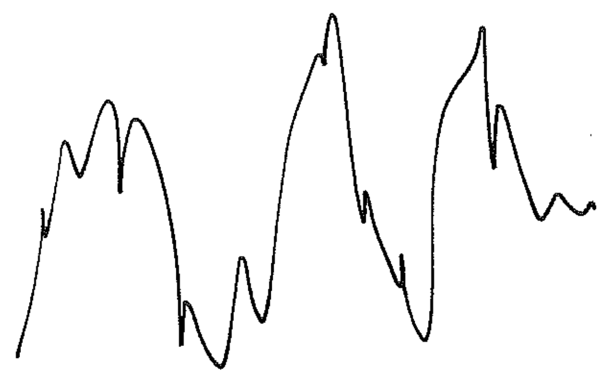
Optimization

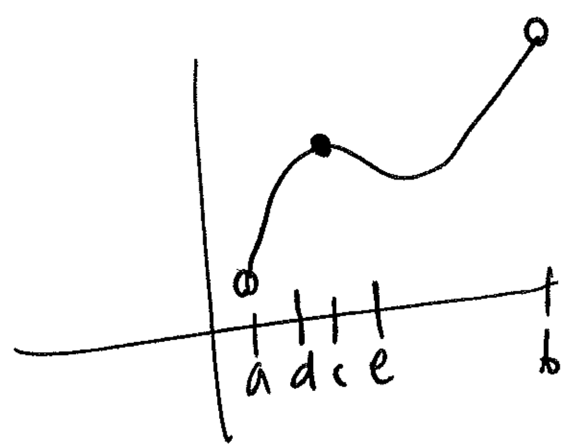
Suppose f is a function defined on an interval (a,b).

$x=c$ is a local maximum of f if:

"decreasing on both sides"



There's a smaller interval (d,e) with $d \le c \le e$ and if $x \in (d,e)$ then $f(x) \le f(c)$.



$x=c$ is a global maximum of f: for any $x \in (a,b)$ $f(x) \le f(c)$.

Theorem

Suppose f is a differentiable function defined on (a,b). If c is a local max/min of f, then $f'(c)=0$.

Pf. We know that $\displaystyle\lim_{x\to c} \underbrace{\frac{f(x)-f(c)}{x-c}}_{} = f'(c).$

Suppose $f'(c)>0$ for contradiction.

This means there is an interval (d,e) containing c so that if $x\in(d,e)$

then

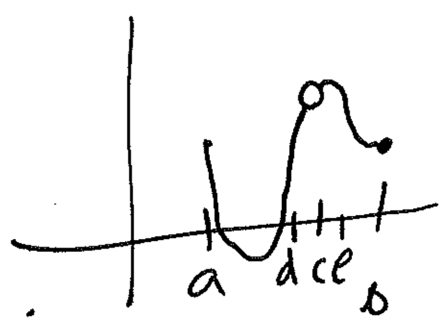
$$\frac{f(x)-f(c)}{x-c}>0.$$

~~I claim~~ if Suppose $x\in(d,c)$

then $x-c<0$ so $f(x)-f(c)<0$.

So $f(x)<f(c)$, so then c not a local min..

draw a picture of this as a function of x



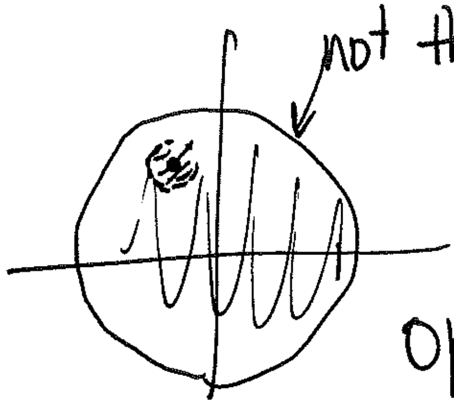
If $x\in(c,e)$ then $f(x)>f(c)$ so c not local max.

A set $S \subset \mathbb{R}^2$ is *open*

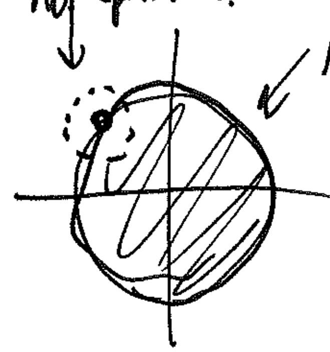
if for ~~any~~ every point $x \in S$, there's a small disk around x that's entirely contained in S.

