This week: knot theory

What is a knot?

- "A closed loop in $\mathbb{R}^3$

- A function $f: \mathbb{R} \to \mathbb{R}^3$ periodic.

~~$f: \mathbb{R} \to$~~

$0 \le t \le 1$

- $f: [0,1] \longrightarrow \mathbb{R}^3$

  $f(t) = (x(t), y(t), z(t))$

  f must be continuous (infinitely differentiable)

  (you can draw without lifting pen)

  $f(0) = f(1)$

  (so it joins up)

  otherwise $f(a) \neq f(b)$ for any a and b.

  (so it doesn't cross itself)

Definition

A knot is an infinitely differentiable function

$f : [0,1] \to \mathbb{R}^3$ given by $f(t) = (x(t), y(t), z(t))$

satisfying:

1) $f(0) = f(1)$

2) $f(a) \neq f(b)$ unless $a = b$ (with exception of $\{0, 1\}$).

Two knots are isotopic if:
(think: the same)

Say $f:[0,1] \to \mathbb{R}^3$
$g:[0,1] \to \mathbb{R}^3$

there exists a family of knots

~~$f_t : [0,1] \to$~~ $f_s : [0,1] \to \mathbb{R}^3 \quad (0 \le s \le 1)$

~~such~~ smoothly varying as s varies and with

$f_0 = f,\ f_1 = g.$

Question - How can we tell if two knots are isotopic?

- How can we tell if a knot can be untied? (I.e. isotopic to the unknot O)

From playing with knots:

- The trefoil is <u>not</u> isotopic to its reverse.

overhand knot

mirror image

"chiral"

- The figure eight knot <u>is</u> isotopic to its reverse.

"amphichiral"

It's hard to tell whether two knots are isotopic!

It's hard to tell whether a knot is isotopic to the unknot!

Those are the "Perko pair": for 75 years thought to be different knots.

How to prove left-handed trefoil is different from right-handed and both are different from unknot?

Idea: Assign numbers to a knot.

If two knots give different answers, they must be different knots.

→ But: need a number that doesn't depend on the exact way the knot is presented.

The <u>crossing number</u> of a knot is the minimum number of times the rope crosses itself in a 2D picture

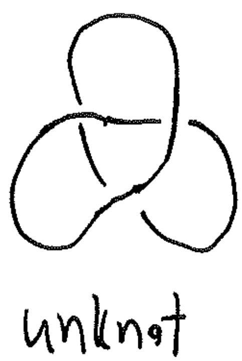
<u>Ex</u> Crossing number of (left- or right-) trefoil is 3.

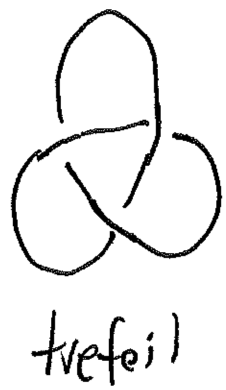
↳ a) How can you prove it's 3? ($\Rightarrow$ not unknot, or figure eight) which has crossing number 4.

b) This can't distinguish handedness.

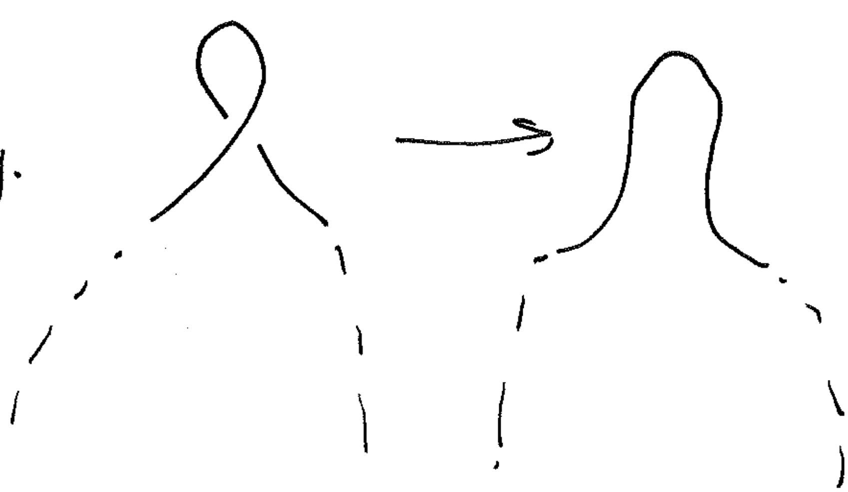
"Knot invariants"

A <u>knot diagram</u> is a 2D picture of a knot, showing which strand goes on top at every crossing:



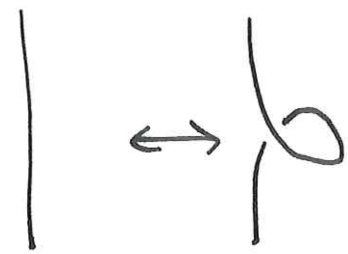


How can we draw isotopies in knot diagrams? What moves can we perform of a knot diagram without changing the knot?

