Negative Answers To Some Positivity Questions

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Nefness in families

Let $\mathcal{X} \to (\mathbb{P}^2)^{10}) \setminus \Delta$ be the family of blow-ups of $\mathbb{P}^2$ at 10 distinct points. There exists an $\mathbb{R}$-divisor D on $\mathcal{X}$ such that $D_{\mathbf{p}}$ is nef on $X_{\mathbf{p}}$ for very general $\mathbf{p}$, but is not nef for $\mathbf{p}$ in countably many codimension-1 subvarieties of the base. Thus nefness is not an open condition under deformation.

Sequences of Cremona maps

- Suppose C is a curve in $\mathbb{P}^2$ with degree d and multiplicities m_1, m_2, and m_3 at three points.
- The strict transform of C under a Cremona transformation centered at those points has degree $2d - m_1 - m_2 - m_3$ and multiplicities $m_1 = d - m_2 - m_3, \ldots$.
- Cremona transformation + permutation of the points generates an action of a Coxeter group on the space of k-tuples of points in $\mathbb{P}^2$.
- Example element: move the last three points to beginning of list, then make a Cremona transformation at these.
- If $\mathbf{p}$ is a k-tuple, this induces a map $M_\sigma^{\mathbf{pq}} : N^1(X_{\mathbf{p}}) \to N^1(X_{\mathbf{q}})$, where $\mathbf{q}$ is a new configuration.
- $M_\sigma^{\mathbf{pq}}$ preserves the nef cone.

Nefness of an eigenvector

- $\Phi : N^1(X_{\mathbf{q}}) \to N^1(X_{\mathbf{p}})$ equating d and m_i preserves the nef cone if $\mathbf{p}$ is very general, *but not otherwise*.
- If $k \geq 10$, then $M_\sigma = \Phi \circ M_\sigma^{\mathbf{pq}}$ has an eigenvalue $\lambda > 1$.
- The dominant eigenvector of M_σ is nef for very general $\mathbf{p}$. In the example,
$$D_\lambda \approx h - 0.451e_1 - 0.440e_2 - 0.408e_3 - \cdots.$$
- This is not nef if:
  - p_1, p_2, and p_3 are collinear
  - $p_1, \ldots, p_6$ lie on a conic
  - $\cdots$
  - There exists a curve of class $M_\sigma^n(h - e_1 - e_2 - e_3)$ on $X_{\mathbf{p}}$ (a codimension-1 condition on the base for each n)
- The reason is simple: if there is such a curve,
$$D_\lambda \cdot C = \frac{1}{\lambda^n}(M_\sigma^n D_\lambda) \cdot (M_\sigma^n(h - e_1 - e_2 - e_3)) = \frac{1}{\lambda^n} D_\lambda \cdot (h - e_1 - e_2 - e_3) < 0.$$

[1] Arthur B. Coble, *Algebraic geometry and theta functions*, American Mathematical Society, 1929.

[2] I. Dolgachev and D. Ortland, *Point sets in projective spaces and theta functions*, Astérisque (1988).

[3] Curtis T. McMullen, *Dynamics on blowups of the projective plane*, Publ. Math. IHES (2007).

The diminished base locus

Let X be the blow-up of $\mathbb{P}^3$ at 9 very general points. There exists a pseudoeffective $\mathbb{R}$-divisor D which has negative intersection with an infinite sequence of curves C_n, which are Zariski dense on X. In particular $\mathbf{B}_-(D)$ is not Zariski closed.

A Cremona transformation

- The standard Cremona transformation on $\mathbb{P}^3$ is defined by $[W;X,Y;Z] \mapsto [W^{-1};X^{-1};Y^{-1};Z^{-1}]$
- Has a resolution

$$\begin{tikzcd} & Y \arrow[dl, "p"'] \arrow[dr, "p'"] & \\ X \arrow[rr, dashed, "\overline{\mathrm{Cr}}"] \arrow[d, "\pi"'] & & X' \arrow[d, "\pi'"] \\ \mathbb{P}^3 \arrow[rr, dashed, "\mathrm{Cr}"] & & \mathbb{P}^3 \end{tikzcd}$$

where π and π' are the blow-up of $\mathbb{P}^3$ at four points, and $\overline{\mathrm{Cr}}$ is the flop of the strict transforms of the six lines through two points.

