Today: Linear transformations (linear algebra part 2)

Knot theory ✓

Two problems:

1) Exact Fibonacci formula

$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} \approx \frac{1+\sqrt{5}}{2} \leftarrow \text{golden ratio } 1.618\ldots$$

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \underbrace{\left(\frac{1-\sqrt{5}}{2}\right)^n}_{\sim(-0.6)^n,\text{ so goes }\to 0\text{ as }n\to\infty.}\right)$$

2) Formula for substitution in a double integral (Jacobian determinant)

A <u>linear transformation</u> is a function

$\mathbb{R}$ real

$\mathbb{Q}$ rational numbers

$\mathbb{Z}$ integers

$\mathbb{C}$ complex #s

$T : \mathbb{R}^n \to \mathbb{R}^m$

vectors with n real entries

input: vector of length n

output: vector of length m

T must have two properties:

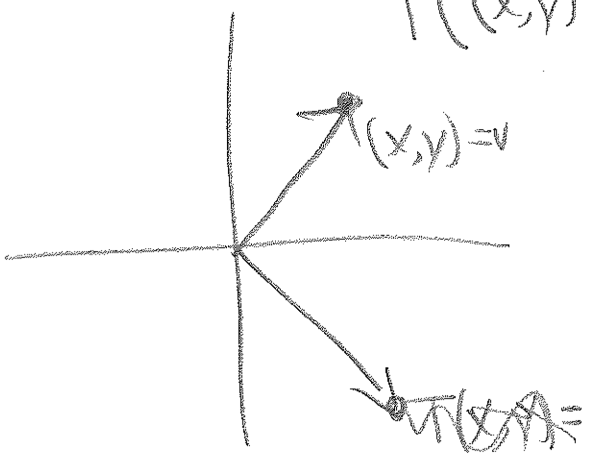
1) $T(v+w) = T(v) + T(w)$ for any v, w

2) $T(cv) = c \cdot T(v)$

scalar vector

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$

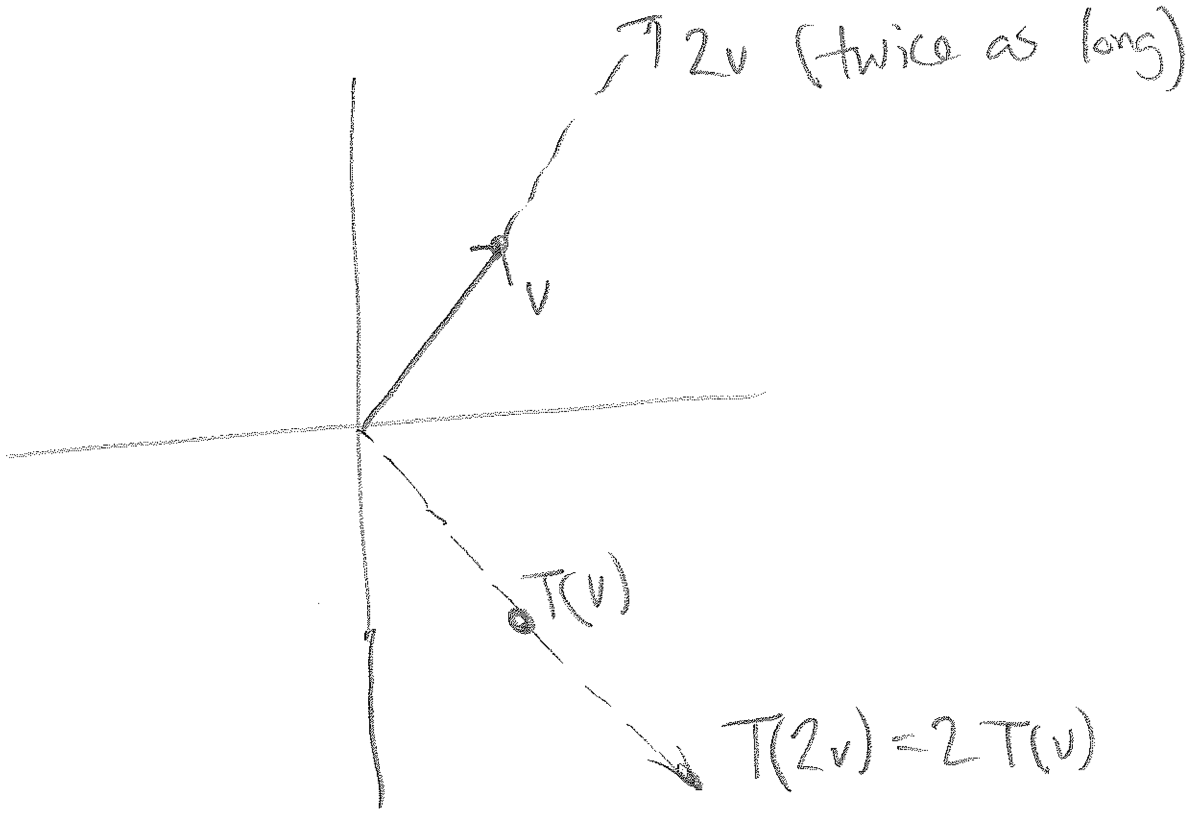
$T((x,y)) = (x,-y)$



$T(v) = (x,-y)$

"T of a vector is the vector reflected vertically."

why is $T(2v) = 2T(v)$? $(c=2)$



Non-Example

$$T(x,y)=(x+1,\ y+1)$$

$$T(10,10)=(11,11)$$

$$T(2\cdot(10,10))=T(20,20)=(21,21)$$

so $2\cdot T(10,10)\neq T(20,20)$

Doesn't work that $T(cu)=cT(u)$

