Today: More complex analysis, maybe some wallpaper groups / tesselations

Thursday: I'm gone

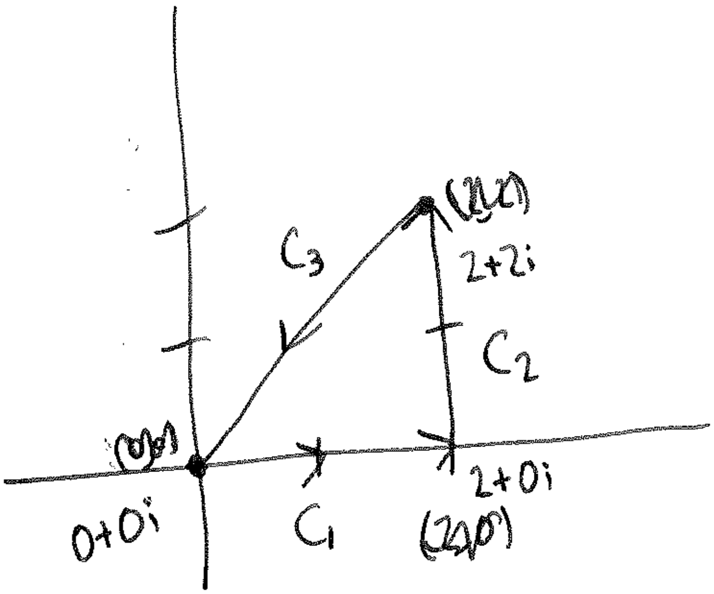
$f(z)$ = holomorphic function (complex-differentiable)

Last time: Green's theorem + Cauchy-Riemann eqns

$$\oint f(z)\,dz = 0$$

$f(z)=z^3$

$$\oint_C f(z)\,dz$$



$\gamma(t)=(2-t)+(2-t)i$

$0\le t\le 2$

$$\oint \quad \int_{C_3} z^3\,dz$$

$$\int_{t=0}^{2} f(\gamma(t))\,\gamma'(t)\,dt$$

$$=\int_{t=0}^{2}\left[(2-t)+(2-t)i\right]^3 (-1-i)\,dt$$

C_2: $\gamma(t) = 2 + ti \quad 0 \leq t \leq 2$ $\qquad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$$\int_{t=0}^{2} (2+ti)^3 \cdot i \, dt = i \int_{t=0}^{2} 8 + 12 \cdot ti - 6t^2 - t^3 i \, dt$$

$u = 2 + ti$

$du = i \, dt$

$$= i\left(16 + 12i \cdot 2 - 6 \cdot \frac{8}{3} - 4i\right)$$

$$= i(20i) = -20$$

$u = 2 - t$

$du = -dt$

$$= \int_{u=2}^{0} (u+ui)^3 (1+i)\,du = -(1+i) \int_{u=0}^{2} (u+ui)^3\,du$$

$$= -(1+i)(1+i)^3 \int_{u=0}^{2} u^3\,du = -(1+i)(1+i)^3 (4) =$$

$$= (-4)(1+i)^4 = (-4)(-4) = 16$$

$$\oint_C z^3\,dz = \int_{C_1} z^3\,dz + \int_{C_2} z^3\,dz + \int_{C_3} z^3\,dz$$

$$= 4 + (-20) + 16$$

$$= 0$$

Residue theorem

$f(z)$ meromorphic (holomorphic, but finite number of "poles" where $= \infty$)

$f(z) = 1/z$

$$\oint f(z)\,dz = 2\pi i \sum_{\text{poles } a_k} \text{Res}(f, a_k)$$

residue of f at a_k: the coefficient of $\frac{1}{z-a_k}$ if you expand as series at a_k.

