\square$

Figure 3. The union $a_{k,L} \cup a_{k,R} \cup \Delta$ forms a 3-component unlink in S^3, similarly for $b_{k,L} \cup b_{k,R} \cup \Delta$.

When applying Proposition 2.14, we suppress conjugation by an isotopy arranging the candidate surface so that it intersects an $S^3 \times [0,1]$ in the cylinder on an unlink.

Hirose [Hir02] proved that unknotted surfaces in S^4 are spin flexible (the genus one case is due to Montesinos [Mon83]). In more detail, Hirose constructed generating sets for the stabilizer of Rochlin's form (an *even* quadratic form), using the methods above to exhibit each generator as an extendable mapping class.

We conclude this section by observing that spin-flexibility for unknots extends in a straightforward way to the *rel boundary* setting. This result is used to lift mapping classes along an unknotted subsurface in Lemma 4.6.

Proposition 2.15. *For $g \geq 1$, let $\Sigma_g^1 \subset B^4$ denote an unknotted genus g Seifert surface for an unknot in $S^3 = \partial B^4$ with its interior pushed into the interior of B^4. Then $\mathcal{E}_\partial(B^4, \Sigma_g^1) = \mathrm{Mod}(\Sigma_g^1)[q_{\Sigma_g^1}]$.*

Proof. Cap off the boundary component Δ of Σ_g^1 with a closed embedded disk in S^3. Let Σ_g denote the resulting surface. Rochlin's form $q := q_{\Sigma_g^1}$ extends over Σ_g (inclusion induces an isomorphism $H_1(\Sigma_g^1; \mathbb{Z}/2\mathbb{Z}) \cong H_1(\Sigma_g; \mathbb{Z}/2\mathbb{Z})$). Therefore, by (the restriction of) *disk-pushing* [FM12], there is a short exact sequence

$$1 \to \pi_1(UT\Sigma_g) \to \mathrm{Mod}(\Sigma_g^1)[q] \to \mathrm{Mod}(\Sigma_g)[q] \to 1,$$

where $UT\Sigma_g \to \Sigma_g$ denotes the unit tangent bundle of Σ_g. Hirose [Hir02] constructs, for $g \geq 2$, a generating set for $\mathrm{Mod}(\Sigma_g)[q]$ consisting of multitwists supported in Σ_g^1. The methods in [Hir02] show each element has a lift to $\mathcal{E}_\partial(B^4, \Sigma_g^1)$. The same is true for $g = 1$ (see [Hir15]).

It now suffices to prove that

$$\pi_1(UT\Sigma_g) \leq \mathcal{E}_\partial(B^4, \Sigma_g^1).$$

If a loop $\widetilde{\gamma} \in \pi_1(UT\Sigma_g)$ has image a loop $\gamma \in \pi_1(\Sigma_g)$ under projection $UT\Sigma_g \to \Sigma_g$, the image of $\widetilde{\gamma}$ in $\mathrm{Mod}(\Sigma_g^1)$ is the multitwist $T_{\gamma_L} T_{\gamma_R}^{-1} T_\Delta^k$ for some $k \in \mathbb{Z}$, where γ_L (resp. γ_R) is obtained by pushing γ to the left (resp. right). The curves $\gamma_L \cup \gamma_R \cup \Delta$ bound a pair of pants on Σ_g^1.

Let $\{a_k, b_k\}_{k=1}^g$ denote the standard generating set for $\pi_1(\Sigma_g, *)$, $* \in \partial\Sigma_g^1$. In Figure 3, $a_{k,L} \cup a_{k,R} \cup \Delta$ and $b_{k,L} \cup b_{k,R} \cup \Delta$ are 3-component separating unlinks in an S^3 slice of a collar about the boundary. By Proposition 2.14, the multitwists along the boundary components of the pair of pants are in $\mathcal{E}_\partial(B^4, \Sigma_g^1)$.

By the tube isotopy (Figure 1), the square of each Dehn twist along a_k, b_k is in $\mathcal{E}_\partial(B^4, \Sigma_g^1)$. By Proposition 2.14, we have $T_\Delta \in \mathcal{E}_\partial(B^4, \Sigma_g^1)$. As disjoint twists commute, it follows that a generating set for $\pi_1(UT\Sigma_g)$ is contained in $\mathcal{E}_\partial(B^4, \Sigma_g^1)$. $\qquad\square$

3. Framed and Spin mapping class groups

3.1. Framings. Recall that a *framing* of a manifold is a trivialization of its tangent bundle. Let $\Sigma = \Sigma_g^b$ be a compact oriented surface of genus g with $b \geq 1$ boundary components. Fix a framing $\phi : T\Sigma \cong \Sigma \times \mathbb{R}^2$. Let $\mathcal{S}(\Sigma)$ denote the set of oriented isotopy classes of simple closed curves on Σ. The framing ϕ determines a *winding number function* (again denoted) $\phi : \mathcal{S}(\Sigma) \to \mathbb{Z}$ by

$$\phi(\gamma) = \deg(\phi \circ \overrightarrow{\gamma} : S^1 \to \mathbb{R}^2 - \{0\}) \in \mathbb{Z}.$$

Here $\overrightarrow{\gamma}$ is the forward pointing velocity vector field of γ. Winding number functions depend only on the isotopy class of framing. We abuse terminology, identifying winding number functions with their respective framings. As Σ is oriented, we frequently specify an isotopy class of framing by means of a nowhere vanishing vector field. Winding number functions satisfy the following basic properties as established by Humphries-Johnson in [HJ89].

Proposition 3.1. *For a framed surface (Σ, ϕ) the function $\phi : \mathcal{S}(\Sigma) \to \mathbb{Z}$ satisifes*

(a) *(Twist-linearity) For any $a, b \in \mathcal{S}(\Sigma)$,*

$$\phi(T_a(b)) = \phi(b) + \langle b, a \rangle \phi(a)$$

where $\langle -, - \rangle : H_1(\Sigma; \mathbb{Z}) \times H_1(\Sigma; \mathbb{Z}) \to \mathbb{Z}$ denotes the oriented intersection pairing.

(b) *(Homological coherence) Let $S \subset \Sigma$ be a subsurface with boundary components $c_1, \ldots, c_k$ oriented with S to their left. Then*

$$\sum_{i=1}^{k} \phi(c_k) = \chi(S)$$

where $\chi(S)$ denotes the Euler characteristic of S.

A framing of a surface may be extended across a stabilization of its boundary. Precisely, we have

Lemma 3.2 (Lemma 3.20 in [BS25]). *Let (Σ, ϕ) be a framed surface. Let Σ^+ be a surface obtained from Σ by attaching a 1-handle along $\partial \Sigma$. Fix $w \in \mathbb{Z}$. Let $c \subset \Sigma^+$ be a curve for which $c \cap \Sigma$ is a single arc. Then there is a unique isotopy class of framing ϕ^+ of Σ^+, extending ϕ, for which $\phi^+(c) = w \in \mathbb{Z}$.*

Framed mapping class groups. Let $\phi : \mathcal{S}(\Sigma) \to \mathbb{Z}$ be a framing of Σ. Let $\mathrm{Mod}(\Sigma)$ act on the set of (isotopy classes of) framings by pullback on the determined winding number functions. Precisely, we set $(f \cdot \phi)(x) = \phi(f^{-1}(x))$ for $x \in \mathcal{S}(\Sigma)$.

Definition 3.3. Let (Σ, ϕ) be a framed surface. The stabilizer of ϕ is denoted

$$\mathrm{Mod}(\Sigma)[\phi] = \{ f \in \mathrm{Mod}(\Sigma) \,|\, (f \cdot \phi) = \phi \}.$$

We call $\mathrm{Mod}(\Sigma)[\phi]$ a *framed mapping class group*.

Following Calderon-Salter [CS21], we list a variety of foundational results regarding the structural properties of framed mapping class groups to be used in the sequel.

