<u>Today</u>: Last day of dynamical systems

+ chaos

Next:
- Inequalities, least-squares regressions
- Abstract algebra (Galois theory...)
- Topology
- Calculus of variations
- Probability

Dimension of fractals.

→ Suppose S is a shape (any dimension)

→ How to define $\dim(S)$?

"size of"

$$\mu(\underbrace{k \cdot S}) = k^d \cdot \mu(S)$$

d is the dimension

S scaled by factor of k in every direction

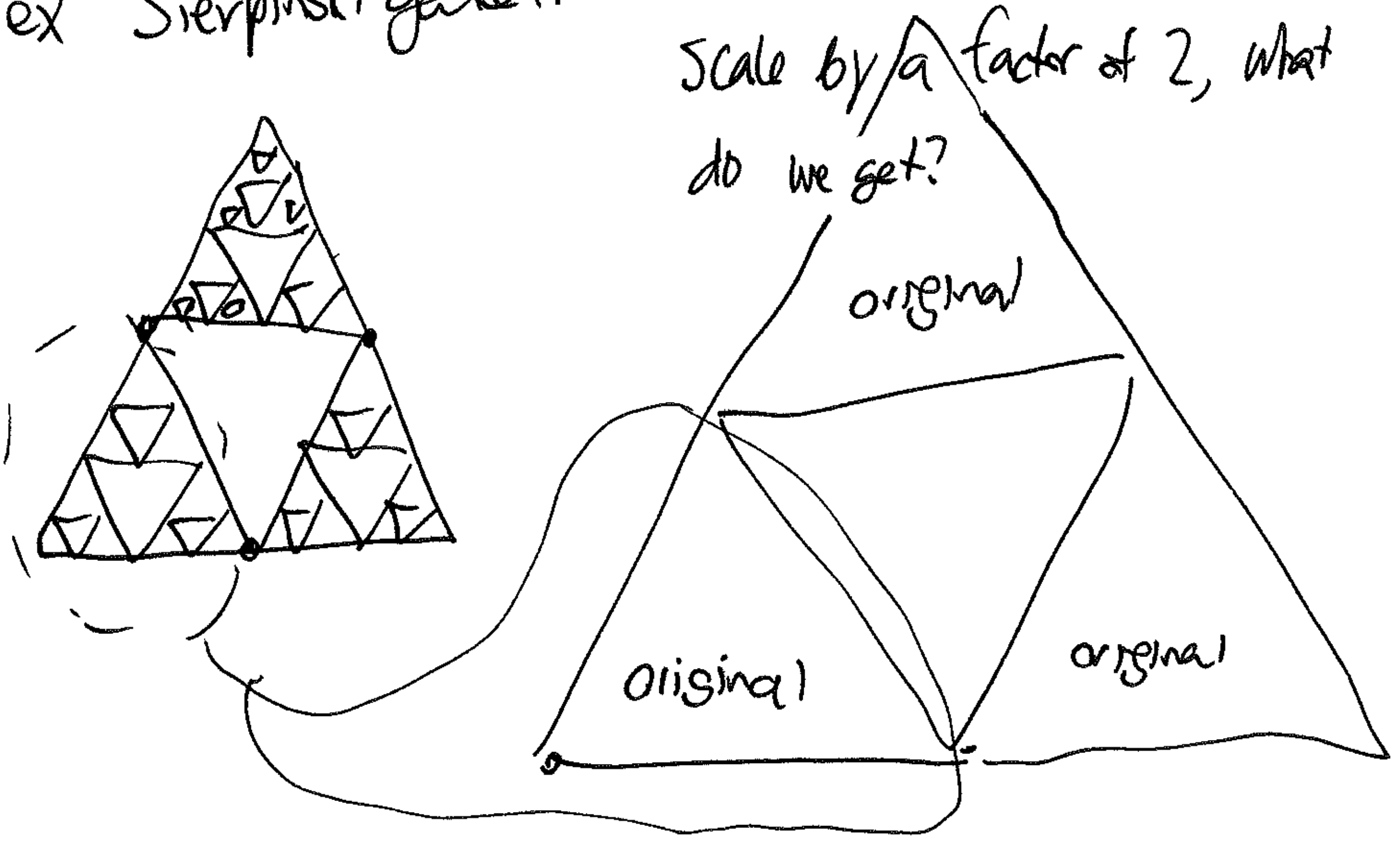
$\mu(2 \cdot S) = 3 \cdot \mu(S)$

$2^d = 3 \rightsquigarrow d = \log_2 3$

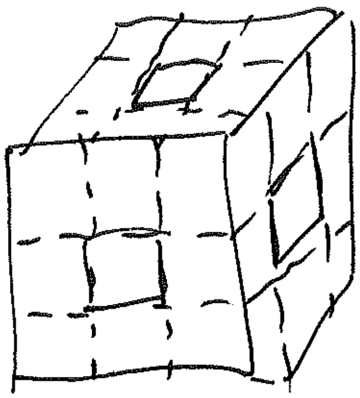
$1 < d < 2.$

ex Sierpinski gasket:

Scale by a factor of 2, what do we get?



Menger sponge



dimension?

scale by factor of 3:

20 = 8+4+8 copies of original.

$$\mu(3\cdot S) = 3^d \cdot \mu(S)$$
$$\shortparallel$$
$$20 \cdot \mu(S)$$

$$3^d = 20$$

$$d = \log_3 20$$

"Hausdorff dimension"

Hausdorff dimension

Two problems with what we've done:

1) Doesn't work if not perfectly self-similar
   (e.g. Mandlebrot)

2) What is "μ" anyway? (hard!)

size

The official definition (avoids these problems)

Suppose we have a set S.

