Total 83/100

File read in SVD fails, though I'll ignore that.

Code compiles and runs: +20

Final Part 1: Report

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Question 1a.
Using the singular value reconstruction with the $k \in \{20, 40, 80, 160, 320, 640, 1280, 2560, 3355\}$ largest singular values in the Σ matrix, we obtain the following compressed images:

SVD 25/30

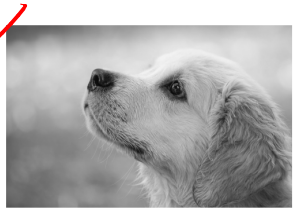
(a) $k = 20$

(b) $k = 40$

(c) $k = 80$

(a) $k = 160$

(b) $k = 320$



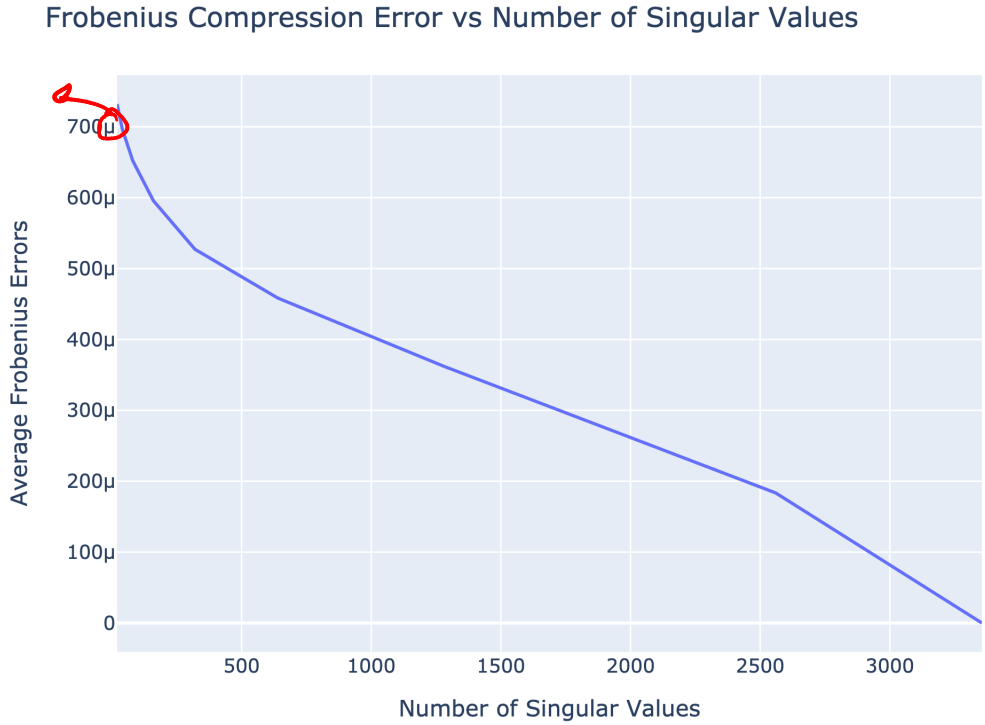
(c) $k = 640$

(a) $k = 1280$

(b) $k = 2560$

(c) $k = 3355$ (original image)

With more values in the Σ matrix of the SVD set to zero ($k = 20$ being the fewest nonzero), the visual distortion is the highest. However, by just increasing $k = 80$, the visual distortion is quite minimal, and for $k > 80$ I cannot see any visual difference with a small png. Now, defining the average Frobenius error of an $m \times n$ matrix W by $\|W\|_F/(mn)$, we plot the errors of the difference between the original image and the kth SVD compression, i.e. $\|A - A_{k\sigma}\|_F/(mn)$.



With $k = 20$, we obtain an error less than 10^{-3}, and for subsequent k (with less distortion), we clearly do as well.