Definition 3.4. Let (Σ, ϕ) be a framed surface. A nonseparating simple closed curve $\gamma \subset \Sigma$ is called *admissible* if $\phi(\gamma) = 0$.

By twist linearity, if $\gamma \subset \Sigma$ is admissible, $T_\gamma \in \mathrm{Mod}(\Sigma)[\phi]$. The *admissible subgroup* is defined as

$$\mathcal{T}_\phi := \langle T_\gamma \,|\, \gamma \text{ admissible} \rangle \leq \mathrm{Mod}(\Sigma)[\phi].$$

A foundational result is

Theorem 3.5 (Proposition 5.11 in [CS23]). *For a framed surface (Σ_g^b, ϕ), where $g \geq 5$ and $b \geq 1$, we have*

$$\mathrm{Mod}(\Sigma_g^b)[\phi] = \mathcal{T}_\phi.$$

We record the following instances of the *framed* change-of-coordinates principle (a corollary of [CS23, Proposition 2.15]).

Lemma 3.6. *Let (Σ, ϕ) be a framed surface. Let α and β be two nonseparating simple closed curves with $\phi(\alpha) = \phi(\beta)$. Then there exists $f \in \mathrm{Mod}(\Sigma)[\phi]$ for which $f(\alpha) = \beta$.*

Lemma 3.7. *Let (Σ, ϕ) be a framed surface. Let $\alpha \subset \Sigma$ be a nonseparating simple closed curve. Then there exists an admissible curve $\beta \subset \Sigma$ which is geometrically dual to α, that is $i(\alpha, \beta) = |\alpha \cap \beta| = 1$.*

A result that plays a fundamental role in this paper is a stabilization lemma for framed mapping class groups.

Lemma 3.8 (Lemma 5.13 in [CS23]). *Let (Σ, ϕ) be a framed surface. Let $S \subset \Sigma$ be a subsurface of genus at least 5 and $a \subset \Sigma$ an admissible curve such that $a \cap S$ is a single arc. Let S^+ be a regular neighborhood of $S \cup a$. Then*

$$\mathrm{Mod}(S^+)[\phi] = \langle \mathrm{Mod}(S)[\phi], T_a \rangle.$$

3.2. $\mathbb{Z}/r\mathbb{Z}$ **winding number functions.** Recall that $\mathcal{S}(\Sigma)$ denotes the set of oriented isotopy classes of simple closed curves on Σ.

Definition 3.9. A $\mathbb{Z}/r\mathbb{Z}$ *winding number function* ψ on Σ is a function

$$\psi : \mathcal{S}(\Sigma) \to \mathbb{Z}/r\mathbb{Z}$$

satisfying twist-linearity and homological coherence modulo r (cf. Proposition 3.1).

Remark 3.10. As a consequence of homological coherence, winding number functions satisfy *reversibility*. For $c \in \mathcal{S}(\Sigma)$, let $\bar{c} \in \mathcal{S}(\Sigma)$ denote c with the opposite orientation. Then $\psi(\bar{c}) = -\psi(c) \in \mathbb{Z}/r\mathbb{Z}$.

Example 3.11. A $\mathbb{Z}/r\mathbb{Z}$ winding number function may be constructed as follows [CS23, Definition 2.6]. Consider a framed surface (Σ, ϕ). Let $\overline{\Sigma}$ denote the surface obtained by capping off boundary components $\Delta_1, \ldots, \Delta_n$ of Σ with closed disks (note $\overline{\Sigma}$ may still have boundary). Orient each component Δ_k so that Σ is to the left. Isotoping a curve on Σ across the disk bounding Δ_k changes the ϕ-framing by $\phi(\Delta_k) + 1$ (homological coherence). Let $r = \gcd(\phi(\Delta_1) + 1, \ldots, \phi(\Delta_n) + 1)$ and consider $\overline{\phi} : \mathcal{S}(\overline{\Sigma}) \to \mathbb{Z}/r\mathbb{Z}$ defined by

$$\overline{\phi}(\gamma) = \phi(\gamma) \mod r.$$

We say the winding number function $\overline{\phi}$ is obtained by *reducing* ϕ mod r.

Definition 3.12. A nonseparating simple closed curve $\gamma \subset \Sigma$ is *admissible* under $\psi : \mathcal{S}(\Sigma) \to \mathbb{Z}/r\mathbb{Z}$ if $\psi(\gamma) = 0 \in \mathbb{Z}/r\mathbb{Z}$.

As in the framed settting, we define the admissible subgroup

$$\mathcal{T}_\psi = \langle T_\gamma \,|\, \gamma \subset \Sigma \ \psi-\text{admissible} \rangle.$$

The set of $\mathbb{Z}/r\mathbb{Z}$ winding number functions carries an action of the mapping class group via pullback.

Definition 3.13. The stabilizer of a $\mathbb{Z}/r\mathbb{Z}$ winding number function ψ on Σ is denoted

$$\mathrm{Mod}(\Sigma)[\psi] = \{ f \in \mathrm{Mod}(\Sigma) \,|\, (f \cdot \psi) = \psi \}.$$

Remark 3.14. The stabilizer of a $\mathbb{Z}/r\mathbb{Z}$ winding number function is called an *r-spin mapping class group*. The terminology is motivated by a bijective correspondence between $\mathbb{Z}/r\mathbb{Z}$ winding number functions and *r-spin structures* on Σ [BS25, Proposition 3.6]. Let $UT\Sigma \to \Sigma$ be the unit tangent bundle of Σ.[6] An *r-spin structure* on Σ is a cohomology class in $H^1(UT\Sigma; \mathbb{Z}/r\mathbb{Z})$ which assigns the value $1 \in \mathbb{Z}/r\mathbb{Z}$ to the class of a positively oriented fiber.

For closed surfaces we have an analogue of Theorem 3.5.

Theorem 3.15 (Proposition 5.1 in [CS21]). *Assume that Σ is closed and has genus $g(\Sigma) \geq 3$. Let ψ be a $\mathbb{Z}/r\mathbb{Z}$ winding number function on Σ. Then*

$$\mathrm{Mod}(\Sigma)[\psi] = \mathcal{T}_\psi.$$

Quadratic forms and $\mathbb{Z}/2\mathbb{Z}$ winding number functions. In the sequel we require passage between quadratic forms and winding number functions on surfaces with boundary. Following [Joh80] and Section 2.3 of [RW14], a calculation gives

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[6]Relative to an auxillary Riemannian metric.

Proposition 3.16. *A winding number function* $\psi : \mathcal{S}(\Sigma) \to \mathbb{Z}/2\mathbb{Z}$ *defines a quadratic form* $q_\psi : H_1(\Sigma; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ *by the requirement that*

$$q_\psi(\gamma) = \psi(\gamma) + 1 \mod 2$$

for $\gamma \in \mathcal{S}(\Sigma)$. *Moreover, quadratic forms on* $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$ *are determined by their values on a generating set for* $H_1(\Sigma; \mathbb{Z}/2\mathbb{Z})$.

Remark 3.17. In the presence of a quadratic form $q : H_1(\Sigma; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$, a nonseparating simple closed curve $\gamma \subset \Sigma$ is called *admissible* if $q(\gamma) = 1 \mod 2$.

4. (Spin) Flexible Surfaces in $\mathbb{C}P^2$ and $S^2 \times S^2$

We begin by considering a family of framed Seifert surfaces, following [Fer25]. After stabilizing with an unknotted surface of sufficiently large genus, we use the inductive techniques recalled in Section 3 to prove Lemma 4.6. As a consequence we obtain Theorem A.

Seifert surfaces for homogeneous braid closures. Recall that a braid β on n strands can be expressed as a word in n generators $\sigma_1, \ldots, \sigma_{n-1}$, where the letter σ_i consists of a single over-crossing between the $(i+1)$-st and i-th strands.

Definition 4.1 ([Sta78]). A word $\beta = \sigma_{\iota_1}^{\varepsilon_1} \cdots \sigma_{\iota_k}^{\varepsilon_k}$, $\varepsilon_i = \pm 1$, is *homogeneous* if every letter σ_i occurs at least once, and the exponents of all occurences of a given letter σ_i are the same.

