Complex analysis (= calculus with complex numbers!)

→ Good news: usually easier than real analysis.

+ we'll practice multivariable calculus.

→ We'll be able to do some previously impossible ~~~~ integrals.

Functions f whose domain and ~~range~~ codomain are complex numbers. $f: \mathbb{C} \to \mathbb{C}$.

Ex $f(z) = z^2$ ← z = complex variable (any other polynomial) $f(z) = \frac{1}{1-z}$

$f(z) = iz$

$f(z) = e^z$

Reminder:

FOIL:

$(3+4i)^2 = (3+4i)(3+4i)$

$= 9 + 12i + 12i + 16i^2$

$= -7 + 24i$

$f(x+iy) = e^{x+iy} = e^x e^{iy}$

$= e^x(\cos y + i \sin y)$

$= (e^x \cos y) + i(e^x \sin y)$

You can plug in complex number to most familiar functions!

what if z is complex?

$\sin(z)$

$\sin(i) = ??$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$

$$= \cos\theta - i\sin(\theta)$$

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

$$\cos(2+3i) = \frac{e^{i(2+3i)} + e^{-i(2+3i)}}{2}$$

$$= \frac{e^{-3+2i} + e^{+3-2i}}{2} =$$

$$\cancel{= e^{-3}\left(\frac{e^{2i} + e^{-2i}}{2}\right)}$$

$$\cancel{= e^{-3}(\cos 2}$$

$\cos(A+B)$ $\cos A \cos B - \sin A \sin B$

$$\frac{e^{(A+B)i} + e^{-(A+B)i}}{2} \quad \text{vs} \quad \left(\frac{e^{Ai}+e^{-Ai}}{2}\right)\left(\frac{e^{Bi}+e^{-Bi}}{2}\right)$$

How to plot $\cos(z)$ where z is complex?

$$\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$$

hard! but we could graph

~~$|\cos(z)|$~~

$|\cos(x+iy)|$ where x, y real.

output is real.

$1/z$

z^2

$\sin(z)$

also plot: what does f do to a grid? [CoCalc]

$$\frac{e^{-3}(\cos 2 + i\sin 2) + e^{3}(\cos 2 - i\sin 2)}{2}$$

$$= \frac{(e^{-3}\cos 2 + e^{3}\cos 2) + (e^{-3}\sin 2 - e^{3}\sin 2)i}{2}$$

$$\cos^2\theta = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)$$

$$= \frac{e^{2i\theta} + 1 + 1 + e^{-2i\theta}}{4}$$

$$\boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

$$= \frac{1}{2} + \frac{e^{2i\theta} + e^{-2i\theta}}{4} = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$$

Observations

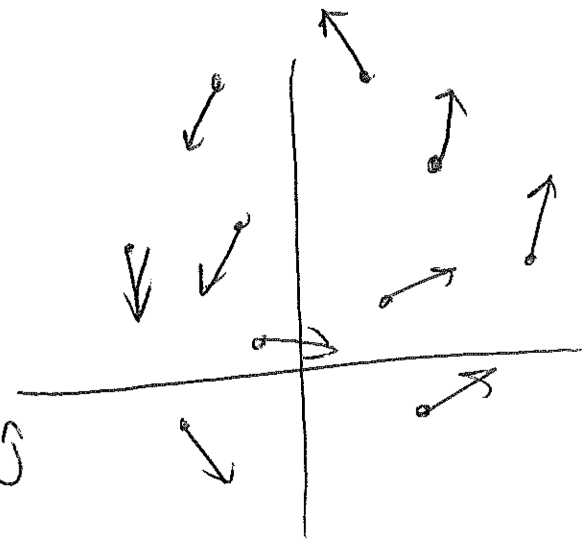
It's <u>conformal</u>: red and blue lines meet at angle

even after we plot!

Recap: path integrals (for work)

$$\int \vec{F} \cdot d\vec{s}$$

$\vec{F}(x,y)$ vector field

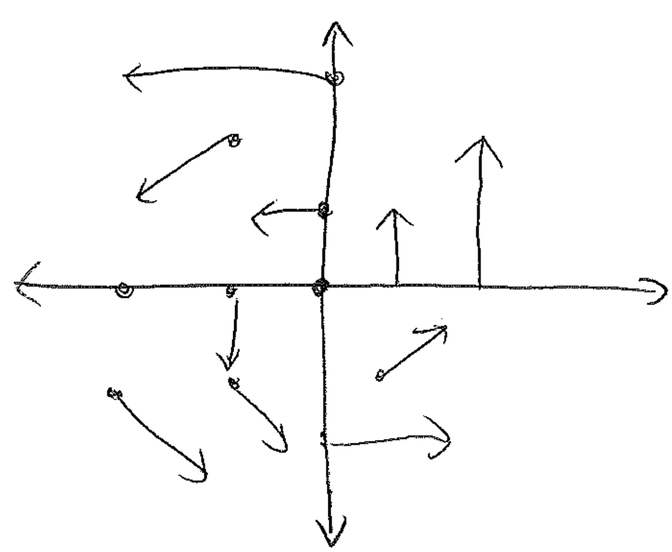


$\vec{F}(x,y) = (3x+2y)\hat{\imath} + (x-y^2)\hat{\jmath}$

e.g. $\vec{F}(1,1) = 5\hat{\imath} + 0\hat{\jmath}$

e.g. imagine it's current in water.