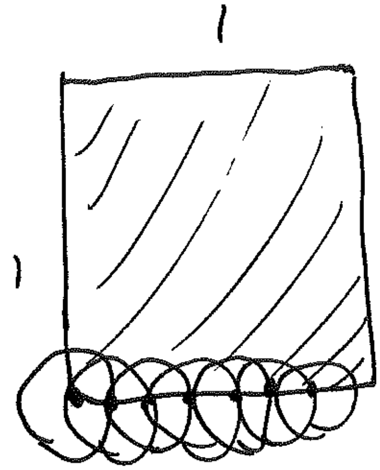
"the d-dimensional size of S"

"what would be the size of S if we think of S as being d-dimensional?"

$$\mu_d(S) = \lim_{\epsilon \to 0} \min_{\substack{\text{covers} \\ U_i}} \sum (\mathrm{diam}(U_i))^d$$

where U_i are balls that cover S and have diameter at most ϵ. (they can overlap) ex

$S =$



square

U_i
"
{put balls of radius 0.1 in a 0.1×0.1 grid, it will cover S}

here we have 11×11 grid

121 balls U_i

$\epsilon = 0.2$

$$\cancel{\mu_\epsilon(S) = 121}$$

$$\sum (\operatorname{diam}(U_i))^d = 121 \cdot (0.2)^d.$$

What about a different ϵ?

Cover U_i could be grid of balls of radius $\epsilon/2$ spaced every $\epsilon/2$. There are $\sim \left(\frac{2}{\epsilon}\right) \cdot \left(\frac{2}{\epsilon}\right) = \frac{4}{\epsilon^2}$ balls.

$$\mu_d(\text{square}) = \lim_{\epsilon \to 0} \sum_{\text{balls } U_i} \text{diam}(U_i)^d$$

$$= \lim_{\epsilon \to 0} \frac{4}{\epsilon^2} \cdot \left(\frac{\epsilon}{2}\right)^d$$

$$= \lim_{\epsilon \to 0} \frac{4}{\epsilon^2} \left(\frac{\epsilon}{2}\right)^d$$

try:

$d=3$

$$\lim_{\epsilon \to 0} \frac{4}{\epsilon^2} \left(\frac{\epsilon}{2}\right)^3 = \lim_{\epsilon \to 0} \frac{1}{2}\epsilon = 0$$

$d=1$

$$\lim_{\epsilon \to 0} \frac{4}{\epsilon^2} \left(\frac{\epsilon}{2}\right)^1 = \lim_{\epsilon \to 0} \frac{2}{\epsilon} = \infty.$$

$$\rightsquigarrow \mu_d(S) = \begin{cases} 0 \text{ if } d>2 \\ 1 \text{ if } d=2 \\ \infty \text{ if } d<2 \end{cases}$$

Def

If S is any shape, there is a value ~~of~~ d_0 so that

$$\mu_d(S) = 0 \quad \text{if } d > d_0$$

$$\mu_d(S) = \infty \quad \text{if } d < d_0$$

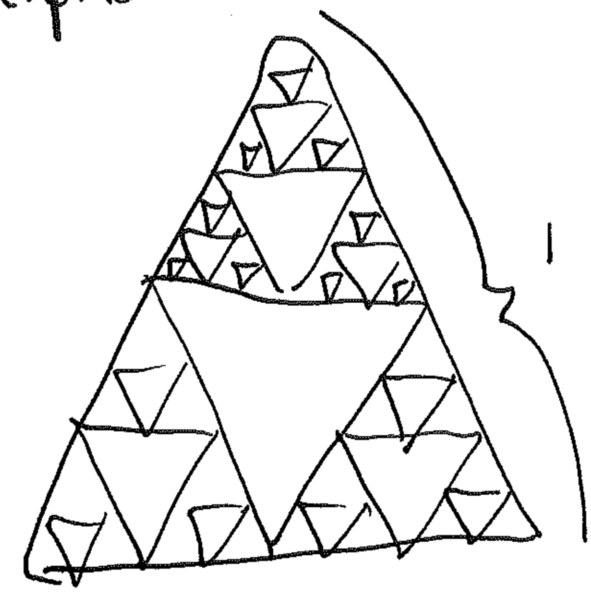
d_0 is the Hausdorff dimension of S.

(looked at pictures of the coast of Britain, pictures of broccoli, ...)

up next... topological entropy

one to try.

Sierpinski



2 circles of diameter δ

cover a triangle of side δ

U_i:

$\epsilon = 1/2 \rightsquigarrow 3$ tri $\rightsquigarrow 6$

$\epsilon = 1/4 \rightsquigarrow 9$ tri $\rightsquigarrow 18$

$\epsilon = 1/8 \rightsquigarrow 27$ tri $\rightsquigarrow 54$

$\epsilon = 1/2^n \longrightarrow 3^n$ tri $\longrightarrow 2\cdot 3^n$

$$\mu_d(S) = \sum \operatorname{diam}(U_i)^d$$

to cover it with balls of diameter $\frac{1}{2^n} = \epsilon$

~~we need:~~

$$\mu_d(S) = \lim_{\epsilon\to 0} \sum \operatorname{diam}(U_i)^d = \lim_{n\to\infty} \left(\frac{1}{2^n}\right)^d \cdot (2\cdot 3^n)$$

$$= \lim_{n\to\infty} \left(\frac{3}{2^d}\right)^n \cdot 2 = \begin{cases} 0 & \text{if } d > \log_2 3 \\ \infty & \text{if } d < \log_2 3 \end{cases} \Rightarrow$$

Hausdorff is $\log_2 3$.