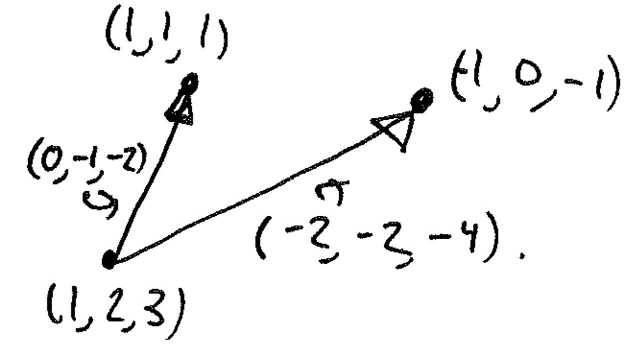
A couple other forms of linear equations.

Consider the plane through $(1,2,3)$, $(1,1,1)$ and $(-1,0,-1)$. Is $(2,1,2)$ on the plane?

→ one method:

1) find eqn of plane



2) cross product $(0,-1,-2)\times(-2,-2,-4)$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & -2 \\ -2 & -2 & -4 \end{vmatrix} = 0\hat{i} + 4\hat{j} - 2\hat{k}$$

that's the normal vector!

Plane is: $0x + 4y - 2z = C$ some constant C.

$(1,1,1)$ on plane $\Rightarrow C = 2$.

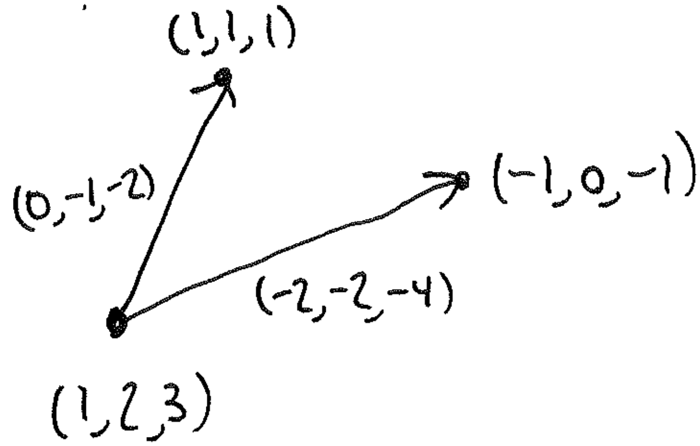
Plane is $4y - 2z = 2$.

Is $(2,1,2)$ there?

$4y - 2z = 0$ $\boxed{\text{NO}}$

Other method!

Plane through (1,2,3), (1,1,1), (-1,0,-1).



Some other pts:

$$\begin{pmatrix}1\\2\\3\end{pmatrix}+2\begin{pmatrix}-2\\-2\\-4\end{pmatrix} \qquad \begin{pmatrix}1\\2\\3\end{pmatrix}+3\begin{pmatrix}0\\-1\\-2\end{pmatrix}.$$

Every pt is:

$$\begin{pmatrix}1\\2\\3\end{pmatrix}+s\begin{pmatrix}-2\\-2\\-4\end{pmatrix}+t\begin{pmatrix}0\\-1\\-2\end{pmatrix}.$$

Is $\begin{pmatrix}2\\1\\2\end{pmatrix}$ on the plane? Try to solve for s&t

$$\begin{pmatrix}1\\2\\3\end{pmatrix}+s\begin{pmatrix}-2\\-2\\-4\end{pmatrix}+t\begin{pmatrix}0\\-1\\2\end{pmatrix}=\begin{pmatrix}2\\1\\2\end{pmatrix}$$ solve for set!

(oops, I switched a sign! Let's use this plane.)

$$\begin{pmatrix}1-2s+0t\\2-2s-t\\3-4s+2t\end{pmatrix}=\begin{pmatrix}2\\1\\2\end{pmatrix}$$

$$\left.\begin{aligned}1-2s+0t&=2\\2-2s-t&=1\\3-4s+2t&=2\end{aligned}\right)\longrightarrow\begin{aligned}-2s+0t&=1\\-2s-t&=-1\\-4s+2t&=-1\end{aligned}$$

Augmented:

$$\left(\begin{array}{cc|c}-2&0&1\\-2&-1&-1\\-4&2&-1\end{array}\right)$$

$$\left(\begin{array}{cc|c}-2&0&1\\-2&-1&-1\\-4&2&-1\end{array}\right)\xrightarrow[R3-=2R1]{R2-=R1}\left(\begin{array}{cc|c}-2&0&1\\0&-1&-2\\0&2&-3\end{array}\right)\xrightarrow{R3+=2R2}\left(\begin{array}{cc|c}-2&0&1\\0&-1&-2\\0&0&-7\end{array}\right)$$

not ref, but I can already see no solutions!