Not a linear transformation!

$T(x,y)=(1,0)$ not linear

$$\left.\begin{array}{l} T(v+w)=(1,0) \\ T(v)+T(w)=(2,0) \end{array}\right\}\text{ not equal!}$$

More examples

$T : \mathbb{R}^2 \to \mathbb{R}^2$

$T(x, y) = (2x+3y, 5x-7y)$

$T(1,1) = (5, -2)$

$T(2\cdot(1,1)) = T(2,2) = (10, -4) = 2 \cdot T(1,1)$.

Another way to write it:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 5 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+3y \\ 5x-7y \end{pmatrix}$$

Fact: Any time you have an $\overset{\text{rows}}{n} \times \overset{\text{columns}}{m}$ matrix M, it gives you a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$

$T(v) = Mv$

$\uparrow$

Ex $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightsquigarrow$ linear transformation?

$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$ reflecting over x-axis

Here are a bunch of matrices:

For each one
1) Find the formula for transformation
2) Describe what it's doing geometrically

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

input size 3
output size 2

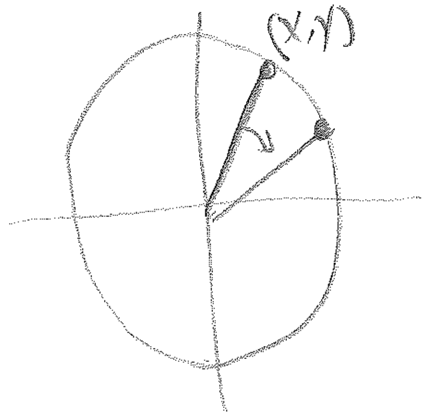
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix}$$

If you have a transformation in mind, how to find the matrix?

e.g. matrix for rotation by $\pi/6$ clockwise?



