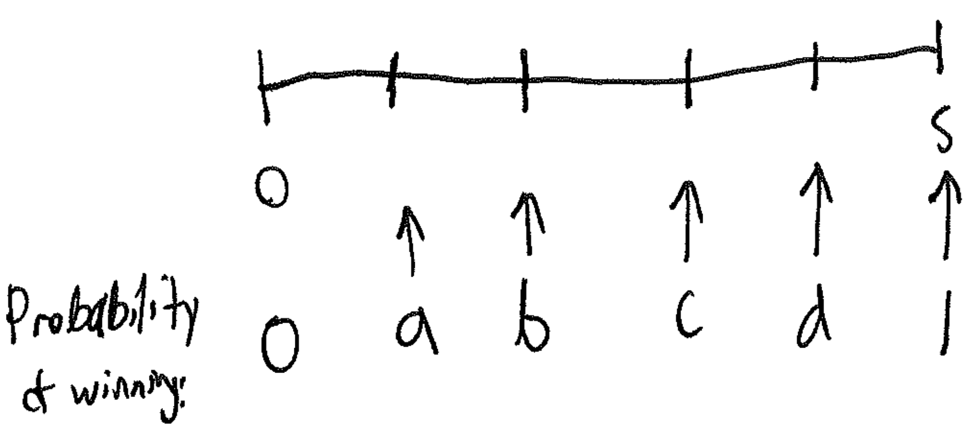
Today: Finish random walks, start calculus of variations

Try: Gambler's ruin with weights



Probability

losing $= p$

winning $= 1-p$

$$a = p \cdot 0 + (1-p)\,b$$

$$b = p \cdot a + (1-p)\,c$$

$$c = p \cdot b + (1-p)\,d$$

$$d = p \cdot c + (1-p) \cdot 1$$

solve linear eqns

$p = 0.6$

$$a = \frac{16}{211}, \quad b = \frac{40}{211}, \quad c = \frac{76}{211}, \quad d = \frac{130}{211}$$

3D random walk.

Compute

$u_{2n} = (\cdots)$ probability of coming back, by counting paths.

Answer is a mess, double $\sum\sum$, can't simplify it.

Q. Does $\sum u_{2n}$ converge?

Inequality tricks (Jensen's):

$u_{2n} \leq \frac{c}{n^{3/2}}$ so because $\sum \frac{1}{n^{3/2}}$ converges

$\Rightarrow \sum u_{2n}$ converges, and so random walk may never come back!

Polya's Random Walk Theorem

In dimension 1, 2, a fair random walk always returns to start.

In dim ≥ 3, it might not!

You can define a continuous-time version of random walk, a "Wiener process."

This is a good model for Brownian motion, maybe the stock market.

Reminder

Multivariable chain rule.

$$f(x,y) = x^3y + y$$

$$x(s) = 1+2s$$

$$y(s) = 3-4s.$$

What's $\frac{df}{ds}$? If you change s by a little, how much does f change?

$$\boxed{\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}}$$

Increase s by Δs, then x increases by $\underbrace{\frac{dx}{ds}\Delta s}_{= \Delta x}$

y increase by $\underbrace{\frac{dy}{ds}\Delta s}_{= \Delta y}.$

For f: x increases by Δx, which makes f increase by $\frac{\partial f}{\partial x} \Delta x$

y increases by Δy, which makes f increase by $\frac{\partial f}{\partial y} \Delta y$.

Total change in f is:

$$\frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y = \frac{\partial f}{\partial x} \frac{dx}{ds} \Delta s + \frac{\partial f}{\partial y} \frac{dy}{ds} \Delta s$$

$$\boxed{\frac{df}{ds} = \frac{\partial f}{\partial x} \frac{dx}{ds} + \frac{\partial f}{\partial y} \frac{dy}{ds}.}$$

