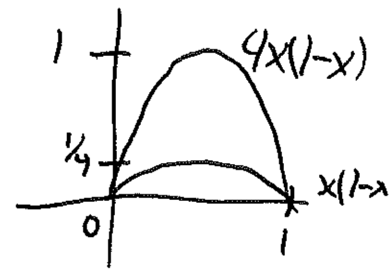
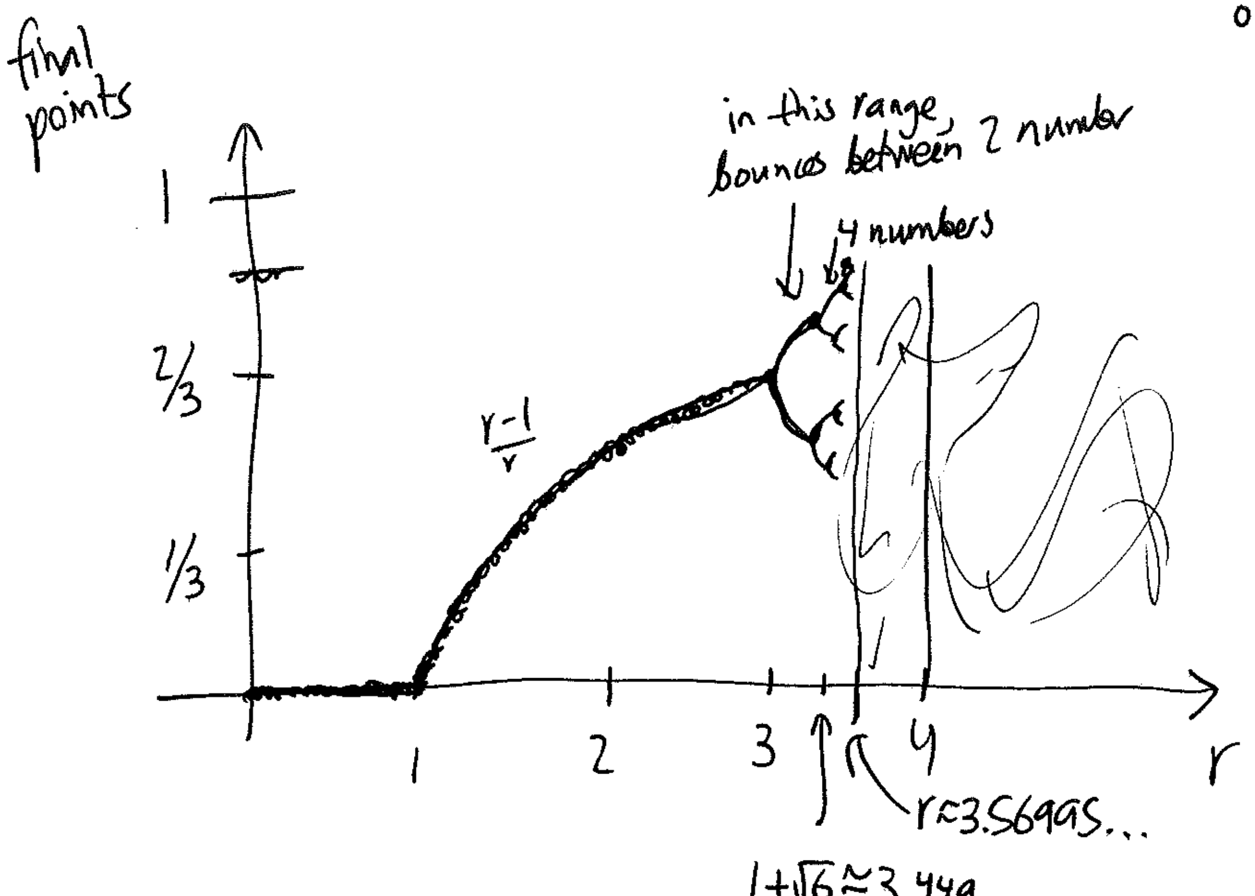
Today: More dynamical systems

$0<r<4$ and $x_0 \in [0,1]$



$f(x) = rx(1-x)$ "logistic map"



Past $r = 3.56995\ldots$ the map devolves into chaos: typically it doesn't repeat any trajectory at all.

There a handful of special r values where not chaotic: $r = 1+\sqrt{8}$: $\rightarrow$ period 3 cycle.

Warm-up

$f(x) = \frac{1}{2}x(1-x)$

Suppose $r = 1/2$, $x_0 = 1/2$.

$f(f(f(f(f(x)))))$ n times

Can you prove $\lim_{n \to \infty} f^n(x_0) = 0$? "$n^{th}$ iterate"

(Hint: use calculus theorems about limits)

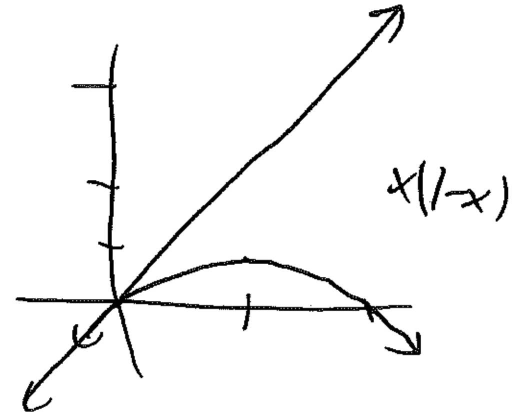
$\boxed{r = 3}$

Suppose $r = 3/2$. $x_0 = 1/2$.

