→ Two sets are the "same size" if you can find a one-to-one correspondence between them.

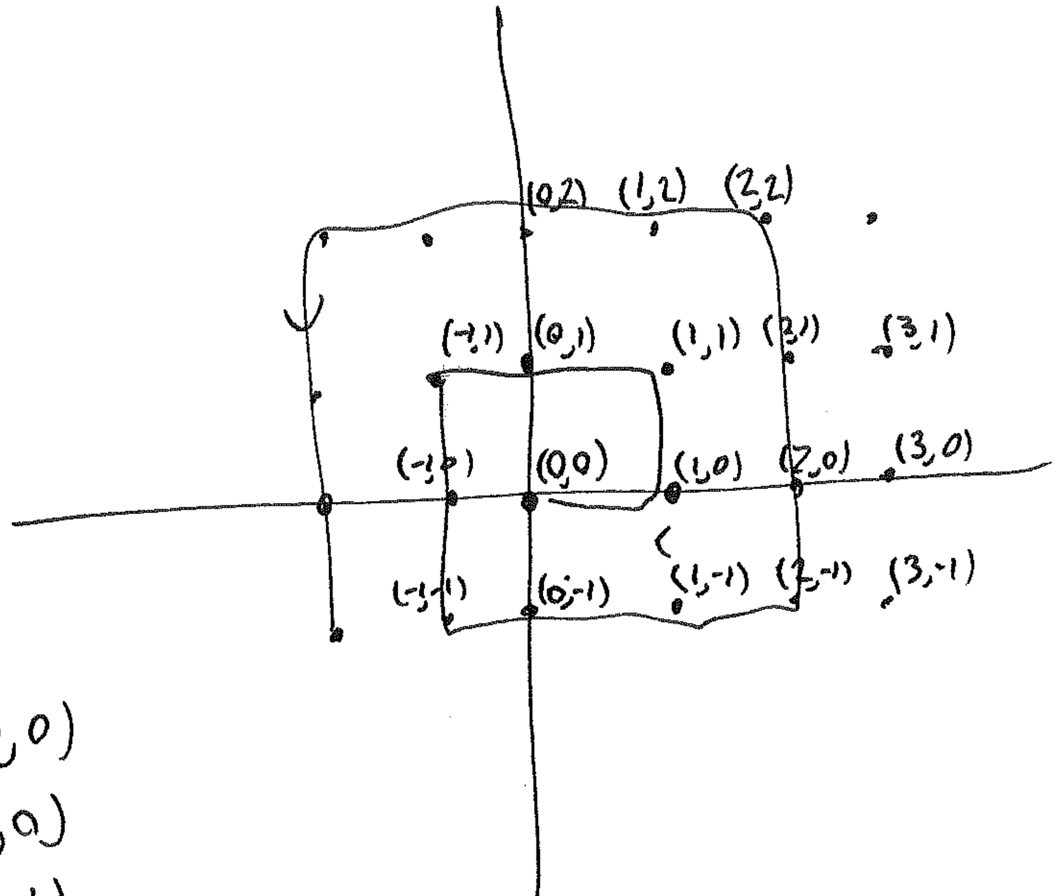
→ A set is called "countable" if it's the same size as $\mathbb{N}$.

$$\begin{array}{ccccccccccc} 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 0, & 1, & -1, & 2, & -2, & 3, & -3, & 4, & -4, & \ldots \end{array}$$

integers are countable!

What about $\mathbb{Z}^2$, ordered pairs of integers?

also countable!



$(0,0)$
$(1,0)$
$(1,1)$
$(0,1)$
$(-1,1)$
$(-1,0)$
$\vdots$

Last time:

