DYNAMICAL MORDELL–LANG AND AUTOMORPHISMS OF BLOW-UPS

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Abstract. Suppose that $\phi : X \to X$ is an étale endomorphism of a smooth projective variety X over a field k of characteristic zero. We show that if Y and Z are two closed subschemes of X, then the set $A_\phi(Y, Z) = \{n : \phi^n(Y) \subseteq Z\}$ is a finite union of arithmetic progressions, and we examine the dependence of the lengths of these progressions on the subscheme Y.

We apply this result to show that if $\phi : X \to X$ is an automorphism of a smooth variety and $D \subset X$ is a divisor for which the set $\{d \in D : \phi^n(d) \in D$ for some nonzero $n\}$ is not Zariski dense, then ϕ admits an equivariant rational fibration to a curve. As further applications, we extend results of Bayraktar–Cantat on automorphism groups of varieties constructed by blow-ups, and results of Arnold on the growth of multiplicities of the intersection of a variety with the iterates of some other variety under an automorphism.

1. Introduction

We have two goals in this note: first, to give an extension of the dynamical Mordell–Lang conjecture to the orbits of non-reduced closed subschemes of a variety; and second, to give some geometric applications of this generalization to questions about automorphisms of higher-dimensional varieties. The usual form of the conjecture is the following.

Conjecture 1 (Dynamical Mordell–Lang)**.** *Suppose that $\phi : X \to X$ is endomorphism of a quasiprojective variety X, with $V \subset X$ is a subvariety and $p \in X$ a point. Then the set*

$$A_\phi(p, V) = \{n : \phi^n(p) \in V\}$$

is a union of a finite set and a finite number of arithmetic progressions.

When ϕ is étale, the conjecture is a theorem of Bell–Ghioca–Tucker [4]. Our first result extends this result in the étale case to the setting in which p and V are replaced by arbitrary closed subschemes of X.

Theorem 1.1. *Suppose that X is a smooth projective variety over a field of characteristic zero, and that $\phi : X \to X$ is an étale endomorphism of X.*

1. *Let Y and Z be closed subschemes of X. Then the set $A_\phi(Y, Z) = \{n : \phi^n(Y) \subseteq Z\}$ is a union of a finite set and a finite number of arithmetic progressions.*
2. *Suppose that $Y_1^\circ, \dots, Y_r^\circ$ is a finite set of closed subschemes of X, defined by ideal sheaves $\mathscr{I}_{Y_i^\circ}$, and that $\{Y_j\}$ is the infinite collection of subschemes defined by the ideal sheaves*

$$\sum_{i=1}^{r} \mathscr{I}_{Y_i^\circ}^{n_i},$$

$n_i \geq 0$. Then there exists a constant $N = N(\{Y_i^\circ\}, Z)$ such that the lengths of the arithmetic progressions appearing in $A_\phi(Y_j, Z)$ are bounded by N (and in particular are independent of j).

Part (1) of the theorem was proved by Bell–Lagarias for affine X [5]. We are primarily motivated by geometric applications of part (2), which does not have an analog in the setting where Y and Z are reduced. The uniform bound on the period makes it possible to control the multiplicities of the intersection of a subvariety V with the iterates $\phi^n(V)$ under an automorphism. We use this control to prove:

Theorem 1.2. *Let X be a smooth projective variety over $\mathbb{C}$ and $\phi : X \to X$ an automorphism. Suppose that $D \subset X$ is a divisor, containing a codimension-2 closed subset V with $\phi(V) = V$. Then there exists a birational morphism $\pi : Y \to X$ such that after replacing ϕ by a suitable iterate:*

1. *Y is smooth;*
2. *some iterate of ϕ lifts to an automorphism $\psi : Y \to Y$;*
3. *$\pi(\psi^m(\tilde{D}) \cap \psi^n(\tilde{D}))$ does not contain V for any $m \neq n$.*

(Here $\tilde{D}$ is the strict tranform of D on Y.)

Repeated applications of this local separation result make it possible to blow up X several times and simultaneously make the strict transforms of all of the divisors $\phi^n(D)$ disjoint. Global geometric considerations then yield:

Theorem 1.3. *Let X be a smooth variety over $\mathbb{C}$ and $\phi : X \to X$ an automorphism. Suppose that $D \subset X$ is a divisor which is not periodic under ϕ and for which the set*

$$V(D) = \{d : \phi^n(d) \in D \textit{ for some nonzero } n\} = \bigcup_{n \neq 0} D \cap \phi^n(D)$$

is Zariski closed. Then there exists a birational morphism $\pi : Y \to X$ such that

1. *Y is smooth;*
2. *ϕ lifts to an automorphism $\psi : Y \to Y$;*
3. *the divisors $\psi^n(\tilde{D})$ are pairwise disjoint;*
4. *there exists a curve C, a morphism $f : Y \to C$, and an automorphism $\tau : C \to C$ such that $f \circ \psi = \tau \circ f$.*

Moreover, the normal bundle of D satisfies $H^0(D, N_{D/X}) > 0$.

Theorem 1.3 can then be used to address concrete questions about automorphisms of varieties constructed using sequences of blow-ups.

Theorem 1.4. *Suppose that X is a smooth projective variety of dimension n, and that Y is a variety constructed by a sequence of smooth blow-ups of X at centers of dimension $\leq r$. If either*

1. *$2r + 3 \leq n$, or*
2. *$n = 4$ and $r = 1$*

there exists $N \in \mathbb{Z}_{\geq 1}$ such that for any automorphism ϕ of Y, ϕ^N descends to X.

This was proved in case (1) by Bayraktar and Cantat under the additional hypothesis that $\rho(X) = 1$ [3]. Here we present a slightly different proof, which does not require this hypothesis. Although the proof of (1) is independent of our main results, the handling of case (2) relies crucially on the arithmetic techniques of the preceding sections.

The statement is false when $r = n - 2$ for every value of n (see example 2.4 below), but the situation for $\lfloor \frac{n-1}{2} \rfloor \leq r \leq n - 3$ remains unclear. It seems natural to ask:

Question 1. *Does there exist a variety X with $\mathrm{Aut}(X)$ a finite group, and a blow-up $\pi : Y \to X$ along a smooth center of codimension at least 3 such that $\mathrm{Aut}(Y)$ is infinite?*

Corollary 6.4 shows that the answer is negative for $n \leq 4$. Truong has obtained a variety of results in this direction, under some additional hypotheses on the nef cone of Y (e.g. [10, Lemma 6]). In the case $n = 3$ and $r = 1$, infinite order automorphisms of Y may exist, but it is possible to give fairly precise control on these automorphisms using explicit methods from threefold geometry and some classification theorems from the minimal model program [8].

The next section contains some geometric motivation for extending the conjecture to non-reduced schemes, and provides some simple examples illustrating the relations between the main results. In §3, we discuss and prove Theorem 1.1, the dynamical Mordell–Lang statement for non-reduced schemes. In §4, we prove Theorems 1.2 and 1.3. In §6, we address concrete questions about automorphisms of blow-ups of higher-dimensional varieties. At last, in §7 we explain how the results of the paper can be used to control the growth of intersection multiplicities of properly intersecting subvarieties under automorphisms, and show that the results of Arnold in [2] can be interpreted as an instance of dynamical Mordell–Lang in the non-reduced setting.

2. Some examples

Before giving the proof of Theorem 1.1, we give an indication of its geometric content in the case that Y is a non-reduced subscheme. For simplicity, suppose that X is a smooth algebraic surface, and that Y and Z are two smooth curves on X which intersect at x, a fixed point of an automorphism $\phi : X \to X$.

Let $Y^{(1)}$ be the first-order germ of Y at x, isomorphic to $\mathrm{Spec}\,\mathbb{C}[\epsilon]/(\epsilon^2)$. We have $\phi^n(Y^{(1)}) \subseteq Z$ exactly when $\phi^n(Y)$ is tangent to Z at the fixed point x.

Theorem 1.1 applied to the scheme $Y^{(1)}$ then asserts that

$$A_\phi(Y^{(1)}, Z) = \{n : \phi^n(Y) \text{ is tangent to } Z \text{ at } x\}$$

is a union of a finite set and finitely many arithmetic progressions.

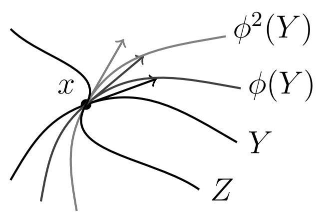

FIGURE 1. $\phi^n(Y)$ is tangent to Z at x when $\phi^n(Y^{(1)}) \subset Z$.

