Homework 1

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Question 1 Suppose A is both unitary and upper-triangular, that is, $A^*A = AA^* = UU^{-1} = I$, Therefore, $a_{ij} = 0$ for $i > j$, that is, A is upper triangular. Then we have that A^*, the conjugate transpose, is a lower triangular matrix and that $a^*_{ij} = 0$ for $j < i$. Then for the ith row, $A^*A_{i,} = \sum_{j=1}^{m} A_{i,j}A^*_{i,j} = A_{i,i}A^*_{i,i} + 0 + ... + 0 = AA^*1_{,}$. So, $A_{i,j} = 0$ for $j \neq i$, so A is diagonal.
Question 2.

a. Let x be such that $Ax = \lambda x$. Then

$$\begin{aligned} A^{-1}Ax &= A^{-1}(\lambda x) \\ \implies x &= A^{-1}(\lambda x) \\ \implies x &= A^{-1}\lambda x \\ \implies A^{-1}x &= 1/\lambda x \end{aligned}$$

Therefore, $1/\lambda$ is an eigenvalue of A^{-1}.

b. Suppose $AB = \lambda x$. Then $BABx = B\lambda x$. Since linear maps are associative, we have that $(BA)Bx = \lambda(Bx)$, that is, the eigenvalue of BA is the same as AB with a different eigenvector. Therefore, the eigenvalues of AB and BA are the same

c. Since A is real, $A^* = A^T$, therefore $det(A - \lambda I) = det((A - \lambda I)^T)$. Since the characteristic polynomials are the same, the root (eigenvalues) are the same.

Question 3.

a. We have that $A = A^*$, so A is Hermitian. Then $Ax = \lambda x$. Taking the conjugate transpose of this relation, we have that $x^*A^* = \lambda^* x^*$. Then, $x^*A^*x = \lambda^* x^* x \iff x^*\lambda x = \lambda^* x^* x \iff \lambda x^* x = \lambda^* x^* x \implies \lambda = \lambda^* \implies \lambda \in \mathbb{R}$.

b.

Question 4. Proof ($\implies$). Suppose that A is positive-definite and Hermitian. Let v be a a nonzero vector such that $Av = \lambda v$ where $\lambda \in \mathbb{R}$. Then $(Av, v) = (\lambda v)^* v = \lambda v^* v = \lambda\sqrt{(v, v)} > 0$ by assumption. Since v is nonzero, we have that the inner product $(x, x) > 0$ and hence $\lambda > 0$.

Proof ($\Longleftarrow$). Suppose $\lambda > 0$, $\lambda \in \Lambda(A)$. Then we have that

Question 5.

a. Consider that we have the following two facts: $Ax = \lambda x$ and $(Ax)^* = (\lambda x)^* \iff x^* A^* = \lambda^* x^*$. Then we have that

$$x^* A^* A x = \lambda^* x^* A x$$
$$x^* \underbrace{A^* A}_{I} x = \lambda^* \lambda x^* x$$
$$x^* x = \lambda^* \lambda x$$

But if $\lambda = a + bi$, then $\lambda^* \lambda = a^2 - b^2 = |\lambda|^2$. Therefore, we must have that $|\lambda|^2 = 1 \implies |\lambda| = 1$ by the equality we derived above.

b. This is false, since have that $|A|_F = \sqrt{tr(A^* A)} = \sqrt{tr(I)} = n$ where n is the number of columns or rows in the matrix.

Question 6.

a. We have that

$$(Ax)^* x = (\lambda x)^* x$$
$$\iff x^* A^* x = \lambda^* x^* x$$
$$\iff -x^* A x = \lambda^* x^* x$$
$$\iff -x^* \lambda x = \lambda^* x^* x$$
$$\iff -\lambda(x^* x) = \lambda^*(x^* x)$$
$$\iff -\lambda = \lambda^*$$

Then letting $\lambda = a + bi \iff -\lambda = -a - bi$, we have that $a + bi = -a - bi \iff 2a + bi = -bi$ so $a = 0$, and hence λ is purely imaginary.

b. Suppose A is singular. Then we have that for some $x \neq 0$, $(I - A)x = 0$. But this means that $Ix - Ax = 0 \iff Ax = x$, so x is an eigenvector with eigenvalue 1. This is a contradiction, since the eigenvalues of A are purely imaginary. Hence, $I - A$ is nonsingular

Question 7. Suppose that $Av = \lambda v$ for some nonzero vector v such that $\|v\| = 1$. Then we have that $\|Av\| = \|\lambda v\| = |\lambda| \|v\|$. Also, since

$$\|A\| = \sup_{\|x\|=1} \|Ax\|$$

We have that

$$\sup_{\|x\|=1} \|Ax\| \geq \|Av\|$$
$$= \|\lambda x\| = |\lambda| \|x\| = |\lambda|$$

Then choose $\lambda = \rho(A)$ since this inequality holds for arbitrary eigenvalues. Therefore, $\|A\| \geq \rho(p)$

Question 8.

Question 9.

Question 11.

a. Since f is differentiable, we have that

$$\kappa(f) = \frac{\|J(x)\|}{\|f(x)\|/\|x\|} = \frac{\|x\|\|J(x)\|}{\|f(x)\|}$$

We have that $J = [1, 1]^T$, and choosing the ∞ norm we have that

$$\kappa(f) = \frac{\max\{x_1, x2\}}{|x_1 + x_2|}$$

Therefore, as $x_1, x_2 \longrightarrow 0$, f becomes ill conditioned since $\kappa(f) \longrightarrow \infty$

b. We have that $J(f) = [x_2, x_1]^T$, therefore the ∞ norm on J is $\|J\|_\infty = \max\{x_1, x2\}$. This gives us that the condition number is

$$\frac{2\max\{x_1, x_2\}}{|x_2 x_1|}$$

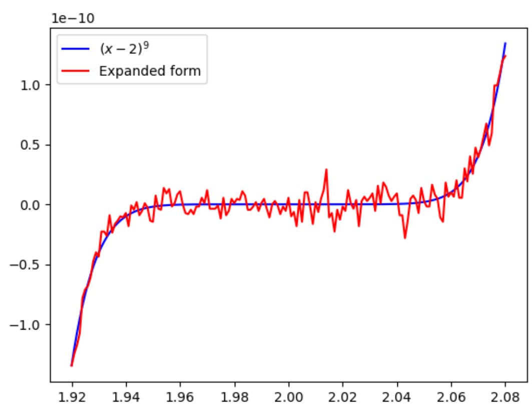
Therefore, as $x_1 \longrightarrow 0$ or $x_2 \longrightarrow 0$, f becomes ill conditioned since $\kappa \longrightarrow \infty$ as the denominator goes to zero.

c. Here, we have that $f'(x) = 9(x-2)^8$, therefore, the condition number is given by

$$\kappa(f) = \frac{9(x-2)^8|x|}{|x-2|^9}$$

So as $x \longrightarrow 2$, f becomes ill conditioned since $k \longrightarrow \infty$

Question 12.



a and b See figure

c. Since we are plotting f around