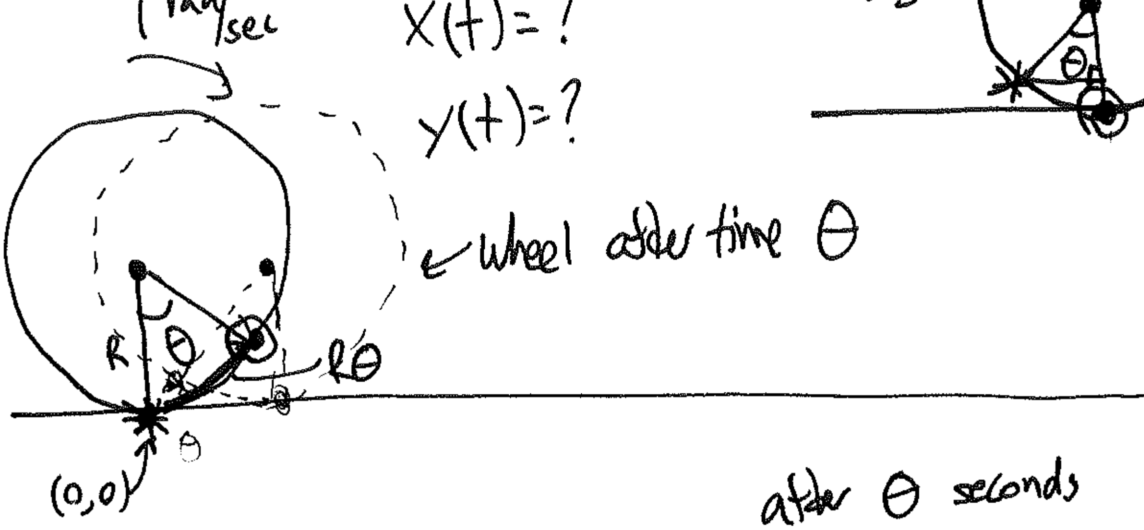
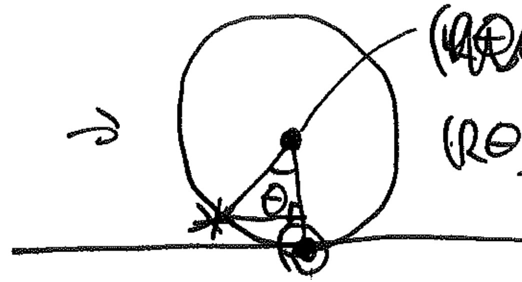
# Cycloid

$x(t) = ?$

$y(t) = ?$




after θ:



