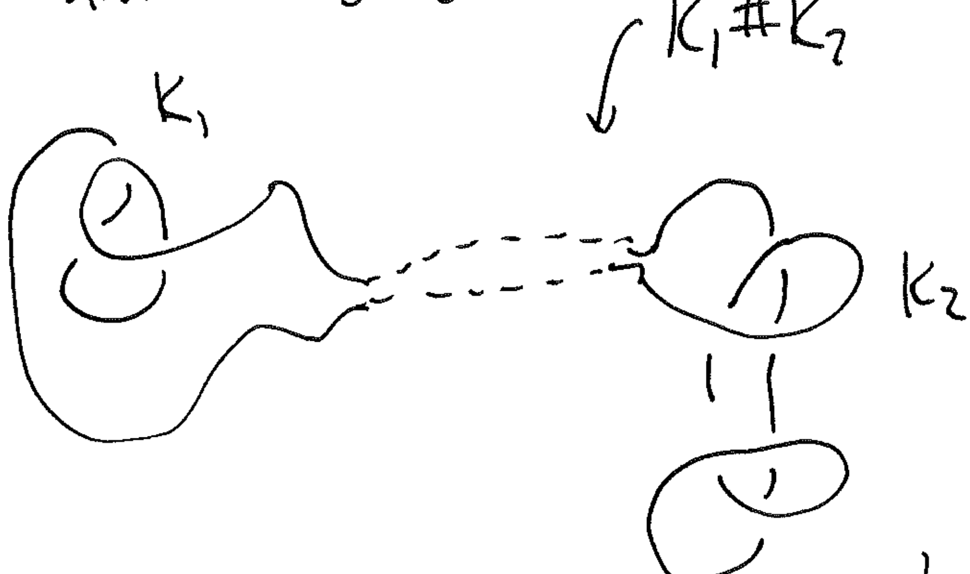
Connected Sum

If K_1, K_2 are two knots,

$K_1 \# K_2$ is obtained by cutting both open and connecting together.



Seems like it shouldn't depend on where we cut it.

this does depend on having an orientation on the knot: a chosen direction to walk along the knot.

1) Is (right-handed trefoil) # (left-handed trefoil)

$\shortparallel$ $\nwarrow$ "square knot"

(right-hand trefoil) # (right-handed trefoil)

$\nwarrow$ "granny knot"

2) What's $cr(K_1 \# K_2)$ in terms of $cr(K_1)$ and $cr(K_2)$

3) Could $K_1 \# K_2$ be the unknot even if neither knot is?

1) A knot is a prime knot if it can't be written as a sum of two other nontrivial knots.

→ square/granny knots are not prime.

→ figure 8/trefoil are prime.

Every knot can be broken down into prime knots, in a unique way. (a bit hard to prove.)

2) $cr(K_1 \# K_2) \leq cr(K_1) + cr(K_2)$

<u>Open problem</u>: Is it always equal?

(probably yes)

3) Could $K_1 \# K_2 = \bigcirc$?

If 2) true, couldn't happen.

Let's prove it's impossible.

Classic wrong proof.

$$1 = 1 + 0 + 0 + 0 + \cdots$$

$$= 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + \cdots$$

$$= (1 + -1) + (1 + -1) + (1 + -1) + (1 + -1) + \cdots$$

$$= 0 + 0 + 0 + 0 + \cdots = 0$$

rebracketing non-convergent infinite series doesn't work.

Another warning:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots$$ converges to $\log 2$.

AKA $\ln 2$

But! You can reorder terms and make it convergent to anything you want.

For Knots, this proof is correct(able)!

