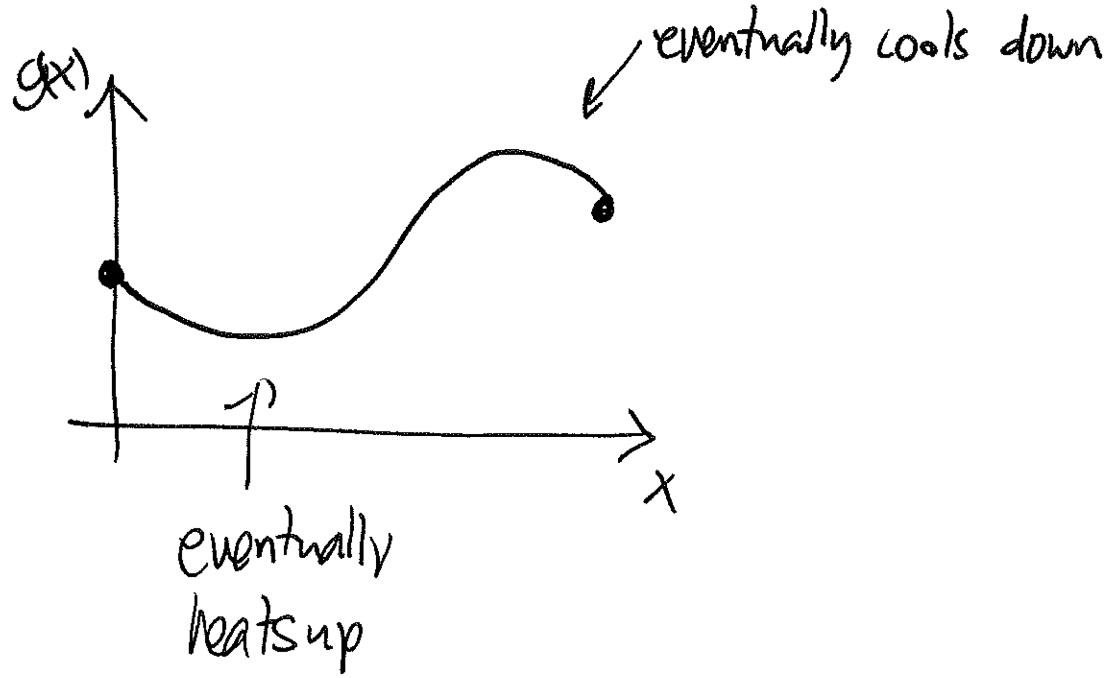
The heat equation

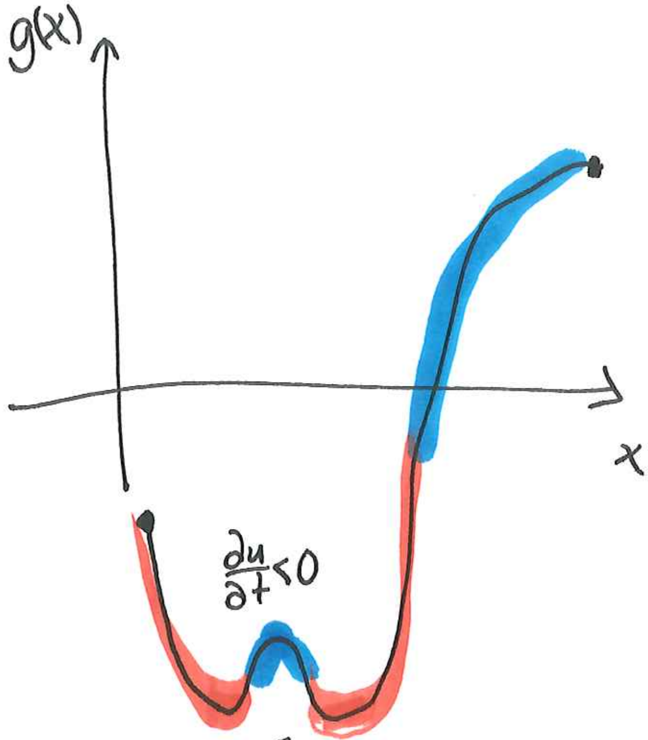
heat initially distributed as a function

$g(x) =$ temperature at position
x on the rod.

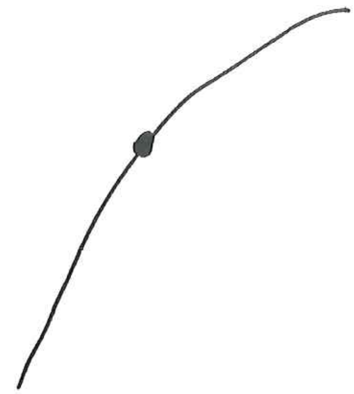
what happens when heat starts to flow?

