Complex integrals

Complex-differentiable functions

complex

$f(z) = z^2 + e^z$

When is it differentiable?

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \text{ exists?}$$

h could be real, imaginary, ...

How to tell?

Write $f(x+iy) = u(x,y) + v(x,y)i$

Differentiable if $\left.\begin{array}{l} u_x = v_y \\ v_x = -u_y \end{array}\right\}$ Cauchy-Riemann eqns.

$f(z)=z^2$

$f(x+iy)=x^2+(iy)^2+2ixy$

$$=\underbrace{(x^2-y^2)}_{u(x,y)}+\underbrace{(2xy)}_{v(x,y)}i$$

$\cancel{u_x=2x}$ $\quad u_x=2x \quad$, $\quad v_x=2y$

$\cancel{v_x=2x}$ $\quad v_y=2x \; \checkmark \quad\quad u_y=-2y \; \checkmark$

$f(z)=z^3$

$f(z)=z+\overline{z}$

$f(z)=e^z$

$$f(x+iy) = (x+iy)^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$$

$$= x^3 + (3x^2y)i + (-3xy^2) + (-y^3)i$$

$$= \underbrace{(x^3 - 3xy^2)}_{u(x,y)} + \underbrace{(3x^2y - y^3)}_{v(x,y)}i$$

$$u_x = 3x^2 - 3y^2$$
$$v_y = 3x^2 - 3y^2 \quad \checkmark$$

$$f(x+iy) = (x+iy) + (x-iy) = \underbrace{2x}_{u(x,y)} + \underbrace{0}_{v(x,y)}i$$

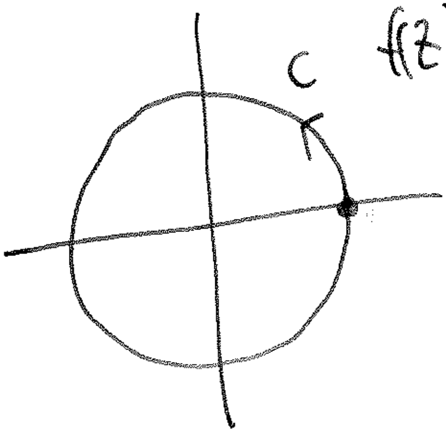
$$u_x = 2$$
$$v_y = 0 \quad \times$$

$$f(x+iy) = e^{x+iy} = e^x e^{iy}$$

$$= (e^x \cos y) + (e^x \sin y) i$$

$$u_x = e^x \cos y \qquad u_y = -e^x \sin y$$

$$v_y = e^x \cos y \qquad v_x = e^x \sin y.$$



$$f(z) = \frac{1}{z^2} + 3z$$

paramaterize path

$$\gamma(t) = \cdots$$

$$\oint_c f(z)\,dz$$

$$\int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

$$\gamma(t) = e^{it}$$

$$\int_0^{2\pi} \left(\frac{1}{(e^{it})^2} + 3(e^{it})\right)(ie^{it})\,dt = \int_0^{2\pi} ie^{-it} + 3ie^{2it}\,dt$$

$$= 0$$

$$\oint_C \frac{1}{z}\,dz = 2\pi i$$

Suppose $f(z)$ holomorphic. $f(x+iy) = u(x,y) + iv(x,y)$

Parametrize path

$$\gamma(t) = (\overset{r(t)+is(t)}{r(t),\ s(t)})$$

$\uparrow$ x component, y component

$$\oint_C f(z)\,dz = \int_a^b$$

$$= \oint_a^b \underbrace{\Big(u(r(t),s(t)) + i\, v(r(t),s(t))\Big)}_{f(\gamma(t))} \cdot \Big(r'(t) + i\, s'(t)\Big)\, dt$$

$$= \oint_a^b \Big[u(r(t),s(t))\, r'(t) - v(r(t),s(t))\, s'(t)\Big]$$

$$+ \Big[v(r(t),s(t))\, r'(t) + u(r(t),s(t))\, s'(t)\Big] i \; dt$$

just real part

$$\oint_a^b \left[u(r(t), s(t))\, r'(t) - v(r(t), s(t))\, s'(t) \right] dt$$

$$= \oint_a^b \langle u(r,s), -v(r,s) \rangle \cdot \langle r'(t), s'(t) \rangle \, dt$$

$$= \oint_C \vec{F} \cdot d\vec{r} \quad \text{when } \vec{F} = \langle u(x,y), -v(x,y) \rangle$$

$$\overset{\text{Green's}}{=} \iint_R \operatorname{curl} \vec{F} \, dA = \iint_R \text{"} N_x - M_y \text{"} = \iint_R (-v_x - u_y)\, dA$$

$$= \iint_R 0 \, dA = 0$$

by Cauchy-Riemann!

