

EFFECTIVE CONES OF CYCLES ON BLOW-UPS OF PROJECTIVE SPACE

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ABSTRACT. In this paper, we study the cones of higher codimension (pseudo)effective cycles on point blow-ups of projective space. We determine bounds on the number of points for which these cones are generated by the classes of linear cycles, and for which these cones are finitely generated. Surprisingly, we discover that for (very) general points, the higher codimension cones behave better than the cones of divisors. For example, for the blow-up X_r^n of \mathbb{P}^n , $n > 4$, at r very general points, the cone of divisors is not finitely generated as soon as $r > n + 3$, whereas the cone of curves is generated by the classes of lines if $r \leq 2^n$. In fact, if X_r^n is a Mori Dream Space and $(r, n) \neq (8, 4)$, then all the effective cones of cycles on X_r^n are finitely generated.

1. INTRODUCTION

Let Γ be a set of r distinct points on \mathbb{P}^n . Let X_Γ^n denote the blow-up of \mathbb{P}^n along Γ . When Γ is a set of r very general points, we denote X_Γ^n by X_r^n . For a variety Y , we write $\overline{\text{Eff}}^k(Y)$ for the pseudoeffective cone of codimension- k cycles on Y , and $\overline{\text{Eff}}_k(Y)$ for the pseudoeffective cone of dimension- k cycles. In this paper, we study the cones $\overline{\text{Eff}}_k(X_\Gamma^n)$ when the points of Γ are either in linearly general or very general position. We also investigate the cones when Γ contains points in certain special configurations.

In recent years, the theory of cones of cycles of higher codimension has been the subject of increasing attention [CC], [DELV], [DJV], [FL1], [FL2]. However, these cones have been computed only for a very small number of examples, mainly because the current theory is hard to apply in practice. The goal of this paper is to provide some much needed examples.

Cones of positive divisors on X_Γ^n provide an important source of examples in the study of positivity. These cones are particularly attractive since they have concrete interpretations in terms of subvarieties of projective space, yet still have very complicated structure. However, even the cones of divisors on blow-ups of \mathbb{P}^2 at 10 or more points are far from well-understood, and several basic questions remain open, including the Nagata [N] and Segre–Harbourne–Gimigliano–Hirschowitz (SHGH) conjectures [G], [Har], [Hi]. We expect the cones of higher codimension cycles on X_Γ^n to be an equally rich source of examples.

Surprisingly, these cones are simpler than one might expect. Effective cones of low-dimensional cycles are generated by the classes of linear spaces for r well into the range for which X_r^n ceases to be a Mori Dream Space. For example, $\overline{\text{Eff}}_1(X_r^n)$ is generated by classes

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of lines for $r \leq 2^n$ even though $\overline{\text{Eff}}^1(X_r^n)$ is not finitely generated for $r \geq n + 4$ when $n \geq 5$. We now describe our results in greater detail.

Definition 1. We say that $\overline{\text{Eff}}_k(X_\Gamma^n)$ is *linearly generated* if it is the cone spanned by the classes of k -dimensional linear spaces in the exceptional divisors and the strict transforms of k -dimensional linear subspaces of \mathbb{P}^n , possibly passing through the points of Γ . We say $\overline{\text{Eff}}_k(X_\Gamma^n)$ is *finitely generated* if it is a rational polyhedral cone.

Theorem (3.1). *Let Γ be a set of r points in \mathbb{P}^n in linearly general position. If $r \leq \max(n + 2, n + \frac{n}{k})$, then $\overline{\text{Eff}}_k(X_\Gamma^n)$ is linearly generated.*

There exist configurations of $2n + 2 - k$ points in linearly general position in \mathbb{P}^n for which $\overline{\text{Eff}}_k(X_\Gamma^n)$ is not linearly generated (see Example 3.1). In particular, Theorem 3.1 is sharp for 1-cycles. We expect that this bound can be improved to $r \leq 2n + 1 - k$, and prove this in the case that Γ is a very general configuration of points (Theorem 4.3). We obtain the following consequence.

Corollary (4.4). *If X_n^r is a Mori Dream Space, then $\overline{\text{Eff}}_k(X_n^r)$ is finitely generated with the possible exception of $\overline{\text{Eff}}_2(X_8^4)$.*

Mori Dream Spaces, may have cones of intermediate dimensional cycles which are not finitely generated. A good example is [DELV, Example 6.10] attributed to Tschinkel. Let X_b be the blow-up of \mathbb{P}^4 along a smooth quartic K3 surface $Y_b \subset \mathbb{P}^3 \subset \mathbb{P}^4$. Then X_b is Fano, hence, by [BCHM], a Mori Dream Space. On the other hand, $\overline{\text{Eff}}_2(X_b)$ has infinitely many extremal rays when $\overline{\text{Eff}}_1(Y_b)$ does. Quartic K3 surfaces may have infinitely many (-2) -curves or even a round cone of curves. This example also shows that the property of having finitely generated higher codimension cones can fail countably many times in a family.

The bounds can be exponentially improved (at least for 1-cycles) if we assume that Γ is a set of very general points.

Proposition (4.1). *The cone $\overline{\text{Eff}}_1(X_r^n)$ is linearly generated if and only if $r \leq 2^n$.*

As a consequence of Proposition 4.1, we conclude that $\overline{\text{Eff}}_k(X_r^n)$ is not linearly generated if $r \geq 2^{n-k+1} + k$ (Corollary 4.2). This specializes to the fact that the cone of divisors of X_r^n is not linearly generated as soon as $r > n + 2$ (see Theorem 2.5).

Work of Mukai [Mu] shows that the cone of divisors of X_r^n is not finitely generated if $r \geq n + 4$ and $n \geq 5$ (one needs $r \geq 8$ for $n = 2, 4$ and $r \geq 7$ for $n = 3$). Mukai explicitly constructs infinitely many extremal divisors on $\overline{\text{Eff}}^1(X_r^n)$, as the orbit of one of the exceptional divisors under the action of Cremona transformations. However, in higher codimensions it is more difficult to prove that the corresponding cones become infinite.

Many questions about cones of higher codimension cycles appear to be intractable, quickly reducing to difficult questions about cones of divisors. For example, the interesting part of the cone of curves of \mathbb{P}^3 blown up at 9 points is given by curves lying on the unique quadric Q through the 9 points. This blow-up is isomorphic to the blow-up of \mathbb{P}^2 at 10 points, and the curves which are extremal on X_9^3 are certain K_Q -positive ones contained in Q . Hence understanding $\overline{\text{Eff}}_1(X_9^3)$ requires understanding the $K_{X_{10}^2}$ -positive part of $\overline{\text{Eff}}_1(X_{10}^2)$, running immediately into the SHGH conjecture (see Conjecture 1). We are able to show this non-finiteness only for cones of codimension-2 cycles, and then assuming the SHGH conjecture on the cone of curves of \mathbb{P}^2 blown up at 10 points.

Corollary (5.5). *Assume the SHGH conjecture holds for blow-ups of \mathbb{P}^2 at 10 points. Then $\overline{\text{Eff}}^2(X_r^n)$ is not finitely generated if $r \geq n + 6$ and $n \geq 3$.*

Finally, in the last section, we compute $\overline{\text{Eff}}_k(X_\Gamma^n)$ when Γ is a set of points in certain special positions. Using these computations, we show that linear and finite generation of $\overline{\text{Eff}}_k(X_\Gamma^n)$ are neither open nor closed in families (see Corollary 6.5 and Corollary 6.6). This generalizes analogous jumping behavior exhibited for divisors and Mori Dream Spaces to all codimensions.

The organization of the paper. In §2, we collect basic facts concerning the cohomology of X_Γ^n , cones of divisors, the action of Cremona transformations, and some preliminary lemmas. In §3, we prove Theorem 3.1 and study the linear generation of the cones $\overline{\text{Eff}}_k(X_\Gamma^n)$ when Γ is a linearly general set of points. In §4, we study the linear generation of the cones $\overline{\text{Eff}}_k(X_r^n)$. In §5, we prove that $\overline{\text{Eff}}^2(X_r^n)$ is not finitely generated for $r \geq n + 6$ assuming the SHGH conjecture. In §6, we discuss the cones X_Γ^n when Γ contains points in certain special configurations and study the variation of $\overline{\text{Eff}}_k(X_\Gamma^n)$ in families.

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2. PRELIMINARIES

In this section, we recall basic facts about the cohomology of X_Γ^n and cones of codimension-1 cycles. We will work over the complex numbers \mathbb{C} .

The cohomology of X_Γ^n . Let Γ be a set of r points p_1, \dots, p_r in \mathbb{P}^n , and let

$$\pi : X_\Gamma^n = \text{Bl}_\Gamma \mathbb{P}^n \rightarrow \mathbb{P}^n$$

denote the blow-up of \mathbb{P}^n along Γ . Let H denote the pullback of the hyperplane class and let E_i denote the class of the exceptional divisor over p_i . The exceptional divisor E_i is isomorphic to \mathbb{P}^{n-1} and $\mathcal{O}_{E_i}(E_i) \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. Consequently, we have the following intersection formulas:

$$H^n = (-1)^{n-1} E_i^n = 1, \quad H \cdot E_i = 0, \quad E_i \cdot E_j = 0, \quad i \neq j.$$

Notation 2.1. In order to simplify notation, we make the convention that H_k is the class of a k -dimensional linear space in \mathbb{P}^n and $E_{i,k}$ is the class of a k -dimensional linear space contained in the exceptional divisor E_i . We then have the relations

$$H^{n-k} = H_k, \quad (-1)^{n-k+1} E_i^{n-k} = E_{i,k}, \quad E_i \cdot E_{i,k} = -E_{i,k-1}.$$

On X_Γ^n homological, numerical and rational equivalence coincide. For $0 < k < n$, we write $N_k(X_\Gamma^n)$ for the \mathbb{R} -vector space of k -dimensional cycles on X_Γ^n , modulo numerical equivalence. Dually, $N^k(X_\Gamma^n)$ denotes the space of codimension- k cycles modulo numerical equivalence. They are both $(r + 1)$ -dimensional vector spaces.

A class V in $D_k(X_\Gamma^n)$ is said to be pseudoeffective if it is the limit of classes of effective cycles. We write $\overline{\text{Eff}}_k(X_\Gamma^n)$ for the closed convex cone in $N_k(X_\Gamma^n)$ containing pseudoeffective classes. If V is an (irreducible) k -dimensional subvariety of X_Γ^n , we write $[V]$ for the class of V in $N_k(X_\Gamma^n)$, although when confusion seems unlikely we omit the brackets.

A set of points in \mathbb{P}^n is said to be *linearly general* if no $k + 2$ points are contained in a linear subspace $\mathbb{P}^k \subset \mathbb{P}^n$ for $1 \leq k \leq n - 1$. A claim holds for a *very general* configuration of points

if it holds for all points in the complement of a countable union of proper configurations of points.

Convention 2.1. It is occasionally useful to compare the cones $\overline{\text{Eff}}_k(X_\Gamma^n)$ and $\overline{\text{Eff}}_k(X_\Delta^m)$, where X_Γ^n and X_Δ^m are the blow-ups of \mathbb{P}^n and \mathbb{P}^m along sets of points Γ and Δ , respectively. If $n > r$, we can identify $N_k(X_\Gamma^n)$ with the abstract vector space spanned by H_k and $E_{i,k}$ for $1 \leq i \leq r$, irrespective of n and Γ provided that Γ has cardinality r . We can thus view the cones $\overline{\text{Eff}}_k(X_\Gamma^n)$ as cones in the same abstract vector space and compare the effective cones of different blow-ups after this identification. In the rest of the paper, we will do so without further comment.

We will often use the following easy lemma implicitly.

