Dynamical and Arithmetic Degrees

John Lesieutre

January 21, 2020

Rational maps

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$$f(x, y) = \left(\frac{p_1(x, y)}{q_1(x, y)}, \dots, \frac{p_n(x, y)}{q_n(x, y)} \right).$$

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- What happens when we iterate? How dynamically complex is f?

An example

- Formulas for the iterates

$$\begin{aligned}(x, y) &\mapsto (y, xy) \mapsto (xy, xy^2) \mapsto (xy^2, x^2y^3)\\ &\mapsto (x^2y^3, x^3y^5) \mapsto (x^3y^5, x^5y^8) \mapsto \cdots\end{aligned}$$

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- You can guess the pattern: $(x^{F_n}y^{F_{n+1}}, x^{F_{n+1}}y^{F_{n+2}})$.

Degree of a map

- Assume that all denominators are equal, and terms are coprime.

$$f(x, y) = \left(\frac{p_1(x, y)}{q(x, y)}, \ldots, \frac{p_n(x, y)}{q(x, y)} \right).$$

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- (This is what the equation looks like on projective space)

- The degree of f is the maximum of the degree of q and any of the p_i.

Degree sequences

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- E.g.: Fibonacci numbers
- How about $(x, y) \mapsto (x, xy)$?
- $(x, y) \mapsto (x, xy) \mapsto (x, x^2y) \mapsto (x, x^3y) \mapsto (x, x^4y)$
- Here $\deg(f^{\circ n}) = n$.

Degree sequences

- One more: $(x, y) \mapsto \left(\frac{1}{x}, \frac{1}{y}\right) = \left(\frac{y}{xy}, \frac{x}{xy}\right)$.

Degree sequences

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- $f^{\circ 2} = \mathrm{id}$
- $\mathrm{DS}(f) = (2, 1, 2, 1, 2, 1, \ldots)$.

Dynamical degrees

- Define $\lambda_1(f) = \lim_{n\to\infty} \deg(f^{\circ n})^{1/n}$.
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- Theorem (Dinh–Sibony): The limit exists.
- In our examples: $\frac{1+\sqrt{5}}{2}$, 1, 1.

Theorems and questions

- Old Q: Is $\lambda_1(f)$ always an algebraic number?
- T (Hasselblatt–Propp): DS(f) does not always satisfy a linear recurrence relation.
- T (Bell–Diller-Jonsson 2019): It can be transcendental!

Theorems and questions

- Q: If $\mathrm{DS}(f)$ is not bounded, does it grow at least linearly?
- T (Cantat–Xie '18): Bounded subsequence implies bounded.

Theorems and questions

- Q: Are there constants so:

$$C_1 n^a (\lambda_1(f))^n < \mathrm{DS}(f) < C_2 n^a (\lambda_1(f))^n$$

- Q: If $\mathrm{DS}(f)$ is roughly linear, does f preserve a meromorphic fibration?

The arithmetic side

- The height of a point $P = \left(\frac{X_1}{Y}, \ldots, \frac{X_n}{Y}\right) \in \mathbb{Q}^n$ is $h(P) = \log \max(|X_i|, |Y|)$.
- How does $h(f^n(P))$ grow if pick a P and iterate a rational map?
- Experiments: usually it's exponential.

An example

- Example: $(x, y) \mapsto (y, xy)$.
- If $P = (3, 5)$, then $f^n(P) = (3^{F_n} 5^{F_{n+1}}, 3^{F_{n+1}} 5^{F_{n+2}})$.
- $h(f^n(P)) \approx C \cdot F_n$

Arithmetic degree

- Set $\alpha_f(P) = \lim_{n\to\infty} h(f^n(P))^{1/n}$
- In our example, it's $\frac{1+\sqrt{5}}{2}$.
- Coincidence?

Arithmetic degree

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- In our example, it's $\frac{1+\sqrt{5}}{2}$.
- Coincidence?
- Careful: what if $P = (1, 1)$?