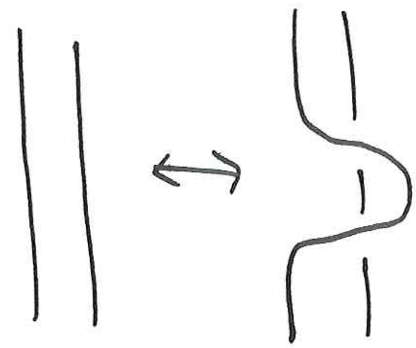
e.g. 

Any isotopy can be made as a combination of three basic moves:

("Reidemeister moves")

1) 

untwist

2) 

cross over

3) 

pass over a crossing
(or under)

Every isotopy can be broken down into these steps! (Annoying to prove, though.)

Def A knot is <u>tricolorable</u> if (True/false invariant, not a number)

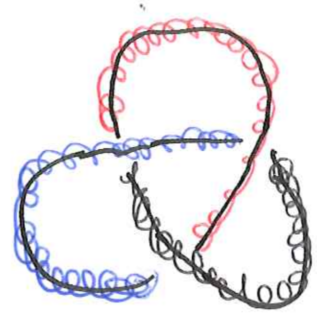
in a knot diagram for the knot,

we can color each arc of the knot

with one of three colors, such that:

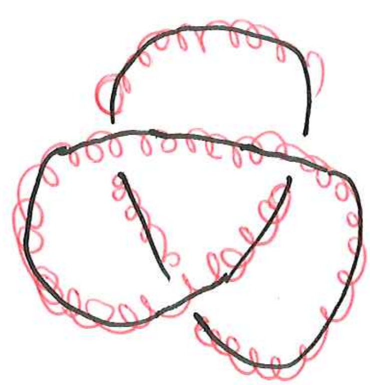
1) At each crossing, either only one color, or all three colors appear.

2) At least two colors are used.



trefoil

tricolorable

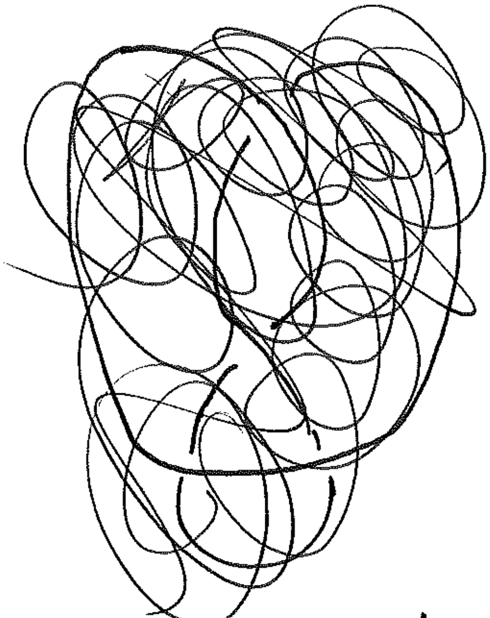


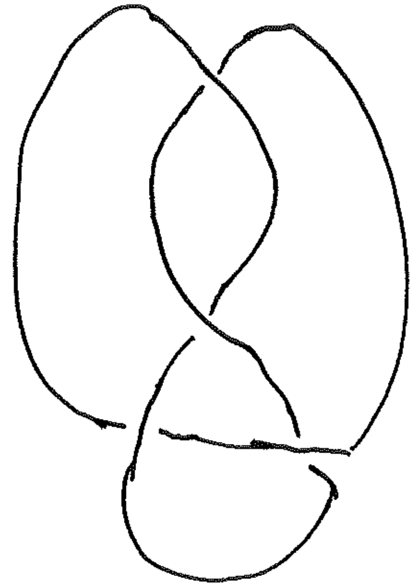
unknot

not tricolorable

(we still need to check that Reidemeister moves don't affect tricolorability!)

Is figure eight tricolorable?





No

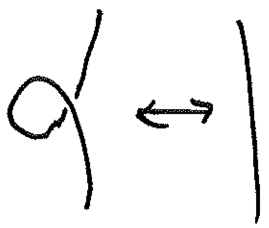
→ not the same as trefoil

→ but maybe the same as unknot?

Recap

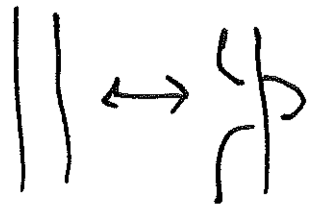
→ We defined a knot and a knot diagram.