Why is $\oint_{\text{western Europe}} f(z)\,dz = 0$?

all the poles are in Eastern Europe, ha ha.

$$f(z) = \frac{1}{z^2+1}$$

Where are poles, what are residues?

$$\left.\begin{matrix} z = i \\ z = -i \end{matrix}\right) \text{ makes it infinite}$$

solve for a, b

$$\frac{1}{z^2+1} = \frac{a}{z+i} + \frac{b}{z-i} = \frac{a(z-i) + b(z+i)}{z^2+1}$$

$$= \frac{(a+b)z + (-ia + bi)}{z^2+1}$$

$a + b = 0$

$bi - ai = 1$

$bi + bi = 1$

$\cancel{b = \frac{1}{2i} = \frac{-i}{2}}$

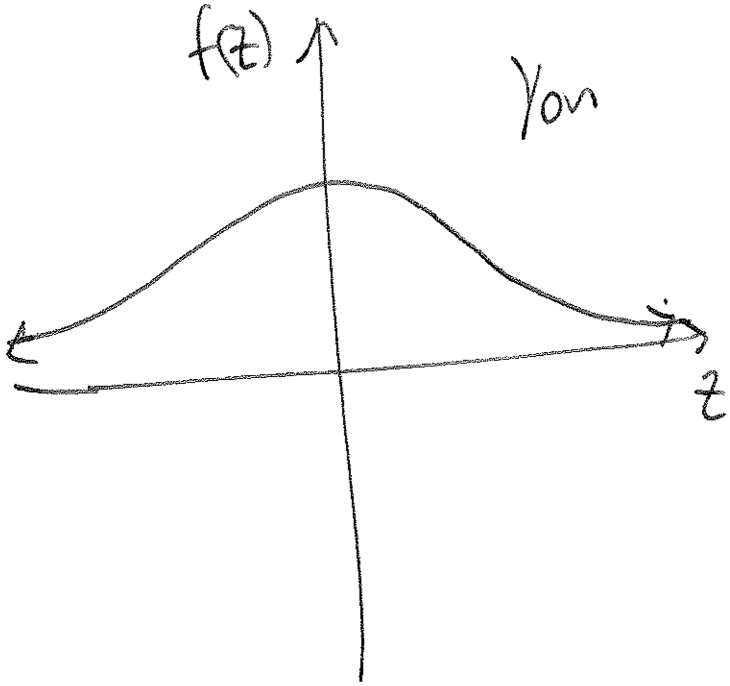
$2bi = 1$

$b = \frac{1}{2i} = -\frac{i}{2}$

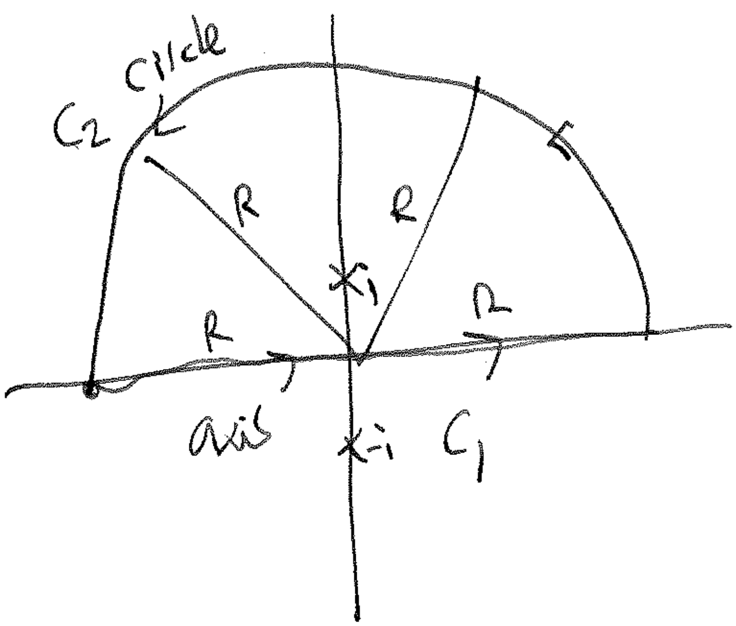
$a = \frac{i}{2}$

$$\frac{1}{z^2+1} = \frac{i/2}{z+i} + \frac{-i/2}{z-i}$$

$$f(z) = \frac{1}{z^2+1}$$



Residue theorem:



Residue at i: $-i/2$

Residue at $-i$: $i/2$.

Residue theorem:

$$\oint_C \frac{1}{z^2+1}\,dz = 2\pi i \cdot \operatorname{Res}(f, i) = 2\pi i\left(-\tfrac{i}{2}\right) = \pi$$

$$\|$$

$$\oint_{C_1} \frac{1}{z^2+1}\,dz + \int_{C_2} \frac{1}{z^2+1}\,dz$$

$$\|$$

$$\int_{-R}^{R} \frac{1}{x^2+1}\,dx + \int_{C_2} \approx \frac{1}{R^2}\,dz$$

goes to 0 as $R \to \infty$!