Eigenvector intersections

- As in the first example, we can repeatedly make a Cremona transformation at the first four points and then move the last four to the front.
- The induced action $M_\sigma : N^1(X_{\mathbf{p}}) \to N^1(X_{\mathbf{p}})$ has an eigenvalue bigger than 1 as long as at least 9 points; let D_λ be the eigenvector.
- If C_0 is the line between p_1 and p_2, its strict transforms C_n are disjoint from the indeterminacy loci. Thus C_n is a curve of class $N_\sigma^n([C_0])$.
- We have
$$D_\lambda \cdot C_n = \left(\frac{1}{\lambda^n} M_\sigma^n D_\lambda\right) \cdot (N_\sigma^n C_0) = \frac{1}{\lambda^n} D_\lambda \cdot C_0 < 0.$$
- The diminished base locus is $\mathbf{B}_-(D) = \bigcup_{A \text{ ample}} \mathbf{B}(D + A)$, a countable union of subvarieties.
- Since $D_\lambda \cdot C_n < 0$, $C_n \subset \mathbf{B}_-(D_\lambda)$, and D_λ is a countable union of curves.
- Can construct a similar 4-dimensional example with D' big and X' a $\mathbb{P}^1$-bundle over X. Here $\mathbf{B}_-(D')$ is an infinite set of curves, dense in a codimension-1 subvariety.

[1] Turgay Bayraktar, *Green currents for meromorphic maps of compact Kähler manifolds*, pre-print (2012).

[2] Ein, Lazarsfeld, Mustaţă, Nakamaye, and Popa, *Asymptotic invariants of base loci*, Ann. Inst. Fourier (Grenoble) **56** (2006).

[3] Antonio Laface and Luca Ugaglia, *Elementary (-1)-curves of $\mathbb{P}^3$*, Comm. Algebra **35** (2007).

Fourier-Mukai partners

There is an infinite set W of configurations of 8 points in $\mathbb{P}^3$ such that if $\mathbf{p}$ and $\mathbf{q}$ are distinct elements of W, then $D^b\operatorname{Coh}(\mathrm{Bl}_{\mathbf{p}}(\mathbb{P}^3)) \cong D^b\operatorname{Coh}(\mathrm{Bl}_{\mathbf{q}}(\mathbb{P}^3))$, but $\mathrm{Bl}_{\mathbf{p}}(\mathbb{P}^3)$ and $\mathrm{Bl}_{\mathbf{q}}(\mathbb{P}^3)$ are not isomorphic.

Reconstruction problems

- The derived category $D(X) = D^b\operatorname{Coh}(X)$ is a fairly strong invariant of X.
- Two varieties with equivalent derived categories have the same dimension, Kodaira dimension, etc.
- X and Y are said to be *Fourier-Mukai partners* if $D(X) \cong D(Y)$.
- Question (Kawamata): is the number of Fourier-Mukai partners of X always finite?
- Yes, for curves, surfaces, abelian varieties, toric varieties, Fano varieties, varieties with K_X ample.
- The example shows this is not the case for all threefolds!

Cremona orbits

- If $\mathbf{p}$ is a configuration of 8 points in $\mathbb{P}^3$, we can make a Cremona transformation centered at the first four.
- This gives a new configuration of points $\mathbf{q}$, and the blow-ups differ by a rational map $\mathrm{Bl}_{\mathbf{p}}(\mathbb{P}^3) \dashrightarrow \mathrm{Bl}_{\mathbf{q}}(\mathbb{P}^3)$ which flops six curves.
- Bondal-Orlov: if $X \dashrightarrow X^+$ is the flop of a rational curve with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, then $D(X) \cong D(X^+)$.
- The claimed example then follows from three observations:

1. $\mathrm{Bl}_{\mathbf{p}}(\mathbb{P}^3)$ and $\mathrm{Bl}_{\mathbf{q}}(\mathbb{P}^3)$ are isomorphic if and only if $\mathbf{p}$ and $\mathbf{q}$ coincide, up to permutation and an automorphism of $\mathbb{P}^3$.
2. If $\mathbf{q}$ can be obtained from $\mathbf{p}$ by a sequence of standard Cremona transformations, then $\mathrm{Bl}_{\mathbf{p}}(\mathbb{P}^3)$ and $\mathrm{Bl}_{\mathbf{q}}(\mathbb{P}^3)$ are connected a sequence of flops of rational curves with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, and so $D(X_{\mathbf{p}}) \cong D(X_{\mathbf{q}})$.
3. The orbit of a sufficiently general configuration $\mathbf{p}$ of 8 points under standard Cremona transformations is infinite.

[1] A. Bondal and D. Orlov, *Derived categories of coherent sheaves*, Proceedings of the ICM, 2002.

[2] Yujiro Kawamata, *D-equivalence and K-equivalence*, J. Differential Geom. **61** (2002).

[3] Mathieu Anel and Bertrand Toën, *Dénombrabilité des classes d'équivalences dérivées de variétés algébriques*, J. Algebraic Geom. **18** (2009).