Some theorems

differentiable everywhere

1) If $f(z)$ is holomorphic, then

$$\oint_C f(z)\,dz = 0$$

Ex $\quad f(z) = \frac{2}{z^2} + \frac{7}{z} + e^{e^z} \cos z$

$$\oint_{\text{unit circ}} f(z)\,dz = \oint \frac{2}{z^2}\,dz + \oint \frac{7}{z}\,dz + \oint e^{e^z} \cos z\,dz$$

$$0 + 7(2\pi i) + 0$$

$$= 14\pi i.$$

Residue theorem

Suppose $f(z)$ is meromorphic: it is mostly differentiable, but maybe ∞ at ~~example~~ some points

e.g. $\frac{1}{z} + \frac{1}{z-1} - \frac{2}{\sin z}$

$$\oint_C f(z)\,dz = 2\pi i \left(\sum_{a_k \text{ pole}} \text{residue of } f(z) \text{ at } a_k \right)$$

the $\frac{1}{z}$ coefficient

$$g(z) = \frac{1}{2z + az^2 + a} \quad \text{at} \quad r_+ = \frac{-1 + \sqrt{1-a^2}}{a}$$

To find residue:

c_{-1} is the residue.

$$g(z) = \frac{c_{-1}}{z - r_+} + c_0 + c_1(z - r_+) + c_2(z - r_+)^2 + \cdots$$

$$c_{-1} = \lim_{z \to r_+} (z - r_+) g(z) = \lim_{z \to r_+} \frac{z - r_+}{2z + az^2 + a}$$

$$= \lim_{z \to r^+} \frac{1}{2 + 2az} = \frac{1}{2 + 2ar_+}.$$

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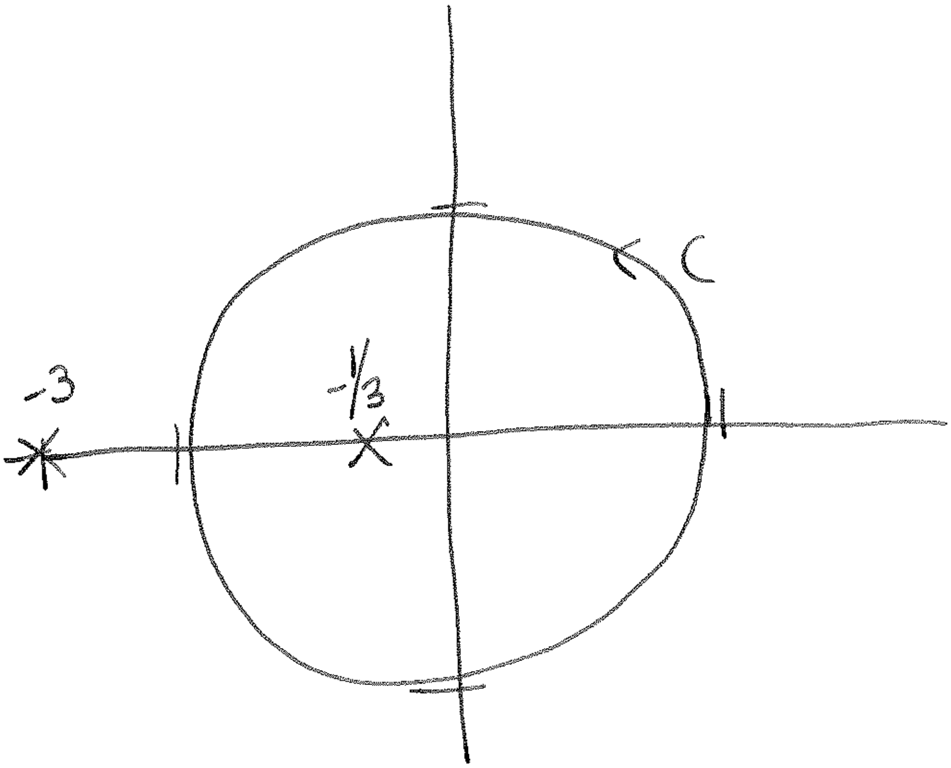
substitute

