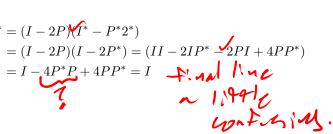
Homework 3: Theory Questions

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Question 1. We want to show that if P is an orthogonal projector, that is $P^2 = P$ and $P = P^*$, then B = (I - 2P) is unitary, that is $B^* = B^{-1}$. Then we have that

 $(I-2P)(I-2P)^* = (I-2P)(I^*-P^*2^*)$



As desired.

- Mission 2. Whith a. Let $P^2=P$ and $P\neq 0$. Then by the Cauchy-Schwartz inequality, we have that $\|P^2\|_2^2<\|P\|_2\|P\|_2$ But since $\|P^2\|_2^2$ have that $||P^2||_2^2 \le ||P||_2 ||P||_2$. But since $||P^2||_2 = ||P||_2$, $||P|| \le ||P||_2^2$ so $||P||_2 \ge 1$. This holds for orthogonal projectors, since that is an extra condition on the proof. Now let P be an orthogonal projector, so $P^* = P$.
 - b. First, consider that if $Px = \lambda x$, and $P^2x = \lambda x$, then $P^2x = P(Px) =$ $P(\lambda x) = \lambda \lambda x = \lambda^2 x$. So $\lambda x = \lambda^2 x$, therefore, $\lambda^2 - \lambda = 0 \iff (\lambda - 1)\lambda = 0$ $0 \implies \lambda = 0, 1,$ so the eigenvalues are zero or one.

Question 3.

- a. Let $R \equiv \hat{R}$. Proof (\Longrightarrow). If A is full rank n (since rank is at most $\min m, n$), then $A^T A$ is an $m \times m$ matrix with rank m, and is hence invertible. Therefore, consider the QR decomp of A, and we must have that $A^T A = (QR)^T (QR) = R^T Q^T QR = R^T R$ since Q is orthogonal. Hence $R^T R$ is invertible. Since R is by construction upper-triangular, we must have that the columns of R are linearly independent. Therefore, if any column i has a zero on the diagonal, then it is a linear combination of the i-1th row. Therefore, the diagonal entries of R are nonzero. Proof (\iff). Suppose the diagonal entries of R are nonzero and let QR be the QR decomposition Q A. Then R^*R is invertible, so R^*R $(QR)^*(QR) = A^*A$ is invertible. Therefore, A is full rank, i.e. rank n.
- b. Since the rank of R is the dimension of its image, the vectors corresponding to the nonzero entries will be in the basis for the image of R. Since we have k nonzero entries, then $rank(A) \geq k$. Also, since there are k-n-1other linearly independent vectors in the span of R, we have that $k \leq$ $rank(A) \le n - 1.$

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Question 4. Consider the Householder transformation given by $H = I - 2vv^T$, where vv^T is the outer product.

Then from (1), we have that if $P = vv^T$, then H = I - 2P is an orthogonal projector. We know that orthogonal projectors have eigenvalues ± 1 . Additionally, since if $\sigma_1, ..., \sigma_n$ are the singular values of H, then $\sigma_1^2, ..., \sigma_n^2$ are the singular values of H^TH (see my derivation in (5)), but since $H^TH = I$, we have that $\sigma = 1$, that is, the singular values are 1. This also makes geometric sense, since the hyperellipse given by the set S is taken to HS = S, that is, the principal axis are not scaled at all. Therefore, the eigenvalues are ± 1 , and the determinant is either 1 or 0, which we know since H is an orthogonal projector.

Question 5. First, consider the SVD of A as $A = U\Sigma V^T$. Then we have that

Question 5. First, consider the SVD of A as $A = U\Sigma V^T$. Then we have that $A^TA = V\Sigma U^TU\Sigma V^T = V(\Sigma\Sigma^T)V^T = V(\Sigma^2)V^T$. So if the singular values of A are σ_{\min} , ..., σ_{\max} , then the singular values of σ_{\min}^2 , ..., σ_{\max}^2 . Then

$$\left(\operatorname{cond}(A) \right)^{2} = \frac{(\sigma_{\max}A)^{2}}{(\sigma_{\min}A)^{2}} = \frac{\sigma_{\max}A^{T}A}{\sigma_{\min}A^{T}A} = \operatorname{cond}(A^{T}A)$$