A PROJECTIVE VARIETY WITH DISCRETE, NON-FINITELY GENERATED AUTOMORPHISM GROUP

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Abstract. We construct a projective variety with discrete, non-finitely generated automorphism group.

1. Introduction

Suppose that X is a projective variety over a field K. The set of automorphisms of X can be given the structure of a scheme by realizing it as an open subset of $\mathrm{Hom}(X, X)$. In general, $\mathrm{Aut}(X)$ is locally of finite type, but it may have countably many components, arising from components of the Hilbert scheme. Write $\pi_0(\mathrm{Aut}(X)) = (\mathrm{Aut}(X)/\mathrm{Aut}^0(X))_{\bar{K}}$ for the group of geometric components.

Examples.

1. Let $X = \mathbb{P}^r$. Then $\mathrm{Aut}(X) \cong \mathrm{Aut}^0(X) \cong \mathrm{PGL}_{r+1}(K)$, and $\pi_0(\mathrm{Aut}(X))$ is trivial.
2. Let E be a general elliptic curve over K. Then $\pi_0(\mathrm{Aut}(E \times E)) \cong \mathrm{GL}_2(\mathbb{Z})$ is an infinite discrete group.
3. Let X be a general hypersurface of type $(2, 2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then X is a K3 surface, and the covering involutions associated to the three projections $X \to \mathbb{P}^1 \times \mathbb{P}^1$ generate a subgroup of $\pi_0(\mathrm{Aut}(X))$ isomorphic to $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ [4].

According to a result of Brion [3], any connected algebraic group over a field of characteristic 0 can be realized as $\mathrm{Aut}^0(X)$ for some smooth, projective variety. In contrast, very little seems to be known in general about the component group $\pi_0(\mathrm{Aut}(X))$. In what follows, let K be a field of characteristic 0, not necessarily algebraically closed, and let $\bar{K}$ be an algebraic closure. All varieties are defined over K, except where noted otherwise, and by a point we mean a K-point. Our main result is the following.

Theorem 1. *There exists a smooth, geometrically simply connected, projective variety X over K for which $\pi_0(\mathrm{Aut}(X))$ is not finitely generated.*

The question of finite generation of $\pi_0(\mathrm{Aut}(X))$ has been raised several times in various arithmetic [12],[1] and geometric [3],[6] contexts.

The component group $\pi_0(\mathrm{Aut}(X))$ is an algebraic analog of the mapping class group $\pi_0(\mathrm{Diff}(M))$ of a smooth manifold M. In general, the mapping class group is not finitely generated, with an example provided by tori in dimension at least five [9]. However, at least in high dimensions, the failure of $\pi_0(\mathrm{Diff}(M))$ to be finitely generated is attributable to the fundamental group of M: according to a theorem of Sullivan [15], if $\dim M \geq 5$ and $\pi_1(M) = 0$, then $\pi_0(\mathrm{Diff}(M))$ is finitely generated.

The related group $\mathrm{Bir}(X)$ of birational automorphisms of X can also be very complicated, even for simple varieties over $\mathbb{C}$. To begin with, the birational automorphism group does not admit a reasonable scheme structure in general [2]. If the canonical class K_X has some positivity, the situation is somewhat better. If X is of general type, then $\mathrm{Bir}(X)$ is finite. As long as X is not uniruled, $\mathrm{Bir}(X)$ admits the structure of a group scheme of locally finite type [8]. If the canonical class of X is nef, then the Kawamata–Morrison cone conjecture places additional constraints on $\mathrm{Aut}(X)$ and $\mathrm{Bir}(X)$ [11]. In dimension 2, the automorphism group of a K3 surface is known to be finitely generated [14], but need not be commensurable with an arithmetic group [16]. Little seems to be known for Calabi–Yau varieties in higher dimensions.

Before giving the example, we sketch the technique. If X is a variety and Z is a closed subscheme of X, then the automorphisms of X that lift to automorphisms of the blow-up $\mathrm{Bl}_Z(X)$ are precisely those that map Z to itself (not necessarily fixing Z pointwise). Our approach, roughly speaking, is to find a variety X with a subscheme Z so that $\mathrm{Stab}(Z) \subset \mathrm{Aut}(X)$ is not finitely generated, and then to pass to the blow-up $\mathrm{Bl}_Z(X)$ to obtain a variety realizing $\mathrm{Stab}(Z)$ as an automorphism group. There are two main difficulties. The first is to find X and Z for which the stabilizer of Z in $\mathrm{Aut}(X)$ is not finitely generated. The second is to ensure that $\mathrm{Bl}_Z(X)$ does not have any automorphisms other than those that lift from X.

