0.1 Analyzing the scattering distribution

0.2 Creating a PDF that fits the scattering distribution

0.2.1 Fitting the Gaussian functions

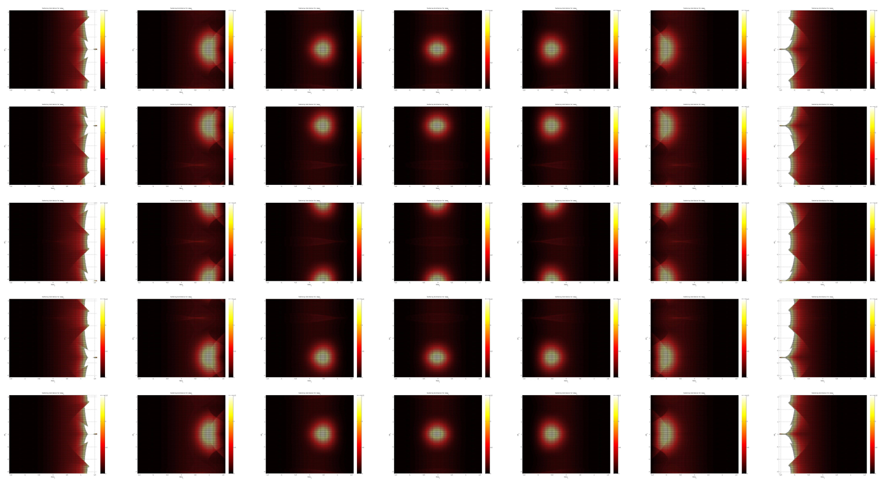

Figure 1: The scattering distribution when no hair strands are between the viewer and the shading point (direct scattering). Each graph represent a different setting for $\omega_r = (\theta_r, \phi_r)$. Going from left to right (7 graphs), the value of θ_r is -90, -60, -30, 0, 30, 60 and 90 degrees. Going from top to bottom (5 graphs), the value of ϕ_r is 180, 90, 0, -90 and -180 degrees. Each of the smaller graphs represent variations in $\omega_i = (\theta_i, \phi_i)$. In this way, the collection of graphs show what incident light directions have a strong contribution for different directions of the viewer ω_r.

The scattering distribution shows which incident light directions ω_i are preferred to be sampled for a given viewer direction ω_r. To be able to sample a direction, three steps should be performed.

1. A probability density function (PDF) should be created that accurately matches the scattering distribution.
2. The PDF needs to be integrated to a normalized cumulative distribution function (CDF).
3. The CDF needs to be inverted to an inverted CDF.

In this section, the creation of the PDF is explained. The PDF should accurately match the scattering distribution. The more accurate it matches

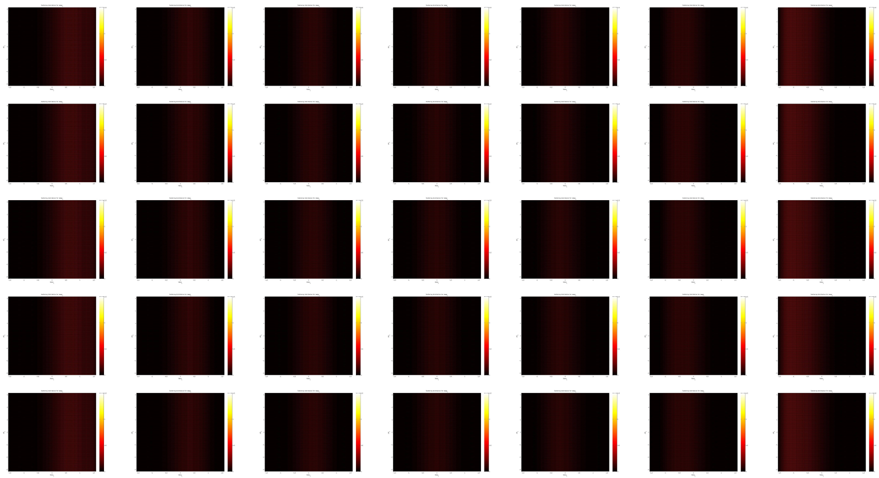

Figure 2: Similarly to figure 1, this graph depicts the same information but now for two hair strands (indirect scattering).

the distribution, the better the sampling will become. Since the scattering distribution differs between directly illuminated strands, versus indirectly illuminated strands, a choice is made to create two different probability density functions that are combined together. These are the direct scattering PDF and the multiple scattering PDF.

The direct scattering is more complex to be matched.