$$\frac{\partial f}{\partial x} = 3x^2 y$$

$$\frac{dx}{ds} = 2$$

$$\frac{\partial f}{\partial y} = x^3 + 1$$

$$\frac{dy}{ds} = -4$$

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

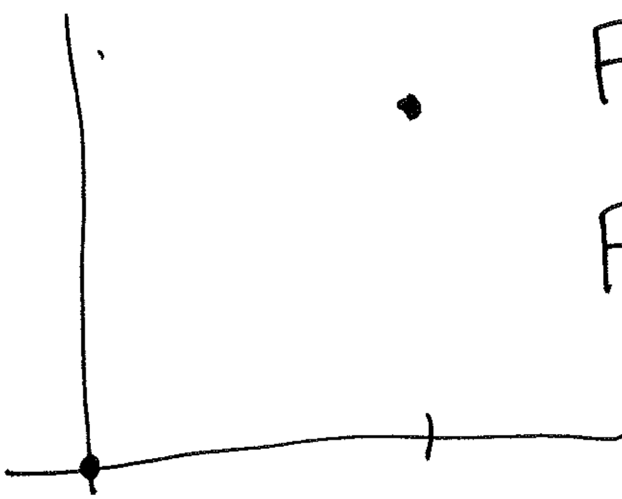
$$= (3x^2 y)(2) + (x^3+1)(-4).$$

~~Next of variations~~

Next: Calculus of variations.

Let's say you're trying to find a function $y(x)$ that minimizes a functional F.

takes a function as input, gives a number as output.



Find y that minimizes

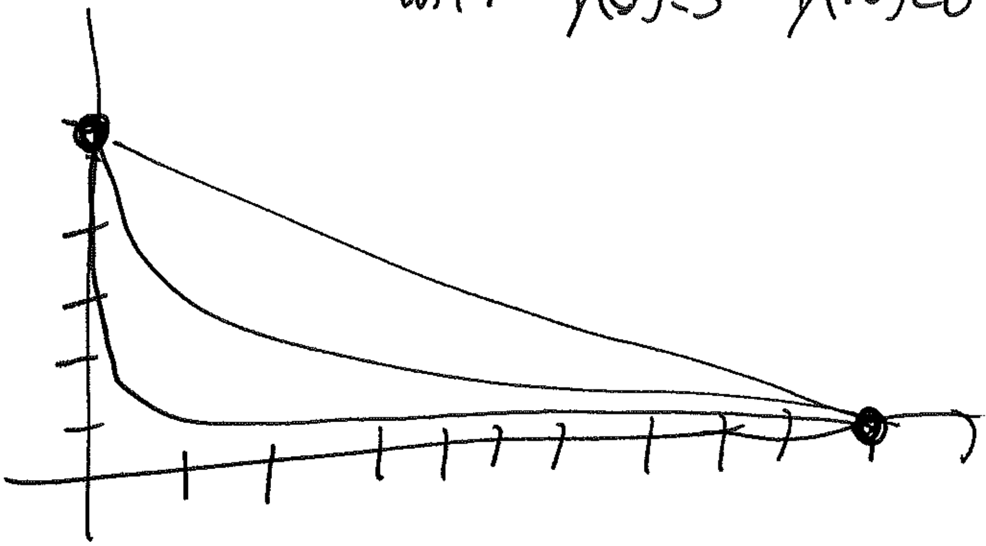
$$F(y) = \int_0^1 \sqrt{1+y'(x)^2}\, dx$$

with $y(0)=0$ and $y(1)=1$.

i.e. Find shortest path from $(0,0)$ to $(1,1)$.

or: let $F(y) = \left(\begin{array}{l} \text{the amount of time it takes a} \\ \text{ball to roll from } (0,5) \text{ to } (10,0) \\ \text{along the graph of } y \end{array} \right)$