"Mazur's swindle"

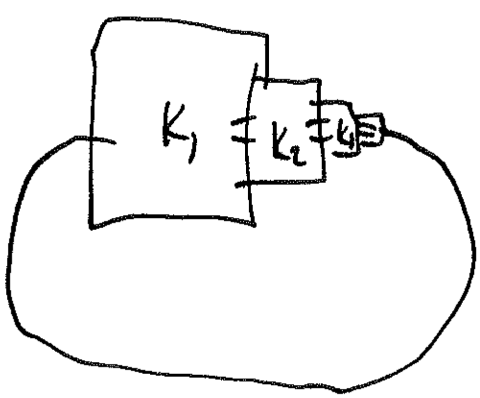
Suppose $K_1 \# K_2 = 0$

$$K_1 = K_1 \# (K_2 \# K_1) \# (K_2 \# K_1) \# (K_2 \# K_1) \# \cdots$$

$$= (K_1 \# K_2) \# (K_1 \# K_2) \# (K_1 \# K_2) \# \cdots$$

$$= 0 \# 0 \# 0 \# 0 = 0$$ so $K_1 = 0$. (ditto K_2)

You can always take an infinite sum of knots by making them smaller and smaller and smaller

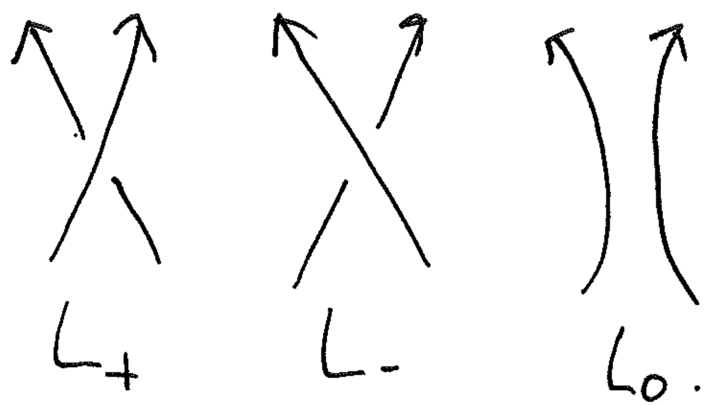


size of $K_n = 1/2^n$

Alexander polynomial

Definition using "Skein relations".

Suppose we have three oriented knots, that only differ in one crossing:



The Alexander polynomial $\Delta_K(t)$ is defined recursively by two rules:

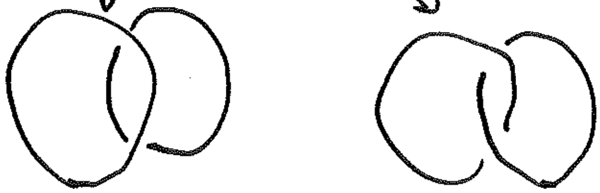
1) $\Delta_K(\text{unknot}) = 1$

2) $\Delta_{L_+} - \Delta_{L_-} + (t^{+1/2} - t^{-1/2})\Delta_{L_0} = 0$

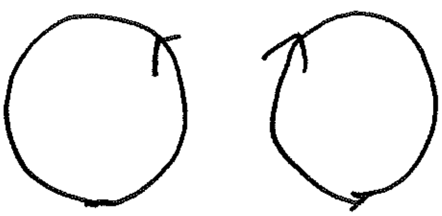
(so if you know Δ for two, you can get the third)

This is defined not just for knots, but also links:

a link



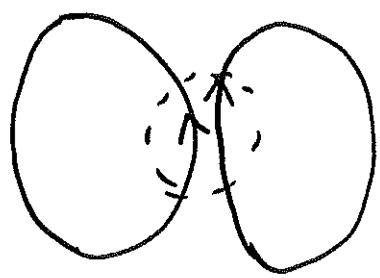
Warm-up: Δ for two disjoint unknots



$$\Delta_{L_+} - \Delta_{L_-} + (t^{1/2} - t^{-1/2})\Delta_{L_0} = 0$$

$$1 - 1 + (t^{1/2} - t^{-1/2})\Delta_{L_0} = 0$$

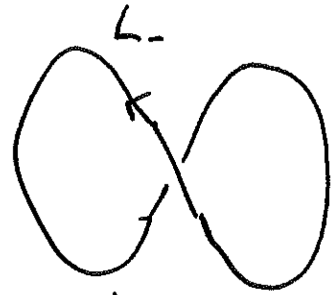
L_0



L_+

unknot

L_-

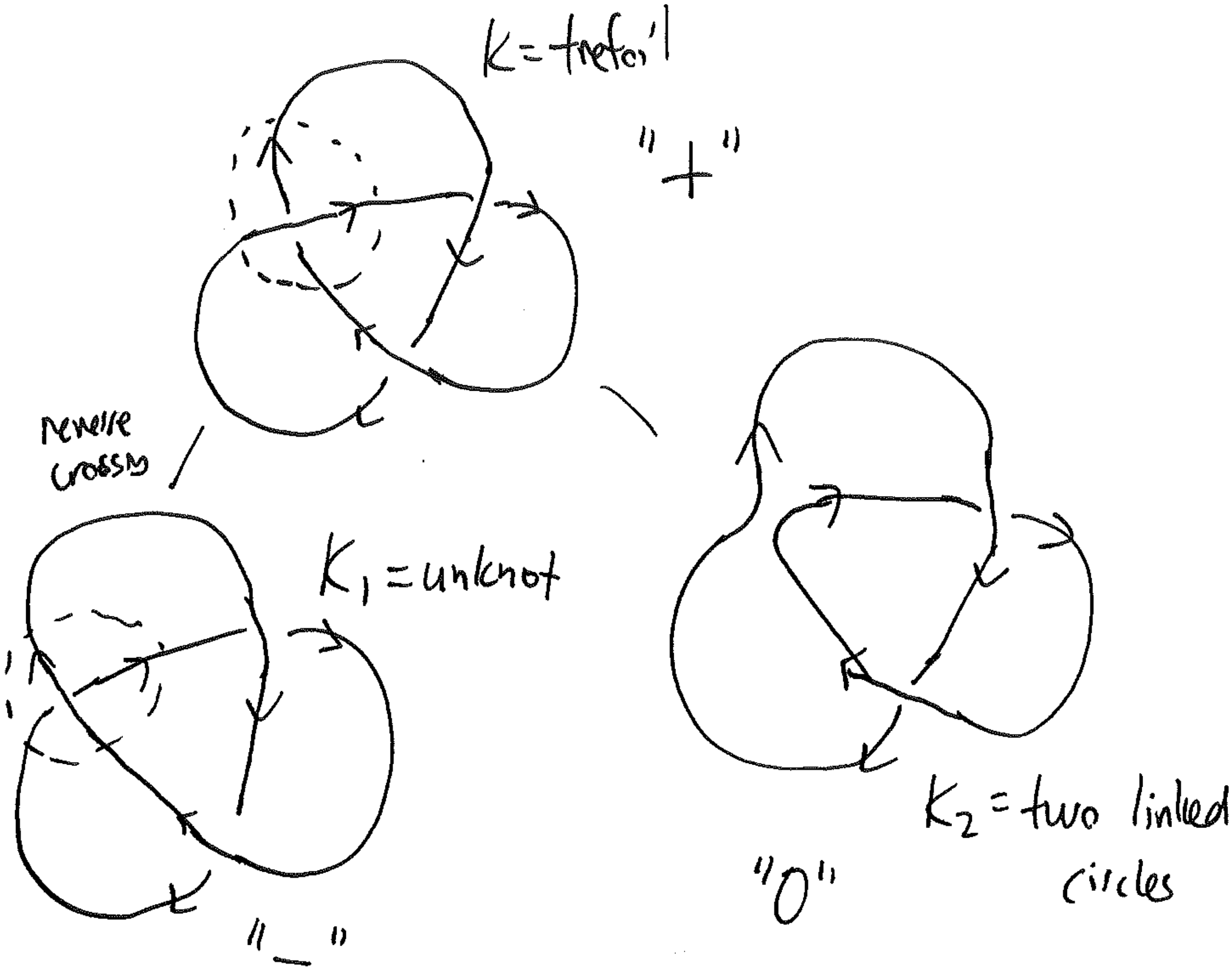


unknot

$$\boxed{\Delta_{L_0} = 0}$$

really

$$\left(\Delta_{L_0}(t) = 0\right)$$



$$\cancel{\Delta_K(t)} \qquad \overset{+}{\Delta_K} - \overset{-}{\Delta_{K_1}} + (t^{1/2} - t^{-1/2})\overset{0}{\Delta_{K_2}} = 0$$

