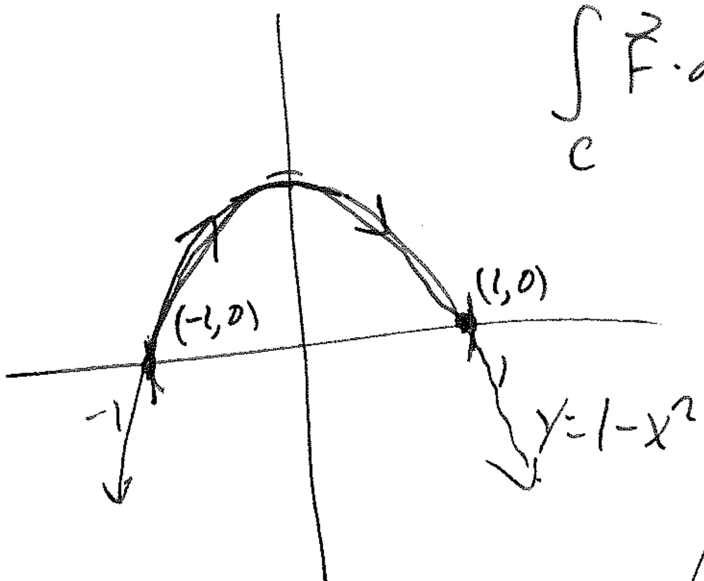
$$\vec{F} = (x^2 - y)\hat{\imath} + 2x\hat{\jmath}$$

not a loop, no $\oint$.

$$\int_C \vec{F} \cdot d\vec{r}$$



$x(t)$ $y(t)$

$$\vec{r}(t) = (t,\ 1 - t^2)$$

$$-1 \le t \le 1$$

x y

$$\frac{d\vec{r}}{dt} = (1, -2t)$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=-1}^{1} (x^2 - y, 2x) \cdot (1, -2t)\, dt$$

$$= \int_{t=-1}^{1} (t^2 - (1 - t^2),\ 2t) \cdot (1, -2t)\, dt$$

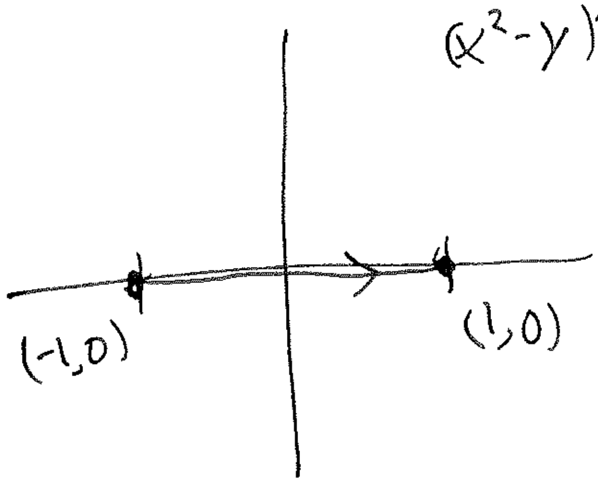
$$= \int_{t=-1}^{1} (2t^2 - 1) + (-4t^2)\, dt = \int_{t=-1}^{1} \cancel{2t^2 - 4t^2\, dt}$$

$$-2t^2 + 1\, dt$$

$= \cancel{\int 2t^3} \qquad = \int_{t=-1}^{1} -2t^2 - 1 \, dt$

$$= \left(-\frac{2t^3}{3} - t \right) \Bigg|_{-1}^{1} = \left(-\frac{2}{3} - 1 \right) - \left(\frac{2}{3} + 1 \right)$$

$$= -\frac{10}{3}.$$



$(x^2 - y)\hat{i} + 2x\hat{j}$ $\qquad \int_C \vec{F} \cdot d\vec{r}$

$$\vec{r}(t) = (\underset{\downarrow}{\overset{x}{t}}, \underset{\downarrow}{\overset{y}{0}}) \qquad -1 \le t \le 1$$

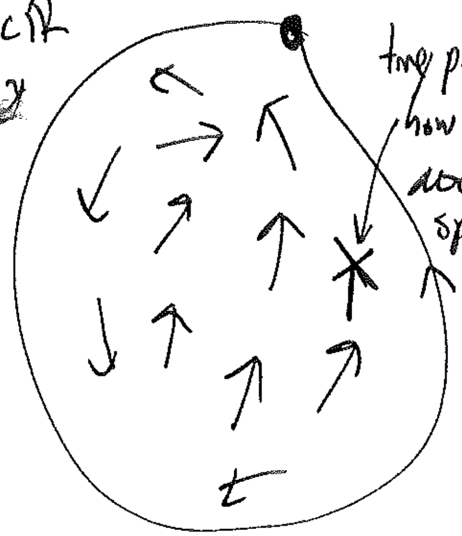
$$\frac{d\vec{r}}{dt} = (1, 0)$$

$$\int \vec{F} \cdot d\vec{r} = \int_{t=-1}^{1} (t^2, 2t) \cdot (1, 0) \, dt = \int_{t=-1}^{1} t^2 \, dt$$

$$= \left(\frac{t^3}{3} \right) \Bigg|_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3} \right) = \frac{2}{3}.$$