Kawaguchi–Silverman conjecture

- Conj: Suppose $f : X \dashrightarrow X$ is a rational map and P is a $\overline{\mathbb{Q}}$-point of X. Then the limit defining $\alpha_f(P)$ exists, and $\{\alpha_f(P)\}$ is a finite set of algebraic numbers.
- Moreover, if the orbit of P is Zariski dense, then $\alpha_f(P) = \lambda_1(f)$.

Oops

- [Lesieutre-Satriano, '18] A counterexample to the finiteness claim is given by the four-dimensional quadratic map

$$(W, X, Y, Z) \mapsto \left(W, Y + W, \frac{YZ}{X} + 1, Z\right)$$

- The "algebraic numbers" part isn't looking too good either.
- But the other parts are open!

Height growth

- Consider the map $x \mapsto x + 1$.
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- Consider the map $x \mapsto x + 1$.
- Then $h(f^n(P)) \sim \log n$.
- Conjecture: for any f,

$$h(f^n(P)) \sim n^a (\log n)^b \lambda^n$$

Height growth of cycles

- What if we have a map $\mathbb{A}^2 \to \mathbb{A}^2$ and we ask about height growth of a curve?
- e.g. take the orbit of the x-axis: how does the defining equation grow?
- New conjecture in [DGHLS '20], but we know hardly any cases!

Need more geometry!

- The case of $\mathbb{C}^n$ might be the hardest – from the point of view of algebraic geometry, there are no geometric features to work with.
- A general feature of arithmetic geometry: the geometry of a variety controls the set of rational points (e.g. Faltings' theorem)

Need more geometry!

- [KS] Conjecture true for abelian varieties.
- [MSS] True for surfaces.
- [LS] True for X a hyper-Kähler variety (simply connected, and $h^0(X, \Omega^2)$ is spanned by a holomorphic symplectic form).
- ...

Varieties and their automorphisms

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- $\phi_1(x, y) = (x, -y)$ (a reflection)
- $\phi_2(x, y) = \left(\frac{4}{5}x - \frac{3}{5}y, \frac{3}{5}x + \frac{4}{5}y\right)$ (a rotation)

Varieties and their automorphisms

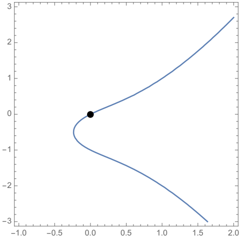
- Projective variety: zero locus of a list of polynomials in n variables, plus points at infinity.
- Automorphism of variety: invertible set of rational functions sending zero locus to itself.

Example: $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$

- Automorphisms: $z \mapsto \frac{az+b}{cz+d}$ $(ad - bc \neq 0)$
- This is the group of Möbius transformations, $\mathrm{PGL}_2(\mathbb{C})$
- Watch out: $\mathrm{Diff}(\mathbb{CP}^1)$ is much less rigid

Application: Rational solutions

- An elliptic curve: $y^2 + y = x^3 + x$.



- Translation by P in the group law gives an automorphism.

Application: Rational solutions

- The equation for this map is:

$$\phi(x, y) = \left(\frac{y^2 - x^3}{x^2}, \frac{x^3 y - x^3 - y^3}{x^3} \right).$$

- Complex picture: translation on a torus

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- $\phi(3, 5) = (-{}^{2}/_{9}, -{}^{17}/_{27})$
- $\phi^2(3, 5) = ({}^{33}/_{4}, -{}^{195}/_{8})$
- The points $\phi^n(3, 5)$ give infinitely many solutions.

Application: Complex dynamics

- We can also ask questions about the dynamics of automorphisms: periodic points, measures of maximal entropy, etc.

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- The topological *entropy* of a map $\phi : X \to X$ is a measure of the complexity of the dynamics of the map.
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- We can also ask questions about the dynamics of automorphisms: periodic points, measures of maximal entropy, etc.
- The topological *entropy* of a map $\phi : X \to X$ is a measure of the complexity of the dynamics of the map.
- Roughly, it measures the tendency of iterates of the map to separate general points.
- Gromov–Yomdin theorem: can compute entropy from algebraic data.

