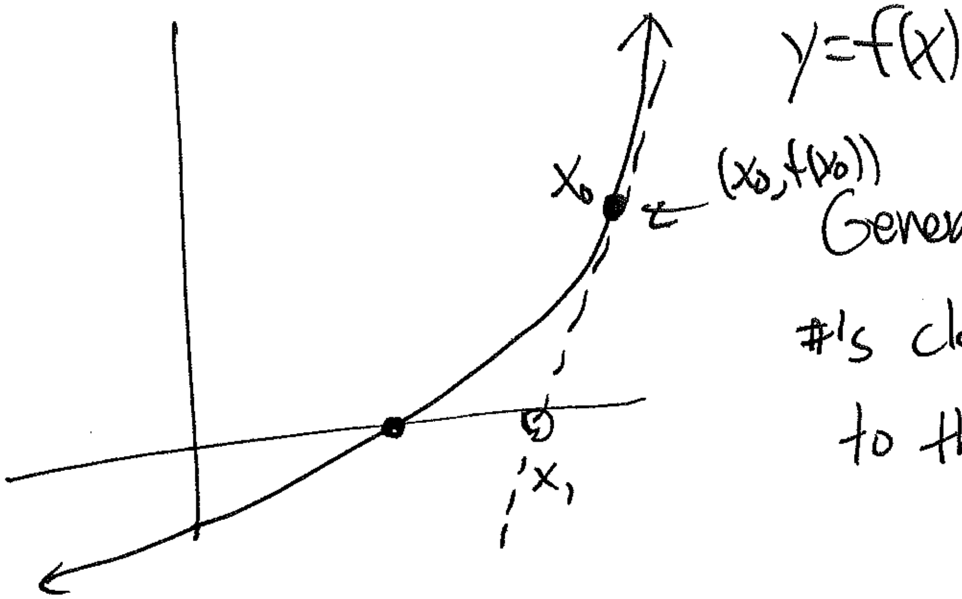
Newton's method

"numerical root finding"



Generate a sequence of #'s closer and closer to the actual root.

Tangent line: $\quad y = f'(x_0)(x-x_0) + f(x_0)$

Where does it hit axis?

$$0 = f'(x_0)(x-x_0) + f(x_0)$$

$$\Rightarrow x - x_0 = -\frac{f(x_0)}{f'(x_0)}$$

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Adv Topics 2 (Lesieutre)
Newton's method
November 11, 2021

Problem 1. Use Newton's method to approximate the cube root of 3. For the first five values of x_i, compute the error $\left|\sqrt{3} - x_i\right|$. How fast is the error decreasing?

it seems to be squared every time!

Problem 2. Repeat the first problem, this time to approximate the value of π.

use sin x = f(x).

Problem 3. Try Newton's method with $f(x) = x^3 - 1$ starting with $x_0 = 0$. What happens? What if you try other starting values?

horizontal tgt line → failure

Problem 4. How about $f(x) = x^3 - 2x + 2$, starting with $x_0 = 0$? What about other starting values?

bounces 0 → 1 → 0 → 1 → 0

Problem 5. What happens if you try Newton's method for $f(x) = x^2 + 1$? Why?

there's no root

Problem 6. What happens if you try Newton's method for $f(x) = \sqrt[3]{x}$? Why?

gets further and further away!

Problem 7. Consider $f(x) = \frac{1}{1+x^2} - \frac{1}{2}$. Try a few values of x_0. Which root of f do you find? Is there a pattern?

Problem 8. Now we want to derive the steepest descent algorithm for minimization. We use a similar idea to Newton's method: start with a guess x_0, and keep improving it one step at a time.

Do you have a guess? How should we choose a next x-value?

Problem 9. What is your proposal for the step size?

Problem 10. Write down a function with multiple maxima and minima. Apply the algorithm, with a couple options for step size. What do you find?