Lemma 2.1. *Let $Y \subset X_\Gamma^n$ be a k -dimensional subvariety.*

- (1) *If $Y \subset E_i$ for some $1 \leq i \leq r$, then $[Y] = b_i E_{i,k}$ for $b_i > 0$.*
- (2) *Otherwise, $[Y] = a H_k - \sum_{i=1}^r b_i E_{i,k}$ with $a \geq b_i \geq 0$. The coefficient b_i is equal to the multiplicity of $\pi(Y)$ in \mathbb{P}^n at the point p_i .*

Proof. If $Y \subset E_i$, then Y is a subvariety of $E_i \cong \mathbb{P}^{n-1}$. Hence, its class is a positive multiple of the class of a k -dimensional linear space. The linear system $H - E_i$ defines the projection from the point p_i and is a base-point-free linear system. Hence, the intersection of k general members of $H - E_i$ with Y is either empty or finitely many points. Therefore, $(H - E_i)^k \cdot [Y] = a - b_i \geq 0$. Similarly, the intersection $Y \cap E_i$ is a (possibly empty) effective cycle of dimension $k - 1$ contained in E_i . Hence, by the first part of the lemma, $b_i \geq 0$. That b_i in fact coincides with the multiplicity is [Fu, Cor. 6.7.1]. \square

The cones $\overline{\text{Eff}}_k(X_\Gamma^n)$ satisfy a basic semicontinuity property under specialization.

Lemma 2.2. *Suppose that $V \subset \mathbb{P}^n \times T$ is a closed subvariety, flat over T , and let $p : T \rightarrow \mathbb{P}^n$ be a section. Then $\text{mult}_{p(t)}(V_t)$ is an upper semicontinuous function of T .*

Proof. It suffices to prove this in the case that T has dimension 1. Let $\pi : Y \rightarrow \mathbb{P}^n \times T$ be the blow-up along $p(T)$, with exceptional divisor E , and let \tilde{V} be the strict transform of V on Y . Since \tilde{V} is irreducible and dominates T , this family is flat. The intersection of a flat family of cycles with a Cartier divisor is constant in t [Fu, Prop. 10.2.1], and so $(-1)^k E^k \cdot V_t$ is independent of t .

The general fiber \tilde{V}_t is irreducible, but a special fiber \tilde{V}_0 may have additional components in the exceptional divisor E_0 . Write $\tilde{V}_0 = V^0 \bigcup \cup_i W_i$, where the W_i are contained in E_0 . Then $(-1)^k E^k \cdot V^0 = \text{mult}_{p(0)} V$. The class $(-1)^k E_0^k$ is a hyperplane in E_0 , and so $(-1)^k E_0^k \cdot W_i \geq 0$. This shows that $\text{mult}_{p(0)} V_0 \geq (-1)^k E^k \cdot \tilde{V}_t = \text{mult}_{p(t)} V_t$, and so the multiplicity is upper semicontinuous. \square

Corollary 2.3. *Let Γ be a configuration of r distinct points on \mathbb{P}^n . Then $\overline{\text{Eff}}_k(X_\Gamma^n) \subseteq \overline{\text{Eff}}_k(X_\Gamma^n)$.*

Proof. Let Γ_t be a very general one-parameter family of configurations of points in \mathbb{P}^n with $\Gamma_0 = \Gamma$. If a k -cycle class W is effective for very general T , then by a Chow variety argument there exists a flat family $V_t \subset \text{Bl}_{\Gamma_t} \mathbb{P}^n$ over T with $[V_t] = W$ for general T . Since the multiplicity of W_t can only increase at $t = 0$ by Lemma 2.2, the class W is also effective on X_Γ . \square

Cones. Taking cones will be a useful method to generate interesting cycles. Let Γ' be a very general configuration of $r+1$ points p'_0, \dots, p'_r in \mathbb{P}^{n+1} . The projection of the points p'_1, \dots, p'_r from p'_0 is then a set Γ of r very general points p_1, \dots, p_r in \mathbb{P}^n . Suppose that V is a k -cycle on X_Γ^n , with class $aH_k - \sum_{i=1}^r b_i E_{i,k}$. The image of V in \mathbb{P}^n has degree a and multiplicity b_i at the points of Γ . We may form the cone CV over V inside \mathbb{P}^{n+1} with vertex at p'_0 . This is a $(k+1)$ -dimensional variety, of degree a . It has multiplicity a at the cone point, and multiplicity b_i along the lines spanned by p_i and p'_i for $1 \leq i \leq r$. In particular, the cycle CV has degree a , and multiplicities a, b_1, \dots, b_r at the points of Γ' . Its proper transform has class $aH_{k+1} - aE_{0,k+1} - \sum_{i=1}^r b_i E_{i,k+1}$.

We define a map $C : N_k(X_r^n) \rightarrow N_{k+1}(X_{r+1}^{n+1})$ by $C(H_k) = H_{k+1} - E_{0,k+1}$ and $C(E_{i,k}) = E_{i,k+1}$. With this definition, $C([V])$ is the class of the cone over V with vertex p'_0 , and so $C(\overline{\text{Eff}}_k(X_r^n)) \subseteq \overline{\text{Eff}}_{k+1}(X_{r+1}^{n+1})$ with respect to the identification discussed in Convention 2.1.

A numerical lemma. The following simple computation of a dual cone will be useful on a number of occasions.

Lemma 2.4. *Suppose that $\mathbf{v} = (a, b_1, \dots, b_r) \in \mathbb{Z}^{r+1}$ is a vector satisfying*

- (1) $a, b_i \geq 0$;
- (2) $a \geq b_i$ for every i ;
- (3) $na \geq \sum_{i=1}^r b_i$.

Then \mathbf{v} is a positive linear combination of the vectors e_i ($1 \leq i \leq r$) and $h_I = e_0 - \sum_{i \in I} e_i$ with $|I| = n$.

Proof. Note first that the vectors $h_I = e_0 - \sum_{i \in I} e_i$ with $|I| < n$ are positive linear combinations of the given vectors. We now proceed by induction on a . The case $a = 1$ is clear: since by (2) $a \geq b_i$ for each i , each b_i is either 0 or 1. By (3) there are at most n nonzero b_i , and the vector is of the form claimed.

Suppose instead that $a > 1$. Let j be the minimum of n and the number of nonzero b_i , and let J the the set of indices of j nonzero b_i . Then the vector h_J is a nonnegative linear combination of the given vectors. Set $v' = (a', b'_i) = \mathbf{v} - h_J$. If $j \geq n$, v' still satisfies all of the inequalities in question since $a' = a - 1$ and $\sum b'_i = \sum b_i - n$. If $j < n$, then in view of inequality (2), the inequality (3) can be improved to $ja \geq \sum_{i=1}^r b_i$. Then v' satisfies these improved inequalities. This completes the proof by induction on a . \square

The codimension-1 cones and Cremona actions. We are primarily interested in the question of when the cones of cycles on X_Γ^n are linearly or finitely generated. For cones of divisors, the answers to these questions are well-known.

Theorem 2.5 ([Mu]). *Let Γ be a set of r very general points in \mathbb{P}^n . The cone $\overline{\text{Eff}}^1(X_\Gamma^n)$ is linearly generated if and only if $r \leq n+2$, and finitely generated if and only if*

- (1) $n = 2$ and $r \leq 8$,
- (2) $n = 3$ and $r \leq 7$,
- (3) $n = 4$ and $r \leq 8$,
- (4) $n \geq 5$ and $r \leq n+3$

The characterization of cases when the effective cone of divisors is finitely generated is based on the study of the action of Cremona transformations on the pseudoeffective cone. The Coxeter group W corresponding to a T -shaped Dynkin diagram of type $T_{2,n+1,r-n-1}$ acts on $N^1(X_\Gamma^n)$ and preserves the pseudoeffective cone $\overline{\text{Eff}}^1(X_\Gamma^n)$. This is an infinite group if

$\frac{1}{2} + \frac{1}{n+1} + \frac{1}{r-n-1} < 1$, which happens as soon as $n \geq 5$ and $r \geq n+4$. (When $n = 2$ or 4 , we need $r \geq 9$, while when $n = 3$, we require $r \geq 8$). The orbit of a single exceptional divisor class gives an infinite set of divisors spanning other extremal rays. For details on this group action, we refer to [Do].

Unfortunately, there does not seem to be a simple way to use the Cremona action to understand cones of cycles of higher codimension. The standard Cremona involution acts on X_r^n by a map with codimension 2 indeterminacy, so it does not define an action preserving the cone $\overline{\text{Eff}}^k(X_r^n)$ for any $k > 1$. For example, suppose that L is a line through two blown up points. The class of L defines an extremal ray on $\overline{\text{Eff}}_1(X_r^n)$. The strict transform of L under a Cremona transformation centered at $n+1$ other points is a rational normal curve in \mathbb{P}^n passing through $n+3$ points. If $n \geq 3$, this is no longer an extremal ray on $\overline{\text{Eff}}_1(X_r^n)$, since it is in the interior of the subcone generated by classes of lines through 2 of the $n+3$ points.

One might attempt to construct interesting codimension-2 cycles on X_r^n by taking the intersections of a fixed divisor with an infinite sequence of (-1) -divisors of increasing degree. However, the next lemma shows that the intersection of a (-1) -divisor with any other effective divisor on X_r^n is in the span of the classes of codimension-2 linear cycles.

Lemma 2.6. *Suppose that D_1 is a (-1) -divisor and that D_2 is an irreducible effective divisor distinct from D_1 . Then $D_1 \cdot D_2 \geq 0$ and $[D_1 \cap D_2]$ is in the span of linear codimension-2 cycles.*

Proof. Consider the pairing on $N^1(X_r^n)$ defined by $H \cdot H = n-1$, $H \cdot E_i = 0$, $E_i \cdot E_i = -1$, and $E_i \cdot E_j = 0$ if $i \neq j$. This pairing is invariant under the action of W on $N^1(X_r^n)$ [Mu], [Do].

We first show that $D_1 \cdot D_2 \geq 0$. Since the pairing \cdot is invariant under the action of W on $N^1(X_r^n)$, we may apply a suitable element of W and assume that $D_1 = E_1$ is an exceptional divisor. If $D_2 = E_j$ is an exceptional divisor different from E_1 , then $D_1 \cdot D_2 = 0$. Otherwise, $[D_2] = aH - \sum_{i=1}^r b_i E_i$, with $b_i \geq 0$, in which case $D_1 \cdot D_2 = b_1 \geq 0$.

For the second part, write $D_1 = aH - \sum_{i=1}^r b_i E_i$ and $D_2 = cH - \sum_{i=1}^r d_i E_i$. That $D_1 \cdot D_2 \geq 0$ yields

$$(n-1)ac \geq \sum_{i=1}^r b_i d_i.$$

By Lemma 2.4, this means that the codimension-2 cycle $[D_1 \cap D_2] = acH - \sum_{i=1}^r b_i d_i$ is contained in the span of linear cycles. \square

Easy Lemmas. Here we collect a couple of geometric lemmas that we will use repeatedly.