$\mathbb{Q}$ countable

rational numbers

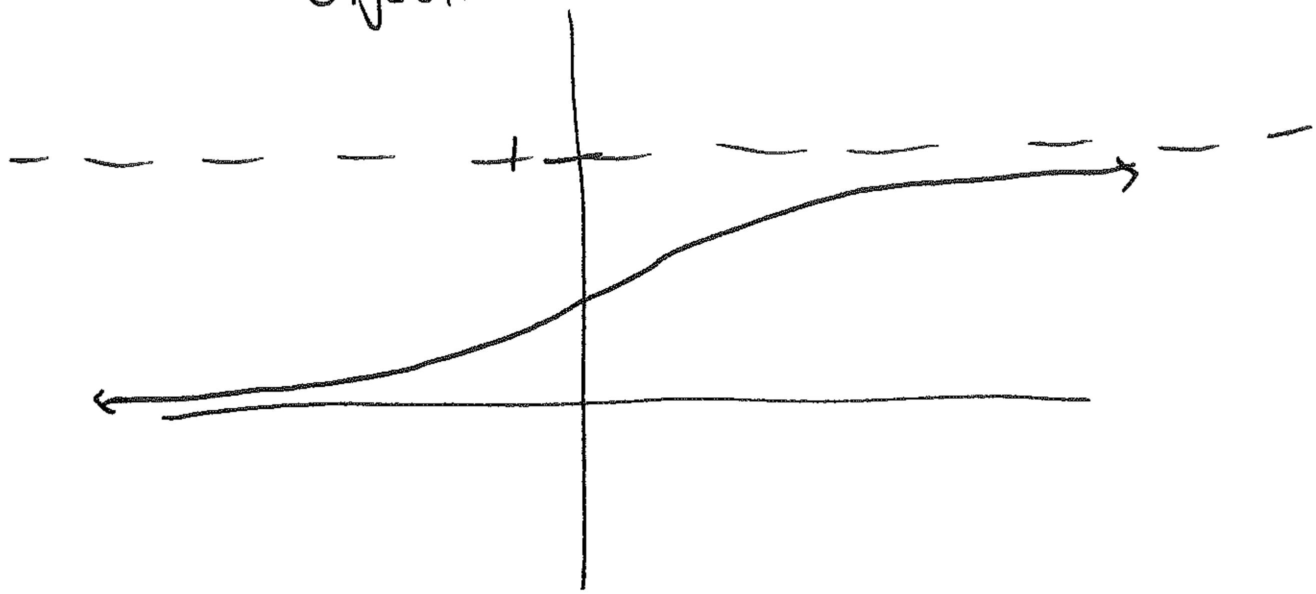
Digression on "-jective" terminology.

Cantor-Schröder-Bernstein:

If you have two sets X and Y and you can find ~~one-to-one~~ injective maps $f: X \to Y$, $g: Y \to X$, then there exists a ~~one-to~~ bijection $h: X \to Y$

$\mathbb{R}$ and $(0,1)$?

bijection:



$f(x) = \frac{1}{\pi}\tan^{-1} x + \frac{1}{2}$ (or something like that)

But are there infinite sets not of the same size? Are some infinities bigger than others?

Cantor-Schröder-Bernstein:

If there exist injective maps

$f: X \to Y$ and $g: Y \to X$

then there exists a bijection $h: X \to Y$.

$[0,1)$ and $[0,1) \times [0,1)$

$(0.314159 2653\ldots,\ 0.1234567891011\ldots)$

$\updownarrow$ interweave

$(0.31124315\ldots)$

$\mathbb{R}$ and $\mathbb{N}$.

No bijection is possible!

"Cantor's diagonalization argument"

<u>Proof</u>: It's good enough to show no bijection between $(0,1)$ and $\mathbb{N}$. (since $(0,1)$ is in bijection with $\mathbb{R}$).

<u>I</u>magine you could make a list of <u>every</u> real number:

0.21371246613...

0.11712345882...

0.317812456123...

0.459897891112...

0.31415926535...

0.2718281828459045...

⋮

We'll prove that the list must have missed some number.

Circle the n^{th} digit of the n^{th} number, and consider the real number formed by adding one to each digit ($9 \to 0$)

$\alpha = 0.328969\ldots$

I claim that α is not in our list.

It can't be the n^{th} number in the list, since it has a different n^{th} digit from that number!

<u>Corollary:</u>

There exists a real number that is not rational.

<u>Pf.</u> $\mathbb{R}$ is uncountable, but $\mathbb{Q}$ is countable, so there must be elements of $\mathbb{R}$ that aren't in $\mathbb{Q}$.

"non-constructive proof"

Theorem

$\pi^{(\log_\pi 10)} = 10$

$\log_\pi 10$: probably irrational but who knows

It's possible to have $\text{irrational}^{\text{irrational}} = \text{rational}$.

($\text{rational}^{\text{rational}} = \text{irrational}$ is easier: $2^{1/2} = \sqrt{2}$)

($e^{i\pi}$ works, but let's say real).