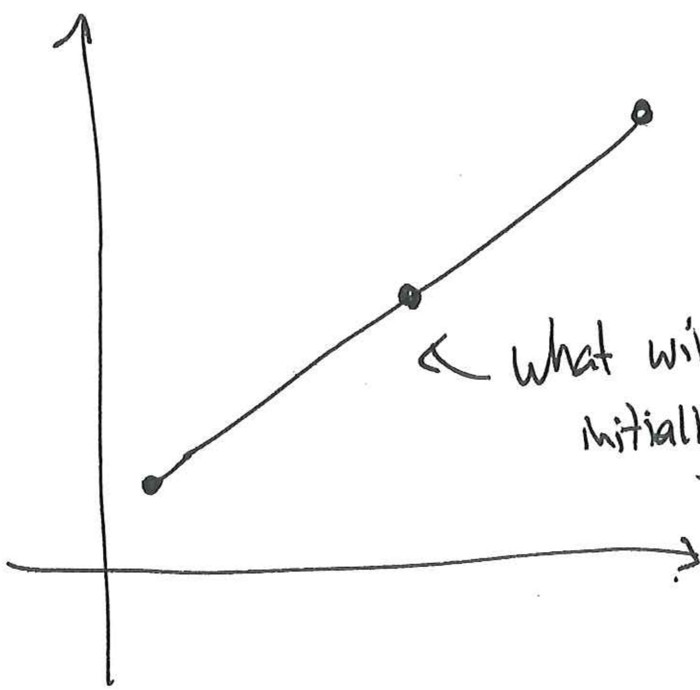
e.g.





this point will initially cool off, then eventually warm up.





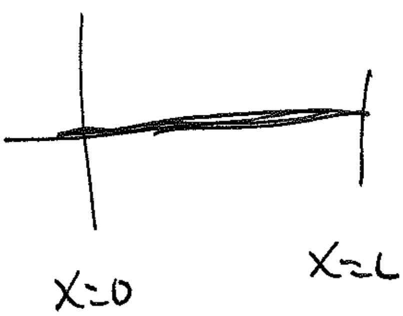
what will happen here, initially

Let's hold ends at constant temperature 0.

let $u(x,t) =$ temperature at position x at time t.

how do we find $u(x,t)$?

we know: $u(x,0) = g(x)$ ← initial condition



$$\left.\begin{array}{l} u(0,t) = 0 \\ u(L,t) = 0 \end{array}\right\} \text{ ends at constant temp.}$$

we need an equation for how $u(x,t)$ changes as time increases.

$$\underbrace{\frac{\partial u}{\partial t}(x,t)}_{\text{rate of increase at pt } x \text{ at time } t.} = \underbrace{\alpha^2}_{\text{some constant}} \frac{\partial^2 u}{\partial x^2}(x,t).$$

$u_t = \alpha^2 u_{xx}$ "Heat equation"

"thermal diffusivity"; depends on what rod is made of.

this is a "partial differential equation" has partial derivatives

	cm^2/s
Copper	1.14
aluminum	0.86
iron	0.12
brick	0.0038

$$u_t = \alpha^2 u_{xx}$$

Want to find $u(x,t)$ satisfying this eqn.

Let's try to find some solutions.

To simplify, let's look for "seperable" solutions.

<u>try</u> $u(x,t) = f(x)\,g(t)$

$$\left.\begin{aligned} u_t &= f g_t \\ u_{xx} &= f_{xx} g \end{aligned}\right]$$

<u>Heat eqn</u>

$$f g_t = \alpha^2 f_{xx} g$$

$$\frac{f_{xx}}{f} = \frac{1}{\alpha^2}\frac{g_t}{g}$$

t only

x only

Must both be constant!

Call constant $-\lambda$. (assume $\lambda > 0$)

$\cancel{f_{xx} = -\lambda f}$ $\quad$ $f_{xx} + \lambda f = 0$ $\leftarrow$ left $= -\lambda$

$\cancel{g_t}$ $\quad$ $g_t + \alpha^2 \lambda g = 0$ $\leftarrow$ right $= -\lambda$

Solve separately.

$f(x)$ has

$$f'' = -\lambda f$$

$$f(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$$

But! We need $f(0) = 0$ and $f(L) = 0$.

For most λ, this is impossible.

But! if $\lambda = \frac{n^2 \pi^2}{L^2}$ then

$$f(x) = \sin\left(\frac{n\pi x}{L}\right) \text{ works.}$$

For those λ values,

$$\lambda = \frac{n^2\pi^2}{L^2}$$

the equation

$g_t + \alpha^2 \lambda T = 0$ has solution

$$e^{-n^2\pi^2\alpha^2/L^2)t}$$

So a solution to the heat eqn is:

$$u_n(x,t) = e^{-\frac{n^2\pi^2\alpha^2}{L^2}t} \sin\left(\frac{n\pi}{L}x\right).$$

for $n = 1, 2, 3, \ldots$

e.g. $L=\pi$, $\alpha^2 = 1$, $n=1$

$$u(x,t) = e^{-t}\sin(x)$$