Lemma 2.7. *Suppose that E is an effective divisor and that P is a nef divisor. If Y is an irreducible, effective variety of dimension k which is not contained in E , then $P^{k-1} \cdot E \cdot Y \geq 0$.*

Proof. The intersection $E \cdot Y$ is a (possibly empty) cycle of dimension $k-1$ by assumption. Since P is nef, it follows that $P^{k-1} \cdot E \cdot Y \geq 0$. \square

Lemma 2.8. *Let $Y \subset X_r^n$ be an irreducible variety of dimension k , not contained in any exceptional divisor E_i , with class $aH_k - \sum_{i=1}^e b_i E_{i,k}$. If $b_i + b_j > a$ for two indices $i \neq j$, then Y contains the line through p_i and p_j with multiplicity at least $b_i + b_j - a$.*

Proof. The base locus of the linear system $|H - E_i - E_j|$ is the line $l_{i,j}$ spanned by p_i and p_j . Consequently, the intersection $(H - E_i - E_j)^{k-1} \cdot Y$ is an effective 1-cycle Z . Express

$$Z = \alpha l_{i,j} + u,$$

where u is a 1-cycle not containing $l_{i,j}$. Since

$$-\alpha \leq (H - E_i - E_j) \cdot Z = a - b_i - b_j < 0,$$

we conclude that $\alpha \geq b_i + b_j - a$. Hence, Y must have multiplicity at least $b_i + b_j - a$ at every point of $l_{i,j}$. \square

3. LINEARLY GENERATED CONES

In this section, we study $\overline{\text{Eff}}_k(X_\Gamma^n)$ when the cardinality of Γ is small and the points of Γ are in linearly general position. Our main theorem is the following.

Theorem 3.1. *Let Γ be a set of r points in \mathbb{P}^n in linearly general position. If*

$$r \leq \max \left(n + 2, n + \frac{n}{k} \right),$$

then $\overline{\text{Eff}}_k(X_\Gamma^n)$ is linearly generated.

The proof will be by induction on k and n . We first single out the case $k = 1$.

Lemma 3.2. *Let Γ be a set of $r \leq 2n$ points in \mathbb{P}^n in linearly general position. Then $\overline{\text{Eff}}_1(X_\Gamma^n)$ is linearly generated.*

Proof. Let B be an irreducible curve. By Lemma 2.1, we may assume that B is not contained in any of the exceptional divisors and has class $aH_1 - \sum_{i=1}^r b_i E_{i,1}$ with $a \geq b_i \geq 0$. Any $r \leq 2n$ points in linearly general position are cut out by quadrics [H, Lecture 1]. Consequently, there is a quadric whose proper transform has class $[Q] = 2H - \sum_{i=1}^r E_i$ in X_Γ^n and does not contain B . Hence, B has nonnegative intersection with Q and satisfies $2a \geq \sum_{i=1}^r b_i$. By Lemma 2.4, the class of B is spanned by the classes of lines. \square

Next, we study the case when $r \leq n + 1$. In this case, X_Γ^n is toric and the effective cones are generated by torus-invariant cycles (see e.g. [Hu, Theorem 33]). For the reader's convenience we will give a simple independent proof.

Lemma 3.3. *Let Γ be a set of $r \leq n + 1$ linearly general points in \mathbb{P}^n . The cone $\overline{\text{Eff}}_k(X_\Gamma^n)$ is linearly generated for any k .*

Proof. Let $\Gamma' \subset \Gamma$ be two sets with cardinality r and $n + 1$, respectively. Then the proper transform of any effective cycle in X_Γ^n is an effective cycle in $X_{\Gamma'}^n$. Consequently, if $\overline{\text{Eff}}_k(X_\Gamma^n)$ is linearly generated, then $\overline{\text{Eff}}_k(X_{\Gamma'}^n)$ is also linearly generated. Hence, without loss of generality, we may assume that $r = n + 1$. Let Y be an irreducible k -dimensional variety in X_Γ^n with class

$$[Y] = aH_k - \sum_{i=1}^{n+1} b_i E_{i,k}.$$

By Lemma 2.1, we may assume that Y is not contained in an exceptional divisor and that $a \geq b_i \geq 0$. We proceed by induction on k and n . Up to reordering the points, we may assume $b_1 \geq b_2 \geq \dots \geq b_{n+1}$. Let L be the proper transform of the \mathbb{P}^{n-1} spanned by the first n points. First, suppose Y is contained in L . Since L is isomorphic to the blow-up of \mathbb{P}^{n-1} in n points, by induction on n with base case Theorem 2.5, we conclude that the class of Y is in the span of linear spaces. Otherwise, $Y \cap L$ is an effective cycle of dimension $k - 1$ in L .

Write $H_{L,k-1}$ and $E_{L,i,k-1}$ for the restriction of H_k to L and the $(k-1)$ -dimensional linear space in the exceptional divisor $E_{L,i}$ of the blow-up of L at p_i . Then we have

$$[Y \cap L] = aH_{L,k-1} - \sum_{i=1}^n b_i E_{L,i,k-1}.$$

By induction on n with base case Lemma 3.2, $Y \cap L$ is in the span of linear spaces. In particular, $ka \geq \sum_{i=1}^n b_i$. Hence, $(k+1)a \geq \sum_{i=1}^{n+1} b_i$. By Lemma 2.4, the class of Y is in the span of linear spaces. This concludes the proof. \square

We can now complete the proof of Theorem 3.1.

Proof of Theorem 3.1. We preserve the notation from the proof of Lemma 3.3 and argue similarly. Suppose that Y is an irreducible k -dimensional variety on X_{Γ}^n with class

$$[Y] = aH_k - \sum_{i=1}^r b_i E_{i,k}.$$

We may assume that Y is not contained in an exceptional divisor and, by reordering the points, we have that

$$a \geq b_1 \geq \cdots \geq b_r \geq 0.$$

Let L be the \mathbb{P}^{n-1} passing through the points p_1, \dots, p_n . If Y is contained in L , then its class is linearly generated by Lemma 3.3. Otherwise, $Y \cap L$ is an effective cycle of dimension $k-1$ with class

$$[Y \cap L] = aH_{L,k-1} - \sum_{i=1}^n b_i E_{L,i,k-1}$$

which is also linearly generated by Lemma 3.3. Therefore, the class $[Y \cap L]$ can be written as a combination of linear classes $H_{L,k-1} - \sum_{|I|=k} E_{L,i,k-1}$ and $E_{L,i,k-1}$

$$[Y \cap L] = \sum_{j=1}^a \alpha_j \left(H_{L,k-1} - \sum_{|I|=k} E_{L,i,k-1} \right) + \sum_{j=1}^n \beta_j E_{L,j,k-1}.$$

Each of the classes in this sum is effective, with those on the left the classes of \mathbb{P}^{k-1} through k of the points in L . By taking cones over these classes, we obtain a \mathbb{P}^k on X , passing through an additional one of the points p_i with $i > n$. Since there are a planes available, if $\sum_{i=n+1}^r b_i \leq a$, the class Y can be expressed as a sum of linear cycles.

Observe that

$$ak \geq \sum_{i=1}^n b_i \geq nb_n, \text{ and so } b_j \leq b_n \leq \frac{ak}{n} \text{ for } j \geq n.$$

This implies that if $(r-n)\frac{k}{n} \leq 1$ or equivalently if $r \leq n + \frac{n}{k}$, the classes of all effective cycles are in the span of the classes of linear spaces.

If $k \leq \frac{n}{2}$, then $n+2 \leq n + \frac{n}{k}$ and the theorem is proved. If $k > \frac{n}{2}$, then $n+1 < n + \frac{n}{k} < n+2$ and we need to settle the case $r = n+2$. There is a rational normal curve through any $n+3$ points in linearly general position in \mathbb{P}^n [H, Lecture 1]. Consequently, given an effective divisor D , there exists a rational normal curve C containing the points but not contained in D . Hence, $C \cdot D \geq 0$ and all effective divisors satisfy $na \geq \sum_{i=1}^{n+2} b_i$. We recover the linear generation result of Theorem 2.5. By Lemma 3.2, the curve classes are also linearly generated. By induction assume that for all $m < n$ and all $k < m$, the effective cone of k

cycles of the blow-up of \mathbb{P}^m in $m + 2$ linearly general points is linearly generated. We carry out the inductive step for \mathbb{P}^n . Let Y, L be as above. By Lemma 3.3, we may assume that Y is not contained in L . If $b_{n+1} + b_{n+2} \leq a$, then we already proved that the class of Y is linearly generated. If $b_{n+1} + b_{n+2} > a$, then, by Lemma 2.8, Y contains the line $l_{n+1,n+2}$ spanned by p_{n+1} and p_{n+2} with multiplicity at least $b_{n+1} + b_{n+2} - a$. Let p_0 denote the point of intersection $L \cap l_{n+1,n+2}$. Then the proper transform of $L \cap Y$ is an effective cycle in the blow-up of L in p_0, p_1, \dots, p_n with class

$$aH_{L,k-1} - (b_{n+1} + b_{n+2} - a)E_{L,0,k-1} - \sum_{i=1}^n b_i E_{L,i,k-1}.$$

By induction on n , this class is linearly generated. Hence,

$$ka \geq b_{n+1} + b_{n+2} - a + \sum_{i=1}^n b_i, \quad \text{therefore} \quad (k+1)a \geq \sum_{i=1}^{n+2} b_i.$$

By Lemma 2.4, the class of Y is linearly generated. This concludes the proof. \square

Example 3.1. Lemma 3.2 is sharp in the sense that there exist sets Γ of $r > 2n$ points in general linear position such that $\overline{\text{Eff}}_1(X_\Gamma^n)$ is not linearly generated. For example, let Γ be $r > 2n$ points on a rational normal curve C in \mathbb{P}^n . Points on a rational normal curve are in general linear position [H]. Then the class of the proper transform of C is

$$nH_1 - \sum_{i=1}^r E_{i,1}.$$

Since $r > 2n$, this class cannot be in the span of the classes of lines. In the next section, we will see that we can improve the bounds for linear generation exponentially if instead of assuming that Γ is linearly general, we assume Γ is a set of very general points in \mathbb{P}^n .

More generally, let Y be the cone over a rational normal curve of degree $n - k + 1$ with vertex V a \mathbb{P}^{k-2} . Let Γ be the union of a set of $k - 1$ general points p_1, \dots, p_{k-1} in V and a set of $r - k + 1$ general points p_k, \dots, p_r on Y . Then Γ is in general linear position. The class of the proper transform of Y is

$$(n - k + 1)H_k - \sum_{i=1}^{k-1} (n - k + 1)E_{i,k} - \sum_{i=k}^r E_{i,k},$$

which cannot be in the span of linear spaces if $r > 2n - k + 1$. Consequently, we conclude the following.

Proposition 3.4. *There exists sets Γ of $r > 2n - k + 1$ points in general linear position in \mathbb{P}^n such that $\overline{\text{Eff}}_k(X_\Gamma^n)$ is not linearly generated.*

In view of Proposition 3.4, it is natural to ask whether the bound in Theorem 3.1 can be improved to $r \leq 2n - k + 1$.

Question 3.1. Assume that Γ is a set of r linearly general points in \mathbb{P}^n such that

$$\max\left(n + 2, n + \frac{n}{k}\right) < r \leq 2n - k + 1.$$

Is $\overline{\text{Eff}}_k(X_\Gamma^n)$ linearly generated?

The answer is affirmative for curves and divisors. We will shortly check that for 2-cycles in \mathbb{P}^4 the answer is also affirmative. In Theorem 4.3 we will see that the answer is also affirmative if the points are very general. In view of this evidence, we expect the answer to Question 3.1 to be affirmative.

Remark 3.1. The dimension of the space $\mathcal{S}_{n-k,k+1}(\mathbb{P}^n)$ of scrolls of dimension $n-k$ and degree $k+1$ in \mathbb{P}^n is

$$2n + 2nk - k^2 - 2$$

[C1, Lemma 2.4]. There are scrolls in $\mathcal{S}_{n-k,k+1}(\mathbb{P}^n)$ passing through $2n-k+2$ points (see [C2] for the surface case). Hence, the family of scrolls passing through $2n-k+1$ points covers \mathbb{P}^n . By Lemma 2.4, an affirmative answer to Question 3.1 is equivalent to the statement that every effective k -dimensional cycle intersects the proper transform of a scroll passing through the $2n-k+1$ points non-negatively.

