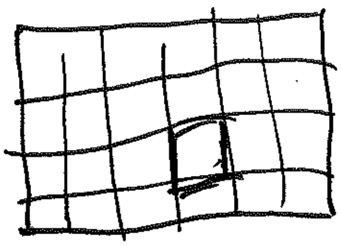
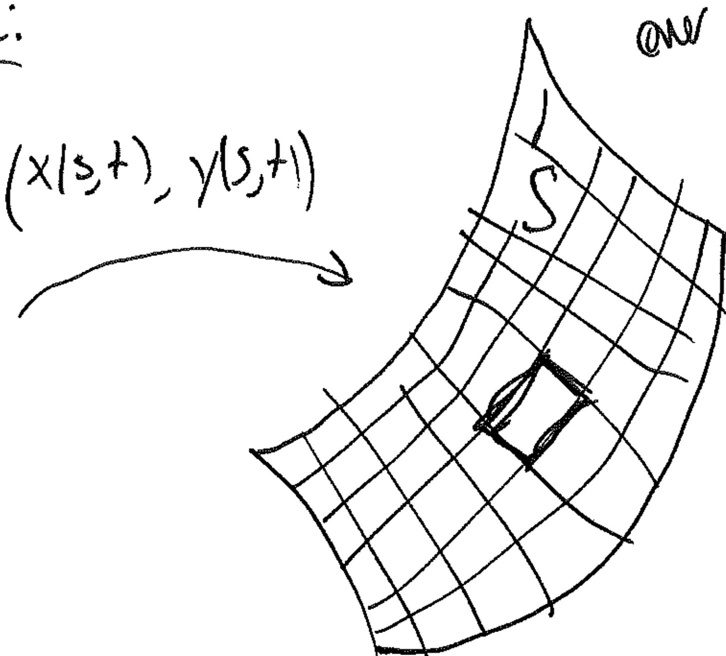
Last time:

Want to integrate over S:



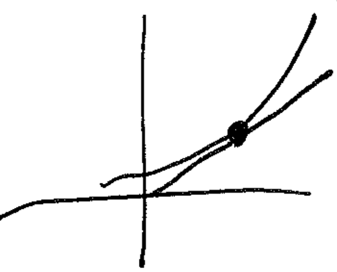


$a \le s \le b$

$c \le t \le d$

$$f(x+\Delta x) \approx \underline{f(x)} + \underline{\Delta x} f'(x)$$

2x2 matrix "Jacobian"



$$\begin{pmatrix} x(s+ds, t+dt) \\ y(s+ds, t+dt) \end{pmatrix} \approx \begin{pmatrix} x(s,t) \\ y(s,t) \end{pmatrix} + \begin{pmatrix} x_s & x_t \\ y_s & y_t \end{pmatrix} \begin{pmatrix} ds \\ dt \end{pmatrix}$$

$$\iint_S f(x,y)\,dx\,dy = \int_{s=a}^{b} \int_{t=c}^{d} f(x(s,t), y(s,t)) \underline{\underline{(x_s y_t - x_t y_s)}}\, ds\, dt$$

Example:

Let $f\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = \begin{pmatrix}x^2y+x\\ xy^3-7y\end{pmatrix}$.

Approximate $f\left(\begin{pmatrix}2.01\\2.99\end{pmatrix}\right)$?

Renamed:

$$f\left(\begin{pmatrix}s\\t\end{pmatrix}\right) = \begin{pmatrix}s^2t+s\\ st^3-7t\end{pmatrix} \begin{matrix}\leftarrow x(s,t)\\ \leftarrow y(s,t)\end{matrix}$$

