Types of games:

- number of players
- turn-taking vs simultaneous moves
- complete information (both players know where on the game tree they are)
- randomness ("nature"/"fate" player?)

...

(check Wiki!)

Tic-tac-toe: 2-player, complete information, not randomness.

Prisoner's dilemma/RPS: 2-player, not complete information

Battleship

Yahtzee!

Sorry! Incomplete + random

[...]

Theorem (Zermelo's theorem)

In a 2-player game with complete information, no randomness, only win/loss outcomes, then either Player 1 or Player 2 has a forced win strategy.

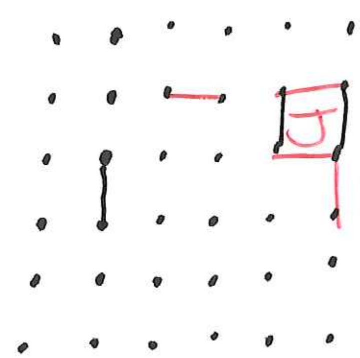
(a choice of move at every node on the game tree)

Example

If game also allows draws, there's a third possibility: either player can force a draw.

What does this apply to?

- Chomp
- Dots & Boxes
- Nim
- Hex



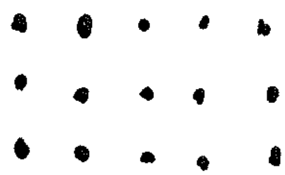
Nim (one version):

2 player

→ Put N rocks in middle of table (maybe $N=15$?)

→ Each player picks up 1, 2, or 3 rocks as a turn.

→ Whoever takes last rock loses.



Zermelo: either first or second player has a forced win! But which is it?

Who wins? $N=15$. First player should pick up $\underline{\underline{2}}$.

1st!

13 left.

If second player picks up j, first player picks $4-j$.] total is 4

9 left.

repeat. 2×

5 left, 1 left, and 2nd player picks it up and loses.

First player loses if $N \equiv 1 \bmod 4$.

Hex (see board on next page)

- draws are impossible! so forced win for either 1st or 2nd player

(John Nash proved in 1949)

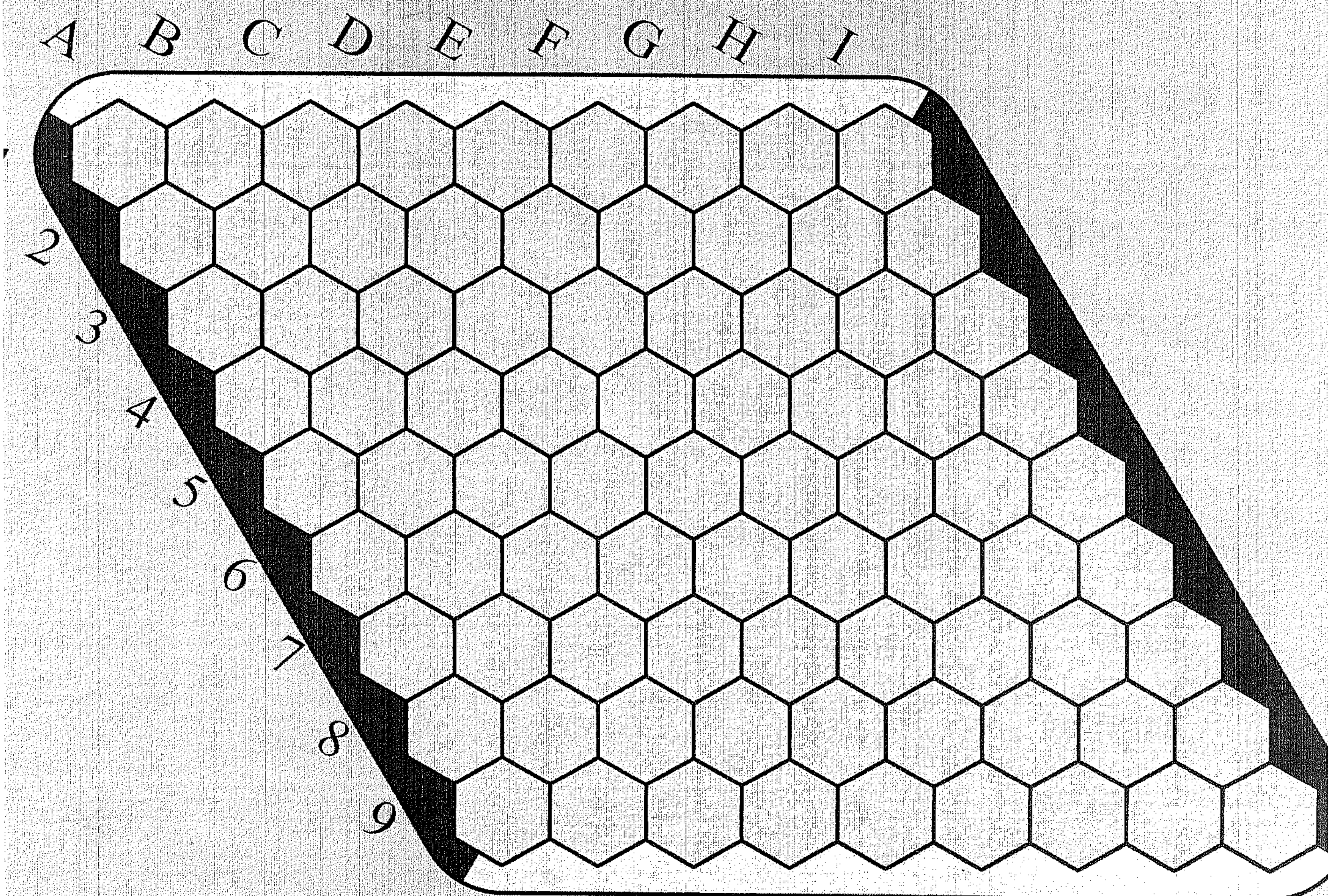
"Brouwer fixed point theorem"

