Nash Equilibrium

Two player game.

Two Strategy

	a	b
	c	d

Mixed strategy: Player 1 does Strategy 1 with prob x
Strategy 2 with prob $1-x$

Player 2 does Strategy 1 with prob y
strategy 2 with prob $1-y$.

---

e.g.

$$\begin{array}{cc} & \begin{array}{cc} y & 1-y \\ \downarrow & \downarrow \end{array} \\ \begin{array}{c} x \rightarrow \\ 1-x \rightarrow \end{array} & \begin{pmatrix} 1 & 5 \\ 6 & 4 \end{pmatrix} \end{array}$$

Player 1's matrix

Expected payoff for P1: $xy(1) + x(1-y)(5) + (1-x)y(6) + (1-x)(1-y)(4)$

$$= (x \quad 1-x)\begin{pmatrix} 1 & 5 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} y \\ 1-y \end{pmatrix}$$

$$= -6xy + 2y + x + 4$$

Nash equilibrium occurs at saddle point!

$f(x,y) = -6xy + 2y + x + 4$

$f_x = -6y + 1 \qquad f_y = -6x + 2.$

Saddle point: $\left(\frac{1}{3}, \frac{1}{6}\right)$ is the equilibrium!

A catch: sometimes the equilibrium isn't at a saddle pt; it could be on the "edge" of region ($x = 0$ or 1, $y = 0, 1$).

Here's a game:

Imagine $f(x,y) = x + y$. What's Nash equilibrium?

$x = 1$ (otherwise increasing x improves player 1)

$y = 0$ (otherwise decreasing y would improve player 2)

Try another: chicken.

		y DS	$1-y$ S
x	DS	-10	1
$1-x$	S	-1	-10

	DS	S
DS	-10	-1
S	1	-10

Nash equilibria?

$$f_1(x,y) = -10xy + x(1-y) - y(1-x) = -10xy + x - y$$

$$f_2(x,y) = -10xy + (1-x)y - x(1-y) = -10xy - x + y$$

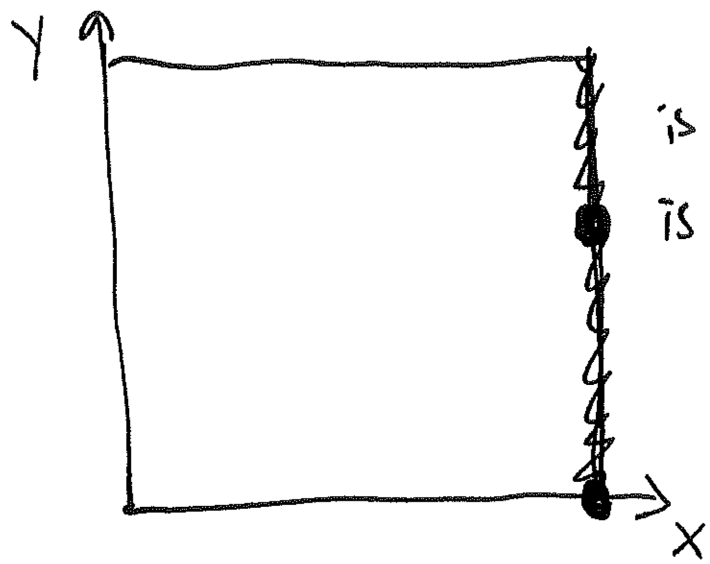
Is there one in the "middle"?

$$\frac{\partial}{\partial x} f_1 = 0 \quad \text{and} \quad \frac{\partial}{\partial y} f_2 = 0$$

$$-10y + 1 = 0 \qquad -10x + 1. \rightsquigarrow \left(\frac{1}{10}, \frac{1}{10}\right)$$

Nash equilibrium

On "edges"



is there an equilibrium here?

is $(1, y_0)$ an equilibrium for some y?

if $\left(\frac{\partial}{\partial x} f_1\right)(1, y_0) \geq 0$ then P1 ok.

$\left(\frac{\partial}{\partial y} f_2\right)(1, y_0) = 0$ then P2 ok.

$$(-10x+1)\Big|_{(1,y_0)} = 0 \qquad \cancel{x = \tfrac{1}{10}}$$

No way!

No equilibrium in the middle of an edge!

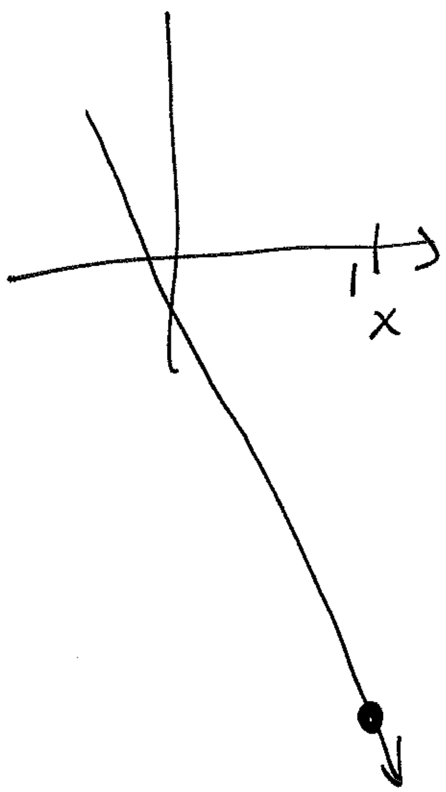
What about $\cancel{(1,0)}$?

(1,1)? If P1 decreases x a little bit, does f_1 increase or decrease?