More generally, take $Y_1^\circ = x$ and $Y_2^\circ = C$, both with the reduced induced scheme structures. Then the scheme $Y^{(1)}$ is defined by the ideal sheaf

$$\mathscr{I}_{Y_1^\circ}^2 + \mathscr{I}_{Y_2^\circ},$$

which is in the collection $\{Y_j\}$ appearing in part (2) of the theorem. The same argument can be applied to higher-order germs $Y^{(k)}$, of Y at x, defined by the ideal sheaves

$$\mathscr{I}_{Y_1^\circ}^{k+1} + \mathscr{I}_{Y_2^\circ},$$

corresponding to a kth order germ of the curve C at x.

The theorem then yields that

$$A_\phi(Y^{(k)}, Z) = \{n : \phi^n(Y) \text{ is tangent to } Z \text{ at } x \text{ to order } \geq k\}$$

is a semilinear set, and there is a bound periods of the arithmetic progressions occurring in these sets which is independent of k.

Now consider Theorem 1.2 in this setting: the claim is that it is possible to blow up above the point x a finite number of times to reach a model X' such that ϕ lifts to an automorphism of X', and the curves $\phi^n(Y)$ simultaneously become disjoint above x. There are two obstacles to constructing the requisite blow-ups:

(1) Suppose, for example, that C in tangent to $\phi^n(C)$ at x precisely when n is a perfect square. After blowing up the point x, the curves $C, \phi(C), \phi^4(C), \ldots$ all meet at a single point x' on the exceptional divisor, while the other $\phi^n(C)$ do not. The point x' can not be fixed by $\psi : \mathrm{Bl}_x X \to \mathrm{Bl}_x X$. It is not possible to separate the curves passing through x', because the automorphism will not lift to the blow-up at x'. In general, x' might not be periodic under the induced map on $\mathrm{Bl}_x X$.

This issue can be avoided only if C is tangent to $\phi^n(C)$ for either all n or for no n. More generally, since the statement of the theorem allows us to replace ϕ by a positive iterate, it is necessary that the set of n for which C is tangent to $\phi^n(C)$ is an arithmetic progression. Similar considerations on higher blow-ups suggest that it is also necessary that the set of n for which $\phi^n(C)$ is tangent to C to order k is an arithmetic progression, for any value of k.

(2) Suppose that C is tangent to $\phi^{2^k n}(C)$ to order k when k is odd. This is consistent with the requirement of (1) that the set of iterates tangent to a given order is an arithmetic progression. However, no finite sequence of blow-ups can separate all the curves, because the order of tangency between C and $\phi^{2^k}(C)$ grows without bound.

To overcome these two obstacles, we must show that: (1) C is tangent to $\phi^n(C)$ to order k for all n in some arithmetic progression; (2) the bound on the length of the progression is independent of k. This is precisely the content of Theorem 1.1 where we take the schemes Y_i to be k^{th} order neighborhoods $C^{(k)}$ of x in C and $Z = C$.

The next two examples give automorphisms to which Theorem 1.3 is applicable, in which conclusions of the theorem can easily be seen.

Example 2.1. Let $X = \mathbb{P}^2$, and let $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ be an automorphism with three distinct fixed points. Let D be a general line through a fixed point x. Then D has infinite order under ϕ, but $V_D = \bigcup_{n \neq 0} D \cap \phi^n(D) = \{x\}$ is not Zariski dense.

This is consistent with the last part of Theorem 1.3: take $\pi : Y \to X$ to be the blow-up at x. Then ϕ lifts to an automorphism ψ of Y, and the divisors $\psi^n(\tilde{D})$ are pairwise disjoint. The pencil of lines through x induces a map $f : Y \to \mathbb{P}^1$, and there is an automorphism of $\tau : \mathbb{P}^1 \to \mathbb{P}^1$ with $f \circ \psi = \tau \circ f$. Note that $H^0(D, N_{D/X}) = H^0(D, \mathscr{O}_D(1)) > 0$ as required.

The next example shows that although $N_{D/X}$ must be effective, it need not satisfy any stronger sort of positivity, e.g. nefness.

Example 2.2. Let $\psi : \mathbb{P}^3 \to \mathbb{P}^3$ be a linear automorphism with 4 distinct fixed points p_i. Let $\pi : X \to \mathbb{P}^3$ be the blow-up of $\mathbb{P}^3$ at the points p_1 and p_2, so that ψ induces an automorphism $\phi : X \to X$. Let D be the strict transform on X of a general plane in $\mathbb{P}^3$ passing through the points p_1 and p_2, and let L be the strict transform of the line between these two points. Then $V_D = L$ is not Zariski dense.

In this case, $N_{D/X} \cong (\pi^*\mathcal{O}_{\mathbb{P}^3}(1) \otimes \mathscr{O}_X(-E_1 - E_2))\,|_D$ is not nef, since it has negative intersection with L. This shows that the conclusion of Theorem 1.3 that $N_{D/X}$ is effective can not be strengthened to include the conclusion that it is nef.

It bears noting that Theorem 1.1 contains the classical Skolem–Mahler–Lech theorem on arithmetic progressions as a special case. As a result, it comes as no surprise that the proof relies on methods of p-adic analysis.

Example 2.3 ([4])**.** Let $X = \mathbb{P}^2$ with coordinates $[W_0, W_1, W_2]$, and let $M : \mathbb{P}^2 \to \mathbb{P}^2$ be the linear automorphism given by the matrix

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ c_3 & c_2 & c_1 \end{bmatrix}$$

Let Y be the point $[x_0, x_1, x_2]$, and let Z be the hyperplane $W_0 = 0$. Then $\phi^n(Y)$ is the point $[x_n, x_{n+1}, x_{n+2}]$, where x_n is defined by the linear recurrence sequence $x_n = c_1x_{n-1} + c_2x_{n-2} + c_3x_{n-3}$. The fact that $\phi^n(Y) \subset Z$ for all n in the union of a finite set and an arithmetic progression is the classical Skolem–Mahler–Lech theorem.

Example 2.4. There are many examples of blow-ups of $\mathbb{P}^2$ at configurations of 9 or more points which admit automorphisms such that no iterate descends to an automorphism of $\mathbb{P}^2$, and so the dimensional hypothesis of Theorem 1.4 is needed. For example, let C_1 and C_2 be two general cubics in $\mathbb{P}^2$, and let X be the blow-up of $\mathbb{P}^2$ at the 9 points of $C_1 \cap C_2$, with exceptional divisors $E_1, \ldots, E_9$. The pencil of cubics induces an elliptic fibration $\pi : X \to \mathbb{P}^1$, and the E_i are sections of the pencil. The divisor E_1 determines a section of π, which we take to be the base point on each fiber. Fiberwise addition using the group law of the fibers the gives automorphisms $\tau_i : X \to X$ $(2 \leq i \leq 9))$ induced by addition of E_i.

Taking products $X = \mathbb{P}^2 \times \mathbb{P}^{n-2}$, we obtain examples of automorphisms of blow-ups of X along codimension 2 subsets which admit automorphisms of infinite order.

3. Dynamical Mordell–Lang in the non-reduced case

Definition 1. A subset $A \subset \mathbb{Z}$ is said to be a a one-sided semilinear set there is a decomposition

$$A = F \cup \left(\bigcup_{i=1}^{m} P_i \right),$$

where F is a finite set and each $P_i = a_i + b_i\mathbb{Z}_{\geq 0}$ is a one-sided arithmetic progressions. Similarly, $A \subset \mathbb{Z}$ is called a (two-sided) semilinear set if

$$A = F \cup \left(\bigcup_{i} P_i \right),$$

where F is a finite set and each $P_i = a_i + b_i\mathbb{Z}$ is a two-sided arithmetic progression.

We say that a one-sided or two-sided semilinear set A is N-periodic if there exists a decomposition for which the period b_i of each P_i divides N.

Lemma 3.1. *Suppose that $\{A_i\}$ is a (possibly infinite) collection of N-periodic semilinear sets. Then $\bigcap_i A_i$ is an N-periodic semilinear set.*

Proof. Observe that if A is an N-periodic semilinear set, then it can be written as the union of a finite set and a finite set of arithmetic progressions of length exactly N: indeed, if k divides N, an arithmetic progression of length k can be subdivided into N/k progressions of length N.

