Continuity

Intuition: a continuous function is a function you can graph without lifting your pen.

How to make this a precise definition?

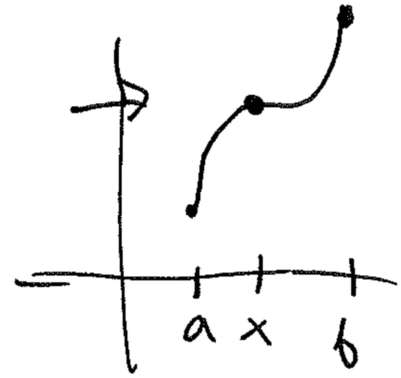
We want a definition of "continuous" that guarantees all the important theorems are true:

- Intermediate Value Theorem:

  If $f(x)$ is continuous, and $a < b$, then c is between $f(a)$ and $f(b)$, there exists an x with $a \le x \le b$ so $f(x) = c$.

- FTC:

$$\frac{d}{dx} \int_a^x \cancel{f(t)\,dx}\, f(t)\,dt = f(x)$$



can you even integrate every continuous fct?

Old, defunct def of continuous:

$f(x)$ is continuous if it satisfies the intermediate value theorem: given c with $f(a) \le c \le f(b)$, there exists x so $f(x) = c$.

A crazy function: Conway's Base-13 function. (it satisfies above definition, but shouldn't count as continuous.)

Given x to compute $f(x)$:

1) Write x in base 13, using 0-9, A, B, C.
2) Turn A into "+", B into "−", C into ".".
3) If ends with:

$\pm$ (some numbers) . (some numbers, maybe infinite)

then $f(x)$ = that.

4) If it doesn't end like that, $f(x) = 0$.

If

numbers

$x_{13} = 1AB.32CA9134C1347\ldots$

$\rightarrow 1+-.32.+9134.1347\ldots$

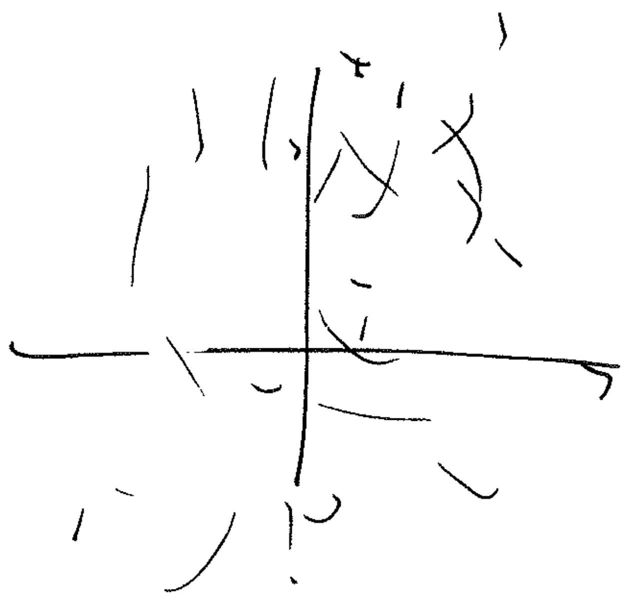
then $f(x) = +9134.1347\ldots$

$x_{13} = AB.12A12A12A12A\ldots$

$\rightarrow +-.12+12+12+12+\ldots$

$f(x) = 0$

No way to graph it!



Why does it satisfy IVT?

→ Given any a&b, and any target value C whatsoever (doesn't have to be between a and b), there's $x \in [a,b]$ so $f(x) = C$.

Challenge: $a = \pi = 3.141592$

$b = 3\sqrt{2} = 4.2\ldots$

$c = e = 2.71828\ldots$

can put anything here and be $\in [a,b]$

$$x_{13} = 4.0000\underset{+}{A}2\underset{\cdot}{C}71828\;182845\ldots$$

then $f(x) = e$!

Modern definition (Cauchy, 1800s?)

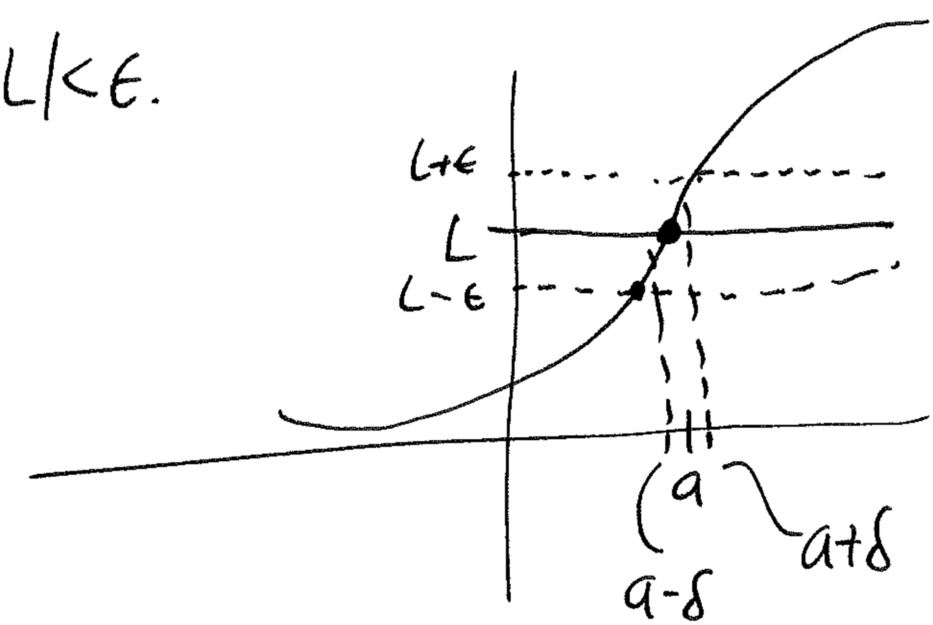
First need to define limit of a function.