Fitting the direct scattering distribution

The direct scattering distribution is a bit trickier to fit, because of the dependency on both θ_r and ϕ_r. We want to approximate the elliptical highlight, clearly visible in the scattering distribution (figure 1). The idea is to break down the probability density function into two Gaussian functions: one for the variation in the longitudinal direction (θ_i) and one for the azimuthal variation (ϕ_i). The Gaussian functions can then be fitted separately for each direction and combined by multiplication to form the approximation of the scattering distribution.

As the light rotates around the hair fiber, so does the highlight rotate around the fiber. It turns out that the highlight is always opposite to the incident azimuthal direction ϕ_i. Assuming that the azimuthal component of the incident light direction $\phi_i = x°$, then the highlight will always be observed at $\phi_r = x + 180°$. This means that we can combine [TODO: Propose general function]

Figure 1 shows that as the viewer moves around the hair fiber, then the highlight rotates around it as well, but the shape remains the same. Using

ϕ_i / θ_i	$-90°$	$-60°$	$-30°$	$0°$	$30°$	$60°$
$-180°$	$(90°, -178°)$	$(56°, -14°)$	$(27°, -3°)$	$(-2°, -1°)$	$(-31°, -3°)$	$(-59°, -15°)$
$-90°$	$(90°, -91°)$	$(56°, 104°)$	$(27°, 90°)$	$(-2°, 90°)$	$(-31°, 88°)$	$(-59°, 108°)$
$0°$	$(90°, -1°)$	$(56°, -166°)$	$(27°, -176°)$	$(-2°, -178°)$	$(-31°, -176°)$	$(-59°, -162°)$
$90°$	$(90°, 93°)$	$(56°, -102°)$	$(27°, -93°)$	$(-2°, -91°)$	$(-31°, -93°)$	$(-59°, -106°)$
$180°$	$(90°, -178°)$	$(56°, -14°)$	$(27°, -3°)$	$(-2°, -1°)$	$(-31°, -3°)$	$(-59°, -15°)$

Table 1: The locations in the scattering distribution graphs containing the maximum value. The idea is that the maximum value corresponds to the center location of the elliptical highlight as visible in figure 1.

this knowledge, the PDF becomes much simpler to construct. That means that we only need to look at the difference to the gaussian functions when varying θ_r. A change of ϕ_r will thus have no impact on the shape of the gaussians.

To fit the function we consider the center point (θ_i^C, ϕ_i^C) of the elliptical highlight. Instead of choosing the center point of the highlight manually, we choose the center point by finding the maximum value of the distribution. The table below shows the maximum values and their corresponding incident light direction ω_i.

From the center points it is clear that varying ϕ_r does not change the θ_i-angle of the peak. This means that

[TODO: finish this section]

The only thing that is important to consider is changing the longitudinal direction θ_r. What you can see from the data in table 2 are two remarkable observations:

- The shift in the azimuthal direction ϕ_i remains constant at exactly 0 degrees. This means that, whatever the longitudinal angle θ_r is, the highlight stays at exactly $phi_i = \phi_r + 180°$ (directly opposite to the fiber).
- The standard deviation remains constant for the longitudinal case. This means that as the longitudinal angle θ_r increases, the width of the lobe or highlight will not stretch in the longitudinal direction θ_i. As you look at the fiber at increasing angles, then the highlight will not stretch. The center location of the fiber will only shift by a few degrees.

The normalized gaussian function for the direct scattering scenario can now be completed. We still need to know what values to fill in for the standard deviations and the shift of the mean (e.g. the shift of the center of the

θ_r	s_V	σ_V	$\mu - shift_V$	s_H	σ_H	$\mu - shift_H$
$-90°$	-	-	-	-	-	-
$-70°$	3.50	1.10	0°	0.90	0.29	$-5°$
$-60°$	2.40	0.90	0°	0.84	0.29	$-5°$
$-50°$	2.00	0.80	0°	0.80	0.29	$-5°$
$-40°$	1.75	0.70	0°	0.78	0.29	$-4.5°$
$-30°$	1.58	0.62	0°	0.83	0.30	$-4°$
$-20°$	1.55	0.61	0°	0.79	0.29	$-3.5°$
$-10°$	1.52	0.60	0°	0.82	0.30	$-3°$
$0°$	1.50	0.60	0°	0.75	0.27	$-3°$
$+10°$	1.50	0.59	0°	0.79	0.29	$-2°$
$+20°$	1.48	0.60	0°	0.79	0.30	$-1°$
$+30°$	1.58	0.62	0°	0.80	0.30	$-1°$
$+40°$	1.72	0.70	0°	0.78	0.30	0°
$+50°$	2.00	0.80	0°	0.74	0.29	1°
$+60°$	2.40	0.90	0°	0.80	0.29	1°
$+70°$	3.50	1.10	0°	0.89	0.29	2°
$+90°$	-	-	-	-	-	-