Example: a rational surface

- Draw six general lines in the plane $\mathbb{R}^2$.
- Mark them into three pairs: red, green and blue.
- Define an involution on the pencil of lines through the intersection of the two red lines.
- (Use Desargues' involution theorem)

Since a projectivity is fixed by three elements of one structure and the homologous elements of the other, an involution is determined by two pairs A, A' and B, B' of conjugate elements insofar as the elements A, A', B of the one structure correspond to the elements A', A, B' of the other.

Construction of an involution, i.e., construction of an element P' corresponding to an arbitrary element P, is most effectively accomplished by means of Desargues' involution theorem (where conic sections do not enter into the picture). Let us say, for example, that we are concerned with the involution of two ranges of points. Let (A, A') and (B, B') be the given point pairs of the involution, C an additional given point of the base $\mathfrak{T}$, and C' the homolog of C we are looking for. We draw through A, B, C three lines that form a triangle 1 2 3 (A on 23, B on 31, C on 12), connect A' with 1, B' with 2, and the point of intersection 4 of these connecting lines with 3. Then 34 touches the base at C'. (The opposite side pairs 23 and 14, 31 and 24, 12 and 34 of the tetragon 1 2 3 4 cut $\mathfrak{T}$ at the point pairs (A, A'), (B, B'), and (C, C') of the Desargues involution.) The

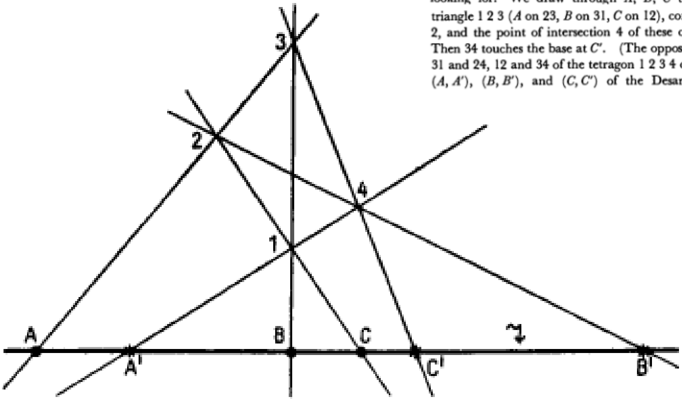

FIG. 72.

Example: the Bertini involution

- Let $S \subset \mathbb{P}^3$ be a cubic hypersurface, and fix a line L.
- Given a point z, form the plane $P = P_{Lz}$.
- Look at the curve $S \cap P$; use that to define the involution.
- $P \cap S$ is a plane containing an elliptic curve.

NOTES ON THE BERTINI INVOLUTION

ETHEL I. MOODY[1]

1. **Introduction.** Given a pencil of plane cubic curves

$$\lambda w(x) + \mu w'(x) = 0 \tag{1}$$

with the vertices of the reference triangle among its base points. Arranged as to (0, 0, 1) the equations may be written

$$w(x) = x_3^2 u_1 + x_3 u_2 + u_3,$$
$$w'(x) = x_3^2 u_1' + x_3 u_2' + u_3',$$

with

$$u_1 = a_1 x_1 + a_2 x_2, \qquad u_1' = a_1' x_1 + a_2' x_2,$$
$$u_2 = b_1 x_1^2 + b_2 x_1 x_2 + b_3 x_2^2, \quad u_2',$$
$$u_3 = c_1 x_1^2 x_2 + c_2 x_1 x_2^2, \quad u_3',$$

and $a_i, a_i', b_i, b_i', c_i, c_i'$ generic constants.

A point y of the plane fixes the curve of the pencil (1) passing through it, hence

$$w(x)w'(y) - w'(x)w(y) = 0, \tag{2}$$

which may be written in the form

$$\begin{aligned} W_3(x) = x_3(A_1 x_1 + A_2 x_2) + x_3(B_1 x_1^2 + B_2 x_1 x_2 + B_3 x_2^2) \\ + C_1 x_1^2 x_2 + C_2 x_1 x_2^2 = 0 \end{aligned} \tag{3}$$

in which $A_i = a_i w'(y) - a_i' w(y)$, and similarly for B_i and C_i. The tangent to $W_3(x) = 0$ at (0, 0, 1) is

$$A_1 x_1 + A_2 x_2 = 0, \tag{4}$$

which meets the curve again at $R = (r_1, r_2, r_3)$,

Received by the editors August 31, 1942.