Consider $\underset{\substack{\shortparallel \\ x}}{}\sqrt{2}^{\sqrt{2}} \approx 1.6325269\ldots$

Case I: x is irrational

~~done!~~

$x^{\sqrt{2}} = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^2 = 2$

$\text{irrat}^{\text{irrat}} = \text{rat}$

Case II: x is rational

done! $\sqrt{2}^{\sqrt{2}} = x$ is

$\text{irrat}^{\text{irrat}} = \text{rat}$

So either $\sqrt{2}^{\sqrt{2}}$ or $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ works, but which is it? Non-constructive.

The Banach-Tarski Paradox

It's possible to take a 3D ball D^3,

~~a~~ cut it into five disjoint parts:

$$D^3 = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$$

Rotate and translate the parts and rearrange them into two identical copies of the ball:

$$A_1 \cup A_2 = D^3, \quad A_3 \cup A_4 \cup A_5 = D^3.$$

Q What's an anagram for Banach-Tarski?

A Banach-Tarski Banach-Tarski.

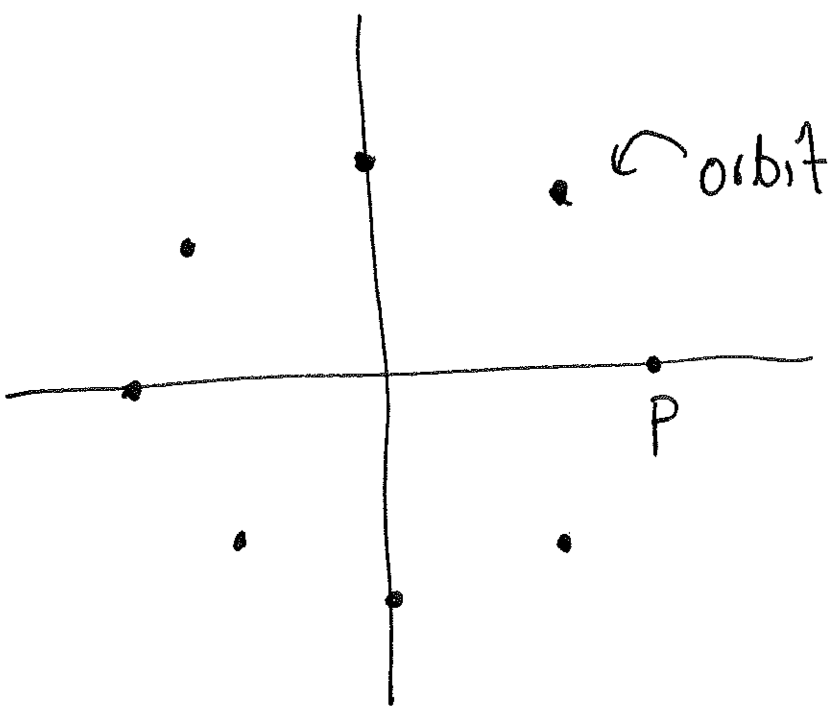
Let G be a set of transformations

e.g. $G = \{$ rotations by a multiple of $\pi/4$ radians $\}$
around origin

If P is a point, the "orbit" of P under G"

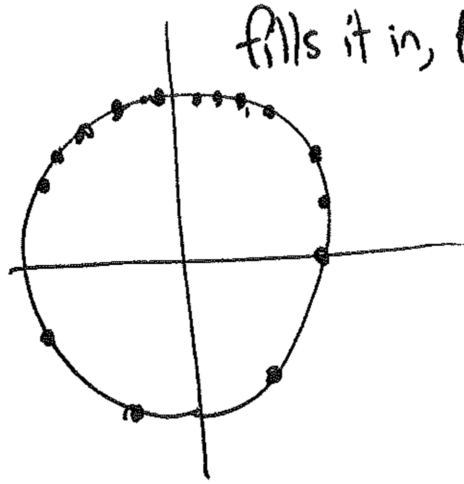
is the set of all things you can get by applying

some member of G to P.



what if $G = \left\{ \begin{array}{l} \text{rotations by a multiple} \\ \text{of 1 radian} \\ \quad\quad \text{SS} \\ \quad\quad 57.29° \end{array} \right\}$