Question 3.2. Let Γ be $2n-k+1$ linearly general points in \mathbb{P}^n . For every effective k cycle Y in X_Γ^n , does there exist a scroll S of dimension $n-k$ and degree $k+1$ such that the proper transform \bar{S} in X_Γ^n intersects Y in finitely many points?

By Remark 3.1, an affirmative answer to Question 3.2 implies an affirmative answer to Question 3.1.

Effective 2-cycles on the blow-up of \mathbb{P}^4 at 7 points. We now verify that the answer to Question 3.1 is affirmative for two-cycles in \mathbb{P}^4 . The argument is subtle because we need to verify linear generation for *every* configuration of 7 points in linear general position.

Theorem 3.5. *Let Γ be 7 linearly general points on \mathbb{P}^4 . Then the cone $\overline{\text{Eff}}_2(X_\Gamma^4)$ is linearly generated.*

Proof. There is a unique rational normal quartic curve R containing 7 linearly general points in \mathbb{P}^4 [H]. The secant variety $\text{Sec}(R)$ to R is a cubic hypersurface which is double along R . Hence, its proper transform $\bar{\text{Sec}}(R)$ in X_Γ^4 has class $3H - \sum_{i=1}^7 2E_i$.

Let Y be an irreducible surface in X_Γ^4 . Without loss of generality, we may assume that Y is not contained in an exceptional divisor and has class $aH_2 - \sum_{i=1}^7 b_i E_{i,2}$ with $a \geq b_i \geq 0$. First, suppose that Y is not contained in $\bar{\text{Sec}}(R)$. The class of a quadric $[Q] = H - \sum_{i=1}^7 E_i$ is nef, and so by Lemma 2.7 we have

$$Y \cdot Q \cdot \bar{\text{Sec}}(R) = 6a - \sum_{i=1}^7 2b_i \geq 0.$$

Lemma 2.4 implies that $[Y]$ is in the span of the classes of planes.

We are reduced to showing that if $Y \subset \bar{\text{Sec}}(R)$, then $[Y]$ is in the span of the classes of planes. Let \mathcal{S}_3 denote the space of cubic surface scrolls containing the points of Γ . We will show the following.

Theorem 3.6. *The proper transform \bar{S} of a general member $S \in \mathcal{S}_3$ intersects $\bar{\text{Sec}}(R)$ in an irreducible curve B whose projection to $\text{Sec}(R)$ is a degree 9 curve double at the points of Γ . Furthermore, the curve B can be made to pass through a general point of $\bar{\text{Sec}}(R)$.*

Assume Theorem 3.6. Let $p \in \bar{\text{Sec}}(R)$ be a general point not contained in Y . Hence, an irreducible curve B passing through p intersects Y in finitely many points. Let \bar{S} be the proper transform of a scroll $S \in \mathcal{S}_3$ containing p and intersecting $\bar{\text{Sec}}(R)$ in an irreducible

curve. We conclude that \overline{S} and Y intersect in finitely many points, hence their intersection number is positive. Therefore,

$$[\overline{S}] \cdot [Y] = (3H_2 - \sum_{i=1}^7 E_{i,2}) \cdot (aH_2 - \sum_{i=1}^7 b_i E_{i,2}) = 3a - \sum_{i=1}^7 b_i \geq 0.$$

By Lemma 2.4, we conclude that $[Y]$ is in the span of the classes of planes.

There remains to prove Theorem 3.6, which we will do via a series of claims. We first set some notation.

Notation 3.1. Let $l_{i,j}$ denote the line spanned by $p_i, p_j \in \Gamma$ and let Π_{i_1, \dots, i_l} denote the linear space spanned by $p_{i_1}, \dots, p_{i_l} \in \Gamma$. Let Γ_{i_1, \dots, i_l} denote the set of points p_{i_1}, \dots, p_{i_l} . Let l be the line of intersection $\Pi_{1,2,3,4} \cap \Pi_{5,6,7}$ and, for $5 \leq i < j \leq 7$, let $z_{i,j}$ denote the point of intersection $\Pi_{1,2,3,4} \cap l_{i,j}$.

Next, we recall a compactification $\overline{\mathcal{S}}_3$ of \mathcal{S}_3 . Every irreducible cubic scroll induces a degree 3 rational curve in the Grassmannian $\mathbb{G}(1, 4)$ of lines in \mathbb{P}^4 . We can compactify the space of degree 3 rational curves in $\mathbb{G}(1, 4)$ via the Kontsevich moduli space. Hence, we can take the closure of \mathcal{S}_3 in the Kontsevich moduli space (see [C2, §3] for details). More precisely, let $\overline{\mathcal{M}}_{0,7}(\mathbb{G}(1, 4), 3)$ denote the Kontsevich moduli space of 7-pointed genus-0 maps of degree 3 to $\mathbb{G}(1, 4)$. It is equipped with 7 evaluation morphisms $\text{ev}_i : \overline{\mathcal{M}}_{0,7}(\mathbb{G}(1, 4), 3) \rightarrow \mathbb{G}(1, 4)$, $1 \leq i \leq 7$. Define

$$\overline{\mathcal{S}}_3 = \bigcap_{i=1}^7 \text{ev}_i^{-1}(\Sigma_3(p_i)),$$

where $\Sigma_3(p_i)$ denotes the Schubert variety of lines containing p_i .

Claim 3.7. *The space $\overline{\mathcal{S}}_3$ is irreducible of dimension 4.*

Proof. The locus $\overline{\mathcal{T}} = \bigcap_{i=1}^4 \text{ev}_i^{-1}(p_i)$ in the Kontsevich moduli space $\overline{\mathcal{M}}_{0,4}(\mathbb{P}^3, 3)$ of 4-pointed genus 0 maps of degree 3 to \mathbb{P}^3 provides a compactification of the space of twisted cubic curves in $\Pi_{1,2,3,4}$ containing $\Gamma_{1,2,3,4}$. Since $\overline{\mathcal{M}}_{0,4}(\mathbb{P}^3, 3)$ is irreducible of dimension 16, every component of $\overline{\mathcal{T}}$ has dimension at least 4.

If a twisted cubic T is irreducible, then any finite set of points on T is linearly general. Furthermore, given 6 linearly general points in \mathbb{P}^3 , there is a unique twisted cubic curve containing them. Consider the incidence correspondence $I = \{(T, q_1, q_2) | q_1, q_2 \in T\}$, where T is a twisted cubic curve containing the set of points $\Gamma_{1,2,3,4}$ and q_1, q_2 are points such that $\Gamma_{1,2,3,4} \cup \{q_1, q_2\}$ are in linearly general position. The incidence correspondence I is irreducible of dimension 6 since it is isomorphic to an open subset of $\mathbb{P}^3 \times \mathbb{P}^3$. It dominates the space of twisted cubic curves containing p_1, \dots, p_4 via the first projection. Since the fibers of the first projection are two-dimensional, we conclude that the space of irreducible twisted cubics containing $\Gamma_{1,2,3,4}$ is irreducible of dimension 4.

Since there are no connected curves of degree two or one containing 4 points in linearly general position in \mathbb{P}^3 , any map in $\overline{\mathcal{T}}$ is birational to its image. If there is a reducible curve of degree 3 containing $\Gamma_{1,2,3,4}$, either a degree two curve must contain 3 of the points or a line must contain two of the points. In either case, it is easy to see that there is a 3-dimensional family of reducible cubics containing $\Gamma_{1,2,3,4}$. Hence, these cannot form a component of $\overline{\mathcal{T}}$ and $\overline{\mathcal{T}}$ is irreducible.

Furthermore, 2 additional points q_1, q_2 impose independent conditions on twisted cubics unless they are coplanar with three of the points in $\Gamma_{1,2,3,4}$ or one of the points is collinear with two of the points in $\Gamma_{1,2,3,4}$. If q_1 is collinear with p_1, p_2 , then there is a 1-parameter family of reducible cubics containing the line $l_{1,2}$. Similarly, if q_1 and q_2 are in $\Pi_{1,2,3}$ but no 4 of the points are collinear, then there is a 1-parameter family of reducible cubics containing the conic through $\Gamma_{1,2,3} \cup \{q_1, q_2\}$. If q_1, q_2 are collinear with p_1 and p_2 , there is a three-parameter family of reducible cubics containing $l_{1,2}$. Recall that $l = \Pi_{1,2,3,4} \cap \Pi_{5,6,7}$. In particular, the subset of $\overline{\mathcal{T}}$ that parameterizes twisted cubics incident (respectively, secant) to l has dimension 3 (respectively, 2) since any pair of distinct points impose independent conditions on twisted cubics. Similarly, the locus of twisted cubics in $\overline{\mathcal{T}}$ passing through $z_{5,6}$ has dimension 2.

Since $\overline{\mathcal{M}}_{0,7}(\mathbb{G}(1,4), 3)$ is irreducible of dimension 25 [C2, §2], every irreducible component of $\overline{\mathcal{S}}$ has dimension at least 4. Let T be a twisted cubic curve containing $\Gamma_{1,2,3,4}$ and not secant to the line l and not containing the points $z_{5,6}, z_{5,7}$ and $z_{6,7}$. Then there is a unique cubic scroll S containing T and passing through p_5, p_6, p_7 [C2, Example A1]. Briefly, take a general \mathbb{P}^3 containing $\Pi_{5,6,7}$. This \mathbb{P}^3 intersects T in 3 points r_1, r_2, r_3 . There is a unique twisted cubic curve T' containing r_1, r_2, r_3 and $\Gamma_{5,6,7}$. The curves T and T' are both isomorphic to \mathbb{P}^1 and there is a unique isomorphism ϕ taking $r_i \in T$ to $r_i \in T'$. Then the surface $S_{T,T'}$ swept out by lines joining the points that correspond under ϕ is the unique cubic scroll containing T and $\Gamma_{5,6,7}$. If T contains the point $z_{5,6}$ or is secant to the line l , then there is a 1-parameter family of choices for T' . Once we fix T and T' , the scroll is uniquely determined by a similar construction. Since the locus of T containing $z_{5,6}$ or secant to l has codimension 2, this locus cannot form a component of $\overline{\mathcal{S}}_3$. Finally, reducible cubic surfaces containing Γ must contain a plane through 3 of the points and a quadric surface through the remaining 4 points. There is a 2-dimensional family of such surfaces and they do not give rise to a component in $\overline{\mathcal{S}}_3$ (see [C2]). We conclude that $\overline{\mathcal{S}}_3$ is irreducible of dimension 4. \square

Claim 3.8. *There exists a dense open set $U \subset \overline{\mathcal{S}}_3$ such that $S \not\subset \text{Sec}(R)$ for $S \in U$. Furthermore, we may assume that S contains a general point of $\text{Sec}(R)$.*

Proof. It suffices to exhibit one $S \in \overline{\mathcal{S}}_3$ such that $S \not\subset \text{Sec}(R)$. Given 7 points in general linear position and 2 general additional points, [C2, Example A1] shows that there are 2 cubic scrolls containing these nine points. In particular, if we take one of the two additional points outside $\text{Sec}(R)$ and one of the points a general point of $\text{Sec}(R)$, the claim follows. Alternatively, one can explicitly construct such a scroll using the proof of Claim 3.7. Take a twisted cubic containing $\Gamma_{1,2,3,4}$, but not contained in $\text{Sec}(R)$ and not intersecting l . Then the irreducible scroll $S_{T,T'}$ constructed in the proof of Claim 3.7 provides an explicit point in \mathcal{S}_3 not contained in $\text{Sec}(R)$. \square

Claim 3.9. *There exists a dense open set $U \subset \overline{\mathcal{S}}_3$ such that for $S \in U$ the following hold:*

- (1) *The intersection $\overline{S} \cap \overline{\text{Sec}}(R) \cap E_i$ is a finite set of points in X_Γ^4 for every $1 \leq i \leq 7$.*
- (2) *The scroll S does not contain any lines $l_{i,j}$ for $1 \leq i < j \leq 7$.*
- (3) *The scroll S does not contain any conics through 3 of the points in Γ .*
- (4) *The scroll S does not contain a twisted cubic curve through 5 of the points of Γ .*
- (5) *The scroll S does not contain the rational normal quartic R .*
- (6) *The scroll S does not contain a quintic curve double at one of the points of Γ and passing through the others.*
- (7) *The directrix of the scroll does not contain any of the points in Γ .*

Proof. Since each of these conditions are closed conditions and $\overline{\mathcal{S}}_3$ is irreducible, it suffices to exhibit one element $S \in \overline{\mathcal{S}}_3$ satisfying each condition. For (1), there exists a twisted cubic containing $\Gamma_{1,2,3,4}$ with any tangent line at p_1 (for example, the reducible twisted cubic consisting of any line through p_1 and a conic through $\Gamma_{2,3,4}$). Hence, the tangent spaces to the scrolls at p_1 sweep out E_1 and there exists S such that $\overline{S} \cap E_1 \not\subset \overline{\text{Sec}}(R)$. By permuting indices, we conclude (1).

