Adv Topics 2 (Lesieutre)
September 2, 2021

Problem 1. For each of the following matrices T, choose a couple sample vectors $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and compute $T\mathbf{v}$. What does the matrix do to a vector, geometrically? What does it do to the unit square?

a) $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}$

d) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

e) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

f) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

g) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

h) $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

i) $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

Problem 2. For each of the linear maps described, write down the matrix for the corresponding transformation.

a) Reflection about the y-axis.

b) Rotation by $45°$ clockwise. What about other angles θ?

c) In 3D: rotation by an angle θ around the z-axis.

Problem 3. Let A and B be the first two matrices above, and let f and g be the corresponding linear maps.

a) Compute the composition $g \circ f$, using the formulas for the functions. What is the matrix for the composition?

b) Compute the product of the matrices BA. Notice anything?

c) Compute $f \circ g$ and AB. Do these match your earlier answers?

Problem 4. Suppose that a parallelogram has vertices at $(0,0)$, (a,c), and (b,d). What is its area? (Try to do this using basic geometry. To make life easy, you can assume (a,c) is in the first quadrant and (b,d) is in the second.)

Problem 5. One property of a linear map is that it rescales all areas by the same scaling factor.

a) For each of the 2×2 maps in the first problem, what is the scaling factor?

b) The determinant of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $ad - bc$. Compute the determinants of the 2×2 matrices.

c) Can you guess a formula for the determinant of the product of two matrices?

Problem 6. What is the determinant of the 3×3 matrix

$$\begin{pmatrix} 4 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 2 & 2 \end{pmatrix}?$$

Can you guess what the geometric meaning of this might be? (We'll learn another way to compute determinants later on.)

Problem 7. One application of this is in the formula for change of variables in multiple integrals. Suppose you want to integrate a function $f(x, y)$ over a non-rectangular region S. Maybe you can parametrize the region by $x(s, t)$ and $y(s, t)$ where $a \leq s \leq b$ and $c \leq t \leq d$.

a) One case of this would be integrating over a circle. How would you parametrize a circle, using the description above?

b) Imagine the rectangle $a \leq s \leq b$ and $c \leq t \leq d$ is covered with a "mesh" of rectangles of width ds and height dt. Applying $x(s, t)$ and $y(s, t)$, we get a mesh covering S, but the elements won't be rectangles anymore.

Imagine a small rectangle based at (s_0, t_0) with sides of length ds and dt, parallel to the axes. When you apply $x(-, -)$ and $y(-, -)$, what happens to this small rectangle? Imagine ds and dt are so small that the map looks locally linear there. What is the area of the image?

c) How would you compute an integral over your area, using this formula?

d) I always find this formula confusing. Use it to compute the area of a circle to make sure everything works.

Problem 8. Let's do another integral over a strange region.

a) Consider the area S between $y = 0$ and $y = x^2$ satisfying $0 \leq x \leq 2$. How could you parametrize this region by functions $x(s,t)$ and $y(s,t)$ where s, t range over a rectangular region?

b) Compute

$$\iint_S xy\, dA$$

using change of variables.

c) Compute

$$\iint_S xy\, dA$$

directly, by choosing suitable bounds for the intgral.

Problem 9. For each of the maps in the first problem, describe the eigenvectors if you can.

Problem 10. The map in part (h) has a bit of a different flavor. I don't think it's especially exciting geometrically, but it has another useful property.

a) Compute $T(0,1)$, $T^2(0,1)$, $T^3(0,1)$, $\ldots$ until you find a pattern.

b) Can you find the eigenvectors and eigenvalues for this map?

c) Can you use this to determine a formula for the pattern you noticed in (a)? (Hint: write $(1,0)$ in terms of the eigenvectors you found in (b).