Today: Linear maps, determinants

Let $M = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}$ $\quad N = \begin{pmatrix} -2 & 0 \\ 3 & 1 \end{pmatrix}$.

What are the linear transformations $f : \mathbb{R}^2 \to \mathbb{R}^2$

$M \rightsquigarrow f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = ?$ $\quad N \rightsquigarrow g : \mathbb{R}^2 \to \mathbb{R}^2$

What's the composite function $(f \circ g)\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)$? $= f\left(g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)\right)$

What's MN?

An $\quad f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ -x + 4y \end{pmatrix}$

$g\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} -2 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x \\ 3x + y \end{pmatrix}$

$(f\circ g)\left(\begin{pmatrix}x\\y\end{pmatrix}\right)$? Take $g\left(\begin{pmatrix}x\\y\end{pmatrix}\right)=\begin{pmatrix}-2x+0y\\3x+y\end{pmatrix}$

Plug in to $f\left(\begin{pmatrix}x\\y\end{pmatrix}\right)$:

$$\begin{pmatrix}2(-2x+0y)+3(3x+y)\\-1(-2x+0y)+4(3x+y)\end{pmatrix}=\begin{pmatrix}5x+3y\\14x+4y\end{pmatrix}$$

$$\begin{pmatrix}2&3\\-1&4\end{pmatrix}\begin{pmatrix}-2&0\\3&1\end{pmatrix}=\begin{pmatrix}5&3\\14&4\end{pmatrix}\longleftrightarrow\begin{pmatrix}5&3\\14&4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

Matrix for composition is product of matrices!

Adv Topics 2 (Lesieutre)
September 2, 2021

$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$

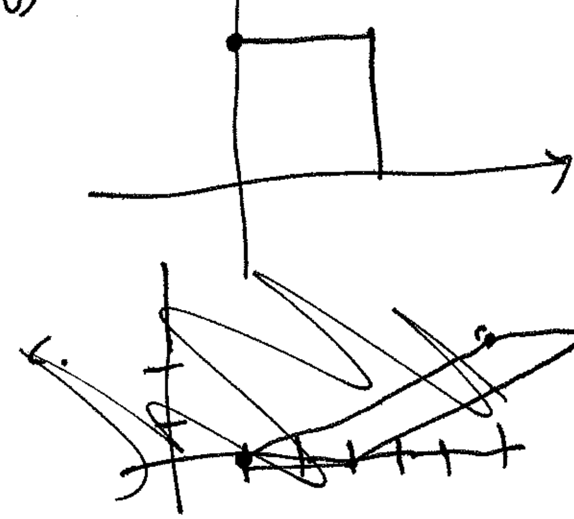
Problem 1. For each of the following matrices T, choose a couple sample vectors $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and compute $T\mathbf{v}$. What does the matrix do to a vector, geometrically? What does it do to the unit square?

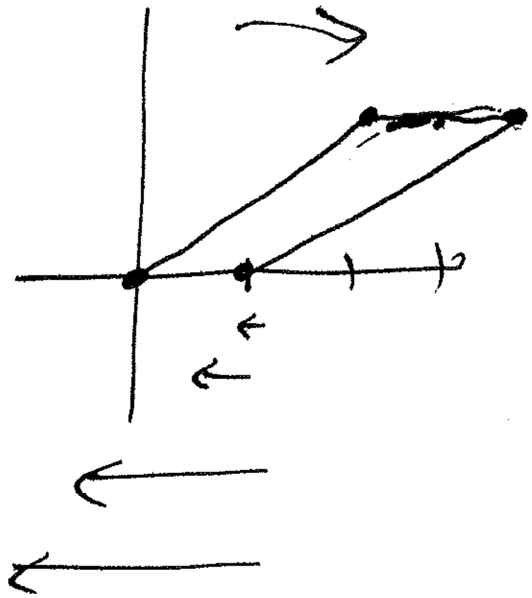
6 a) $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

a) Stretch horizontally by 3, vertically by 2. | area multiplies by 6

1 b) $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

b) area doesn't change!