For (2) and (3), take the scroll $S_{T,T'}$ constructed in the proof of Claim 3.7. Since $\Pi_{1,2,3,4} \cap S_{T,T'} = T$, this scroll does not contain any of the linear $l_{i,j}$ with $1 \leq i < j \leq 4$ or any conic passing through any of the three points in $\Gamma_{1,2,3,4}$. By permuting indices, we conclude (2) and (3).

Since a twisted cubic curve spans a \mathbb{P}^3 and the points are in linearly general position (4) is clear. For (5), (6) and (7), it is more convenient to exhibit a reducible scroll satisfying these properties. Let S be the union of the plane $\Pi_{5,6,7}$ and a general quadric surface Q containing l and $\Gamma_{1,2,3,4}$. After choosing a point of l , this surface determines a point p of $\overline{\mathcal{S}}_3$ [C2]. The directrix line is then the unique line on the quadric Q intersecting l at p . Hence, (7) holds. Since R is irreducible and nondegenerate, it cannot be contained in this surface. Suppose there is a quintic curve F in S containing Γ and double at p_1 . Since p_5, p_6, p_7 are not collinear, F must intersect $\Pi_{5,6,7}$ in a curve of degree at least 2. Hence, F intersects Q in a curve of degree 3 containing $\Gamma_{2,3,4}$ and double at p_1 . Any cubic double at p_1 must contain the line of ruling through p_1 . Since $\Gamma_{1,2,3,4}$ are linearly general there cannot be a degree 2 curve through these points on Q . After permuting indices, we conclude (6) holds. \square

Claim 3.10. *There exists a dense open set $U \subset \overline{\mathcal{S}}_3$ such that for $S \in U$ the intersection $S \cap \text{Sec}(R)$ is an irreducible degree 9 curve double along Γ .*

Proof. By Claim 3.8 and Claim 3.9, we can find a scroll $S \not\subset \text{Sec}(R)$ and satisfying the conclusions of Claim 3.9. The intersection $S \cap \text{Sec}(R)$ is a curve B of degree 9 double along Γ . We need to show that B is irreducible. Recall that a smooth cubic scroll is isomorphic to the blow-up of \mathbb{P}^2 at a point. Its Picard group is generated by the directrix e (the curve of self-intersection -1) and the class of a fiber line f . The intersection numbers are

$$e^2 = -1, \quad e \cdot f = 1, \quad f^2 = 0.$$

The effective cone is spanned by e and f . The canonical class is $-2e - 3f$ and the class of B is $3e + 6f$. The degree of a curve $ke + mf$ is $k + m$. If $k > m$, then any representative contains e with multiplicity $k - m$. By adjunction, the arithmetic genus of a curve in the classes $e + mf$, $2e + mf$ and $3e + mf$ are 0, $m - 2$ and $2m - 5$, respectively.

It is now straightforward, but somewhat tedious to check that B cannot be reducible. Suppose B is reducible. Unfortunately, B can have many components. Write $B = B_1 \cup B_2$, where the class of B_1 is $ke + mf$ with $2 \leq k \leq 3$ and assume that B_1 does not contain any fibers as components. Furthermore, if $k = 3$, we may assume that B_1 is irreducible. Otherwise, we can regroup a component with class $e + m'f$ with B_2 . Then the class of B_2 is $(3 - k)e + (6 - m)f$ and every fiber component of B is included in B_2 . By Claim 3.9 (2), a curve with class mf can be double at most in $0 \leq d \leq \frac{m}{2}$ points of Γ in which case it can contain at most $m - 2d$ of the remaining points of Γ . We tabulate the possibilities for curves with class $e + mf$.

class	# double points of Γ	# remaining points of Γ resp.	Reason
e	0	0	Claim 3.9 (7)
$e + f$	0	2	Claim 3.9 (3,2)
$e + 2f$	0 or 1	4 or 1	Claim 3.9 (4,3,2)
$e + 3f$	0, 1 or 2	6, 3 or 0	Claim 3.9 (7,5,4,3,2)
$e + 4f$	0, 1 or 2	7, 5 or 2	Claim 3.9 (7,5,4,3,2)

First, suppose B_1 has class $3e + mf$. By assumption, it is irreducible and by arithmetic genus considerations can have at most $2m - 5$ nodes. On the other hand, B_2 can pass through at most $6 - m$ of the points. We have $2m - 5 + 6 - m = m + 1 < 7$ if $m < 6$. Hence, such a curve cannot be double at all the points of Γ .

We may therefore assume that the class of B_1 is $2e + mf$ and the class of B_2 is $e + (6 - m)f$. If B_1 is reducible, then it can have at most 2 components with classes $e + m_1f$ and $e + m_2f$. An inspection of the above table shows that it is not possible to make B double at all points of Γ . If B_1 is irreducible, then $m \geq 2$ and its arithmetic genus is $m - 2$. Hence, the maximal number of double points on B_1 is $m - 2$. If $m = 2$, B_1 can contain at most 6 of the points of Γ by Claim 3.9 (5) and it is smooth at those points. Hence, B cannot be made double at all points of Γ by the last line of the table. If $m = 3$ and B_1 has a double point, then by Claim 3.9 (6) B_1 contains at most 5 other points of Γ . By the second to last row of the table, B cannot be double at all points of Γ . If $m \geq 4$, an easy inspection of the first three rows of the table show that B can have at most 6 double points. We conclude that B is irreducible. \square

This concludes the proof of Theorem 3.6 and consequently of Theorem 3.5. \square

4. NON-LINEARLY GENERATED CONES

Recall that X_r^n denotes the blow-up of \mathbb{P}^n in r very general points. In this section, we study the cones of effective cycles on X_r^n . Our first result completely characterizes when the cone of curves is linearly generated.

Proposition 4.1. *The cone $\overline{\text{Eff}}_1(X_r^n)$ is linearly generated if and only if $r \leq 2^n$.*

Proof. We first observe that the linear system of quadrics through 2^n very general points is nef. Choose n general quadrics Q_1, \dots, Q_n in \mathbb{P}^n . By Bertini's theorem, the intersection of these quadrics is a set of 2^n points in \mathbb{P}^n . Let X_0 be the blow-up of \mathbb{P}^n at these points. We claim that $D = 2H - \sum_{i=1}^{2^n} E_i$ is nef on X_0 . Note that the proper transforms of Q_i have class D and D has positive degree on curves contained in exceptional divisors E_i . Since the intersection $Q_1 \cap \dots \cap Q_n$ is finite, if B is a curve on X_0 not contained in an exceptional divisor, there is a quadric Q_i whose proper transform does not contain B . Consequently, $D \cdot B \geq 0$ and D is nef. By [La, Prop. 1.4.14], $2H - \sum_{i=1}^{2^n} E_i$ is nef for very general configurations of 2^n points as well. We conclude that if $r \leq 2^n$, an effective curve class in X_r^n satisfies the inequalities in the assumptions of Lemma 2.4, and so every curve class is a linear combination of classes of lines.

The top self-intersection of the class $Q = 2H - \sum_{i=1}^r E_i$ on X_r^n is given by $2^n - r$. Hence, if $r > 2^n$, then the top self-intersection of Q is negative and Q cannot be nef by Kleiman's Theorem [La, Theorem 1.4.9]. Suppose the class of every effective curve is in the span of the classes of lines. The cone generated by the classes of lines is a closed cone. Hence, the effective and the pseudoeffective cones coincide. Since every line has nonnegative intersection

with Q , we conclude that Q is nef. This contradiction shows that there must exist effective curves whose classes are not spanned by the classes of lines. \square

We obtain the following consequence.

Corollary 4.2. *If $r \geq 2^{n-k+1} + k$, then $\overline{\text{Eff}}_k(X_r^n)$ is not linearly generated.*

Proof. Let Γ be a set of r very general points. Project Γ from the first $k-1$ points p_1, \dots, p_{k-1} and let Γ' be the set of points in \mathbb{P}^{n-k+1} consisting of the images of the remaining points. Then Γ' is a set of $r-k+1$ very general points in \mathbb{P}^{n-k+1} . If $r-k+1 > 2^{n-k+1}$, the cone $\overline{\text{Eff}}_1(X_{r-k+1}^{n-k+1})$ is not linearly generated by Proposition 4.1. Fix a 1-cycle B with class $aH_1 - \sum_{i=k}^r b_i E_{i,1}$ that is not in the span of linear spaces. In particular, $2a < \sum_{i=k}^r b_i$. Then the class

$$aH_k - \sum_{i=1}^{k-1} aE_{i,k} - \sum_{i=k}^r b_i E_{i,k}$$

is represented in X_r^n by the proper transform of the cone over B with vertex the span of p_1, \dots, p_{k-1} . The resulting k -cycle is not in the span of k -dimensional linear spaces since $(k+1)a < (k-1)a + \sum_{i=k}^r b_i$. \square

Question 4.1. If $r < 2^{n-k+1} + k$, is $\overline{\text{Eff}}_k(X_r^n)$ linearly generated?

The answer to Question 4.1 is affirmative for curves and divisors. For cycles of intermediate dimension, we do not know any examples with $r = 2^{n-k+1} + k - 1$ where the cone is linearly generated. We can, however, give a linear bound.