Ex

$$S = \{(x,y) : x^2 + y^2 < 1\}$$



$$S = \{(x,y) : x^2 + y^2 \leq 1\}$$



S is *closed* if its complement is open.

- for any $p \notin S$, there's a disk around p entirely not in S.

<u>Quiz:</u> Open, closed, both, neither.

a) $\{(x,y): 0 < x < 1\}$

open but not closed

b) $\{(x,y): 0 \le x < 1\}$

neither open nor closed

c) $\{(x,y): x \text{ and } y \text{ are both integers}\}$

closed

g) $\{(x,y): x \text{ and } y \text{ are both rational}\}$

neither

"clopen"; "sets are not doors"

d) $S = \mathbb{R}^2$

both!

e) $S = \phi$.

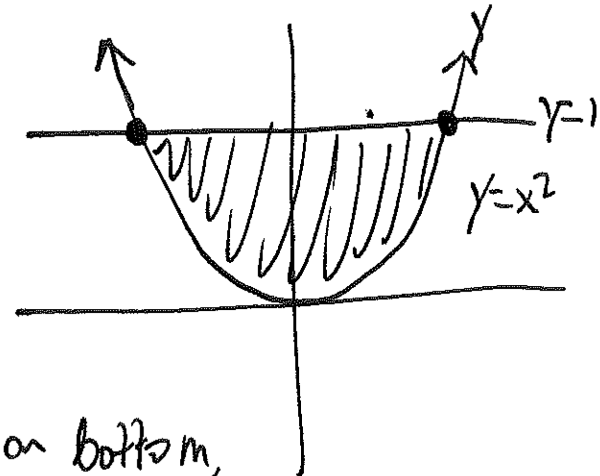
both!

f) All of $\mathbb{R}^2$ except $(0,0)$

open but not closed.

Find global max+min

$$f(x,y)=5x-7y$$



Max/min either in middle, on top, on bottom, or at corner.

Candidates:

(x,y)	$f(x,y)$	
$(1,1)$	-2	
$(-1,+1)$	-12	MIN
$\left(\frac{5}{14},\frac{25}{196}\right)$	$\frac{175}{196}$	MAX

Critical pts:

$f_x=5$ $f_y=-7$, none!

top edge: plug in $y=1$, find x $(-1\leq x\leq 1)$ to max/min $5x-7$.

bottom edge: plug in x^2. max/min $5x-7y=5x-7x^2$ for $-1\leq x\leq 1$.

$$\frac{d}{dx}(5x-7x^2)=5-14x \rightsquigarrow x=\frac{5}{14},\ y=\frac{25}{196}$$

Who cares?

- Suppose $f(x,y)$ has a local max / min at c on an <u>open</u> set S. Then c is a critical pt:

$$\frac{\partial f}{\partial x}(c) = \frac{\partial f}{\partial y}(c) = 0.$$

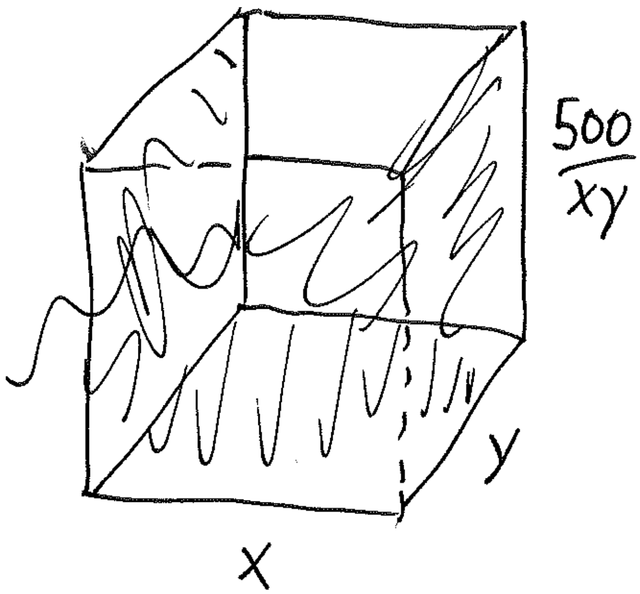
(You already know this from Calc I!
a max/min on non-open set might be at an endpt)

↳ Suppose you want to maximize f on a set T that isn't open! Split T into

$T = S \cup B$ where S is open, B is "boundary".

