Homework 1

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Question 1 Suppose A is both unitary and upper-triangular, that is, $A^*A = AA^* = UU^{-1} = I$, Therefore, $a_{ij} = 0$ for $i > j$, that is, A is upper triangular. Then we have that A^*, the conjugate transpose, is a lower triangular matrix and that $a^*_{ij} = 0$ for $j < i$. Then for the ith row, $A^*A_{i,} = \sum_{j=1}^{m} A_{i,j}A^*_{i,j} = A_{i,i}A^*_{i,i} + 0 + ... + 0 = AA^*1,$. So, $A_{i,j} = 0$ for $j \neq i$, so A is diagonal.
Question 2.

a. Let x be such that $Ax = \lambda x$. Then

$$\begin{aligned} A^{-1}Ax &= A^{-1}(\lambda x) \\ \implies x &= A^{-1}(\lambda x) \\ \implies x &= A^{-1}\lambda x \\ \implies A^{-1}x &= 1/\lambda x \end{aligned}$$

Therefore, $1/\lambda$ is an eigenvalue of A^{-1}.

b. Suppose $AB = \lambda x$. Then $BABx = B\lambda x$. Since linear maps are associative, we have that $(BA)Bx = \lambda(Bx)$, that is, the eigenvalue of BA is the same as AB with a different eigenvector. Therefore, the eigenvalues of AB and BA are the same

c. Since A is real, $A^* = A^T$, therefore $det(A - \lambda I) = det((A - \lambda I)^T)$. Since the characteristic polynomials are the same, the root (eigenvalues) are the same.

Question 3.

a. We have that $A = A^*$, so A is Hermitian. Then $Ax = \lambda x$. Taking the conjugate transpose of this relation, we have that $x^*A^* = \lambda^* x^*$. Then, $x^*A^*x = \lambda^* x^* x \iff x^*\lambda x = \lambda^* x^* x \iff \lambda x^* x = \lambda^* x^* x \implies \lambda = \lambda^* \implies \lambda \in \mathbb{R}$.

b. Let $Au = \lambda u$ and $Av = \tau v$ where $\lambda \neq \tau$. Then consider

$$(Au)^* = (\lambda u)^* \implies u^*A^* = \lambda u^* \implies u^*A = \lambda u^*$$

Since A is Hermitian. Then multiplying on the right by v, we have that

$$\begin{aligned} u^*Av &= \lambda u^* v \\ \tau u^* v &= \lambda u^* v \\ \tau(u,v) &= \lambda(u,v) \end{aligned}$$

Since $\lambda \neq \tau$, we must have that $(u, v) = 0$, that is, the eigenvectors are orthogonal.

Question 4. Proof ($\implies$). Suppose that A is positive-definite and Hermitian. Let v be a a nonzero vector such that $Av = \lambda v$ where $\lambda \in \mathbb{R}$. Then $(Av, v) = (\lambda v)^* v = \lambda v^* v = \lambda \sqrt{(v, v)} > 0$ by assumption. Since v is nonzero, we have that the inner product $(x, x) > 0$ and hence $\lambda > 0$.

Proof ($\impliedby$). Suppose $\lambda > 0$, $\lambda \in \Lambda(A)$. Then we have that for $x \neq 0$, $(x, x) > 0 \iff (\lambda x, x) > 0 \iff (Ax, x) > 0$ therefore A is positive-definite.

Question 5.

a. Consider that we have the following two facts: $Ax = \lambda x$ and $(Ax)^* = (\lambda x)^* \iff x^* A^* = \lambda^* x^*$. Then we have that

$$x^* A^* Ax = \lambda^* x^* Ax$$
$$x^* \underbrace{A^* A}_{I} x = \lambda^* \lambda x^* x$$
$$x^* x = \lambda^* \lambda x$$

But if $\lambda = a + bi$, then $\lambda^* \lambda = a^2 - b^2 = |\lambda|^2$. Therefore, we must have that $|\lambda|^2 = 1 \implies |\lambda| = 1$ by the equality we derived above.

b. This is false, since have that $|A|_F = \sqrt{tr(A^* A)} = \sqrt{tr(I)} = n$ where n is the number of columns or rows in the matrix.

Question 6.

a. We have that

$$\begin{aligned} (Ax)^* x &= (\lambda x)^* x \\ \iff x^* A^* x &= \lambda^* x^* x \\ \iff -x^* Ax &= \lambda^* x^* x \\ \iff -x^* \lambda x &= \lambda^* x^* x \\ \iff -\lambda(x^* x) &= \lambda^*(x^* x) \\ \iff -\lambda &= \lambda^* \end{aligned}$$

Then letting $\lambda = a + bi \iff -\lambda = -a - bi$, we have that $a + bi = -a - bi \iff 2a + bi = -bi$ so $a = 0$, and hence λ is purely imaginary.

b. Suppose A is singular. Then we have that for some $x \neq 0$, $(I - A)x = 0$. But this means that $Ix - Ax = 0 \iff Ax = x$, so x is an eigenvector with eigenvalue 1. This is a contradiction, since the eigenvalues of A are purely imaginary. Hence, $I - A$ is nonsingular

Question 7. Suppose that $Av = \lambda v$ for some nonzero vector v such that $\|v\| = 1$. Then we have that $\|Av\| = \|\lambda v\| = |\lambda|\|v\|$. Also, since

$$\|A\| = \sup_{\|x\|=1} \|Ax\|$$

We have that

$$\begin{aligned}\sup_{\|x\|=1} \|Ax\| &\geq \|Av\| \\ &= \|\lambda x\| = |\lambda|\|x\| = |\lambda|\end{aligned}$$

Then choose $\lambda = \rho(A)$ since this inequality holds for arbitrary eigenvalues. Therefore, $\|A\| \geq \rho(A)$.

