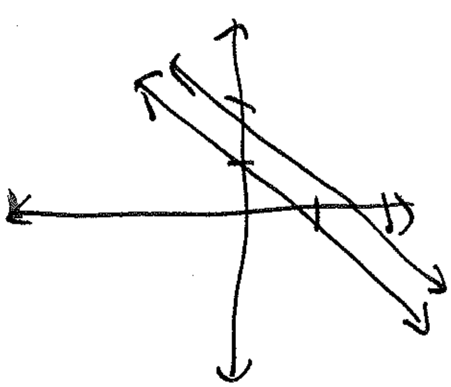
Solving systems of linear equations in many variables (Gaussian elimination).

2 linear equations in two variables.

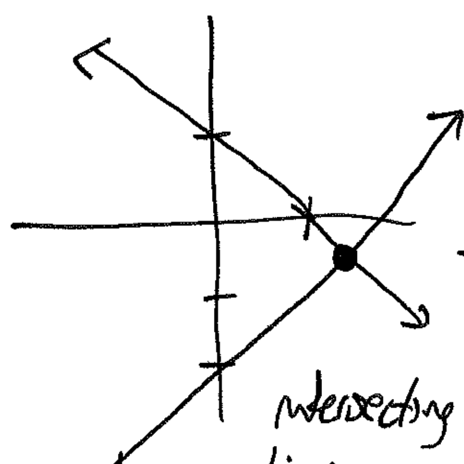
$3x+5y=-2, \quad 6x-7y=1$

~~Usually only~~ How many solutions?
three options

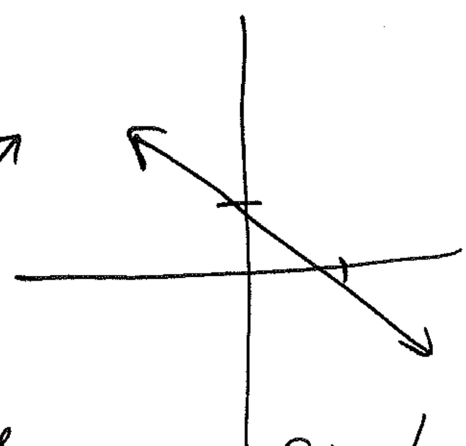
No solutions:	Just one: (common)	Infinitely many:
$x+y=1$	$x+y=1$	$x+y=1$
$2x+2y=3$	$x-y=2$	$2x+2y=2$



parallel lines



intersecting lines



same line twice

3 variables, 2 equations.

How many solutions?

$x+2y+3z=7$

(intersections of two planes)

No solutions

$x+y+z=1$

$x+y+z=3$

parallel planes!

Infinitely many solutions

$x=0$

$y=0$ $\longrightarrow$ $(0,0,z)$ solution

(two planes intersecting in a line)

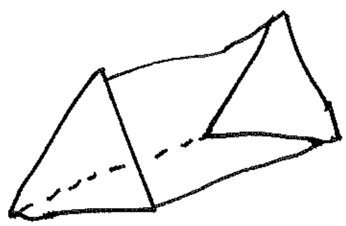
Can't have just one solution!

3 variables, 3 equations.

(three intersecting planes)

Zero solutions | One solution (meet in a pt) | infinitely many

possible now even if they're not parallel!



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m equations, n variables? ← Goal: find a foolproof method.

$m=n$: 0, 1, or ∞ solutions ("usually" 1)

$m>n$: 0, 1, or ∞ solution (usually 0)

$m<n$: 0, or ∞ (usually ∞)

Solve this:

$$\begin{aligned} 2x+y&=1 \\ -x+y&=3 \end{aligned} \quad \leadsto \quad \begin{aligned} x&=-2/3 \\ y&=7/3 \end{aligned}$$

First step: represent equations as a matrix ("augmented matrix")

$$\begin{array}{l} 2x+y=1 \leadsto \\ -x+y=3 \to \end{array} \left(\begin{array}{cc|c} 2 & 1 & 1 \\ -1 & 1 & 3 \end{array}\right) \xrightarrow{R1 \mathrel{+}= (-1)*R2} \left(\begin{array}{cc|c} 3 & 0 & -2 \\ -1 & 1 & 3 \end{array}\right) \xrightarrow[R1 \mathrel{*}= 1/3]{\text{mult by } 1/3} \left(\begin{array}{cc|c} 1 & 0 & -2/3 \\ -1 & 1 & 3 \end{array}\right)$$

subtract second from first

$$\begin{aligned} 2x+y&=1 \\ -x+y&=3 \end{aligned} \leadsto \begin{aligned} 3x&=-2 \\ -x+y&=3 \end{aligned} \xrightarrow[\text{by 3}]{\text{divide first}} \begin{aligned} x&=-2/3 \\ -x+y&=3 \end{aligned}$$

add first equation to second

$$\xrightarrow{R2 \mathrel{+}= 1 \cdot R1} \left(\begin{array}{cc|c} 1 & 0 & -2/3 \\ 0 & 1 & 7/3 \end{array}\right).$$

Solved!