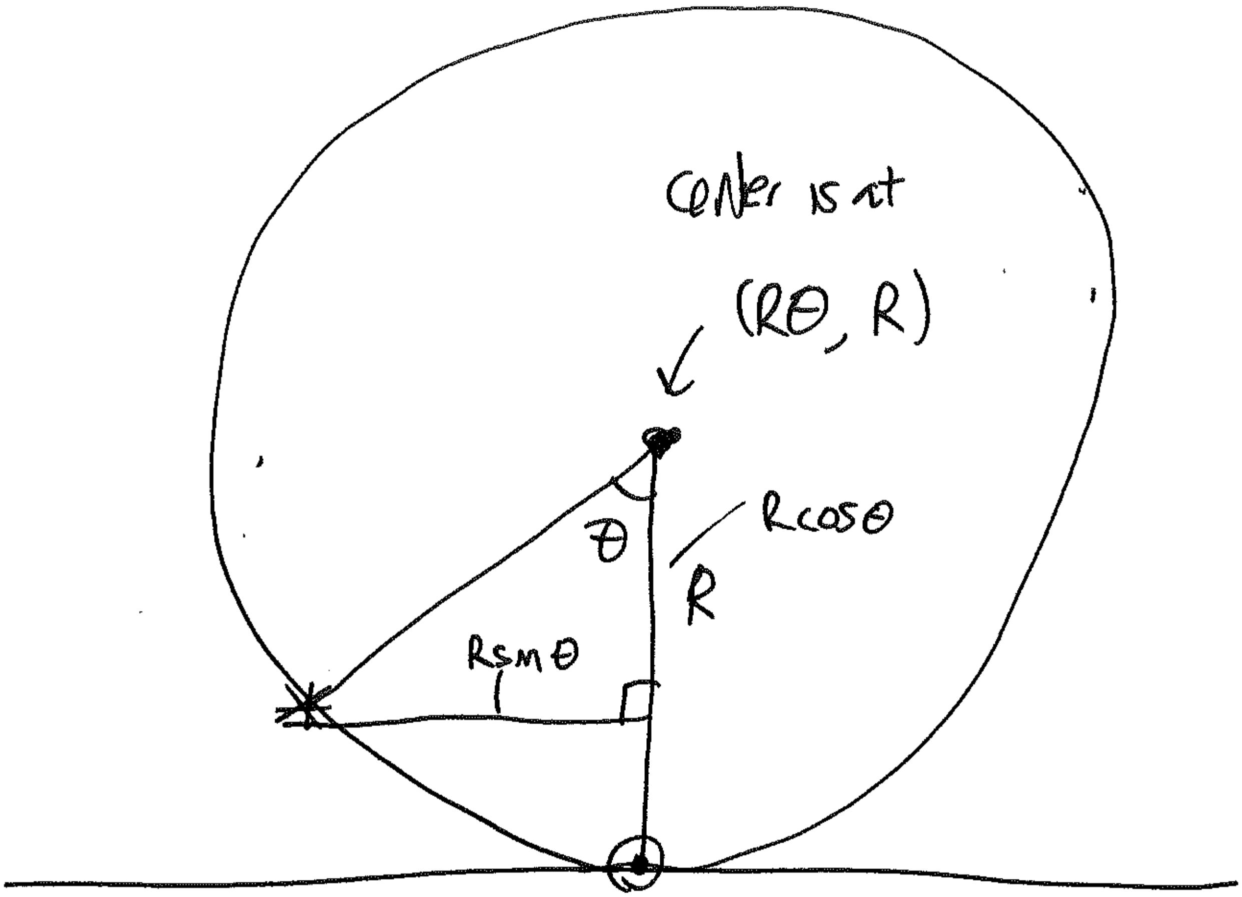

Center is at $(R\theta, R)$

after θ seconds

after we roll an angle θ,

Where is the middle of the wheel?

- moves forward by $R\theta$
- height is still $R$




$$x(\theta) = R\theta - R\sin\theta = R(\theta - \sin\theta)$$
$$y(\theta) = R - R\cos\theta = R(1 - \cos\theta)$$

"cycloid"

Last time:

We used Euler-Lagrange eqn to get a differential equation for brachistochrone problem!

It turns out an upside-down cycloid solves this eqn!

(movie from Wikipedia)

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 4 \qquad \text{(positive integers)}$$

$$a(c+a)(a+b) + b(b+c)(a+b) + c(b+c)(c+a)$$

$$= 4(b+c)(c+a)(a+b)$$

$$a^3 + b^3 + c^3 - 3(a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2) - 5abc = 0.$$

Degree 3 equation!

Remember completion of squares:

To solve $x^2 + bx + c = 0$

~~to~~ rewrite as $\left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right) = 0.$

If we may a new variable $\tilde{x} = x + \frac{b}{2}$,

you get an equation with no "$\tilde{x}$" term:

$$\tilde{x}^2 + \left(c - \frac{b^2}{4}\right) = 0.$$

Substitute a la completion of squares:

$$x = \frac{-28(a+b+2c)}{6a+6b-c}$$

$$y = \frac{364(a-b)}{6a+6b-c}$$

our equation becomes

$$y^2 = x^3 + 109x^2 + 224x.$$

If you can find a solution $(x,y)$ to that equation, you can get back $(a,b,c)$:

$$a = \frac{56-x+y}{56-14x} \qquad b = \frac{56-x-y}{56-14x} \qquad c = \frac{-28-6x}{28-7x}.$$

This might give rational number, but in that case, clear denominators of $a, b, c$ and get integer solution.