[1] Miss Moody, Ph.D. Cornell University, an instructor in mathematics at Pennsylvania State College, was killed in an automobile accident April 11, 1941. I had suggested that she compare my cumbersome method of derivation of the equations of this transformation (Amer. J. Math. vol. 33 (1911) pp. 327–336) with that of employing a pencil of cubic curves. The following notes were found among her posthumous papers sent me recently. The equations of the Bertini involution are simpler than those previously known, and other properties found may be extended by others.

VIRGIL SNYDER

THE BERTINI INVOLUTION

ALEX DEGTYAREV

Abstract. We summarize and extend E. Moody's results on the explicit equations related to the Bertini involution.

These notes are the result of my attempt to understand E. Moody's paper [1]. I correct a few misprints in [1] and take the computation a bit further.

I express my admiration to Ethel I. Moody, who managed to perform this tedious computation in the pre-`Maple` era. A `Maple` implementation of most equations is found at `http://www.fen.bilkent.edu.tr/~degt/papers/Bertini.zip`.

This text is not intended as an 'official' publication; it is distributed in the hope that it may be useful. It can be cited by its `arXiv` location.

Whenever possible, I try to keep the original notation of [1].

1. The Bertini involution

1.1. The results of [1]. Consider the pencil of cubics

$$\lambda w(x) + \mu w'(x) = 0, \tag{1.1}$$

where

$$w(x) = x_3^2(a_1x_1 + a_2x_2) + x_3(b_1x_1^2 + b_2x_1x_2 + b_3x_2^2) + (c_1x_1^2x_2 + c_2x_1x_2^2)$$

and similar for w', so that the coordinate vertices are amongst the basepoints of the pencil. The point $(0 : 0 : 1)$ will play a special rôle.

The curve of the pencil passing through a point y is given by

$$W_3(x) := w(x)w'(y) - w'(x)w(y) = 0. \tag{1.2}$$

Example: a K3 surface

- X a $(2,2,2)$ hypersurface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
- Example: $3x^2y^2z^2 + xz^2 + 13xyz + x^2 + y^2 + 1 = 0.$

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- Example: $3x^2y^2z^2 + xz^2 + 13xyz + x^2 + y^2 + 1 = 0$.
- If we fix x and y, there are two solutions z.
- $x = 1$, $y = 1$, then $3z^2 + z^2 + 13z + 3 = 0$.
- Then $z = -3$, $z = -1/4$.

Example: a K3 surface

- $3x^2y^2z^2 + xz^2 + 13xyz + x^2 + y^2 + 1 = 0$
- Involution $\phi_z : X \to X$: send (x, y, z) to (x, y, z').

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- In our example, $\phi_z(1, 1, -3) = (1, 1, -1/4)$.

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- In our example, $\phi_z(1, 1, -3) = (1, 1, -1/4)$.
- Similar maps ϕ_x, ϕ_y
- These generate $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \subset \mathrm{Aut}(X)$.
- The composition of all three has positive entropy!

Classification?

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- Which algebraic varieties can have positive entropy automorphisms?
- No one-dimensional examples.
- Fairly precise classification in dimension 2 by Cantat.
- In higher dimensions, we don't know!
- Very few examples, but also few constraints.
- Thm (L–): If X is smooth and dimension 3, then any positive entropy automorphism has an invariant divisor (excluding some special cases).

Applications: algebraic geometry

- Great source of examples in algebraic geometry.
- Most varieties are either too simple, or hard to compute with.
- Varieties with automorphisms strike a balance.

Fourier–Mukai partners

- The derived category $\mathrm{D}^b\mathrm{Coh}(X)$ is an invariant of X.
- Two varieties can be different, but nevertheless have the same derived category!
- Conjecture (Kawamata '06): A variety has only finitely many Fourier–Mukai partners.
- Counterexample (L–): X the blow-up of $\mathbb{P}^3$ at 8 points.

