Eigenstuff

$T: \mathbb{R}^2 \to \mathbb{R}^2$ linear transformation
(or matrix)

A vector v is an eigenvector if $T(v) = \lambda v$
for some scalar λ. This means $T(v)$ points in
same direction as v (or opposite if $\lambda < 0$).

If T given by $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ so

$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2x \\ 3y \end{pmatrix}$ then $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenvectors.

$T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad T\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, not parallel to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$!

$T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Consider the matrix $M = \begin{pmatrix}0&1\\1&1\end{pmatrix}$.

Compute $\begin{pmatrix}0&1\\1&1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \boxed{\begin{pmatrix}1\\1\end{pmatrix}}$

$$\begin{pmatrix}0&1\\1&1\end{pmatrix}^2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0&1\\1&1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \boxed{\begin{pmatrix}1\\2\end{pmatrix}}$$

$$\begin{pmatrix}0&1\\1&1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \boxed{\begin{pmatrix}2\\3\end{pmatrix}} = \begin{pmatrix}0&1\\1&1\end{pmatrix}^3\begin{pmatrix}0\\1\end{pmatrix}$$

What's this, in terms of n?

$\dots$

$$\begin{pmatrix}0&1\\1&1\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix} = \boxed{\begin{pmatrix}3\\5\end{pmatrix}} \qquad = \begin{pmatrix}0&1\\1&1\end{pmatrix}^n\begin{pmatrix}0\\1\end{pmatrix}.$$

Can you prove it?

$$\begin{pmatrix}0&1\\1&1\end{pmatrix}\begin{pmatrix}3\\5\end{pmatrix} = \begin{pmatrix}5\\8\end{pmatrix}.$$

Claim: $M^n\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}F_n\\F_{n+1}\end{pmatrix}$

Where $F_1 = 1$, $F_2 = 1$, $\dots$ $F_{n+1} = F_{n-1} + F_n$

Fibonacci

Pf. Induction. $n=1$

$$\begin{pmatrix}0&1\\1&1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}F_1\\F_2\end{pmatrix} \quad ✓$$

Inductive step:

Assume $M^n \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}F_n\\F_{n+1}\end{pmatrix}$

Then $M^{n+1}\begin{pmatrix}0\\1\end{pmatrix} = M\left(M^n\begin{pmatrix}0\\1\end{pmatrix}\right) = \overset{M}{\begin{pmatrix}0&1\\1&1\end{pmatrix}}\begin{pmatrix}F_n\\F_{n+1}\end{pmatrix}$

$$= \begin{pmatrix}F_{n+1}\\F_n+F_{n+1}\end{pmatrix} = \begin{pmatrix}F_{n+1}\\F_{n+2}\end{pmatrix} \quad \checkmark$$

Idea: Let's try to get an exact formula for F_n by computing $\begin{pmatrix}0&1\\1&1\end{pmatrix}^n\begin{pmatrix}0\\1\end{pmatrix}$ using linear algebra.

Suppose v and w are two eigenvectors of M:

$Mv = \lambda v$ some λ, $Mw = \mu w$ some μ.

Suppose too we can write $\begin{pmatrix}0\\1\end{pmatrix}$ as a combination of v and w:

a, b constants

$$\begin{pmatrix}0\\1\end{pmatrix} = av + bw$$

$$M^n \begin{pmatrix}0\\1\end{pmatrix} = M^n\big((av + bw)\big)$$

$$\begin{pmatrix}F_n\\F_{n+1}\end{pmatrix} = M^n \begin{pmatrix}0\\1\end{pmatrix}$$

$$= a M^n v + b M^n w$$

$$= \underbrace{a\lambda^n v + b\mu^n w}_{\text{(a vector)}}.$$

F_n is top entry of the vector!

What's

$M^n v ? = \lambda^n v$

$Mv = \lambda v$

$M^2 v = M(Mv)$

$= M(\lambda v) = \lambda \cdot Mv$

$= \lambda^2 v$

$M^3 v = M(M^2 v)$

$= M(\lambda^2 v) = \lambda^3 v$

$= \cancel{\lambda^3 v}$

Let's try it!

$M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. First: find eigenvectors.