~~Algorithm~~

~~To find~~

<u>Fun fact</u> The determinant of a diagonal matrix

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \text{ is } \lambda_1 \lambda_2 \lambda_3.$$

In fact, the determinant of an ~~up~~ upper triangular matrix is the product of the diagonal entries:

$$\det \begin{pmatrix} 4 & 1 & 191 \\ 0 & 7 & -95 \\ 0 & 0 & 2 \end{pmatrix} = 4 \cdot 7 \cdot 2 = 56.$$

To find det(M):

Use row reduction to make M upper triangular.

Try to only use $R_i \mathrel{+}= c \cdot R_j$. (no swaps, no mult by constants).

Then det(M) = det(upper triangular)

$$\det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 7 \\ 1 & 2 & -1 \end{pmatrix} ?$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 7 \\ 1 & 2 & -1 \end{pmatrix} \xrightarrow{R2 \mathrel{+}= -R1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 1 & 2 & -1 \end{pmatrix} \xrightarrow{R3 \mathrel{+}= R1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 1 & -2 \end{pmatrix}$$

$$\xrightarrow{R3 \mathrel{+}= -\frac{1}{2} R2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & -5 \end{pmatrix}$$

$$\boxed{\det = (1)(2)(-5) = -10}$$

Other applications of row reduction

- How does applying each row operation change the determinant of a matrix?

Try it: use $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. How does each of the three row operations change determinant? (why does the same thing work for other matrices?)

Reminder: $\det = ad - bc$
= area of parallelogram with legs the rows (or cols) of the matrix.

How to find determinant of $\begin{pmatrix} 3 & 5 \\ 1 & 3 \end{pmatrix}$ using row reduction?

Try algebraically:

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ determinant = 1

a) Multiply a row by constant λ: $\Longrightarrow$ det multiplied by λ.

$\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ det $= \lambda$.

b) Add c times a row to another row: $\Rightarrow$ no change

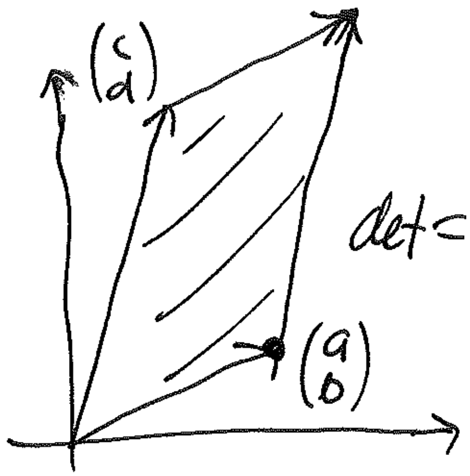
$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ det $= 1$

c) Swap rows $\Longrightarrow$ determinant mult by -1.

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ det $= -1$

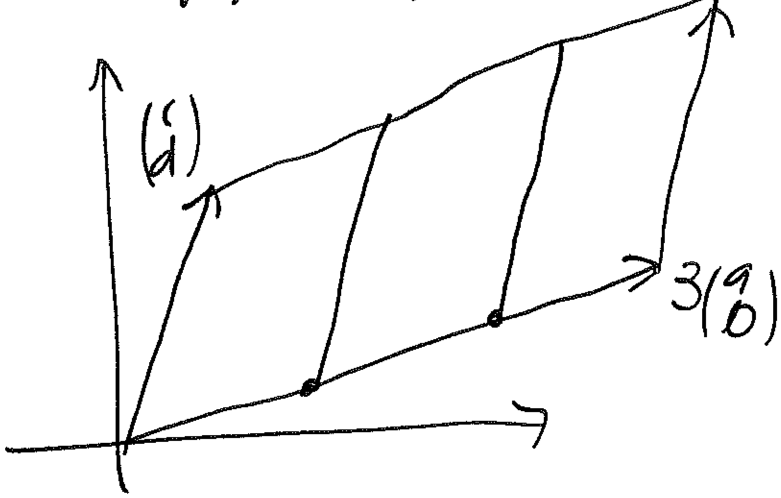
Or geometrically:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$