Question b.1

The Gauss-Jacobi and Gauss-Seidel are both iterative methods for solving exact systems of equations of the form $Ax = b$, where A is an $m \times m$ matrix and b is an m-vector. For the former, we factorize $A = D + L + U$ where D contains the diagonal elements and zeros elsewhere, and L and U contain the strictly lower and upper diagonal elements of A, respectively. Then we simply apply the iteration $x^{k+1} = D^{-1}\left(b - (L + U)x^k\right)$.

Note that for the Gauss-Jacobi method, the values x_i^k obtained in the kth it-

eration remain completely unchanged until the $k+1$th iteration has been fully calculated. Instead, the Gauss-Seidel uses the values x_i^{k+1} as soon as they're known. That is, once we calculate x_j^k in the jth equation we use it in the x_{j+1}^kth equation. This difference allows us to only store one solution vector x, instead of having to store two for the current and previous.

For the Gauss-Jacobi method to converge, it is sufficient (but not necessary) that A must be strictly diagonally dominant. More generally, we must have that $p(D^{-1}(L+U)) < 1$, that is, the spectrum is less than 1. The Gauss-Seidel method is known to converge if A is symmetric positive-definite, or if A is strictly or irreducibly diagonally dominant.

In our case, we define a 10×10 matrix A such that $a_{ij} = 1$ for $i \neq j$ and $a_{ii} = D$ for $D \in \{2, 5, 10, 100, 1000\}$. Below, visualize the convergence for the two methods, where $b_i = i$ for $i = 1, ..., 10$.

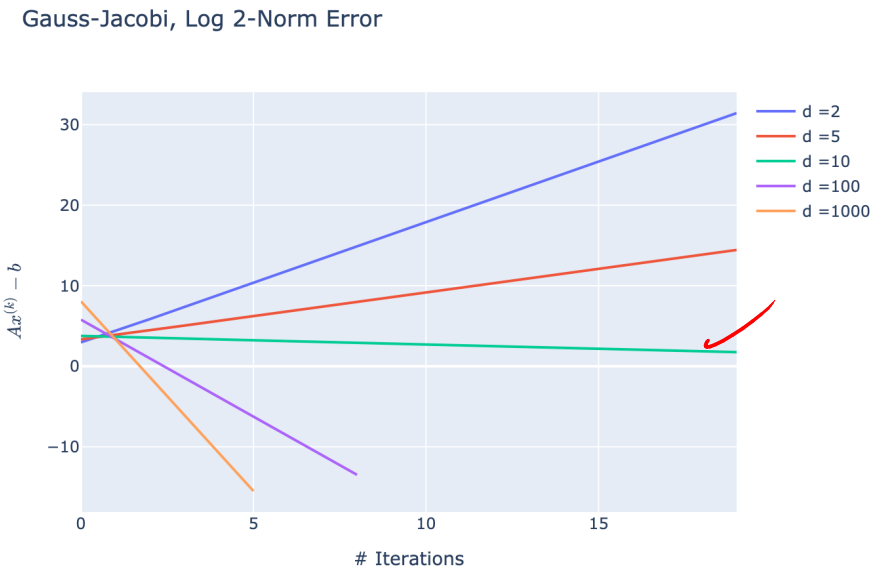


Figure 4: Convergence of Gauss-Jacobi, number of iterations vs log norm error.