Following Stallings [Sta78], a Seifert surface F_β is constructed for the closure of β using a stack of n disks, joining adjacent disks with half-twisted bands. See Figure 4. We prefer to use half curls in our bands. Our conventions are that positive letters correspond to down rightward curls and negative letters correspond to down leftward curls.

Remark 4.2. For a homogeneous braid word β, F_β is a *fibered surface* in S^3. In fact, F_β is obtained by *plumbing* a sequence of positive/negative Hopf bands (see [Sta78] for details).

Any adjacent pair of bands attached to the same pair of disks determines a simple closed curve on F_β by connecting the cores of the bands with simple arcs in the disks. We write $\mathcal{C}_{F_\beta}$ for the finite collection of all such (isotopy classes of) simple closed curves. See Figure 4. We call $\mathcal{C}_{F_\beta}$ the *braid word assemblage* on F_β.

Definition 4.3 ([Fer25]). Define

$$\mathcal{T}_{\mathcal{C}_{F_\beta}} = \langle T_\gamma \mid \gamma \in \mathcal{C}_{F_\beta} \rangle.$$

We call $\mathcal{T}_{\mathcal{C}_{F_\beta}}$ the *geometric monodromy group* of the braid word β.

Orient curves in $\mathcal{C}_{F_\beta}$ as in Figure 4. As β is homogeneous, each curve in $\mathcal{C}_{F_\beta}$ is a Hopf band curve (Definition 2.12). By construction, the surface F_β deformation retracts onto the union of the curves in $\mathcal{C}_{F_\beta}$.

Framing the surface F_β. Following [Fer25], we frame the surface $F_\beta \subset S^3$ through the nowhere vanishing vector field depicted in Figure 4. Let
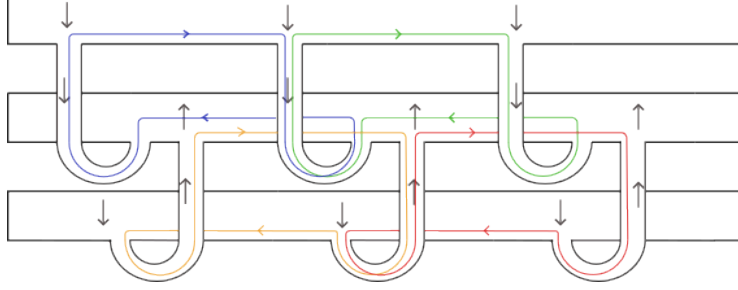
$$\phi_\beta : \mathcal{S}(F_\beta) \to \mathbb{Z}$$

be the associated winding number function on the set of oriented isotopy classes of simple closed curves on F_β. The constructed vector field is parallel to the core of the 1-handles and so contribution to the winding number ϕ_β takes place only in the 0-handles. Consider an oriented simple closed curve $\gamma \subset F_\beta$. Let a_R^γ (resp. a_L^γ) denote the number of 'rightward' (resp. 'leftward') arcs of γ in the 0-handles of F_β. We compute

$$\phi_\beta(\gamma) = \frac{1}{2}(a_L^\gamma - a_R^\gamma).$$

Any curve in $\mathcal{C}_{F_\beta}$ is therefore admissible under the framing and there is the sequence

$$\mathcal{T}_{\mathcal{C}_{F_\beta}} \leq \mathcal{T}_{\phi_\beta} \leq \mathrm{Mod}(F_\beta)[\phi_\beta].$$

Figure 4. The framed Seifert surface F_β associated to the word $\beta = (\sigma_1\sigma_2^{-1})^3$ together with its braid word assemblage of Hopf band curves.

Ferretti proves [Fer25, Theorem 2] that for a sufficiently general collection of positive braids[7] β whose closure is a knot, the above is a sequence of equalities, namely

$$\mathcal{T}_{\mathcal{C}_{F_\beta}} = \mathrm{Mod}(F_\beta)[\phi_\beta].$$

Corollary 4.4. *Let F_β' denote the result of pushing the interior of F_β into the interior of B^4. Pull back the framing ϕ_β to a framing ϕ_β' of F_β'. Then, for a braid β to which Theorem 2 of [Fer25] applies, we have*

$$\mathrm{Mod}(F_\beta')[\phi_\beta'] \le \mathcal{E}_\partial(B^4, F_\beta').$$

We use Corollary 4.4 in proving Theorem B.

4.1. **An integral lift of the linking form on a stabilization of F_β.** Let $\Sigma_g \subset S^3 \backslash F_\beta$ be the boundary of a genus g handlebody. Consider a standard internal connect sum of F_β with Σ_g along a smoothly embedded disk $D_0 \subset F_\beta$. We assume the disk D_0 sits inside a 0-handle of F_β which has at least a pair of bands attached along its boundary (if no such 0-handle exists the surface F_β is a disk). Denote the resulting surface

$$S_{\beta,g} = F_\beta \# \Sigma_g \subset S^3.$$

Let $\widehat{F}_\beta = F_\beta - \mathrm{int}(D_0) \subset S_{\beta,g}$ and $\widehat{\Sigma}_g = \Sigma_g - \mathrm{int}(D_0) \subset S_{\beta,g}$. The framing ϕ_β of F_β restricts to the subsurface $\widehat{F}_\beta$, denoted again by ϕ_β. This framing does not extend over the surface $S_{\beta,g}$. Indeed, winding number functions satisfy homological coherence. On F_β the curve ∂D_0 bounds a disk, whilst on $S_{\beta,g}$ it bounds a surface of genus g.

We circumvent this obstruction by removing a small disk D_1 in the interior of the unknotted piece $\widehat{\Sigma}_g$. Write

$$P_{\beta,g} = S_{\beta,g} - \mathrm{int}(D_1).$$

The framing ϕ_β now extends, by Lemma 3.2, to a framing ϕ_β^+ of $P_{\beta,g}$ using the chain of curves $\mathcal{U}_g = \{c_1, \ldots, c_{2g-1}\} \subset \widehat{\Sigma}_g$ and the curve $\mu \subset S_{\beta,g}$ depicted in Figure 5, setting $\phi_\beta^+(\mu) = 0$ and $\phi_\beta^+(c_k) = 1$ for each $k \in \{1, \ldots, 2g-1\}$. The curve μ is a homologically the sum of a curve $\gamma \subset \mathcal{C}_{F_\beta}$ and a meridian in $\widehat{\Sigma}_g$. The essential property of this extension is that the resulting framing is an integral lift of the linking form on the Seifert surface $S_{\beta,g}$. We now consider the winding number function induced by the framing ϕ_β^+ on the surface $S_{\beta,g}$ (as in Example 3.11).

Henceforth suppress notational dependency on β and g, writing $S = S_{\beta,g}$, $P = P_{\beta,g}$ and $F = F_\beta$. The framing ϕ^+ induces a winding number function
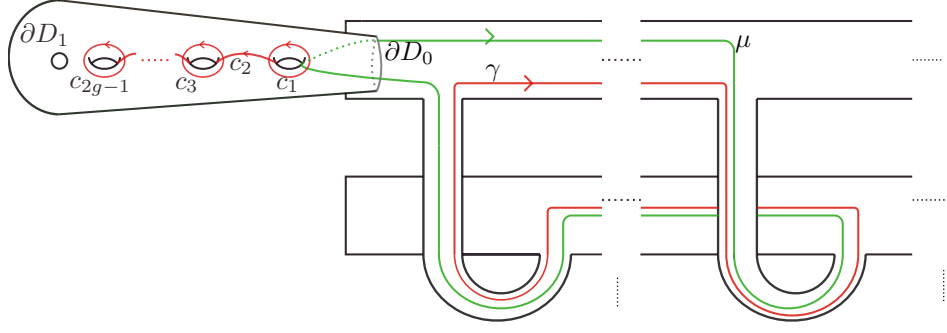
$$\psi : \mathcal{S}(S) \to \mathbb{Z}/2g\mathbb{Z}$$

on S. To see this, orient the curves ∂D_0 and ∂D_1 so that $\widehat{\Sigma}_g$ is to the left. By homological coherence,

$$\phi^+(\partial D_1) + \phi^+(\partial D_0) = 2 - 2g - 2 = -2g$$

and so $\phi^+(\partial D_1) + 1 = -2g$. Let $\overline{\psi} : \mathcal{S}(S) \to \mathbb{Z}/2\mathbb{Z}$ denote the mod 2 reduction of ψ; in particular, $\overline{\psi} = \phi^+ \mod 2$ for curves in P. Recall that $S = F \# \Sigma_g \subset S^3$ is a Seifert surface for the link ∂F,

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[7]All but finitely many prime positive braids not of type A_n.

