Today: The projection formula

vector space: a bunch of things where you can add them, multiply by scalars.

subspace: a subset of a vector space closed under addition, scaler mult.

inner product: a rule like dot product, but for a different vector space

$\underline{ex}$ $(f,g) = \int_a^b fg\,dx$

basis: A (probably finite) list of vectors that can be combined to make any other vector. (no "redundancy")

$\underline{Ex}$ $W = \{\text{polynomials of deg} \leq 2\}$ $\quad 1, x, x^2$

or $\quad x^2+1, x+1, x^2+x$

(many possibilities for basis!)

$W=$ all polynomials, no restriction of degree:

$1, x, x^2, x^3, x^4, x^5, \dots$ *

$W=$ 2×2 symmetric matrices $\begin{pmatrix} 1 & 7 \\ 7 & -3 \end{pmatrix}$

dim=3 $\rightarrow$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

dim=4 $\rightarrow$ 2×2 matrices: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

eg. $\begin{pmatrix} 2 & 3 \\ 4 & -5 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 4\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + (-5)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

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dimension of subspace = # of vectors in a basis

<u>Problem</u> V a vector space with an inner product

W a subspace

$v \in V$ a vector, maybe not in W.

minimize $\|v-w\|$

Key problem → How to find vector in W as close as possible to v?

ignore this dot

<u>Def</u> A basis $v_1, \dots, v_n$ is <u>orthonormal</u>

if 1) $\langle v_i, v_j \rangle = 0$ if $i \neq j$ (basis vectors are orthogonal)

for any i, j

2) $\|v_i\| = 1$ i.e. $\langle v_i, v_i \rangle = 1$. (length 1)

Examples

$V = \mathbb{R}^3$

$W = (xy\text{-plane}) = \{(x, y, 0)\}.$

orthonormal basis for W? $(1,0,0), (0,1,0)$

not unique! another is $(-1,0,0), (0,1,0)$

another is $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

$\sqrt{2}$ stuff is to make length 1.

non-orthonormal basis: $(2,0,0), (0,2,0)$

length not 1!

$V = \mathcal{P}(\mathbb{R})$ polynomials, any degree.

W = polynomials of degree ≤ 1. $\langle f, g\rangle = \int_{-1}^{1} fg\,dx$

orthonormal basis for W?

A basis is $1, x$. Let's check it orthonormal

then adjust it if not.

length not 1!
not orthonormal

$\langle 1, 1\rangle = \int_{-1}^{1} 1\cdot 1\,dx = 2$ so $\|1\| = \sqrt{2}$

$\langle 1, x\rangle = \int_{-1}^{1} 1\cdot x\,dx = 0$ ✓

$\langle x, x\rangle = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$ $\|x\| = \sqrt{\frac{2}{3}}$

To make orthonormal: make lengths $= 1$!

$\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\,x$ $\leftarrow$ orthonormal basis.

$$\int \underset{u}{x}\, \underset{dv}{\sin x\, dx} = (-x\cos x) - \int (-\cos x)\, dx$$

$$= -x\cos x + \sin x$$

$$= \cancel{-x\cos x + \cos x}$$

$u = x \qquad v = -\cos x$

$du = dx \qquad dv = \sin x\, dx$

$$\int_{-1}^{1} \sqrt{\frac{3}{2}}\, x\sin x\, dx = \sqrt{\frac{3}{2}} \int_{-1}^{1} x\sin x\, dx$$

$$= \sqrt{\frac{3}{2}} \left(-x\cos x + \sin x\right)\Big|_{-1}^{1}$$

$$= \left[\sqrt{\frac{3}{2}}\left(-\cos 1 + \sin 1\right)\right] - \left[\sqrt{\frac{3}{2}}\left(\cos(-1) + \sin(-1)\right)\right]$$

$$= \sqrt{\frac{3}{2}} \cdot \left(-2\cos 1 + 2\sin 1\right)$$

$$= \sqrt{\frac{3}{2}} \cdot 2\left(\sin 1 - \cos 1\right) = \sqrt{6}\left(\sin 1 - \cos 1\right)$$

<u>Solution</u> Find orthonormal basis for W (how?)

$e_1, \dots, e_n$.

The vector in W closest to a given v is

$$\underbrace{\langle v, e_1\rangle}_{\text{a number}} \underbrace{e_1}_{\text{a vector}} + \langle v, e_2\rangle e_2 + \cdots + \langle v, e_n\rangle e_n$$

Ex. What vector in $\overbrace{\text{xy-plane}}^{W}$ is closest to

$\underbrace{(5,7,-2)}_{v}$?