To prove that our variety X has non-finitely generated automorphism group, we will exhibit a smooth rational curve C which is fixed by every automorphism of X. Restriction of automorphisms then determines a map $\rho : \mathrm{Aut}(X) \to \mathrm{Aut}(C) \cong \mathrm{PGL}_2(K)$. We arrange that the image of ρ is contained in an abelian subgroup of $\mathrm{PGL}_2(K)$ and exhibit an explicit non-finitely generated subgroup of $\mathrm{Im}(\rho)$. It follows that $\mathrm{Aut}(X)$ is not finitely generated.

We turn now to the construction. Given a subvariety $V \subset X$, write

$$\mathrm{Aut}(X; V) = \{\phi \in \mathrm{Aut}(X) : \phi(V) = V\}$$

There are three main steps. First, we describe a family of elliptic rational surfaces S for which $\mathrm{Aut}(S)$ is a large discrete group, and there is a rational curve C on S with $\mathrm{Aut}(S; C)$ of finite index. Second, we specialize the surface S in order to control the image of $\mathrm{Aut}(S; C) \to \mathrm{Aut}(C) \cong \mathrm{PGL}_2(K)$. We show that there is a point p on C so that the subgroup $\mathrm{Aut}_{\mathrm{par}}(S; C, p)$ of automorphisms ϕ so that $\phi|_C$ is parabolic with fixed point p is not finitely generated. At last, by some auxiliary constructions, we arrive at a six-dimensional variety X whose automorphisms are precisely given by $\mathrm{Aut}_{\mathrm{par}}(S; C, p)$.

Step 1: Automorphisms of surfaces with prescribed action on a curve

If z_1, z_2, z_3, and z_4 are four distinct points in $\mathbb{P}^1$, there is a unique involution $\imath : \mathbb{P}^1 \to \mathbb{P}^1$ with $\imath(z_1) = z_2$ and $\imath(z_3) = z_4$. Figure 1 shows how this map can be constructed geometrically when $\mathbb{P}^1$ is embedded as a conic in $\mathbb{P}^2$.

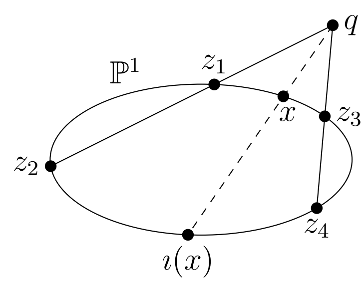

Figure 1. Geometric construction of $\imath$

Given an ordered 5-tuple $P = (p_1, p_2, p_3, p_4, p_5)$ of points in $\mathbb{P}^1$, let $\Gamma_P \subset \mathrm{PGL}_2(K)$ be the subgroup generated by the involutions $\imath_{ij,kl} : \mathbb{P}^1 \to \mathbb{P}^1$ satisfying $\imath(p_i) = p_j$ and $\imath(p_k) = p_l$, where i, j, k and l are distinct indices. For a given configuration P, there are 15 such involutions for different choices of points.

Theorem 2. *Suppose that P is a configuration of five distinct points in $\mathbb{P}^1$. There exists a smooth rational surface S containing a rational curve $C \subset S$ such that*

(1) $\mathrm{Aut}(S)$ *is discrete;*
(2) $\mathrm{Aut}(S; C)$ *has finite index in* $\mathrm{Aut}(S)$*;*
(3) The image of $\rho : \mathrm{Aut}(S; C) \to \mathrm{Aut}(C)$ *contains* Γ_P.

Proof. Let $L_0, \dots, L_5$ be six lines in $\mathbb{P}^2$ intersecting at 15 distinct points, and let S be the blow-up of $\mathbb{P}^2$ at these 15 points. Write R for a partition of the six lines into two sets of three, with a distinguished line in each set. Given such a labelling, denote by $L_{R,0}$, $L_{R,1}$, $L_{R,2}$ and $L'_{R,0}$, $L'_{R,1}$, $L'_{R,2}$ the two triples, with $L_{R,0}$ and $L'_{R,0}$ the two distinguished lines. Let O_R be the point of intersection of $L_{R,0}$ and $L'_{R,0}$.