Table 2: This table shows the optimal values for the standard deviation σ and the shift of the highlight μ-shift, found by fitting the gaussian functions to the original scattering distribution.Longitudinal angles above $\pm 70°$ are ignored, because for these orientations the gaussian function is hard to match with the distribution.

highlight). The shift of the lobe for the azimuthal variation remains constant at 0° and the standard deviation of the longitudinal variation remains a constant value of approximately 0.30.

$$\mu s_V = 0^\circ \quad (1)$$
$$\sigma_H = 0.29 \quad (2)$$

The standard deviation of the azimuthal variation and the shift of the lobe for the longitudinal variation are plotted in figure **??** in green. This figure also contains plots for functions that approximate the data. These functions are found easily and are as follows:

$$\sigma_V(\theta_r) = \frac{1}{4}\theta_r^4 + 0.59 \quad (3)$$
$$\mu s_H(\theta_r) = 3\theta_r - 2 \quad (4)$$

The direct scattering gaussian function

Put together, the normalized gaussian function for the direct scattering equation $g(\mu, \sigma)$ can now be constructed.

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot \exp\left(-\frac{(\theta-\mu)^2}{2\sigma^2}\right) \begin{cases} \theta = \theta_i + \theta_r \\ \sigma(\theta_r) = \frac{1}{4}\theta_r^4 + 0.59 \\ \mu(\theta_r) = 3\theta_r - 2 \end{cases} \quad (5)$$

$$g(\omega_i, \omega_r) = g_A() \cdot g_B() \quad (6)$$

Fitting the standard deviation and peak shift

After matching the graph for different orientations of $\omega_r = (\theta_r, \phi_r)$ we obtain a function definition that more or less fits the scattering distribution. The only variation that still needs to encapsulated is the optimal setting for the standard deviation σ_H and σ_V and the shift of the peak μs_H and μs_V. The idea is to propose a function that encapsulates the variation for every combination of (θ_r, ϕ_r). For this, we need to fit the scattering distribution for smaller increments of θ_r so that a function can be found. Table **??** shows the optimal settings for σ and μs, the same as done in table **??**.

Fitting the multiple scattering distribution

For the multiple scattering scenario, the likelihood of direction ω_i to be sampled is only dependent on the longitudinal directions θ_r and θ_i. The azimuthal variation has no impact on the response, as can be seen in figure 2.

The shape looks a lot like a Gaussian function and therefore the distribution is matched by using a single normalized Gaussian function.

In order to fit the Gaussian functions to the scattering distribution, we will need to adjust the mean and the standard deviation to find an accurate match. Table 3 shows optimal values for different orientations of the viewer θ_r. As light scatters through multiple fibers, light is absorbed leading to a smaller response. For this reason, a scale factor is introduced so that the normalized Gaussian function can be scaled up or down to match the scattering distribution. Eventually the scale factor is ignored in the PDF, because for the sampling process, the scale factor is not relevant. Since we need a normalized PDF, we can ignore the scale factor and still have a PDF that matches the shape of the scattering distribution.

# strands	θ_r	s	σ	μ-shift	# strands	θ_r	s	σ	μ-shift
1	-90	1	0.75	$0°$	2	-90	$\frac{4}{10}$	0.90	$0°$
	-60	$\frac{10}{67}$	0.55	$-34.5°$		-60	$\frac{100}{965}$	0.55	$-38°$
	-30	$\frac{1}{9}$	0.49	$-20°$		-30	$\frac{1}{12}$	0.49	$-20°$
	0	$\frac{2}{19}$	0.49	$-1.2°$		0	$\frac{10}{125}$	0.49	$-1.2°$
	30	$\frac{1}{9}$	0.49	$17.5°$		30	$\frac{1}{12}$	0.49	$17.5°$
	60	$\frac{10}{67}$	0.55	$29°$		60	$\frac{100}{965}$	0.55	$34°$
	90	2	0.65	$0°$		90	$\frac{4}{10}$	0.90	$0°$
4	-90	$\frac{1}{15}$	0.51	$-61°$	8	-90	$\frac{1}{60}$	0.41	$-70°$
	-60	$\frac{10}{205}$	0.47	$-43°$		-60	$\frac{1}{70}$	0.38	$-49°$
	-30	$\frac{1}{25}$	0.41	$-22°$		-30	$\frac{1}{73}$	0.38	$-25°$
	0	$\frac{10}{255}$	0.41	$-1.2°$		0	$\frac{1}{78}$	0.36	$-1.2°$
	30	$\frac{1}{25}$	0.41	$20°$		30	$\frac{1}{75}$	0.38	$23°$
	60	$\frac{10}{205}$	0.47	$40°$		60	$\frac{1}{73}$	0.38	$46°$
	90	$\frac{1}{14}$	0.52	$54°$		90	$\frac{1}{63}$	0.41	$68°$

