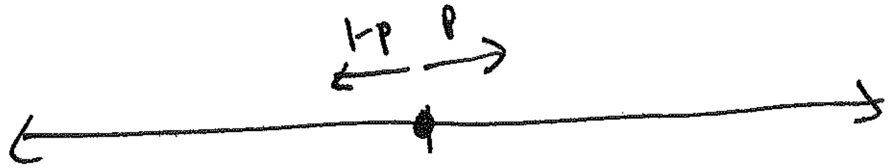
More random walks

$p = 1/2$ comes back.

What about other p?



Chance of return after n steps is u_n:

$$u_{2n} = \binom{2n}{n} p^n (1-p)^n$$

total number of possible paths.

$$\text{LRLLLRRLRR}$$

$\leftarrow$ $2n$ symbols, with equal number left & right.

$$(1-p)\cdot p\,(1-p)(1-p)(1-p)\,p\;p = p^n(1-p)^n$$

$$\frac{(2n)!}{n!\,n!} = \binom{2n}{n}$$

Does

$$\sum_{n=1}^{\infty} \binom{2n}{n} p^n (1-p)^n \text{ converge?}$$

$\frac{(2n)!}{n!\,n!}$

$$\sum_{n=1}^{\infty} \binom{2n}{n} p^n (1-p)^n$$

$(2n)^{2n} = 2^{2n} \cdot n^{2n}$

Stirling: $n! = n^n e^{-n} \sqrt{2\pi n}$

$(n^n)^2 = n^{2n}$

$$\sum_{n=1}^{\infty} \frac{(2n)^{2n} \cdot e^{-2n} \sqrt{2\pi \cdot 2n}}{(n^n e^{-n} \sqrt{2\pi n})^2} p^n (1-p)^n$$

$$= \sum_{n=1}^{\infty} 2^{2n} \frac{1}{\sqrt{n}} p^n (1-p)^n \frac{1}{\sqrt{\pi}}$$

$$= \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} (4p(1-p))^n \frac{1}{\sqrt{n}}$$

Ratio test: For a sum $\sum_{n=1}^{\infty} a_n$,

if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, converges.

(if > 1, diverges, if $= 1$, try another test)

If $a_n = (4p(1-p))^n \frac{1}{\sqrt{n}}$

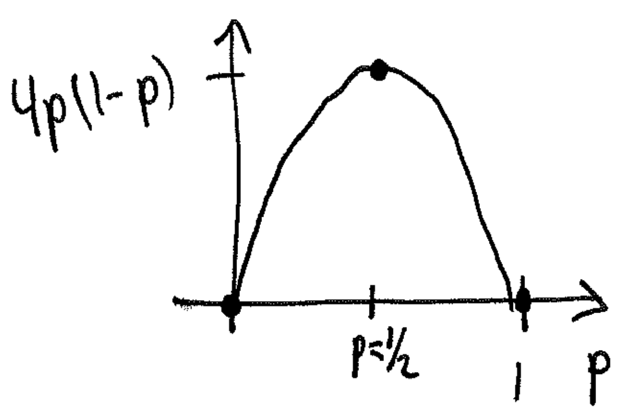
then

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{(4p(1-p))^{n+1}\frac{1}{\sqrt{n+1}}}{(4p(1-p))^n \frac{1}{\sqrt{n}}} = \lim_{n\to\infty} 4p(1-p)\sqrt{\frac{n}{n+1}}$$

$$= 4p(1-p).$$

If $4p(1-p) < 1 \Rightarrow \sum u_{2n}$ converges $\Rightarrow$ lost forever

But which values of p is that? When is $4p(1-p) < 1$?

