The zeta function

(Hi Tony!)

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

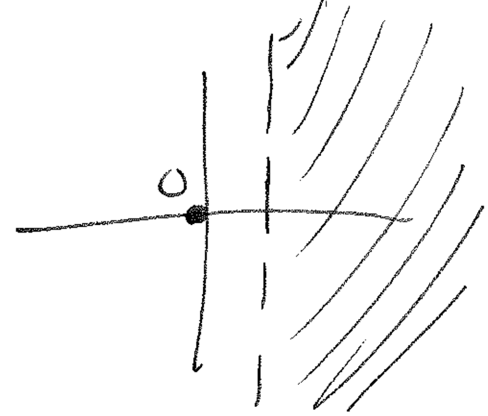
input s

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty \text{ diverges.}$$

It converges for $s > 1$.

In fact, you can plug in a complex number s and it will converge if $Re(s) > 1$.



Holomorphic!

Connected to prime numbers.

Euler product formula

$$\prod_{\substack{p \\ \text{prime}}} \frac{1}{1-p^{-s}} = \left(\frac{1}{1-2^{-s}}\right)\left(\frac{1}{1-3^{-s}}\right)\left(\frac{1}{1-5^{-s}}\right)\left(\frac{1}{1-7^{-s}}\right) \cdots$$

$$= \prod_{p \text{ prime}} \left(\sum_{n=0}^{\infty} \left(\frac{1}{p^s}\right)^n \right)$$

$$= \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \frac{1}{2^{3s}} + \frac{1}{2^{4s}} + \cdots\right)\left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \cdots\right)\left(1 + \frac{1}{5^s} + \frac{1}{5^{2s}} + \cdots\right) \cdots$$

$$\overset{!}{=} \sum_{n=1}^{\infty} \frac{1}{n^s}$$

What if $s<1$?

How to define it?

Recall the Γ function:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt .$$

(we used it to define factorial for non-integers)

Γ satisfies the rule $\Gamma(z+1) = z \cdot \Gamma(z)$ (integrate by parts.

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt$$

e.g $\Gamma(2) = \int_0^\infty e^{-t} \cdot t \, dt$

converges if $z > 1$ (or for complex z, if $\operatorname{Re} z > 1$)

Fact:

$\Gamma(n) = (n+1)!$ or $(n-1)!$

I forget.

$$\Gamma(3/2) = \int_0^\infty e^{-t} t^{1/2}\, dt = \frac{\sqrt{\pi}}{2}$$

$\Gamma(1/2)$ doesn't make sense,

but we can define it anyway.

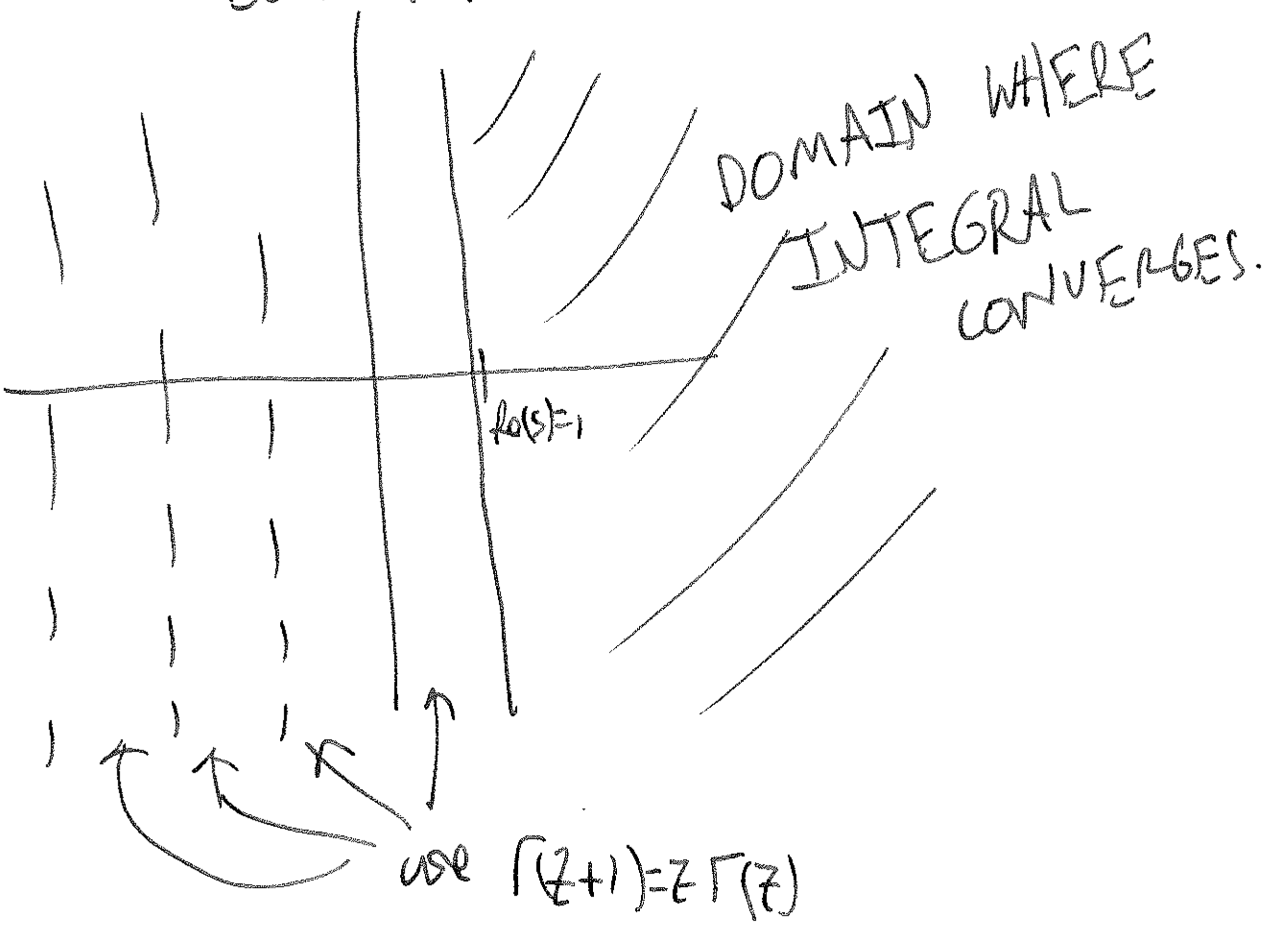
$$\Gamma(1/2) = \int_0^\infty e^{-t} t^{-1/2}\, dt \text{ diverges.}$$

We can't use integral to define it, but we can use the rule $\Gamma(z+1) = z\Gamma(z)$

$$\Gamma(3/2) = 1/2 \cdot \Gamma(1/2)$$

so $\Gamma(1/2) = 2\Gamma(3/2) = \sqrt{\pi}$

Domain of Γ:



this defines it everywhere!

We can do something similar with ζ-function and extend domain even where sum doesn't converge.

It turns out:

$$\zeta(z) = \pi^{(z-\frac{1}{2})} \frac{\Gamma\left(\frac{1-z}{2}\right)}{\Gamma\left(\frac{z}{2}\right)} \zeta(1-z)$$

Problem: Where is this 0?

tells us value even for $z<0$!