AM 213A, Winter 2022
Homework 1 (100 points)

Posted on Thu, Jan 20, 2022
Due 11:59 pm, Mon, January 31, 2022

Submit your homework to your Git repository by 11:59 pm

- You are encouraged to use LaTex or MS-words like text editors for homework. A scanned copy of handwritten solutions will be acceptable on the condition that your handwriting is clean and well-organized, and your scanned copy is fully readable.
- Your report needs to have relevant discussions on each problem to describe what you demonstrate. For all coding problems, showing the screen output from your code execution is considered to be not sufficient.
- To disprove, you need to provide a counter-example.

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1. (7 pts) Let $A \in \mathbb{C}^{m\times m}$ be both upper-triangular and unitary. Show that A is a diagonal matrix. Does the same hold if $A \in \mathbb{C}^{m\times m}$ is both lower-triangular and unitary?

2. (15 pts) Prove the following in each problem.

(a) (5 pts) Let $A \in \mathbb{C}^{m\times m}$ be invertible and $\lambda \neq 0$ is an eigenvalue of A. Show that $1/\lambda$ is an eigenvalue of A^{-1}.

(b) (5 pts) Let $A, B \in \mathbb{C}^{m\times m}$. Show that AB and BA have the same eigenvalues.

(c) (5 pts) Let $A \in \mathbb{R}^{m\times m}$. Show that A and A^* have the same eigenvalues. (Hint 1: Use $\det(M) = \det(M^T)$ for any square matrix $M \in \mathbb{R}^{m\times m}$ in connection to the definition of characteristic polynomials. Hint 2: When a real-valued matrix A has a complex eigenvalue λ, then $\bar{\lambda}$ is also an eigenvalue of A.)

3. (8 pts) Let $A \in \mathbb{C}^{m\times m}$ be hermitian. Suppose that for nonzero eigenvectors of A, there exist corresponding eigenvalues λ satisfying $Ax = \lambda x$.‘

(a) (4 pts) Prove that all eigenvalues of A are real.

(b) (4 pts) Let x and y be eigenvectors corresponding to distinct eigenvalues. Show that $(x, y) = 0$, i.e., they are orthogonal. (Hint: Use the result of Part (a).)

4. (6 pts) A matrix A is called *positive definite* if and only if $(Ax, x) > 0$ for all $x \neq 0$ in $\mathbb{C}^m$. Show that A is Hermitian and positive definte if and only if $\lambda_i > 0$, $\forall \lambda_i \in \Lambda(A)$, the spectrum of A.

(Hint 1: Use the following Theorem: If $A \in \mathbb{C}^{m \times m}$ is Hermitian, then A has real eigenvalues $\lambda_i, i = 1, \cdots, m$, not necessarily distinct, and m corresponding eigenvectors u_i form an orthonormal basis for $\mathbb{C}^m$. Hint 2: Now, you realize that for any arbitrary $x \neq 0$, you can write $x = \sum_{i=1}^{m} \alpha_i u_i$. Use Hint 1 to show $(Ax, x) = \sum_{i=1}^{m} \lambda_i |\alpha_i|^2$ and draw your conclusion.) Note that in the lecture, we discussed that a Hermitian matrix is non-defective, which implies that a Hermitian matrix has m linearly independent eigenvectors u_i and is diagonalizable with the eigenvector matrix. The above theorem states that, in addition, the linearly independent eigenvector matrix $U = [u_1|\dots|u_m]$ "orthogonally diagonalizes" A. The theorem is referred to as the "Principal axis theorem."

5. (6 pts) Suppose A is unitary.

(a) (3 pts) Let (λ, x) be an eigenvalue-vector pair of A. Show λ satisfies $|\lambda| = 1$.

(b) (3 pts) Prove or disprove $||A||_F = 1$.

6. (6 pts) Let $A \in \mathbb{C}^{m \times m}$ be skew-hermitian, i.e., $A^* = -A$.

(a) (3 pts) Show that the eigenvalues of A are pure imaginary.

(b) (3 pts) Show that $I - A$ is nonsingular.

7. (7 pts) Show that $\rho(A) \leq ||A||$, where $\rho(A)$ is the spectral radius of A.

8. (6 pts) Let A be defined as an outer product $A = uv^*$, where $u \in \mathbb{C}^m$ and $v \in \mathbb{C}^n$.

(a) (3 pts) Prove or disprove $||A||_2 = ||u||_2||v||_2$.

(b) (3 pts) Prove or disprove $||A||_F = ||u||_F||v||_F$. (Note: For a vector u, $||u||_F = ||u||_2$ by definition.)

9. (9 pts) Let $A, Q \in \mathbb{C}^{m \times m}$, where A is arbitrary and Q is unitary.

(a) (3 pts) Show that $||AQ||_2 = ||A||_2$ (note that we did $||QA||_2 = ||A||_2$ in class).

(b) (6 pts) Show that $||AQ||_F = ||QA||_F = ||A||_F$.

10. (8 pts) We say that $A, B \in \mathbb{C}^{m \times m}$ are unitarily equivalent if $A = QBQ^*$ for some unitary $Q \in \mathbb{C}^{m \times m}$.

(a) (4 pts) Show that if A and B are unitarily equivalent, then they have the same singular values. (Hint. Note that every matrix has a SVD and their singular values are uniquely defined.)

(b) (4 pts) Show that the converse of Part (a) is not necessarily true.

11. (9 pts) Find the relative condition number of the following functions and discuss if there is any concern of being ill-conditioned. If so, discuss when.

(a) (3 pts) $f(x_1, x_2) = x_1 + x_2$

(b) (3 pts) $f(x_1, x_2) = x_1 x_2$

(c) (3 pts) $f(x) = (x - 2)^9$

12. (13 pts) Note that the function $f(x) = (x-2)^9$ in Part (c) in Problem 9 can also be expressed as $g(x) = x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512$. Note that, mathematically, the two functions f and g are identical. (Note: Use Matlab or Python for this problem, particularly for plotting purposes. Fortran coding is not necessary.)

(a) (4 pts) Plot $f(x)$ by evaluating discrete function values of f at $1.920, 1.921, 1.922, \ldots, 2.080$, which are equally spaced with the distance of 0.001.

(b) (4 pts) Over-plot $g(x)$ at the same set of discrete points in Part (a).

(c) (5 pts) Draw your conclusion from your results of Part (c) in Prob. 11 and Parts (a) and (b) in this problem.