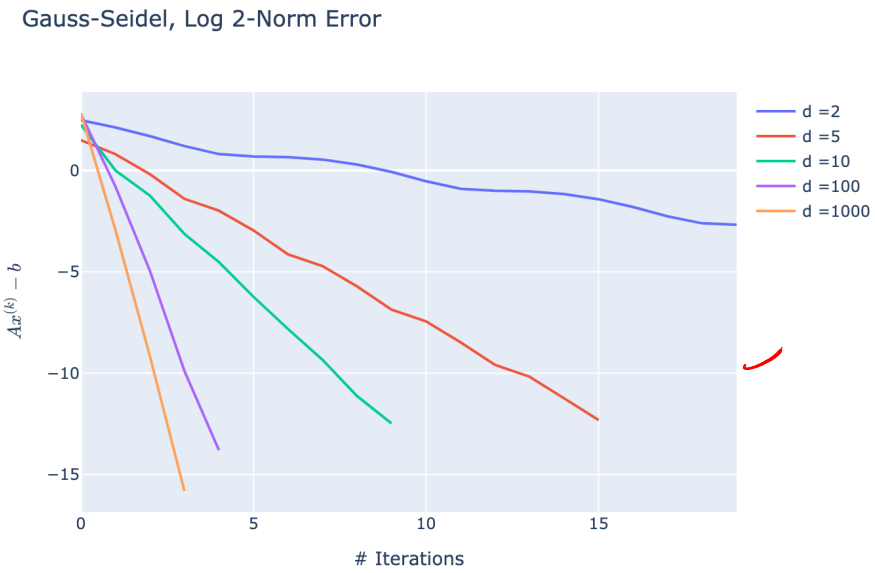


Figure 5: Convergence of Gauss-Seidel, number of iterations vs log norm error

For the Jacobi method, we can see that for $D = 2, 5$, the method diverges, wheres for the Gauss-Seidel method, these values converge (albiet quite slowly). For the Gauss-Seidel algorithm, since $a_{ii} = D$ and $a_{ij} = 1$, $i \neq j$, we have that $a_{ji} = a_{ij}$ for all i and j. Therefore, A is symmetric. Now, consider $\hat{x_i}$ to be the ith element of x under the image of our matrix A. Then $\hat{x_i} = Dx_i + \sum_{j=1\ j\neq i}^{m} x_j$. Therefore, $x_i\hat{x_i} = D(x_i)^2 + \sum_{j=1\ j\neq i}^{m} x_j x_i$. For large enough D, this means A is positive-definite, so the algorithm converges.

Question b.2 The conjugate gradient algorithm numerically solves an exact system of equations $Ax = b$ where A is symmetric and positive-definite. Then, this is equivalent to minimizing the function $f(x) = \frac{1}{2}x^T Ax - x^T b$, since $\nabla f = Ax - b$ and solving $\nabla f = 0$ is equivalent to solving $Ax = b$. The conjugate gradient method therefore chooses n conjugate directions, taking a step size towards the global minimum of f. The algorithm begins by choosing the direction like the gradient descent algorithm, i.e. opposite the direction of greatest ascent, or the gradient vector itself. So, $p_0 = b - Ax_0$. We then successively choose directions p_k orthogonal to p_{k-1}, and search along our surface until we find the minima.

Here, we show the number of iterations convergence takes for the Gauss-Jacobi, Gauss-Seidel, and Conjugate-Gradient methods.

Your plots don't show this either..

D	Conjugate Gradient	Gauss-Jacobi	Gauss-Seidel
2	2	Not reached	58
5	2	Not reached	16
10	2	11	10
100	2	11	5
1000	2	11	4

In general, the diagonal preconditioner is particularly efficient for diagonally dominant matrices. However, in the case of the conjugate gradient method, multiplying by the diagonal preconditioner given by $M = diag(\frac{1}{D}, ..., \frac{1}{D})$ means scaling each row by $\frac{1}{D}$, which in the case of our matrix is really just scaling the entire matrix down by a constant since the diagonals are constant. Therefore, we aren't actually changing the system in any way, and therefore the preconditioning isn't helpful at all. In the case where A is a 10×10 matrix with $a_{ii} = i$, we have that the conjugate gradient algorithm converges in 10 iterations, and for A being 100×100 with $a_{ii} = i$, it actually converges on the first iteration! This means that the initial vector x given by $x = [1, ..., 1]^T$ (how I initialized x_0) was a particularly good guess in this case.

That is a bit unusual. I suppose we should mandate a particular starting guess in the future...