Theorem 4.3. *The cone $\overline{\text{Eff}}_k(X_r^n)$ is linearly generated if $r \leq 2n - k + 1$.*

Proof. The theorem is true for $k = 1$ by Lemma 4.1 and for divisors by Theorem 3.1. We will prove the general case by induction on n . Assume that the theorem is true for $\overline{\text{Eff}}_k(X_r^m)$ for $r \leq 2m - k + 1$ and all $k < m < n$. Let Γ be a set of r points such that Γ consists of $r-2$ very general points p_1, \dots, p_{r-2} in a hyperplane $L = \mathbb{P}^{n-1}$ and two very general points p_{r-1}, p_r not contained in L . Let L' denote the proper transform of L in X_Γ^n . Note that $L' \cong X_{r-2}^{n-1}$. Let Y be an irreducible k -dimensional subvariety of X_Γ^n not contained in an exceptional divisor with class $aH_k + \sum_{i=1}^r b_i E_i$ on X_Γ^n . If Y is contained in L' , then $Y \subset X_{r-2}^{n-1}$. Since $r-2 \leq 2(n-1) - k + 1$, by the induction hypothesis the class of Y is linearly generated and $(k+1)a \geq \sum_{i=1}^r b_i$. If Y is not contained in L' , then $Z = Y \cap L'$ is an effective cycle of dimension $k-1$. Let p_0 denote the intersection of the line $l_{1,2}$ spanned by p_{r-1} and p_r with L . Let $\beta = \max(0, b_{r-1} + b_r - a)$. Consider the blow-up X_{r-1}^{n-1} of L along p_0, p_1, \dots, p_{r-2} . Then the proper transform of Z is an effective cycle in X_{r-1}^{n-1} with class

$$aH_{k-1} - \beta E_{0,k-1} - \sum_{i=1}^{r-2} b_i E_{i,k-1}.$$

Since $r \leq 2n - k + 1$, the inductive hypothesis $r-1 \leq 2(n-1) - (k-1) + 1$ is satisfied. We conclude that this class is linearly generated. Consequently,

$$ka \geq \beta + \sum_{i=1}^{r-2} b_i \quad \text{and hence} \quad (k+1)a \geq \sum_{i=1}^r b_i.$$

By Lemma 2.4, the class of Y is linearly generated. By Corollary 2.3, $\overline{\text{Eff}}_k(X_r^n) \subset \overline{\text{Eff}}_k(X_\Gamma^n)$ and $\overline{\text{Eff}}_k(X_\Gamma^n)$ is linearly generated. This concludes the induction and the proof of the theorem. \square

We immediately deduce the following corollary.

Corollary 4.4. *If X_r^n is a Mori Dream Space, then $\overline{\text{Eff}}_k(X_r^n)$ is finitely generated with the possible exception of $\overline{\text{Eff}}_2(X_8^4)$.*

Question 4.2. Is the cones $\overline{\text{Eff}}_2(X_8^4)$ finitely generated? Is it linearly generated?

It is not easy to find explicit curves in X_r^n which are not in the span of lines. The following example gives a construction in the case of 9 very general points in \mathbb{P}^3 .

Example 4.1. By [CM], the class $C_{CM} = 57H_1 - \sum_{i=1}^{10} 18E_{i,1}$ on X_{10}^2 is represented by a unique irreducible plane curve of genus 10. On X_9^3 , there is a unique divisor Q in the class $2H_1 - \sum_{i=1}^9 E_{i,1}$, given by the strict transform of the unique quadric through the 9 points. There is a morphism $i : X_{10}^2 \rightarrow X_9^3$ identifying the proper transform of Q with the blow-up of \mathbb{P}^2 at 10 points.

A quick calculation shows that the pushforward of the class of C_{CM} to X_9^3 is

$$i_*(C_{CM}) = 78H_1 - 21E_{1,1} - \sum_{i=2}^9 18E_{i,1}.$$

We have $21 + 8(18) = 165$, while $2 \cdot 78 = 156$. Hence, this curve is not in the span of the lines. It does not, however, define an extremal ray on $\overline{\text{Eff}}_1(X_9^3)$. In the next section, we will use a similar construction to show that assuming the SHGH conjecture, the cone $\overline{\text{Eff}}_1(X_9^3)$ is not finitely generated.

By repeatedly taking cones over $i_*(C_{CM})$, we obtain explicit non-linearly generated codimension-two cycles on X_{n+6}^n for every $n \geq 3$.

Complete intersections also provide examples of nonlinearly generated pseudoeffective curve classes, provided that the number of points is large.

Example 4.2. Assume that $d^n \geq r > 2d^{n-1}$ for some integer $d > 2$. Then the divisor class $D = dH - \sum_{i=1}^r E_i$ is nef on X_r^n by the argument given in the proof of Proposition 4.1. The $(n-1)$ -fold self-intersection of the class is

$$D^{n-1} = d^{n-1}H_1 - \sum_{i=1}^r E_{i,1}.$$

Since $r > 2d^{n-1}$, this class is not in the span of lines. On the other hand, the class is pseudoeffective. A small perturbation of D is ample. Hence, a sufficiently high multiple is very ample and the $(n-1)$ -fold self-intersection is an effective curve. It follows that the class D^{n-1} is pseudoeffective.

5. NON-FINITELY GENERATED CONES

The cone of curves the blow-up of \mathbb{P}^2 at 10 or more very general points is not entirely understood, and we will find it useful to assume the following standard conjecture.

Conjecture 1 (Segre–Harbourne–Gimigliano–Hirschowitz (SHGH) conjecture, [G]). *Suppose that $r \geq 10$ and that $m_1 \geq m_2 \geq \dots \geq m_r$ and $d > m_1 + m_2 + m_3$. Then*

$$H^0(X_r^2, dH_1 - \sum_{i=1}^r m_i E_{i,1}) = \binom{d+2}{2} - \sum_{i=1}^r \binom{m_i+1}{2}$$

We next prove that the cone of codimension-2 cycles on X_r^n is not finitely generated for $r \geq n+6$, assuming the SHGH conjecture. The calculation relies on the following observation of de Fernex.

Theorem 5.1 ([DF, Prop. 3.4]). *Assume the SHGH conjecture holds for 10 points. Let $P \subset N^1(X_{10}^2)$ be the positive cone*

$$P = \{D \in N^1(X_{10}^2) : D^2 \geq 0, D \cdot H \geq 0\},$$

where H is an ample divisor. Then

$$\overline{\text{Eff}}_1(X_{10}^2) \cap K_{\geq 0} = P \cap K_{\geq 0}.$$

Let $Q \subset X_9^3$ be the strict transform of the unique quadric passing through the 9 points. Note that Q is isomorphic to X_{10}^2 , and so Conjecture 1 provides some information about the cone $\overline{\text{Eff}}_1(Q)$. However, the map $N_1(Q) \rightarrow N_1(X_9^3)$ is not injective, since the two rulings of the quadric both map to the class of a line in \mathbb{P}^3 . The next lemma gives a criterion to show that certain extremal rays on $\overline{\text{Eff}}_1(Q)$ nevertheless push forward to extremal rays on $\overline{\text{Eff}}_1(X_9^3)$.

Write r_1 and r_2 for the classes of the two rulings on the quadric, and let $f_i = E_i|_Q$ be the exceptional curves. Let $\ell_{ij} = r_1 - f_i - f_j \in N_1(Q)$; this class is not effective on Q , but $i_*\ell_{ij}$ is effective in $N_1(X_9^3)$ since it is the class of a line through the points p_i and p_j .

Theorem 5.2. *Suppose that D is a class in $N_1(Q)$ which satisfies:*

- (1) D is nef, and if $\gamma \in \overline{\text{Eff}}_1(Q)$ has $D \cdot \gamma = 0$, then γ is a multiple of D ;
- (2) $D \cdot K_Q > 0$;
- (3) $D \cdot r_1 = D \cdot r_2$;
- (4) $D \cdot \ell_{ij} > 0$ for all i and j .

Then i_*D is extremal on $\overline{\text{Eff}}_1(X_9^3)$.

Proof. We claim first that D lies on a two-dimensional extremal face of the cone

$$\Sigma = \overline{\text{Eff}}_1(Q) + \sum_{i,j} \mathbb{R}_{\geq 0} \ell_{ij} + \mathbb{R}(r_1 - r_2) \subset N_1(Q).$$

More precisely, if $D = \alpha + \beta$ with $\alpha, \beta \in \Sigma$, then

$$\alpha = a_1 D + b_1(r_1 - r_2) \quad \text{and} \quad \beta = a_2 D + b_2(r_1 - r_2),$$

where a_1 and a_2 are positive. Note that D is nef, $D \cdot (r_1 - r_2) = 0$, and $D \cdot \ell_{ij} > 0$ for all i, j . Hence, D is contained in the dual cone of Σ . By conditions (1) and (3), the classes in Σ with $D \cdot C = 0$ are precisely $\mathbb{R}_{\geq 0} D + \mathbb{R}(r_1 - r_2)$.

We claim next that

$$\overline{\text{Eff}}_1(X_9^3) = i_* \overline{\text{Eff}}_1(Q) + \sum_{i,j} \mathbb{R}_{\geq 0} [\ell_{ij}] = i_* \Sigma.$$

Suppose that Γ is an irreducible effective cycle on $\overline{\text{Eff}}_1(X_9^3)$. If $Q \cdot \Gamma < 0$, then Γ must be contained in Q , and so $[\Gamma]$ is contained in $i_* \overline{\text{Eff}}_1(Q)$. If $Q \cdot \Gamma \geq 0$, then Γ satisfies $2a \geq \sum_i b_i$, which means that $[\Gamma]$ is in the span of classes of lines by Lemma 2.4.

Suppose now that $i_*D = \alpha + \beta$, where α and β are pseudoeffective classes on $\overline{\text{Eff}}_1(X_9^3)$. Using the decomposition above, we can write $\alpha = i_*\alpha_Q + \sum c_{ij}[\ell_{ij}]$ and $\beta = i_*\beta_Q + \sum d_{ij}[\ell_{ij}]$. Because $D \cdot K_Q > 0$ and $K_Q \cdot \ell_{ij} = 0$, it must be that $\alpha_Q \neq 0$.

We claim now that

$$D = \alpha_Q + \beta_Q + \sum (c_{ij} + d_{ij})(r_1 - e_i - e_j) + f(r_1 - r_2)$$

for some constant f . Since $i_*(r_1 - e_i - e_j) = [\ell_{ij}]$, the two sides differ by an element of the kernel of i_* . The kernel of $i_* : N_1(Q) \rightarrow N_1(X_9^3)$ is generated by $r_1 - r_2$, giving rise to the constant f .

Since $D^2 = 0$ and $D \cdot (r_1 - e_i - e_j) > 0$ for any i and j , we must have $c_{ij} = 0$ and $d_{ij} = 0$ for all i, j . We conclude that

$$\alpha_Q = a_1 D + b_1(r_1 - r_2) \quad \text{and} \quad \beta_Q = a_2 D + b_2(r_1 - r_2).$$

Hence, $\alpha = i_* \alpha_Q = a_1 i_* D$ and $\beta = i_* \beta_Q = a_2 i_* D$. This shows that $i_* D$ is extremal. \square

The requirement that $K_Q \cdot D > 0$ makes it tricky to explicitly exhibit such classes, since the K_Q -positive part of $\overline{\text{Eff}}_1(Q)$ is difficult to determine without assuming the SHGH conjecture.

Theorem 5.3. *Assume that the SHGH conjecture holds for blow-ups of \mathbb{P}^2 at 10 very general points. Then there exist infinitely many classes D satisfying the hypotheses of Theorem 5.2.*

Proof. It is convenient to fix an identification $Q \cong X_{10}^2$ and rewrite the hypotheses in the basis for $N^1(Q)$ arising from this identification. The strict transforms of the two rulings through the point p_1 give disjoint (-1) -curves on Q , and these can be contracted. The other 8 exceptional curves f_i can then be contracted to give a map to \mathbb{P}^2 . Let e_0 and e_1 be the first two (-1) -curves contracted, and let $e_j = f_j$ for $2 \leq j \leq 8$. With respect to this new basis, we have $r_1 - f_1 = e_0$ and $r_2 - f_1 = e_1$, and $f_1 = h - e_0 - e_1$, where h denotes the class of a line on \mathbb{P}^2 .

While the first two conditions in Theorem 5.2 are independent of the basis, the last two can be rewritten as

$$(3) \quad D \cdot e_0 = D \cdot e_1.$$

$$(4) \quad D \cdot e_0 > D \cdot e_j \text{ for any } j > 1, \text{ and } D \cdot (h - e_1 - e_i - e_j) > 0 \text{ for any } i, j > 1.$$

The first part of (4) arises when $i = 1 < j$, while the second case is when $1 < i < j$.

