Optimization

1) One more problem like last time

2) "Constrained optimization":   Lagrange multipliers

   maximize $x^2+3xy+7$

   subject to constraint: $x^2+y^2=25$

3) Linear programming

4) Gradient descent

Example:

Max/min $f(x,y,z)=2x^2+y^2+z^2-2z+xz$

on a cylinder of radius 3 and height 5

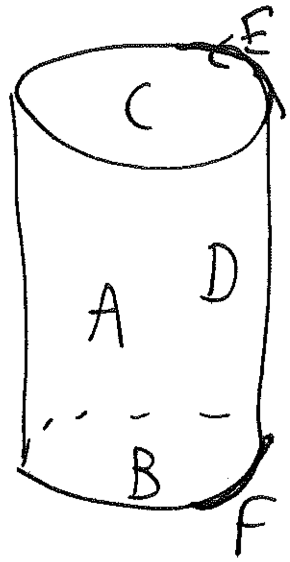
~~centered on (0,0,0)~~

with base in xy-plane around origin

Things to check:

A - Inside cylinder

critical points: $f_x = f_y = f_z$



B - Bottom face

parametrize in polar

or: plug in $z=0$

C - Top face

D - Outer ~~edge~~ face

parametrize

E - Top edge

parametrize with one variable

F - Bottom edge

parametrize with one variable

$x^2+y^2 \le 9$

$0 \le z \le 5$

Inside the cylinder:

Look for critical points

$f_x = 4x + z \rightarrow z = -4x$

$f_y = 2y$ ~~[illegible]~~

$f_z = 2z - 2 + x \qquad -8x - 2 + x = 0$

$7x = 2$

$x = -2/7,\ y = 0,\ z = +8/7$

Candidate points.

~~f~~(x,y,z)	f(x,y,z)
$(-2/7, 0, 8/7)$	$-8/7$

Bottom face

Parametrize it!

$x = r\cos\theta$

$y = r\sin\theta$

$z = 0$

$0 \le \theta \le 2\pi$

$0 \le r \le 3$

$f(x,y,z) =$

$2x^2 + y^2 + z^2 - 2z + xz$

$= 2r^2\cos^2\theta + r^2\sin^2\theta + 0 - 0 + 0$

$= r^2 + r^2\cos^2\theta = (r^2+1)\cos^2\theta$

Max on outside face of cylinder.

Parametrize it by θ, z:

$x = 3\cos\theta$

$y = 3\sin\theta$

$z = z$

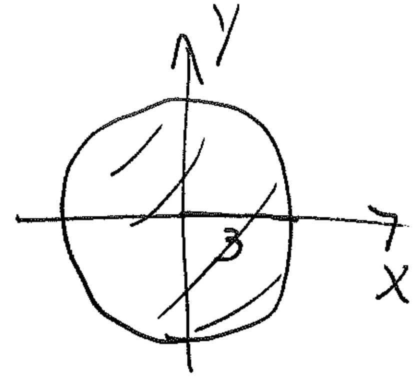
plug in as x,y,z.

find θ, z to max/min function.

$f(x,y,z) = 2x^2 + y^2 + z^2 - 2z + xz$

Bottom face:

Plug in $z=0$



what x,y make max/min?

$f(x,y) = 2x^2 + y^2$.

$f_x = 4x$ $\Rightarrow$ $x=0,\ y=0,\ z=0.$

$f_y = 2y$

Edge of bottom face:

$(0,0,0)$

$x^2+y^2=9$, substitute in:

$x=-3 \Rightarrow (-3,0,0)$

$f(x,y) = 9+x^2$. max at $x=3 \Rightarrow (3,0,0)$

min at $x=0 \Rightarrow (0,3,0)$

$(0,-3,0)$

Lagrange multipliers

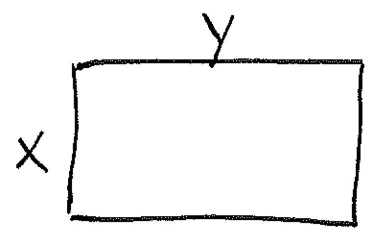
Suppose you want to maximize/minimize $f(x,y)$ subject to a constraint $g(x,y)=c$.

Then the max & min satisfy

$$\nabla f(x,y) = \overset{\text{scalar}}{\lambda}\, \nabla g(x,y).$$

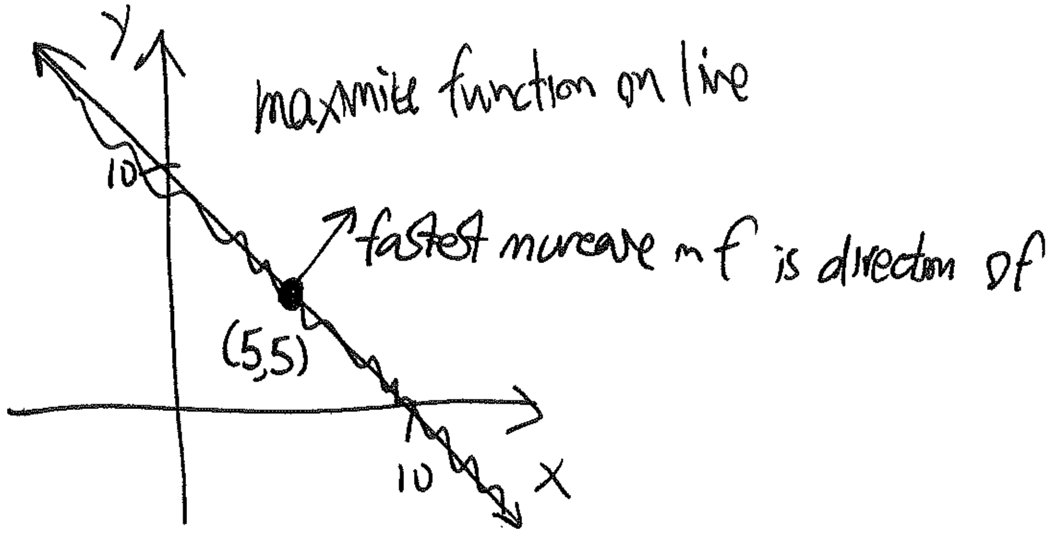
∇f parallel to ∇g!