Green's theorem
- tangential
- normal

$U \subset \mathbb{R}^2$



tiny propeller: how fast does it spin? (counterclockwise = +)

$\vec{F}(x,y)$ = vector field on U

$= M(x,y)\,\hat{\imath} + N(x,y)\,\hat{\jmath}$

e.g. $\vec{F} = \underbrace{xy}_{M}\,\hat{\imath} + \underbrace{\cos x}_{N}\,\hat{\jmath}$

$$\operatorname{curl} \vec{F} = N_x - M_y$$

(2D)

$\operatorname{curl}\vec{F} = (-\sin x) - x$

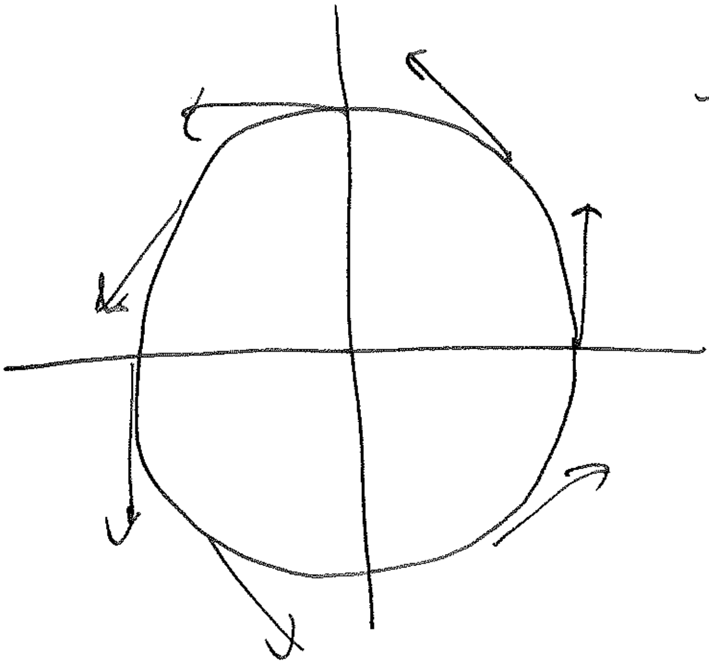
curl $\vec{F}$ is a function! (2D)

= how fast does propellor spin? at (x,y)

Other objecti

$\oint_C \vec{F} \cdot d\vec{r}$ = how much work does $\vec{F}$ do on you when you go around C?

= how much help do you get from current in lake when swimming a lap around C?



$$\vec{F}(x,y) = -y\hat{\imath} + x\hat{\jmath}$$

To compute $\oint_C \vec{F}\cdot d\vec{r}$:

1) Parametrize path: $\vec{r}(t) = (x(t), y(t))$

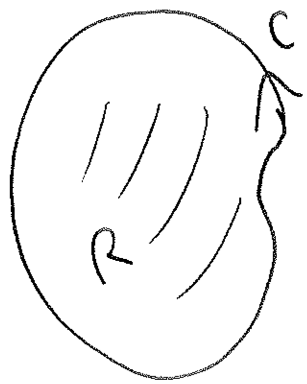
2) Compute $\dfrac{d\vec{r}}{dt} = (x'(t), y'(t))$

3) Integrate

substitute in $x(t)$, $y(t)$ to x, y.