$f_1(x,y) = -10xy + x - y.$

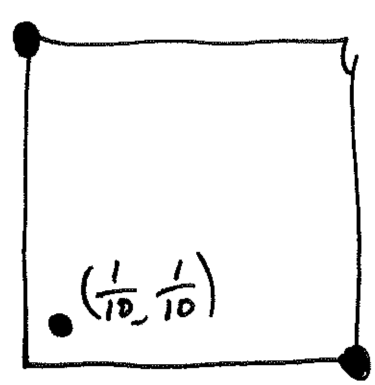
If $y=1$:

$-9x - 1.$



<u>Not</u> a Nash equilibrium!

Player 1 can change strategy and do better.



3 equilibria!

Main part of proof

Brouwer Fixed Point Thm

If you have a continuous function

$f: K \to K$ where K is a closed convex subset of $\mathbb{R}^n$,

then there exists an x with $f(x) = x$

ex: $f: [0,1] \times [0,1] \to [0,1] \times [0,1]$

"If you crumple up a sheet of paper.

map and set

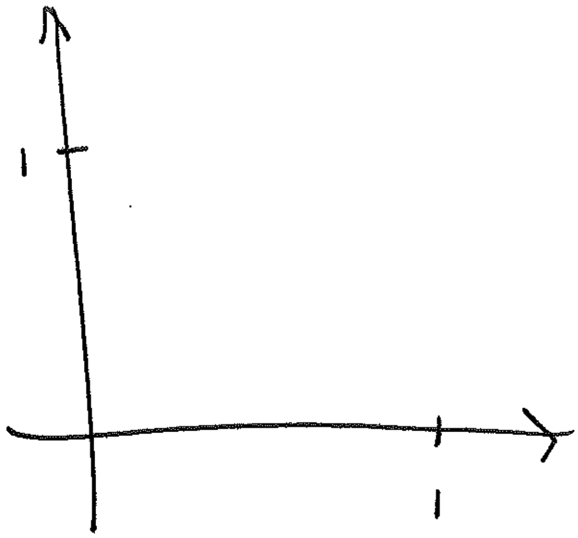
- If you crumple a sheet of paper and place it on top of another sheet of paper, some point ends up directly above corresponding point on other paper.

- If you place a map of USA on the ground inside USA, some point on map is directly above the corresponding real-life point.

  (Why: Let K be the map, maybe $0 \le x \le 2$, $0 \le y \le 1$

  $$f((x,y)) = \begin{pmatrix} \text{the new } (x,y) \text{ on map where} \\ \text{original pt ended map} \end{pmatrix}$$

- If you stir a cup of coffee, some coffee particle ends up in the same place it was before you stirred!



This doesn't work if we relax the requirements!

→ If not closed, $x^2 : (0,1) \to (0,1)$.

→ If not continuous, $f(x) = x + 0.1 \mod 1$

→ If not convex, rotate

"algebraic topology"

Borsuk-Ulam thm

Suppose $f: S^2 \to \mathbb{R}^2$ is continuous.

sphere (no cutting!)

opposite pt.

There exists an x so $f(x) = f(-x)$.

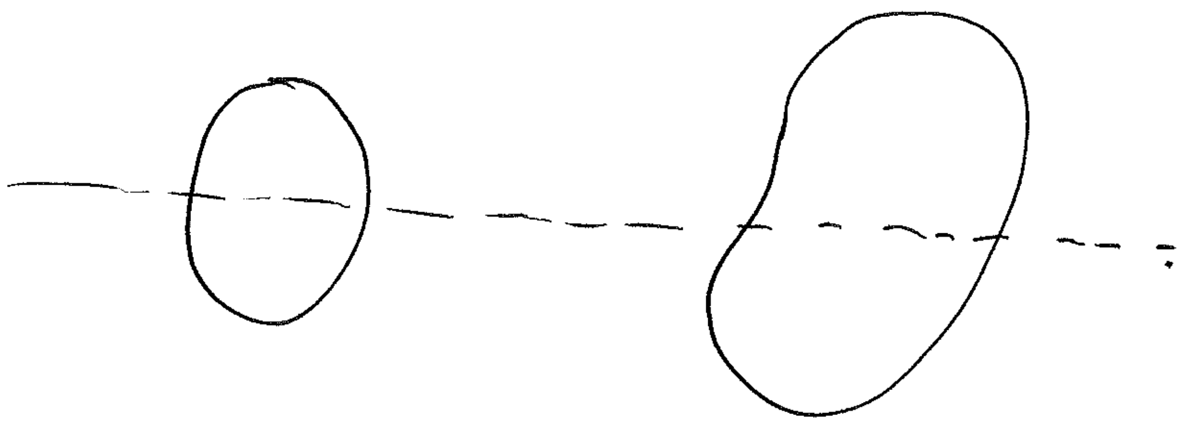
"There are always two opposite points on earth with the same temperature and the same humidity."

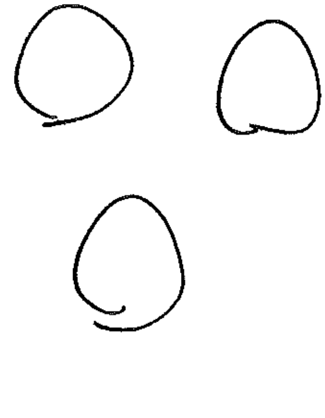
$$f(\text{pt on earth}) = \begin{pmatrix} \text{temp at pt} & \text{humidity at pt} \end{pmatrix}$$

Ham Sandwich Theorem

Suppose you have n regions in $\mathbb{R}^n$.

There exists an $(n-1)$-dimensional plane which cuts all your regions in half.





Hairy ball theorem: ("you can't comb a tennis ball")

Let $f : S^2 \to \mathbb{R}^3$ is a function such that

sphere

$f(x)$ is ~~parallel~~ ~~perpendicular~~ tangent to sphere

at every point.

Then $f(x) = 0$ for some x.

"There's always a point on Earth where the wind isn't blowing."