For each A_i, write

$$A_i = F_i \cup \left(\bigcup_j P_{i,j} \right)$$

where each $P_{i,j}$ is an arithmetic progression with period N. Then

$$\bigcap_i A_i = \left(\bigcup_{0 \le m \le N} (m + n\mathbb{Z}) \right) \cap \bigcap_i A_i = \bigcup_{0 \le m \le N} \left((m + N\mathbb{Z}) \cap \bigcap_i A_i \right).$$

Note that $(m+N\mathbb{Z}) \cap A_i = m+N\mathbb{Z}$ if $m+N\mathbb{Z}$ is among the arithmetic progressions $P_{i,n}$, while $(m+N\mathbb{Z}) \cap A_i = (m+N\mathbb{Z}) \cap F_i$ is a finite set otherwise. It follows that $(m+N\mathbb{Z}) \cap A_i = m+N\mathbb{Z}$ is either $m + N\mathbb{Z}$ or finite, and so

$$\bigcup_{0 \le m \le N} \left((m + N\mathbb{Z}) \cap \bigcap_i A_i \right)$$

is a union of finite sets and arithmetic progressions of length N. $\square$

Lemma 3.2 (Local Dynamical Mordell-Lang)**.** *Let K be a finite extension of $\mathbb{Q}_p$ with valuation ring R, uniformizer π, and residue field k, and let $f \in R\langle x_1, \cdots, x_n \rangle^n$ be an n-tuple of convergent power series (i.e. the p-adic absolute values of the coefficients tend to zero) inducing a topological isomorphism*

$$R[[x_1, \cdots, x_n]] \to R[[x_1, \cdots, x_n]].$$

Suppose that f is affine-linear mod π.

Then for any two closed formal subschemes Y, Z of $\mathrm{Spf}(R[[x_1, \cdots, x_n]])$, the set of $m \in \mathbb{Z}$ such that $f^m(Y) \subset Z$ is a two-sided semi-linear set. Furthermore, the semilinear set is N-periodic for some N depending only on $\#|k|$ and $|\pi|_p$ (and not on Y, Z, or f).

Proof. By the affine-linearity assumption,

$$f(\mathbf{x}) = A\mathbf{x} + \mathbf{b} \bmod \pi$$

for some

$$A \in GL_n(k), \ \mathbf{b} \in k^n.$$

As the group of affine-linear transformations of k^n is finite, there exists some N such that

$$f^N = \mathbf{x} \bmod \pi.$$

(Observe that this N depends only on $\#|k|$.) After possibly increasing N (say, replacing it with $p^M N$ for some $M \gg 0$ depending only on $|\pi|_p$), we may assume

$$f^N = \mathbf{x} \bmod \pi^a$$

for any fixed a. (See e.g. Remark 4 of [9].)

We may now apply the main result of [9] to obtain an element

$$g \in R\langle x_1, \cdots, x_n, m \rangle$$

such that

$$g(\mathbf{x}, r) = f^{Nr}(\mathbf{x})$$

for any $r \in \mathbb{Z}$.

Now let $\gamma \in R$ and consider the composition

$$g_\gamma : \mathscr{I}_Z \to R[[x_1, \cdots, x_n]] \xrightarrow{g(\mathbf{x},\gamma)} R[[x_1, \cdots, x_n]].$$

Fixing any element $h \in \mathscr{I}_Z$, the function

$$g_h : \gamma \mapsto g_\gamma(h)$$

is p-adic analytic. Let $q : R[[x_1, \cdots, x_n]] \to \mathscr{O}_Y$ be the natural quotient map.

Let $m = Nr + s$. Now $f^m(Y) \subset Z$ if and only $q \circ f^s \circ g_h(r) = 0$ for all $h \in \mathscr{I}_Z$. But this last is a p-adic analytic function of r; hence it either has finitely many zeros or is identically zero. It follows that the set of zeros of $q \circ f^s \circ g_h$ is an N-periodic set for every $h \in \mathscr{I}_Z$. By Lemma 4.2 this proves the theorem. $\square$

Before proceeding to the global situation (Theorem 1.1), we need two lemmas which will allow us to reduce to Lemma 3.2.

Lemma 3.3 ([4, Proposition 2.2])**.** *Let K be a finite extension of $\mathbb{Q}_p$ with valuation ring R, uniformizer π, and residue field k. Let X be a smooth, finite-type R-scheme with geometrically connected fibers.*

Let $\phi : X \to X$ be an étale R-map, and let $x \in X(k)$ be a $\phi(x) = x$. Then there exists an isomorphism

$$\widehat{\mathscr{O}_{X,x}} \simeq R[[x_1, \cdots, x_n]]$$

with $n = \dim(X)$ such that

1. *The automorphism of $R[[x_1, \cdots, x_n]]$ induced by ϕ is given by some $f \in R\langle x_1, \cdots, x_n\rangle^n$, and*
2. *f is affine-linear* mod *π.*

Proof. The proof is exactly the same as [4, Proposition 2.2], replacing p with π. $\square$

Lemma 3.4. *Let R be a discrete valuation ring and X/R a separated, finite-type R-scheme. Let Y, Z be closed subschemes of X. Suppose that Y has exactly one associated point (in particular, Y is irreducible). Let $y \in Y$ be any closed point with ideal sheaf $\mathfrak{m}_y$; let $y_n \subset Y$ be the closed subscheme associated to $\mathfrak{m}_y^n$. Then $Y \subset Z$ if and only if $y_n \subset Z$ for all y and n.*

Proof. The hypothesis is that $\mathscr{O}_X \to \widehat{\mathscr{O}_{Y,y}}$ has $\mathscr{I}_Z$ in its kernel for all n; we wish to show that $\mathscr{I}_Z$ is in the kernel of the map to $\mathscr{O}_Y$. But consider the diagram

$$\begin{tikzcd}
\mathscr{O}_X \arrow[r] \arrow[ddr] & \mathscr{O}_Y \arrow[d, "\pi_1"] \\
& \mathscr{O}_{Y,y} \arrow[d, "\pi_2"] \\
& \widehat{\mathscr{O}_{Y,y}}.
\end{tikzcd}$$

We observe that π_1 is injective as Y has one associated point, and π_2 is injective by Krull's intersection theorem. Thus

$$\ker(\mathscr{O}_X \to \widehat{\mathscr{O}_{Y,y}}) = \ker(\mathscr{O}_X \to \mathscr{O}_Y),$$

completing the proof. ☐

Proposition 3.5. *Let K be a finite extension of $\mathbb{Q}_p$ with valuation ring R, uniformizer π, and residue field k. Let X be a smooth, finite-type R-scheme with geometrically connected fibers. Let Y, Z be closed subschemes of X; let $f : X \to X$ be an étale R-endomorphism. Then the set of $m \geq 0$ such that $f^m(Y) \subset Z$ is a one-sided semilinear set, whose period can be bounded purely in terms the support of the associated points of Y. Furthermore, if $f : X \to X$ is an automorphism, then the set of m such that $f^m(Y) \subset Z$ is a two-sided semilinear set, with length bounded in the same way.*

Proof. Let r be such that for each associated point y of Y, $\bar{y}$ contains a $\mathbb{F}_{q^r}$-point. The map $X(\mathbb{F}_{q^r}) \to X(\mathbb{F}_{q^r})$ induced by f has eventual image

$$I = \bigcap_n f^N(X(\mathbb{F}_{q^r}))$$

which is permuted by f; hence f^N for some $N \gg 0$ fixes the eventual image I. Let R' be an unramified extension of R such that each point of I is the image of some R' point of X; replace R with R'.

Now given $x \in I$ with associated ideal sheaf $\mathfrak{m}_x$, let

$$\widehat{\mathscr{O}_{X,x}} = \varprojlim \mathscr{O}_X/\mathfrak{m}_x^n \simeq R[[x_1, \cdots, x_n]].$$

As f is étale, one has that f^N induces an isomorphism

$$\widehat{\mathscr{O}_{X,x}} \to \widehat{\mathscr{O}_{X,x}}$$

for each $x \in I$. By Lemma 3.3, the hypotheses of Lemma 3.2 are satisfied for this map. Now applying Lemma 3.2 to

$$Y \cap \operatorname{Spf}(\widehat{\mathscr{O}_{X,x}}), Z \cap \widehat{\mathscr{O}_{X,x}}$$

and using Lemma 3.4 on the closure of each associated point of Y gives the result. ☐

Remark. In fact, the period of the semilinear sets in the statement above does not depend on the supports of the associated points of Y, Z, but rather on X and the minimal k such that

$$\overline{x}(\mathbb{F}_{q^k}) \neq \emptyset$$

for all associated points x of Y, Z. Here $\overline{x}$ denotes the Zariski-closure of x.