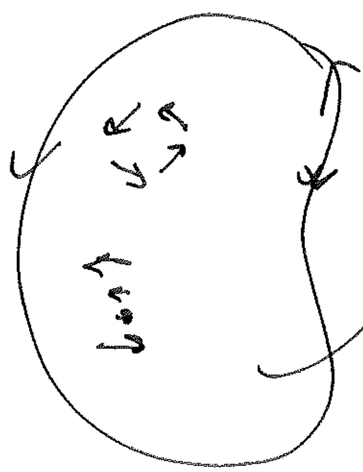
vector depending on t

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt = \dots$$



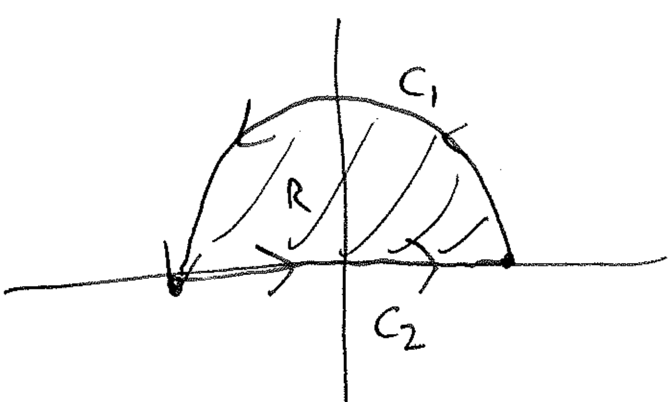
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\,dA$$

Green's theorem, tangential



imagine current $\vec{F}$ has curl $\vec{F} > 0$

A test: $\vec{F} = (x^2 - y)\hat{\imath} + 2x\hat{\jmath}$



$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\,dA$$

Let's see if the sides match.

C = loop: C_1, then C_2.

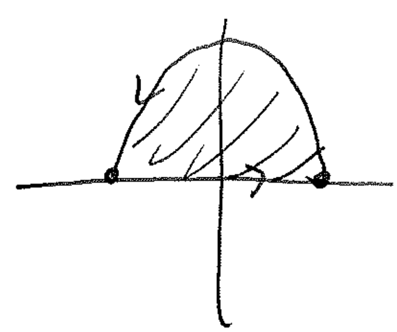
$$\oint_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$$

$$= \left(\frac{10}{3}\right) + \left(\frac{2}{3}\right) = 4$$

flip the sign, other direction!

$$\iint_R \text{curl}\,\vec{F}\,dA \qquad \vec{F} = \overbrace{(x^2-y)}^{M}\hat{\imath} + \overbrace{2x}^{N}\hat{\jmath}$$

$$\text{curl}\,\vec{F} = N_x - M_y = 2-(-1) = 3$$



$$\int_{x=-1}^{1}\int_{0}^{1-x^2} 3\,dy\,dx = \int_{x=-1}^{1} 3(1-x^2)\,dx$$

$$= \int_{-1}^{1} 3-3x^2\,dx = \left(3x-\frac{3x^3}{3}\right)\Big|_{-1}^{1} = (3-1)-(-3+1)$$

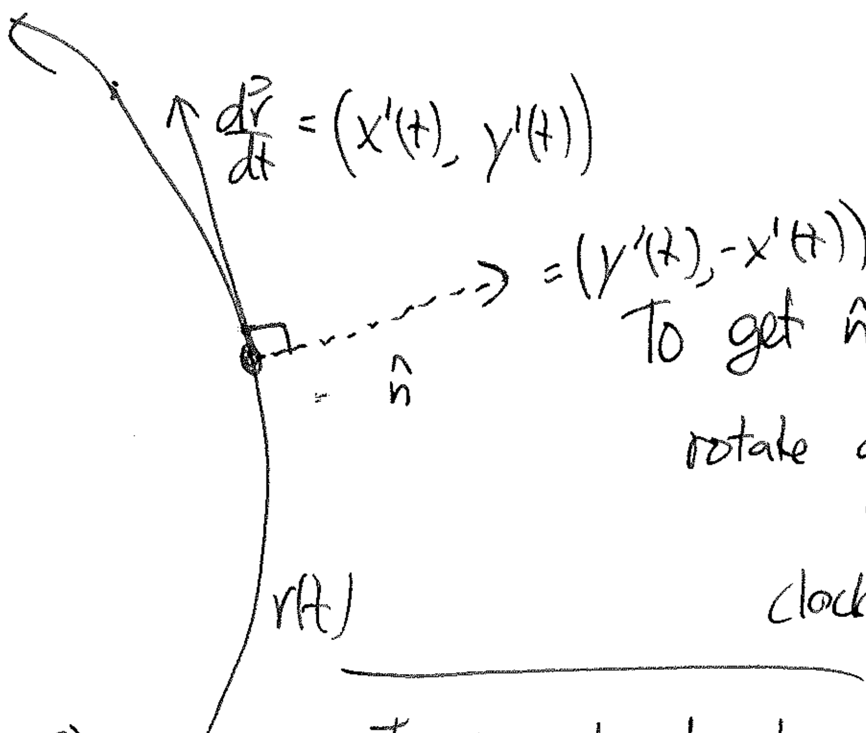
$$= 4$$

How to compute $\oint_C \vec{F}\cdot\hat{n}\,ds$?

