AM 213A, Winter 2022
Homework 4 (100 points)

Posted on Mon, Feb 28, 2022
Due Mon, Mar 7, 2022

IMPORTANT: Read the instructions carefully.
Submit your homework to your Git repository by 11:59 pm

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1. General Guidelines

1.1. Two parts

You have two parts in this homework set:

- **Part 1**: Numerical coding problems (Make sure to use double precision unless you are told otherwise!!!)
- **Part 2**: Theory problems

Your final set of answers should consist of

- For Part 1: a pdf file with your written answers. This pdf *must* be created using a word processor.
- For Part 2: solutions to the theory problems in a pdf file (a cleanly hand-written scanned pdf is also allowed).
- For both Part 1 and Part 2: Please submit all your answers to your git. Any submission to Canvas is not going to be graded. Use subdirectories named as **Part1** and **Part2** under your top directory called **HW4**.

Please organize and present the material in the best possible way, in particular, for your Part 1 solutions:

- For any written material, be informative but concise. Mathematical derivations, if appropriate, should contain enough pertinent steps to show the reader how you proceeded. All included figures should be clearly annotated and have an informative caption. Each problem should be addressed in its own Section (e.g., "Section 1: Problem 1", "Section 2: Problem 2", etc.). References should be included if you are using any material as part of your discussions and/or calculations.
- For the codes: All codes must be written in Fortran 90 (or more recent Fortran, but not Fortran 77) or C, but *not* in any other language. MATLAB or Python is to be used to (i) generate figures, and/or (ii) perform simple analysis (e.g., calculating errors) the data produced by your Fortran/C codes. Put all the codes relevant for each problem in separate directories

called Prob1, Prob2, etc, as needed. Annotate your codes carefully so the reader knows what each part of the code does. For each routine and function in your Fortran/C, include a header (i.e., comment sections at the beginning explanations) explaining the inputs and outputs of the code, and what the routine does. In the main program in each directory, include a header that contains (i) the command required to compile the code, and (ii) the structure of the inputs and outputs of the main program.

- Template codes are provided in Fortran 90 only, which means you need to translate the template Fortran codes to C if your language choice is C. Be careful about the column-wise convention of arrays in Fortran vs. the row-wise convention in C.

Part 1: Coding Problems (30 points; 10 points each)

In your report, include inputs (e.g., initial guess) and outputs of your codes. Describe sufficiently the behaviors of your codes in each problem. Clarify what parts of your code implementations are corresponding to each individual problem (e.g., subroutines foo1 & foo2 in `LinAl.f90` are for Problem 1, etc.).

To debug your code, make sure you test your code to solve at least one of worked examples in the lecture note.

1. Write a program to reduce a symmetric matrix to tridiagonal form using Householder matrices for the similarity transformations. Use the program to reduce the following matrix to tridiagonal form:

$$\mathbf{A} = \begin{bmatrix} 5 & 4 & 1 & 1 \\ 4 & 5 & 1 & 1 \\ 1 & 1 & 4 & 2 \\ 1 & 1 & 2 & 4 \end{bmatrix}. \tag{1}$$

2. Write a program of the QR algorithm (i) without shift and (ii) with shift, to calculate the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}. \tag{2}$$

3. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & -3 & 1 & 5 \\ 3 & 1 & 6 & -2 \\ 4 & 5 & -2 & -1 \end{bmatrix}, \tag{3}$$

which has three eigenvalues $\lambda_1 = -8.0286, \lambda_2 = 7.9329, \lambda_3 = 5.6689$, and $\lambda_4 = -1.5732$. Write a program of the inverse iteration to calculate the corresponding eigenvectors.

Part 2: Theory Problems (70 points; 10 points each)

1. The Schur decomposition theorem states that every square matrix $\mathbf{A} \in \mathbb{C}^{m\times m}$ has a Schur decomposition, $\mathbf{A} = \mathbf{QUQ}^*$, where Q is unitary and T is upper-triangular. Use this theorem to prove that, for an arbitrary norm $||\cdot||$,

$$\lim_{n\to\infty} ||\mathbf{A}^n|| = 0 \iff \rho(\mathbf{A}) < 1. \tag{4}$$

(Note: Show the claim first with the 2-norm or the Frobenius norm and use the fact that all norms are equivalent in a finite vector space.)

2. Let $\mathbf{A}$ be $m \times n$ and $\mathbf{B}$ be $n \times m$. Show that the matrices $\begin{bmatrix} \mathbf{AB} & 0 \\ B & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ \mathbf{B} & \mathbf{BA} \end{bmatrix}$ have the same eigenvalues.

3. Show that for a real valued square matrix the Gerschgorin theorem also holds with the bounds r_i which are given by the partial column sums (instead of the partial row sums):

$$r_i = \sum_{i=1, i\neq j}^{m} |a_{i,j}|, \quad i = 1, \ldots, m. \tag{5}$$

(Hint: Recall Prob 2(c) in HW1 and use the result.)

4. Use the Gerschgorin theorem to show that the matrix

$$\mathbf{A} = \begin{bmatrix} 1.0 & 0.3 & 0.1 & 0.4 \\ 0.0 & 2.0 & 0.0 & 0.1 \\ 0.0 & 0.4 & 3.0 & 0.0 \\ 0.1 & 0.0 & 0.0 & 4.0 \end{bmatrix} \tag{6}$$

has exactly one eigenvalue in each of the four circles

$$|z - k| \leq 0.1, \quad k = 1, 2, 3, 4. \tag{7}$$

5. Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be real and symmetric that is positive definite. Let $y \in \mathbb{R}^m$ be nonzero. Prove that the limit

$$\lim_{k\to\infty} \frac{y^T \mathbf{A}^{k+1} y}{y^T \mathbf{A}^k y} \tag{8}$$

exists and is an eigenvalue of $\mathbf{A}$.

6. Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be real with nonnegative enties such that

$$\sum_{j=1}^{m} a_{ij} = 1 \ \ (1 \le i \le m). \tag{9}$$

Prove that no eigenvalue of $\mathbf{A}$ has an absolute value greater than 1.

7. Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be a non-defective matrix with its eigenvalues $\{\lambda_i\}_{i=1}^m$ and its singular values $\{\sigma_i\}_{i=1}^m$, satisfying

$$|\lambda_1| \ge |\lambda_2| \ge \cdots \ge |\lambda_m|, \tag{10}$$
$$\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_m. \tag{11}$$

Let $\rho(\mathbf{A})$ be the spectral radius of $\mathbf{A}$ and $\mathrm{cond}(\mathbf{A}) = ||\mathbf{A}||_2||\mathbf{A}^{-1}||_2$ be the condition number of $\mathbf{A}$. Let $\mathbf{A}$ be normal, i.e., $\mathbf{A}^T\mathbf{A} = \mathbf{A}\mathbf{A}^T$. Show that:

(a) $\sigma_i = |\lambda_i|$, $1 \le i \le m$.

(b) $||\mathbf{A}||_2 = |\lambda_1| = \rho(\mathbf{A})$.