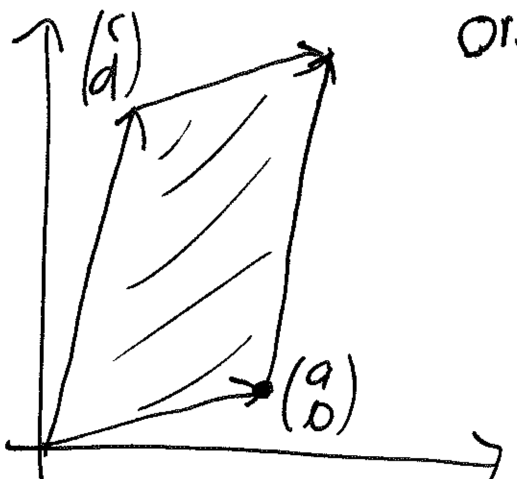
det = area

a) Multiply row by constant



determinant multiplied by c!

b) Row swap



orientation, det mult by -1.

c) Add $c \cdot row_i$ to row_j:

Apply shear transform $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$, no change.

$$\begin{pmatrix} 3 & 5 \\ 1 & 3 \end{pmatrix} \qquad \det = d$$

$$\Big\downarrow \text{swap}$$

$$\begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix} \qquad \det = -d$$

$$\Big\downarrow R1 \mathrel{+}= -3R2$$

$$\begin{pmatrix} 1 & 3 \\ 0 & -4 \end{pmatrix} \qquad \det = -d$$

$$\Big\downarrow R2 \mathrel{+}= \tfrac{3}{4}R1$$

$$\begin{pmatrix} 1 & \cancel{3}\,0 \\ 0 & -4 \end{pmatrix} \qquad \det = -d$$

$$\Big\downarrow R2 \mathrel{*}= -\tfrac{1}{4}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \det = \tfrac{1}{4}d$$

so $\frac{1}{4}d = 1$

$d = 4$

Imagine you want to solve

$$\begin{aligned} x+2y+3z&=5 \\ y+4z&=4 \\ 5x+6y \quad &=-1 \end{aligned} \qquad \Bigg\| \qquad \begin{aligned} x+2y+3z&=6 \\ y+4z&=-2 \\ 5x+6y \quad &=4 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 4 & 4 \\ 5 & 6 & 0 & -1 \end{array}\right) \quad \text{row reduce} \ldots \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & ?? \\ 0 & 0 & 1 & \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 4 & -2 \\ 5 & 6 & 0 & 4 \end{array}\right) \quad \text{row reduce} \ldots \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & ?? \\ 0 & 0 & 1 & \end{array}\right)$$

Only the left side matters! we do the same steps both ways. just use two augmented columns, and row reduce that.

$$\left(\begin{array}{ccc|cc} 1 & 2 & 3 & 5 & 6 \\ 0 & 1 & 4 & 4 & -2 \\ 5 & 6 & 0 & -1 & 4 \end{array}\right) \xrightarrow{\text{row red}} \left(\begin{array}{ccc|cc} 1 & 0 & 0 & -53 & -160 \\ 0 & 1 & 0 & 44 & 134 \\ 0 & 0 & 1 & -10 & -34 \end{array}\right)$$

First system. $x=-53$ $y=44$ $z=-10$

Second: $x=-160$ $y=134$ $z=-34$

Related problem:

Say you want to find the inverse of a matrix.

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

How to find all the letters? How to find a, d, g?

(top left

$$\begin{aligned} 1a+2d+3g&=1 \\ 0a+1d+4g&=0 \\ 5a+6d+0g&=0 \end{aligned}$$

b, e, h

$$\begin{aligned} 1b+2e+3h&=0 \\ 0b+1e+4h&=1 \\ 5b+6e+0h&=0 \end{aligned}$$

$\Downarrow$ $\Downarrow$ $\Downarrow$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 4 & 0 \\ 5 & 6 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 4 & 1 \\ 5 & 6 & 0 & 0 \end{array}\right) \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 5 & 6 & 0 & 1 \end{array}\right)$$

Row reduce together!

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R3 \mathrel{+}= (-5)R1} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R1 \mathrel{+}= (-2)R2} \left(\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -2 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & -4 & -15 & -5 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R3 \mathrel{+}= 4R2} \left(\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -2 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right)$$

$$\longrightarrow \quad \cdots$$

tell us a,d,g

b,h,e

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -24 & 18 & 5 \\ 0 & 1 & 0 & 20 & -15 & -4 \\ 0 & 0 & 1 & -5 & 4 & 1 \end{array}\right)$$

this is the inverse!