Figure 5. We stabilize F_β along the chain of curves $\{\mu, c_1, \ldots, c_{2g-1}\}$.

$\theta_S : H_1(S; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ denotes the mod 2 linking form on S, and that $\overline{\psi} + 1 : H_1(S; \mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ defines a quadratic form (Proposition 3.16).

Proposition 4.5. *The quadratic form $\overline{\psi} + 1$ agrees with the linking form θ_S on $H_1(S; \mathbb{Z}/2\mathbb{Z})$. Precisely*

$$\overline{\psi}(\gamma) + 1 = \theta_S(\gamma) \mod 2$$

for a simple closed curve $\gamma \subset S$.

Proof. By Proposition 3.16, it suffices to prove that the quadratic forms $\overline{\psi} + 1$ and θ_S agree on a generating set for $H_1(S; \mathbb{Z}/2\mathbb{Z})$. The collection $\mathcal{C}_F \cup \{\mu\} \cup \mathcal{U}_g$ is a basis for $H_1(S; \mathbb{Z}/2\mathbb{Z})$. Each curve in $\mathcal{C}_F$ is a Hopf band curve and has ϕ^+-framing zero. The curve μ is a Hopf band curve on the Seifert surface S and has ϕ^+-framing zero. Each curve in $\mathcal{U}_g$ has self-linking number zero and ϕ^+-framing one. $\square$

Extending admissible twists rel boundary. Let $S' \subset B^4$ be obtained by pushing the interior of S into the interior of B^4. For lifting diffeomorphisms along subsurfaces and extending by the identity, an explicit model is convenient. Let $S^3 \times [0, 1]$ be a collar about $\partial B^4 = S^3 \times \{0\}$, and set

$$S' = \partial F \times [0, 1/4] \cup \widehat{F} \times \{1/4\} \cup \partial D_0 \times [1/4, 1/2] \cup \widehat{\Sigma}_g \times \{1/2\}.$$

We suppress identification between S and S' and henceforth pull back all data on S to the surface S'. A subsurface N of S pulled back to S' will be denoted N', the same rule applying to curves and configurations thereof. Recall that the linking form on $H_1(S; \mathbb{Z}/2\mathbb{Z})$ pulls back to Rochlin's form $q_{S'} : H_1(S'; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ (Proposition 2.10). We now exploit the unknotted piece of the surface S', in conjunction with the inductive framework behind the theory of framed mapping class groups, to manufacture more extendable mapping classes in $\mathrm{Mod}(S')$.

Lemma 4.6. *Fix $g \geq 5$ and a homogeneous braid word β. Let $\gamma \subset S'_{\beta,g}$ be a nonseparating simple closed curve with $q_{S'_{\beta,g}}(\gamma) = 1$. Then*

$$T_\gamma \in \mathcal{E}_\partial(B^4, S'_{\beta,g}).$$

Proof. Recall that we suppress subscripts indicating dependency on β and g. By an isotopy if necessary, we may assume that $\gamma \subset P'$. By Propositions 4.5 and 2.10,

$$\phi^+ = q_{S'} + 1 \mod 2$$

on P'. After orienting γ, $\phi^+(\gamma) = 2k$ for some $k \in \mathbb{Z}$. Let $p_{D'_1} : \mathrm{Mod}(P') \to \mathrm{Mod}(S')$ be the homomorphism induced by extension by identity over the closed 2-disk D'_1. We claim that

$$p_{D'_1}(\mathrm{Mod}(P')[\phi^+]) \leq \mathcal{E}_\partial(B^4, S').$$

Let $V' = P' \cap \widehat{\Sigma}_g \times \{1/2\} \subset S^3 \times \{1/2\}$. The surface V' is our genus g unknot with two standard boundary components. The surface P' is obtained by stabilizing V' along the ϕ^+-admissible configuration $\mathcal{Z}' = \{\mu'\} \cup \mathcal{C}'_F \backslash \{\gamma'\}$. As the genus of V' is at least 5, it follows from Lemma 3.8 that

$$\mathrm{Mod}(P')[\phi^+] = \langle \mathrm{Mod}(V')[\phi^+], T_c : c \in \mathcal{Z}' \rangle.$$

Each curve $c \in \mathcal{Z}'$ is (after an isotopy of S' rel $\partial S'$) a Hopf band curve in an S^3 slice of the collar and so $T_c \in \mathcal{E}_\partial(B^4, S')$. By Theorem 3.5, $\mathrm{Mod}(V')[\phi^+]$ is generated by Dehn twists on admissible curves. Fix an admissible curve $\zeta \subset V'$. By Proposition 4.5, it follows that

$$q_{S'}(\zeta) = \mathrm{lk}(\zeta, \zeta^+) = 1 \mod 2,$$

the linking number computed in $S^3 \times \{1/2\}$. By Proposition 2.15, $T_\zeta \in \mathcal{E}_\partial(B^4, S')$.

Fix a nonseparating simple closed curve $\delta \subset \widehat{\Sigma}_g \times \{1/2\} \subset S'$ with $\phi^+(\delta) = 2k$. By the framed change of coordinates principle (Proposition 5.4), there exists $f \in \mathrm{Mod}(P')[\phi^+]$ such that $f(\gamma) = \delta$. Then

$$T_\delta = f T_\gamma f^{-1},$$

and by spin-flexibility of $\widehat{\Sigma}_g \times \{1/2\}$ rel boundary (Proposition 2.15), we have $T_\delta \in \mathcal{E}_\partial(B^4, S')$ and $T_\gamma \in \mathcal{E}_\partial(B^4, S')$. $\qquad \square$

We now construct (spin) flexible representatives in $\mathbb{C}P^2$ and $S^2 \times S^2$.

4.2. **Surfaces in $\mathbb{C}P^2$ and $S^2 \times S^2$.** Work with the standard handlebody descriptions for the manifolds $\mathbb{C}P^2$ and $S^2 \times S^2$ - after removing a 4-ball in each, $\mathbb{C}P^2$ is obtained by attaching a 2-handle along a $+1$-framed unknot in ∂B^4, and $S^2 \times S^2$ by attaching a pair of 2-handles along a 0-framed Hopf link in ∂B^4.

Let $\mathbb{C}P^1 \subset \mathbb{C}P^2$ be a line. Recall that a representative for the class $d[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2; \mathbb{Z})$ may be constructed as follows (see [GS99] for a thorough treatment). Let $B^4 \subset \mathbb{C}P^2$ be the 0-handle. In ∂B^4 take d parallel copies of the attaching circle for the 2-handle, an oriented (d, d) torus link, which we represent as the closure of the braid $\beta_d = (\sigma_1 \cdots \sigma_{d-1})^d \in B_d$. We cap off the pushed in Seifert surface $S'_{\beta_d, g}$ for β_d constructed in Subsection 4.1 with d disjoint 2-disks, which are parallel copies of the core of the 2-handle. This yields a closed oriented surface $Z_d = Z_{d,g} \subset \mathbb{C}P^2$ with $[Z_d] = d[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2; \mathbb{Z})$ and genus $g(Z_d) = \frac{1}{2}(|d| - 1)(|d| - 2) + g$.