Ex. Find a rectangle of perimeter 20 with maximize area.



maximize $f(x,y) = xy$

constraint: $g(x,y) = 2x + 2y = 20.$



Solve: $\nabla f = \lambda \nabla g$ and $g(x,y) = c$.

$$\langle y, x \rangle = \lambda \langle 2, 2 \rangle \qquad 2x + 2y = 20.$$

Three equations:

$$\left.\begin{aligned} y &= 2\lambda \\ x &= 2\lambda \\ 2x + 2y &= 20 \end{aligned}\right]$$

Solve:

$x = y$ so

$4x = 20$ so

$\boxed{x = 5}$ $\boxed{y = 5}$

$\boxed{\lambda = 5/2}$

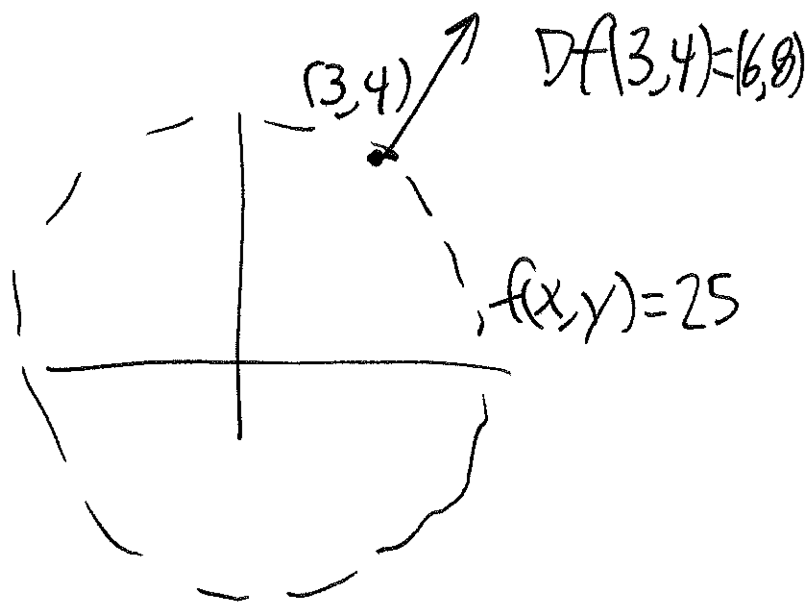
Two ways to think about Df:

1) Direction of fastest increase of f

2) Perpendicular to level curves of f:

$f(x,y) = x^2 + y^2$

$Df = (2x, 2y)$



Maximize:

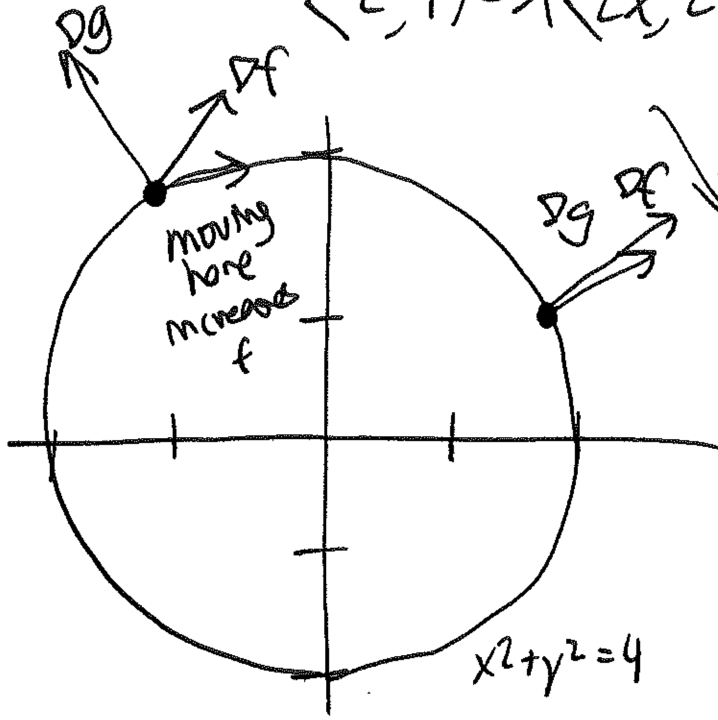
$f(x,y)=2x+y$

Subject to:

$\underbrace{x^2+y^2=4}_{g(x,y).}$

$\nabla f=\lambda \nabla g$

$\langle 2,1\rangle=\lambda\langle 2x,2y\rangle$



$$\left.\begin{aligned} 2&=2\lambda x\\ 1&=2\lambda y\\ x^2+y^2&=4\end{aligned}\right]$$

$\frac{2}{x}=\frac{1}{y} \quad 2y=x$

$(2y)^2+y^2=4$

$y^2=4/5 \quad y=\pm 2/\sqrt{5}$

$x=\pm 4/\sqrt{5}$