Suppose $f(x)$ is a function. We say

$\lim_{x \to a} f(x) = L$ if... for any $\epsilon > 0$,

there exists a $\delta > 0$ such that if $\underbrace{|x-a| < \delta}_{\text{x is within } \delta \text{ of } a}$

then $|f(x) - L| < \epsilon$.



Prove that $\lim_{x\to 2} x^2 = 4$.

Suppose $\epsilon > 0$. You need to find δ so that if $|x-2| < \delta$, then $|x^2-4| < \epsilon$.

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want:

$|x^2-4| < \epsilon$

$|x-2|\,|x+2| < \epsilon$.

We can make this as small as we want! by picking δ.

As long as we pick a δ that's ≤ 1, x will be between 3 and 5, so $|x+2| \leq 5$.

Suppose $\epsilon > 0$.

<u>Pf.</u> Take $\delta = \min(1, \epsilon/5)$.

This means $\delta \leq 1$ and $\delta \leq \epsilon/5$.

Then if $|x-2| < \delta$

$$|x^2 - 4| = \underbrace{|x-2|}_{\leq (\epsilon/5)} \cdot \underbrace{|x+2|}_{(5)} \leq (\epsilon/5) \cdot (5) = \epsilon.$$

we know $|x-2| < \delta$ so $|x-2| < 1$. This means $1 < x < 3$, so $3 < |x+2| < 5$

Another: Prove $\lim_{x \to 1} 3x = 3$

Suppose $\epsilon > 0$. You need a δ so that if $|x-1| < \delta$ then $|3x-3| < \epsilon$.

<u>Pf.</u> Suppose $\epsilon > 0$. Let $\delta = \epsilon/3$. If $|x-1| < \delta = \epsilon/3$, then $3|x-1| < \epsilon$ so $|3x-3| < \epsilon$, which is what we wanted.

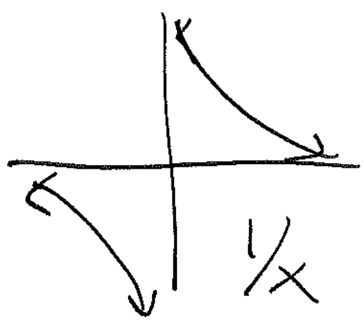
$\leftarrow |f(x) - L|$

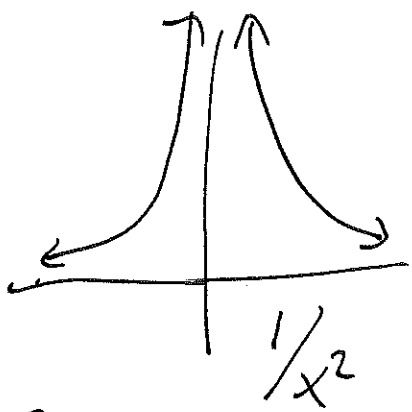
<u>Def</u> $f(x)$ is <u>continuous</u> if for any value of a,

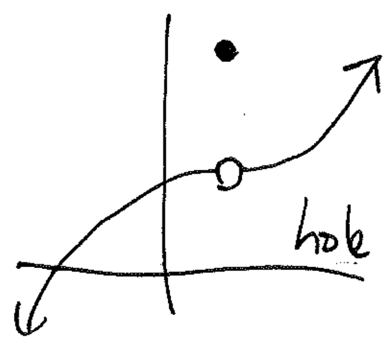
$\lim_{x \to a} f(x)$ exists (and is equal to $f(a)$.)