Fix $x \in S^2$. Consider

$$(n, m) = n\left[\{x\} \times S^2\right] + m\left[S^2 \times \{x\}\right] \in H_2(S^2 \times S^2; \mathbb{Z}),$$
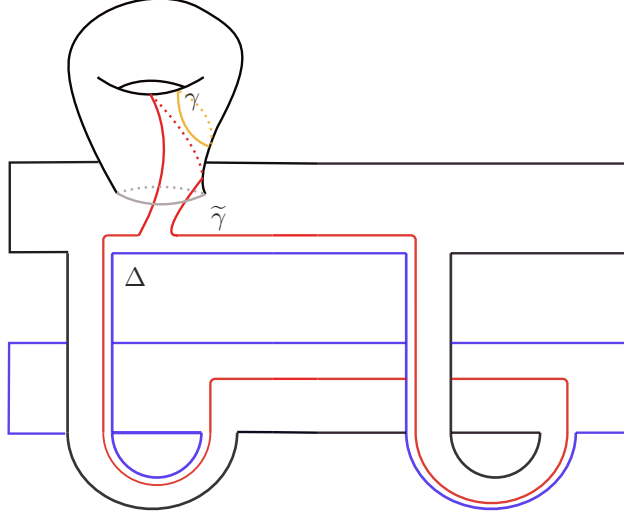
the class of n vertical and m horizontal spheres. We call a class (n, m) *positive* if both coefficients are strictly positive, and *negative* if one of n or m is strictly negative. Note that, if either of n or m is zero, the class admits a genus zero representative, obtained by tubing together a disjoint collection of 2-spheres.

We start by constructing representatives for positive classes. Fix $n, m > 0$. In the boundary of the 0-handle, take n parallel copies of one (vertical) attaching circle and m parallel copies of the other (horizontal), the closure of a positive braid $\beta_{n,m}$ on $n + m$ strands. Let $Z_{n,m,g} = Z_{n,m} \subset S^2 \times S^2$ be the closed surface obtained by capping off $S'_{\beta_{n,m},g}$ (Subsection 4.1) with parallel copies of the cores of the 2-handles. The surface $S'_{\beta_{n,m},g}$ has genus $(n-1)(m-1)+g$ and $[Z_{n,m}] = (n, m) \in H_2(S^2 \times S^2; \mathbb{Z})$. For the negative classes, we apply diffeomorphisms of $S^2 \times S^2$ that reverse orientation in the relevant factors of $S^2 \times S^2$ to the surface $Z_{n,m}$. Our convention shall be to drop the subscript g.

Remark 4.7. By construction, the surfaces representing negative classes in $H_2(S^2 \times S^2; \mathbb{Z})$ are of the form $\varphi(Z_{n,m})$ for a diffeomorphism $\varphi : S^2 \times S^2 \to S^2 \times S^2$ and $n, m > 0$. In particular, if $Z_{n,m}$ is (spin) flexible, so too is $\varphi(Z_{n,m})$.

Flexibility. We isolate the origin of flexibility, referring the reader to Figure 6 for an immediate explanation. Recall that a link in S^3 is *proper* if the linking number of any component with the remaining components is even. If $S \subset S^3$ is a Seifert surface, recall that $\theta_S : H_1(S; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ denotes the mod two linking form on S.

Proposition 4.8. *Let $\Sigma \subset M^4$ be a closed surface in a 4-manifold M. Assume that there exists a 4-ball $B^4 \subset M$ for which $\Sigma \cap B^4 = \Sigma \cap \partial B^4 = S$, a Seifert surface for a non-proper link in S^3. Assume that $\Sigma \backslash int(S)$ is a collection of closed 2-disks. Then there is a component Δ of ∂S such*

Figure 6. Consider $S_{\beta_2,1} \subset S^3$, the connect sum of a Hopf band and unknotted torus, a subsurface of $Z_2 \subset \mathbb{C}P^2$. By construction, the boundary components of the Hopf band bound disks in the 2-handle. Sliding γ across one of these disks yields a curve $\widetilde{\gamma}$ on $S_{\beta_2,1}$, where $\mathrm{lk}(\gamma, \gamma^+) = 0$ and $\mathrm{lk}(\widetilde{\gamma}, \widetilde{\gamma}^+) = 1 \mod 2$ (in fact $\widetilde{\gamma}$ is a Hopf band curve). Sliding a standard pair of dual curves on the torus across disks in the 2-handle yields a pair of dual Hopf band curves. Applying Proposition 2.11 yields flexibility of Z_2.

that isotoping a simple closed curve $\gamma \subset S$ across the disk that Δ bounds yields a simple closed curve $\widetilde{\gamma} \subset S$ for which

$$\theta_S(\widetilde{\gamma}) = \theta_S(\gamma) + 1 \mod 2.$$

Proof. By assumption, ∂S is not proper. Let Δ denote a component of ∂S whose linking number with the remaining components $\partial S - \Delta$ is odd. Then

$$0 = \mathrm{lk}(\partial S, \Delta^+) = \mathrm{lk}(\partial S - \Delta, \Delta^+) + \mathrm{lk}(\Delta, \Delta^+) = \mathrm{lk}(\partial S - \Delta, \Delta) + \mathrm{lk}(\Delta, \Delta^+).$$

Therefore $\theta_S(\Delta) = \mathrm{lk}(\Delta, \Delta^+)$ is odd. By construction, the curves $\gamma \cup \widetilde{\gamma} \cup \Delta$ bound a pair of pants on S and so

$$\theta_S(\gamma) + \theta_S(\widetilde{\gamma}) + \theta_S(\Delta) = 0 \mod 2.$$

Therefore $\theta_S(\widetilde{\gamma}) = \theta_S(\gamma) + 1 \mod 2$. $\qquad\square$

We are now in a position to prove

Theorem A. *Let $M = \mathbb{C}P^2$ or $S^2 \times S^2$. Fix $x \in H_2(M; \mathbb{Z})$. If x is characteristic, then x admits a spin-flexible representative, otherwise x admits a flexible representative.*

Proof. First, if $M = S^2 \times S^2$, it suffices to work with positive classes (Remark 4.7). Given $x \in H_2(M; \mathbb{Z})$, let $Z_x \subset M^4$ be a representative for x constructed in Section 4.2 with $Z_x \cap B^4 = S'_{\beta_x,g}$ and $g \geq 5$. We start by assuming that x is characteristic. Then $\mathcal{E}(M^4, Z_x) \leq \mathrm{Mod}(Z_x)[q_{Z_x}]$ where $q_{Z_x} : H_1(Z_x; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ is Rochlin's quadratic form. Recall that a simple closed curve $\gamma \subset Z_x$ is admissible if it is nonseparating and $q_{Z_x}(\gamma) = 1$ (Remark 3.17). By Theorem 3.15,

$$\mathrm{Mod}(Z_x)[q_{Z_x}] = \langle T_\gamma \,|\, \gamma \subset Z_x \text{ admissible} \rangle.$$

Fix an admissible curve $\gamma \subset Z_x$. By an isotopy of γ if necessary, we may assume that $\gamma \subset S'_{\beta_x,g}$. Then

$$q_{Z_x}(\gamma) = q_{S'_{\beta_x,g}}(\gamma) = 1.$$

By Lemma 4.6, $T_\gamma \in \mathcal{E}_\partial(B^4, S'_{\beta_x,g})$. Therefore $T_\gamma \in \mathcal{E}(M, Z_x)$ and $\mathcal{E}(M, Z_x) = \mathrm{Mod}(Z_x)[q_{Z_x}]$.

