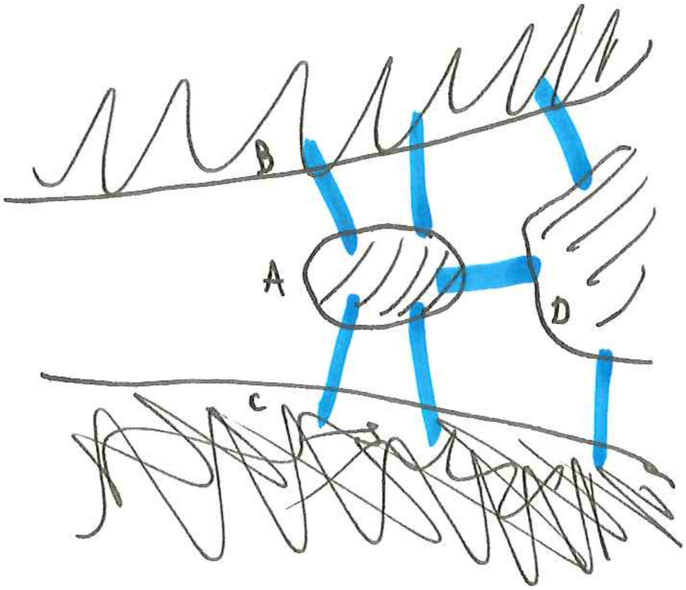
Graph theory

Kaliningrad

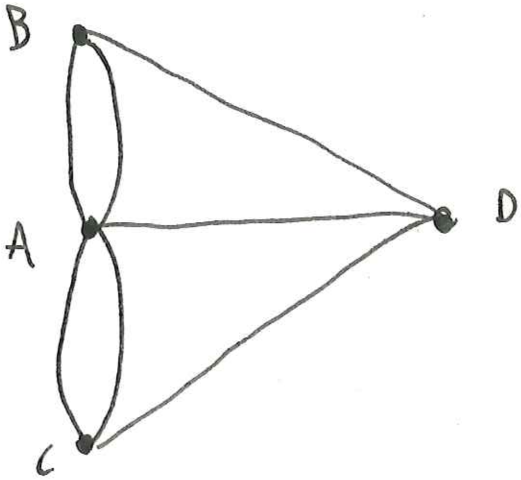
↓

Bridges of Königsberg



Can you find a path that hits every ~~island~~ bridges exactly once?

Euler:



if a vertex has an odd number of bridges, you have to either start there or end there.

but in Königsberg, all landmasses have an odd number!

No tour is possible.

Follow-up: if a city has all land masses with even number of bridges, is there always a tour?
(or maybe ≤ 2 odd)

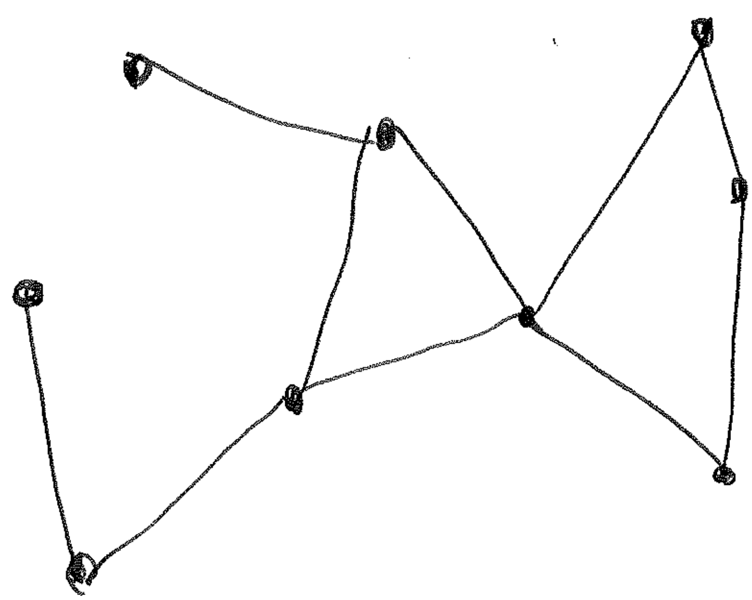
Harris "Combinatorics and Graph Theory."
(Google it.)

A graph is a set of vertices V and a set of edges E, each of which connects two vertices.

(In a graph: undirected, at most one edge between two vertices).

[Königsberg: not a graph, two bridges in some places.]