(and eigenvalues)

want $v = \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$

so $\begin{pmatrix} y \\ x+y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$

$y = \lambda x, \quad x + y = \lambda y \Rightarrow x + \lambda x = \lambda(\lambda x) = \lambda^2 x$

$(\lambda^2 - \lambda - 1)x = 0$

so $\lambda^2 - \lambda - 1 = 0$

If $x=0$, $y=0$ too, boring.

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2} = \underbrace{\frac{1+\sqrt{5}}{2}}_{\lambda} \text{ or } \underbrace{\frac{1-\sqrt{5}}{2}}_{\mu}$$

What are the eigenvectors?

We can use any number for x, and solve for y! $x=1$ for simplicity. Then $y=\lambda x$.

So: eigenvectors are $v=\begin{pmatrix}1\\ \lambda\end{pmatrix}$ with eigenvalue $\lambda=\frac{1+\sqrt{5}}{2}$

$w=\begin{pmatrix}1\\ \mu\end{pmatrix}$ with eigenvalue $\mu=\frac{1-\sqrt{5}}{2}$

$M\begin{pmatrix}1\\ \lambda\end{pmatrix}=\lambda\begin{pmatrix}1\\ \lambda\end{pmatrix}$

$M\begin{pmatrix}1\\ \mu\end{pmatrix}=\mu\begin{pmatrix}1\\ \mu\end{pmatrix}$.

Want: $M^n\begin{pmatrix}0\\ 1\end{pmatrix}$. How to write $\begin{pmatrix}0\\ 1\end{pmatrix}$ as combo of v, w?

Find a & b so

$$\underbrace{a\begin{pmatrix}1\\ \lambda\end{pmatrix}}_{v}+\underbrace{b\begin{pmatrix}1\\ \mu\end{pmatrix}}_{w}=\begin{pmatrix}0\\ 1\end{pmatrix} \Rightarrow \begin{matrix}a+b=0\\ \lambda a+\mu b=1\end{matrix}$$

$b = -a$ so

$\lambda a + \mu(-a) = 1$

$a(\lambda - \mu) = 1 \quad a = \frac{1}{\lambda - \mu} = \frac{1}{\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$

$b = -\frac{1}{\sqrt{5}}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}} v - \frac{1}{\sqrt{5}} w$ (v & w are the eigenvectors).

So...

$Mv = \lambda v \quad Mw = \mu w$

$M^n v = \lambda^n v \quad M^n w = \mu^n w$

$M^n \begin{pmatrix} 0 \\ 1 \end{pmatrix} = M^n \left(\frac{1}{\sqrt{5}} v - \frac{1}{\sqrt{5}} w \right)$ (only really care about top entry)

$= \frac{1}{\sqrt{5}} M^n v - \frac{1}{\sqrt{5}} M^n w = \frac{1}{\sqrt{5}} \lambda^n v - \frac{1}{\sqrt{5}} \mu^n w$

$= \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n \begin{pmatrix} 1 \\ \lambda \end{pmatrix} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n \begin{pmatrix} 1 \\ w \end{pmatrix}.$

Top entry:

$$\cancel{F_n} \quad \boxed{F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n}.$$

this term goes to 0 as $n \to \infty$ since $\left|\frac{1-\sqrt{5}}{2}\right| \sim 0.6 < 1$.

So $$F_n \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

Linear Recurrence sequence.

Ex Def Define a sequence by

$S_0 = 1$

$S_1 = 2$ and $S_{n+1} = 3S_n - 4S_{n-1} + 5S_{n-2}$

$S_2 = 3$

$$1, 2, 3, \underbrace{6}_{3\cdot 3-4\cdot 2+5\cdot 1}, \underbrace{26}_{3\cdot 6-4\cdot 3+5\cdot 2}, \dots$$

"Fibonacci-like sequence" You can always analyze it in the same way!

For that one, consider the matrix:

remember the two terms

top is "oldest term"

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -4 & 3 \end{pmatrix}$$

then

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -4 & 3 \end{pmatrix}\begin{pmatrix} S_{n-2} \\ S_{n-1} \\ S_n \end{pmatrix} = \begin{pmatrix} S_{n-1} \\ S_n \\ 5S_{n-2} - 4S_{n-1} + 3S_n \end{pmatrix}$$

$$= \begin{pmatrix} S_{n-1} \\ S_n \\ S_{n+1} \end{pmatrix}.$$

to find S_n:

compute:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -4 & 3 \end{pmatrix}^n \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

find eigenvalues & eigenvectors!