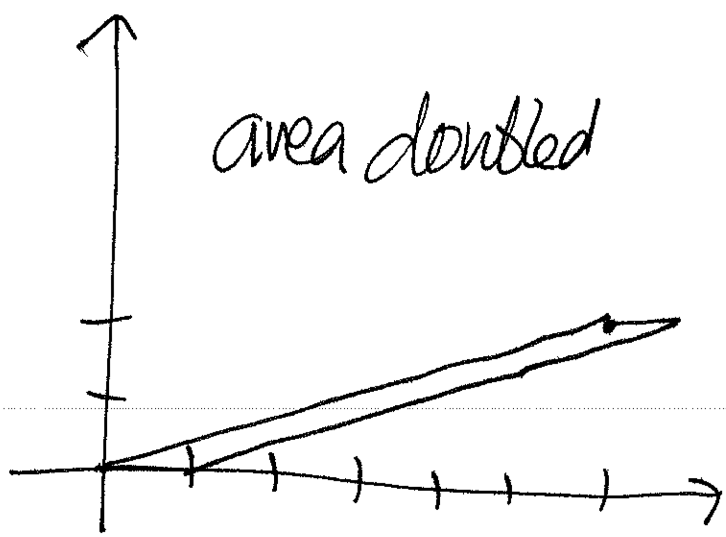
2 c) $\begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}$

−1 d) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

−1 e) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

0 f) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

c)



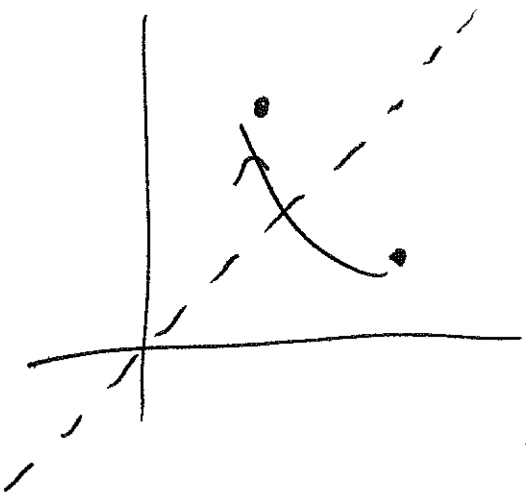
area doubled

g) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

h) $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

i) $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

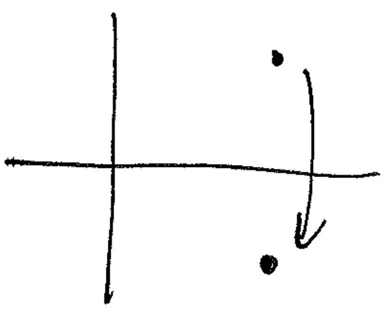
d) Reflects over $y=x$



area doesn't change!

but reverses orientation

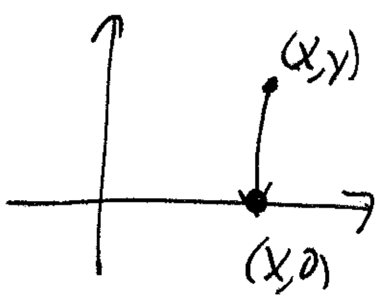
e



flip over x-axis.