Corollary 3.6 (Uniform Mahler-Skolem-Lech)**.** *Let $f : \mathbb{Z}^n \to \mathbb{Z}^n$ be a linear automorphism, and let $x, y \in \mathbb{Z}^n$ be two elements. Then*

$$A(x, y) := \{n \mid f^n(x) = y\}$$

is two-sided semilinear with period bounded only in terms of n.

Proof. This is immediate from the remark above, after extending scalars to $\mathbb{Z}_3$. ☐

Definition 2 (Finite-type diagram of schemes with conditions)**.** Let I be a small category and $F : I \to \mathrm{Sch}/S$ a functor; as usual we call such data a diagram of S-schemes. A *condition* is a subcategory of Sch; for example, the subcategory consisting of schemes with étale morphisms, or the full subcategory consisting only of proper schemes.

A *diagram with conditions* is a diagram in which every object O and morphism f of I is decorated with the data of a subcategory $\mathscr{C}_O$ or $\mathscr{C}_f$, such that $F(O) \in \mathscr{C}_O$ and $F(f) \in \mathscr{C}_f$.

We say that a diagram with conditions is of *of finite type* if there exists a finite-type $\mathbb{Z}$-scheme X, a flat morphism $\iota : S \to X$, and a functor $F' : I \to \mathrm{Sch}/X$ so that F is obtained from F' by base change along ι.

Remark. In the case that I consists of a single object, with no conditions, one recovers the usual notion of a finite-type S-scheme.

Any finite diagram of finite-type schemes is of finite type; likewise, any diagram constructed from finitely many finite-type schemes by operations that commute with flat base change (e.g. scheme-theoretic image, fiber product, etc.) is of finite-type. The purpose of this notion is to formalize situations in which one can apply the so-called Lefschetz principle to infinite diagrams of schemes.

Theorem 3.7. *Let k be a field of characteristic zero and X a smooth (geometrically connected) k-variety. Let $\{Y_i\}_{i\in I}, \{Z_j\}_{j\in J}$ be a collection of closed subschemes of X. Let f be an étale endomorphism of X.*

Then for any i, j, the set

$$\{n \in \mathbb{Z}_{\geq 0} \mid f^n(Y_i) \subset Z_j\}$$

is the union of a finite set and finitely many arithmetic progressions.

Suppose in addition that the diagram consisting of all the Y_i, Z_j, X and the embeddings of Y_i, Z_j in X (with the condition that these embeddings are closed) is of finite type, and that the set of associated points of the $\{Y_i\}$ is finite. Then the common difference in the arithmetic progressions in question is independent of i, j.

Proof. For the first statement, by the Lefschetz principle there exists a finite-type integral $\mathbb{Z}$-algebra R, R-schemes X', Y_i', Z_j', an étale endomorphism f' of X', and a flat map $\iota : \mathrm{Spec}(k) \to \mathrm{Spec}(R)$ such that X, Y_i, Z_j, f are obtained by base change along ι. Choose a finite $\mathbb{Z}_p$-algebra R' and a map $R \to R'$ such that

1. $X'_{R'}$ is smooth with geometrically connected fibers,
2. All associated points of $Y_{i,R'}$ are contained on the generic fiber.

Then we may conclude the result by applying Proposition 3 to $X'_{R'}$.

For the second statement, we may by assumption find a model of all of the Y_i, Z_i over a fixed $\mathbb{Z}$-algebra R and then proceed as above. □

This completes the proof of:

Theorem 1.1. *Suppose that X is a smooth projective variety over a field of characteristic zero, and that $\phi : X \to X$ is an étale endomorphism of X.*

1. *Let Y and Z be closed subschemes of X. Then the set $A_\phi(Y, Z) = \{n : \phi^n(Y) \subseteq Z\}$ is a union of a finite set and a finite number of arithmetic progressions.*
2. *Suppose that $Y_1^\circ, \ldots, Y_r^\circ$ is a finite set of closed subschemes of X, defined by ideal sheaves $\mathscr{I}_{Y_i^\circ}$, and that $\{Y_j\}$ is the infinite collection of subschemes defined by the ideal*

sheaves

$$\sum_{i=1}^{r} \mathscr{I}_{Y_i^\circ}^{n_i},$$

$n_i \geq 0$. *Then there exists a constant* $N = N(\{Y_i^\circ\}, Z)$ *such that the lengths of the arithmetic progressions appearing in* $A_\phi(Y_j, Z)$ *are bounded by* N *(and in particular are independent of* j*).*

This is because the diagram of closed subschemes of X defined by

$$\sum_i \mathscr{I}_{Y_i^\circ}^{n_i}$$

is of finite type.

The next example shows that the period of the arithmetic progressions in $A_\phi(Y_j, Z)$ does not in general depend only on the support of Y; it is sensitive to the presence of embedded points on Y.

Example 3.1. Let $g : S \to S$ be an automorphism of a variety for which there exist periodic points of all periods $k \geq 0$. For example, we may take E to be an elliptic curve and let $S = E \times E$. Then the map $g : S \to S$ given by the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ has periodic points of all possible periods: if x_k is an k-torsion point on E, then $\phi^n(0, x_k) = (nx_k, x_k)$, so this point has period k.

Now, let $X = S \times S \times \mathbb{P}^1$ and consider the automorphism $\phi = \mathrm{id}_S \times \phi \times \mathrm{id}_{\mathbb{P}^1} : X \to X$. Consider the subscheme $Z = (S \times S \times 0) \cup (\Delta^{(2)} \times 0)$ which is $S \times S \times 0$ with nilpotents in the structure sheaf along the diagonal $\Delta \subset S \times S$. Consider the subschemes $Y_k = (S \times S \times 0) \cup (x_k, x_k, 0)^{(2)}$, where x_k is a g-periodic point of period k. The associated points of Y_k are the generic point of $S \times S \times 0$, together with the embedded point $(x_k, x_k, 0)$.

Observe that $\phi^n(Y_k) = (S \times S \times 0) \cup \phi(x_k, x_k, 0)^{(2)} = (S \times S \times 0) \cup (x_k, \phi^n(x_k), 0)^{(2)}$, since $\phi^n((x_k, x_k, 0) = (x_k, \phi^n(x_k), 0)$. Then $\phi^n(Y_k)$ is contained in Z if and only if the embedded point $(x_k, \phi^n(x_k), 0)^{(2)}$ is contained in Z, which is the case exactly when $(x_k, \phi^n(x_k), 0)$ lies on the diagonal $\Delta \times 0 \subset X$.

4. Local separation by blow-ups

We now deduce some geometric consequences of the results of the previous section. Before beginning the proof of Theorem 1.2, we need some simple lemmas. For each $k \geq 0$, let $D^{(k)}$ be the closed subscheme of X defined by the ideal sheaf $\mathscr{I}_V^{k+1} + \mathscr{I}_D$. This is the k^{th} order germ of D inside V. Then consider the set

$$A_k = \left\{ n : \phi^n(D^{(k)}) \subseteq D \right\}.$$

By Theorem 1.1, there exists an N so that each set A_k is an N-periodic semilinear set. Note that when ϕ is replaced by an iterate ϕ^n, the set A_k is replaced by $A_k \cap n\mathbb{Z}$. To begin, we give some lemmas showing that after replacing ϕ by a suitable iterate the sets A_k take a particularly simple form.

The next lemma is the key application of Theorem 1.1, part (2): in the two-dimensional setting, it implies that there is a uniform bound on the order of tangency between C and the curves $\phi^n(C)$. The key observation is that because of the uniform bound N on the periods of the arithmetic progressions in A_k, if any $\phi^k(D)$ is tangent to D to order j, this tangency must occur for some $0 < k \leq N$.

Lemma 4.1. *There exists some k for which A_k is a finite set.*

Proof. If A_k is an infinite set, then it contains an infinite arithmetic progression P with step size dividing N by Theorem 1.1(2). In particular, there exists some i with $1 \leq i \leq N$ for which $\phi^i(D^{(k)}) \subseteq D$.

