Today: Some linear algebra.

personal.psu.edu/jdl249/courses/topics2f21/

1) Formula for Fibonacci: $1, \overset{F_n}{\overset{\shortparallel}{1}}, 2, 3, 5, 8, 13, 21, 34, \ldots$

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

2) Formula for substition in double integrals ("Jacobian determinant").

Def A function $T: \mathbb{R}^n \to \mathbb{R}^m$ $\leftarrow$ vectors of length m.

$\uparrow$ vector of length n

is called a <u>linear transformation</u> if

1) $T(v+w) = T(v) + T(w)$ any two vectors $v, w \in \mathbb{R}^n$

2) $T(cv) = c \cdot T(v)$

Ex $T: \mathbb{R}^2 \to \mathbb{R}^3$

defined by

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x - 7y \\ 2x + 5y \\ x + y \end{pmatrix}$$

linear equations

check:

$$T\left(\underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{v}\right) = \begin{pmatrix} -13 \\ 12 \\ 3 \end{pmatrix}$$

$$T\left(\underbrace{\begin{pmatrix} 2 \\ -3 \end{pmatrix}}_{w}\right) = \begin{pmatrix} 23 \\ -11 \\ -1 \end{pmatrix}$$

$$T\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -3 \end{pmatrix}\right) = T\left(\underbrace{\begin{pmatrix} 3 \\ -1 \end{pmatrix}}_{v+w}\right) = \begin{pmatrix} 10 \\ 1 \\ 2 \end{pmatrix}$$

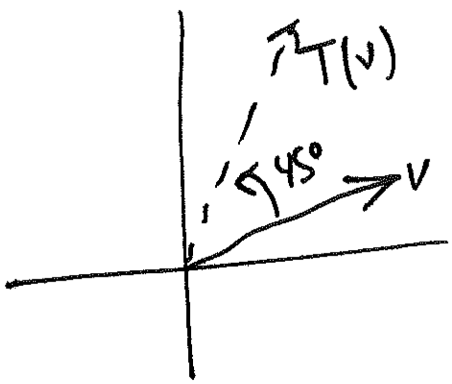
Did it work?

$T(v) + T(w) = T(v+w)$?

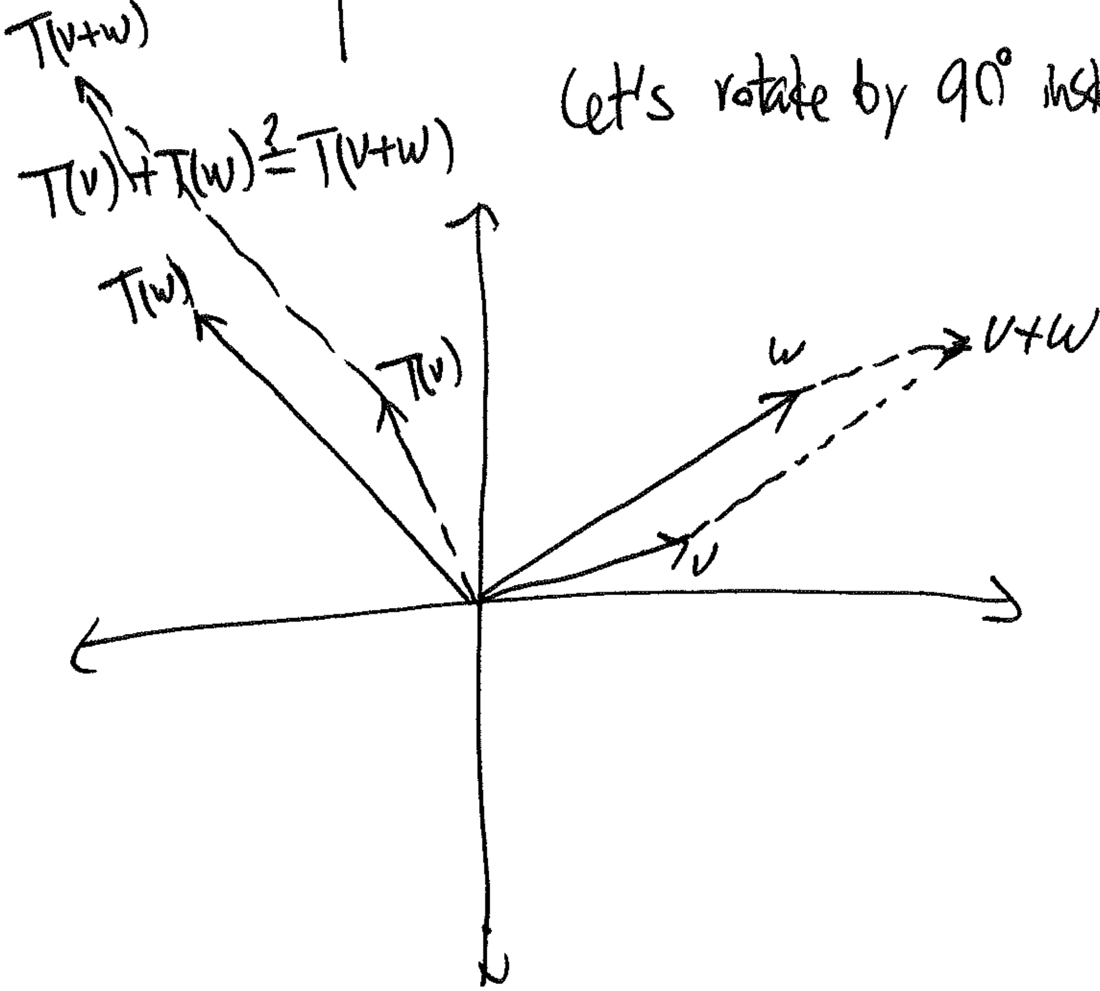
$$\begin{pmatrix} -13 \\ 12 \\ 3 \end{pmatrix} + \begin{pmatrix} 23 \\ -11 \\ -1 \end{pmatrix} = \begin{pmatrix} 10 \\ 1 \\ 2 \end{pmatrix} \quad \checkmark$$