$$\text{so} \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \frac{2}{i} \oint \frac{1}{2z + az^2 + a} dz$$

$$= \frac{2}{i}(2\pi i \cdot \text{residue at } r_+)$$

$$= \frac{2}{i}\left(2\pi i \cdot \frac{1}{2 + 2ar_+}\right) = \frac{2\pi}{1 + ar_+} = \frac{2\pi}{\sqrt{1-a^2}}$$



$-1 < a < 1,$

$a = 3/5:$

$$\frac{-1 \pm \sqrt{1-a^2}}{a}$$

$$\| $$

$$\frac{-1 \pm \sqrt{16/25}}{3/5} = \frac{-1 \pm \frac{4}{5}}{\frac{3}{5}}$$

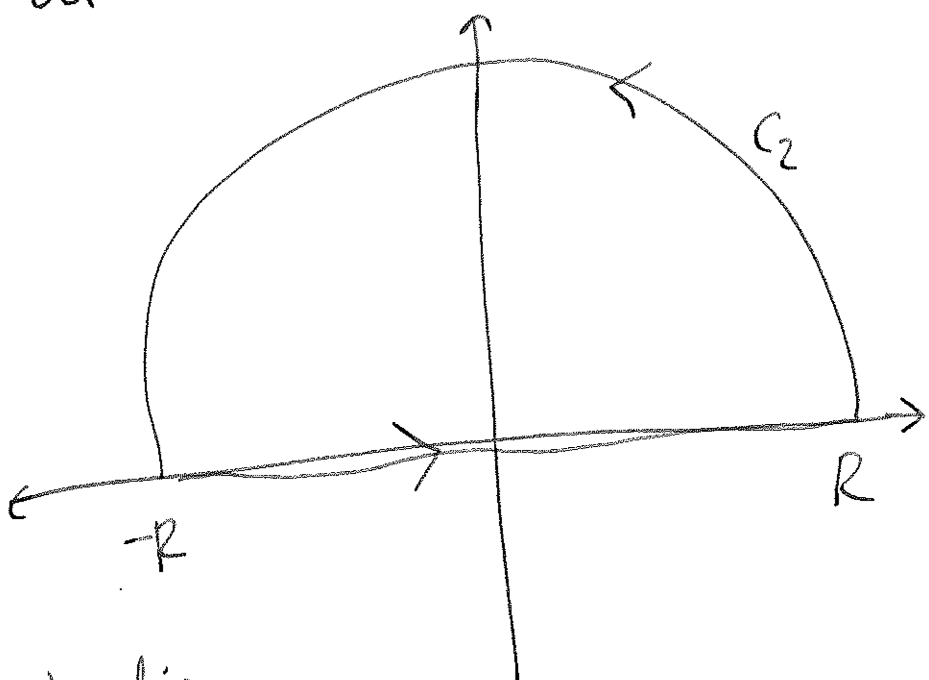
Residue theorem

$$\oint_C \frac{1}{2z + az^2 + a}\, dz = 2\pi i \left(\begin{array}{l} \text{residue of that} \\ \text{at the pole inside circle} \end{array} \right)$$

$$= 2\pi i \left(\text{residue at } \frac{-1+\sqrt{1-a^2}}{a} \right)$$

$$\int_0^{2\pi} \frac{d\theta}{1+\frac{1}{2}\cos\theta} = \frac{2\pi}{\sqrt{1-(1/2)^2}} = \frac{2\pi}{\sqrt{3/4}} = \frac{4\pi}{\sqrt{3}}$$

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$$



R really big

$$\int_{-R}^{R} \frac{1}{x^4+1}\,dx = \oint \frac{1}{z^4+1} - \int_{C_2} \frac{1}{z^4+1}\,dz$$

↑

want to know