A vector space is a collection of things V:

1) you can add them

2) you can mult. by (real) scalars

3) (+ ... axioms: associative, distributive, ...)

Examples:

$C^0(\mathbb{R})$ ← continuous functions, domain $\mathbb{R}$

$C^0([a,b])$ ← " domain $[a,b]$

$\mathbb{R}^n$ ← regular vectors of length n

$\mathcal{M}^{m\times n}$ ← $m \times n$ matrices

$\mathcal{P}(\mathbb{R})$ ← polynomials whose coefficients are real

A subspace $\overset{\swarrow W}{}$ of a vector space V is

just a subset of V closed under addition & scalar mult.

if $w_1, w_2 \in W$ then $w_1 + w_2 \in W$

if $w \in W$ and $a \in \mathbb{R}$, then $aw \in W$.

ex

$\mathbb{R}^3 = \{(x,y,z) : x,y,z \in \mathbb{R}\}$.

subspaces: • xy-plane $= \{(x,y,0) : x,y \in \mathbb{R}\}$

• $W = \{(x,x,x)$: all coords are equal$\}$

• $\{(x,y,z) : x+y+z=0\}$ is a subspace.

non-subspaces: $\{(x,y,1) : x,y \in \mathbb{R}\}$

not a subspace: $(2,3,1)+(-2,1,1)=(0,4,2)$.

first ~~quadrant~~ octant: $\{(x,y,z) : x,y,z \geq 0\}$.

$(1,1,1) \in W$, but $(-1)\cdot(1,1,1)$ isn't.

$V = C^0(\mathbb{R})$.

subspaces?

- polynomials
- linear functions $mx+b$
- even functions 
- functions with period 2π.
- functions with $f(7)=0$

Other Operations you can do to regular vectors in $\mathbb{R}^3$:
(besides addition, scaler mult).

1) dot product
2) cross product

Our definition of a vector space didn't require those operations.

An <u>inner product space</u> is a vector space that also requires something like dot product (satisfying some axioms):

An <u>inner product</u> on a vector space V is a rule that takes two elements of V as input, and outputs a number.

Write it as $\langle v, w \rangle$.

<u>Axioms for inner product:</u>

- $\langle v, v \rangle \geq 0$ for any v
- $\langle v, v \rangle = 0$ only happens if $v = 0$
- $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
- $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$ if λ is a scalar
- $\langle u, v \rangle = \langle v, u \rangle$

<u>Example</u>: Dot product of vectors in $\mathbb{R}^n$ is an inner product!

Prove from inner product axioms that

$$\langle u, \lambda v\rangle = \lambda\langle u,v\rangle.$$

Pf. $\langle u, \lambda v\rangle \overset{\text{axiom 5}}{=} \langle \lambda v, u\rangle \overset{\text{axiom 4}}{=} \lambda\langle v,u\rangle \overset{\text{axiom 5}}{=} \lambda\langle u,v\rangle$

The other important inner product space:

for any a,b, $C^0([a,b])$ ← continuous fcts, is an inner product space.

$$\langle f,g\rangle = \int_a^b f(x)\,g(x)\,dx$$

(this will give us Fourier)

In $\mathbb{R}^n$, the length of a vector is defined to be: $\sqrt{\langle v, v \rangle}$.

e.g. length of $(1,2,3)$ is

$$\sqrt{\underbrace{1\cdot 1 + 2\cdot 2 + 3\cdot 3}_{(1,2,3)\cdot(1,2,3)}} = \sqrt{1^2+2^2+3^2}$$

Def: In any inner product space, the <u>norm</u> of v is $\|v\| = \sqrt{\langle v, v\rangle}$.

What's the norm of x^2 in $C^0([0,1])$?

$$\|x^2\| = \sqrt{\langle x^2, x^2\rangle} = \sqrt{\int_0^1 x^4\,dx} = \sqrt{\left(\frac{x^5}{5}\right)\Big|_0^1}$$

$$= \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}}.$$

What's the norm of $2x^2$?