Ex $T : \mathbb{R}^2 \to \mathbb{R}^2$

$T(\overset{\binom{x}{y}}{v}) =$ v rotated counterclockwise by 45°.



Let's rotate by 90° instead



↑

we can rewrite this using matrices.

$$\begin{pmatrix} x-7y \\ 2x+5y \\ x+y \end{pmatrix} = \begin{array}{c} (x \quad y) \; x \\ \begin{pmatrix} 1 & -7 \\ 2 & 5 \\ 1 & 1 \end{pmatrix} \end{array} \begin{pmatrix} x \\ y \end{pmatrix}$$

Our first map can be written as:

$T(v) = Mv$ where M is matrix $\begin{pmatrix} 1 & -7 \\ 2 & 5 \\ 1 & 1 \end{pmatrix}$

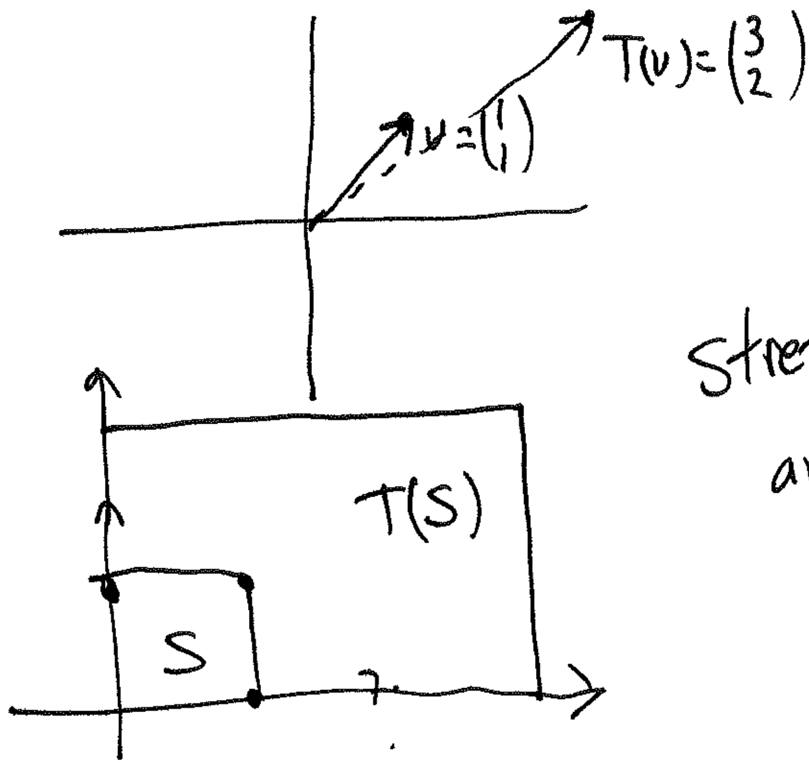
$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 1 & -7 \\ 2 & 5 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

In fact, <u>every</u> linear map is determined by a matrix: if $T: \mathbb{R}^n \to \mathbb{R}^m$ satisfies the axioms, then there's a $m \times n$ matrix M so $T(v) = Mv$ for any $v \in \mathbb{R}^n$.

1a. $M = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

Gives us a map $T : \mathbb{R}^2 \to \mathbb{R}^2$.

$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 2y \end{pmatrix}$$



stretches horizontally 3x and vertically 2x.