The choice of a labelling R determines two completely reducible cubics $C = L_{R,0} \cup L_{R,1} \cup L_{R,2}$ and $C' = L'_{R,0} \cup L'_{R,1} \cup L'_{R,2}$, which span a pencil in $\mathbb{P}^2$. The base locus of the pencil is the nine points $L_{R,i} \cap L'_{R,j}$. Let $\pi_R : S_R \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at these points, so that the pencil gives rise to an elliptic fibration $\gamma_R : S_R \to \mathbb{P}^1$. Note that the fibration γ_R must be relatively minimal (i.e. there are no (-1)-curves contained in the fibers): a general fiber is linearly equivalent to $-K_{S_R}$, and so a (-1)-curve on S_R must have intersection 1 with every fiber.

The exceptional divisor of π_R above the point O_R provides a section E of γ_R. Let $\imath_R : S_R \dashrightarrow S_R$ be the birational involution induced by the γ_R-fiberwise action of $x \mapsto -x$ on the smooth fibers, with the section E as the identity. Since γ_R is a relatively minimal fibration, $\imath_R$ extends to a regular map on the entire surface S_R (see e.g. [10, II.10, Theorem 1]). Such a map necessarily permutes the three nodes on each of the fibers C and C', and so lifts to a biregular involution on the fifteen point blow-up S.

There is a simple geometric description of $\imath_R$ as a rational map of $\mathbb{P}^2$, and in particular of its action on $L_{R,0}$. Suppose that ℓ is a line in $\mathbb{P}^2$ passing through the point O_R, and that x is a point on ℓ lying on a smooth fiber C_x of γ_R. Then ℓ meets C_x at x, O_R, and the third point $\imath_R(x)$. Thus $\imath_R$ acts on ℓ so that the two points $L_{R,1} \cap \ell$ and $L_{R,2} \cap \ell$ are exchanged, as are $L'_{R,1} \cap \ell$ and $L'_{R,2} \cap \ell$. This also holds on the singular fibers: if ℓ is any line through O_R for which the four points $L_{R,1} \cap \ell$, $L_{R,2} \cap \ell$, $L'_{R,1} \cap \ell$ and $L'_{R,2} \cap \ell$ are distinct, including $L = L_{R,0}$, then $\imath_R$ restricts to ℓ as the unique involution exchanging these two pairs of points.

The rational surface S claimed by the theorem can now be constructed by choosing the lines in special position. Fix a line $C \subset \mathbb{P}^2$, and choose five other lines $L_1, \dots, L_5$ so that $L_i \cap C = p_i$, where the p_i are the points of the configuration P. Since the field K is infinite, for general choices of the L_i, the fifteen points of intersection are distinct. The involution of C exchanging p_i with p_j and p_k with p_l is realized as the restriction of $\imath_R : S \to S$ for a suitable labelling R: let m be the unique index which does not appear among i, j, k, and l, and take $L_{R,0} = C$, $L_{R,1} = L_i$, $L_{R,2} = L_j$, $L'_{R,0} = L_m$, $L'_{R,1} = L_k$, and $L'_{R,2} = L_l$. Thus each involution $\imath_{ij,kl}$ on C is the restriction of an automorphism of $\imath_R : S \to S$ fixing C, as claimed.

A blow-up X of $\mathbb{P}^2$ at four points with no three collinear satisfies $H^0(X, TX) = 0$, and so $\mathrm{Aut}^0(S)$ is trivial since $S \to \mathbb{P}^2$ factors through such a blow-up. It remains only to check that the subgroup $\mathrm{Aut}(S; C)$ has finite index in $\mathrm{Aut}(S)$. This is a consequence of the fact that S

is a Coble rational surface [7], [5]: the linear system $|-2K_S|$ has a unique element, the union of the strict transforms of the six lines L_i. Indeed, each line satisfies $-2K_S \cdot L_i = -4$, and so must be contained in the base locus. An automorphism preserves the anticanonical class, so the six lines are permuted by any element of $\mathrm{Aut}(S)$, giving rise to a map $\mathrm{Aut}(S) \to S_6$. The subgroup $\mathrm{Aut}(S; C)$ is the preimage of the subgroup of permutations fixing C, and thus of finite index. □

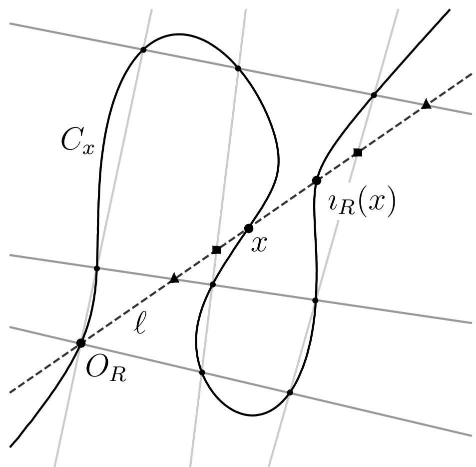

FIGURE 2. Construction of $\imath_R : S \to S$

Figure 2 illustrates the geometry of the map $\imath_R$. The restriction of $\imath_R$ to the line ℓ is the unique involution exchanging the two points marked "▲" and the two points marked "■".