$$\begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}$$

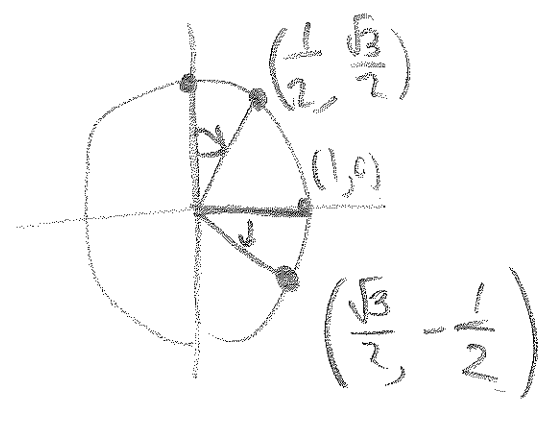
2x2 matrix

image of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

put image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ here, 1st col

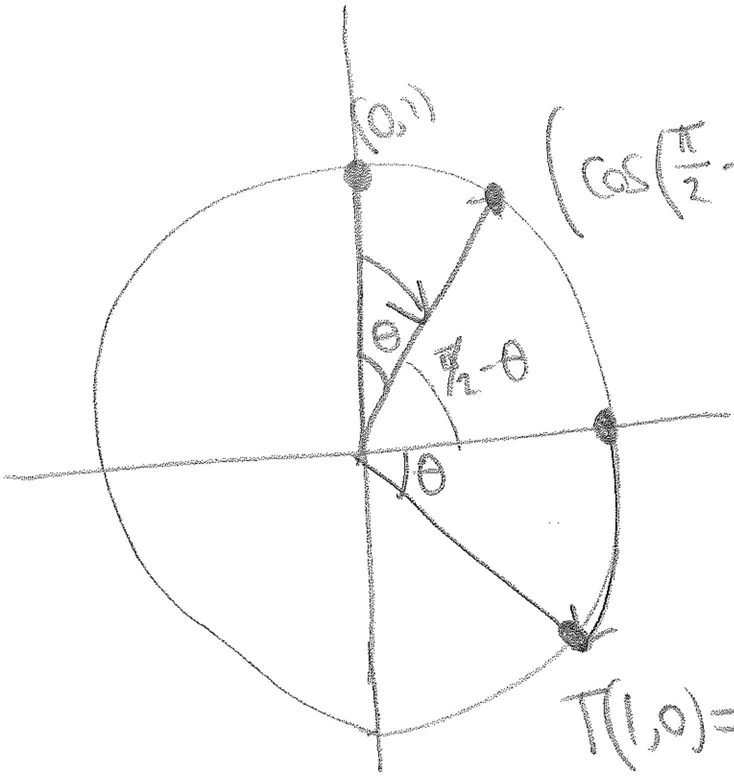
$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$



any angle

Rotation by Θ, ~~counter~~ clockwise.



$\left(\cos\left(\frac{\pi}{2}-\Theta\right), \sin\left(\frac{\pi}{2}-\Theta\right)\right)$

