Today: Inequalities.

I'm going to follow: notes by Charles Martin "A ~~Primer~~ Primer on Inequalities"

also more notes by Kiran Kedlaya

Four important inequalities:

0) $x^2 \geq 0$ for any x

1) Triangle inequality

2) AM / GM inequality

3) Cauchy-Schwarz inequality

4) Jensen's inequality

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1) Problem: Prove that for any real a, b,

$$\frac{a^2+b^2}{2} \geq ab.$$

Sol:
$$(a-b)^2 \geq 0$$
$$a^2-2ab+b^2 \geq 0$$
$$a^2+b^2 \geq 2ab$$
$$\frac{a^2+b^2}{2} \geq ab \quad \checkmark.$$

Try this:

Prove that $a^2+b^2+c^2 \geq ab+bc+ca$ any a,b,c.

Sol: $(a-b)^2+(b-c)^2+(c-a)^2 \geq 0$

$2a^2+2b^2+2c^2-2ab-2bc-2ca \geq 0$

$a^2+b^2+c^2 \geq ab+bc+ca$.

AM / GM / HM inequality: arithmetic mean / geometric mean.

Suppose $a_1, \dots, a_n$ are positive.

arithmetic mean: $\dfrac{a_1+a_2+\cdots+a_n}{n}$

geometric mean: $\sqrt[n]{a_1 a_2 \cdots a_n}$

harmonic mean: $\left(\dfrac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}{n}\right)^{-1}$

AM/GM $\dfrac{a_1+\cdots+a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \geq$

Harmonic mean

You have to paint 1000 ft of fence.

You can paint ~~40~~ it in 4 hours

Your friend can do it in 5 hours.

How long if you work together?

You: $250 \text{ ft}/\text{hr} = \frac{1000}{4}$

Friend: $200 \text{ ft}/\text{hr} = \frac{1000}{5}$

Together: $450 = 1000\left(\frac{1}{4} + \frac{1}{5}\right)$

Total: $\dfrac{1000}{1000\left(\frac{1}{4}+\frac{1}{5}\right)} = \left(\frac{1}{4}+\frac{1}{5}\right)^{-1}$

like the harmonic mean.

Proof of ~~an~~ AM/GM.

Induction. Base case. $n=1$:

$$\frac{a_1}{1} \geq (a_1)^{1/1} \quad \checkmark$$

Base case $n=2$:

$$\frac{a_1+a_2}{2} \geq \sqrt{a_1 a_2}$$

Pf. Let $a=\sqrt{a_1}$, $b=\sqrt{a_2}$. By first inequality we did:

$$\frac{a^2+b^2}{2} \geq ab$$

$$\frac{a_1+a_2}{2} \geq \sqrt{a_1 a_2} \quad \checkmark$$

$\ddots$

Crazy induction:

Let's prove that AMGM with n things implies it with $2n$ things. Assume it works with n things.

Let $a_1, \ldots, a_{2n}$ be positive.

n^{th} root

$$\left.\begin{aligned} \frac{a_1+\cdots+a_n}{n} &\geq (a_1\cdots a_n)^{1/n} \\ \frac{a_{n+1}+\cdots+a_{2n}}{n} &\geq (a_{n+1}\cdots a_{2n})^{1/n} \end{aligned}\right] \text{ by inductive hypothesis.}$$

$$\begin{aligned} a_1+\cdots+a_n+a_{n+1}+\cdots+a_{2n} &\geq n(a_1\cdots a_n)^{1/n}+n(a_{n+1}\cdots a_{2n})^{1/n} \\ &= 2n\,\frac{(a_1\cdots a_n)^{1/n}+(a_{n+1}\cdots a_{2n})^{1/n}}{2} \\ &\geq 2n\sqrt{(a_1\cdots a_n)^{1/n}(a_{n+1}\cdots a_{2n})^{1/n}} \\ &= 2n(a_1\cdots a_{2n})^{1/2n} \end{aligned}$$

so

$$\frac{a_1+\cdots+a_{2n}}{2n} \geq (a_1\cdots a_{2n})^{1/2n}.$$

This proves AM/GM, but only when n is a power of 2!

Next: Prove that AM/GM for n things implies AM/GM for $n-1$ things!

This will cover all cases:

Suppose we have $a_1, \ldots, a_{n-1}$.