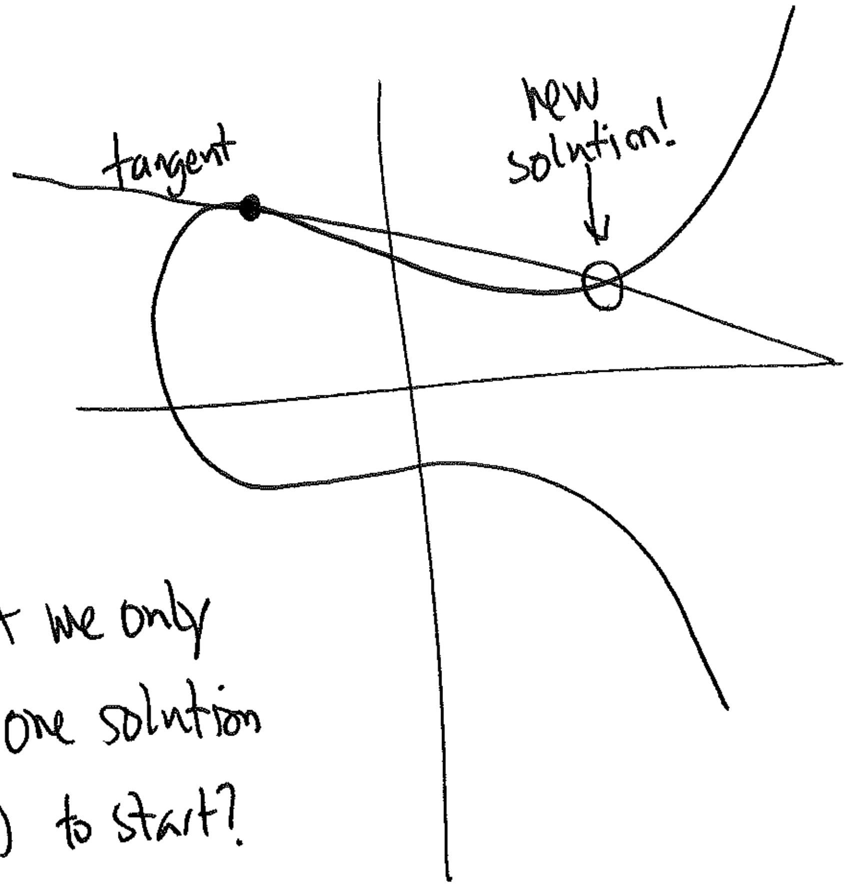
This is an <u>elliptic curve</u>.

$$y^2 = x^3 + ax^2 + bx + c$$

Sometimes you make one more substitution to put in <u>Weierstrass form</u>

$$y^2 = x^3 + bx + c.$$

What do elliptic curves look like?




$y^2 = x^3 - 7x + 10$

what if we only
had one solution
$(x,y)$ to start?

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In the fruit equation: $y^2 = x^3 + 109x^2 + 224x$

Let's start with solution we know

$x = -100 \quad y = 260,$

draw lines, solve for intersection,
find more solutions a, b, c of fruit,
hope they're positive.

$$y^2 = x^3 + 109x^2 + 224x$$

solves fruit eqn

Brute force on computer:

$x = -100$ $\rightsquigarrow$ $a = 2/7$ $\xrightarrow{\times 14}$ $a = 4$

$y = 260$ $\qquad$ $b = -1/14$ $\qquad$ $b = -1$

$\qquad\qquad\qquad$ $c = 11/14.$ $\qquad$ $c = 11.$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 4$$

Does it work?