Fourier–Mukai partners

- $\mathcal{P}$ = parameter space for 8-tuples of points
- Find $\phi : \mathcal{P} \dashrightarrow \mathcal{P}$ that doesn't change $\mathrm{D}^b\mathrm{Coh}(X)$
- Show that ϕ has positive entropy

Automorphism group

- We saw the groups $\mathrm{PGL}_2(\mathbb{C})$, $(\mathbb{Z}/2\mathbb{Z})^{*3}$, $\mathrm{SL}_2(\mathbb{Z})$.

Automorphism group

- We saw the groups $\mathrm{PGL}_2(\mathbb{C})$, $(\mathbb{Z}/2\mathbb{Z})^{*3}$, $\mathrm{SL}_2(\mathbb{Z})$.
- In general $\mathrm{Aut}(X)$ has a continuous part (finite dimensional, arising from holomorphic vector fields) and a discrete part.
- Look at $\pi_0(\mathrm{Aut}(X)) = \mathrm{Aut}(X)/\mathrm{Aut}^0(X)$, a countable group.
- What groups can appear?

The discrete part of $\mathrm{Aut}(X)$

- Question (Mazur '92): Is $\pi_0(\mathrm{Aut}(X))$ always finitely generated?
- Theorem (L, '18): There exists a variety with non-finitely generated automorphism group.
- To get it: start with a simple variety, make a sequence of geometric constructions that change the group in predictable ways.
- In the end, it's a very complicated variety!
- Theorem (Dinh–Oguiso '19) There exists a surface with non-finitely generated automorphism group.

Application: Twists

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- Sure: $(x, y) \mapsto (ix, iy)$.

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- Are the varieties $x^2 + y^2 = 1$ and $x^2 + y^2 = -1$ isomorphic?
- (This means there's a bijection between the solution sets)
- Sure: $(x, y) \mapsto (ix, iy)$.
- That's cheating! Our equations only had real numbers.
- These two varieties are called *twists*: they are isomorphic as varieties over $\mathbb{C}$, but not as varieties over $\mathbb{R}$.

Application: Twists

- K/k-twists $\leftrightarrow H^1(\mathrm{Gal}(K/k), \mathrm{Aut}\, X_K)$.
- Theorem: A Riemann surface has only finitely many real forms.
- Theorem (Cartan, 1890): A connected, semisimple compact Lie group X has only finitely many real forms.
- Theorem (Borel–Serre, '64): Only finitely many real forms if $\mathrm{Aut}(X_K)$ is an arithmetic group.
- Theorem (Degtyarev–Itenberg–Kharlamov, '00): Only finitely many real forms if X is a minimal surface.

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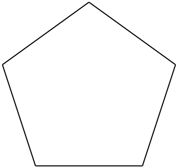
- Upshot: to get infinitely many twists, $\mathrm{Aut}(X_K)$ has to be really bad.
- Theorem (L–): There exists a projective variety with infinitely many real forms.
- Need to construct a variety for which $\mathrm{Aut}(X_K)$ contains infinitely many conjugacy classes of involutions.

Cutting down Aut(X)

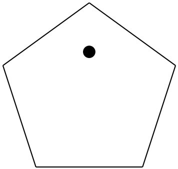
- Suppose we have a figure in the plane:



- $\mathrm{Aut}(R) = D_5$

Cutting down Aut(X)

- Now mark the figure:



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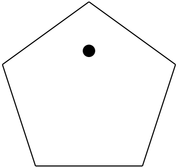

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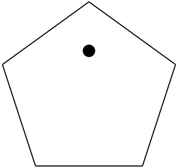

- $\mathrm{Aut}(R') = \mathbb{Z}/2\mathbb{Z}$
- $\mathrm{Aut}(R') = \mathrm{Stab}(\bullet) \subset \mathrm{Aut}(R)$

Varieties

- Start with a variety with $\mathrm{Aut}(X) = \mathbb{Z} * \mathbb{Z} = \langle f, g \rangle$
- Subgroup of a finitely generated group might not be finitely generated! Classic example: $\langle f^{-n} g f^{n} \rangle$.