→ we defined three Reidemeister moves:



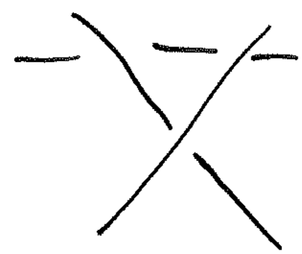
twist

I



crossover

II



passover

III

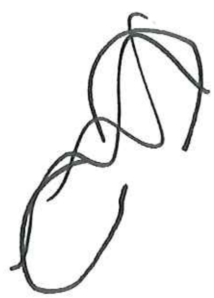
Properties of knots:

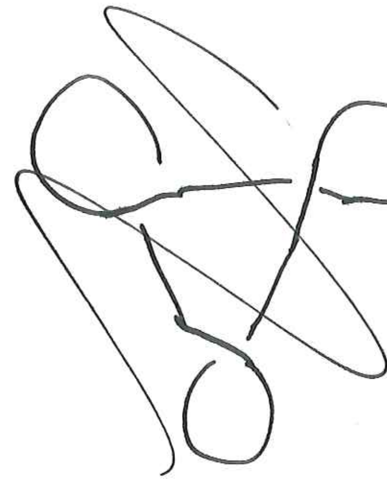
- crossing number (minimal number of crossings in a diagram)
- tricolorability
- chirality (is knot same as reverse?)

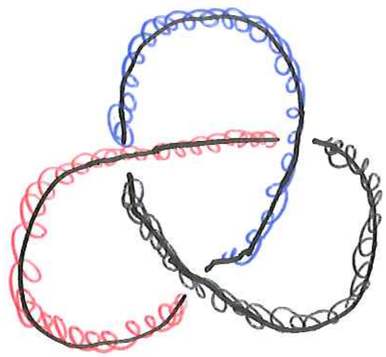
. . .

Today:
- bridge number
- Alexander polynomial

Trefoil knot







trefoil

tricolorable:

Crossing number?

3

tricolorable?

yes

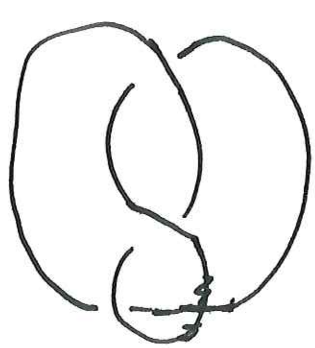
chiral? (different from reverse, or same)

yes (different)

Figure eight

crossing number?

4



tricolorable?

no

chiral?

no (amphichiral/achiral)

Unknot

crossing number?

0

tricolorable?

no

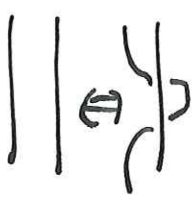
chiral?

amphichiral

- Tricolorability unaffected by Reidemeister moves
- Proofs for crossing numbers.

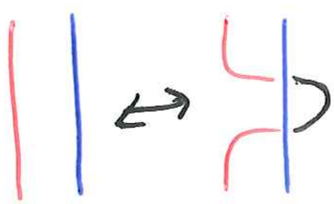
Suppose K is tricolorable. If we perform any Reidemeister moves, it stays that way. (needs checked for all three moves)

ex. <u>Type II</u>. existing tricolor could be either:



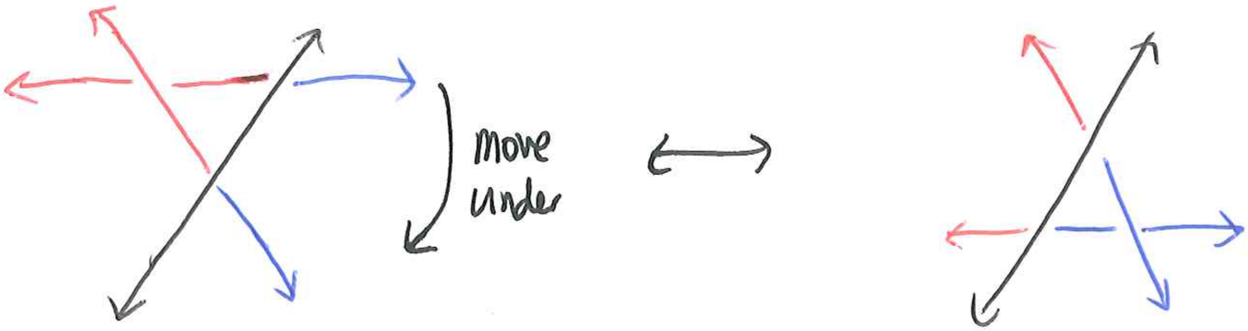
a) 