Same area, orientation reversed

f) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$



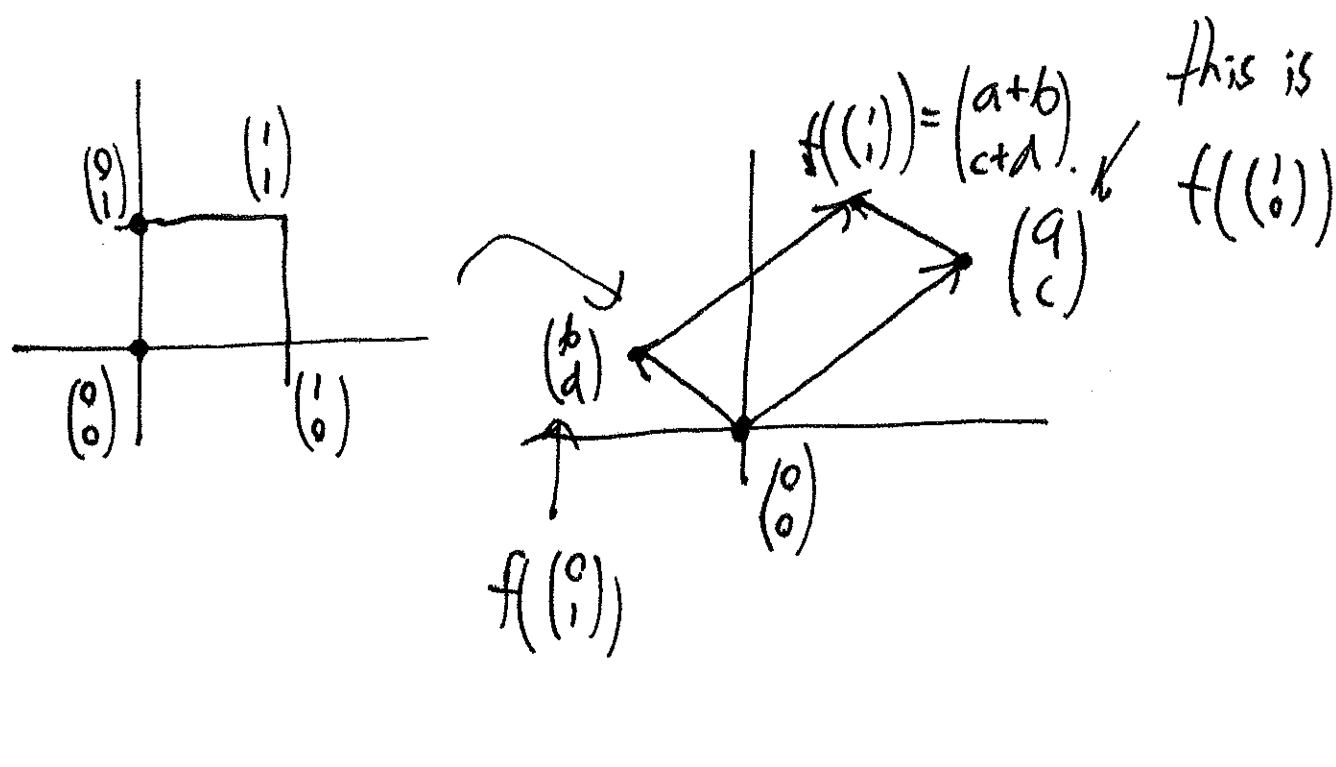
multiplies area by 0.

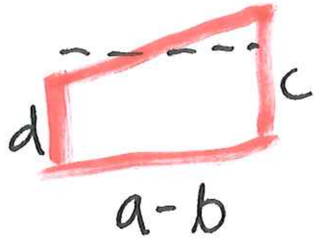
If $f: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, corresponding to a 2×2 matrix M, then f rescales areas by a factor of $\det(M) = ad-bc$

(If this is negative, it means f reverses orientations).

Why does $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ scale areas by $ad-bc$?

What happens to unit square?