Assume now that x is not a characteristic element. To prove that $\mathcal{E}(M, Z_x) = \mathrm{Mod}(Z_x)$, it suffices, in light of Lemma 4.6, to prove that if $\gamma \subset S'_{\beta_x,g}$ is a nonseparating simple closed curve with $q_{S'_{\beta_x,g}}(\gamma) = 0$, then $T_\gamma \in \mathcal{E}(M, Z_x)$. By inspection, the link $\partial S'_{\beta_x,g}$ is not proper. By Proposition

4.8, there exists a boundary component Δ of $S'_{\beta_x,g}$ for which isotoping γ across the disk that Δ bounds yields a curve $\widetilde{\gamma} \subset S'_{\beta_x,g}$ with

$$q_{S'_{\beta_x,g}}(\widetilde{\gamma}) = 1.$$

By Lemma 4.6, $T_{\widetilde{\gamma}} \in \mathcal{E}_\partial(B^4, S'_{\beta_x,g})$. As γ and $\widetilde{\gamma}$ are isotopic on Z_x, $T_\gamma = T_{\widetilde{\gamma}} \in \mathcal{E}(M, Z_x)$. As $\mathrm{Mod}(Z_x)$ is generated by Dehn twists on nonseparating simple closed curves, Z_x is a flexible surface. $\qquad\square$

5. A SINGLE STABILIZATION FOR SMOOTH PLANE CURVES

Let $X_d \subset \mathbb{C}P^2$ be a smooth plane curve of degree d. Recall that any two such curves are smoothly isotopic in $\mathbb{C}P^2$. Again, we consider the standard handle decomposition of $\mathbb{C}P^2$ with 0-handle $B^4 \subset \mathbb{C}P^2$. Following [AK80] and [GS99], up to isotopy, we may assume $X_d \cap B^4 = X_d \cap \partial B^4 = F_{\beta_d}$ where $\beta_d = (\sigma_1 \cdots \sigma_{d-1})^d$, and F_{β_d} is the Seifert surface for the closure of β_d constructed in Section 4. Furthermore, $X_d = F_{\beta_d} \cup D$ where D is a disjoint collection of 2-disks (parallel copies of the core of the 2-handle).

Given this description of smooth plane curves, the methods used to establish Theorem A prove that stabilizing X_d with an unknot in S^4 of genus at least 5 yields a (spin) flexible surface. We now show that in fact a single stabilization suffices, starting with a general argument for curves of degree $d \geq 5$ and a sequence of ad hoc arguments dealing with $d \leq 4$. We continue to use notation and conventions established in Subsection 4.1.

Let $Z_d = X_d \# T \subset \mathbb{C}P^2 \# S^4$ where $T \subset S^4$ is an unknotted torus. Up to isotopy, we may assume $Z_d \cap \partial B^4 = S_{\beta_d,1}$ where $S_{\beta_d,1} = F_{\beta_d} \# T$. Recall that we have equipped F_{β_d} with a framing $\phi_{\beta_d} : S(F_{\beta_d}) \to \mathbb{Z}$, which induces a framing on $\widehat{F}_{\beta_d} = F_{\beta_d} - \mathrm{int}(D_0)$, where D_0 is the surgery disk for the connect sum $F_{\beta_d} \# T$.

Remark 5.1. In clarifying remarks made from the introduction, we note that there is a winding number function $\phi_d : \mathcal{S}(X_d) \to \mathbb{Z}/(d-3)$ arising from a $(d-3)^{rd}$ root of the adjoint line bundle of X in $\mathbb{C}P^2$ (we refer the reader to [Sal19] for more details). The framing ϕ_{β_d} induces a $(d-3)$-spin structure on X_d which *coincides* with ϕ_d. In this case, $\mathcal{C}_{F_{\beta_d}}$ consists of *vanishing cycles* for nodal degenerations of the plane curve X_d. We do not need these facts and so we omit details.

Recall that $P_{\beta_d,1} = S_{\beta_d,1} - \mathrm{int}(D_1)$ where D_1 is a closed 2-disk supported in the torus T. We extend the framing ϕ_{β_d} to a framing $\phi^+_{\beta_d}$ of $P_{\beta_d,1}$ by declaring the curves $\delta_1, \delta_2 \subset P$ depicted in Figure 7 admissible (Lemma 3.2). By construction, this extension is an integral lift of the linking form on the Seifert surface $S_{\beta_d,1}$.
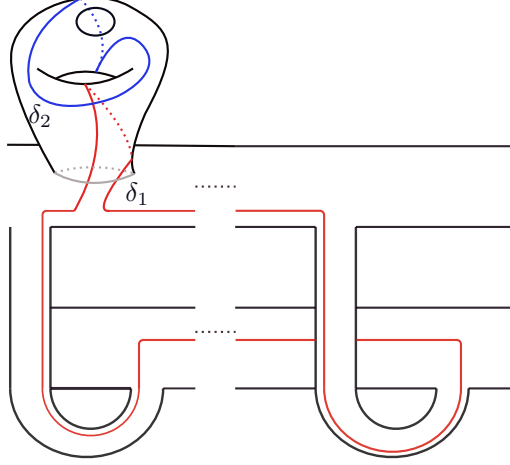
In Section 4, we relied on a sufficiently large unknotted piece of the surface to initiate an inductive framework (Lemma 3.8). This propagated into a lower bound on the extendable subgroup. We now rely on the *geometric monodromy group* of the braid β_d, using work of Ferretti in [Fer25]. Recall, from Section 3, that on the Seifert surface $F_{\beta_d} \subset S^3$ there is the *braid word assemblage* $\mathcal{C}_{F_{\beta_d}}$ (a filling configuration of Hopf band curves). The geometric monodromy group of the braid β_d is

$$\mathcal{T}_{\mathcal{C}_{F_{\beta_d}}} = \langle T_c \, : \, c \in \mathcal{C}_{F_\beta} \rangle.$$

Lemma 5.2. *For $d \geq 5$, $\mathrm{Mod}(F_{\beta_d})[\phi_{\beta_d}] = \langle T_c \, : \, c \in \mathcal{C}_{F_\beta} \rangle$.*

Proof. This is due to Ferretti following from [Fer25, Theorem 2] supplemented with [Fer25, Remark 5.1]. It also follows directly from [Fer25, Theorem 2] as follows. Let $\nu = (\sigma_1 \cdots \sigma_{d-1})^{d-1}$. Then $(F_\nu, \phi_\nu) \subset (F_{\beta_d}, \phi_{\beta_d})$ (an inclusion of framed surfaces) and F_ν is a Seifert surface for the $(d, d-1)$ torus knot. By [Fer25, Theorem 2], $\mathrm{Mod}(F_\nu)[\phi_\nu] = \langle T_c \, : \, c \in \mathcal{C}_{F_\nu} \rangle$. The surface F_{β_d} is obtained by stabilizing F_ν along the collection of admissible curves in $\mathcal{C}_{F_\beta} \setminus C_{F_\nu}$. As the genus of F_ν is at least 5, Lemma 3.8 yields $\mathrm{Mod}(F_\beta)[\phi_\beta] = \langle T_c \, : \, c \in \mathcal{C}_{F_\beta} \rangle$. $\qquad\square$

Henceforth suppress all subscripts, writing $F = F_{\beta_d}$, $S = F_{\beta_d} \# T$, $P = P_{\beta_d,1}$ and $\phi^+ = \phi^+_{\beta_d}$. Stabilization (Lemma 3.8) gives

Figure 7. The framing ϕ_{β_d} on $\widehat{F}_{\beta_d}$ is extended over the surface $P_{\beta_d,1} = S_{\beta_d,1} - \mathrm{int}(D_1)$ by declaring the curves δ_1, δ_2 admissible (winding number zero). Both δ_1, δ_2 are Hopf band curves.

Proposition 5.3. *For $d \geq 5$ we have* $\mathrm{Mod}(P)[\phi^+] = \langle \mathrm{Mod}(\widehat{F})[\phi], T_{\delta_1}, T_{\delta_2} \rangle$.

Let $p : \mathrm{Mod}(P) \to \mathrm{Mod}(Z_d)$ be the homomorphism induced by extension as the identity. Recall that $Z_d \backslash P$ is a collection of disks.