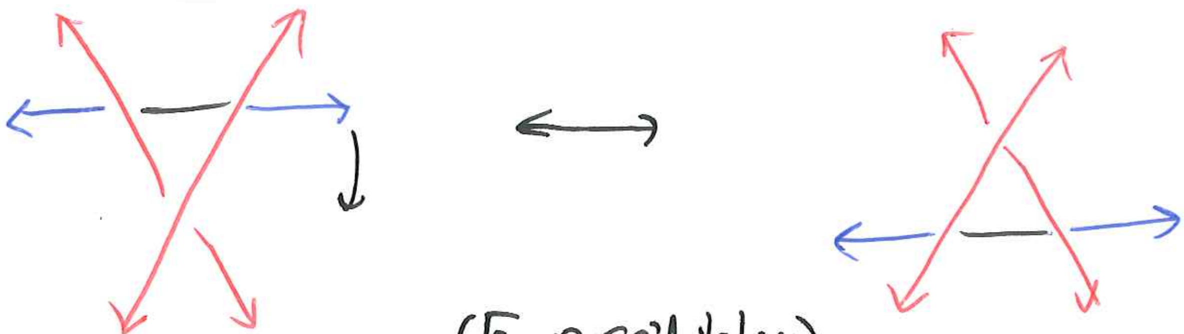
all reversible!

b) 

we need to get a tricoloring on new diagram without changing colors of "outbound" strands.

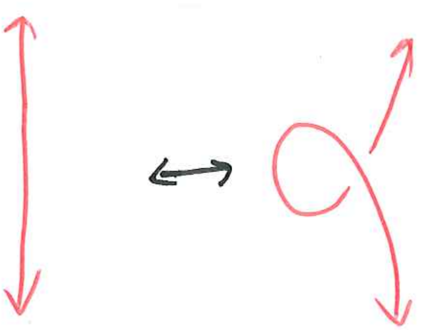
Type III





(5 possibilities)