$=$

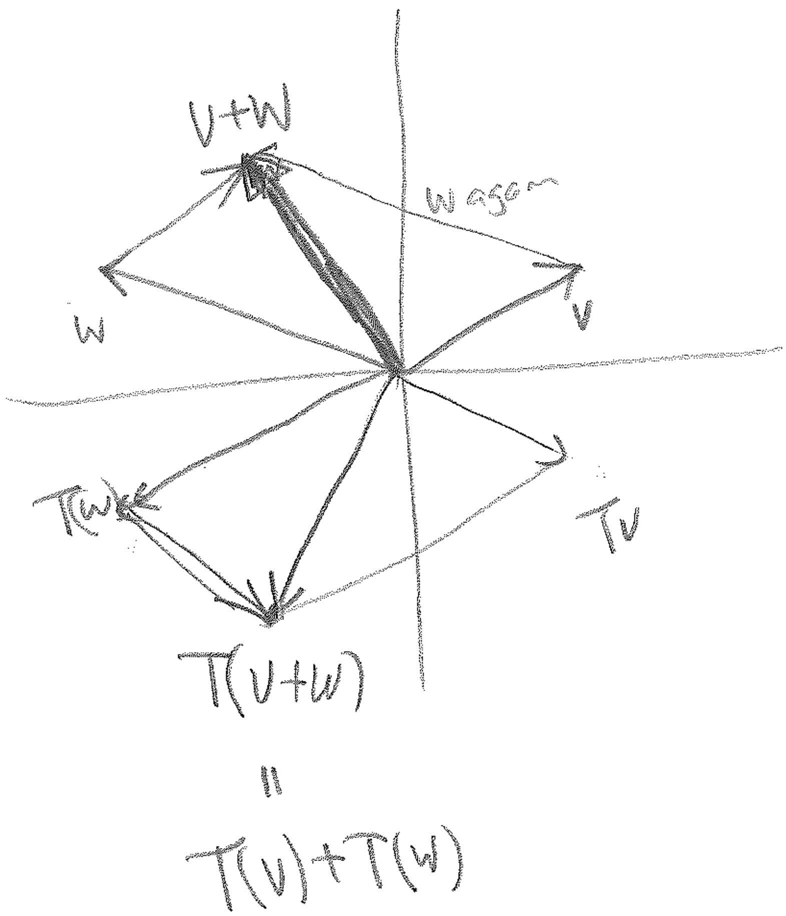
$(\sin\Theta, \cos\Theta)$

$T(1,0) = (\cos(-\theta), \sin(-\theta))$

$= (\cos\theta, -\sin\theta)$

$$\begin{pmatrix} \cos\Theta & \sin\Theta \\ -\sin\Theta & \cos\Theta \end{pmatrix}$$

Is $T(v+w) = T(v) + T(w)$?



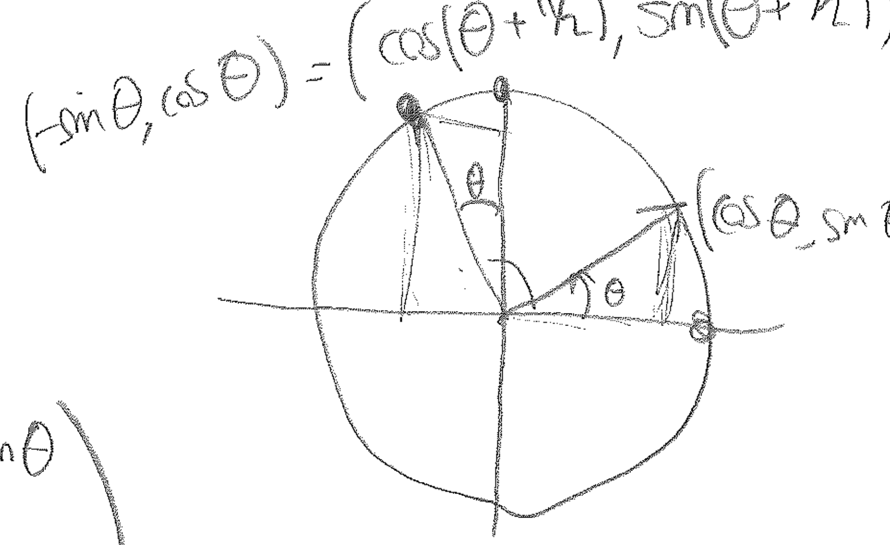
Other examples: $T(x,y) = (-x, y)$ (reflection about y-axis)

$T(x,y) = (3x, 3y)$ (stretch by factor of 3)

$T(x,y) = (y, -x)$ (rotation by $90°$ clockwise)

Rotation by θ counterwise

$T: \mathbb{R}^2 \to \mathbb{R}^2$



$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Followup: What do you get if you rotate $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ by $30°$ counterclockwise?

$$M_{30° c.c.} = \begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

HW: What's $\cos(75°)$?

$$\begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 - 1/2 \\ \sqrt{3}/2 + 1/2 \end{pmatrix}$$

Reminders

An $m \times n$ matrix gives you a function

$T: \mathbb{R}^n \to \mathbb{R}^m$.

e.g. $M = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ 2×2 matrix

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ 3x+2y \end{pmatrix}$$

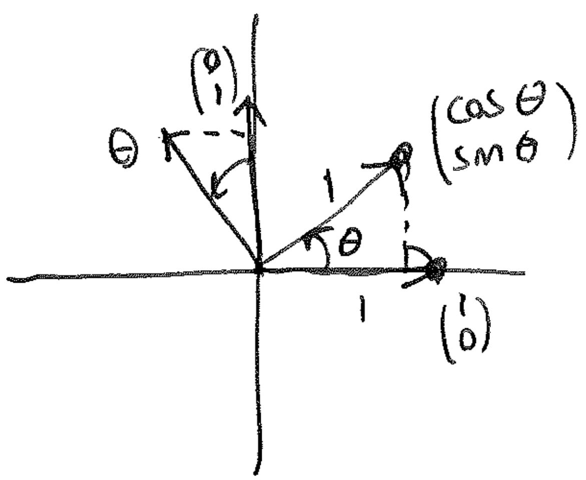
input output

What if we want our function to be rotation by θ counterclockwise? What's M?