You can draw a picture:

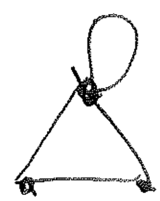


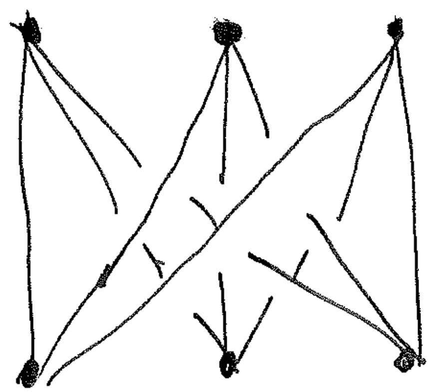
Variations on definition:

- In a "directed graph": edges have direction

- Multiple edges between a pair of vertices: "multigraph"
  (Königberg)

- Edges between vertex & self
  "pseudograph"





3 houses

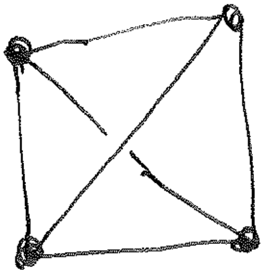
3 utilities
electric
gas
water

Can you connect without crossing?

No

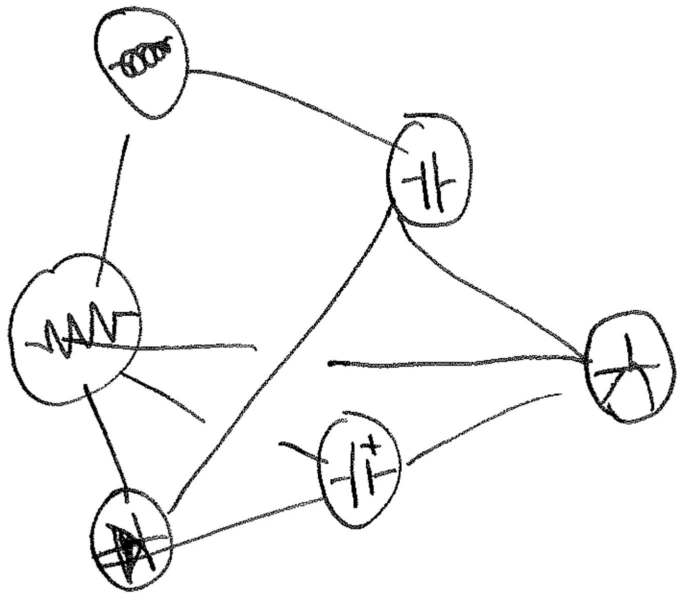
Can we fit that into the plane?

Q. Can edges intersect?

Yes:  counts.

A graph is "planar" if you can draw it in the plane. (that one is)  Figuring out if a graph is planar is a useful question.

e.g. designing a circuit board?



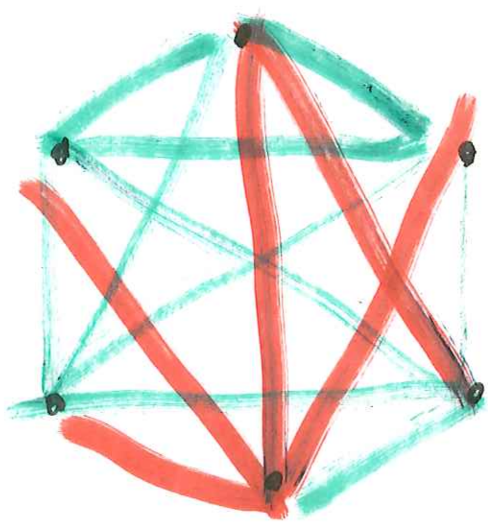
Imagine you have a party with six people.

Prove that either:

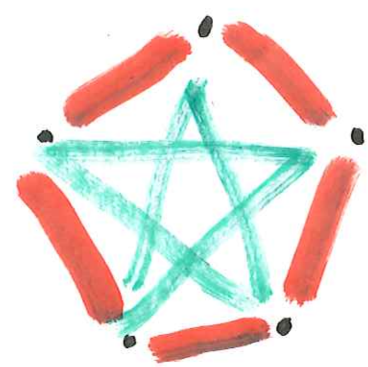
1) There's a group of 3 who all know each other.

2) There's a group of 3 none of whom know each other.

What happens with five people?



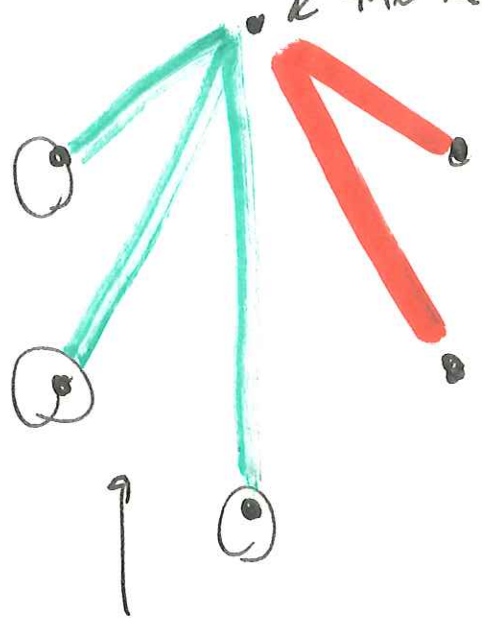
With 5



green = know each other

red = don't

must there be a red or green triangle?



if any of these two know each other, there's a green triangle with top.
if none of them know each other, they form a red triangle.