Take limit $R \to \infty$

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \pi$$

Tip:

To find resume of $\frac{f(z)}{g(z)}$ at $z=a$, $\left(\begin{array}{c}\text{say}\\ g(a)=0\end{array}\right)$

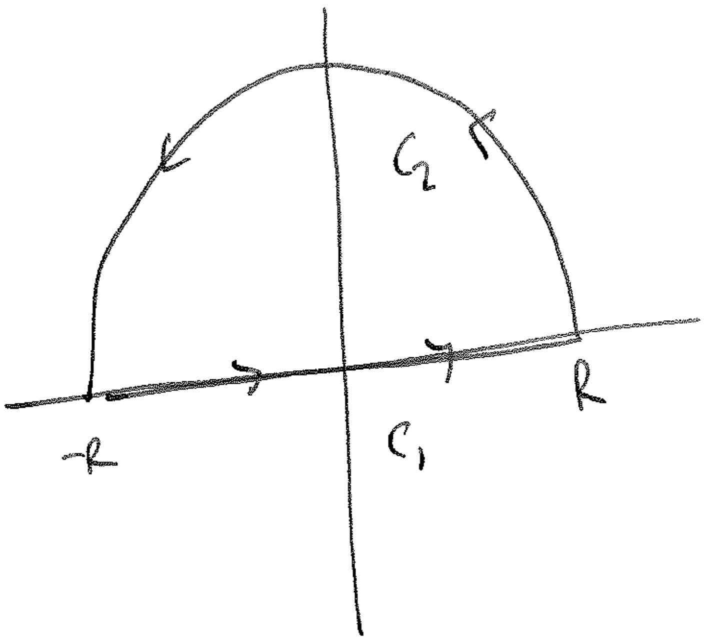
you can do:

$$\text{Res} = \lim_{z\to a} (z-a)\frac{f(z)}{g(z)} = \lim_{z\to a} \frac{(z-a)f(z)}{g(z)} = \frac{f(z)+(z-a)f'(z)}{g'(z)}$$

$$\boxed{= \frac{f(a)}{g'(a)}}$$

$$\oint_{C_R} \cancel{f}\ \frac{1}{z^4+1}\,dz$$

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$$



$$\oint_{C_R} \frac{1}{z^4+1}\,dz$$

When is denominator 0?

$$z^4 = -1$$

Poles	Residue
$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$	
$-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$	
$-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$	
$\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i$	

$$\frac{f(z)}{g(z)} = \frac{1}{z^4+1} \quad \text{at} \quad a = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

$$\text{use } \frac{f(a)}{g'(a)} = \frac{1}{4a^3} = \frac{1}{4\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^3}$$

$$= \frac{1}{4} \frac{1}{\left(e^{i\pi/4}\right)^3} = \frac{1}{4} \frac{1}{e^{3i\pi/4}}$$

$$= \frac{1}{4} e^{-\frac{3\pi}{4}i} = -\frac{\sqrt{2}}{8} - \frac{\sqrt{2}}{8}i$$

$$\text{at} \quad a = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

$$\frac{1}{4\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^3} = \frac{\sqrt{2}}{8} - \frac{\sqrt{2}}{8}i$$

$$\oint_{C_R} \frac{1}{z^4+1}\,dz = 2\pi i \left(\sum \text{residues} \right)$$

$$= 2\pi i \left(\left(-\frac{\sqrt{2}}{8} - \frac{\sqrt{2}}{8} i \right) + \left(\frac{\sqrt{2}}{8} - \frac{\sqrt{2}}{8} i \right) \right)$$

$$= 2\pi i \left(-\frac{\sqrt{2}}{4} i \right) = \frac{\pi\sqrt{2}}{2}$$

$$\int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz = \frac{\pi\sqrt{2}}{2}$$

take $R \to \infty$ that's $\to 0$

$$\boxed{\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = \frac{\pi\sqrt{2}}{2}}$$

This also lets you do other hopeless integrals:

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx = \pi/e$$