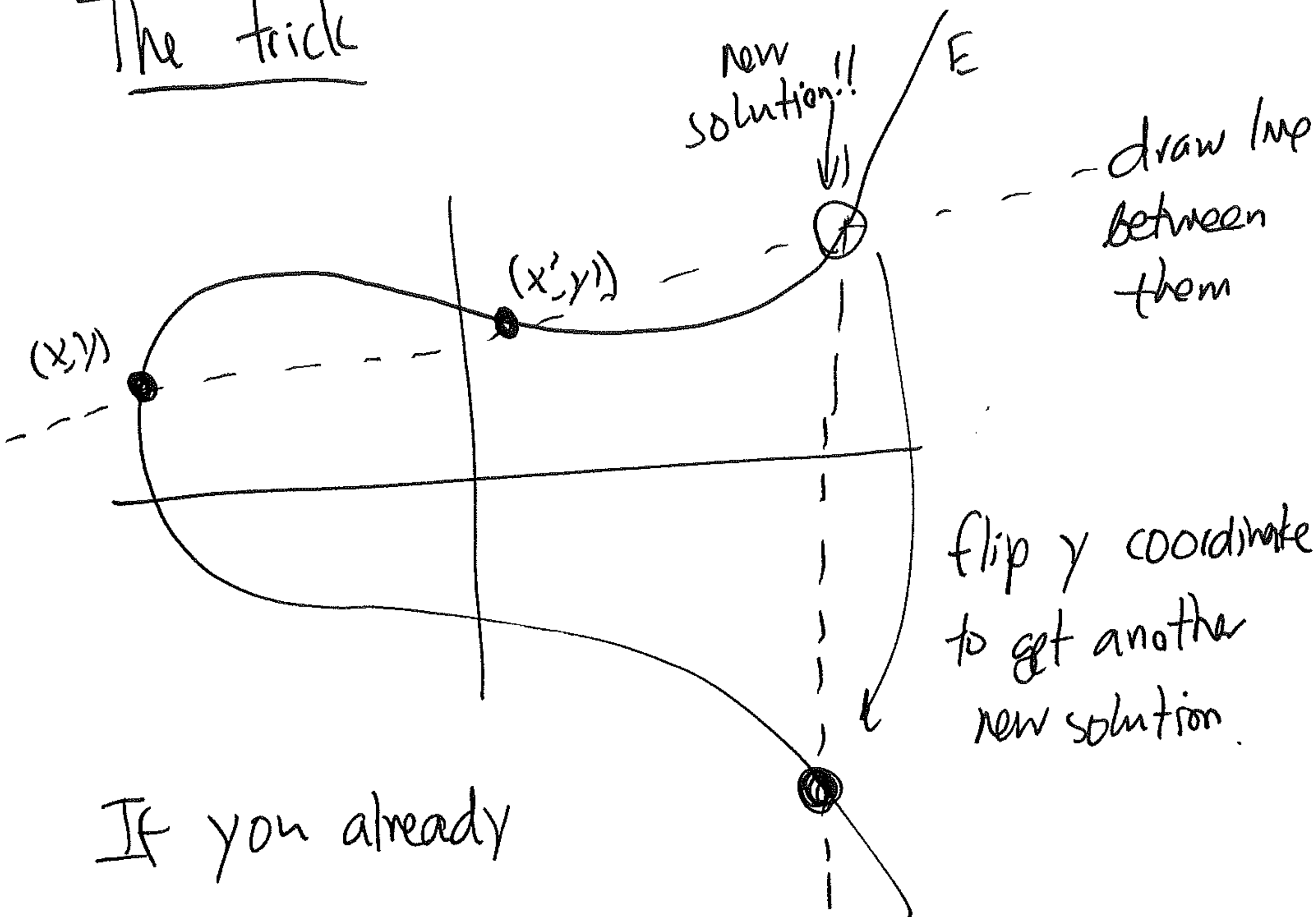
$$\frac{4}{10} + \frac{-1}{15} + \frac{11}{3}$$

$$= \frac{12}{30} + \frac{-2}{30} + \frac{110}{30} = \frac{120}{30} = \boxed{4}$$