$$f\left(\begin{pmatrix}2.01\\2.99\end{pmatrix}\right)$$

$x_s(2,3)$

$x_s = 2st+1 \quad y_s = t^3$

$x_t = s^2 \quad y_t = 3st^2-7$

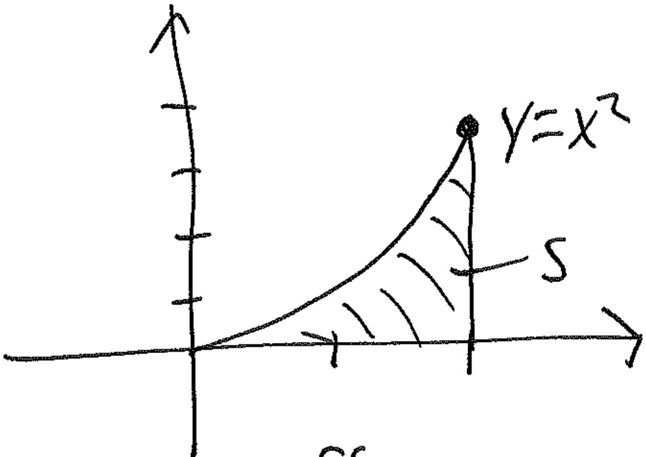
$x_s(2,3) = 13 \quad y_s(2,3) = 27$

$x_t(2,3) = 4 \quad y_t(2,3) = 47$ (?)

$$f\left(\begin{pmatrix}2.01\\2.99\end{pmatrix}\right) = \cancel{f\left(\begin{pmatrix}2\\3\end{pmatrix}\right) + \begin{pmatrix}x_s(2,3)\\x_t(2,3)\end{pmatrix} \quad y_s(2}$$

$$\approx f\left(\begin{pmatrix}2\\3\end{pmatrix}\right) + \begin{pmatrix}x_s(2,3) & x_t(2,3)\\ y_s(2,3) & y_t(2,3)\end{pmatrix}\begin{pmatrix}0.01\\-0.01\end{pmatrix}$$

$$= \begin{pmatrix}14\\33\end{pmatrix} + \begin{pmatrix}13 & 4\\27 & 47\end{pmatrix}\begin{pmatrix}0.01\\-0.01\end{pmatrix} = \begin{pmatrix}14.09\\32.80\end{pmatrix}$$



$f(x,y) = xy$

$0 \le x \le 2$

$$\iint_S xy \, dx \, dy$$

$$\left(\frac{y^2}{2}\right)\Big|_0^{x^2} = \frac{x^4}{2}$$

$$\int_{x=0}^{2} \int_{y=0}^{x^2} xy \, d\cancel{y} \, d\cancel{x} = \int_{x=0}^{2} x \cdot \left(\int_{y=0}^{x^2} y \, dy \right) dx$$

$$= \int_{x=0}^{2} \frac{x^5}{2} \, dx = \frac{x^6}{12}\Big|_0^2 = \frac{64}{12} = \frac{32}{6} = \frac{16}{3}.$$