Type I

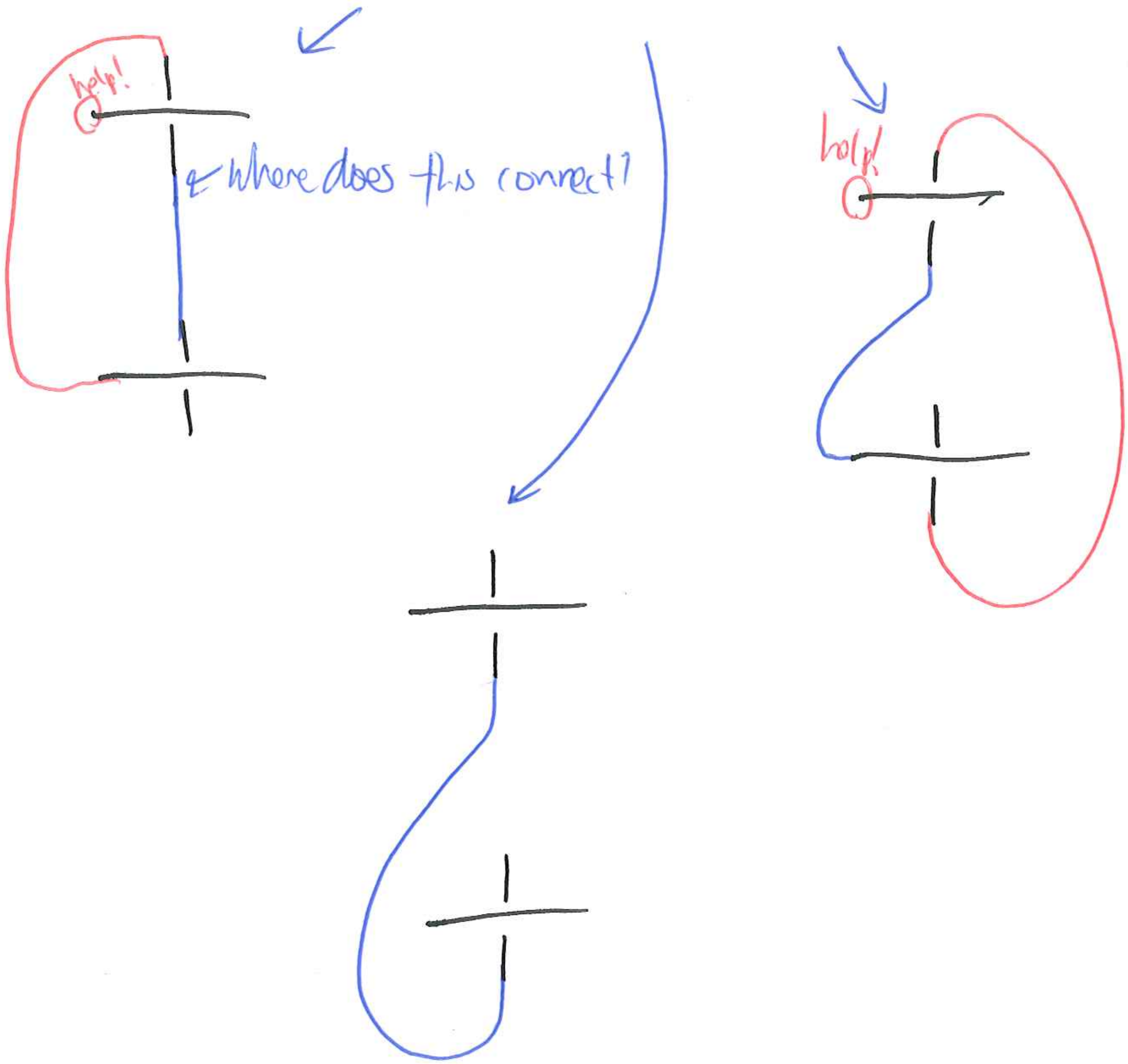


Two-crossing knots

/ first step

/ 2nd

Is there a knot with crossing number two?



No! All are unknots.

→ Crossing number

Theorem Trefoil is ~~actually~~
not isotopic to the unknot, and $cr(\text{trefoil})=3$.

Pf. It's tricolorable, and the unknot isn't.
(and tricolorability is isotopy invariant)

$cr(K) \leq 3$ because we can draw it with 3 crossings.

$cr(K) \neq 2$ because any knot with crossing number 2 is unknot (but trefoil is not unknot)

$cr(K) \neq 1, 0$ for same reason.

How to prove crossing number of fig 8 is 4?

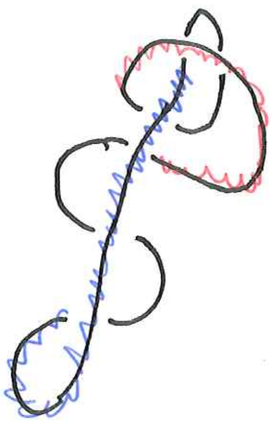
(Prove anything with 3 is trefoil or unknot
and prove figure eight is not unknot)

Crossing number too hard to compute!! Useless invarient. Tricolorability not precise enough.

We need more invariants!

(these have same defect as crossing #; hard to compute)

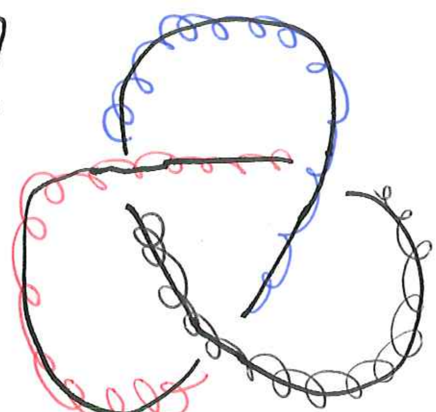
Bridge number



a *bridge* in a knot diagram is an arc that makes 1 or more overcrossings

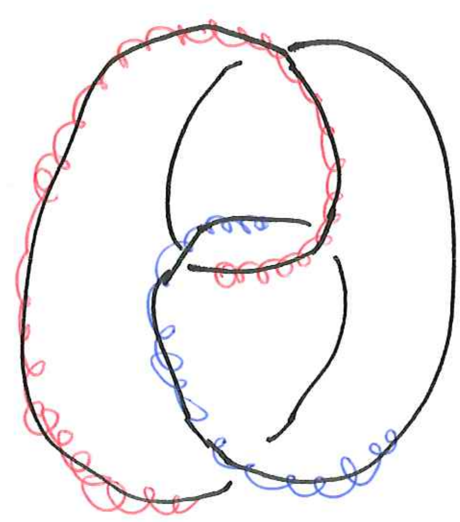
the bridge number of a knot is the minimum number of bridges in a diagram for the knot.

Trefoil?



at most 3...

it's actually just 2!