Multiplicities on CY3's

Let $\pi : X \to S$ be the versal deformation space of a fiber of Kodaira type I_2. There exists a π-pseudoeffective Cartier divisor D on X and a curve Γ for which $\sigma_\Gamma(D; X/S)$ is infinite.

Asymptotic multiplicities

- If D is big, set $\sigma_\Gamma(D) = \inf_{\substack{D' \equiv_{\mathbb{R}} D \\ D' \geq 0}} \{\mathrm{mult}_\Gamma(D')\}$
- This extends to the pseudoeffective boundary as
$$\sigma_\Gamma(D) = \lim_{\epsilon \to 0} \sigma_\Gamma(D + \epsilon A).$$
- This limit is finite and depends only on the numerical class of D.
- Analogous definition in the relative setting – but the example shows this limit can be infinite!

Basic example

- $\pi : X \to S$ has central fiber the union of two smooth rational curves C_1 and C_2 meeting transversally at two points. The base S is two dimensional, one for each node.
- $N^1(X/S)$ is spanned by C_1 and C_2.
- There exists an infinite sequence of flops of curves in central fiber, giving infinitely many chambers in $\overline{\mathrm{Mov}}(X/S) = \bigcup_i \mathrm{Nef}(X_i/S)$.

Multiplicities under a flop

- If we know $D \cdot C_1$, $D \cdot C_2$, $\mathrm{mult}_{C_1}(D)$, and $\mathrm{mult}_{C_2}(D)$, we can find how all of these change when taking strict transform under the flop of C_1.
- Let D_0 be ample, D_n its transform under n flops.

n	$D_n \cdot C_1$	$D_n \cdot C_2$	$\mathrm{mult}_{C_1} D_n$	$\mathrm{mult}_{C_2} D_n$
0	1	1	0	0
1	3	-1	0	1
		$\cdots$		
n	$2n+1$	$-2n+1$	$\frac{n(n-1)}{2}$	$\frac{n(n+1)}{2}$

- Then compute
$$\sigma_C(D) = \lim_{n\to\infty} \mathrm{mult}_C(D + \frac{1}{2n} D_0) = \lim_{n\to\infty} \frac{1}{2n} \mathrm{mult}_C D_n = \lim_{n\to\infty} \frac{n-1}{4} = \infty.$$

[1] Yujiro Kawamata, *On the cone of divisors of Calabi-Yau fiber spaces*, Internat. J. Math. **8** (1997).

[2] Noboru Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, vol. 14, 20043

[3] Miles Reid, *Minimal models of canonical 3-folds*, Algebraic varieties and analytic varieties.

Zariski decompositions

- *The $\mathbb{R}$-divisor D of column two does not admit a weak Zariski decomposition $f^*D = P + N$ (P nef, N effective) on any birational model $f : Y \to X$.*
- *The divisor D of column four does not admit a relative weak Zariski decomposition over S.*

These follow respectively from the fact that D is negative on a dense set of curves, and the fact that $\sigma_C(D; X/S) = \infty$.

A second example

- Let X be a complete intersection of type $(1,1)$, $(1,1)$, $(2,2)$ in $\mathbb{P}^3 \times \mathbb{P}^3$.
- This is a Calabi-Yau threefold of Picard number 2 with infinitely many minimal models.
- Multiplicities can be computed by the same strategy as before.

Let D be a divisor on the pseudoeffective boundary of X and Γ be a flopping curve. Then $\sigma_\Gamma(mD) - \sigma_\Gamma(mD + A)$ is not bounded in m.

- This shows that although σ_Γ must have a finite limit, it may still increase very fast: this function is not Lipschitz at the boundary.

Two conjectures

Conjecture A. If X is a smooth threefold and D is a pseudoeffective $\mathbb{R}$-divisor on X with $\mathbf{B}_-(D)$ closed, then D admits a Zariski decomposition in the sense of Nakayama.

Conjecture B. If X is a terminal threefold, the number of K_X-negative extremal rays on $\overline{\mathrm{NE}}(X)$ is finite.

Question. Does the D above admit a weak Zariski decomposition?

- If no, then Conjecture A is false.
- If yes, then Conjecture B is false.

(Reason: if $f : Y \to X$ is birational, and H is ample on Y, the K_Y-MMP with scaling by H ends up at the model of X on which f_*H is nef. If $f^*D = P + N$, then taking $H = P + tA$ would yield infinitely many models as possible MMP outcomes, all X_i whose chambers accumulate at D.)

Confession. I don't know! (my guess: no)

[1] Caucher Birkar, *On existence of log minimal models and weak Zariski decompositions*, (2009).

[2] Lazić and Peternell, *On the cone conjecture for Calabi-Yau manifolds with Picard number two.*

[3] Keiji Oguiso, *Automorphism groups of Calabi-Yau manifolds of Picard number two*, pre-print (2012).

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