Adv Topics 2 (Lesieutre)
August 31, 2021

$T(v)$

Problem 1. For each of the following matrices T, choose a couple sample vectors $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and compute $T\mathbf{v}$. What does the matrix do to a vector, geometrically? What does it do to the unit square?

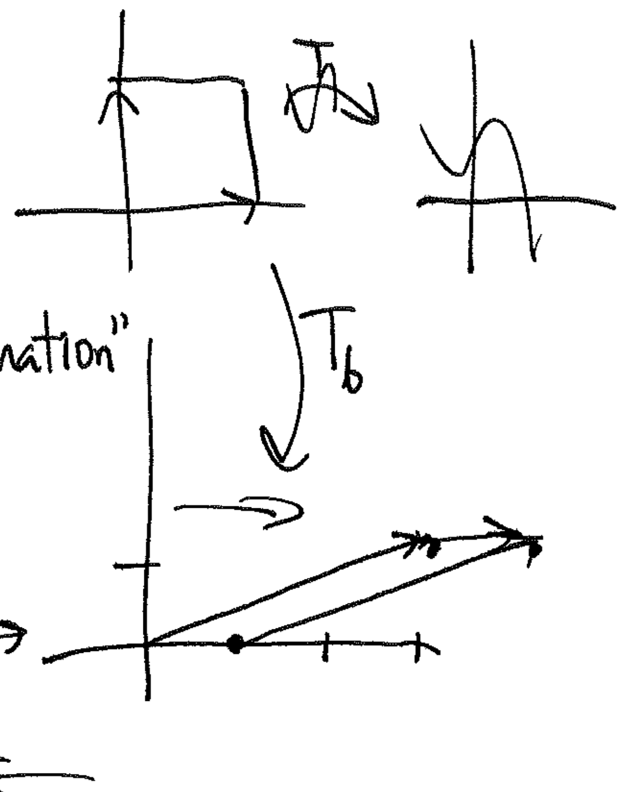
a) $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+3y \\ y \end{pmatrix}$

$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$



c) $\begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}$

"Shear transformation"

d) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

linearity means squares turn into rectangles →

e) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

f) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$T: \mathbb{R}^3 \to \mathbb{R}^2$

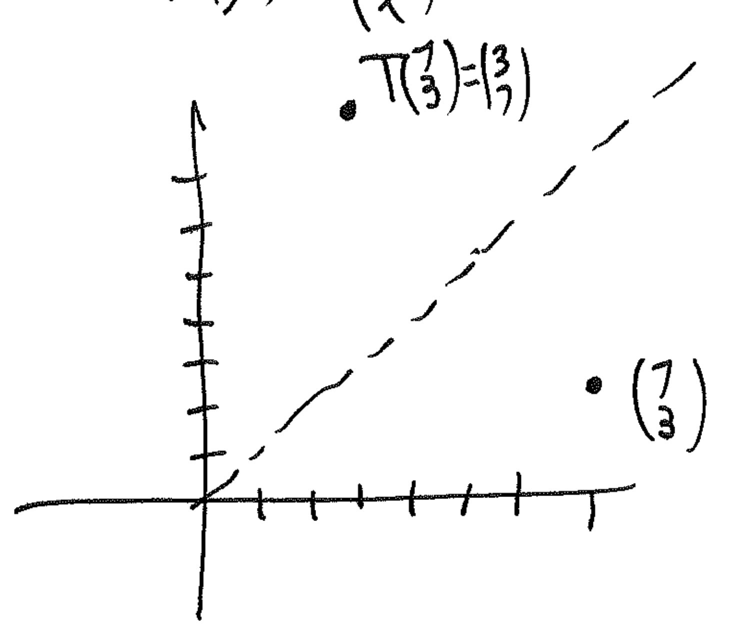
g) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

c) yuck

h) $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

i) $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

d) $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$

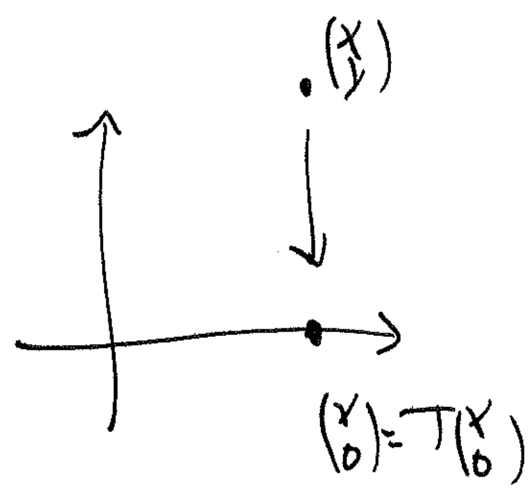


e) $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$

reflect over x-axis

f) $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$

project onto x-axis



g) $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ project onto xy-plane.