To find matrix for a transformation:

The first column is just $T\begin{pmatrix}1\\0\end{pmatrix}$

second column is $T\begin{pmatrix}0\\1\end{pmatrix}$



$$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

Imagine we want to know the result when we rotate $\begin{pmatrix}x\\y\end{pmatrix}$ by an angle $\theta+\varphi$. What do we get?

Method 1: Rotate by θ and then φ:

$$\begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

Method 2: $\begin{pmatrix}\cos(\theta+\varphi) & -\sin(\theta+\varphi)\\ \sin(\theta+\varphi) & \cos(\theta+\varphi)\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$.

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi\cos\theta - \sin\varphi\sin\theta & -\sin\theta\cos\varphi - \cos\theta\sin\varphi \\ \cos\theta\sin\varphi + \sin\theta\cos\varphi) & -\sin\theta\sin\varphi + \cos\theta\cos\varphi \end{pmatrix}$$

So! $\cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$

$\sin(\theta+\varphi) = \cos\theta\sin\varphi + \sin\theta\cos\varphi$

General rule: if $S : \mathbb{R}^2 \to \mathbb{R}^2$, $T : \mathbb{R}^2 \to \mathbb{R}^2$ are two linear transformations, then:

$$\begin{pmatrix} \text{matrix for} \\ S \circ T \end{pmatrix}$$

Function composition:

If $f : S \to T$ is a function.

This means: input is an element of S (whatever S is)

$x \in S$

output is an element of T

$f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$

$\det : M_{2\times 2} \to \mathbb{R}$

output is a number.

input is a 2×2 matrix

If $f : S \to T$ is a function and

$g : T \to U$ is a function, then

$g \circ f : S \to U$ is a function

input: S
output: U

$(g \circ f)(x) = g(f(x))$.

$\gamma : \mathbb{R} \to \mathbb{R}^3$

$\gamma(t) = (t+1,\ t+2,\ t^2)$

γ is parametrizing a path!

$f : \mathbb{R}^3 \to \mathbb{R}$

$f(x,y,z) = x^2 y + z^3$

The composition $f \circ \gamma$ is a function:

$f \circ \gamma : \mathbb{R} \to \mathbb{R}$

$$\overbrace{(f \circ \gamma)}^{\text{take the fct } f\circ\gamma}\overbrace{(t)}^{\text{and plug in } t} = (t+1)^2(t+2) + (t^2)^3$$

Let's say

$$S : \mathbb{R}^2 \to \mathbb{R}^2$$

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

are two linear transformations.

Suppose matrix for S is $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$

matrix for T is $\begin{pmatrix} e & g \\ f & h \end{pmatrix}$.

What's the matrix for $T \circ S$? (S first, then T)

first col:

$$(T \circ S)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = T\left(S\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = T\begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ea+fb \\ ga+hb \end{pmatrix}$$

$$S\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Second col:

$$(T \circ S)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = T\left(S\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = T\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ec+fd \\ gc+hd \end{pmatrix}$$

So matrix for $T \circ S$ is:

$$\begin{pmatrix} ea+fb & ec+fd \\ ga+hb & gc+hd \end{pmatrix}$$

this is just

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}!$$

$$\begin{bmatrix} \text{matrix for} \\ T \circ S \end{bmatrix} = \begin{bmatrix} \text{matrix for } T \end{bmatrix}\begin{bmatrix} \text{matrix for } S \end{bmatrix}$$