Fix any $\frac{1}{\sqrt{10}} < \delta < \frac{1}{3}$, and let $\delta' = \sqrt{\frac{1-2\delta^2}{8}}$. Observe that $\frac{3}{10} < \frac{1}{6}\sqrt{\frac{7}{2}} < \delta' < \delta$ for δ in this range. Consider the divisor

$$D_\delta = h - \delta(e_0 + e_1) - \delta' \sum_{j=2}^9 e_j.$$

We check each of the hypotheses in turn. To simplify notation, for the rest of this proof set $X = X_{10}^2$.

(1) First we check that D_δ is nef. The cone theorem implies that

$$\overline{\text{Eff}}_1(X) = \overline{\text{Eff}}_1(X)_{K_X \geq 0} + \sum_i \mathbb{R}_{\geq 0}[C_i],$$

where the C_i are K_X -negative curves. According to Theorem 5.1,

$$\overline{\text{Eff}}_1(X) \cap \overline{\text{Eff}}_1(X)_{K_X \geq 0} = P \cap K_{\geq 0}.$$

Hence, it suffices to show that $D_\delta \cdot C \geq 0$ if C is K_X -negative, and that $D_\delta \cdot C \geq 0$ if C has $C^2 \geq 0$ and $C \cdot H > 0$.

First, suppose that C is a pseudoeffective class with $K_X \cdot C < 0$. We have

$$3D_\delta - K_X = (3\delta - 1)(e_0 + e_1) + (3\delta' - 1) \sum_{j=2}^9 e_j,$$

and so

$$3D_\delta \cdot C = K_X \cdot C + \left((3\delta - 1)(e_0 + e_1) + (3\delta' - 1) \sum_{j=2}^9 e_j \right) \cdot C$$

However, since $\delta < 1/3$, the number $3\delta - 1$ is negative. It is easy to check that $\delta' < 1/3$ as well, and so the divisor on the right is a sum of exceptional divisors with negative coefficients. If C is any curve other than one of the e_i , then both terms on the right are negative. If C is one of the curves e_i , then $D_\delta \cdot C > 0$ because δ and δ' are both positive.

It remains to check that $D_\delta \cdot C > 0$ if C is a class with positive self-intersection. This follows from the Cauchy-Schwartz inequality: suppose that $d^2 \geq \sum a_i^2$ and $e^2 \geq \sum b_i^2$. Then $de \geq \sum a_i b_i$. Moreover, equality is achieved if and only if C is a multiple of D_δ .

(2) We have

$$D_\delta \cdot K_Q = -3 + 2\delta + 8\delta' > -3 + 10\delta' > 0,$$

since $\delta' > 3/10$.

(3) Since D_δ is of the form $h - \delta e_0 - \delta e_1 - \dots$, we have $D \cdot e_0 = D \cdot e_1$.

(4) Because $\delta > \delta'$, we have $D \cdot e_0 = D \cdot e_j$ for any $j > 1$. We also have $D \cdot (h - e_1 - e_i - e_j) = 1 - \delta - 2\delta' > 0$, since $\delta' < \delta < 1/3$.

□

Remark 5.1. One can even arrange that D_δ is a rational class through judicious choice of δ . For example,

$$D_{\frac{226}{692}} = h - \frac{226}{692}(e_0 + e_1) - \frac{217}{692} \sum_{i=2}^9 e_i.$$

However, in general such classes are not expected to have any effective representatives.

Assuming the SHGH conjecture, we can now conclude that $\overline{\text{Eff}}^2(X_r^n)$ is not finitely generated if $r \geq n + 6$. We need the following lemma, which guarantees that cones over extremal classes in $\overline{\text{Eff}}_k(X_r^n)$ are extremal in $\overline{\text{Eff}}_{k+1}(X_{r+1}^{n+1})$.

Lemma 5.4. *Suppose that $D = aH_k - \sum_{i=1}^r a_i E_{i,k}$ spans an extremal ray on $\overline{\text{Eff}}_k(X_r^n)$. Then $CD = aH_{k+1} - aE_{0,k+1} - \sum_{i=1}^r a_i E_{i,k+1}$ spans an extremal ray on $\overline{\text{Eff}}_{k+1}(X_{r+1}^{n+1})$. In particular, if $\overline{\text{Eff}}_k(X_r^n)$ has infinitely many extremal rays, then so does $\overline{\text{Eff}}_{k+1}(X_{r+1}^{n+1})$.*

Lemma 5.4 immediately implies the following.

Corollary 5.5. *Assume the SHGH Conjecture for the blow-up of \mathbb{P}^2 at 10 points. Then $\overline{\text{Eff}}^2(X_r^n)$ is not finitely generated if $r \geq n + 6$.*

Proof of Lemma 5.4. Given $r + 1$ very general points p_0, \dots, p_r in \mathbb{P}^{n+1} , their projection from p_0 give r very general points in \mathbb{P}^n . Let D_i be effective cycles arbitrarily close to D in $\overline{\text{Eff}}_k(X_r^n)$. Then the classes of the cones CD_i over D_i converge to CD . Hence, $CD \in \overline{\text{Eff}}_{k+1}(X_{r+1}^{n+1})$.

Conversely, we claim that if $CD = aH_{k+1} - aE_0 - \sum_{i=1}^r b_i E_{i,k+1}$ is a pseudoeffective $(k + 1)$ -cycle on X_{r+1}^{n+1} , then $D = aH_k - \sum_{i=1}^r b_i E_{i,k}$ is a pseudoeffective k -cycle on X_r^n . The class $CD + \epsilon H_{k+1}$ is effective for any $\epsilon > 0$. Let V_ϵ be a (rational) cycle representing the

class $CD + \epsilon H_{k+1}$. Let ℓ_{0j} denote the strict transform on X_{n+1}^{r+1} of the line through p_0 and p_j . By Lemma 2.8, V_ϵ contains the line ℓ_{0j} with multiplicity $\beta_j \geq b_j - \epsilon$. Let $L \subset X_{r+1}^{n+1}$ be the proper transform of a general hyperplane in \mathbb{P}^{n+1} . The lines ℓ_{0j} intersect L in r very general points p . The proper transform of the intersection $L \cap V_\epsilon$ gives an effective cycle with class $(a + \epsilon)H_k - \sum_{i=1}^j \beta_j E_{j,k}$. Letting ϵ tend to 0, we see that $aH_k - \sum_{i=1}^r b_i E_{i,k}$ is pseudoeffective in X_r^n , as required.

Now, suppose that $D = aH_k - \sum_{i=1}^r b_i E_{i,k}$ spans an extremal ray of $\overline{\text{Eff}}_k(X_r^n)$. We claim that $CD = aH_{k+1} - aE_{0,k+1} - \sum_{i=1}^r b_i E_{i,k+1}$ spans an extremal ray of $\overline{\text{Eff}}_{k+1}(X_{r+1}^{n+1})$. Suppose $CD = \alpha + \beta$, where α and β are both pseudoeffective $(k+1)$ -cycles on X_{r+1}^{n+1} . Since any pseudoeffective class has $a \geq b_0$, it must be that

$$\alpha = cH_{k+1} - cE_{0,k+1} - \sum_{i=1}^r c_i E_{i,k+1}, \quad \beta = dH_{k+1} - dE_{0,k+1} - \sum_{i=1}^r d_i E_{i,k+1}.$$

Then

$$\alpha_0 = cH_k - \sum_{i=1}^r c_i E_{i,k}, \quad \beta_0 = dH_k - \sum_{i=1}^r d_i E_{i,k}.$$

are pseudoeffective on X_r^n . Hence, α_0 and β_0 are proportional to D . It follows that α and β are proportional to CD and CD is extremal. \square

There are several interesting remaining questions concerning the finite generation of cones of higher codimension.

Question 5.1. Can one show that $\overline{\text{Eff}}_{n-2}(X_r^n)$ is not finitely generated for $r \geq n + 6$ independently of the SHGH Conjecture?

Question 5.2. Fix n and k . Does there exist r for which $\overline{\text{Eff}}_k(X_r^n)$ is not finitely generated? How does r depend on n and k ?

In particular, we have the following fundamental question.

Question 5.3. For every n , does there exist r for which $\overline{\text{Eff}}_1(X_r^n)$ is not finitely generated?

If $\overline{\text{Eff}}_1(X_r^n)$ is not finitely generated for $r \geq r_0$, then by Lemma 5.4 $\overline{\text{Eff}}_k(X_{r+k-1}^{n+k-1})$ is not finitely generated for $r \geq r_0$. Hence, an affirmative answer to Question 5.3 implies an affirmative answer to Question 5.2.

6. BLOW-UPS AT POINTS IN SPECIAL POSITION

Until now we have considered blow-ups of \mathbb{P}^n at linearly general or very general points. It is also interesting to consider cones of effective cycles on blow-ups of \mathbb{P}^n at special configurations of points. The dependence of the cones on the position of the points can be subtle, which makes degeneration arguments difficult. We will see that the property of the effective cone being finite is neither an open nor closed condition, even in families where the vector space of numerical classes of k -dimensional cycles has constant dimension.

Proposition 6.1. *Let Γ be a set of r points whose span is $\mathbb{P}^m \subset \mathbb{P}^n$. Let X_Γ^n and X_Γ^m denote the blow-up of \mathbb{P}^n and \mathbb{P}^m along Γ , respectively.*

- (1) *Then $\overline{\text{Eff}}_k(X_\Gamma^n)$ is linearly generated for $m \leq k \leq n - 1$ and*
- (2) *$\overline{\text{Eff}}_k(X_\Gamma^n) = \overline{\text{Eff}}_k(X_\Gamma^m)$ for $k < m$.*

Proof. Since X_Γ^m embeds in X_Γ^n as the proper transform of the \mathbb{P}^m spanned by Γ , any effective cycle $Z \subset X_\Gamma^m$ is also an effective cycle in X_Γ^n with the same class. Hence, $\overline{\text{Eff}}_k(X_\Gamma^m) \subseteq \overline{\text{Eff}}_k(X_\Gamma^n)$ for $k < m$. Conversely, suppose that $k < m$. Let Z be an effective cycle in \mathbb{P}^n of dimension k . We may assume that Z is not contained in an exceptional divisor. Choose a general point p . Let q_i denote the projection of p_i from p and let Z' be the projection of Z from p . Then Z' and Z have the same degree and the multiplicities of Z' at q_i are greater than or equal to the multiplicities of Z at p_i . Repeatedly projecting Z to \mathbb{P}^m from general points, we obtain an effective cycle contained in \mathbb{P}^m with the same class. Taking closures, we obtain the reverse inclusion $\overline{\text{Eff}}_k(X_\Gamma^n) \subseteq \overline{\text{Eff}}_k(X_\Gamma^m)$.

If $k \geq m$, let L be a k -dimensional linear space containing Γ . Then the proper transform \bar{L} of L has class $H_k - \sum_{i=1}^r E_{i,k}$. Since a k -dimensional variety not contained in an exceptional divisor has class $aH_k - \sum_{i=1}^r b_i E_{i,k}$ with $a \geq b_i \geq 0$, we conclude that any k -dimensional effective cycle is a nonnegative linear combination of $[\bar{L}]$ and $E_{i,k}$, $1 \leq i \leq r$. \square

By taking $m = 1$, we obtain the following corollary.

Corollary 6.2. *Suppose Γ is a set of r collinear points in \mathbb{P}^n . Then $\overline{\text{Eff}}_k(X_\Gamma^n)$ is linearly generated for every $1 \leq k \leq n - 1$.*

Remark 6.1. Ottem [Ot] proved that the blow-up of \mathbb{P}^2 in collinear points is a Mori Dream Space and computed its Cox ring. Consequently, the cone of curves and divisors are finite polyhedral. The previous corollary generalizes this to all cycles.