Question 8.

a. First, note that since $\|A\|_2 = \sigma_1$, the largest singular value, this implies by definition that $\|A\|_2 = \sqrt{\rho(A^*A)}$ where $\rho(A)$ is the spectral radius of A since $\sigma_i = \sqrt{\lambda_i}$ where λ_i is the ith eigenvalue of A^*A. Also, we have that

$$(A^*A)uv^* = vu^*uv^*v^*u = \|u\|_2^2\|v\|_2^2 v^*u$$

Therefore

$$\sqrt{\rho(A^*A)} = \sqrt{\|u\|_2^2\|v\|_2^2} = \|v\|_2\|u\|_2$$

b. We have that

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |u_i v_j|^2} = \sqrt{\sum_{i=1}^{m} |u_i|^2 \sum_{j=1}^{n} |v_j|^2} = \sqrt{\|u\|_2^2\|v\|_2^2} = \|u\|_2\|v\|_2$$

Question 9.

a. To show this, we show that $\|Qx\|_2 = 1$ for any unitary matrix Q. By definition, we have that

$$\begin{aligned}\|Qx\| &= \sqrt{(Qx, Qx)} \\ &= \sqrt{(Qx)^*Qx} \\ &= \sqrt{x^*Q^*Qx} = \sqrt{x^*x} = \sqrt{(x,x)} = \|x\|_2\end{aligned}$$

Therefore, by the associativity of matrix multiplication we have that

$$\|AQ\|_2 = \sup_{\|x\|_2=1} \|AQx\|_2 = \sup_{\|x\|_2=1} \|A(Qx)\|_2 = \sup_{\|x\|_2=1} \|A\|_2 = \|A\|$$

b. We exploit the fact that $\|A\|_F = \sqrt{tr(A^*A)} = \sqrt{tr(AA^*)}$ from lecture. Then, we have that

$$\begin{aligned}\|AQ\|_F &= \sqrt{tr((AQ)(AQ)^*)} \\ &= \sqrt{tr(AQQ^*A^*)} = \sqrt{tr(AA^*)} = \|A\|_F\end{aligned}$$

and

$$\begin{aligned}\|AQ\|_F &= \sqrt{tr((QA)^*(QA))} \\ &= \sqrt{tr(A^*Q^*QA)} = \sqrt{tr(A^*A)} = \|A\|_F\end{aligned}$$

And the equality $\|AQ\|_F = \|QA\|_F$ comes from transitivity.

Question 10.

a. Lemma: The product of two unitary matrices is unitary. Proof: Let A, B be unitary, so $AA^* = I$ and $BB^* = I$. Then $(AB)^*(AB) = B^*A^*AB = B^*IB = B^*B = I$, so AB is unitary. Now let $A = U_A\Sigma_A V_A^*$ be the SVD of A adn $B = U_B\Sigma_B V_B^*$ be the SVD of B. Then $A = (QV_B)\Sigma_B(V_BQ^*)$. But this is also the SVD for A, so $\Sigma_A = \Sigma_B$ by the uniqueness of the SVD. Since the singular values are the square roots of the diagonal of Σ_A and Σ_B, and these are equal, the singular values of A and B are equal.

b. Consider A and $-A$. Then

Question 11.

a. Since f is differentiable, we have that

$$\kappa(f) = \frac{\|J(x)\|}{\|f(x)\|/\|x\|} = \frac{\|x\|\|J(x)\|}{\|f(x)\|}$$

We have that $J = [1, 1]^T$, and choosing the ∞ norm we have that

$$\kappa(f) = \frac{\max\{x_1, x2\}}{|x_1 + x_2|}$$

Therefore, as $x_1, x_2 \longrightarrow 0$, f becomes ill conditioned since $\kappa(f) \longrightarrow \infty$

b. We have that $J(f) = [x_2, x_1]^T$, therefore the ∞ norm on J is $\|J\|_\infty = \max\{x_1, x2\}$. This gives us that the condition number is

$$\frac{2\max\{x_1, x_2\}}{|x_2x_1|}$$

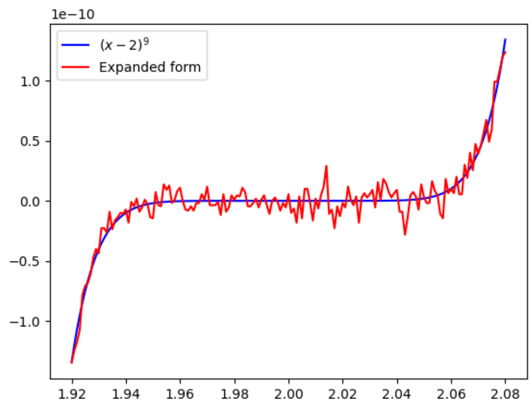
Therefore, as $x_1 \longrightarrow 0$ or $x_2 \longrightarrow 0$, f becomes ill conditioned since $\kappa \longrightarrow \infty$ as the denominator goes to zero.

c. Here, we have that $f'(x) = 9(x-2)^8$, therefore, the condition number is given by

$$\kappa(f) = \frac{9(x-2)^8|x|}{|x-2|^9}$$

So as $x \longrightarrow 2$, f becomes ill conditioned since $k \longrightarrow \infty$

Question 12.



a and b See figure

c. Since we are plotting f around 2, where the conditioning number goes to ∞, we can see the line is perburbed randomly in this domain.