Sol. Basis: $\overset{e_1}{\overset{\shortparallel}{(1,0,0)}}, \overset{e_2}{\overset{\shortparallel}{(0,1,0)}}$

$$\langle v, e_1\rangle e_1 + \langle v, e_2\rangle e_2 = \big((5,7,-2)\cdot(1,0,0)\big)\,(1,0,0)$$
$$\big((5,7,-2)\cdot(0,1,0)\big)\,(0,1,0)$$

$$= 5(1,0,0) + 7(0,1,0) = (5,7,0)$$

$(5, 7, -2)$

Try a different Basis: $\underset{e_1}{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)}$ $\underset{e_2}{\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)}$

$$(5, 7, -2) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = \frac{12}{\sqrt{2}}$$

$$(5, 7, -2) \cdot \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right) = -\frac{2}{\sqrt{2}}.$$

Answer: $\frac{12}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) - \frac{2}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

$$= (6, 6, 0) - (1, -1, 0) = (5, 7, 0).$$

Find the linear function $y = mx + b$

closest to $y = \sin(x)$, using $\langle f, g \rangle = \int_{-1}^{1} fg\,dx$.

V = all functions

W = polynomials of degree ≤ 1

$v = \sin(x)$

Orthonormal basis for W: $\underset{e_1}{\overset{\|}{\frac{1}{\sqrt{2}}}}, \underset{e_2}{\overset{\|}{\sqrt{\frac{3}{2}}\,x}}$

$\langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2$

odd

$\langle \sin(x), \frac{1}{\sqrt{2}} \rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}} \sin(x)\,dx = 0$

parts

$\langle \sin(x), \sqrt{\frac{3}{2}}\,x \rangle = \int_{-1}^{1} \sqrt{\frac{3}{2}}\,x \cdot \sin x\,dx = \cdots = \sqrt{6}(\sin 1 - \cos 1)$

Answer:

$0 \cdot \frac{1}{\sqrt{2}} + \sqrt{6}(\sin 1 - \cos 1) \cdot \sqrt{\frac{3}{2}}\,x = 3(\sin 1 - \cos 1)\,x$

But how to find an orthonormal basis?

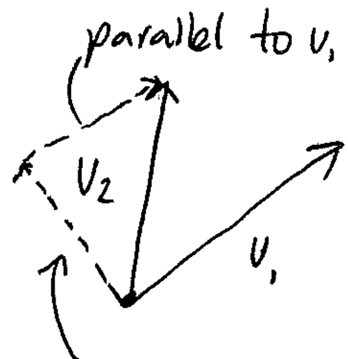
Problem

Find an orthonormal basis for

$W = \{(x,y,z) : x+y+z=0\} \subset \mathbb{R}^3$

Hint: start with a non-orthonormal basis and try to fix it.

$$\left.\begin{array}{l} v_1 = (1,-1,0) \\ v_2 = (1,0,-1) \end{array}\right\} \text{not orthonormal; how to fix it?}$$



take component of v_2 perpendicular to v_1, use that as second basis

Step 1: Make v_1 have length 1

$$v_1 = (1,-1,0).$$

$$/\sqrt{2} \rightsquigarrow e_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right).$$

$v_2 = (1, 0, -1)$ not orthogonal to v_1 (or e_1)

we want to make it orthogonal without leaving plane!

$$v_2 = \underbrace{\underset{\text{scalar}}{\underline{\qquad}}\; e_1}_{\text{parallel to } e_1} + \underbrace{f_2}_{\text{perpendicular to } e_1}$$

$$\uparrow$$

from old formula
$$\frac{\langle v_2, e_1\rangle}{\langle e_1, e_1\rangle} = \frac{(1,0,-1)\cdot\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)}{1} = \frac{1}{\sqrt{2}}$$

perpendicular part

so ~~so~~ $$f_2 = v_2 - ce_1 = (1,0,-1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$$
$$= \left(\frac{1}{2}, \frac{1}{2}, -1\right)$$

But f_2 not length 1!

To get final answer e_2, normalize f_2!

$$e_2 = \frac{f_2}{\|f_2\|} = \frac{\left(\frac{1}{2}, \frac{1}{2}, -1\right)}{\left\|\left(\frac{1}{2}, \frac{1}{2}, -1\right)\right\|} = \sqrt{\frac{2}{3}}\left(\frac{1}{2}, \frac{1}{2}, -1\right)$$

$$= \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right)$$

Gram-Schmidt Orthonormalization.

Starts with any basis $v_1, \dots, v_n$

$$\downarrow$$

Ends with orthonormal basis $e_1, \dots, e_n$

(then use in projection formula!)

1. Fix length of v_1:

$$e_1 = v_1 / \|v_1\|$$

2. Make v_2 orthogonal to v_1

$$f_2 = v_2 - \langle v_2, e_1 \rangle e_1$$

3. Fix length:

$$e_2 = f_2 / \|f_2\|$$

4. Make v_3 perp to v_1, v_2

$$f_3 = v_3 - \langle v_3, e_1 \rangle e_1 - \langle v_3, e_2 \rangle e_2$$

5. Fix length

$$e_3 = f_3 / \|f_3\| .$$

$\dots$