Since D is not periodic under ϕ, the divisors D and $\phi^i(D)$ are distinct for any nonzero i, and so for any i, there is a maximal j for which $D^{(j)} \subset D \cap \phi^i(D)$. The claim of the lemma then holds with

$$k = \max_{1 \leq i \leq N} \left(\max \left\{ j : D^{(j)} \subset D \cap \phi^i(D) \right\} \right). \qquad \square$$

Lemma 4.2. *Suppose that $A_0, \ldots, A_n$ is a finite collection of semilinear sets. Then there exists an integer N so that $A_i \cap N\mathbb{Z}$ is either empty, $\{0\}$, or $N\mathbb{Z}$ for each i.*

Proof. Let k be an integer divisible by the period of every arithmetic progression appearing in any of the sets A_i. Each A_i can then be written as the union of a finite number of residue classes $k\mathbb{Z} + r$ and a finite set. Then $A_i \cap k\mathbb{Z}$ is either all of $k\mathbb{Z}$ are a finite set for each i. Replacing k by a sufficiently large multiple N, we may eliminate any nonzero element of $A_i \cap k\mathbb{Z}$. $\square$

Lemma 4.3. *Suppose that*

$$\mathbb{Z} = A_0 \supseteq \cdots \supseteq A_{k_1} \supseteq A_k \supseteq A_{k+1} \supseteq \cdots$$

is a decreasing chain of semilinear sets, so that A_k is finite for sufficiently large k and each A_k contains 0. Then there exist $n > 0$ and k such that $A_i \cap n\mathbb{Z} = n\mathbb{Z}$ for $i \leq k$ and $A_i \cap n\mathbb{Z} = \{0\}$ for $i > k$.

Proof. According to Lemma **??**, there exists some m so that A_m is a finite set. By Lemma 4.2 applied to the sets $A_0, \ldots, A_m$, there exists N so that for $1 \leq i \leq m$, we have either $A_i \cap N\mathbb{Z} = \{0\}$ or $A_i \cap N\mathbb{Z} = N\mathbb{Z}$, with $A_0 \cap N\mathbb{Z} = N\mathbb{Z}$ and $A_m \cap N\mathbb{Z} = \{0\}$. Since the sets decreasing, the claim follows.

$\square$

It follows that there exists n so that after replacing ϕ by the iterate ϕ^n, there exists k with $A_i = \mathbb{Z}$ for $i \leq k$, and $A_i = \{0\}$ for $i > k$.

Remark. The geometric significance of Lemma 4.3 in the example of Section 2 is this: it might happen, for example, that $\phi^n(C)$ is tangent to C at x whenever n is is 0 or 1 modulo 3, while C is tangent to $\phi^n(C)$ to order 2 only for a finite set $n = 3, 6, 7$. In this setting we have $N = 3$ and take $k_0 = 1$. After replacing ϕ by ϕ^9, we arrange that

(1) C is tangent to $\phi^n(C)$ for all n,
(2) C is not tangent to $\phi^n(C)$ to order 2 for any n.

We now begin the proof of the local result on separation of blow-ups. Let us fix some notation: we write $\mathscr{I}_D$ for the ideal sheaf on X associated to the divisor D, and $\mathscr{I}_V$ for the ideal sheaf of V. We write $I_{D,V}$ for the ideal in the local ring $\mathscr{O}_{X,V}$ determined by $\mathscr{I}_D$, and $\mathfrak{m}_V$ for the maximal ideal in $\mathscr{O}_{X,V}$. The first step in the proof is to show that $I_{D,V} + I_{\phi^n(D),V}$ stabilizes to an ideal independent of n; the means roughly that the multiplicity of intersection of D and $\phi^n(D)$ along V is independent of n.

Lemma 4.4. *Let $\phi : X \to X$ be an automorphism of smooth variety over a field of characteristic 0, and V a codimension 2 subvariety with $\phi(V) = V$. Suppose that D is a divisor* *containing V and smooth at the generic point of V. Then after replacing ϕ by an iterate, there exists a value of k so that:*

(1) *the ideal sheaf $\mathscr{I}_{D,V} + \mathfrak{m}_V^{k+1}$ is invariant under ϕ;*
(2) *$I_{D,V} + I_{\phi^n(D),V} = I_{D,V} + \mathfrak{m}_V^{k+1}$ in $\mathscr{O}_{X,V}$ for every value of n.*

Proof. By Lemma 4.3, we may replace ϕ by an iterate ϕ^d so that there exists a value of k for which $A_k = \mathbb{Z}$ while $A_{k+1} = \{0\}$. This means that $\phi^n(D^{(k)}) \subseteq D$ for all n, while $\phi^n(D^{(k+1)}) \subseteq D$ only for $n = 0$. At the level of ideal sheaves on X, this yields $\mathscr{I}_D \subseteq \mathscr{I}_{\phi^n(D)} + \mathscr{I}_V^{k+1}$, while $\mathscr{I}_D \not\subseteq \mathscr{I}_{\phi^n(D)} + \mathscr{I}_V^{k+2}$. Replacing n by $-n$ we obtain $\mathscr{I}_{\phi^n(D)} \subseteq \mathscr{I}_D + \mathscr{I}_V^{k+1}$, while $\mathscr{I}_{\phi^n(D)} \not\subseteq \mathscr{I}_D + \mathscr{I}_V^{k+2}$.

It follows from the above that

$$\mathscr{I}_D + \mathscr{I}_V^{k+1} = \mathscr{I}_{\phi^n(D)} + \mathscr{I}_V^{k+1}.$$

Observe that

$$\phi(\mathscr{I}_D + \mathscr{I}_V^{k+1}) = \mathscr{I}_{\phi(D)} + \mathscr{I}_{\phi(V)}^{k+1} = \mathscr{I}_{\phi(D)} + \mathscr{I}_V^{k+1} = \mathscr{I}_D + \mathscr{I}_V^{k+1},$$

so this ideal sheaf is invariant under ϕ. Now consider the local ring $\mathscr{O}_{X,V}$ at the generic point of V. Restricting the ideal sheaves in question, we obtain an equality of ideals

$$I_{D,V} + \mathfrak{m}_V^{k+1} = I_{\phi^n(D),V} + \mathfrak{m}_V^{k+1}.$$

Since X is smooth, $\mathscr{O}_{X,V}$ is a two-dimensional regular local ring. As D is smooth at the generic point of V, the ideal $I_{D,V} \subset \mathscr{O}_{X,V}$ is principal, generated by an element not contained in $\mathfrak{m}_V^2$. Then $\mathscr{O}_{X,V}/I_{D,V}$ is a one-dimensional regular local ring, hence a discrete valuation ring. The ideals in $\mathscr{O}_{X,V}$ containing $I_{D,V}$ correspond to ideals in $\mathscr{O}_{X,V}/I_{D,V}$, which are of the form $\mathfrak{m}_V^r$. It follows that every ideal in $\mathscr{O}_{X,V}$ containing $I_{D,V}$ is of the form $I_{D,V} + \mathfrak{m}_V^r$.

We claim next that

$$I_{D,V} + I_{\phi^n(D),V} = I_{D,V} + \mathfrak{m}_V^{k+1}$$

for every nonzero value of n. On one hand, we have

$$I_{D,V} + I_{\phi^n(D),V} \subseteq I_{D,V} + I_{\phi^n(D),V} + \mathfrak{m}_V^{k+1} = I_{D,V} + \mathfrak{m}_V^{k+1}$$

To show that $I_{D,V} + I_{\phi^n(D),V} = I_{D,V} + \mathfrak{m}_V^{k+1}$, it suffices to show that $I_{D,V} + I_{\phi^n(D),V} \not\subseteq I_{D,V} + \mathfrak{m}_V^{k+2}$. Since $\mathscr{I}_{\phi^n(D)} \not\subseteq \mathscr{I}_D + \mathscr{I}_V^{k+2}$, and because neither of these ideal sheaves has embedded points, it must be that the inclusion also fails at the generic point of the support V, so that $I_{\phi^n(D),V} \not\subseteq I_D + \mathfrak{m}_V^{k+2}$ in $\mathscr{O}_{X,V}$. But this shows that $I_{D,V} + I_{\phi^n(D),V} \not\subseteq I_D + \mathfrak{m}_V^{k+2}$, as needed. □

Remark. Some care is needed to prove local separation results in the case that D is not smooth at the generic point of V. Before proving the main lemma, we show that it is possible to reduce to the case in which D is smooth at the generic point of V.

To see the difficulty, suppose that X is a surface and V is a point, as §2. It might be that $D \subset X$ is a curve with a node at D. It is possible that one of the local branches of D has tangent direction fixed by ϕ, while the other does not. However, after blowing up the node of the curve, the problem is reduced to the case that D is smooth. If D had more complicated singularities at x, it might be necessary to blow up several times.