Can you prove $\lim_{n \to \infty} f^n(x_0) = \frac{r-1}{r} = \frac{1}{3}$?

If r is in $[3, 1+\sqrt{6}]$: what are the repeating values?

Pf. for $r = 1/2$



$$x_{n+1} = f(x_n) = \tfrac{1}{2} x_n (1 - x_n)$$

$$\lim_{n \to \infty} x_n = 0.$$

$$x_{n+1} = \frac{1}{2} x_n (1 - x_n) = \frac{1}{2} x_n - \frac{1}{2} x_n^2 \le \frac{1}{2} x_n$$

so $x_1 \le \frac{1}{2} x_0$

$x_2 \le \frac{1}{2} x_1 \le \frac{1}{4} x_0$

...

$\cancel{x_{n+1}}$ $\boxed{x_n \le \frac{1}{2^n} x_0}$

Since

Let $y_n = \frac{1}{2^n} x_0$

$z_n = 0$

$f^n(x_0)$

$$\begin{array}{ccccc} z_n & \le & x_n & \le & y_n \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

by sandwich thm.

This works for $0 < r < 1$

What about $r=1$?

$x_0 = 1/2$.

$f(x) = x(1-x)$

$\lim_{n\to\infty} f^n(x_0) = 0$

$\boxed{\text{Homework!}}$
for me

0.5, 0.25, 0.1875, 0.152343,

0.129135, ...

What about $r = 3/2$?

$$x_{n+1} = \frac{3}{2} x_n (1 - x_n) \qquad x_0 = 1/2$$

Question: $\lim_{n \to \infty} f(x_n) = 1/3$.

Let $e_n = x_n - \frac{1}{3}$.

Try to prove $\lim_{n \to \infty} e_n = 0$.

What is e_{n+1} in terms of e_n?

$$e_{n+1} = x_{n+1} - \frac{1}{3} = \frac{3}{2} x_n (1 - x_n) - \frac{1}{3}$$

$$= \frac{3}{2}\left(e_n + \frac{1}{3}\right)\left(1 - \left(e_n + \frac{1}{3}\right)\right) - \frac{1}{3}$$

$$= \frac{1}{2} e_n - \frac{3}{2} e_n^2$$

goes to

$$= \frac{1}{2} e_n (1 - 3e_n)$$

this means

$x_n < 2/3$

automatic for e_1!

if $e_n < \frac{1}{3}$ then

$0 < e_{n+1} < \frac{1}{2} e_n$

use sandwich thm.

For $r=3$:

$$e_{n+1} = -e_n - 3e_n^2$$

$$\Downarrow$$

$$e_{n+2} = e_n - 18e_n^3 - 27e_n^4$$

if $e_n = \frac{1}{100}$

then $e_{n+2} = \frac{1}{100} - \frac{18}{1000000}$

$$e_n \sim \frac{1}{\sqrt{n}}$$

One more thing: what about period 2 pts?

$3 < r < 1+\sqrt{6}$.

How to find the numbers a & b it oscillates between?

$\searrow$ $ra(1-a) = b$ and $rb(1-b) = a$

two eq^ns, two variables.

$\downarrow$

$\cancel{r(r(b-1)}$.

$$a = \frac{(r+1) - \sqrt{r^2-2r+3}}{2r}$$

$$b = \frac{(r+1) + \sqrt{r^2-2r+3}}{2r}$$

Julia & Mandlebrot sets

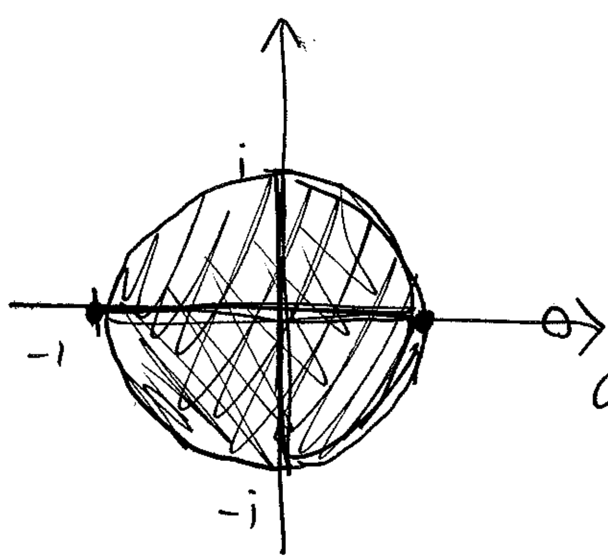
Fix a complex number c. Let $f_c(x) = x^2 + c$

Look at $J(c) = \{z : f_c^n(z)$ stays bounded, doesn't go to $\infty\}$.

Simplest case $c = 0$.

$f_0(z) = z^2$

$J(0) = \{z : f_0^n(z)$ stays bounded$\}$



$(re^{i\theta})^2$

$=$

$r^2 e^{2i\theta}$

$\downarrow$

$r^4 e^{4i\theta}$

$\downarrow$

$r^8 e^{8i\theta} \rightarrow$