Let $L \cong \mathbb{P}^{n-1}$ be a hyperplane in \mathbb{P}^n . Let $\Gamma' \subset L$ be a set of points p_1, \dots, p_r and let $p_0 \in \mathbb{P}^n$ be a point not contained in L . Let $\Gamma = \Gamma' \cup \{p_0\}$. Let X_Γ^n and $X_{\Gamma'}^{n-1}$ denote the blow-up of \mathbb{P}^n and \mathbb{P}^{n-1} along Γ and Γ' , respectively. Taking cones with vertex at p_0 , we generate a subcone $\text{CEff}_k(X_{\Gamma'}^{n-1}) \subset \overline{\text{Eff}}_{k+1}(X_\Gamma^n)$.

Proposition 6.3. *The cone $\overline{\text{Eff}}_{k+1}(X_\Gamma^n)$ is generated by $\text{CEff}_k(X_{\Gamma'}^{n-1})$, $\overline{\text{Eff}}_{k+1}(X_{\Gamma'}^{n-1})$ and $E_{0,k+1}$. Furthermore, the extremal rays of $\text{CEff}_k(X_{\Gamma'}^{n-1})$ are also extremal in $\overline{\text{Eff}}_{k+1}(X_\Gamma^n)$.*

Proof. Let $Z = aH_{k+1} - \sum_{i=0}^r b_i E_{i,k+1}$ be an irreducible $(k+1)$ -dimensional variety in X_Γ^n . We may assume that Z is not contained in any exceptional divisors. Otherwise, its class is a positive multiple of $E_{i,k+1}$. The proper transform of L in X_Γ^n is isomorphic to $X_{\Gamma'}^{n-1}$. If Z is contained in $X_{\Gamma'}^{n-1}$, then the class of Z is in $\overline{\text{Eff}}_{k+1}(X_{\Gamma'}^{n-1})$. Otherwise, $Z \cap X_{\Gamma'}^{n-1}$ is an effective k cycle with class $\alpha = aH_k - \sum_{i=1}^r b_i E_{i,k}$. Then the cone $C(\alpha)$ is an effective class in X_Γ^n and since $b_0 \leq a$, $[Z]$ is in the span of $E_{0,k+1}$ and $C(\alpha)$.

Let Z be a cycle that generates an extremal ray of $\text{CEff}_k(X_{\Gamma'}^{n-1})$. Suppose $[Z] = \alpha + \beta$ in $\overline{\text{Eff}}_{k+1}(X_\Gamma^n)$. Since $b_0 \leq a$ holds on $\overline{\text{Eff}}_{k+1}(X_\Gamma^n)$ and $b_0 = a$ on $\text{CEff}_k(X_{\Gamma'}^{n-1})$, we must have that both α and β satisfy $b_0 = a$. We can perturb α and β by ϵH_{k+1} to obtain rational effective classes. Since the coefficient of $E_{0,k+1}$ of any class contained in $\overline{\text{Eff}}_{k+1}(X_{\Gamma'}^{n-1})$ is 0, the coefficients of any component of the class contained in $\overline{\text{Eff}}_{k+1}(X_{\Gamma'}^{n-1})$ are bounded by ϵ . Taking cones over the classes of the hyperplane sections of the remaining subvarieties and letting ϵ tend to zero, we see that both α and β are contained in $\text{CEff}_k(X_{\Gamma'}^{n-1})$. By the extremality of Z , we conclude that they are both proportional to $[Z]$. \square

Corollary 6.4. *Let Γ be a set of points $\{q_1, \dots, q_9, p_1, \dots, p_{s-1}\}$ such that q_1, \dots, q_9 are general points in a plane $P \subset \mathbb{P}^n$ and p_1, \dots, p_{s-1} are linearly general points with span disjoint from P . Then $\overline{\text{Eff}}_k(X_\Gamma^n)$ is not finitely generated for $k \leq s$ and linearly generated for $k > s$.*

Proof. When $r \geq 9$, the blow-up of \mathbb{P}^2 at r general points has infinitely many (-1) -curves, which span extremal rays of the effective cone of curves. Applying Proposition 6.3 $(k-1)$ -times, the cones over the classes of (-1) -curves with vertex p_1, \dots, p_{k-1} provide infinitely extremal rays of $\overline{\text{Eff}}_k(X_\Gamma^n)$ for $k \leq s$. The linear generation of $\overline{\text{Eff}}_k(X_\Gamma^n)$ for $k > s$ follows from Proposition 6.1 \square

Corollary 6.5. (1) *Linear generation of $\overline{\text{Eff}}_k(X_\Gamma^n)$ is not closed in smooth families.*
 (2) *Finite generation of $\overline{\text{Eff}}_k(X_\Gamma^n)$ is not closed in smooth families.*

Proof. Let $n \geq k+8$. Take a general smooth curve B in $(\mathbb{P}^n)^{k+8}$ which avoids all the diagonals and contains a point $0 \in B$ where 9 of the points are general points in a plane P and the remaining points are in linearly general position with span not intersecting P . Such curves exist by Bertini's Theorem since the diagonals have codimension $n \geq 2$. Consider the family $\mathcal{X} \rightarrow B$, where \mathcal{X}_b is the blow-up of \mathbb{P}^n in the $k+8$ points Γ_b parameterized by $b \in B$. If the points in Γ_b are in linearly general position, then by Lemma 3.3 the cone $\overline{\text{Eff}}_k(X_{\Gamma_b})$ is linearly generated. In particular, the cone is finite. However, by Corollary 6.4, $\overline{\text{Eff}}_k(X_{\Gamma_0})$ is not finitely generated. In particular, the cone is not linearly generated. \square

Corollary 6.6. (1) *Linear generation of $\overline{\text{Eff}}_k(X_\Gamma^n)$ is not open in smooth families.*
 (2) *Finite generation of $\overline{\text{Eff}}_k(X_\Gamma^n)$ is not open in smooth families.*

Proof. Let B be a smooth curve parameterizing 9 general points in a plane P becoming collinear at $0 \in B$. Let Γ' be $k-1$ points in general linear position in \mathbb{P}^n whose span is disjoint from P . Let Γ_b be the union of Γ' and the points parameterized by b . Consider the family $\mathcal{X} \rightarrow B$ obtained by blowing up \mathbb{P}^n along Γ_b . When the points are collinear, $\overline{\text{Eff}}_k(X_{\Gamma_0})$ is linearly generated. However, for the general point of B , $\overline{\text{Eff}}_k(X_{\Gamma_b})$ is not finitely generated by Corollary 6.4. \square

Remark 6.2. Corollary 6.6 is well-known for cones of divisors. For example, Castravet-Tevelev [CT] prove that the blow-up of \mathbb{P}^n at points on a rational normal curve is a Mori Dream Space. In particular, if we specialize a large number of points to lie on a rational normal curve, we see that being a Mori Dream Space is not an open condition.

One can ask for the finite/linear generation of $\overline{\text{Eff}}_k(X_\Gamma^n)$ for Γ any special set of points. Perhaps the following question is the most interesting among them.

Question 6.1. Let Γ be a set of points contained in a rational normal curve in \mathbb{P}^n . Is $\overline{\text{Eff}}_k(X_\Gamma^n)$ finitely generated? Is $\overline{\text{Eff}}_k(X_\Gamma^n)$ generated by the classes of cones over secant varieties of projections of the rational normal curve?

By results of Castravet and Tevelev, the answer to Question 6.1 is affirmative for curves and divisors. The cone of curves $\overline{\text{Eff}}_1(X_\Gamma^n)$ is generated by the class of the proper transform of the rational normal curve $nH_1 - \sum_{i=1}^r E_{i,1}$ and the classes of lines. The rational normal curve is cut out by quadrics. If a curve B has positive intersection with a quadric containing the points, then by Lemma 2.4 the class of B is spanned by the classes of lines. Otherwise, B is contained in the base locus of all the quadrics containing the rational normal curve. Hence, B is a multiple of the rational normal curve. Castravet and Tevelev show that the classes of divisors are generated by linear spaces and codimension-1 cones over secant varieties of the projection of the rational normal curve [CT]. We do not know whether $\overline{\text{Eff}}_2(X_\Gamma^n)$ is generated by the classes of planes and cones over the rational normal curve with vertex a point of Γ .

REFERENCES

- [BCHM] C. Birkar, P. Cascini, C. Hacon and J. McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.* **23** (2010), no. 2, 405–468.
- [CT] A.M. Castravet and J. Tevelev, Hilbert’s 14th problem and Cox rings, *Compositio Math.* **142** no. 6 (2006), 1479–1498.
- [CC] D. Chen, and I. Coskun, Extremal higher codimension cycles on moduli spaces of curves, *Proc. Lond. Math. Soc.* **111** no. 1 (2015), 181–204.
- [CM] C. Ciliberto and R. Miranda, Homogeneous interpolation on ten points, *J. Algebraic Geom.* **20** no. 4 (2011), 685–726.
- [C1] I. Coskun, Gromov-Witten invariants of jumping curves, *Trans. Amer. Math. Soc.*, **360** (2008), 989–1004.
- [C2] I. Coskun, Degenerations of surface scrolls and Gromov-Witten invariants of Grassmannians, *J. Algebraic Geom.* **15** (2006), 223–284.
- [DF] T. de Fernex, On the Mori cone of blow-ups of the plane, arXiv:1001.5243.
- [DELV] O. Debarre, L. Ein, R. Lazarsfeld, and C. Voisin, Pseudoeffective and nef classes on abelian varieties, *Compos. Math.* **147** (2011), no. 6, 1793–1818.
- [DJV] O. Debarre, Z. Jiang, and C. Voisin, Pseudo-effective classes and pushforwards, *Pure Appl. Math. Q.* **9** (2013), no. 4, 643–664.
- [Do] I. Dolgachev, Cremona special sets of points in products of projective spaces, *Complex and differential geometry*, 115134, Springer Proc. Math., 8, Springer, Heidelberg, 2011.
- [FL1] M. Fulger, and B. Lehmann, Positive cones of dual cycle classes, arXiv:1408.5154.
- [FL2] M. Fulger, and B. Lehmann, Kernels of numerical pushforwards, arXiv:1407.6455.
- [Fu] W. Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 1998.
- [G] A. Gimigliano, On Linear Systems of Plane Curves, Ph.D. Thesis, Queens University, 1987.
- [Har] B. Harbourne, The geometry of rational surfaces and Hilbert functions of points in the plane. *Can. Math. Soc. Conf. Proc.* 6 (1986), 95–111.
- [H] J. Harris, *Algebraic geometry: A first course*, Graduate Texts in Mathematics, 133, Springer-Verlag, New York, 1995.
- [Ha] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, 52, Springer-Verlag, New York-Heidelberg, 1977.
- [Hi] A. Hirschowitz, Une conjecture pour la cohomologie des diviseurs sur les surfaces rationnelles génériques, *J. Reine Angew. Math.* **397** (1989), 208–213.
- [Hu] J. Huh, Rota’s conjecture and positivity of algebraic cycles in permutohedral varieties, Ph.D. Thesis, University of Michigan, 2014.
- [La] R. Lazarsfeld, *Positivity in algebraic geometry I, Classical setting: line bundles and linear series*, Springer-Verlag, 2004.
- [Mu] S. Mukai, *Geometric realization of T-shaped root systems and counterexamples to Hilbert’s fourteenth problem*. Algebraic transformation groups and algebraic varieties, 123–129, *Encyclopaedia Math. Sci.*, 132, Springer, Berlin, 2004.
- [N] M. Nagata, On the 14-th problem of Hilbert, *Amer. J. Math.* **33** (1959), 766–772.
- [Ot] J.C. Ottem, On the Cox ring of \mathbb{P}^2 blown up in points on a line. *Math. Scand.* 109 (2011), no. 1, 22–30.

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