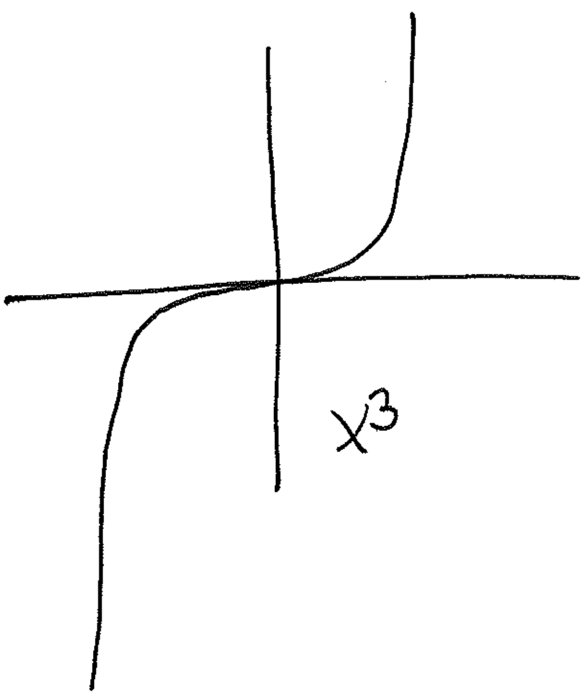
Problem 2: to get π, solve for a root of $\sin(x)$.

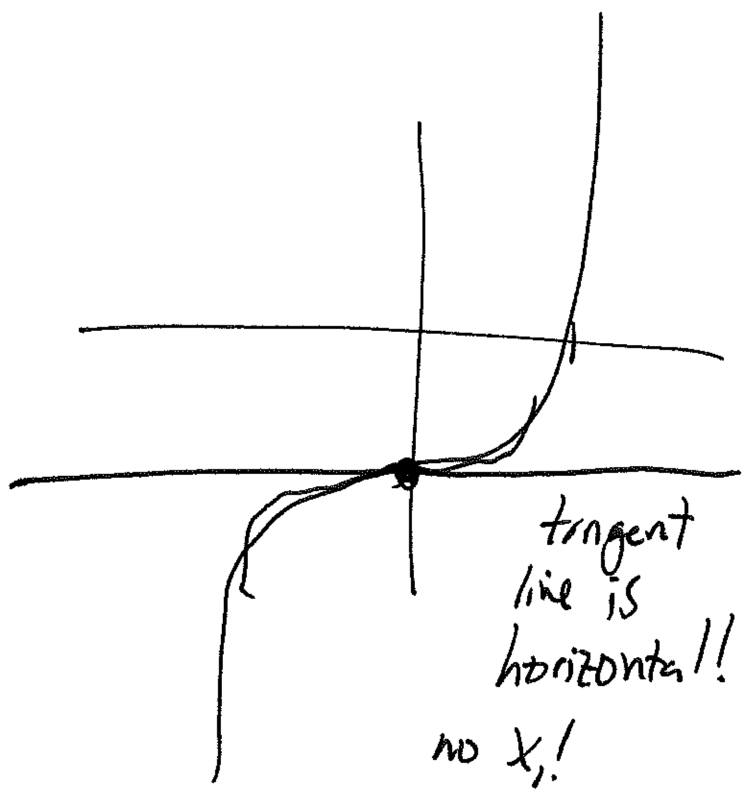
start with $x_0 = 3$ (this is to avoid getting $x=0$).

or: look $\sin(x) - 1$, and it'll converge to $\pi/2$.

(or $5\pi/2$, etc)

Problem 3: $f(x) = x^3 - 1$ with $x_0 = 0$



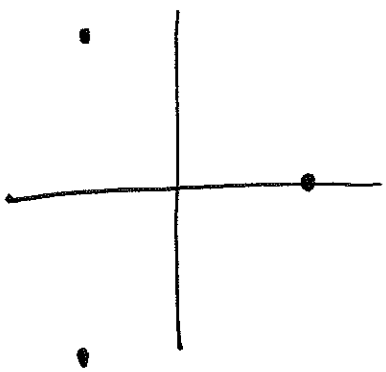


no x_1!

Question:

for $f(x)=x^3-1$, this has 3 complex roots:

If $1, e^{\frac{4\pi}{3}i}, e^{\frac{8\pi}{3}i}$.



