Today

- Finish topology
- Construct $\mathbb{R}$

The belt trick

$SO(3) = \{$ all 3×3 rotation matrices $\}$

what does it do to a vector? rotate it!

you can specify axis + angle

as a matrix
$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$T((1,0,0))$, $T((0,1,0))$, $T((0,0,1))$

rotation matrix?
$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{pmatrix}$$

↖ not rotation matrix! det is -2.

$SO(3)$ is a subset of all 3×3 matrices.

A twisted up belt with buckle + end in fixed orientations can be thought of as specifying a path ion $SO(3)$

$$f : [0,1] \to SO(3)$$

ie it specifies a matrix M_t for any $t \in [0,1]$.

$M_t = \{$what rotation happens to action figure glued at position t on the belt$\}$

Moving the belt changes the path in a continuous way $\rightsquigarrow$ homotopy.

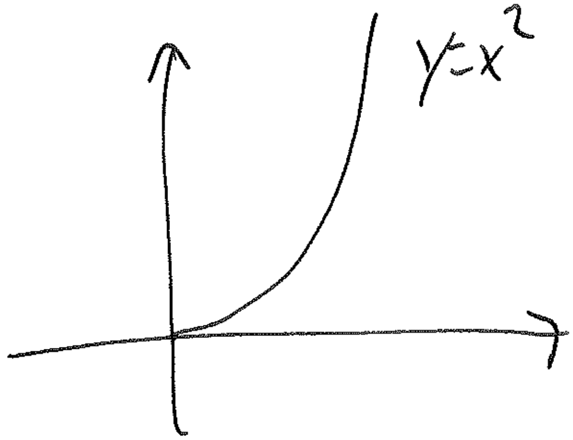
the path from 1x twisted belt $\rightarrow$ can't be homotoped to constant path / not contractable

2x twisted belt $\rightsquigarrow$ can be homotoped to identity

Fundemental group?

Constructing $\mathbb{R}$

- How to prove intermediate value theorem?
- It works for $\mathbb{R}$ but not $\mathbb{Q}$, that's weird.



$f(0)=0$

$f(2)=4$

but there's no $0<a<2$ with $f(a)=2$...

(if you only know $\mathbb{Q}$)

- To prove it, you really need to know what a real number is.

But what's a real number?

- Everything that can be expressed as a (maybe infinite) decimal.

  ° What's an infinite decimal?

  ° 1.29999...

  1.30000...

We're going to start with natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$

How to define them?

<u>Peano's Postulates</u> There exists a set $\mathbb{N}$ with an element $1 \in \mathbb{N}$ and a function $s: \mathbb{N} \to \mathbb{N}$ with three properties:

"successor"

a) There is no $n \in \mathbb{N}$ such that $s(n) = 1$.

b) The function s is injective (i.e. "one-to-one")

[if $s(a) = s(b)$ then $a = b$]

induction

c) Let $G \subseteq \mathbb{N}$ be a subset. Suppose $1 \in G$ and that ~~if~~ for all $g \in G$ ~~then~~ we have $s(g) \in G$. Then $G = \mathbb{N}$.

Q's:

- Are these even consistent?
- What's a "set" or a "function" anyway?

How to define addition of natural numbers?

$\mathbb{N} = \{1, s(1), s(s(1)), s(s(s(1))), \dots\}$

$s(1) + s(s(1)) = ?$

<u>Theorem</u> (we'll skip).

Assuming Peano axioms, there's a unique binary operation $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

such that

some element of $\mathbb{N}$ ie. $s(s(s(s(\cdots(1)))))$

a) $n + 1 = s(n)$

b) $n + s(m) = s(n+m)$

associative, but we don't know that yet!

Why not $n \overset{\text{by a)}}{+} (m+1) = (n+m) + 1$

Now we prove basic facts about addition, ...

Thm ~~Suppose (a+b)+c~~ =

$(a+b)+c = a+(b+c)$ associativity

Pf. Suppose $a, b \in \mathbb{N}$ are fixed.

Let $G = \{c \in \mathbb{N} : (a+b)+c = a+(b+c)\}$

(goal: show $G = \mathbb{N}$ using 3rd Peano Postulate)

First show: $1 \in G$ i.e. $(a+b)+1 = a+(b+1)$

$$(a+b)+1 \overset{\text{def of } +}{=} s(a+b) \overset{(b)}{=} a+s(b) \overset{(a)}{=} a+(b+1)$$

Suppose $c \in G$. Let's prove $s(c) \in G$ (then done)

Need to show: $(a+b)+s(c) = a+(b+s(c))$

Pf

$(a+b)+s(c) \overset{(b)}{=}$

$$s((a+b)+c) = s(\underset{\text{since } c \in G.}{a+(b+c)})$$

$$\overset{(b)}{=} a+(s(b+c)) \overset{(b)}{=} a+(b+s(c))$$

so $s(c) \in G$.

What about $a+b=b+a$?

Next time: - Define $\mathbb{Z}$, from $\mathbb{N}$

- Define $\mathbb{Q}$, from $\mathbb{Z}$

- Define $\mathbb{R}$, from $\mathbb{Q}$

Is $R \in R$?

Sets: - Russell's Paradox:

let $R = \{\text{all sets } X \text{ such that } X \notin X\}$

a set can contain other sets: $\{3, \{1,2\}\}$

- countability

- Banach-Tarski paradox.