STEP 2: SPECIALIZING THE CONFIGURATION P

We now exhibit a configuration $P = (p_1, p_2, p_3, p_4, p_5)$ for which the group Γ_P contains parabolic and hyperbolic elements with a common fixed point. Fix coordinates on $\mathbb{P}^1$.

Lemma 3. *For the configuration*

$$P = (0, 1, 2, 3, 6)$$

the group Γ_P contains the two elements

$$\sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. We claim that $\sigma = \imath_{12,34} \circ \imath_{13,45} \circ \imath_{13,25}$ and $\tau = \imath_{13,45} \circ \imath_{24,35} \circ \imath_{15,23}$. Indeed,

$$(\imath_{12,34} \circ \imath_{13,45} \circ \imath_{13,25})(p_1) = (\imath_{12,34} \circ \imath_{13,45})(p_3) = \imath_{12,34}(p_1) = p_2,$$
$$(\imath_{12,34} \circ \imath_{13,45} \circ \imath_{13,25})(p_2) = (\imath_{12,34} \circ \imath_{13,45})(p_5) = \imath_{12,34}(p_4) = p_3,$$
$$(\imath_{12,34} \circ \imath_{13,45} \circ \imath_{13,25})(p_3) = (\imath_{12,34} \circ \imath_{13,45})(p_1) = \imath_{12,34}(p_3) = p_4.$$

For the configuration P, this yields $(\imath_{12,34} \circ \imath_{13,45} \circ \imath_{13,25})(0) = 1$, $(\imath_{12,34} \circ \imath_{13,45} \circ \imath_{13,25})(1) = 2$, and $(\imath_{12,34} \circ \imath_{13,45} \circ \imath_{13,25})(2) = 3$, so the composition must be the automorphism σ given by

$z \mapsto z+1$. Similarly,

$$\begin{aligned}
(\imath_{13,45} \circ \imath_{24,35} \circ \imath_{15,23})(p_1) &= (\imath_{13,45} \circ \imath_{24,35})(p_5) = \imath_{13,45}(p_3) = p_1,\\
(\imath_{13,45} \circ \imath_{24,35} \circ \imath_{15,23})(p_2) &= (\imath_{13,45} \circ \imath_{24,35})(p_3) = \imath_{13,45}(p_5) = p_4,\\
(\imath_{13,45} \circ \imath_{24,35} \circ \imath_{15,23})(p_3) &= (\imath_{13,45} \circ \imath_{24,35})(p_2) = \imath_{13,45}(p_4) = p_5.
\end{aligned}$$

For the configuration P, this map sends 0 to 0, 1 to 3, and 2 to 6, and so must be the automorphism τ given by $z \mapsto 3z$. □

Let S be the rational surface constructed in Theorem 2 corresponding to the configuration P given by Lemma 3, so that the image of $\mathrm{Aut}(S;C) \to \mathrm{Aut}(C)$ contains the elements σ and τ. Write $\mathrm{Aut}_{\mathrm{par}}(S;C,p)$ for the subgroup of $\mathrm{Aut}(S;C)$ given by automorphisms which restrict to C as parabolic transformations fixing the point p; if coordinates are chosen so that $p=\infty$, these are the automorphisms restricting to translations $z \mapsto z+c$. Equivalently, we ask that $\phi|_C$ is given by a unipotent upper triangular matrix $\left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right)$.

Lemma 4. *Let $p=\infty$ on the curve C. Then $\mathrm{Aut}_{\mathrm{par}}(S;C,p)$ is not a finitely generated group.*

Proof. An automorphism ϕ in $\mathrm{Aut}(S;C)$ lies in $\mathrm{Aut}_{\mathrm{par}}(S;C,p)$ if and only if $\rho(\phi)$ lies in the subgroup $B \subset \mathrm{PGL}_2(K)$ comprising matrices of the form