Table 3: For every longitudinal angle θ_r and number of hair strands, the optimal standard deviation σ, shifted mean (μ) and scale factor is displayed so that the normalized Gaussian function matches the scattering distribution. The scale factor is eventually ignored, because it is not important for the PDF.

Figure 3 shows the fitting of the Gaussian functions for multiple scattering through 1, 2, 4 and 8 fibers. As can be seen is that the Gaussian functions match pretty well. As θ_r increases, the distribution becomes a bit distorted and deviates slightly from the generated gaussian function. We deliberately do not take this behaviour into account, although it is of course best to match the underlying distribution perfectly. In this case, it is a compromise between having an efficient function that matches the underlying PDF pretty well and is easy to integrate and evaluate, versus a complex function that is hard to integrate and expensive to evaluate. If the function becomes too expensive to evaluate, then uniform sampling might be more efficient after all.

0.2.2 Comparing the scattering distribution with the generated distribution

[TODO: show the effect of a generated distribution and compare it to the original distribution]

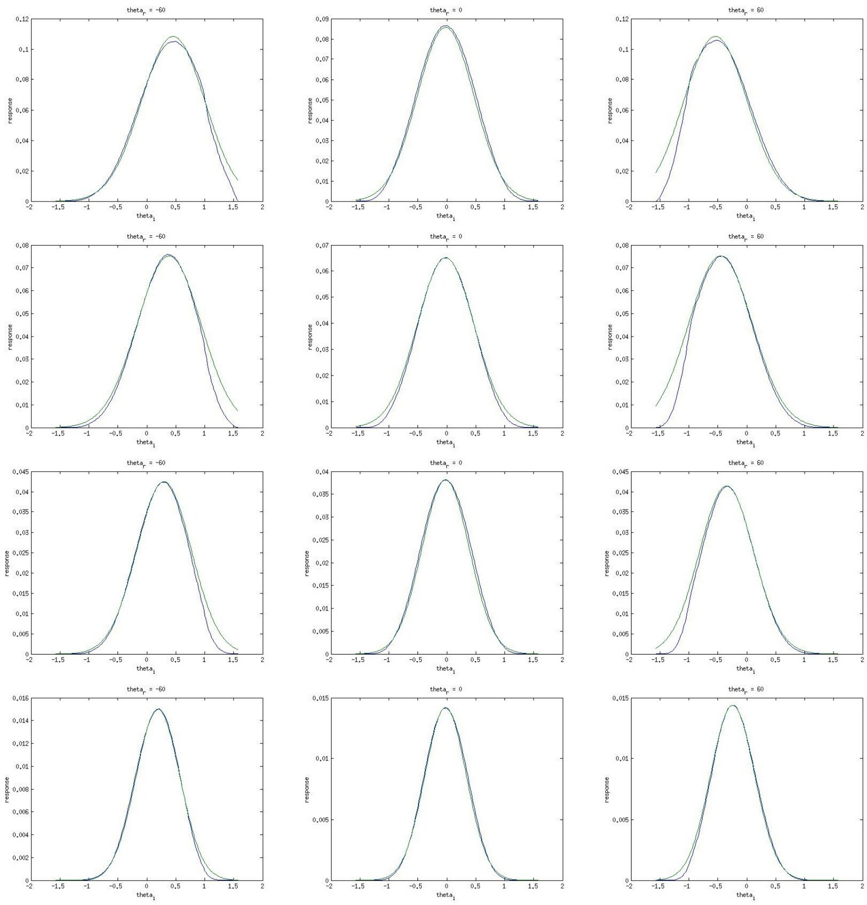

Figure 3: A 2D slice of the scattering distribution for different number of hair strands that the light scatters through versus different θ_r (from left to right: -60, 0 and 60 degrees). The top row corresponds to 1, the second to 2, the third to 4 and the last row to 8 number of hair strands through which the light scatters. The scattering distribution is shown in blue and the generated gaussian function is green.