Last time:

$$\int f(x)\sin(x)\,dx = -f(x)\cos(x) + f'(x)\sin(x) - \int f''(x)\sin(x)\,dx$$

Plug in $\int_0^\pi$ and we get:

$$\int_0^\pi f(x)\sin(x)\,dx = \left(-f(x)\cos x + f'(x)\sin x\right)\Big|_0^\pi - \int_0^\pi f''(x)\sin x\,dx$$

$$= \left((f(\pi)+0) - (-f(0)+0)\right) - \int_0^\pi f''(x)\sin x\,dx$$

$$= \left(f(0)+f(\pi)\right) - \int_0^\pi f''(x)\sin x\,dx.$$

But:

$$\int_0^\pi f''(x)\sin x\,dx = \left(f''(0)+f''(\pi)\right) - \int_0^\pi f''''(x)\sin x\,dx$$

So:

$$\int_0^\pi f(x)\sin x\,dx = \left(f(0)+f(\pi)\right) - \left(f''(0)+f''(\pi)\right) + \int_0^\pi \cancel{f(x)}\, f''''(x)\sin(x)\,dx$$

Again:

$$\int_0^{\pi} f(x)\sin(x)\,dx = \big(f(0)+f(\pi)\big) - \big(f''(0)+f''(\pi)\big) + \big(f^{(4)}(0)+f^{(4)}(\pi)\big)$$
$$- \int_0^{\pi} f^{(6)}(x)\sin(x)\,dx.$$

We could keep going forever.

But if $f(x)$ is a polynomial: eventually the integral is 0.

Suppose $f(x)$ is polynomial of degree $\underbrace{2n}_{\text{even}}$.

$$\int_0^{\pi} f(x)\sin x\,dx = F(0) + F(\pi)$$

where $F(x) = f(x) - f''(x) + f^{(4)}(x) - \cdots (-1)^n f^{(2n)}(x)$

Suppose $\pi = p/q$ is rational

Our plan: pick a clever $f(x)$ so

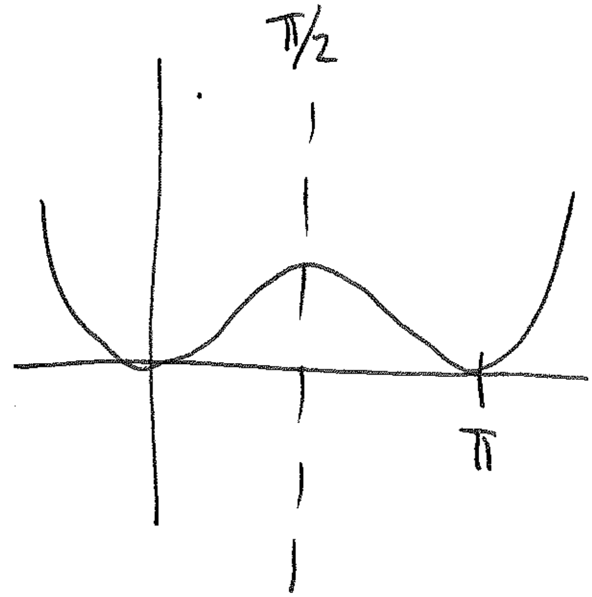
- $0 < \int_0^\pi f(x) \sin x \, dx < 1$

- All derivatives of f at $0, \pi$ are integers

then left side is not integer, but right is integer.

Let $$f_n(x) = q^n \frac{x^n (\pi - x)^n}{n!}$$

Plot:



(this will work for $n >> 0$)

Also: if $0 < x < \pi$, it's positive.

$f(x) = f(\pi - x)$

$f'(x) = -f(\pi - x)$

$f''(x) = f''(\pi - x)$

I claim: all derivatives of $f(x)$ at 0 & π are integers.

$$\cancel{f_n(x) = q^n \frac{x^n(\pi - x)^n}{n!}}$$ why?

$$f_n(x) = q^n \frac{x^n(\overset{\pi = p/q}{\pi} - x)^n}{n!} = \cancel{x^n(\pi -}$$

$$= \frac{x^n(p - qx)^n}{n!}$$

e.g. $n=4$, want to know ~~x~~ x^7 term.

→ What's coefficient on x^j in there?

coefficient is 0 if $j < n$.

if $j \geq n$:

coefficient is $\frac{c_j}{n!}$ for some integer c_j.

you could with binomial thm if desired.

The j^{th} derivative is

$$f_n^{(j)}(0) = \frac{j!}{n!} c_j$$

$$\uparrow$$

$j! \cdot$ (the x^j coefficient)

$$\parallel$$

$$\frac{j!}{n!} c_j$$ ← some integer, don't care

$$\uparrow$$

an integer if $j \geq n$

To know $f^{(3)}(0)$ for

~~f(x) =~~

$$f(x) = 7+3x-2x^2+5x^3-12x^4+x^5$$

$$f'(x) = \cancel{3x-9}$$

$$= 3-2\cdot 2x+3\cdot 5x^2-4\cdot 12x^3+4x^5$$

$$f''(x) = -2\cdot 2+2\cdot 3\cdot 5x-3\cdot 4\cdot 12x^2+20x^4$$

$$f'''(x) = 1\cdot 2\cdot 3\cdot 5-2\cdot 3\cdot 4\cdot 12x+80x^3$$

$$\boxed{f'''(0) = 1\cdot 2\cdot 3\cdot 5}$$

weird stuff about derivatives.

So $$\int_0^{\pi} f_n(x)\sin(x) = F_n(0)+F_n(\pi)$$

$$= \text{integer.}$$

(no matter what n is!)

$$f_n(x) = \frac{x^n(p-qx)^n}{n!}$$ ← when n is big this gets close to 0 function.

$$\int_0^{\pi} f_n(x)\sin(x)\,dx$$ ← small, since $f_n(x) \approx 0$
positive, $f_n(x)$ & $\sin(x)$ are positive

- positive
- less then 1
- integer

Impossible!

How good can rational approximations be?

Approximations for π: $\pi \approx 3/1$

$\approx 22/7 = 3.142857\ldots$

nobody ever says. $\approx 355/113 = $ 6-digits right?

$\pi \approx 40/13 \approx 3.076.$

What would make an approximation "interesting"?

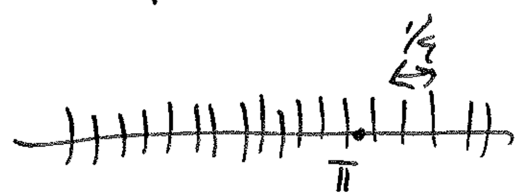
digits correct

vs

how big is q?

For any q, we can find a p so

$$\left|\pi - \frac{p}{q}\right| < \frac{1}{q}$$



random # $\rightarrow$ $\dfrac{3226}{1027} \approx \pi$

$\parallel$

3.1411879

not very impressive

Theorem

Suppose α is any real number.

There are infinitely many p_n, q_n so that

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}$$

(p_n/q_n gets twice as many digits as # digits of q_n).

But: there are some irrational numbers where $1/q^3$ can't be achieved.

Roth's theorem If α is algebraic number, then

$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^r}$ has only finitely many sols if $r > 2$.

Start with $p_1/q_1 = 3/2$.

Define

$$p_{n+1} = p_n + 2q_n.$$

$$q_{n+1} = p_n + q_n.$$

Challenge:

Prove $\lim_{n\to\infty} \frac{p_n}{q_n} = \sqrt{2}$

<u>error:</u>

<u>and</u> $$\left| \frac{p_n}{q_n} - \sqrt{2} \right| < \frac{1}{q_n^2}.$$