Quaternions

You know about complex numbers.

$$z = a + bi$$

$$w = c + di$$

Add them:

$$z + w = (a+c) + (b+d)i$$

Multiply:

$$zw = (a+bi)(c+di) \quad i^2 = -1$$

$$= (ac - bd) + (ad + bc)i$$

What's the point? ↗ solving polynomials!

What about <u>tricomplex</u> numbers (made-up)

$z = a + bi + cj$

$w = d + ei + fj$

Addition: $(a+d) + (b+e)i + (c+f)j$

Can we define multiplication so $z \cdot w$ satisfies usual axioms?

To define multiplication, need to decide on:

$$\left.\begin{array}{l} i^2 = -1 \\ ij, \; ji \; ? = \text{whatever you want!} \\ j^2 = -1 \end{array}\right]$$

no matter what values, we will never even get associativity!

$(ab)c \neq a(bc)$

Quaternians

$z = a + bi + cj + dk$

We can multiply those!
(and it's associative)

$i^2 = -1, \quad ij = k$
$j^2 = -1, \quad jk = i$
$k^2 = -1, \quad ki = j$

If multiplication is associative, what's

ji?

$$\begin{array}{c} (jk)(ki) = ij = k \\ \| \\ j(kk)i \\ \| \\ -ji \end{array}$$

So $ji = -k$!

Multiplication is associative, but not commutative.

What is ijk? $= (ij)k = k \cdot k = -1$

i^{-1}? $= -i$ since $i(-i) = -i^2 = -(-1) = 1$.

$$(1+3i+k)(2i-2j+3k)$$

$$= (1)(2i) + (1)(-2j) + (1)(3k)$$
$$+ (3i)(2i) + (3i)(-2j) + (3i)(3k)$$
$$+ (k)(2i) + (k)(-2j) + (k)(3k)$$

$$= 2i - 2j + 3k$$
$$+ (-6) + (-6k) + (-9j)$$
$$+ (2j) + (2i) + (-3)$$

$$= \boxed{-9 + 4i - 9j - 3k}$$

What's inverse of

$a+bi+cj+dk$?

$$\frac{1}{a+bi+cj+dk}\;\frac{a-bi-cj-dk}{a-bi-cj-dk}$$

$(a+bi+cj+dk)(a-bi-cj-dk)$

$=a^2+b^2+c^2+d^2+$ stuff like

$$\boxed{\begin{aligned}&(+bi)(-cj)+(cj)(-bi)\\&-bc(ij)-bc(ji)=0\end{aligned}}$$

$=a^2+b^2+c^2+d^2$

$\|a+bi+cj+dk\|^2$

Inverse of $1+3i+k$?

$$\frac{1}{1+3i+k} = \frac{1}{1+3i+k}\,\frac{1-3i-k}{1-3i-k} = \frac{1-3i-k}{1^2+(-3)^2+(-1)^2} = \frac{1}{11} - \frac{3}{11}i - \frac{1}{11}k.$$

Using quaternions for rotation

Let's say θ is an angle and $(r_1, r_2, r_3) \in \mathbb{R}^3$ is a unit vector.

Let $q = (\cos\theta) + (\sin\theta)(r_1 i + r_2 j + r_3 k)$

(q knows both angle and vector)

Length is

$\|q\|^2 =$

$q = (\cos\theta) + (r_1 \sin\theta)i + (r_2 \sin\theta)j + (r_3 \sin\theta)k$

$\|q\|^2 = \cos^2\theta + \sin^2\theta\,(r_1^2 + r_2^2 + r_3^2) = \cos^2\theta + \sin^2\theta = 1.$

q is a "unit quaternion"

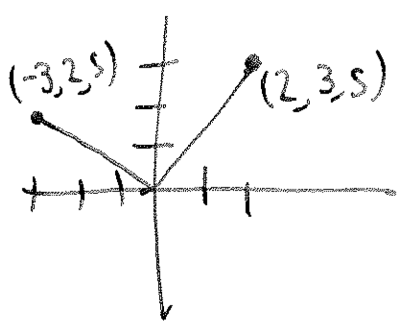
Say you have a 3D point (a,b,c).

To rotate (a,b,c) by an angle 2θ around the axis (r_1, r_2, r_3), you can compute:

$$q(ai+bj+ck)q^{-1} \quad (q \text{ as before})$$

Test: Rotate $(2,3,5)$ by $90°$ around the z-axis. (counterclockwise from above) (Should get

$(-3, 2, 5)$)



For using:

use $\theta = 45^\circ$

$(r_1, r_2, r_3) = (0, 0, 1)$

$q = (\cos\theta) + (\sin\theta)(0i + 0j + 1k)$

$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}k$

to rotate: $q(2i + 3j + 5k)q^{-1}$

$q^{-1} = \frac{\bar{q}}{\|q\|^2} = \bar{q} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}k$

$\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}k\right)(2i + 3j + 5k)\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}k\right)$

$= 3i - 2j + 5k$ ✓

Applications

- Combining rotations:

  Rotation by θ_1 around v_1

  by θ_2 around v_2.

  What if you combine?

  Turn into q_1, q_2, compute product

  $q_2 q_1$, and that tells angle + axis for combined rotation!

- Interpolation of rotations (e.g. for animation)

  To animate an object going from rotation R_1 to R_2.

  Turn them into quaternions q_1, q_2.

  Make a time-dependent quaternion

  plot these with computer

  "LERP" $\quad q_t = (1-t) q_1 + t q_2$.

  $t=0 \to q_t = q_1$

  $t=1 \to q_t = q_2$