Lemma 4.5. *Suppose that $\phi : X \to X$ is an automorphism of a smooth, projective variety over $\mathbb{C}$ and that $V \subset X$ is a codimension 2 subvariety with $\phi(V) = V$. Suppose that $D \subset X$*

is an irreducible divisor containing V. Then there exists a birational map $\pi : Y \to X$ from a smooth variety Y such that after replacing ϕ by a suitable iterate:

(1) *ϕ lifts to an automorphism $\psi : Y \to Y$,*
(2) *for every value of n, the codimension 2 part of $D \cap \phi^n(D) \cap \pi^{-1}(V) = \bigcup_i V_i$ is a union of finitely many ψ-invariant codimension 2 subvarieties V_i of Y, and D is smooth at the generic point of V_i for each i.*

Proof. Let $\pi_0 : Y_0 \to X$ be the blow-up of X along V, and let $\tilde{D}$ be the strict transform of D on Y_0 and $E = \pi_0^{-1}(V)$. Note that Y_0 may not be smooth, and that E may be reducible and have components of different dimensions. Let $\pi : Y \to X$ be a resolution of Y such that ϕ lifts to an automorphism of Y.

Write $V_1, \ldots, V_n$ for the codimension 2 components of $\tilde{D} \cap E$. For every n, the codimension 2 components of $\tilde{D} \cap \psi^n(\tilde{D}) \cap E$ is a union of some subset of the V_i.

Now, each of the sets $A_\phi(V_i, V_j)$ is semilinear. By Lemma 4.2, after replacing ϕ by an iterate, we may assume that $A_\phi(V_i, V_i)$ is either all of $\mathbb{Z}$ or empty, and that $A_\phi(V_i, V_j)$ is empty for $i \neq j$. Suppose that $D \cap \phi^n(D) \cap E$ has a codimension 2 component: then $\phi^n(D_i) = D_j$ for some i and j. But this is possible only if $i = j$.

It is possible that D is still singular along one of the V_i. However, the total multiplicity

$$\sum_{\substack{V_i \in \pi^{-1}(V) \\ \operatorname{codim} V_i = 2 \\ \phi(V_i) = V_i}} \operatorname{mult}_{V_i} D$$

decreases. If there is any codimension 2 subset V_i as above for which D is not smooth at the generic point of V_i, then blowing up V_i and then taking a functorial resolution, we obtain a model on which this total multiplicity decreases. Hence repeating the procedure and blowing up the V_i, we eventually reach a model on which D is smooth at the generic point of each periodic V_i. □

Theorem 1.2. *Let X be a smooth projective variety over $\mathbb{C}$ and $\phi : X \to X$ an automorphism. Suppose that $D \subset X$ is a divisor, containing a codimension-2 closed subset V with $\phi(V) = V$. Then there exists a birational morphism $\pi : Y \to X$ such that after replacing ϕ by a suitable iterate:*

(1) *Y is smooth;*
(2) *some iterate of ϕ lifts to an automorphism $\psi : Y \to Y$;*
(3) *$\pi(\psi^m(\tilde{D}) \cap \psi^n(\tilde{D}))$ does not contain V for any $m \neq n$.*

Proof of Theorem 1.2. By Lemma **??**, we may assume that D is smooth at the generic point of V_i. Now, let $\pi_0 : Y_0 \to X$ be the blow-up of X along the ideal sheaf $\mathscr{I}_D + \mathscr{I}_V^{k+1}$. We claim that

(1) ϕ lifts to an automorphism $\psi_0 : Y_0 \to Y_0$;
(2) $\pi_0(\phi^m(\tilde{D}) \cap \phi^n(\tilde{D}))$ does not contain V for $m \neq n$.

The first claim is immediate since the ideal sheaf $\mathscr{I}_D + \mathscr{I}_V^{k+1}$ is invariant under the automorphism ϕ. Write $\psi_0 : Y_0 \to Y_0$ for this automorphism.

By (2) of Lemma 4.4, we have $I_{D,V} + I_{\phi^n(D),V} = I_{D,V} + \mathfrak{m}_V^{k+1}$ in $\mathscr{O}_{X,V}$ for every value of n. Since this equality holds at the generic point of V, for every n there exists an open set $U_n \subset X$ with $\left(\mathscr{I}_D + \mathscr{I}_{\phi^n(D)}\right)|_{U_n} = \left(\mathscr{I}_D + \mathscr{I}_V^{k+1}\right)|_{U_n}$ for every n. It then follows from [7, Exercise 7.12] that D and $\phi^n(D)$ do not meet above U_n, so that $\pi_0(D \cap \phi^n(D))$ is disjoint from U_n. This shows the necessary disjointness holds for $m = 0$. In general,

$$\pi_0(\psi_0^m(\tilde{D}) \cap \psi_0^n(\tilde{D})) = \pi_0(\psi_0^m(\tilde{D}) \cap \psi_0^{n-m}(\tilde{D})) = \phi^{n-m}(\pi_0(\tilde{D} \cap \psi_0^{n-m}(\tilde{D}))),$$

which is disjoint from $\phi^{n-m}(U_n)$, an open subset of V. Hence (2) and (3) of Theorem 1.2 are satisfied by the model $\pi_0 : Y_0 \to X$. However, the variety Y_0 may not itself be smooth. Applying functorial resolution to $\psi_0 : Y \to Y$, we obtain a smooth model on which all three conditions are satisfied, completing the proof. □

5. Global separation by blow-ups

The following result on divisors containing infinite sets of disjoint divisors will also play a role in the proof of Theorem 1.3.

Proposition 5.1 ([6, Theorem 1.1]). *Suppose that X is a normal projective variety defined over an algebraically closed field, and that $\{D_i\}$ is an infinite set of pairwise disjoint divisors on X. Then there exists a smooth projective curve C and a map $f : X \to C$ such that each D_i is contained in a fiber of f.*

We now prove Theorem 1.3. The strategy is to repeatedly apply Theorem 1.2 to components of $D \cap \phi^n(D)$, eventually forcing these infinitely many divisors to become disjoint.

Theorem 1.3. *Let X be a smooth variety over $\mathbb{C}$ and $\phi : X \to X$ an automorphism. Suppose that $D \subset X$ is a divisor which is not periodic under ϕ and for which the set*

$$V(D) = \{d : \phi^n(d) \in D \text{ for some nonzero } n\} = \bigcup_{n \neq 0} D \cap \phi^n(D)$$

is Zariski closed. Then there exists a birational morphism $\pi : Y \to X$ such that

1. *Y is smooth;*
2. *ϕ lifts to an automorphism $\psi : Y \to Y$;*
3. *the divisors $\psi^n(\tilde{D})$ are pairwise disjoint;*
4. *there exists a curve C, a morphism $f : Y \to C$, and an automorphism $\tau : C \to C$ such that $f \circ \psi = \tau \circ f$.*

Moreover, the normal bundle of D satisfies $H^0(D, N_{D/X}) > 0$.

Proof. The first step is to construct a model Y on which conditions (1)–(3) are satisfied. This will be realized by a sequence of blow-ups $\pi_i : Y_i \to X$ removing the components of $V(D)$ one at a time.

The proof is by induction on the number of components of $V(D)$. If $V(D) = \emptyset$, claims (1)–(3) of the theorem are already satisfied. Observe that replacing ϕ by an iterate can serve only to shrink the set $V(D)$ and decrease the number of components, and so we freely replace ϕ by iterates in what follows.

We claim first that there exists a sequence n_i with $|n_i|$ unbounded, such that $D \cap \phi^{n_i}(D)$ is nonempty for each i. Indeed, suppose that there exists N for which $\phi^n(D) \cap D$ is empty for all n with $|n| > N$. Replacing ϕ by the iterate ϕ^N, we may assume that $\phi^n(D) \cap D$ is empty for all $n \neq 0$. But then $\phi^m(D) \cap \phi^n(D) = \phi^m(D \cap \phi^{n-m}(D))$ is empty for any $m \neq n$. Then claims (1), (2), and (3) hold with π the identity map.

For each i, the intersection $D \cap \phi^{n_i}(D)$ is a non-empty union of codimension-1 subschemes of D. Let $V_0, \ldots, V_r$ be the finitely many components of $\bigcup_{n \neq 0} D \cap \phi^n(D)$, so that each V_i is a codimension-1 subvariety of D. Since the intersections $D \cap \phi^{n_i}(D)$ are nonempty, there must exist some irreducible component $V = V_j$ which is contained in $D \cap \phi^{n_i}(D)$ for infinitely many of the n_i.