To get solution where all positive, we need more solutions to

$$y^2 = x^3 + 109x^2 + 224x$$

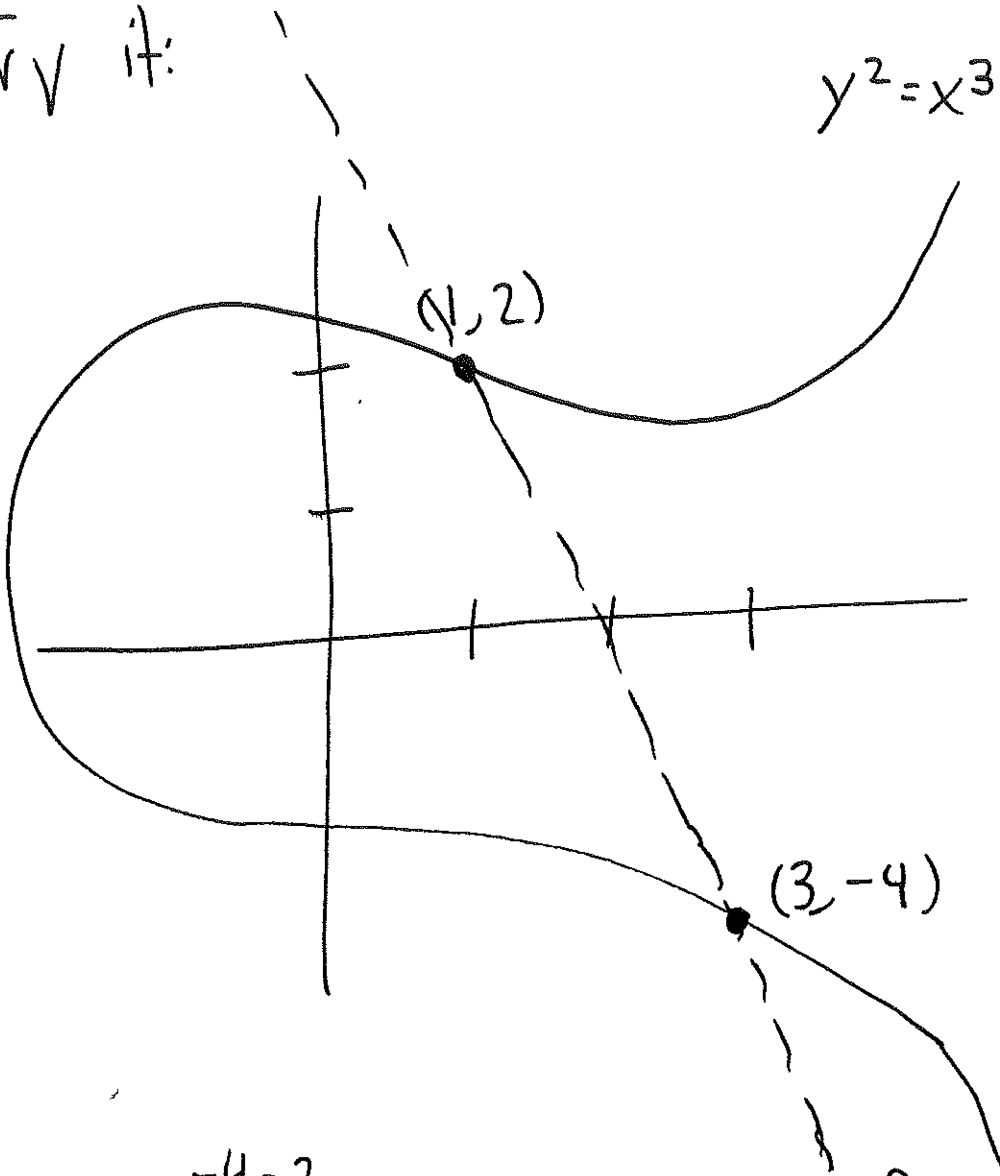
# The trick




If you already have two solutions, you can use them to generate a new solution.

Try it:

$y^2 = x^3 - 7x + 10$




(1,2)
(3,-4) are solutions

find eqn of line, solve for third intersection point

$$\text{slope} = \frac{-4-2}{3-1} = -3$$

$y = -3x + C$ $\quad$ $x = 1$ gives $y = 2$ so $C = 5$

$y = -3x + 5$ is line

$(-3x+5)^2 = x^3 - 7x + 10$

$9x^2 - 30x + 25 = x^3 - 7x + 10$

$x^3 - 9x^2 + 23x - 15 = 0$

$x=1$
$x=3$ are two solutions.

??

$x^3 - 9x^2 + 23x - 15 = (x-1)(x-3)(x-r)$

$= x^3 - (1+3+r)x^2 +$ (don't care)

so $r=5$ is the other root.

$y = -3x+5$ so $y = -10$.

So $(5,10)$ is a root

and so is $(5,-10)$