Zermelo: Either 1st or 2nd player has a winning strategy.

Which is it? It's a forced win for first player!
(But nobody actually knows the winning strategy.)

"Strategy stealing": if there were a forced win for 2nd player, first player could just make a random move and then adopt that strategy.

doesn't apply to chess, dots & boxes, because taking turn can hurt you in those games.


A B C D E F G H I
2
3
4
5
6
7
8
9

Review of last week:

A new game: Battle of the Bismarck Sea

Axis: Must choose to sail either N or S of New Britain.

Allies: Must search either N or S.

<u>Payoff functions:</u>

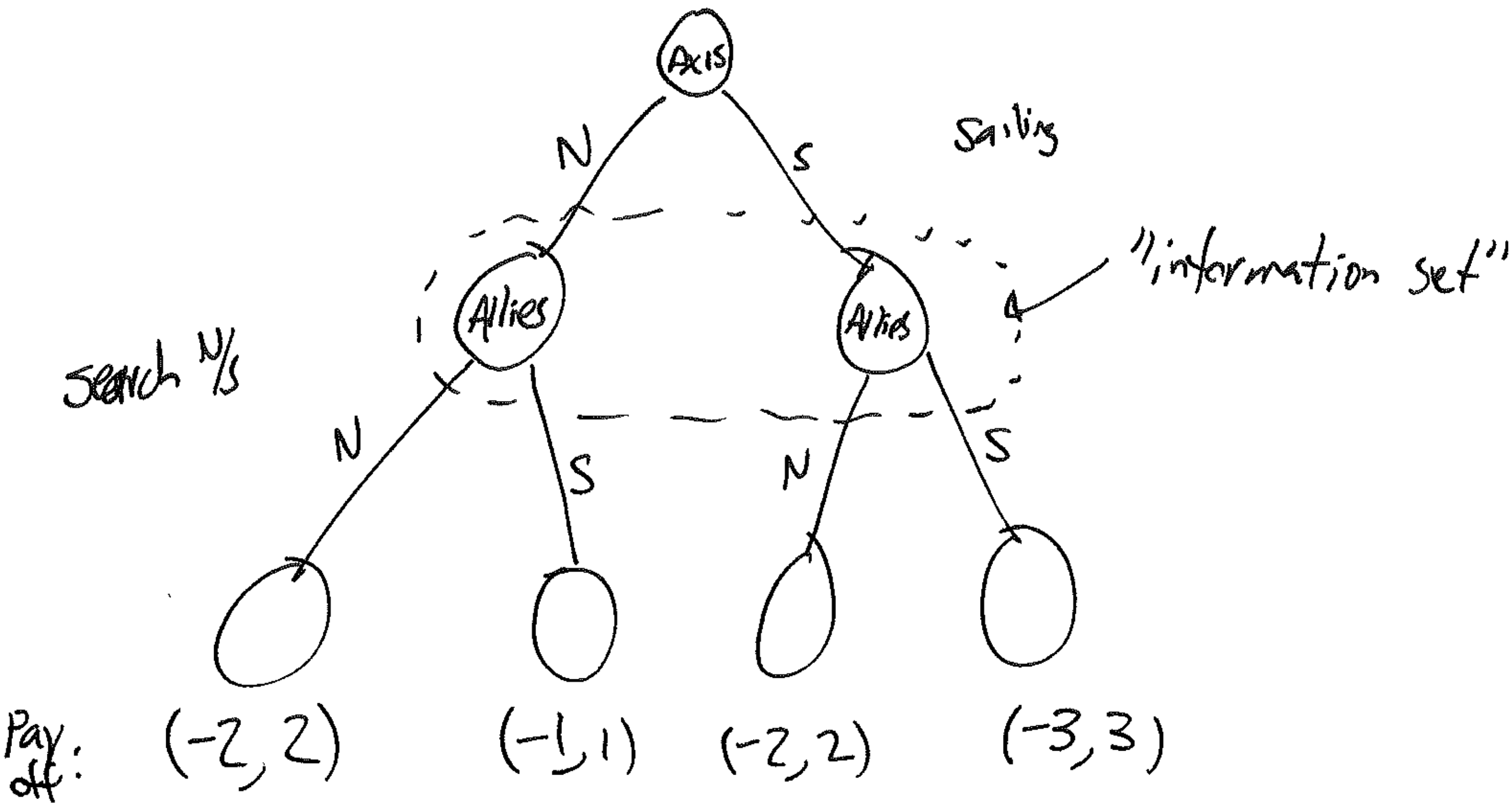
Sail north, search south: bomb for 1 day.