---

x coord $\rightarrow$ $0 \le s \le 2$ $\qquad x(s,t) = s \qquad y(s,t) = s^2 t$

$0 \le t \le 1$ $\qquad \cancel{y(s,t)=}$

proportion of the way you are from $y=0$ to $y=x^2$

$x_s = 1 \quad x_t = 0 \qquad x_s y_t - x_t y_s = s^2$

$y_s = 2st \quad y_t = s^2$

Jacobian

$$\int_{s=0}^{2} \int_{t=0}^{1} (s \cdot s^2 t) \cdot (s^2) \, dt \, ds$$

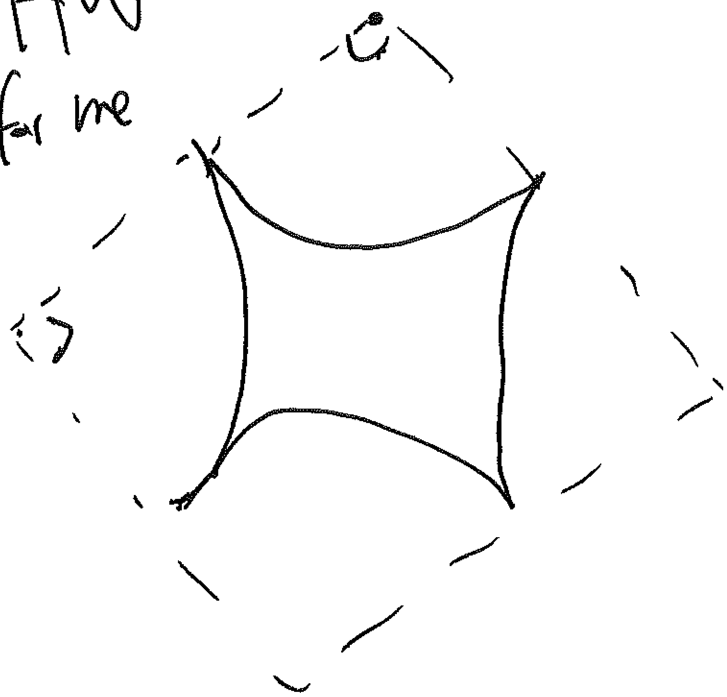
$= xy$, our function

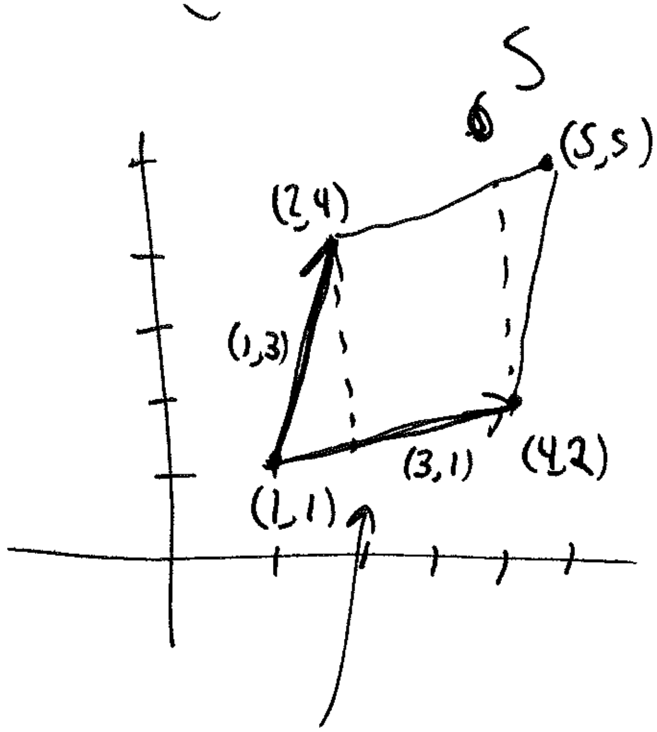
$$= \xcancel{\int_{s=0}^{2} s^5 \, ds \cdot \int_{t=0}^{1} t \, dt}$$

$$= \int_{s=0}^{2} \int_{t=0}^{1} s^5 \, t \, dt \, ds = \int_{s=0}^{2} \frac{1}{2} s^5 \, ds = \left. \frac{s^6}{12} \right|_0^2 = \frac{64}{12} = \frac{16}{3}$$

HW

for me





$$\begin{pmatrix} x \\ y \end{pmatrix}(s,t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + s\begin{pmatrix} 3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

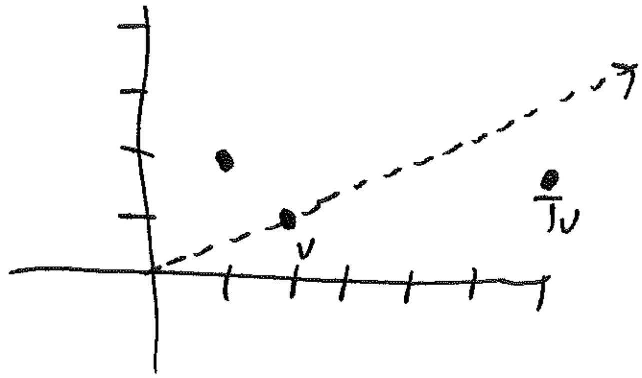
$x(s,t) = 1 + 3s + t \qquad 0 \le s \le 1$

$y(s,t) = 1 + s + 3t \qquad 0 \le t \le 1$

area is

$$\det\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} = 8.$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$



$$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 3x \\ 2y \end{pmatrix}$$

$T(v)$ <u>not</u> parallel to v!

e.g. $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $T(v) = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$ not an eigenvector.

$\cancel{w = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad T(w) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}}$

$\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ is an eigenvector!

$T\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \end{pmatrix} = 3 \cdot \begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

<u>In</u> fact, any vector $\begin{pmatrix} x \\ 0 \end{pmatrix}$! Turns into $\begin{pmatrix} 3x \\ 0 \end{pmatrix}$, so $\lambda = 3$.

The number λ is called the <u>eigenvalue</u>.

$\begin{pmatrix} 0 \\ y \end{pmatrix}$ is also eigenvector: $T\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 2y \end{pmatrix}$, so $\lambda = 2$ for these.

