Adv Topics 2 (Lesieutre)
Real analysis
September 21, 2021

Problem 1. Consider the sequence $b_n = \pi + \frac{1}{\sqrt{n}}$. Does it have a limit? Prove it.

Problem 2. Find, with proof, the limit of the sequence $a_n = \frac{n+1}{n-1}$.

Problem 3. Suppose that the x_n converges to L and y_n converges to M. Prove that $x_n y_n$ converges to LM.

Problem 4. A *limit point* of r of a sequence is a value so that some subsequence $\langle x_{n_i} \rangle$ converges to r.

a) Can you find a sequence with two limit points?

b) Can you find a sequence with three limit points?

c) Can you find a sequence with infinitely many limit points?

d) A sequence with every real number as a limit point?

Problem 5. Prove from the definition that $\lim_{x \to 2} x^2 = 4$.

Problem 6. Compute (with proof) $\lim_{x\to 2} x^2 + 3x$.

Problem 7. Can you find an example of a bounded function $f : (0,1] \to \mathbb{R}$ for which $\lim_{x\to 0^+} f(x)$ does not exist? Do you think your function is continuous?

Problem 8. Suppose that $f_n(x)$ is a sequence of continuous functions, one for each value of n. Suppose too that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for every value of x. Must $f(x)$ be continuous?

Problem 9. Prove from the definition that the derivative of $f(x) = \frac{1}{x}$ at $x = a$ is $-\frac{1}{a^2}$.

Problem 10. This problem is about a function called the "Devil's staircase". Define $f(x)$ by the following procedure:

1. Write x in base 3.
2. If there is a 1 in the expansion, turn every digit after the 1 into a 0.
3. Turn all the 2s into 1s.
4. Interpret the result as a binary number.

a) Compute a few values and try to plot the function.

b) For what values of x is the function continuous? Try to convince yourself, even if you don't write out a careful proof.

c) For what values of x is $f(x)$ differentiable? What is the derivative?

d) Is there anything you find concerning about this function?

Problem 11. Define a function by $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$

Prove that f is infinitely differentiable for all values of x, and check that $f^{(n)}(0) = 0$ for all values of n.

(For this problem, you don't need to give a proof for every limit you use, as long as you make clear what limit you are taking.)

(Hint: prove by induction that for $x > 0$, $f^{(n)}(x) = P_n\left(\frac{1}{x}\right) e^{-1/x^2}$ where P_n is some polynomial. You can assume that $\lim_{x \to 0^+} \frac{1}{x^d} e^{-1/x^2} = 0$, which is not so hard to prove by L'Hôpital's rule.)

(The upshot is that there are nonconstant functions for which all the derivatives are 0. The Taylor series for this function at $\xi = 0$ is the zero function, even though the function itself is not 0; it just grows very very slowly near 0.)