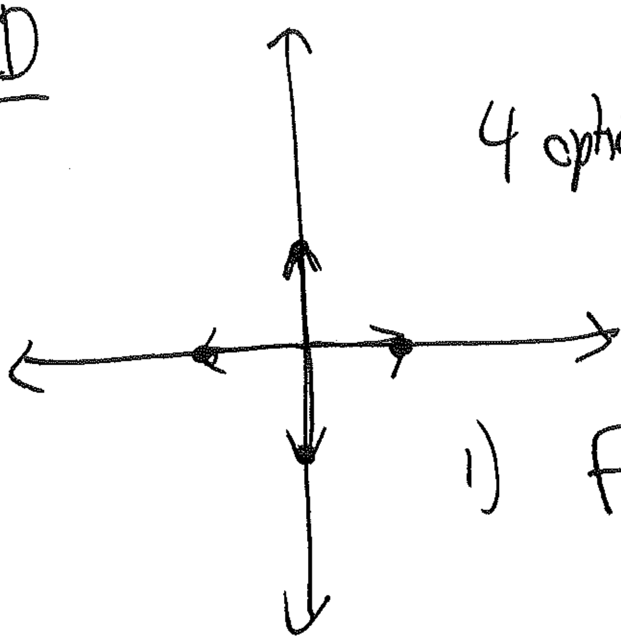


Conclusion about random walks:

$p = \frac{1}{2}$ must come back

any other p values: maybe not!

2D



4 options, equal prob.

1) Find u_{2n} = chance of being back after $2n$ steps

First: how many length $2n$ paths are there that end at $(0,0)$?

How many length $2n$ paths are ending at $(0,0)$ using exactly R rights? $(0 \le R \le n)$

$\rightarrow$ also R lefts

$\rightarrow$ $n-R$ ups

$\rightarrow$ $n-R$ downs.

u_{2n}^{R}

not probability, number of paths, sorry!

$$u_{2n}^{R} = \frac{(2n)!}{R!\,R!\,(n-R)!\,(n-R)!}$$

So

$$\#\text{paths} = \sum_{R=0}^{n} u_{2n}^{R} = \sum_{R=0}^{n} \frac{(2n)!}{R!\,R!\,(n-R)!\,(n-R)!}$$

Probability of following a particular path: $\left(\frac{1}{4}\right)^{2n}$

probability

↓

$$u_{2n} = \left(\frac{1}{4}\right)^{2n} \sum_{R=0}^{n} \frac{(2n)!}{R!\,R!\,(n-R)!\,(n-R)!} = \left(\frac{1}{4}\right)^{2n} \sum_{R=0}^{n} \frac{(2n)!\;n!\;n!}{R!\,R!\,(n-R)!\,(n-R)!\,n!\,n!}$$

$$= \left(\frac{1}{4}\right)^{2n} \sum_{R=0}^{n} \frac{(2n)!\;n!\;n!}{R!\,R!\,(n-R)!\,(n-R)!\,n!\,n!}$$

$$= \left(\frac{1}{4}\right)^{2n} \sum_{R=0}^{n} \binom{2n}{n}\binom{n}{R}\binom{n}{R} = \left(\frac{1}{4}\right)^{2n} \binom{2n}{n} \sum_{R=0}^{n} \binom{n}{R}^{2}$$

$$= \left(\frac{1}{4}\right)^{2n} \binom{2n}{n}^{2}$$

What is

$$\sum_{R=0}^{n} \binom{n}{R}^2$$

1 ⟶ 1

1 1 ⟶ 2

1 2 1 ⟶ 6

1 3 3 1 ⟶ 20

1 4 6 4 1 ⟶ 70

1 5 10 10 5 1 ⟶ 252

1 6 15 20 15 6 1

$$\sum_{R=0}^{n} \binom{n}{R}^2 = \binom{2n}{n}$$

(prove by induction, or generating functions, ...)