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix},$$

which correspond to parabolic Möbius transformations $z \mapsto z+c$. The subgroup B is abelian, isomorphic to $\mathbb{G}_a$; since $\rho(\mathrm{Aut}_{\mathrm{par}}(S;C,p))$ is contained in B, this group is abelian as well. For any integer n, the transformation

$$\tau^{-n} \circ \sigma \circ \tau^n = \begin{pmatrix} 1 & \frac{1}{3^n} \\ 0 & 1 \end{pmatrix}$$

is contained in B. Since σ and τ both lie in $\mathrm{Im}(\rho)$ by the construction of Theorem 2, the elements $\tau^{-n} \circ \sigma \circ \tau^n$ all lie in $\rho(\mathrm{Aut}_{\mathrm{par}}(S;C,p))$, and so $\rho(\mathrm{Aut}_{\mathrm{par}}(S;C,p))$ has a subgroup isomorphic to $\mathbb{Z}\left[\frac{1}{3}\right]$. Since $\rho(\mathrm{Aut}_{\mathrm{par}}(S;C,p))$ is abelian and has a non-finitely generated subgroup, it is not finitely generated. A quotient of a finitely generated group is finitely generated, and we conclude that $\mathrm{Aut}_{\mathrm{par}}(S;C,p)$ itself is not finitely generated. □

The following geometric characterization of elements of $\mathrm{Aut}_{\mathrm{par}}(S;C,p)$ will prove useful. Let $\Delta_S : S \to S \times S$ be the diagonal map.

Lemma 5. *Suppose that $\phi : S \to S$ is an automorphism fixing p. Then ϕ fixes C as well. Furthermore, ϕ lies in $\mathrm{Aut}_{\mathrm{par}}(S;C,p)$ if and only if $\mathrm{id}_S \times \phi : S \times S \to S \times S$ fixes the tangent direction $T_{\Delta_S(p)}(\Delta_S(C))$.*

Proof. Any automorphism of S permutes the components of $|-2K_S|$, which are the strict transforms of the six lines L_i. The only component containing p is C itself, and so C must be invariant under ϕ.

An automorphism fixing C and p lies in $\mathrm{Aut}_{\mathrm{par}}(S;C,p)$ if and only if p is a fixed point of $\phi|_C$ with multiplicity 2, which is the case if and only if $\mathrm{id}_S \times \phi : S \times S \to S \times S$ fixes $\Delta_S(p)$ and the tangent direction $T_{\Delta_S(p)}(\Delta_S(C))$, so that $(\mathrm{id}_S \times \phi)(\Delta_S(C))$ is tangent to the diagonal at $\Delta_S(p)$. □

Remark 1. Let $\bar{\sigma}$ and $\bar{\tau}$ be automorphisms of S which restrict to C as σ and τ, as constructed in Theorem 2. Although the restrictions to C of the automorphisms $\mu_m = \bar{\tau}^{-m} \circ \bar{\sigma} \circ \bar{\tau}^m$ and $\mu_n = \bar{\tau}^{-n} \circ \bar{\sigma} \circ \bar{\tau}^n$ commute and satisfy $\mu_{n-1}|_C \circ \mu_n|_C^{-3} = \mathrm{id}_C$, these maps do not commute as automorphisms of S, and the map $\mathrm{Aut}(S;C) \to \mathrm{Aut}(C)$ is not injective. For example, the commutator $[\mu_0, \mu_1]$ is an automorphism of S which restricts to C as the identity, but a straightforward if somewhat tedious computation of the action of the involutions $\imath_R$ on $\mathrm{NS}(S)$ shows that the induced map $[\mu_0, \mu_1] : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a Cremona transformation of degree 1,944,353. It seems conceivable that $\mathrm{Aut}_{\mathrm{par}}(S;C,p)$ is a free group on the countably many generators μ_n, though this is difficult to prove.

Remark 2. The kernel G of $\mathrm{Aut}(S;C) \to \mathrm{Aut}(C)$ is also of interest: this is the subgroup of automorphisms which fix C pointwise, including the maps $[\mu_m, \mu_n]$ of the remark above. It seems likely that G is not finitely generated; if this is the case, then by choosing a very general point q on C, we might obtain a rational surface $S' = \mathrm{Bl}_q S$ such that $\mathrm{Aut}(S')$ is isomorphic to G and is not finitely generated. However, it is not clear how to prove either that G is not finitely generated, or that the blow-up does not admit automorphisms other than those lifted from S.

Step 3: A variety with non-finitely generated $\mathrm{Aut}(X)$

We now construct a higher-dimensional variety X realizing $\mathrm{Aut}_{\mathrm{par}}(S;C,p)$ as $\mathrm{Aut}(X)$. Although $\mathrm{Aut}_{\mathrm{par}}(S;C,p)$ is not the stabilizer of any closed subscheme of S, it is the stabilizer of a closed subscheme of $S \times S$ under the group of automorphisms of $S \times S$ of the form $\mathrm{id}_S \times \phi$: an automorphism ϕ lies in $\mathrm{Aut}_{\mathrm{par}}(S;C,p)$ and only if $\mathrm{id}_S \times \phi$ fixes both $\Delta_S(p)$ and the tangent direction $T_{\Delta_S(p)}(\Delta_S(C)))$. Our variety X will be realized as a blow-up of $S \times S \times T$, where T is a surface of general type; taking the product with T makes it simpler to control automorphisms of blow-ups.