Next time:

Come up with more invariants that aren't changed by Reidemeister moves, since much easier to calculate

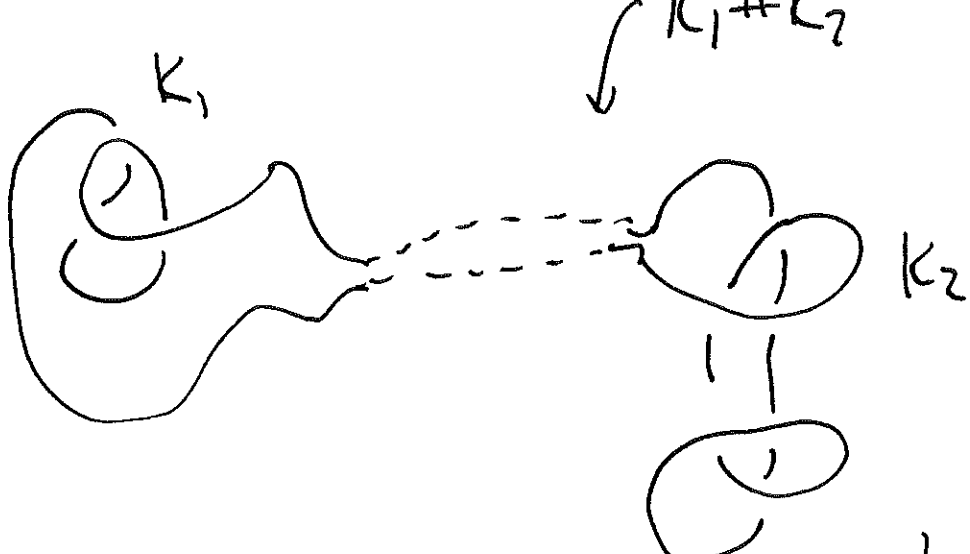
Mostly associate a polynomial to a knot: Alexander poly
Jones poly
HOMFLY poly
...

("Skein relations")

Connected Sum

If K_1, K_2 are two knots,

$K_1 \# K_2$ is obtained by cutting both open and connecting together.



Seems like it shouldn't depend on where we cut it.

this does depend on having an orientation on the knot: a chosen direction to walk along the knot.

1) Is (right-handed trefoil) # (left-handed trefoil)

$\|$ "square knot"

(right-hand trefoil) # (right-handed trefoil)

"granny knot"

2) What's $cr(K_1 \# K_2)$ in terms of $cr(K_1)$ and $cr(K_2)$

3) Could $K_1 \# K_2$ be the unknot even if neither knot is?

1) A knot is a prime knot if it can't be written as a sum of two other nontrivial knots.

→ square/granny knots are not prime.

→ figure 8/trefoil are prime.

Every knot can be broken down into prime knots in a unique way. (a bit hard to prove.)

2) $cr(K_1 \# K_2) \leq cr(K_1) + cr(K_2)$

<u>Open problem</u>: Is it always equal?

(probably yes)

3) Could $K_1 \# K_2 = \bigcirc$?

If 2) true, couldn't happen.

Let's prove it's impossible.

Classic wrong proof:

$$1 = 1 + 0 + 0 + 0 + \cdots$$

$$= 1 + (-1+1) + (-1+1) + (-1+1) + \cdots$$

$$= (1+-1) + (1+-1) + (1+-1) + (1+-1) + \cdots$$

$$= 0 + 0 + 0 + 0 + \cdots = 0$$

rebracketing non-convergent infinite series doesn't work.

Another warning:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots$$ converges to $\log 2$.

AKA $\ln 2$

But! You can reorder terms and make it convergent to anything you want.

For Knots, this proof is correct(able)!

"Mazur's swindle"

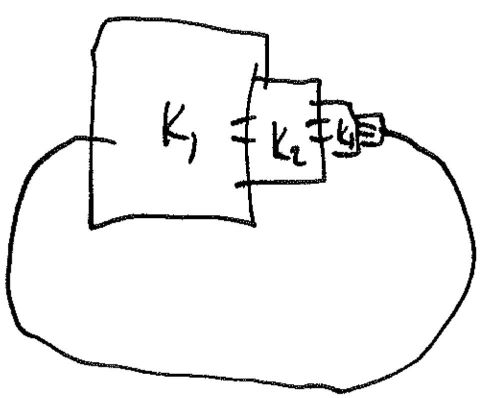
Suppose $K_1 \# K_2 = O$

$$K_1 = K_1 \# (K_2 \# K_1) \# (K_2 \# K_1) \# (K_2 \# K_1) \# \cdots$$

$$= (K_1 \# K_2) \# (K_1 \# K_2) \# (K_1 \# K_2) \# \cdots$$

$$= O \# O \# O \# O = O$$ so $K_1 = O$. (ditto K_2)

You always can take an infinite sum of knots by making them smaller and smaller and smaller

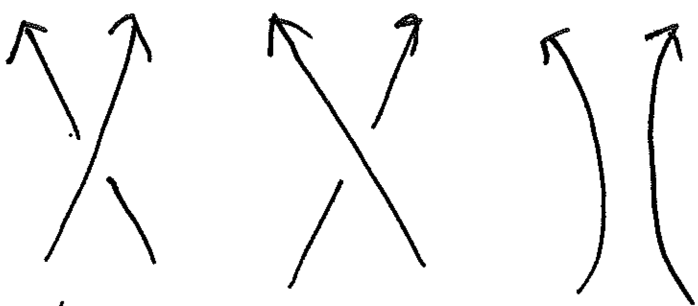


size of $K_n = 1/2^n$

Alexander polynomial

Definition using "Skein relations".