Eigenvectors:

If $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)$ is a linear transformation

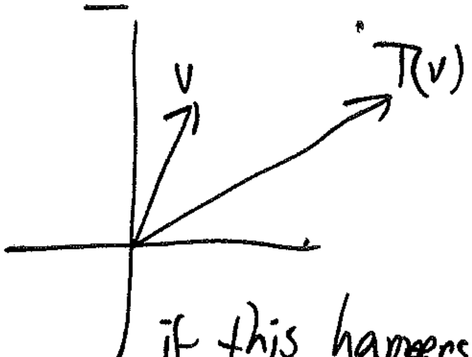
ex $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 3x+2y \\ -x-7y \end{pmatrix}$.

scalar mult

a vector v is called an <u>eigenvector</u> if $T(v) = \lambda v$
for some scalar λ.   for T

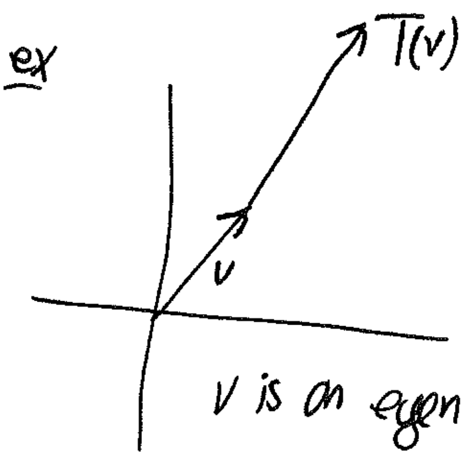
"T(v) is parallel to v"

ex



if this happens,
v is <u>not</u> an eigenvector!

ex



v is an eigenvector

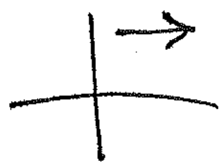
A13B

Adv Topics 2 (Lesieutre)
September 1, 2021

Problem 1. *For each of the following matrices T, choose a couple sample vectors $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and compute $T\mathbf{v}$. What does the matrix do to a vector, geometrically? What does it do to the unit square?*

a) $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

* b) $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$



$T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = t\begin{pmatrix} x \\ y \end{pmatrix}$

c) $\cancel{\begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}}$

* d) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ refleats on $y=x$ (x,y) (x,x) t=1

(x,-x) t=-1.

* e) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ reflection on x-axis $T(x,y)=(x,-y)$ (x,0) t=1

(0,y) t=-1

* f) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ projections (x,0) t=1

(0,x) t=0.

g) $\cancel{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}$

h) $\cancel{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}}$

i) $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$???

Problem 2. *For each of the linear maps described, write down the matrix for the corresponding transformation.*

a) *Reflection about the y-axis.*

b) *Rotation by 45° clockwise. What about other angles θ?*

c) *In 3D: rotation by an angle θ around the z-axis.*

b) $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}$

$(x,0)$ $\lambda = 1$.

that's the only option!

Suppose (a,b) is eigenvector

$T(v) = \lambda v$

$$\begin{pmatrix} a+3b \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

$a + 3b = \lambda a$

$b = \lambda b.$ $\Rightarrow$ $b(\lambda - 1) = 0$

Either: $\lambda = 1$ or $b = 0$

$a + 3b = a$

$3b = 0$

$b = 0$ works!

$a = \lambda a$

either $a = 0$ and $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

or $\lambda = 1$ and we get same again.

$\begin{pmatrix} a \\ 0 \end{pmatrix}$ eigenvector

eg. $\lambda = 1$.

$$\begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}$$

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} a+6b \\ 2b \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix} \Rightarrow$$

$a+6b=\lambda a$

$2b=\lambda b$

$(\lambda-2)b=0$

So... $\lambda=2$ or $b=0$

$\lambda=2$	$b=0$
$a+6b=2a$	$\begin{pmatrix} a \\ 0 \end{pmatrix}$ is eigenvector, $\lambda=1$.
$a=6b$	
eigenvector is $\begin{pmatrix} 6 \\ 1 \end{pmatrix}$ (or a mult of that)	
$\begin{pmatrix} 1 & 6 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 12 \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 6 \\ 1 \end{pmatrix}$	
✓	