Let $b = (a_1 \cdots a_{n-1})^{1/n-1}$

Use AMGM for these n things!

$$\frac{a_1 + a_2 + \cdots + a_{n-1} + b}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_{n-1} b} = b$$

$$a_1 + a_2 + \cdots + a_{n-1} + b \geq nb$$

$$a_1 + \cdots + a_{n-1} \geq (n-1)b$$

$$\frac{a_1 + \cdots + a_{n-1}}{n-1} \geq \overbrace{(a_1 \cdots a_{n-1})^{1/n-1}}^{=b}$$

$0 \to$ 1

$1 \to$ 1 1

$2 \to$ 1 2 1

$3 \to$ 1 3 3 1

$4 \to$ 1 4 6 4 1

$5 \to$ 1 5 10 10 5 1

$6 \to$ 1 6 15 20 15 6 1

a_1 a_2 $\cdots$ a_{n+1}

Question:

If we multiply together everything in n^{th} row, how big could it be?

(Give me an upper bound with no factorials!)

trivia:

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

"Stirling's formula"

Problem: a,b,c positive, prove

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} \geq a+b+c$$

$$\underbrace{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n-1}}_{\text{product of things in row } n,\text{ leaving out the 1's}} \le \left(\frac{\binom{n}{1}+\cdots+\binom{n}{n-1}}{n-1}\right)^{n-1}$$

$$\shortparallel$$

$$\left(\frac{2^n-2}{n-1}\right)^{n-1}$$

Why is sum 2^n?

$\to$ number of ways to choose a subset of $\{1,\dots,n\}$ is 2^n

$$\to \quad = \binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n.$$

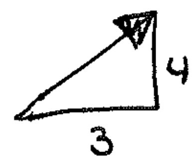
$$2^n=(1+1)^n=\sum_{j=0}^{n}1^j\cdot 1^{n-j}\binom{n}{j}=\sum_{j=0}^{n}\binom{n}{j}.$$

Cauchy - Schwarz - Bunyakovsky inequality

a) Say $V = (v_1, \dots, v_n)$
$W = (w_1, \dots, w_n)$ are vectors (allowed negative numbers)

$$(v_1 w_1 + v_2 w_2 + \cdots + v_n w_n)^2 \leq (v_1^2 + \cdots + v_n^2)(w_1^2 + \cdots + w_n^2)$$

b) If f, g are two functions



$|v| = \sqrt{3^2 + 4^2}$

$$\int_{-\infty}^{\infty} |fg|\, dx \leq \left(\int_{-\infty}^{\infty} |f|^2\, dx\right)^{1/2} \left(\int_{-\infty}^{\infty} |g|^2\, dx\right)^{1/2}$$

(These are actually both cases of the same general theorem! We'll do it when we cover inner product space.)

<u>Pf.</u> of a) at least when $n=2$.

It's saying $(v \cdot w)^2 \leq |v|^2 |w|^2$.

So we need to prove $\underbrace{|v \cdot w|}_{\text{absolute val}} \leq \underbrace{|v|}_{\text{length}} \cdot \underbrace{|w|}_{\text{length}}$

Why is

$|v \cdot w| \leq |v| \cdot |w|$?

Well, $v \cdot w = |v| \cdot |w| \cos \theta$

Since $-1 \leq \cos \theta \leq 1$

we get $|v \cdot w| \leq |v| \cdot |w|$ as needed.

Let $a_1, \dots, a_n$ be positive. Prove that

$$(a_1 + \dots + a_n)\left(\frac{1}{a_1} + \dots + \frac{1}{a_n}\right) \geq n^2$$

Let $v_i = \sqrt{a_i}$

$w_i = 1/\sqrt{a_i}$

CS:

$$\left(\sqrt{a_1}\frac{1}{\sqrt{a_1}} + \dots + \sqrt{a_n}\frac{1}{\sqrt{a_n}}\right)^2 \leq \left(\sqrt{a_i}^2 + \dots \sqrt{a_n}^2\right)\left(\frac{1}{\sqrt{a_1}^2} + \dots + \frac{1}{\sqrt{a_n}^2}\right)$$

If all a_i are 1, then both sides are equal

$$n^2 \leq (a_1 + \dots + a_n)\left(\frac{1}{a_1} + \dots + \frac{1}{a_n}\right)$$

Consider a Taylor series

$$f(x) = \sum_{k=1}^{\infty} a_k x^k$$

Prove that if $0 \le x < 1$.

$$\sum_{k=1}^{\infty} a_k x^k \le \frac{1}{\sqrt{1-x^2}} \left(\sum_{k=1}^{\infty} a_k^2 \right)^{1/2}$$

Pf. $v_k = a_k$

$w_k = x^k$

$$\frac{1}{1-x^2}$$

CS: $$\left(\sum_{k=0}^{\infty} a_k x^k \right)^2 \le \left(\sum_{k=0}^{\infty} a_k^2 \right) \left(\sum_{k=0}^{\infty} x^{2k} \right)$$

so

$$\sum_{k=0}^{\infty} a_k x^k \le \frac{1}{\sqrt{1-x^2}} \left(\sum_{k=0}^{\infty} a_k^2 \right)^{1/2}$$