Sail north, search north: bombed for 2 days.

Sail south, search south: bomb for 3 days.

Sail south, search north: bombed for 2 days.

Challenge: Draw game tree for battle (+payoff functions)



"extensive form" of the game.

"Sail north" <u>weakly dominates</u> "sail south"

(no matter what the other player does, outcome is no worse)

Strategic form of a game.

For a two-player game, let $\Sigma_1 = \left\{ \begin{matrix} \text{strategies for} \\ \text{player 1} \end{matrix} \right\}$

$\Sigma_2 = \left\{ \begin{matrix} \text{strategies for} \\ \text{player 2} \end{matrix} \right\}.$

number of strategies

Suppose $\#\Sigma_1 = N_1$, $\#\Sigma_2 = N_2$.

$|\Sigma_1|$ ″

$\mathrm{card}(\Sigma_1)$ ″

make an $N_1 \times N_2$ matrix, and in ~~each~~ each (i,j) entry put the payoff for player 1 is case ~~of strat~~ player 1 uses i^{th} strategy and player 2 uses j^{th} strategy.

Then do same for player 2.

Ex Battle of Bismarck sea.

Player 1 (Axis)

	search N	search S
sail N	-2	-1
sail S	-2	-3

Player 2 (Allies)

	search N	search S
sail N	2	1
sail S	2	3

$$A_1 = \begin{pmatrix} -2 & -1 \\ -2 & -3 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$$

$A_1 = -A_2$ (a "zero-sum game")

Chicken

Player 1

	swerve	don't swerve
swerve	0	-1
don't swerve	1	-100

$$\begin{pmatrix} 0 & -1 \\ 1 & -100 \end{pmatrix}$$

Player 2

	swerve	don't
swerve	0	1
don't	-1	-100

$$\begin{pmatrix} 0 & +1 \\ -1 & -100 \end{pmatrix}$$

not zero-sum!

For complex games, strategic form is too large, but for simple games, it helps the analysis a lot!

Another example: 2 players, three strategies each

$$A = \begin{pmatrix} -15 & -35 & 10 \\ \boxed{-5} & 8 & 0 \\ -12 & -36 & 20 \end{pmatrix}$$

zero-sum game: payoff for P2 is opposite.

Player 1: "Minimax strategy": Pick option whose worst possible outcomes is least painful.

(that's option 2)

Player 2: analogous option 1

Player 1 doing strategy 2, Player 2 doing strategy 1 is "equilibrium"