with $y(0)=5 \quad y(10)=0$

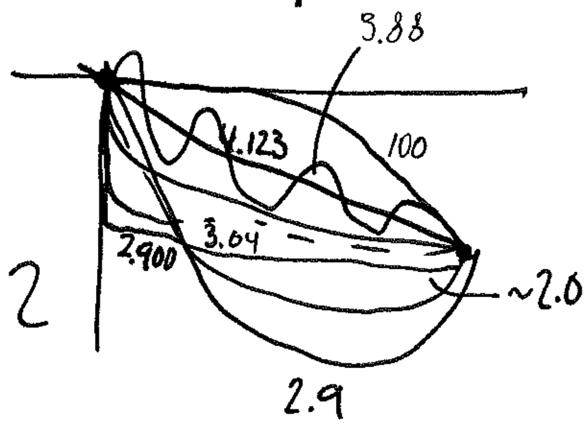


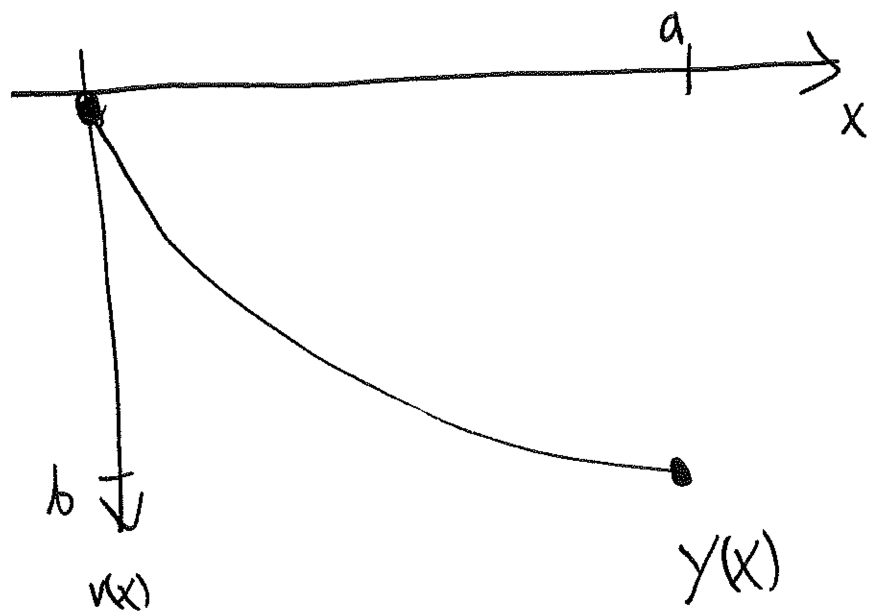
"brachistochrone problem"

$\frac{-5\log(x+1)}{\log(21)}$ | 2.9

(0,0) to (20,−5)

	y	$F(y)$ = time		
$-\sqrt{x}\frac{5}{\sqrt{20}}$	$-x/4$	4.123	$\frac{(x-20)^2}{80}-5$	3.32
[illegible]		3.04	$-\sqrt[4]{x}\frac{5}{\sqrt[4]{20}}$	
	$-\frac{x^2}{80}$	100	$-\sqrt[1000]{x}\frac{5}{\sqrt[1000]{20}}$	~ 2
	$-x^{3.5}/20^3$	10^{15}		





How long to roll from (0,0) to (a,b)

along this graph?

First: velocity as a function of y:

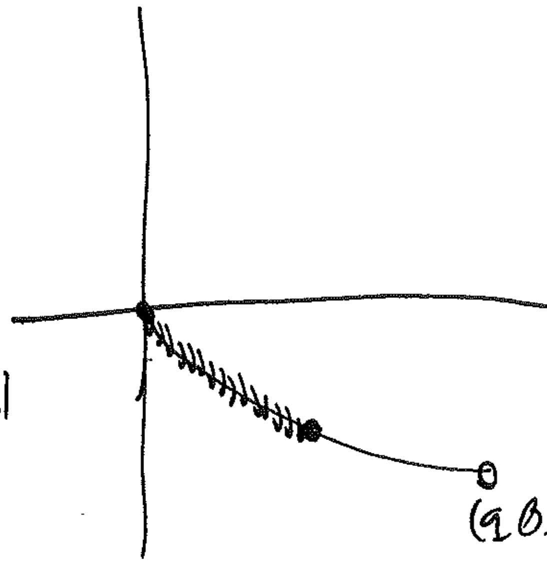
$$\underbrace{\frac{1}{2} m v(x)^2}_{\text{kinetic}} = \underbrace{mg\, y(x)}_{\Delta \text{ potential}}$$

$$v(x) = \sqrt{2g\, y(x)}$$

Let $s(x)$ be length of path from 0 to x.

arc length:

$$s(x) = \int_0^x \sqrt{1+y'(\tilde{x})^2}\, d\tilde{x}$$



$$\frac{ds}{dx} = \sqrt{1+y'(x)^2}$$ fundamental theorem

Total length:

$$L = \int_0^a \sqrt{1+y'(x)^2}\, dx$$

Now: want to take time as our variable!

$$v(t) = \frac{ds}{dt} \to v$$

Total time:

$$T = \int_0^T dt = \int_0^L \frac{1}{v(s)}\, ds = \int_0^a \frac{1}{\sqrt{2g\, y(x)}} \sqrt{1+y'(x)^2}\, dx$$

Final formula:

$$F(y) = \int_0^a \frac{\sqrt{1+y'(x)^2}}{\sqrt{2g\, y(x)}}\, dx$$

want to find y minimizing this