Proposition 5.4. *For all $d \geq 5$, $p(\mathrm{Mod}(P)[\phi^+]) \leq \mathcal{E}(\mathbb{C}P^2, Z_d)$.*

Proof. By Proposition 5.3, $\mathrm{Mod}(P)[\phi^+] = \langle \mathrm{Mod}(\widehat{F})[\phi], T_{\delta_1}, T_{\delta_2} \rangle$. As δ_1 and δ_2 are Hopf band curves, it follows from Proposition 2.11 that their image under p is in $\mathcal{E}(\mathbb{C}P^2, Z_d)$. It remains to prove that

$$p(\mathrm{Mod}(\widehat{F})[\phi]) \leq \mathcal{E}(\mathbb{C}P^2, Z_d).$$

Capping $\widehat{F}$ with D_0 and extending mapping classes by the identity over F induces a short exact sequence

$$1 \to \pi_1(UTF) \to \mathrm{Mod}(\widehat{F}) \to \mathrm{Mod}(F) \to 1,$$

where UTF denotes the unit tangent bundle of F. If a loop $\widetilde{\gamma} \in \pi_1(UTF)$ has a image a loop $\gamma \in \pi_1(F)$ under projection $UTF \to F$, the image of $\widetilde{\gamma}$ in $\mathrm{Mod}(\widehat{F})$ is the multitwist $T_{\gamma_L} T_{\gamma_R}^{-1} T_{\partial D_0}^k$ for some $k \in \mathbb{Z}$, where γ_L (resp. γ_R) is obtained by pushing γ to the left (resp. right).

Using the generating set established in Lemma 5.2, we see that capping is surjective on the level of framed mapping class groups, and we have a short exact sequence

$$1 \to \pi_1(UTF) \cap \mathrm{Mod}(\widehat{F})[\phi] \to \mathrm{Mod}(\widehat{F})[\phi] \to \mathrm{Mod}(F)[\phi] \to 1.$$
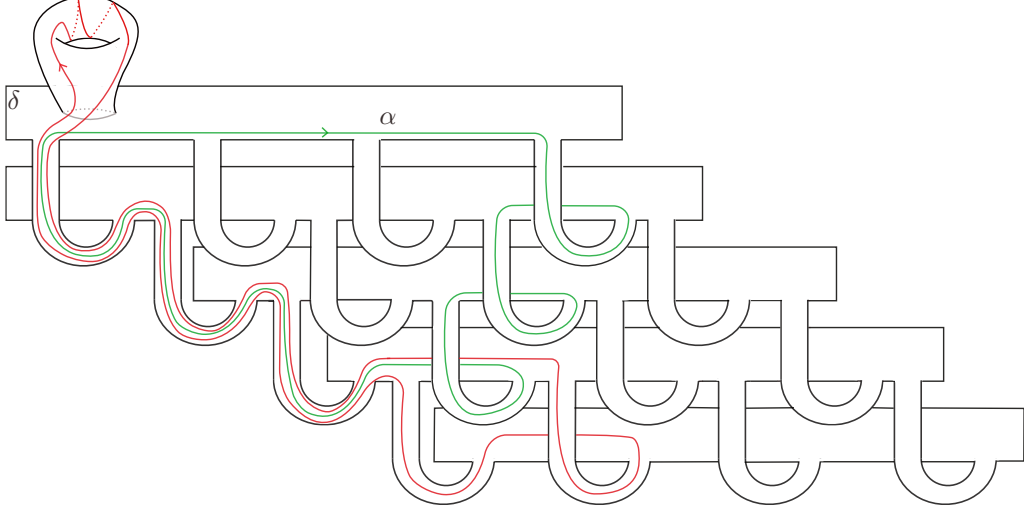
The group $\mathrm{Mod}(\widehat{F})[\phi]$ is generated by lifting a generating set for $\mathrm{Mod}(F)[\phi]$ together with $\pi_1(UTF) \cap \mathrm{Mod}(\widehat{F})[\phi]$. The generating set for $\mathrm{Mod}(F)[\phi]$ supplied by Lemma 5.2 consists of Dehn twists about Hopf band curves supported in $\widehat{F}$. It remains to now show

$$p(\pi_1(UTF) \cap \mathrm{Mod}(\widehat{F})[\phi]) \leq \mathcal{E}(\mathbb{C}P^2, Z_d).$$

In fact we prove

$$p(\pi_1(UTF)) \leq \mathcal{E}(\mathbb{C}P^2, Z_d).$$

Choose a generating set $\{\gamma_k\}_{k=1}^{n_d}$ for $\pi_1(F)$ consisting of simple closed curves based at a point on ∂D_0, each of which is a Hopf band curve on F. We recall that the image of $\pi_1(UTF)$ in $\mathrm{Mod}(\widehat{F})$ is generated by $\mathrm{Push}(\gamma_k) = T_{\gamma_k,L} T_{\gamma_k,R}^{-1}$ for $k = 1, \ldots, n_d$ together with $T_{\partial D_0}$. Both $\gamma_{k,L}$ and $\gamma_{k,R}$ are Hopf band curves so $p_Z(\mathrm{Push}(\gamma_k)) \in \mathcal{E}(\mathbb{C}P^2, Z_d)$, and by Proposition 2.14, $p_Z(T_\Delta) \in \mathcal{E}(\mathbb{C}P^2, Z_d)$. $\qquad \square$

Figure 8. As depicted the calculation takes place on $S_{\nu,1}$ for $\nu = (\sigma_1\sigma_2\sigma_3\sigma_4)^4$, which is naturally a subsurface of $S_{\beta_d,1}$ for all $d \geq 5$.

We now exploit the existence of a compression curve on the surface S to manufacture more extendable mapping classes in $\mathcal{E}(\mathbb{C}P^2, Z_d)$. Such a curve does not exist on the surface X_d, as it minimizes genus in its homology class (Thom Conjecture, [KM94]).

Proposition 5.5. *There exists a nonseparating simple closed curve $\alpha \subset \widehat{F} \subset S$ for which $\phi(\alpha) = \pm 2$ and $p(T_\alpha) \in \mathcal{E}(\mathbb{C}P^2, Z_d)$.*

Proof. Consider the oriented curves $\alpha \subset \widehat{F}$ and $\delta \subset S$ depicted in Figure 8 with $\phi(\alpha) = 2$ and $\mathrm{lk}(\alpha, \alpha^+) = -3$. The curve δ is an unknot with $\mathrm{lk}(\delta, \delta^+) = +1$ i.e. a Hopf band curve (see Definition 2.12). By Proposition 2.11, $T_\delta \in \mathcal{E}(\mathbb{C}P^2, Z_d)$. As $[T_\delta^{-1}(\alpha)] = [\alpha] + [\delta] \in H_1(F; \mathbb{Z})$ it follows that

$$\mathrm{lk}(T_\delta^{-1}(\alpha), T_\delta^{-1}(\alpha)^+) = \mathrm{lk}(\alpha, \alpha^+) + \mathrm{lk}(\alpha, \delta^+) + \mathrm{lk}(\delta, \alpha^+) + \mathrm{lk}(\delta, \delta^+) = -3 + 1 + 0 + 1 = -1.$$

