Final: Theory Questions (extra credit)

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Question 1.

a. We define the iteration to be $x^{(k+1)} = Tx^{(k)} + c$. In the Jacobi iteration case, we have that $T = D^{-1}(L+U)$ and $c = D^{-1}b$

b. In the case of Gauss-Seidel, we have that $T = (D-L)^{-1}U$ and $c = (D-L)^{-1}b$.

Question 2. Let x_0 be an arbitary vector. Since A is not defective, consider that A has an eigenbasis and therefore we can write that $x_0 = \sum_{i=1}^{m} a_i v_i$ where $\{v_i\}$, $i = 1, ...m$ are the set of eigenvectors of A. Then consider

$$\begin{aligned} Ax_0 &= A\left(\sum_{i=1}^{m} a_i v_i\right) = \sum_{i=1}^{m} A a_i v_i = \sum_{i=1}^{m} \lambda_i a_i v_i \\ &= \lambda_1 a_1 v_1 + ... + \lambda_m a_m v_m \end{aligned}$$

Now, we can rewrite this as

$$Ax_0 = a_1\lambda_1 \left(v_1 + \frac{a_2}{a_1}\left(\frac{\lambda_2}{\lambda_1}\right) v_2 + ... + \frac{a_m}{a_1}\left(\frac{\lambda_m}{\lambda_1}\right) v_m\right)$$

Therefore

$$A^k x_0 = a_1\lambda_1^k \left(v_1 + \frac{a_2}{a_1}\left(\frac{\lambda_2}{\lambda_1}\right)^k v_2 + ... + \frac{a_m}{a_1}\left(\frac{\lambda_m}{\lambda_1}\right)^k v_m\right)$$

Since $(\lambda_1/\lambda_i)^k \longrightarrow 0$ as $k \longrightarrow \infty$. Therefore, Since we are normalizing by $\|Ax_0\| \longrightarrow \lambda_1$ as $k \longrightarrow \infty$, we have that the power iteration converges to v_1, the eigenvector associated with the largest eigenvalue of A. Now, consider the case where

Consider the case where $|\lambda_1| = ... = |\lambda_r|$ for $1 < r < m$, specifically where $\lambda_1 = -\lambda_j$ for one or more j for $1 < j < r$. Then, we'll have that the power iteration is

$$A^k x_0 = a_1\lambda_1 \left(v_1 + .. + \frac{a_j}{a_1}(-1)^k v_j + ... + \frac{a_m}{a_1}\left(\frac{\lambda_m}{\lambda_1}\right)^k v_m\right)$$

Then we'll have that the iteration oscillates for $k = 2n$ and $k = 2n+1$ where $(-1)^k \in \{-1, 1\}$.

Question 3.

a.

Question 4.

a. First, we prove that H is symmetric. That is, we have that

$$H^T = (I - 2\frac{vv^T}{v^Tv})^T = I^T - 2\frac{(vv^T)^T}{v^Tv} = I - 2\frac{v^T(T)v^T}{v^Tv} = I - 2\frac{vv^T}{v^Tv} = H$$

Therefore, H is symmetric. Now,

$$\begin{aligned}
H^TH &= \left(I - 2\frac{vv^T}{v^Tv}\right) \\
&= I - 4\frac{vv^T}{v^Tv} + 4\frac{vv^Tvv^T}{(v^Tv)^2} \\
&= I - 4\frac{vv^T}{v} + 4\frac{v(v^Tv)v^T}{(v^Tv)^2} = I - 4\frac{vv^T}{v^Tv} + 4\frac{vv^T}{v^Tv} = I
\end{aligned}$$

Therefore, the Householder transformation is both orthogonal and symmetric.

b. We have that

$$\begin{aligned}
Ha &= \left(I - 2\frac{(a+\alpha e_1)(a^T + \alpha e_1^T)}{(a^T + \alpha e_1^T)(a + \alpha e_1)}\right) a \\
&= a - 2\left(\frac{aa^T + \alpha a e_1^T + \alpha e_1 a^T + \alpha^2 e_1 e_1^T}{a^Ta + \alpha a^T e_1 + \alpha e_1^T a + e_1^T e_1}\right) a \\
&= a - 2\left(\frac{a(a^Ta) + \alpha a(e_1^Ta) + \alpha e_1(a^Ta) + \alpha^2(e_1^Ta)}{a^Ta + \alpha a^T e_1 + \alpha e_1^T a + e_1^T e_1}\right) \\
&= a - 2\left(\frac{(a + \alpha e_1)(a^Ta + 2\alpha a^T e_1 + e_1^T e_1)}{(a^Ta + 2\alpha a^T e_1 + e_1^T e_1)}\right)
\end{aligned}$$