Does

$$\sum u_{2n} = \sum \left(\frac{1}{4}\right)^{2n} \binom{2n}{n}^2 \text{ converge?}$$

Stirling's formula! $\displaystyle \lim_{n\to\infty} \frac{n!}{n^n e^{-n} \sqrt{2\pi n}} = 1$

$$\left(\frac{1}{4}\right)^{2n} \binom{2n}{n}^2 \approx \left(\frac{1}{4}\right)^{2n} \frac{\left((2n)^{2n} e^{-2n} \sqrt{2\pi 2n}\right)^2}{\left(n^n e^{-n} \sqrt{2\pi n}\right)^4}$$

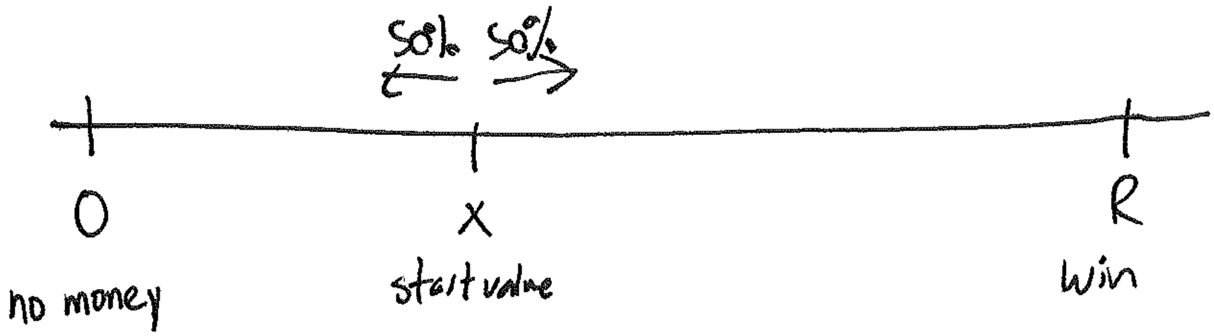
$$= \frac{1}{\pi n}$$

$\sum u_{2n} \approx \sum \frac{1}{\pi n}$ diverges!

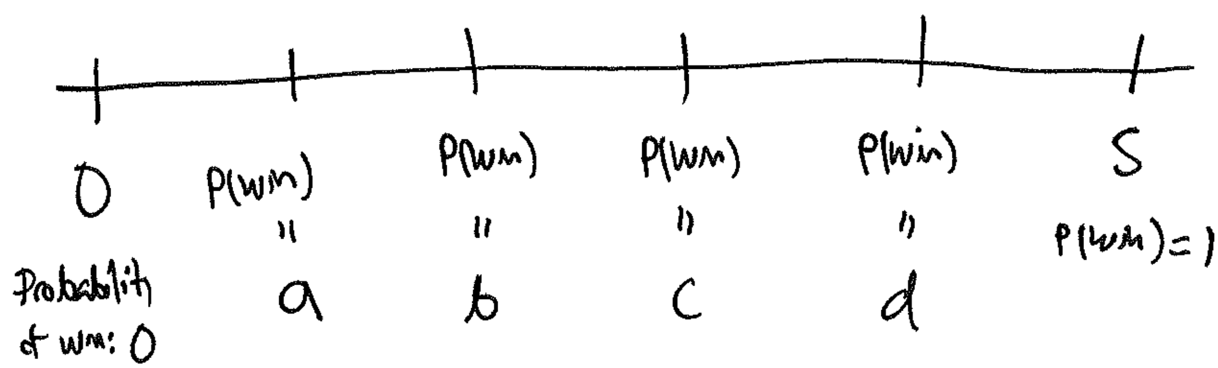
Random walk comes back infinitely many times!

(But it might take a long time!)

Gambler's ruin.



What's the chance the gambler gets to R before getting to 0?



Solve for a, b, c, d!

But we need equations!

$$a = \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot b$$

$$b = \frac{1}{2}\cdot a + \frac{1}{2}\cdot c$$

$$c = \frac{1}{2}\cdot b + \frac{1}{2}\cdot d$$

4 eqns

4 vars

$$d = \frac{1}{2}\cdot c + \frac{1}{2}\cdot 1$$

$$\left.\begin{aligned} 2a - b &= 0 \\ 2b - a - c &= 0 \\ 2c - b - d &= 0 \\ 2d - c - 1 &= 0 \end{aligned}\right\}$$

$$\left(\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 1 \end{array}\right)$$

$a = 1/5$ $b = 2/5$ $c = 3/5$ $d = 4/5$