We begin with a lemma enabling us to show that a blow-up $\mathrm{Bl}_V X$ has no automorphisms except those that lift from X. Say that a variety X is $\mathbb{P}^r$-*averse* if every $\bar{K}$-morphism $h : \mathbb{P}^r_{\bar{K}} \to X_{\bar{K}}$ is constant. Note that if X is $\mathbb{P}^r$-averse, it is also $\mathbb{P}^s$-averse for any $s > r$.

Lemma 6. *Suppose that X is a $\mathbb{P}^{r-1}$-averse variety of dimension n, and $V \subset X$ is a smooth, geometrically irreducible subvariety of codimension r. Write $\pi : \mathrm{Bl}_V X \to X$ for the blow-up, with exceptional divisor E. Then every automorphism of $\mathrm{Bl}_V X$ descends to an automorphism of X, and the induced map $\mathrm{Aut}(\mathrm{Bl}_V X) \to \mathrm{Aut}(X)$ is an isomorphism onto $\mathrm{Stab}(V)$.*

Proof. We first observe that any nonconstant morphism $h : \mathbb{P}^{r-1}_{\bar{K}} \to \mathrm{Bl}_V X_{\bar{K}}$ must have image contained in a geometric fiber of $\pi|_{E_{\bar{K}}}$. Indeed, $\pi \circ h : \mathbb{P}^{r-1}_{\bar{K}} \to X_{\bar{K}}$ must be constant since X is $\mathbb{P}^{r-1}$-averse, and so the image of h is contained in a geometric fiber.

Suppose that $\phi : \mathrm{Bl}_V X \to \mathrm{Bl}_V X$ is an automorphism, and let $h : \mathbb{P}^{r-1}_{\bar{K}} \to \mathrm{Bl}_V X_{\bar{K}}$ be the inclusion of a geometric fiber of $\pi|_{E_{\bar{K}}}$. Then $\phi \circ h$ is an inclusion from $\mathbb{P}^{r-1}_{\bar{K}} \to \mathrm{Bl}_V X_{\bar{K}}$, and so must be the inclusion of some fiber of $\pi|_{E_{\bar{K}}}$. Thus ϕ permutes the fibers of $\pi|_{E_{\bar{K}}}$, and so descends to an automorphism of X fixing $\pi(E) = V$. □

Lemma 7.

1. *Suppose that X_1 and X_2 are $\mathbb{P}^r$-averse. Then $X_1 \times X_2$ is $\mathbb{P}^r$-averse.*
2. *Suppose that X is $\mathbb{P}^r$-averse and $V \subset X$ is a smooth, geometrically irreducible subvariety of codimension $s \le r$. Then $\mathrm{Bl}_V X$ is $\mathbb{P}^r$-averse.*

Proof. For (1), suppose that $h : \mathbb{P}^r_{\bar{K}} \to X_{1,\bar{K}} \times X_{2,\bar{K}}$ is a morphism. Then the projections $p_1 \circ h : \mathbb{P}^r_{\bar{K}} \to X_{1,\bar{K}}$ and $p_2 \circ h : \mathbb{P}^r_{\bar{K}} \to X_{2,\bar{K}}$ must both be constant, so that h is constant. For (2), the map $\pi \circ h$ must be constant, and so if h is nonconstant, its image is contained in a fiber of $\pi|_{E_{\bar{K}}}$. These fibers are isomorphic to $\mathbb{P}^{s-1}$, and since $s - 1 < r$, the map h must be constant. □

We require one more simple lemma before proceding to the construction.

