Homework 4: Theory Questions

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Question 1. Lemma: $A^n = QU^nQ^*$ where U is upper triangular and Q is unitary. Proof (induction on n). For $n = 1$, we have that $A = QUQ^*$ by the Schur decomposition. Then suppose $A^n = QU^nQ^*$, and show $A^{n+1} = QU^{n+1}Q^*$. We have that $A^{n+1} = AA^n = (QUQ^*)(QU^nQ^*) = QUQ^*QU^nQ^* = QUIU^nQ^* = QU^{n+1}Q^*$, as desired. Since A^n is similar to QU^nQ^*, we have that the spectrum of A^n is the same as the spectrum of U^n.

Proof ($\Longrightarrow$). Consider $\|A\|_F$, the Frobenius norm given by $\sqrt{\sum_i \sum_j |a_{ij}|}$. If $\sqrt{\sum_i \sum_j |a_{ij}^n|} \longrightarrow 0$ as $n \longrightarrow \infty$, then we must have that $|a_{ij}| < 0$, since each entry of the matrix must go to zero. Then since $\|A^n\| \longrightarrow 0 \iff \|U^n\| \longrightarrow 0$ as $n \longrightarrow \infty$, and the diagonals of U contain the eigenvalues of A, we have by necessity that $p(A) < 1$.

Proof ($\Longleftarrow$). Suppose that $p(A) < 1$. We prove the following lemma: $\|A^n x\|/\|x\| = p(A)^n$ by induction on n. Suppose $\lambda = p(A)$ and we have that $Ax = \lambda x$. Then $Ax/x = \lambda$, so $\|Ax\| = \|\lambda x\| = \lambda\|x\|$, therefore $\frac{\|Ax\|}{\|x\|}$.

Question 2. Lemma: The eigenvalues of AB are the same as the eigenvalues of BA. Proof: Let $(AB)x = \lambda x$. Then $ABx = BABx = BA(Bx) = \lambda(Bx)$, so λ is an eigenvalue of BA with eigenvalue $y = Bx$. Now,

Question 3. First, note that since $detA = detA^T$, we have that $det(A - \lambda I) = det((A - \lambda I)^T) = det(A^T - \lambda I)$, so the eigenvalues of A and A^T are the same. Therefore, the Gersgorin circles defined by the rows of A^T (columns of A) contain all eigenvalues of A. Equivalently, the theorem holds with column sums.

Question 4. First, consider the absolute row sums given by $r_{1,2,3,4} = 0.8, 0.1, 0.4, 0.1$. Then since we showed the Gershgorin discs can also be found by considering the absolute column sums, consider the column sums of columns 2 and 4, given by $c_{2,4} = 0.1, 0.1$. Therefore, the radius of each circle is 0.1. Additionally, since $k + 0.1 < (k + 1) - 0.1$ the circles are disjoint, and we can conclude that there is exactly one eigenvalue in $|z - k| < 0.1$ for $k = 1, 2, 3, 4$.

Question 5. Lemma: If $Ay = \lambda y$, then $A^n y = \lambda^n y$. Proof (by induction on n). $n = 1$ is handled in the definition. Then suppose $A^n y = \lambda^n y$ and show that $A^{n+1}y = \lambda^{n+1}y$. Then $A^{n+1} = AA^n = A(\lambda^n y) = \lambda^n(Ay) = \lambda^n\lambda = \lambda^{n+1}$. Then we have that $y^T A^k y = y^T y \lambda^k$, so

$$\lim_{k \longrightarrow \infty} \frac{y^T A^{k+1} y}{y^T A^k y} = \frac{y^T y \lambda^{k+1}}{y^T y \lambda^k} \lambda$$

Is an eigenvalue of A.

Question 6. Lemma: $p(A) \leq \|A\|$. Proof: We consider the proof with the 2-norm, since all norms are equivalent in a finite vector space. Let $p(A) = |\lambda|$, and let the corresponding eigenvector be x with $\|x\| = 1$. Then $\|Ax\| = \|\lambda x\| = |\lambda|$. Now consider an arbitary unit vector u. By the Cauchy-Schwartz inequality, we have that $\|Au\| \leq \|A\|\|u\| = \|A\|$, therefore $\|A\| \geq \|Au\|$ for all vectors u. In particular, $\|A\| \geq \|Ax\| = |\lambda|$, so $p(A) \leq \|A\|$.

Now, consider the fact that since A has nonnegative entries, $\sum_j^m a_{ij} = 1 = \|A\|_1$, the 1-norm of A. Therefore, $p(A) < 1$, or equivalently, no eigenvalue has an absolute value greater than one.

Question 7.

a. Consider the SVD of A to be $A = U\Sigma V^T$. Then since $A^T = V\Sigma U^T$, and $A^T A = V\Sigma^2 V^T = AA^T = U\Sigma^2 U^T$, we have that $U = T$. Let σ_i be the ith singular value of A. Since $\sigma_i = \sqrt{\lambda_i(A^T A)}$, $A = U\Sigma U^T$ and $A^T A = U\Sigma^2 U^T$ by virtue of A being normal, $\sigma_i = \sqrt{\sigma_i^2} = |\lambda_i|$.

b. Since $\|A\|_2 = \sqrt{p(A^T A)} = \sigma_{\max}(A)$ by definition, and we just showed that $\sigma_i = |\lambda_i|$, we have that $\|A\|_2 = |\lambda_i|$.