By inspection, $T_\delta^{-1}(\alpha)$ is an unknot and therefore a Hopf band curve. By Proposition 2.11, we have $T_{T_\delta^{-1}(\alpha)} \in \mathcal{E}(\mathbb{C}P^2, Z_d)$. In $\mathrm{Mod}(Z_d)$, we have

$$T_{T_\delta^{-1}(\alpha)} = T_\delta^{-1} T_\alpha T_\delta,$$

and so $p(T_\alpha) \in \mathcal{E}(\mathbb{C}P^2, Z_d)$. $\qquad\qquad\square$

We now use the framed change-of-coordinates principle (Proposition 5.4) to deal with arbitrary nonseparating simple closed curves.
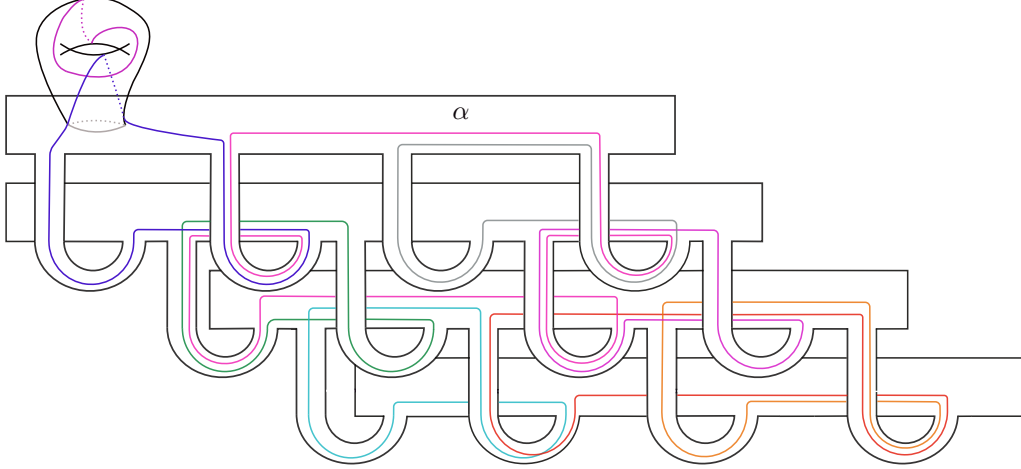
Lemma 5.6. *Let $\gamma \subset P \subset Z_d = X_d \# T$ be a nonseparating simple closed curve with $\phi^+(\gamma) = 0$ mod 2. Then $p(T_\gamma) \in \mathcal{E}(\mathbb{C}P^2, Z_d)$. If d is even, then for any nonseparating simple closed curve $\delta \subset P$, we have $p(T_\delta) \in \mathcal{E}(\mathbb{C}P^2, Z_d)$.*

Proof. Orient γ and write $\phi^+(\gamma) = 2k$. By Proposition 5.5, there exists a nonseparating simple closed curve $\alpha \subset \widehat{F} \subset P$ with $\phi^+(\alpha) = 2$ and $p(T_\alpha) \in \mathcal{E}(\mathbb{C}P^2, Z_d)$. Let β be a geometrically dual admissible curve to α (by Lemma 3.7). Let $\mu = T_\alpha^k(\beta)$. By twist-linearity, $\phi^+(\mu) = 2k$, and

$$T_\mu = T_\alpha^k T_\beta T_\alpha^{-k}.$$

By Proposition 5.4, $p(T_\beta) \in \mathcal{E}(\mathbb{C}P^2, Z_d)$ and so $T_\mu \in \mathcal{E}(\mathbb{C}P^2, Z_d)$. By the framed change-of-coordinates principle on P (Lemma 5.4), there exists $f \in \mathrm{Mod}(P)[\phi^+]$ with $f(\mu) = \gamma$. Now $T_\gamma = fT_\mu f^{-1}$, and by Corollary 5.4, $p(f) \in \mathcal{E}(\mathbb{C}P^2, Z_d)$. It follows that $p(T_\gamma) \in \mathcal{E}(\mathbb{C}P^2, Z_d)$.

Assume now that d is even. By the above, we may assume that $\phi^+(\delta) = 1$ mod 2. By Proposition 4.8, we may isotope δ on Z_d across a disk in D (a parallel copy of the core of the 2-handle of $\mathbb{C}P^2$) obtaining $\widetilde{\delta}$ with $\phi^+(\widetilde{\delta}) = 0$ mod 2. As δ and $\widetilde{\delta}$ are isotopic curves on Z_d, we conclude $p_{Z_d}(T_\delta) = p_{Z_d}(T_{\widetilde{\delta}}) \in \mathcal{E}(\mathbb{C}P^2, Z_d)$. $\qquad\qquad\square$

Figure 9. Dehn twists on the depicted configuration of curves yields a generating set for $\mathrm{Mod}(Z_4)$. The configuration is equivalent to the standard Humphries generating set for the mapping class group of a closed surface of genus four [FM12].

Having constructed an ample supply of Dehn twists in $\mathcal{E}(\mathbb{C}P^2, Z_d)$ we prove Theorem B. Recall that if d is odd $Z_d = X_d \# T \subset \mathbb{C}P^2$ is characteristic, $q_{Z_d} : H_1(Z_d; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ denotes Rochlin's quadratic form, and $Z_d \backslash P$ is a collection of 2-disks.

Theorem B. *Let $X_d \subset \mathbb{C}P^2$ be a smooth plane curve of degree d. Let $T \subset S^4$ be the boundary of a solid torus in S^4. Let Z_d be the stabilization $X_d \# T \subset \mathbb{C}P^2 \# S^4 \cong \mathbb{C}P^2$. If d is even, Z_d is flexible. If d is odd, Z_d is spin-flexible.*

Proof. We start with the general case $d \geq 5$. Assume that d is even. Let $\gamma \subset Z_d$ be a nonseparating simple closed curve. By isotopy if necessary, we may assume that $\gamma \subset P$. Then

$$\phi^+(\gamma) = 0 \text{ or } 1 \mod 2.$$

In either case, by Lemma 5.6, it follows that $T_\gamma \in \mathcal{E}(\mathbb{C}P^2, Z_d)$. As $\mathrm{Mod}(Z_d)$ is generated by Dehn twists on nonseparating simple closed curves, Z_d is a flexible surface.

If d is odd, there is the upper bound

$$\mathcal{E}(\mathbb{C}P^2, Z_d) \leq \mathrm{Mod}(Z_d)[q_{Z_d}].$$

By Theorem 3.15, $\mathrm{Mod}(Z_d)[q_{Z_d}]$ is generated by Dehn twists on nonseparating simple closed curves $\gamma \subset Z_d$ with $q_{Z_d}(\gamma) = 1$. Fix such a curve γ. We may assume $\gamma \subset P$. If θ_S denotes the linking form on $H_1(S; \mathbb{Z}/2\mathbb{Z})$ as a Seifert surface in S^3, then $i^* q_Z = \theta_S$ where $i : S \to Z_d$ denotes inclusion (Proposition 2.9). It follows that $\mathrm{lk}(\gamma, \gamma^+) = 1 \mod 2$ and so $\phi^+(\gamma) = 0 \mod 2$. By Lemma 5.6, $T_\gamma \in \mathcal{E}(\mathbb{C}P^2, Z_d)$, and Z_d is spin-flexible if d is odd.

We now deal with the remaining low degree cases. If $d = 1$, spin flexibility follows from Proposition 2.15. If $d = 2$, flexibility of $Z_2 = X_2 \# \Sigma_1$ follows from Figure 6. If $d = 3$, the surface $Z_3 = X_3 \# \Sigma_1$ is of genus 2 and q_{Z_3} is an *odd* (Arf invariant 1) quadratic form (Theorem 2.7). A generating set for the stabilizer $\mathrm{Mod}(Z_3)[q_{Z_3}]$ is given by a chain of four nonseparating admissible simple closed curves (see [Ham21, Lemma 4.4]), and such a configuration of Hopf band curves on $S_{\beta_3,1} \subset S^3$ is easily constructed. Finally, consider $Z_4 = X_4 \# \Sigma_1$ of genus four. In Figure 9, a Humphries generating set for $\mathrm{Mod}(Z_4)$ is depicted. All but one curve is a Hopf band curve. The curve α has $\mathrm{lk}(\alpha, \alpha^+) = -3$ and arguing as in Proposition 5.5, using a suitably dual positive Hopf band curve, we see that $T_\delta \in \mathcal{E}(\mathbb{C}P^2, Z_4)$. $\square$

Remark 5.7. Hirose proves [Hir05, Theorem 5.1] that, for an unknot $\Sigma_g \subset S^4$ of genus g, the surface $X_3 \# \Sigma_g$ is spin flexible for any $g \geq 1$. Whilst Theorem B generalizes this to any d for $g = 1$, a straightforward generalization of the methods shows that the surface $X_d \# \Sigma_g$ is (spin) flexible for any $g \geq 1$.

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