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- Find a subvariety V whose stabilizer is not finitely generated.
- To "mark" V: blow it up!

Step 1.

- Find a 2-dimensional variety X and a curve $\mathbb{P}^1 \subset X$ fixed by $\mathrm{Aut}(X)$.

- This means we can restrict automorphisms: $\mathrm{Aut}(X) \to \mathrm{Aut}(\mathbb{P}^1)$.

- The first group is extremely complicated, but the second we understand.

- Arrange that there are automorphisms $g : X \to X$ and $f : X \to X$ such that $g|_{\mathbb{P}^1} = (z \mapsto z + 1)$ and $f|_{\mathbb{P}^1} = (z \mapsto 3z)$.

Step 2.

- Let $G \subset \mathrm{Aut}(X)$ be the set of automorphisms that restrict to $\mathbb{P}^1$ as $z \mapsto z + c$.
- g is in G, but f isn't.
- As is

$$f^{-n} \circ g \circ f^n = (z \mapsto 3^n z \mapsto 3^n z + 1 \mapsto z + \frac{1}{3^n}).$$

- So G is (more or less) the non-finitely generated subgroup we mentioned earlier!

Aside: How to come up with automorphisms?

- It's generally extremely hard to find automorphisms.
- It's also extremely hard to prove there are none.

Aside: How to come up with automorphisms?

- It's generally extremely hard to find automorphisms.
- It's also extremely hard to prove there are none.
- Theorem (Poonen): Given X, $V \subset X$, $x \in X$, it is undecidable whether there is an automorphism sending x into V.

Step 2.

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- So the image isn't finitely generated.
- A quotient of a finitely generated group is finitely generated.
- So G isn't finitely generated.

Step 3.

- Unfortunately, G isn't the stabilizer of anything.
- But it is very close: it's automorphisms that fix the point at ∞ and have derivative 1.
- Close enough (but we need some auxiliary constructions).

Smooth analog

- If M is a smooth manifold, the mapping class group is $\pi_0(\mathrm{Diff}(M))$.
- Not finitely generated in general: tori in dim ≥ 5
- Sullivan: in dim ≥ 5, if $\pi_1(M) = 0$ then $\pi_0(\mathrm{Diff}(M))$ is finitely generated.

Infinitely many twists

- $H^1(\mathsf{Gal}(\mathbb{C}/\mathbb{R}), \mathsf{Aut}\, X_{\mathbb{C}})$ is the set of conjugacy classes of involutions (assuming trivial Galois action).
- To find a variety with infinitely many twists, we need a variety with infinitely many non-conjugate real involutions.

Infinitely many twists

- The idea: our X has a *third* special map: an involution h that restricts to $\mathbb{P}^1$ as $z \mapsto 3 - z$.
- Find a variety whose automorphisms are $G' = \{\phi : \phi|_{\mathbb{P}^1} = z \mapsto \pm z + c\}$.
- Infinitely many conjugates of h under the full group $\langle f, g\rangle$ are in G'
- But the conjugating elements are not! So when we pass to a variety with automorphism group G', there are infinitely many conjugacy classes of involutions.

Dynamical Mordell–Lang conjecture

- Suppose that $\phi : X \to X$ is an endomorphism of a variety, $x \in X$ is a point, and $V \subset X$ is a subvariety.
- Then $\{n : \phi^n(x) \in V\}$ is a union of a finite set and finitely many arithmetic progressions.
- Example: $\mathbb{P}^n \approx$ the Skolem–Mahler–Lech theorem.
- Theorem (Bell–Ghioca–Tucker): OK if ϕ étale.
- Theorem (L–Litt): If ϕ is étale, this is true for x and V arbitrary closed subschemes.

Another application

- Conjecture: if X is terminal, then only finitely many K_X-negative extremal rational curves.
- False! (L–) Find X with one such curve, and an automorphism $\phi : X \to X$ positive entropy.
- Orbit of the one gives infinitely many.

Future directions

- Classification of varieties with positive entropy automorphisms (minimal model program+dynamics)
- More counterexamples! (termination of flips?)
- Arithmetic problems (dynamical Mordell–Lang, ...)