← automatic if it exists

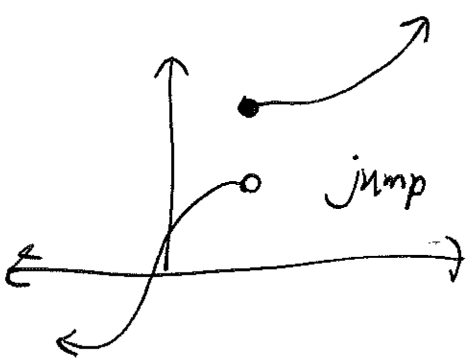
<u>Some non-continuous functions.</u>

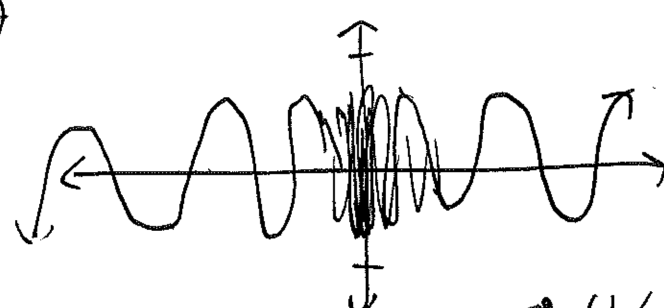






by our def limit doesn't exist





$\sin(1/x)$

$\lim_{x \to 0} \sin(1/x)$ doesn't exist.

Raindrop function

$$f(x) = \begin{cases} 1/q & \text{if } x \text{ is rational and } x = p/q. \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

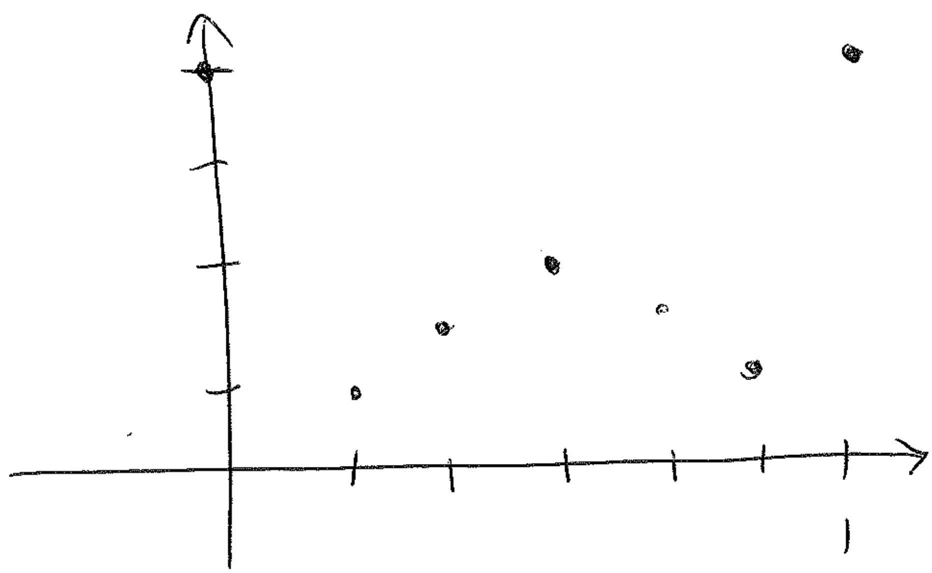
reduced

and $f(0) = 1$

e.g. $f(6/7) = 1/7$

are there a values where it is continuous at $x = a$?

$\lim_{x \to a} f(x)$ exists?



Is it continuous at $x = 1/2$? $f(x) = 1/2$

No! Take $\epsilon = 1/4$?

Can we find a δ so every x between

$(1/2 - \delta, 1/2 + \delta)$ has $|f(x) - f(1/2)| < \epsilon$

i.e. $1/4 < f(x) < 3/4$?

no! no matter how small δ is,

there's an irrational x is $(1/2 - \delta, 1/2 + \delta)$,

which has $f(x) = 0$.

I claim that if ~~a~~ a is irrational then

$f(x)$ is continuous at a.

e.g. $a = \frac{1}{\pi} = 0.318309\ldots$

Suppose we have $\epsilon = \frac{1}{10}$.

Trying to come up with δ so that if

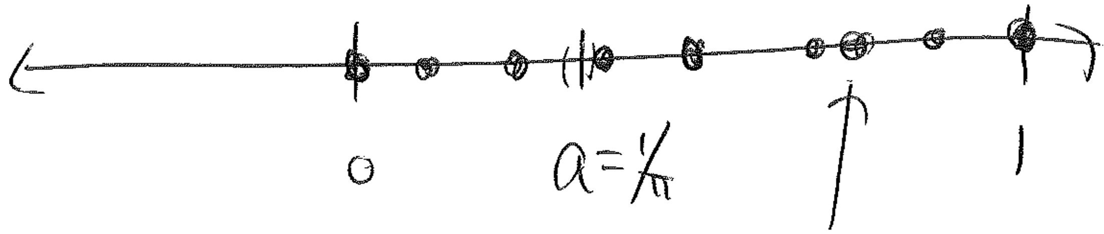
$|x - \frac{1}{\pi}| < \delta$, then $|f(x) - f(1/\pi)| < \epsilon = \frac{1}{10}$.

so $|f(x)| < \frac{1}{10}$

But there are only finitely many

x's for which $|f(x)| \geq \frac{1}{10}$!

$x = \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \ldots, \frac{1}{10}, \ldots, \frac{9}{10}$.



finitely many x values where $f(x) \geq \frac{1}{10}$

pick δ so $(a-\delta, a+\delta)$ avoids these numbers!

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$$

not continuous anywhere!