Suppose we have three oriented knots, that only differ in one crossing:



L_+ $\quad$ L_- $\quad$ L_0.

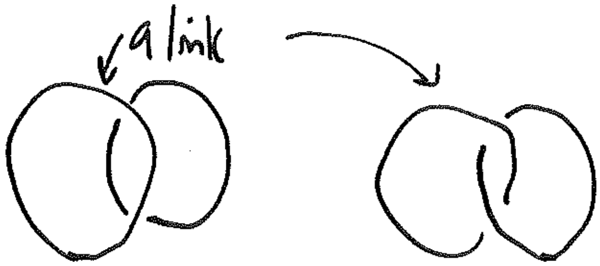
The Alexander polynomial $\Delta_K(t)$ is defined recursively by two rules:

1) $\Delta_K(\text{unknot}) = 1$

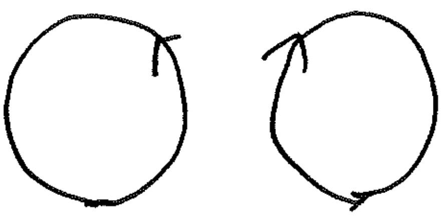
2) $\Delta_{L_+} - \Delta_{L_-} + (t^{+1/2} - t^{-1/2})\Delta_{L_0} = 0$

(so if you know Δ for two, you can get the third)

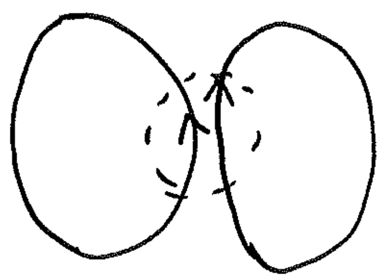
This is defined not just for knots, but also links:



Warm-up: Δ for two disjoint unknots



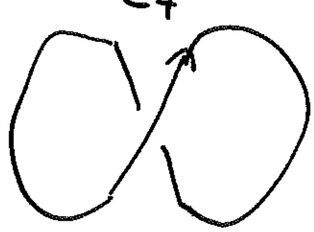
L_0



$$\Delta_{L_+} - \Delta_{L_-} + (t^{1/2} - t^{-1/2})\Delta_{L_0} = 0$$

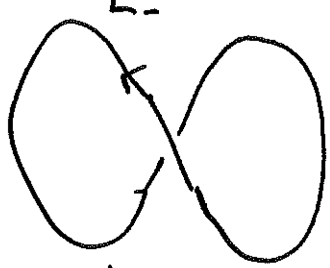
$$1 - 1 + (t^{1/2} - t^{-1/2})\Delta_{L_0} = 0$$

L_+



unknot

L_-



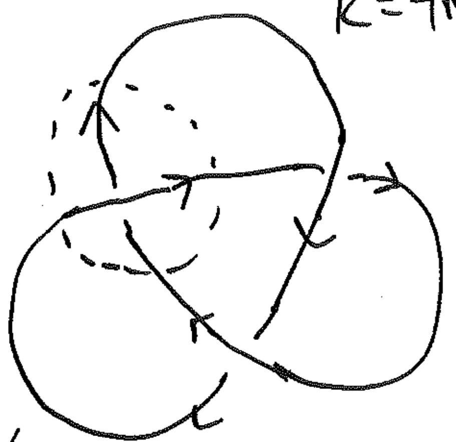
unknot

$$\boxed{\Delta_{L_0} = 0}$$

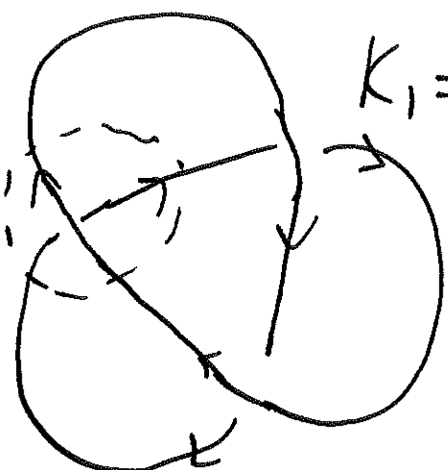
$$\left(\Delta_{L_0}(t) \overset{\text{really}}{=} 0\right)$$

$K = $ trefoil

"+"