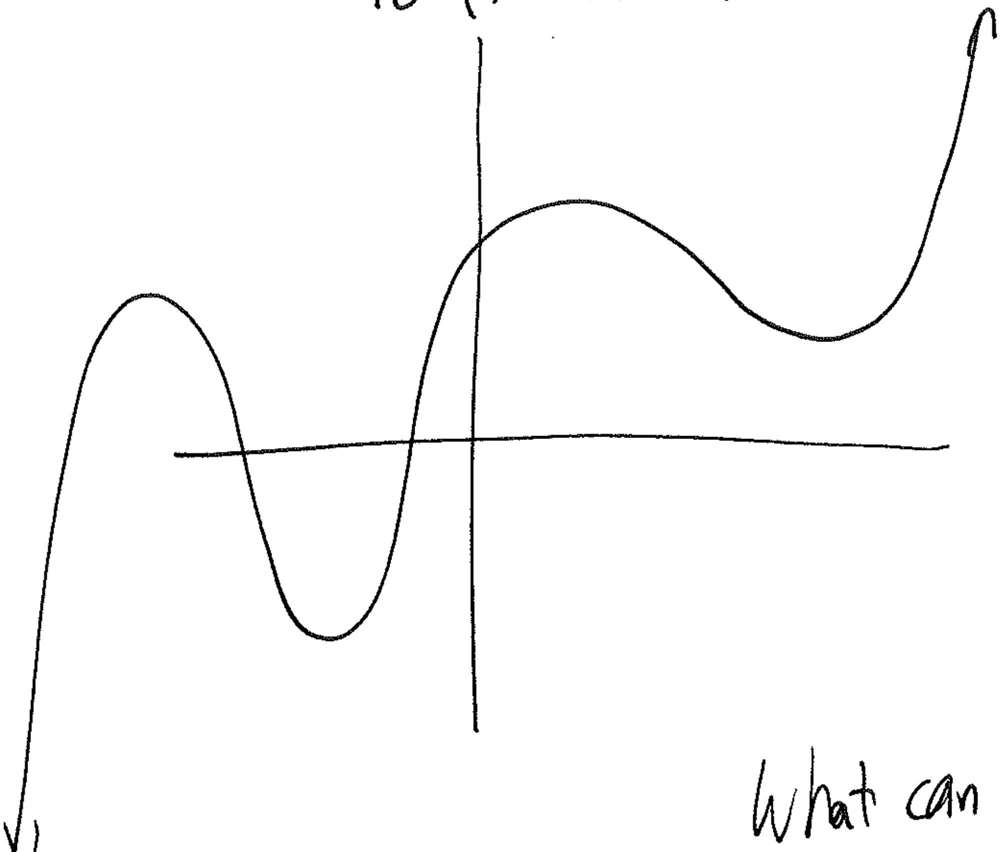
$$\left(e^{\frac{4\pi}{3}i}\right)^3 = e^{4\pi i} = 1.$$

$\cos(4\pi)+i\sin(4\pi)$

Challenge: which complex x_0 find which of the three roots?

Newton's method for optimization.

To find a min:



$y=f(x)$.

You can compute f', f'', but not solve for max/min exactly.

What can we do?

a) Use Newton's method to find a root of f', that's the critical pt

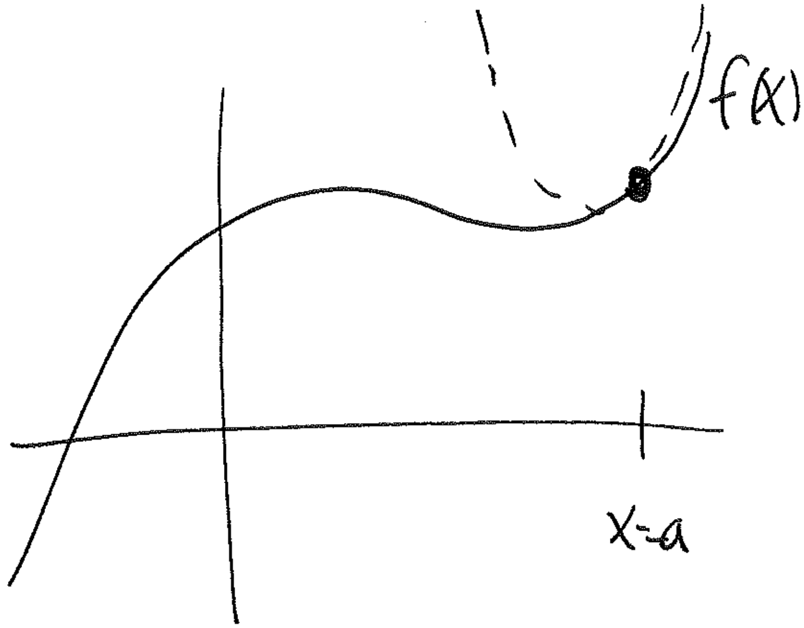
OR

b) Check $f'(x_0)$. If $f'(x_0)<0$, let $x_1=x_0+0.00001$

OR

if $f'(x_0)>0$, let $x_1=x_0-0.00001$

* c) Approximate function by a parabola, let $x_1=$ min of parabola.



<u>Parabola approximation:</u>

$p(x) = \cancel{p}$ 2nd order Taylor series!

(next time)