Thm (Ramsey's theorem)
For any r & s, there's a number $R(r,s)$ so that at any party with $\geq R(r,s)$ people, there's either a group of r who all know each other, or s none of whom know each other.

ex $R(3,3)=6$

Finding $R(r,s)$ is <u>really</u> hard.

$R(4,4)=18$

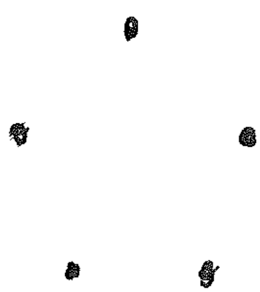
$R(5,5)=??$ $\qquad 43 \leq R(5,5) \leq 48$

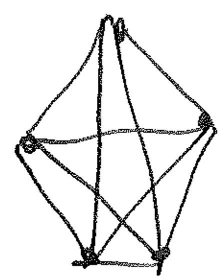
$R(6,6)=$ [weird Erdős story]

The <u>order</u> of a graph is the number of vertices.

If G has order n, how many edges could it have?

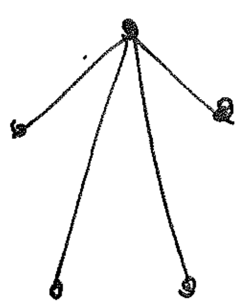
min	max
0	$\binom{n}{2}$





if graph is <u>connected</u>: (all one piece)

min is now $n-1$



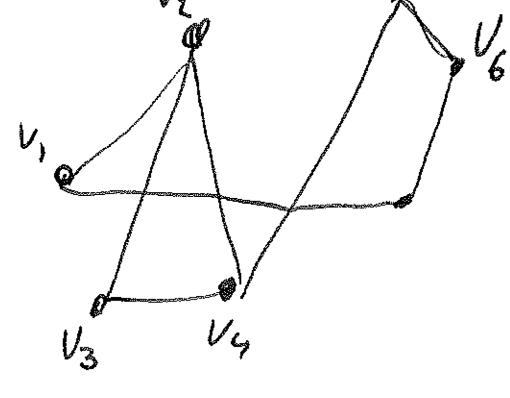
The *degree* of a vertex is the number of edges leaving it.

Thm The sum of degrees of *all* vertices is:

$$\sum_{v \in V} \deg(v) = 2 \cdot \#\text{edges}.$$

Consequently: you can't have just a single vertex of odd degree.

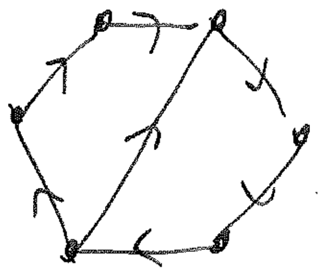
A *walk* on a graph is a sequence of vertices $v_1, \ldots, v_k$ where each v_i, v_{i+1} have an edge between



- If vertices of a walk are distinct, it's called a path.

- If edges of a walk are distinct, it's called a trail.

(Every path is trail, but not every trail is a path.)



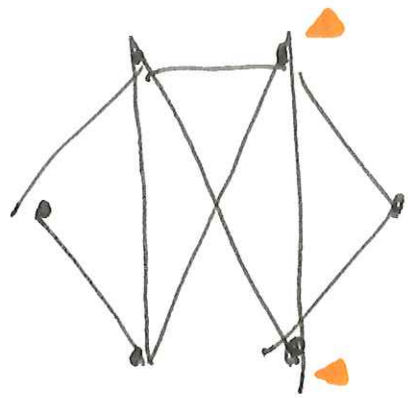
A circuit is a trail that begins and ends at same vertex.

A lot of questions like this: given a graph G, is there a trail that visits every edge?

A graph is connected: any two vertices can be joined by a path.

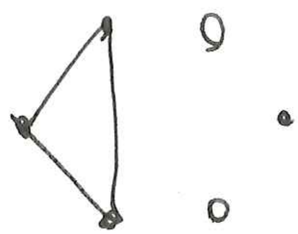
A cutset for G is a set of vertices that you can remove (and remove edges going to vertices) so graph becomes non-connected.

The connectivity of a graph is the size of smallest cut set.



how many vertices need removed for it to have more than one piece?

$K(G)=2$.



Problem: Suppose G is a graph where every vertex has degree $\geq k$.

✓ a) Prove G has path of length $\geq k$. edges.
distinct vertices

b) If $k \geq 2$, prove G has a cycle of length $\geq k+1$. edges