The set $A_{\phi^{-1}}(V, D) = \{n : \phi^{-n}(V) \subset D\} = \{n : V \subset \phi^n(D)\}$ is infinite, so it must contain a non-empty arithmetic progression by Theorem 1.1. Replacing ϕ by a suitable iterate, we may then assume that $V \subset D \cap \phi^n(D)$ for all n. Applying ϕ^{-n}, we see that $\phi^{-n}(V) \subset D \cap \phi^{-n}(D)$ for all n. Since $\phi^{-n}(V)$ has codimension 1 in D, it must coincide with some component V_i, and since there are only finitely many V_i, we must have $\phi^{-n}(V) = \phi^{-m}(V)$ for some distinct m and n. But then $\phi^{n-m}(V) = V$, and replacing ϕ by the iterate ϕ^{n-m}, we may assume that $\phi(V) = V$.

We now apply the local statement of Theorem 1.2 to $\phi : X \to X$ along the invariant subvariety V and obtain a model $\pi : Y \to X$ on which ϕ lifts to an automorphism and for which $\pi(\psi^m(\tilde{D}) \cap \psi^n(\tilde{D}))$ does not contain V. Suppose now that $W \subset V(\tilde{D}, \psi)$ is a codimension-2 subvariety of Y. Then $W \subset \tilde{D} \cap \psi^n(\tilde{D})$ for some n, and $\pi(W) \subset D \cap \phi^n(D)$. This shows that the components of $V(\tilde{D}, \psi)$ are a subset of those of $V(D, \phi)$. Since the component V has been removed, the cardinality decreases. Applying this procedure inductively, we eventually obtain a model $\pi : Y \to X$ satisfying conditions (1)–(3).

Since the divisors $\psi^n(\tilde{D})$ are pairwise disjoint, it follows from Proposition 5.1 that there exists a map $f : Y \to C$ such that all of the divisors $\psi^n(\tilde{D})$ are contained in fibers of f. The map f has only finitely many reducible fibers, and so there are infinitely many m for which $\psi^m(\tilde{D})$ and $\psi^{m+1}(\tilde{D})$ are both irreducible fibers of f.

At last we show that ψ induces an automorphism of C. There is a curve $B \subset \mathrm{Hilb}(Y)$ parametrizing the fibers of f. The automorphism ψ induces an automorphism $\psi_H : \mathrm{Hilb}(Y) \to \mathrm{Hilb}(Y)$. Since $\psi_H([\psi^m(D)]) = [\psi^{m+1}(D)]$, we see that $\psi_H(B)$ meets B infinitely many times. Since B is one-dimensional, it must be that B is fixed by $\psi_H(B)$. Consequently ψ sends all fibers of f to fibers of f, and it follows from the rigidity lemma that ψ induces an automorphism of $\tau : C \to C$ with $\phi \circ f = f \circ \tau$ as required.

At last, the divisor $\tilde{D}$ is a fiber of $f : Y \to C$, and so moves in a 1-parameter family inside Y. It follows that D moves in a 1-parameter family inside X and that $H^0(X, D) > 0$ as claimed. $\square$

6. Applications to automorphism groups

In this section, we will explain how the results of the previous section can be applied in a geometric context to study automorphism groups of varieties constructed by sequences of blow-ups. Before beginning the proof, we collect some preliminary observations about varieties with multiple contractible divisors.

Lemma 6.1. *Suppose that X is a smooth variety of dimension n, and that $\pi_i : X \to Y_i$ ($0 \le i \le m$) are distinct morphisms realizing X as the blow-up of smooth variety along Y_i a center of dimension less than or equal to r.*

1. *If $2r + 3 \le n$, then any two exceptional divisors E_i and E_j are pairwise disjoint.*
2. *If $n = 4$ and $r = 1$, there is an irreducible divisor $W \subset E_0$ such that $E_0 \cap E_j \subseteq W$ for every $j \neq 0$.*

Proof. First we prove (1). Suppose that some E_i and E_j have nonempty intersection, and choose a point x be a point in the intersection, and let $F \cong \mathbb{P}^{n-r-1}$ be the fiber of π_i containing x and $F_j \cong \mathbb{P}^{n-r-1}$ be the fiber of $\pi \circ \phi^{-n}$ containing x. Since $n \ge 2r - 3$, then the intersection of F and F' must contain a curve Γ. The map π contracts F and hence Γ to a point. But it is impossible to contract a positive-dimensional subvariety of F' without contracting all of F'.

Next we prove (2). Let x be any point of $E_0 \cap E_j$, and let F_0 and F_j be the fibers of the maps π_0 and π_j passing through X. Observe that $F_j \cap E_0$ is contracted to a point by π_j, and has dimension $n - r - 2$. Since r is either 0 or 1, this intersection has positive dimension. It follows that the divisor $E_0 \cap E_j \subset E_0$ is contracted by the map π_j.

If $\pi_0 : X \to Y_0$ is the blow-up at a point, then $E_0 \cong \mathbb{P}^3$, and it is impossible to contract any divisor on E_0. If π is the blow-up of a smooth curve, then E is isomorphic to a $\mathbb{P}^2$ bundle over a curve. The Picard rank of such a variety is 2, and one of the two extremal rays of $\mathrm{Nef}(E)$ corresponds to the $\mathbb{P}^2$-bundle contraction, so there is at most one contractible divisor $W \subset E$. Hence the support of the intersection $E \cap \phi^n(E)$ must be either empty or W, for every value of n. □

We will later apply this lemma in the case that X is a variety with an infinite order automorphism $\phi : X \to X$, and the maps $\pi_i : X \to Y_i$ are of the form $\pi_0 \circ \phi^{-n}$.

Lemma 6.2. *Suppose that $\pi : Y \to X$ is the blow-up of a smooth subvariety with exceptional divisor E and that $\phi : Y \to Y$ is an automorphism with $\phi(E) = E$. Then the iterate $\phi^{\rho(E)}$ descends to an automorphism of X.*

Proof. This is a case of [8, Lemma 3.2]. Briefly, it suffices to show that some iterate of ϕ sends fibers of E to fibers of E, for ϕ then descends to an automorphism of X as a consequence of the rigidity lemma. This in turn follows from the fact that E is a $\mathbb{P}^n$-bundle, and a given variety E has at most $\rho(E)$ different $\mathbb{P}^n$-bundle structures [11, Theorem 2.2]. □

Lemma 6.3. *Suppose that $\phi : X \to X$ is an automorphism of a smooth variety. There is a constant e depending only on $\rho(X)$ (and not on ϕ) such that if D is a divisor which is rigid in its cohomology class (in the sense that D is the only divisor with class on the ray $\mathbb{R}_{\geq 0}[D] \subset N^1(X)$), and $\phi^n(D) = D$ for some positive n, there exists $0 < n' \leq n_0$ with $\phi^{n'}(D) = D$.*

Proof. The result does not follow from the bounds on the periods in Theorem 1.1, since the bounds there depend on the field of definition of ϕ. The strategy is instead to conclude the statement from the corresponding statement for the linear map $\phi^* : N^1(X) \to N^1(X)$.

Since ϕ^* and its inverse are both given by integer matrices, the determinant is 1. It follows from Corollary 3.6 that $(\phi^*)^{n'}([D]) = [D]$ for some $n' \leq n_0$. Since D is the only divisor with class $[D]$, it must be that $\phi^n(D) = D$. □

We are now in position to prove the main result.

Theorem 1.4. *Suppose that X is a smooth projective variety of dimension n, and that Y is a variety constructed by a sequence of smooth blow-ups of X at centers of dimension $\leq r$. If either*

(1) $2r + 3 \leq n$, *or*
(2) $n = 4$ *and* $r = 1$

there exists $N \in \mathbb{Z}_{\geq 1}$ such that for any automorphism ϕ of Y, ϕ^N descends to X.

Proof. We denote the sequence of blow-ups used in constructing Y as follows.

$$Y = X_n \xrightarrow{\pi_{n-1}} X_{n-1} \xrightarrow{\pi_{n-2}} \cdots \longrightarrow X_1 \xrightarrow{\pi_0} X_0 = X$$

Let $E_j \subset X_j$ be the exceptional divisor of $X_j \to X_{j-1}$, and $\tilde{E}_j \subset X$ its strict transform. There are two cases:

(1) all of the divisors $\tilde{E}_j$ are periodic under ϕ, and
(2) some $\tilde{E}_j$ is not.

As each $\tilde{E}_j$ is rigid, by Lemma 6.3, there is a constant $e(X)$ so that if $\tilde{E}_j$ is periodic under ϕ, it has period dividing $e(X)$. We claim that in case (1) the iterate ϕ^N descends to an automorphism of X, where

$$N = e(X) \prod_{i=1}^{n} \rho(E_n)$$

It follows from Lemma 6.3, there is a constant $n_0 = n_0(X)$ such that $\phi^{e(X)}(E_j) = E_j$ for some n less than n_0. Hence replacing ϕ by ϕ^n, we may assume that each exceptional divisor E_j is not invariant under ϕ. Now, since $E_n \subset X_n$ is invariant under $\phi : X_n \to X_n$, the iterate $\phi^{\rho(E_n)}$ descends to an automorphism $\phi_{n-1} : X_{n-1} \to X_{n-1}$ by Lemma 6.2. Since by assumption E_{n-1} is periodic under ϕ_{n-1}, the map descends to an automorphism of X_{n-1}. Continuing in this manner, we conclude that ϕ^N descends to an automorphism of X, as claimed.