Δ_{K_2} on next page

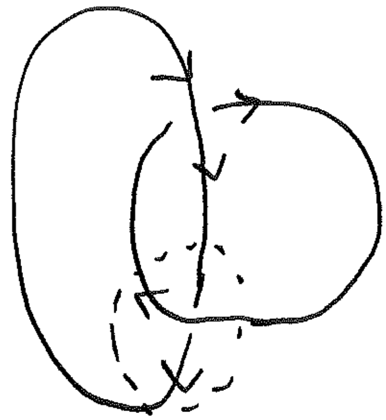
$$\Delta_K - 1 + (t^{1/2} - t^{-1/2})\left(-(t^{1/2} - t^{-1/2})\right) = 0$$

$$\Delta_K = 1 + (t^{1/2} - t^{-1/2})^2 = 1 + (t - 2 + t^{-1})$$

(negative powers of t and half-int) $\longrightarrow$ $\boxed{t - 1 + t^{-1}}$

K_2 = two linked circles

+



$$\Delta_{K_2} - \Delta_{K_3} + (t^{1/2} - t^{-1/2})\Delta_{K_4} = 0$$

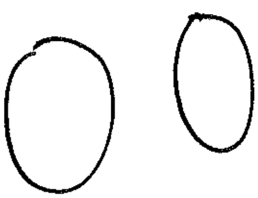
$$\Delta_{K_2} - 0 + (t^{1/2} - t^{-1/2})1 = 0$$

$$\Delta_{K_2} = -(t^{1/2} - t^{-1/2})$$

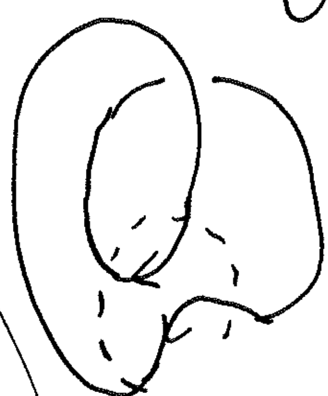
−

=

K_3 = disjoint circles



0



K_4 = unknot

Things to check

- not affected by Reidemeister
- these rules actually define it for any knot

HW Calculate $\Delta_{\text{figure 8}}$.

(you should get $3-t-t^{-1}$)

$\longrightarrow$ not trefoil

$\longrightarrow$ not unknot!

But:

$$\Delta_K = \Delta_{\text{reverse}(K)}$$

(e.g. left- and right-handed trefoil give same answer)

$$\Delta_{K_1 \# K_2} = \Delta_{K_1} \Delta_{K_2} \quad (\text{HW!})$$

So

$$\Delta_{\text{square knot}} = \Delta_{\text{right trefoil}} \cdot \Delta_{\text{left trefoil}}$$

$$= (t-1+t^{-1})(t-1+t^{-1})$$

$$= (t-1+t^{-1})^2$$

$\Delta_{\text{granny knot}} = \Delta_{\text{right trefoil}} \cdot \Delta_{\cancel{\text{left}}\ \text{right trefoil}}$

$$= (t - 1 + t^{-1})^2$$

Δ can't tell the difference between square and granny knots!

Enter the Jones polynomial:

$$\frac{1}{t}\Delta_{K_+} - t\Delta_{K_-} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)\Delta_{K_0}.$$

Calculate in the same way!

→ Jones polynomial <u>can</u> tell difference between square + granny knots.

→ and between left and right trefoils!

but even this can't tell all knots apart...

Need new invariants... Khovanov homology...