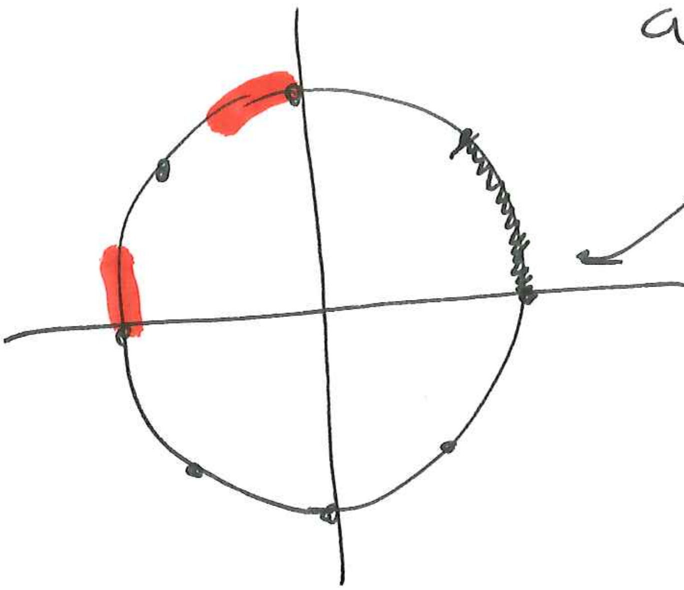
P never rotates back where it started!



Terminology: if G is a set of rotations, a "set of orbit representatives for G" is a set of points M so that any x is obtained by apply some element of G to some element of M.

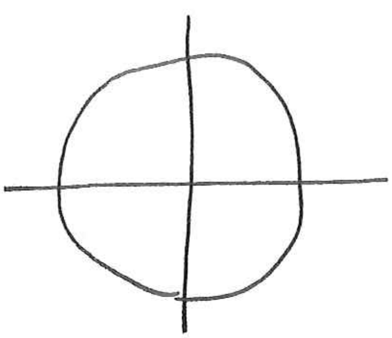
i.e. a set of points M so that any x is obtained by rotating ~~some~~ a unique element of M. e.g. (subset of circle)

e.g. $G = \{\text{rotations by } \pi/4 = 45°\}$



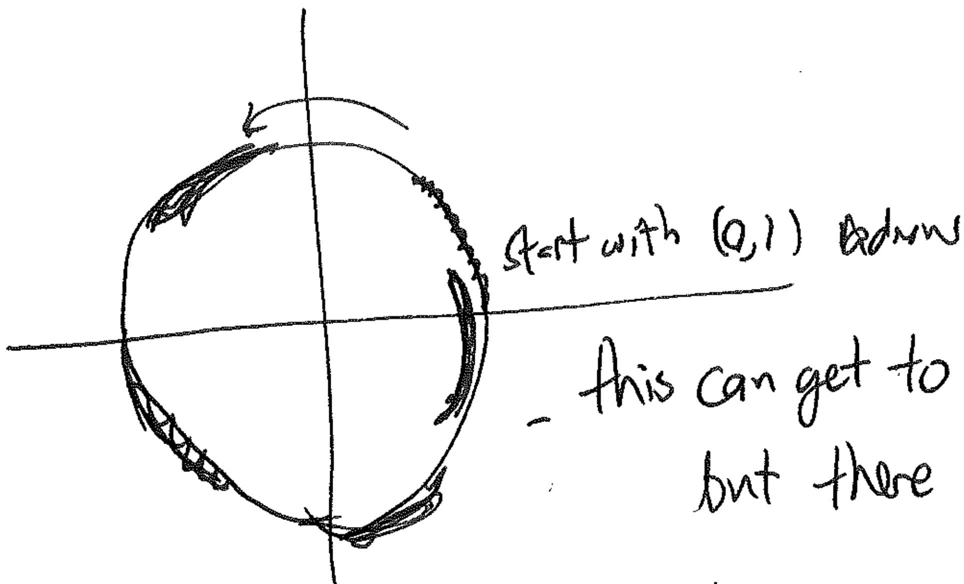
a set of orbit representatives is this arc.

What about orbit representatives for rotation by 1 radian $= \frac{180}{\pi}$ degrees.



does it exist?

A Suggestion



- this can get to every point, but there are duplicates.

- any time it overlaps itself, remove extra points from part that overlapped it.