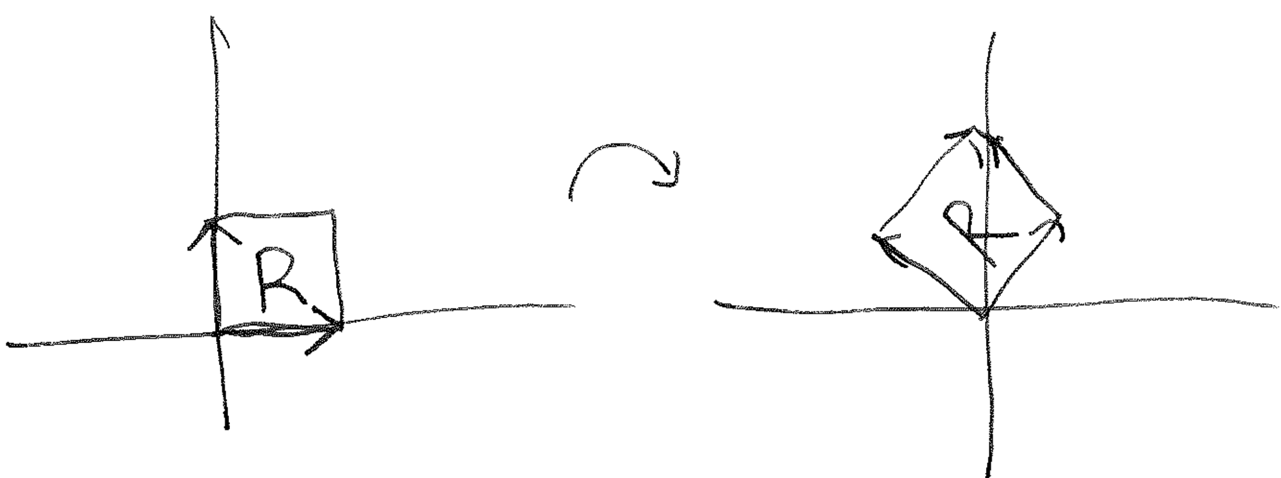
The way we define matrix multiplication is to make this true!

How about determinant?

Any time you have a linear transformation

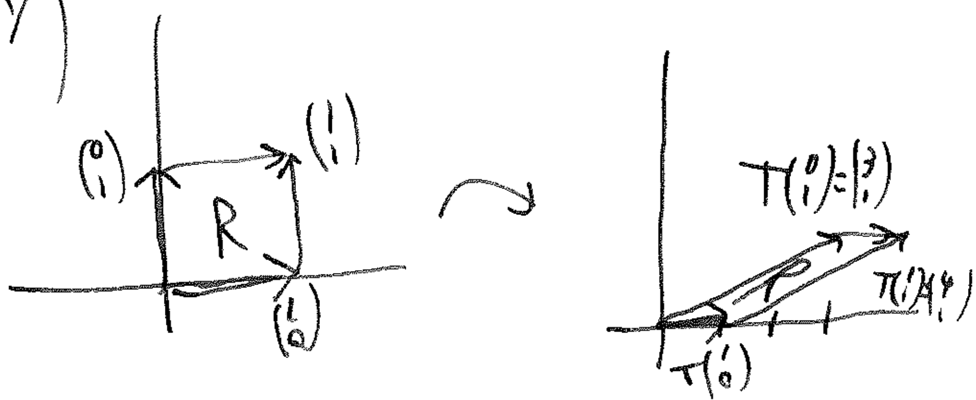
$T: \mathbb{R}^2 \to \mathbb{R}^2$, it maps squares to ~~squares~~ parallelograms.

<u>Ex</u> Rotation by 45°



<u>Ex</u> Shear transformation

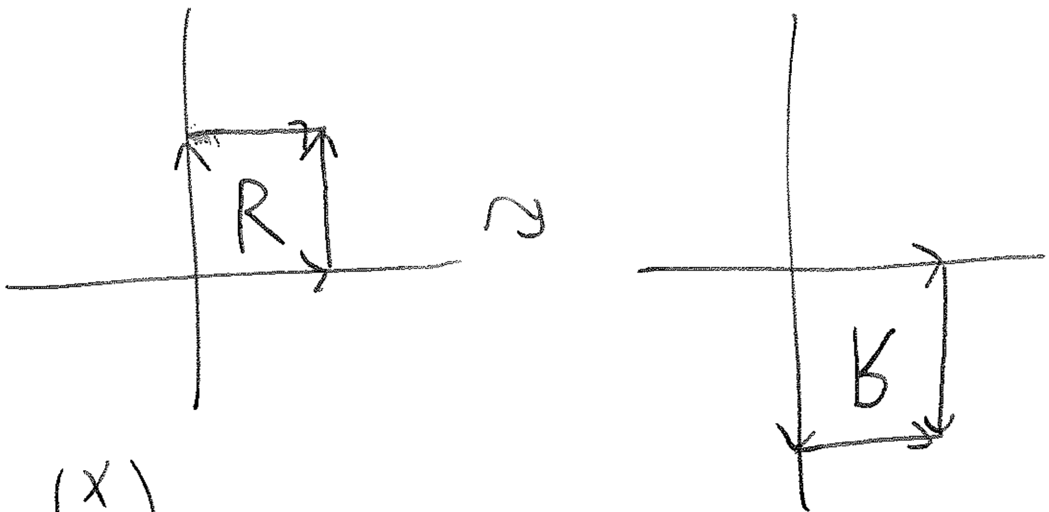
$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+3y \\ y \end{pmatrix}$$