$x=-2/3$

$y=7/3$

If you have a matrix M, there are three "elementary row operations":

1) add a multiple of one row to another row $\longleftrightarrow$ add a multiple of one equation to another

2) multiply a row by a nonzero real number $\longleftrightarrow$ multiply an equation by a number

3) swap two rows $\longleftrightarrow$ reordering the equations.

Given a system of linear equations:*

1) Write the augmented matrix.

2) Perform row operations to make left side the identity matrix. (Then you've solved it!)

*for n equations in n variables.

Try it!

$$2x+4y=-4$$
$$5x+7y=11$$

$$\left(\begin{array}{cc|c} 2 & 4 & -4 \\ 5 & 7 & 11 \end{array}\right) \xrightarrow{R1 \mathrel{+}= -4/5 R2} \left(\quad \right)$$

many ways to start!

"official" way: 1) make everything in left column below the top one 0. ("eliminate x")

$$\left(\begin{array}{cc|c} 2 & 4 & -4 \\ 5 & 7 & 11 \end{array}\right) \xrightarrow{R2 \mathrel{+}= -5/2 R1} \left(\begin{array}{cc|c} 2 & 4 & -4 \\ 0 & -3 & 21 \end{array}\right) \xrightarrow{R1 \mathrel{+}= 4/3 R2} \left(\begin{array}{cc|c} 2 & 0 & 24 \\ 0 & -3 & 21 \end{array}\right)$$

$$\xrightarrow{R1 \mathrel{*}= 1/2} \left(\begin{array}{cc|c} 1 & 0 & 12 \\ 0 & -3 & 21 \end{array}\right) \xrightarrow{R2 \mathrel{*}= -1/3} \left(\begin{array}{cc|c} 1 & 0 & 12 \\ 0 & 1 & -7 \end{array}\right)$$

3 equations, 3 variables

$$x - 3z = 8$$

$$2x + 2y + 9z = 7$$

$$y + 5z = -2$$

make 0 first!

$$\begin{array}{cccc} & 6^{th} & 5^{th} & 4^{th} \\ & \left(\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right) & & \end{array}$$

make 0 second

this is third

1) make first col 0 below diagonal

2) make second col 0 below diagonal

...

3) have an upper triangular matrix!

4) make last col 0 above diagonal

5) then next-to-last

...

6) done!

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right) \xrightarrow{R2 += (-2)\cdot R1} \left(\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{array}\right)$$

$$\Big\downarrow \; R3 += \left(-\tfrac{1}{2}\right) R2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 0 & 1 & -1 \end{array}\right) \xleftarrow{R3 *= -2/5} \left(\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 0 & -5/2 & 5/2 \end{array}\right)$$

$$\Big\downarrow \; \begin{array}{l} R1 += 3\cdot R3 \\ R2 += -15\cdot R3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{R2 *= 1/2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

Try this one:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 3 & 7 & 3 \\ 2 & 4 & 8 & 6 \end{array}\right) \xrightarrow[\substack{R2 \mathrel{+}= (-1)R1 \\ R3 \mathrel{+}= (-2)R1}]{} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 6 & 1 \\ 0 & 2 & 6 & 2 \end{array}\right)$$

$R3 \mathrel{+}= (-1)R2$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 2 & 6 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$ this says $0=1$!

<u>no solutions</u>

We're never going to reach the identity matrix!

But: you can always get to "echelon form", which means you've simplified your equations as much as possible.

You can't always row reduce and make left the identity matrix.

But!

You can always row reduce so matrix is rref.

This corresponds to reducing the linear equations as much as possible. Once you have rref, you can find the solutions.

Ex.

A matrix is in "echelon form" if

1) all rows of 0's are at the bottom
2) the leading entry (first nonzero number) of each row is to the right of leading entry in row above.
3) all entries in a column <u>below</u> the leading entry of any row are 0.

It's in "row reduced echelon form" (rref) if

4) leading entry of any row is 1
5) each leading 1 is the only nonzero thing in its column.

Echelon:

$$\left(\begin{array}{ccccc|c} 1 & 2 & 2 & 3 & 4 & 1 \\ 0 & 0 & 4 & 7 & 1 & 2 \\ 0 & 0 & 0 & 5 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Rref:

$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

rref.

$$\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Say variables are

v, w, x, y, z

$$v + 2w + z = 3$$

$$x + 3z = 4$$

$$y + z = 2$$

v, x, y only in one equation! can't eliminate anything.

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w & z could have any value whatsoever!

Once you pick values for those, the other variable are determined.

w = anything
z = anything $\rightarrow$

$$v = 3 - 2w - z$$

$$x = 4 - 3z$$

$$y = 2 - z$$