$\vec{F}(x,y) = (-y)\hat{\imath} + x\hat{\jmath}$

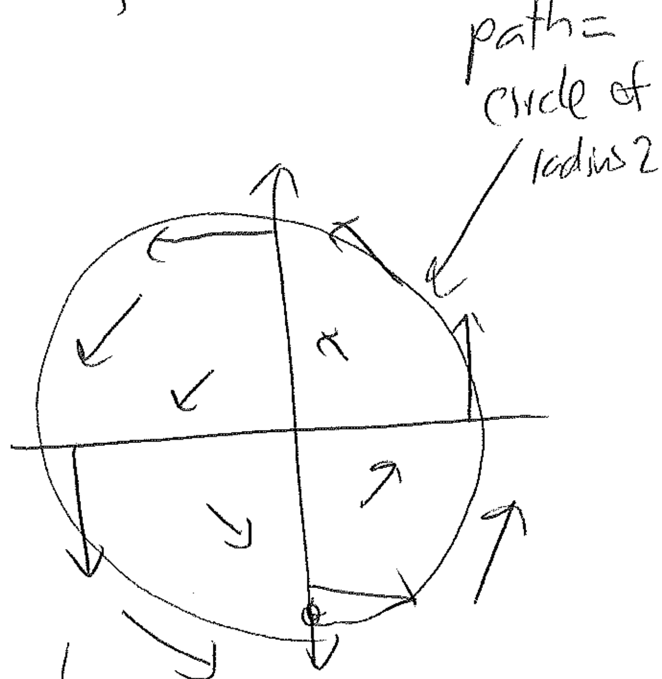


Imagine a path in the plane,

parametriized path

$\vec{r}(t) = \big(2\cos(t), 2\sin(t)\big)$

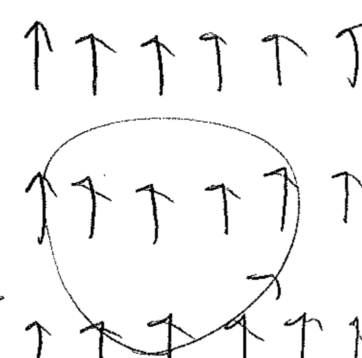
$0 \leq t \leq 2\pi$



path integral

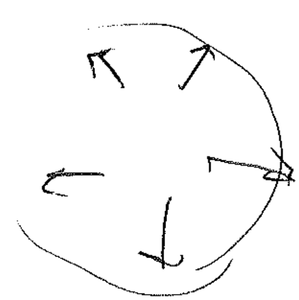
$$\int_C \vec{F} \cdot d\vec{r}$$

$\vec{F}$ $d\vec{r}$ > 0 if same direction



→ probably > 0 $\int_C \vec{F} \cdot d\vec{r}$

$= 0$ $\int \vec{F} \cdot d\vec{r}$



To calculate it:

$$\vec{F}(x,y) = (-y)\hat{\imath} + x\hat{\jmath}$$

$$\int_C \vec{F} \cdot d\vec{r}$$

$$\vec{r}(t) = \Big(\underbrace{2\cos(t)}_{x(t)},\ \underbrace{2\sin(t)}_{y(t)}\Big)$$

$$= \int_C \vec{F} \cdot \frac{d\vec{r}}{dt}\, dt$$

$$\frac{d\vec{r}}{dt} = \big(-2\sin(t),\ 2\cos(t)\big)$$

$$= \int_{t=0}^{2\pi} \underbrace{\langle -y, x \rangle}_{\vec{F}} \cdot \langle -2\sin t, 2\cos t \rangle \, dt$$

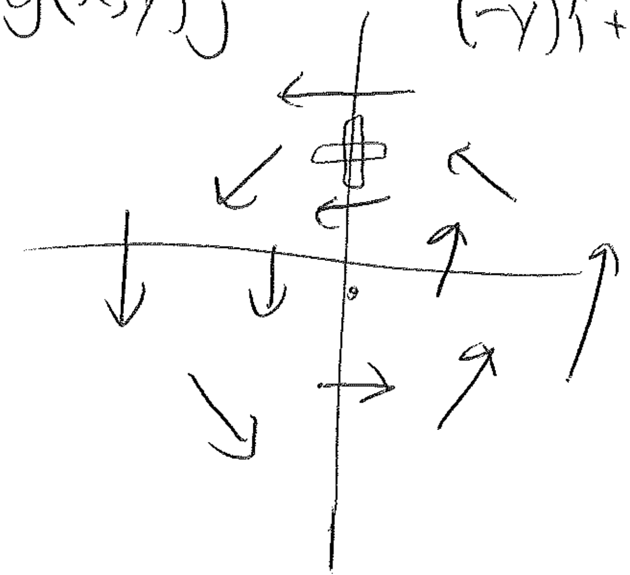
plug in x & y from $\vec{r}$

$$= \int_{t=0}^{2\pi} \langle -2\sin t, 2\cos t \rangle \cdot \langle -2\sin t, 2\cos t \rangle \, dt$$

$$= \int_{t=0}^{2\pi} 4\sin^2 t + 4\cos^2 t \, dt = \int_{t=0}^{2\pi} 4\, dt \boxed{= 8\pi} > 0.$$

$\vec{F}(x,y) = f(x,y)\hat{\imath} + g(x,y)\hat{\jmath}$

$(-y)\hat{\imath} + x\hat{\jmath}$



$(\operatorname{curl} \vec{F})(x,y) = g_x - f_y$

$\uparrow$

a function, in 2D case of x, y

$\operatorname{curl} \vec{F} = 1 - (-1) = 2$