Ex

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$



$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$

The determinant of a transformation T is the factor by which it scales areas when you apply it.

If T is "orientation-reversing" (it turns "R" into "Я"), add in a $-$ sign.

det = ?

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$6 \leftarrow \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 2y \end{pmatrix}$$

$$0 \leftarrow \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$1 \leftarrow \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

$$-1 \leftarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

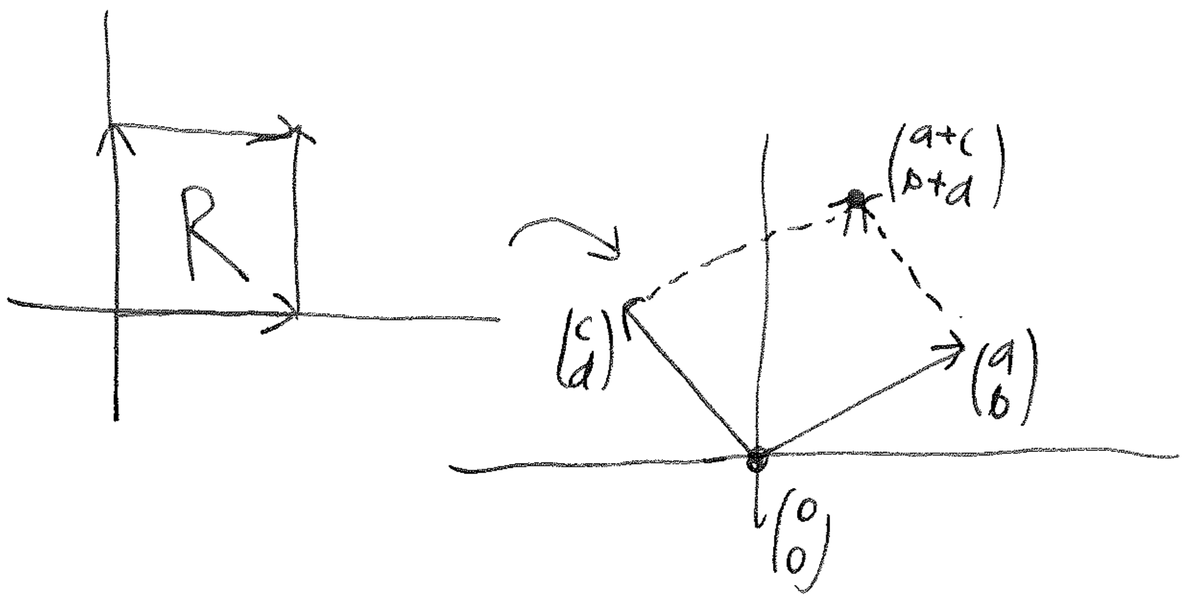
$$\times \leftarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$-1 \leftarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$1 \leftarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

How to compute it?

$$\det\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$



Determinants by row reduction.

1) The row operations ~~add~~ all affect determinant in a simple way:

- Row swap: determinant mults by -1.
- Add $c\cdot$row to another row: no change!
- Multiply row by c: determinant mults by c^n (for $n\times n$ matrix)

2. <u>For an upper triangular matrix.</u>

$$\det \begin{pmatrix} 1 & -3 & 4 & \pi \\ 0 & 2 & -2 & 12 \\ 0 & 0 & 4 & 10 \\ 0 & 0 & 0 & 7 \end{pmatrix} = (1)(2)(4)(7)$$