$\hookrightarrow \frac{2}{\sqrt{5}}$

<u>Problem</u>: Suppose $v \in V$ and c is a scalar.

Prove that $\|cv\| = \underset{\swarrow \text{absolute val.}}{|c|} \cdot \|v\|$

Use only the axioms!

<u>Solutions:</u>

$$\|cv\| = \sqrt{\langle cv, cv \rangle} \overset{\text{axiom 4}}{=} \sqrt{c \langle v, cv \rangle} \overset{\text{proved that}}{=} \sqrt{c^2 \langle v, v \rangle}$$

$$= \sqrt{c^2}\,\sqrt{\langle v, v \rangle} = |c| \cdot \|v\|.$$

By analogy with regular vectors, we say v & w are <u>orthogonal</u> if $\langle v, w\rangle = 0$.

<u>Ex</u> In $C^0([-1,1])$, x and x^2 are orthogonal:

$$\langle x, x^2\rangle = \int_{-1}^{1} \cancel{x\cdot} x^3\,dx = 0.$$

<u>Theorem</u> (Pythagorean)

If V is an inner product space with v, w orthogonal, then $\|v+w\|^2 = \|v\|^2 + \|w\|^2$

Pf.

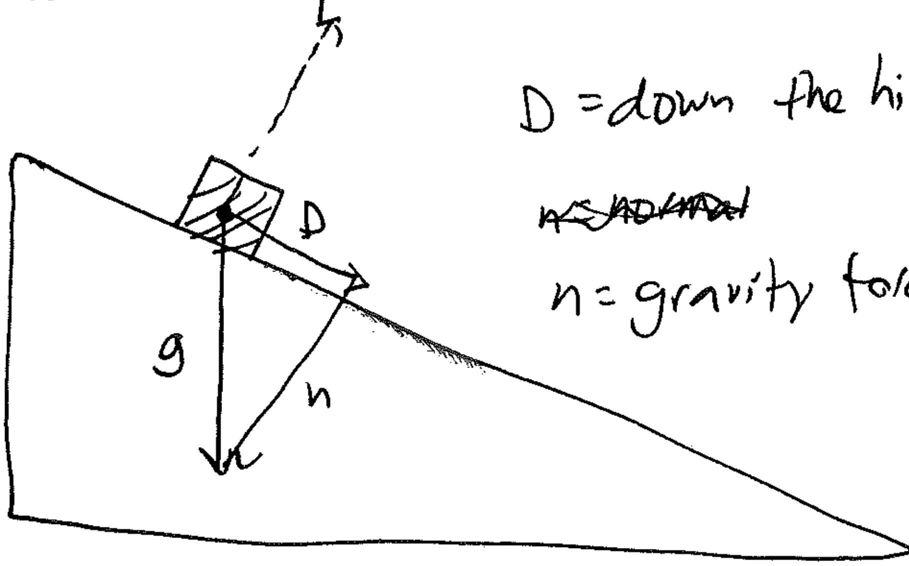
$$\|v+w\|^2 = \langle v+w, v+w\rangle \overset{\text{axiom 3}}{=} \langle v, v+w\rangle + \langle w, v+w\rangle$$

$$= \langle v+w, v\rangle + \langle v+w, w\rangle = \langle v,v\rangle + \langle w,v\rangle + \langle v,w\rangle + \langle w,w\rangle$$

$$= \langle v,v\rangle + \langle w,w\rangle + \underbrace{\langle v,w\rangle + \langle v,w\rangle}_{0\text{: assuming orthogonal}}$$

$$= \|v\|^2 + \|w\|^2$$

Mass on a ramp:



D = down the hill

n = gravity force into ramp

<u>Theorem</u>: Given u, v, you can decompose u as a multiple of v (something in direction of v), plus a vector orthogonal to v.