Lemma 8. *Suppose that X is a smooth projective variety with $\mathrm{Aut}(X)$ discrete. There exists a divisor $G \subset X$ for which $\mathrm{Aut}(X; G)$ is trivial.*

Proof. Choose a very ample linear system $\mathcal{G} \cong \mathbb{P}^N$ on X. By the Lieberman–Fujiki theorem, the subgroup $\mathrm{Aut}(X; \mathcal{G})$ of automorphisms fixing $\mathcal{G}$ is of finite type, and hence finite since $\mathrm{Aut}(X)$ is assumed discrete. If ϕ is any member of $\mathrm{Aut}(X, \mathcal{G})$ other than the identity, it can not act trivially on $\mathcal{G}$. Indeed, suppose that ϕ fixes every element of $\mathcal{G}$. If x is any point of X, then $x = \bigcap_{G \ni x} G$, and so x is fixed by ϕ. It follows that ϕ is the identity map. As a consequence, a general element of $\mathcal{G}$ is not fixed by any automorphisms. □

Let T be a geometrically simply connected surface over K for which $\mathrm{Aut}(T)$ is trivial, T is not geometrically uniruled, and there is at least one K-points t on T; according to [13], we can take T to be the hypersurface in $\mathbb{P}^3$ defined by $x_0^5 + x_0 x_1^4 + x_1 x_2^4 + x_2 x_3^4 + x_3^5$, which has the points $[0, 1, 0, 0]$. (Note that if we work over $K = \mathbb{C}$ or any other uncountable field, then any very general hypersurface in $\mathbb{P}^3$ of degree at least 4 suffices.)

Take $X_0 = S \times S \times T$. The variety X will be constructed by a sequence of four blow-ups of X_0. In each case, the blowup satisfies the hypotheses of Lemma 6, so we may identify its automorphism group with a subgroup of $\mathrm{Aut}(X_0)$.

Lemma 9. *Let $X_0 = S \times S \times T$. Fix a point s on S and a divisor G on S with $\mathrm{Aut}(S; G)$ trivial, as in Lemma 8. Choose three distinct smooth, irreducible curves Γ_1, Γ_2, and Γ_3 in T, and a point t on Γ_3 which does not lie on Γ_1 or Γ_2.*

(1) The variety X_0 is $\mathbb{P}^r$-averse for any $r \geq 2$. The automorphisms of X_0 are of the form $\mathrm{Aut}(S \times S) \times \mathrm{id}_T$.
(2) Let $\pi_1 : X_1 \to X_0$ be the blow-up of X_0 along $s \times S \times \Gamma_1$. The variety X_1 is $\mathbb{P}^r$-averse for any $r \geq 3$. The automorphisms of X_1 are all lifts of $\mathrm{Aut}(S; s) \times \mathrm{Aut}(S) \times \mathrm{id}_T$.
(3) Let $\pi_2 : X_2 \to X_1$ be the blow-up along the strict transform of $G \times p \times \Gamma_2$. The variety X_2 is $\mathbb{P}^r$-averse for $r \geq 4$. The automorphisms of X_2 are given by $\mathrm{id}_S \times \mathrm{Aut}(S; p) \times \mathrm{id}_T$.
(4) Let $\pi_3 : X_3 \to X_2$ be the blow-up along the strict transform of $p \times p \times \Gamma_3$. Then X_3 is $\mathbb{P}^3$-averse for $r \geq 5$, and the automorphisms of X_3 are of the form $\mathrm{id}_S \times \mathrm{Aut}(S; p) \times \mathrm{id}_T$.
(5) Let E_3 be the exceptional divisor of $\pi_3 : X_3 \to X_2$. Then $\Delta_S(C) \times t$ meets E_3 at a single point u. Let $\pi_4 : X_4 \to X_3$ be the blow-up at u. The automorphism group of X_4 is isomorphic to $\mathrm{id}_S \times \mathrm{Aut}_{\mathrm{par}}(S; C, p) \times \mathrm{id}_T$.

Proof. We treat the blow-ups in order.

(1) To show that X_0 is $\mathbb{P}^r$-averse for $r \geq 2$, it suffices to check that S and T are both $\mathbb{P}^2$-averse, according to the first part of Lemma 7. For T this follows since T is not uniruled, while for S we note that a nonconstant morphism $h : \mathbb{P}^2_{\bar{K}} \to S_{\bar{K}}$ must be generically finite, and so induce an injection $h^* : \mathrm{Pic}(S_{\bar{K}}) \to \mathrm{Pic}(\mathbb{P}^2_{\bar{K}})$, which is impossible.

Suppose that $\phi : X_0 \to X_0$ is an automorphism. Let $p_3 : X_0 \to T$ be the third projection. We first claim that ϕ must permute the geometric fibers of p_3. If $p_3 \circ \phi$ contracts any geometric fiber of p_3, it must contract every geometric fiber by the rigidity lemma. So if ϕ does not permute the fibers of p_3, then every fiber of p_3 has image in T of dimension at least 1. Since these fibers are isomorphic to $S \times S$, the image of every geometric fiber is uniruled, which implies that T must be geometrically uniruled, contradicting the choice of T.