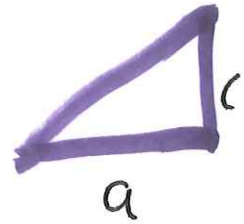


area of 

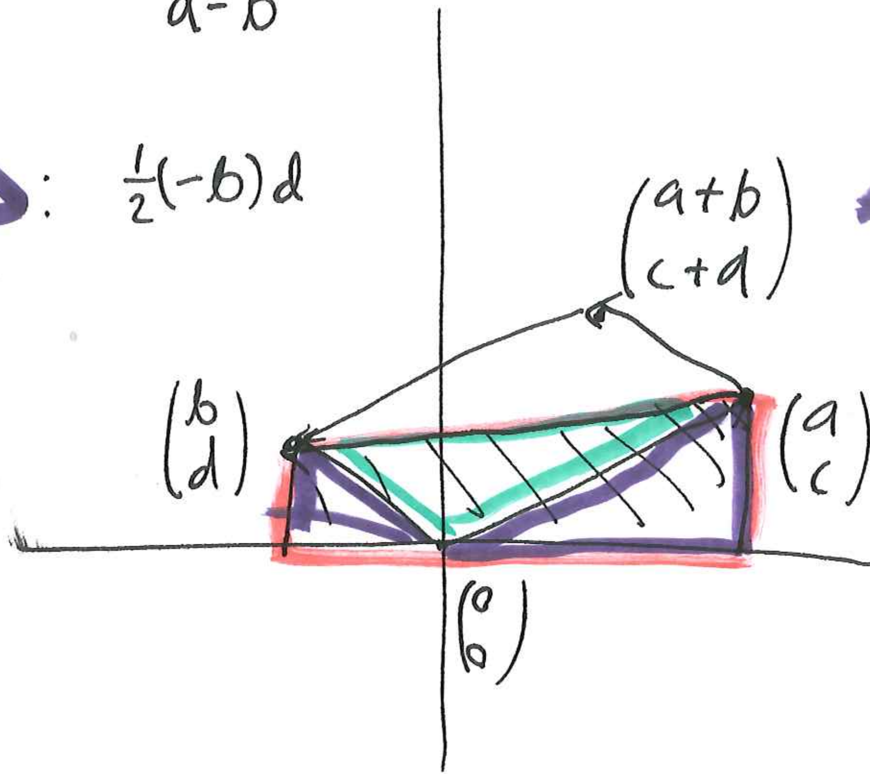
$\rightsquigarrow (a-b)\left(\frac{c+d}{2}\right)$

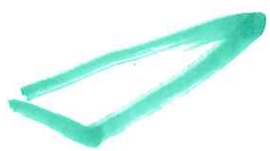
area of 

: $\frac{1}{2}(-b)d$



$A=\frac{1}{2}ac$



Area of  $=(a-b)\left(\frac{c+d}{2}\right)-\frac{1}{2}(-bd)-\frac{1}{2}ac$

$=\frac{1}{2}\left[(ac+ad-bc-bd)+bd-ac\right]$

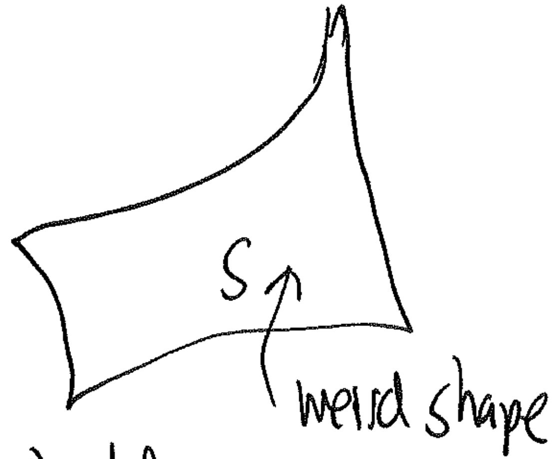
$=\frac{1}{2}(ad-bc).$

Area of 

is $\boxed{ad-bc}$

Jacobian determinant

To compute



$$\iint_S f(x,y)\, dA$$

use multivariable coordinate change!

Imagine you can parametrize the region:

new variables s, t $\quad a \le s \le b$

$\quad c \le t \le d$

and functions

$x(s,t)$ and $y(s,t)$ such that $\begin{matrix} x(s,t) \\ y(s,t) \end{matrix}$

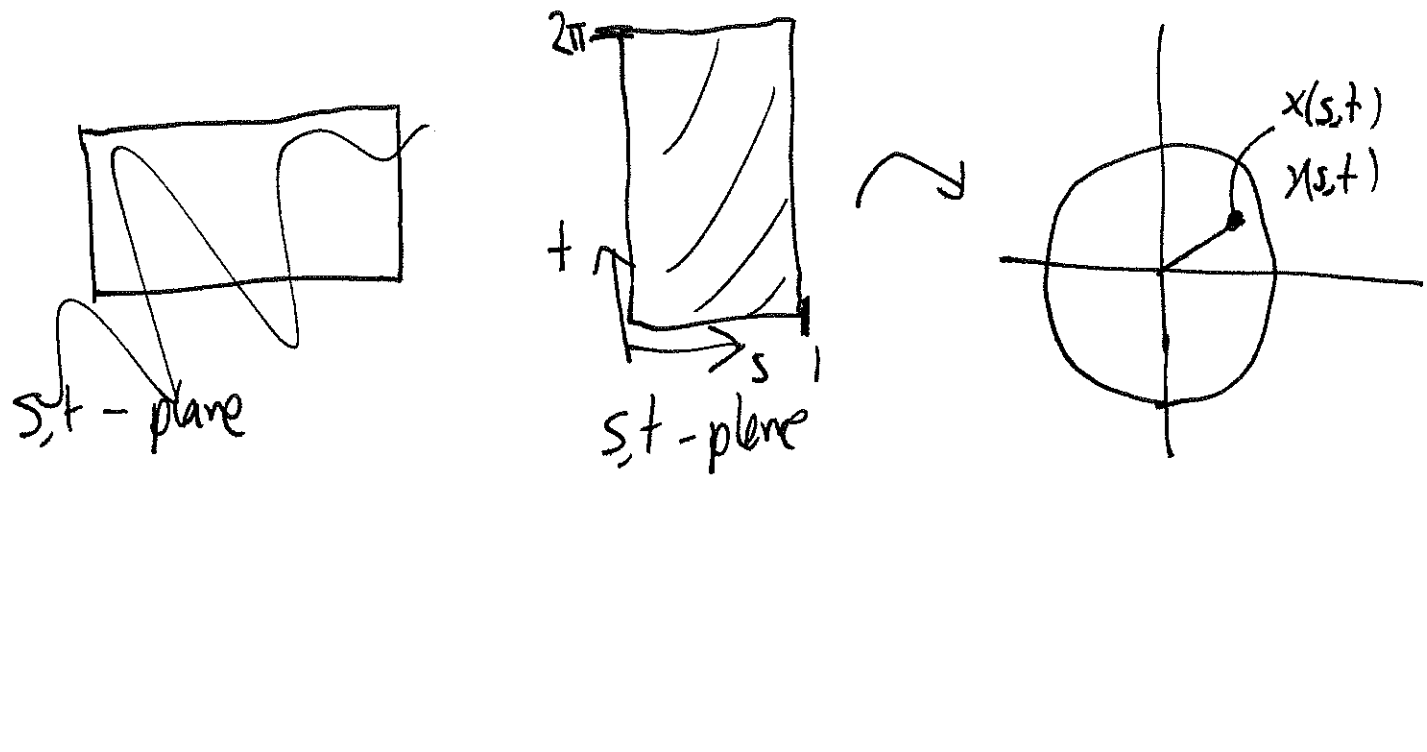
covers our region S when we vary s and t.