Find canditate max/min in S using critical points, then worry about B.

Find dimensions of a box with no top such that volume $= 500$ and surface area is minimized.

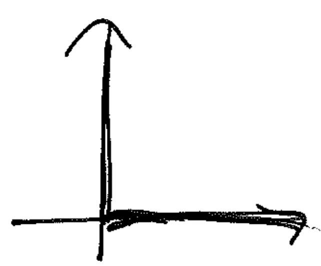


Find x, y to minimize:

$$f(x,y) = xy + \frac{1000}{x} + \frac{1000}{y}$$

constraints:

$x, y \geq 0$



$$f_x = y - \frac{1000}{x^2} \overset{\text{set}}{=} 0, \quad f_y = x - \frac{1000}{y^2} \overset{\text{set}}{=} 0$$

$x=10,\ y=10,\ z=5$

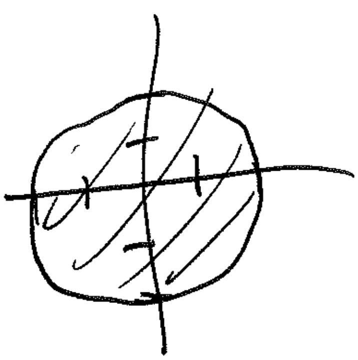
~~Maximize~~

Fact: If T is a set that's closed & bounded, then f has a global maximum on T.
/minimum

Problem:

Minimize $x^2 - 2x + 3y^2 + 2$ on a closed disk of radius 2.

$\{(x,y) : x^2 + y^2 \leq 4\}$



If there's a minimum inside the circle, it will be at a critical pt.

How to check for min on the edge?

Critical pts

$f_x = 2x-2$ $\qquad x=1$

$f_y = 6y$ $\qquad y=0$

(1,0) could be minimum!

Possible minima

(x,y)	$f(x,y)$
$(1,0)$	1 ← global min
$\theta=0$: $(2,0)$	2
$\theta=\pi$: $(-2,0)$	10
$\left(-\frac{1}{2}, \frac{\sqrt{15}}{2}\right)$	~~$\frac{29}{2}$~~ ~~$\frac{29}{4}$~~ $\frac{29}{2}$
$\left(-\frac{1}{2}, -\frac{\sqrt{15}}{2}\right)$	$\frac{29}{2}$

on the edge?

Parametrize it!

$x = 2\cos\theta$

$y = 2\sin\theta$ $\qquad 0 \le \theta \le 2\pi$.

Find the θ that minimizes it:

$$4\cos^2\theta - 4\cos\theta + 12\sin^2\theta + 2$$

$$= 6 - 4\cos\theta + 8\sin^2\theta.$$ ← want to minimize
single var minimization!

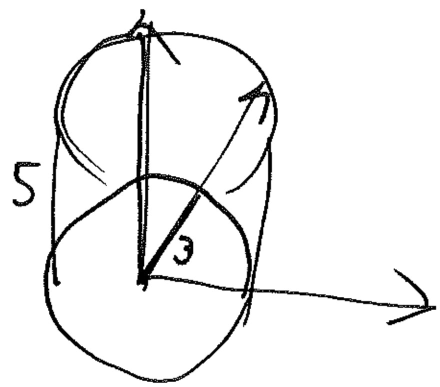
derivative $\frac{d}{d\theta}$ is: $4\sin\theta + 16\sin\theta\cos\theta = 0$

$4(\sin\theta)(1+4\cos\theta) = 0 \qquad \theta = 0, \pi, \cos^{-1}\left(-\frac{1}{4}\right)$ ← all of them let's say ~~so~~ you want to maximize/minimize

$f(x,y,z) = x^2 - y^2 + 2x + z$

on a cylindrical region:

$$\left\{ \begin{array}{l} x^2+y^2 \leq 9 \\ 0 \leq z \leq 5 \end{array} \right\}$$



Candidate pts:

(x,y,z)	$f(x,y,z)$

Regions to search:

- inside
  (find critical pts)

- top
  (parametrize, solve 2-var max/min)

- bottom

- circular rims (top, bottom)