Consequently every automorphism of X_0 is of the form $\phi \times \psi$, where ϕ is an automorphism of $S \times S$ and ψ is an automorphism of T. Since $\mathrm{Aut}(T)$ is trivial, the group $\mathrm{Aut}(X_0)$ can be identified with $\mathrm{Aut}(S \times S) \times \mathrm{id}_T$.

(2) The center of the blow-up π_1 has codimension 3, so it follows from Lemma 7 that X_1 is $\mathbb{P}^r$-averse for $r \geq 3$. According to Lemma 6, since X_0 is $\mathbb{P}^2$-averse, $\mathrm{Aut}(X_1)$ is given by the stabilizer of $s \times S \times \Gamma_1$ in $\mathrm{Aut}(X_0)$, which is isomorphic to the stabilizer of $s \times S$ in $\mathrm{Aut}(S \times S)$.

We claim that an element ϕ of $\mathrm{Aut}(S \times S)$ fixes $s \times S$ only if it is of the form $\phi_1 \times \phi_2$, where ϕ_1 is in $\mathrm{Aut}(S; s)$ and ϕ_2 is in $\mathrm{Aut}(S)$. Indeed, if ϕ fixes one fiber of $p_1 : S \times S \to S$, it must permute the fibers, and so induces an automorphism $\phi_1 : S \to S$ on the base with $p_1 \circ \phi = \phi_1 \circ p_1$. Then $(\mathrm{id}_S \times \phi_1^{-1}) \circ \phi$ is an automorphism of $S \times S$ defined over p_1. This must be given by a map $\mathrm{id}_S \times \phi_2 : S \times S \to S \times S$, since $\mathrm{Aut}(S)$ is discrete, and so ϕ is of the form $\phi_1 \times \phi_2$, where ϕ_1 fixes s.

(3) Since X_1 is $\mathbb{P}^r$-averse for $r \geq 3$ and X_2 is the blow-up of X_1 at a center of codimension 4, it follows that X_2 is $\mathbb{P}^r$-averse for $r \geq 4$. Since the center of π_2 has codimension 4 and X_1 is $\mathbb{P}^3$-averse, the automorphisms of X_2 are given by isomorphisms of X_1 that fix $G \times p \times t_2$. The automorphisms of X_1 are all of the form $\phi_1 \times \phi_2 \times \mathrm{id}_T$, and so this stabilizer is exactly $\mathrm{id}_S \times \mathrm{Aut}(S; p) \times \mathrm{id}_T$.

(4) We have seen that X_2 is $\mathbb{P}^4$-averse, and X_3 is the blow-up of X_2 at a center of codimension 5. It follows that X_3 is $\mathbb{P}^r$-averse for $r \geq 5$, and the automorphisms of X_3 are lifts of automorphisms of X_2 that fix $p \times p \times \Gamma_3$. Every automorphism of X_2 fixes $p \times p \times \Gamma_3$, and so the automorphisms of X_3 are again given by $\mathrm{id}_S \times \mathrm{Aut}(S; p) \times \mathrm{id}_T$.

(5) The centers of the blow-ups π_1 and π_2 are both disjoint from the fiber $S \times S \times t$, since t lies on neither Γ_1 nor Γ_2, while the center of the blow-up π_3 meets $S \times S \times t$ at the single point $p \times p \times t$. As a result, $\Delta_S(C) \times t$ meets E_3 at one point u, as claimed. The restriction of $\pi_3 \circ \pi_2 \circ \pi_1$ to the strict transform of $S \times S \times t$ is the blow-up at the point $p \times p \times t$.

Since X_3 is $\mathbb{P}^5$-averse and the center of π_3 has codimension 6, $\mathrm{Aut}(X_4)$ is isomorphic to the stabilizer of u in $\mathrm{Aut}(X_3)$. These are exactly the automorphisms $\mathrm{id}_S \times \phi \times \mathrm{id}_T$ of X_3 that fix the tangent direction $T_{\Delta(p)}(\Delta_S(C)) \times t$. According to Lemma 5, these are exactly the lifts of automorphisms of the form $\mathrm{id}_S \times \mathrm{Aut}_{\mathrm{par}}(S; C, p) \times \mathrm{id}_T$. □

This completes the construction.

Proof of Theorem 1. Let $X = X_4$ be as in Lemma 9. The variety X is smooth, projective and geometrically simply connected, since it is a blow-up of $S \times S \times T$ where S is a rational surface and T is geometrically simply conneected. The group $\mathrm{Aut}(X)$ is isomorphic to $\mathrm{Aut}_{\mathrm{par}}(S; C, p)$, which is not finitely generated according to Lemma 4. □

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