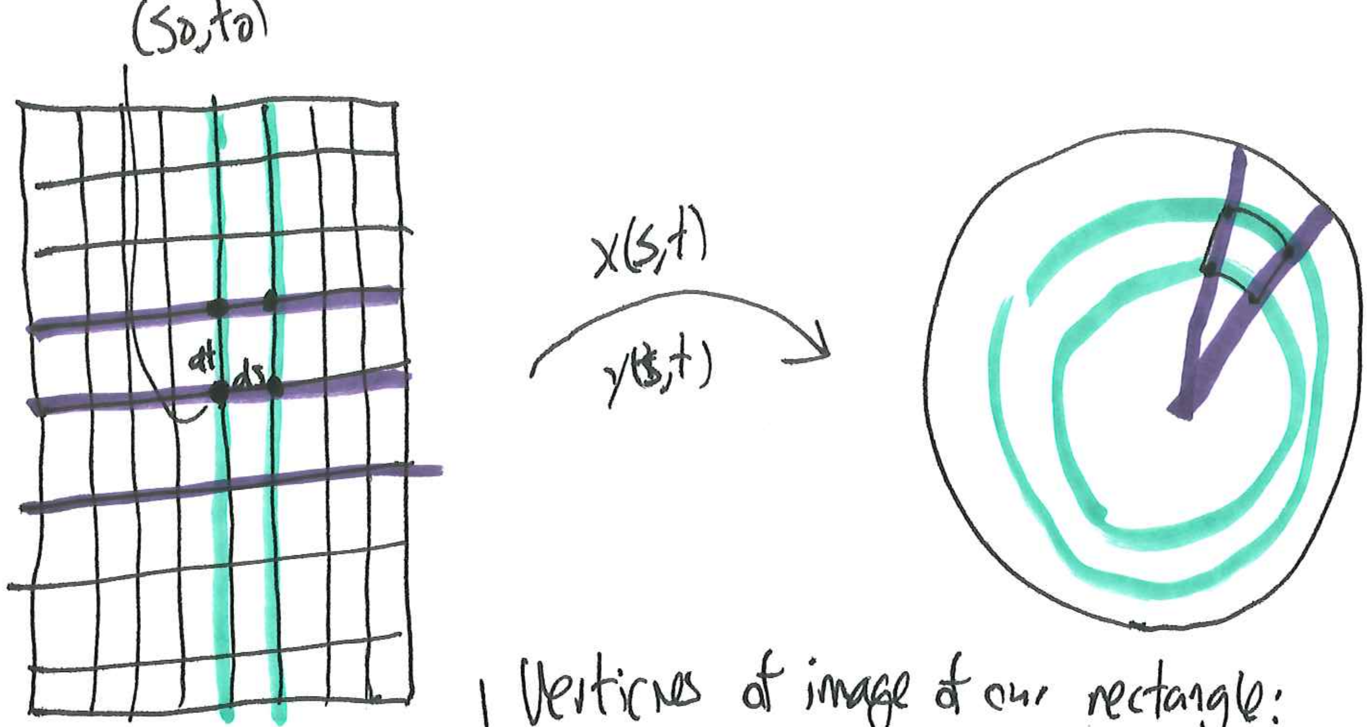
What if S is unit disk? How to parametrize S?

Think of s,t as being new coordinates, and $x(s,t)$ $y(s,t)$ tell you how to convert these new coords into regular old x, y values.

$0 \le s \le 1$ ← (polar "r")　　$x(s,t) = s\cos t$

$0 \le t \le 2\pi$ ← (polar "θ")　　$y(s,t) = s\sin t$





$0 \leq s \leq 1$

$0 \leq t \leq 2\pi$

Vertices of image of our rectangle:

$(s_0, t_0) \longmapsto (x(s_0,t_0), y(s_0,t_0))$

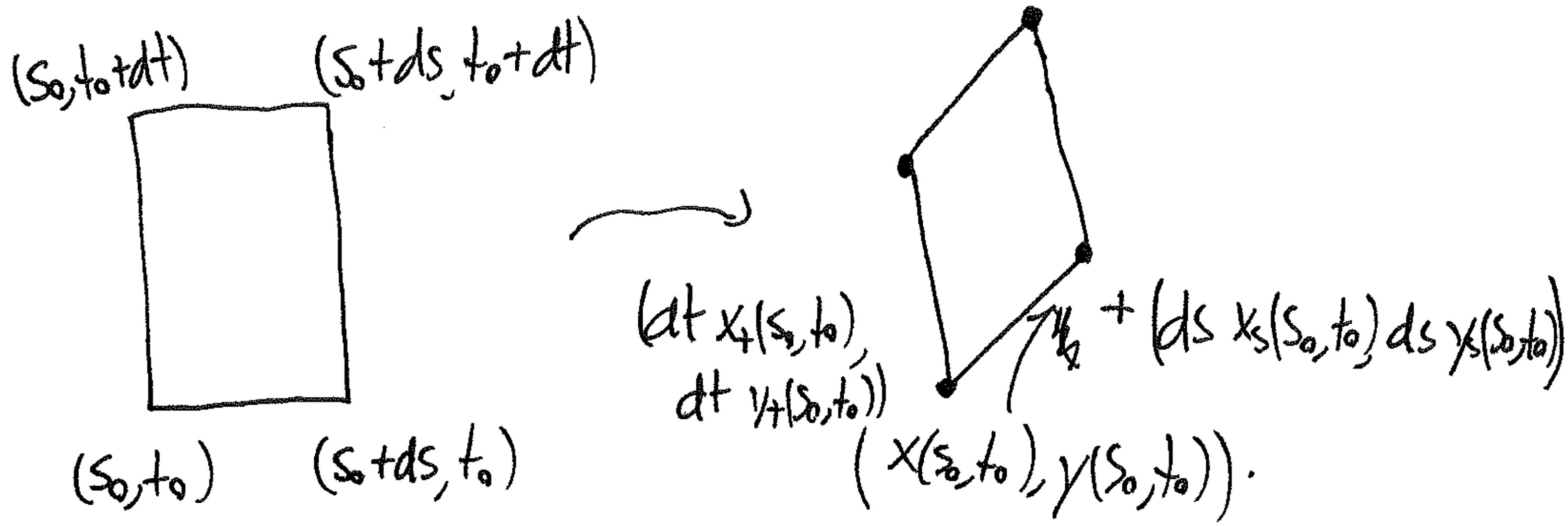
$(s_0+ds, t_0) \longmapsto (x(s_0+ds,t_0), y(s_0+ds,t_0))$

$\approx \Big(x(s_0,t_0) + ds \cdot \underbrace{x_s(s_0,t_0)}_{\frac{\partial x}{\partial s}},$

$y(s_0,t_0) + ds \cdot y_s(s_0,t_0)\Big)$

$(s_0, t_0+dt) \longmapsto (x(s_0,t_0+dt), y(s_0,t_0+dt))$

$\approx \big(x(s_0,t_0) + dt\, x_t(s_0,t_0), y(s_0,t_0) + dt\, y_t(s_0,t_0)\big)$



$$\begin{array}{cc} 1^{st}\text{ leg} & 2^{nd}\text{ leg} \end{array}$$

$$\begin{pmatrix} ds\, x_s(s_0,t_0) & dt\, x_t(s_0,t_0) \\ ds\, y_s(s_0,t_0) & dt\, y_t(s_0,t_0) \end{pmatrix}$$

$$\| \quad \begin{pmatrix} x_s(s_0,t_0) & x_t(s_0,t_0) \\ y_s(s_0,t_0) & y_t(s_0,t_0) \end{pmatrix} \begin{pmatrix} ds \\ dt \end{pmatrix}$$

area of image mesh is:

$$\underbrace{(x_s y_t - x_t y_s)}_{\text{Jacobian determinant.}}(s_0,t_0) \cdot ds \cdot dt$$

To integrate over weird shape.

$$\iint_S f(x,y)\,dx\,dy = \int_{s=a}^{b}\int_{t=c}^{d} f(x(s,t),y(s,t))$$

$$(x_s y_t - x_t y_s)\,ds\,dt$$

<u>Area of circle</u> New coords: $0 \le s \le 1$
$0 \le t \le 2\pi$

$x(s,t) = s\cos t$
$y(s,t) = s\sin t$

$x_s = \cos t$
$x_t = -s\sin t$
$y_s = \sin t$
$y_t = s\cos t$

$$x_s y_t - x_t y_s = (\cos t)(s\cos t) - (-s\sin t)(\sin t)$$
$$= s.$$

$$A = \iint_{\text{circle}} 1\,dx\,dy = \int_{s=0}^{1} \int_{t=0}^{2\pi} 1 \quad s \,\cancel{ds\,dt}\, dt\, ds$$

(that's $r\,dr\,d\theta$!)

$$= \int_{s=0}^{1} 2\pi s\, ds = \pi s^2 \Big|_0^1 = \boxed{\pi}$$