Homework 3: Theory Questions

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Question 1. We want to show that if P is an orthogonal projector, that is $P^2 = P$ and $P = P*$, then $B = (I - 2P)$ is unitary, that is $B^* = B^{-1}$. Then we have that

$$\begin{aligned}(I - 2P)(I - 2P)^* &= (I - 2P)(I^* - P^*2^*) \\ &= (I - 2P)(I - 2P^*) = (II - 2IP^* - 2PI + 4PP^*) \\ &= I - 4P^*P + 4PP^* = I\end{aligned}$$

As desired.

Question 2.

a. Let $P^2 = P$ and $P \neq 0$. Then by the Cauchy-Schwartz inequality, we have that $\|P^2\|_2^2 \leq \|P\|_2\|P\|_2$. But since $\|P^2\|_2 = \|P\|_2$, $\|P\| \leq \|P\|_2^2$ so $\|P\|_2 \geq 1$. This holds for orthogonal projectors, since that is an extra condition on the proof. Now let P be an orthogonal projector, so $P^* = P$.

b. First, consider that if $Px = \lambda x$, and $P^2x = \lambda x$, then $P^2x = P(Px) = P(\lambda x) = \lambda\lambda x = \lambda^2 x$. So $\lambda x = \lambda^2 x$, therefore, $\lambda^2 - \lambda = 0 \iff (\lambda - 1)\lambda = 0 \implies \lambda = 0, 1$, so the eigenvalues are zero or one.

Question 3.

a. Let $R \equiv \hat{R}$. Proof ($\implies$). If A is full rank n (since rank is at most $\min m, n$), then A^TA is an $m \times m$ matrix with rank m, and is hence invertible. Therefore, consider the QR decomp of A, and we must have that $A^TA = (QR)^T(QR) = R^TQ^TQR = R^TR$ since Q is orthogonal. Hence R^TR is invertible. Since R is by construction upper-triangular, we must have that the columns of R are linearly independent. Therefore, if any column i has a zero on the diagonal, then it is a linear combination of the $i - 1$th row. Therefore, the diagonal entries of R are nonzero.
Proof ($\impliedby$). Suppose the diagonal entries of R are nonzero and let QR be the QR decomposition of A. Then R^*R is invertible, so $R^*R = (QR)^*(QR) = A^*A$ is invertible. Therefore, A is full rank, i.e. rank n.

b. Since the rank of R is the dimension of its image, the vectors corresponding to the nonzero entries will be in the basis for the image of R. Since we have k nonzero entries, then $rank(A) \geq k$. Also, since there are $k - n - 1$ other linearly independent vectors in the span of R, we have that $k \leq rank(A) \leq n - 1$.

Question 4. Consider the Householder transformation given by $H = I - 2vv^T$, where vv^T is the outer product.
Then from (1), we have that if $P = vv^T$, then $H = I - 2P$ is an orthogonal projector. We know that orthogonal projectors have eigenvalues ± 1. Additionally, since if $\sigma_1, .., \sigma_n$ are the singular values of H, then $\sigma_1^2, ..., \sigma_n^2$ are the singular values of $H^T H$ (see my derivation in (5)), but since $H^T H = I$, we have that $\sigma = 1$, that is, the singular values are 1. This also makes geometric sense, since the hyperellipse given by the set S is taken to $HS = S$, that is, the principal axis are not scaled at all. Therefore, the eigenvalues are ± 1, and the determinant is either 1 or 0, which we know since H is an orthogonal projector.
Question 5. First, consider the SVD of A as $A = U \Sigma V^T$. Then we have that $A^T A = V \Sigma U^T U \Sigma V^T = V(\Sigma \Sigma^T) V^T = V(\Sigma^2) V^T$. So if the singular values of A are $\sigma_{\min}, ., ..\sigma_{\max}$, then the singular values of $\sigma_{\min}^2, ..., \sigma_{\max}^2$. Then

$$cond(A) = \frac{(\sigma_{\max} A)^2}{(\sigma_{\min} A)^2} = \frac{\sigma_{\max} A^T A}{\sigma_{\min} A^T A} = cond(A^T A)$$