Strategy for finding a matrix. (given a description of map)

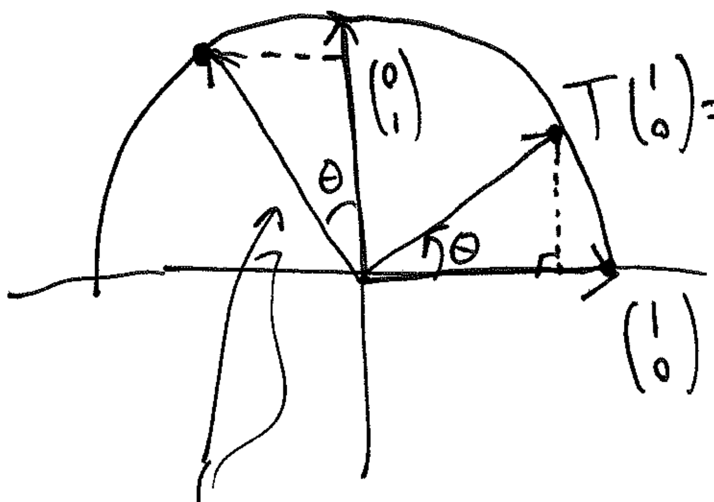
By definition: the first column of M is where $\begin{pmatrix}1\\0\end{pmatrix}$ goes.

the second column is where $\begin{pmatrix}0\\1\end{pmatrix}$ goes.

$$\begin{array}{cc} 1 & 0 \end{array}$$
$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}a\\c\end{pmatrix} \qquad \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}b\\d\end{pmatrix}.$$

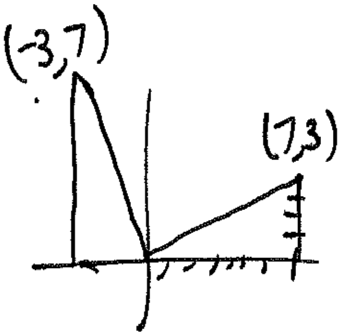
ex

Rotate θ counterclockwise. What's the matrix?



$\cancel{\cos(\theta+90^\circ)=\cos\theta}$

$$T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$$

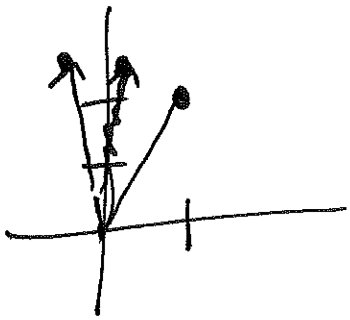


$$T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$

matrix for rotation:

$$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

What's $\begin{pmatrix}1\\2\end{pmatrix}$ rotated 30° counterclockwise?



$$M = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}^{\theta=30^\circ} = \begin{pmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{pmatrix}$$

$$\begin{pmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{3}}{2} - 1\\ \frac{1}{2} + \sqrt{3}\end{pmatrix}$$

$$\begin{pmatrix}3 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}1 & 3\\0 & 1\end{pmatrix} = \begin{pmatrix}3 & 9\\0 & 2\end{pmatrix}.$$

$$\begin{pmatrix}3 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}3\\0\end{pmatrix} \qquad \begin{pmatrix}3 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}9\\2\end{pmatrix}$$

$$\begin{array}{c}\begin{array}{cc}3 & 0\end{array}\\ \begin{pmatrix}1 & 3\\0 & 1\end{pmatrix}\end{array}\begin{pmatrix}3 & 0\\0 & 2\end{pmatrix} = \begin{pmatrix}3 & 6\\0 & 2\end{pmatrix}$$

Why is this the formula?

$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ gives the function $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$

$$T_1\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 2y \end{pmatrix}$$

$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ gives the function $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$.

$$T_2\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+3y \\ y \end{pmatrix}$$

What's $T_1 \circ T_2$? (composition)

$$(T_1 \circ T_2)\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = T_1\left(T_2\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)\right)$$

$$= T_1\begin{pmatrix} x+3y \\ y \end{pmatrix} = \begin{pmatrix} 3x+9y \\ 2y \end{pmatrix}$$

This is just $\begin{pmatrix} 3 & 9 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$!

If $S: \mathbb{R}^2 \to \mathbb{R}^2$ ~~to~~

and $T: \mathbb{R}^2 \to \mathbb{R}^2$ are linear transformations:

~~matrix~~ $(\text{matrix for } \underset{\text{composition}}{T \circ S}) = (\text{matrix for } T)(\text{matrix for } S)$

Matrix multiplication is not commutative!

$$M_2 M_1 \neq M_1 M_2.$$

This just reflects the fact that function composition not commutative.

$f(x) = \log x$ $\qquad$ $f(g(x)) = \log(\sqrt{x})$

$g(x) = \sqrt{x}$ $\qquad$ $g(f(x)) = \sqrt{\log(x)}$