It remains only to consider case (2), in which some $\tilde{E}_j$ is not periodic under ϕ. We will see that this case is impossible due to the dimensional hypotheses on the centers of the blow-ups.

Let j be the largest index for which this is the case. By the argument above, ϕ^N descends to an automorphism $\phi : X_j \to X_j$. Replace X by X_j, and let $\pi : X_j \to X_{j-1}$ be the contraction of E_j, for which we now write E. Then $\pi \circ \phi^{-n} : X_j \to X_{j-1}$ give an infinite set of contractions with distinct exceptional divisors $\phi^n(E)$.

First suppose that we are in the case $2r + 3 \leq n$. According to Lemma 6.1 applied to the maps $\pi \circ \phi^{-n}$, the divisors $\phi^n(E)$ must be pairwise disjoint. Let f be a line contained in a fiber of E, so that $E \cdot f = -1$. Then $\phi^n(E) \cdot \phi^n(f) = -1$, but $\phi^n(E) \cdot \phi^m(f) = 0$ since E is disjoint from $\phi^n(E)$. This implies that the classes of the $\phi^n(E)$ are linearly independent in $N^1(X)$, which is impossible since these divisors are all distinct and $N^1(X)$ is finite-dimensional.

Instead, suppose that $r = 1$ and $n = 4$. Appling Lemma 6.1 as before, we conclude that $\phi^n(E) \cap E$ is supported on a single divisor $W \subset E$, independent of n. This means that the set $V(E) = \bigcup_{n \neq 0} E \cap \phi^n(E)$ is either empty or contains only W. By Theorem 1.3, it must be that $H^0(E, N_{E/X}) > 0$. This is a contraction, since E is the exceptional divisor of a blow-up. □

Corollary 6.4. *Let X and Y be as in Theorem 1.4(1). Then if* $\mathrm{Aut}(X)$ *has finite component group, the same is true for* $\mathrm{Aut}(Y)$.

Proof. Write $Y = \mathrm{Bl}_Z(X)$ and let $G \subset \underline{\mathrm{Aut}}(X)$ be the closed subgroup scheme consisting of automorphisms of X that preserve Z. By assumption, $\underline{\mathrm{Aut}}(X)$ is of finite type, so the same is true for G. Let $M = \#|G/G^0|$, where G^0 is the connected component of the identity in G.

Note that G is naturally a subgroup scheme of $\underline{\mathrm{Aut}}(Y)$. By Theorem 1.4, there exists an integer N such that for any $\phi \in \mathrm{Aut}(Y)$, $\phi^N \in G$; thus ϕ^{NM} is in the identity component of $\underline{\mathrm{Aut}}(Y)$. In particular, $\underline{\mathrm{Aut}}(Y)/\underline{\mathrm{Aut}}^0(Y)$ is a group of bounded exponent.

Thus by Lemma 6.5, the natural map

$$\mathrm{Aut}(Y) \to GL(H^*(Y, \mathbb{C}))$$

has finite image. Let $\mathrm{Aut}_\omega(Y)$ be the kernel of this map. By a well-known result of Lieberman–Fujiki [1, Proposition 2.2], $\mathrm{Aut}^0(Y)$ has finite index in $\mathrm{Aut}_\omega(Y)$, which we've just shown has finite index in $\mathrm{Aut}(Y)$. Hence $\mathrm{Aut}(Y)$ has finite component group, as desired.

$\square$

Lemma 6.5. *Let $H \subset GL_n(k)$ be a group of bounded exponent, with k a field of characteristic zero. Then H is finite.*

Proof. Let $\overline{H}$ be the Zariski closure of H in GL_n. $\overline{H}$ is a subgroup scheme of GL_n which is of bounded exponent in the sense that if N is the exponent of H, we have that the map

$$\overline{H} \xrightarrow{x \mapsto x^N} \overline{H}$$

is the same as the constant map $x \mapsto 1$. But now consider the group

$$T := \ker(\overline{H}(k[\epsilon]/\epsilon^2) \to \overline{H}(k)).$$

This is a torsion-free group (indeed a k-vector space) of exponent N; hence it must equal zero. Thus $\overline{H}$ is zero-dimensional, hence finite.

$\square$

7. Bounds on intersection multiplicities

The applications in the preceding sections have focused on intersections $D \cap \phi^n(D)$ of a divisor with its own iterates under an automorphism of a variety. In fact, many of the same arguments can be applied to intersections of smaller dimensional varieties. In the case that the intersection is 0-dimensional, this is a result of Arnold.

Lemma 7.1. *Let $\phi : X \to X$ be an automorphism of a smooth projective variety, and let $V \subset X$ be a subvariety with $\phi(V) = V$. Suppose that Y and Z are two irreducible closed subvarieties of X containing V and with $\operatorname{codim}_X Y + \operatorname{codim}_X Z = \operatorname{codim}_X V$. Then there exists some k with $\mathfrak{I}_V^k \subseteq \mathscr{I}_D + \mathscr{I}_{\phi^n(D)}$ for all nonzero n.*

Proof. That, I do not know. $\square$

Theorem 7.2 (cf. [2])**.** *Let $\phi : X \to X$ be an automorphism of a smooth projective variety, and let $V \subset X$ be a subvariety with $\phi(V) = V$. Suppose that Y and Z are two Cohen-Macaulay subvarieties of X containing V and with $\operatorname{codim}_X Y + \operatorname{codim}_X Z = \operatorname{codim}_X V$. Then*

$$\{\mathrm{mult}_V(Y \cap \phi^n(Z))\}$$

is uniformly bounded in n, among n with V a component of $Y \cap \phi^n(Z)$ (i.e. those n for which this intersection has the expected dimension along V.

Proof. For n for which the intersection is dimensionally correct, the intersection multiplicity along V is defined by

$$\mathrm{mult}_V(Y, \phi^n(Z)) = \sum_{i \geq 0} (-1)^i \operatorname{len}_{\mathscr{O}_{X,V}} \operatorname{Tor}_i^{\mathscr{O}_{X,V}}(\mathscr{O}_{Y,V}, \mathscr{O}_{\phi^n(Z),V})$$

However, by the Cohen-Maculay assumption on Y and Z, all terms except that with $i = 0$ vanish, so that

$$\mathrm{mult}_V(Y, \phi^n(Z)) = \operatorname{len}_{\mathscr{O}_{X,V}} \mathscr{O}_{X,V}/(\mathscr{I}_{D,V} + \mathscr{I}_{\phi^n(D),V})$$

Since

$$\mathfrak{m}_V^k \subset \mathscr{I}_{D,V} + \mathscr{I}_{\phi^n(D),V}$$

for all nonzero n, we have

$$\mathrm{mult}_V(Y, \phi^n(Z)) = \mathrm{len}_{\mathscr{O}_{X,V}} \mathscr{O}_{X,V}/(\mathscr{I}_{D,V} + \mathscr{I}_{\phi^n(D),V}) \leq \mathrm{len}_{\mathscr{O}_{X,V}} \mathscr{O}_{X,V}/\mathfrak{m}_V^k,$$

a bound independent of n. This completes the proof. □

It may happen that there are some values of n for which $Y \cap \phi^n(Z)$ has larger than expected dimension, so that M_n does not compute the multiplicity. It does not seem easy to control this set.

Question 2. *Is the set of n for which $Y \cap \phi^n(Z)$ has larger than expected dimension along V a semilinear set?*

Example 7.1. The most naive version of Theorem 1.2 for which D is a subvariety of X with codimension greater than 1 does not hold.

Let $\phi : X \to X$ be an automorphism of a smooth fourfold, with x a fixed point of X. Suppose that Y is a codimension 2 subvariety of X, smooth at x. It is possible that for every value of n, the intersection of $\tilde{Y}$ and $\phi^n(\tilde{Y})$ is a distinct point

8. Acknowledgements

We are grateful to Serge Cantat and Paul Reschke for useful conversations. The first author was supported by NSF Research Training Group grant #DMS-1246844. The second author was supported by an NSF Postdoctoral Fellowship.

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