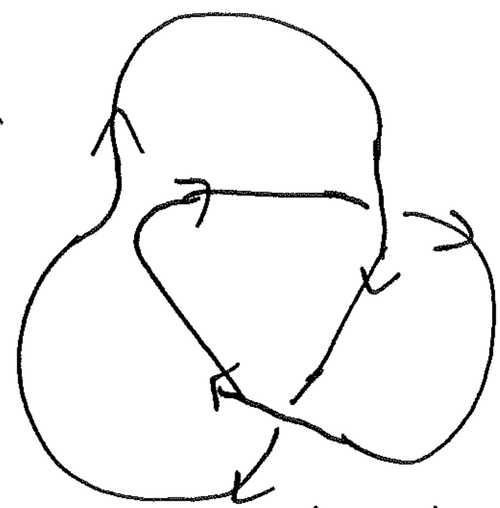


reverse crossing



$K_1 = $ unknot

"−"



$K_2 = $ two linked circles

"0"

$$\cancel{\Delta_K} \qquad \overset{+}{\Delta}_K - \overset{-}{\Delta}_{K_1} + (t^{1/2} - t^{-1/2})\overset{0}{\Delta}_{K_2} = 0$$

Δ_{K_2} on next page

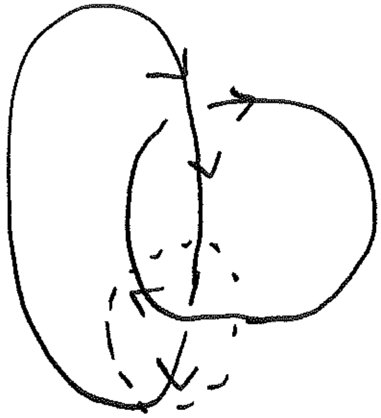
$$\Delta_K - 1 + (t^{1/2} - t^{-1/2})\left(-(t^{1/2} - t^{-1/2})\right) = 0$$

$$\Delta_K = 1 + (t^{1/2} - t^{-1/2})^2 = 1 + (t - 2 + t^{-1})$$

(negative powers of t and half-int) $\longrightarrow$ $\boxed{t - 1 + t^{-1}}$

K_2 = two linked circles

$+$



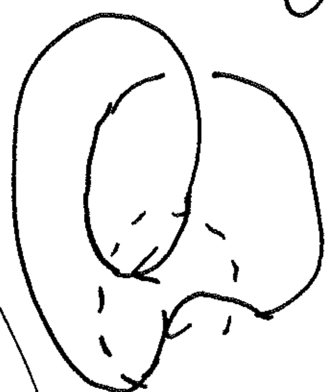
$$\Delta_{K_2} - \Delta_{K_3} + (t^{1/2} - t^{-1/2})\Delta_{K_4} = 0$$

$$\Delta_{K_2} - 0 + (t^{1/2} - t^{-1/2})1 = 0$$

$$\Delta_{K_2} = -(t^{1/2} - t^{-1/2})$$

$-$

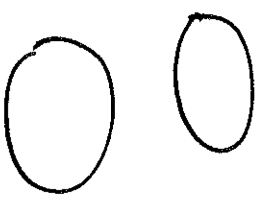
0



K_3 = disjoint circles

K_4 = unknot

$\shortparallel$



Things to check

- not affected by Reidemeister
- these rules actually define it for any knot

HW Calculate $\Delta_{\text{figure } 8}$.

(you should get $3-t-t^{-1}$)

$\longrightarrow$ not trefoil

$\longrightarrow$ not unknot!

But:

$$\Delta_K = \Delta_{\text{reverse}(K)}$$

(e.g. left- and right-handed trefoil give same answer)

$$\Delta_{K_1 \# K_2} = \Delta_{K_1} \Delta_{K_2} \quad \text{(HW!)}$$

So

$$\Delta_{\text{square knot}} = \Delta_{\text{right trefoil}} \cdot \Delta_{\text{left trefoil}}$$

$$= (t-1+t^{-1})(t-1+t^{-1})$$

$$= (t-1+t^{-1})^2$$

$$\Delta_{\text{granny knot}} = \Delta_{\text{right trefoil}} \cdot \Delta_{\cancel{\text{left}}\ \text{right trefoil}}$$

$$= (t - 1 + t^{-1})^2$$

Δ can't tell the difference between square and granny knots!

Enter the Jones polynomial:

$$\frac{1}{t}\Delta_{K_+} - t\Delta_{K_-} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)\Delta_{K_0}.$$

Calculate in the same way!

→ Jones polynomial <u>can</u> tell difference between square + granny knots.

→ and between left and right trefoils!

but even this can't tell all knots apart...

Need new invariants... Khovanov homology...