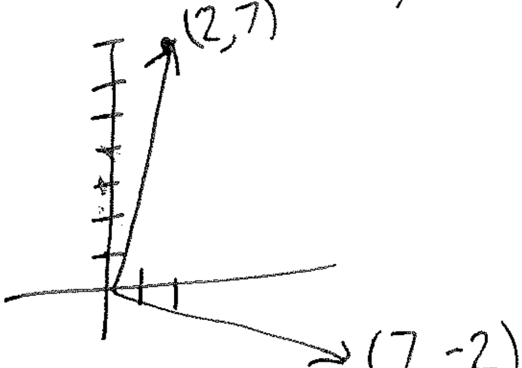
Almost the same as $\oint \vec{F}\cdot d\vec{r}$.

But instead of tangent vector $\frac{d\vec{r}}{dt}$ to our path,

we want a normal vector $\hat{n}$.



To get $\hat{n}$, just rotate $\frac{d\vec{r}}{dt}$ by 90° clockwise!

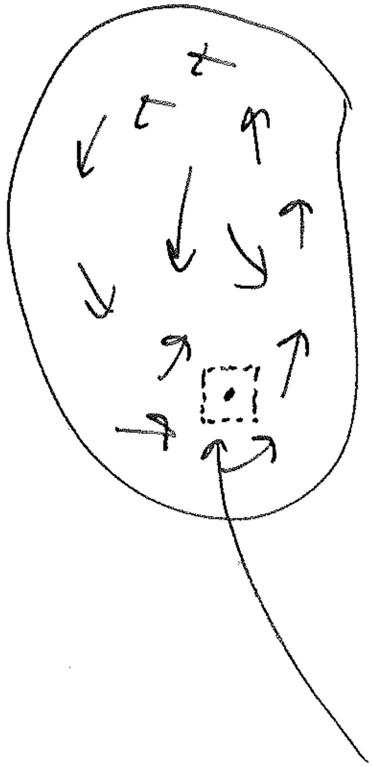


To compute integral: same as before, but replace

$\frac{d\vec{r}}{dt} = (x'(t), y'(t))$ with $\hat{n} = (y'(t), -x'(t))$.

Normal form.

$$\vec{F}(x,y) \text{ `` } M(x,y)\hat{\imath} + N(x,y)\hat{\jmath} \quad \text{vector field}$$



think: currents in a lake, but with spring at the bottom.

$$\operatorname{div} \vec{F} = M_x + N_y$$

physical meaning: imagine you "fence" a small square around (x,y)

$$\operatorname{div} \vec{F} = \frac{\text{water flowing out of box}}{\text{area}} = \text{intensity of spring at } (x,y)$$

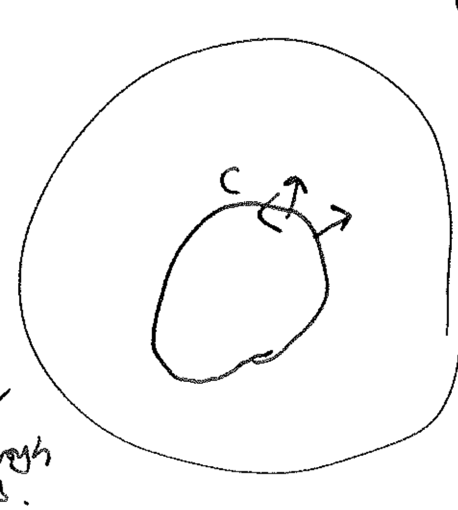
$$\oint_C \vec{F} \cdot \hat{n}\, ds = \text{flux across } C = \text{how much water flows across } C.$$

Green's thm

total water flowing out of fenced area

$$\oint_C \vec{F} \cdot \hat{n}\, ds = \iint_R \operatorname{div} \vec{F}\, dA$$

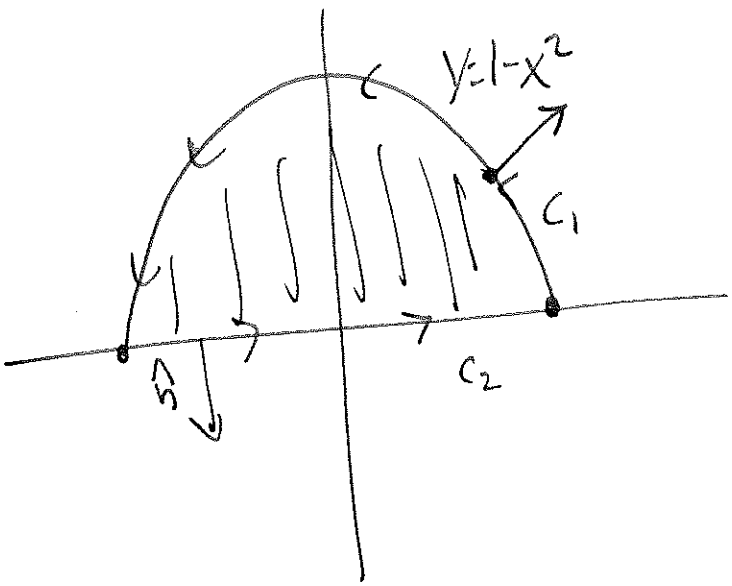
total water entering through springs.



$$\vec{F} = (x^2 - y)\hat{i} + 2x\hat{j}$$

$$\oint \vec{F}\cdot\hat{n}\,ds = \iint_R \operatorname{div} F\, dA$$

$=$



$$\vec{r}(t) = (-t,\ 1-(-t)^2)$$

$$= (-t,\ 1-t^2) \qquad -1 \le t \le 1$$

$$\frac{d\vec{r}}{dt} = (-1,\ -2t)$$

Flux on C_1:

$$\hat{n} = (-2t,\ 1)$$

$$\oint_C \vec{F}\cdot\hat{n}\,ds = \int_{t=-1}^{1} (x^2 - y,\ 2x)\cdot(-2t,\ 1)\,dt$$

$$= \int_{t=-1}^{1} \left((-t)^2 - (1-t^2),\ -2t\right)\cdot(-2t,\ 1)\,dt$$

$$= \int_{t=-1}^{1} (2t^2 - 1,\ -2t)\cdot(-2t,\ 1)\,dt = \int_{t=-1}^{1} (-4t^3 + 2t) + (-2t)\,dt$$

$$= \int_{t=-1}^{1} -4t^3\,dt = 0.$$

Flux on C_2:

$$\vec{r}(t) = (t, 0)$$

$$\frac{d\vec{r}}{dt} = (1, 0) \quad (y, -x)$$

$$\hat{n} = (0, -1)$$

$$\int_{C_2} \vec{F} \cdot \hat{n}\, ds = \int_{t=-1}^{1} (x^2 - y, 2x) \cdot (0, -1)\, dt = \int_{t=-1}^{1} (t^2 - 0, 2t) \cdot (0, -1)\, dt$$

$$= \int_{t=-1}^{1} -2t\, dt = 0!$$

$$\oint_C \vec{F} \cdot \hat{n}\, ds = 0 + 0 = 0.$$

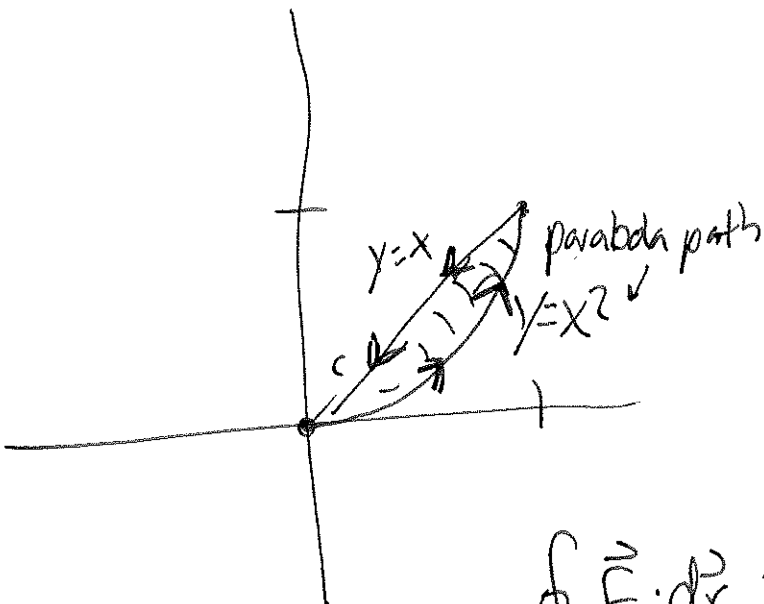
Right hand side

$$\iint_R \operatorname{div} \vec{F}\, dA = \int_{x=-1}^{1} \int_{y=0}^{1-x^2} 2x\, dy\, dx = \int_{x=-1}^{1} 2x(1 - x^2)\, dx \overset{\text{odd}}{=} 0$$

$$\vec{F} = (x^2 - y, 2x) \rightsquigarrow \operatorname{div} \vec{F} = 2x + 0 = 2x$$

Compute both sides:

$$\vec{F} = (xy)\hat{i} + y^2\hat{j}$$



$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\,dA$$