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Coverings and Ball Quotients

with special emphasis on the 3-dimensional case

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§-∞ Introduction

If V is a smooth projective hypersurface in $\mathbb{P}^N$, the projection ρ_p from a point p not on V displays V as a branched cover of $\mathbb{P}^{N-1}$, $\rho_{p|V}$: $V \to \mathbb{P}^{N-1}$. This form of branched covering is very old and classical. The properties of the map ρ_p, however, are not good: in general ρ_p is not Galois. If ρ_p happens to be a double cover, then it is automatically Galois, and this situation has long been used to relate information on V to information concerning the branch locus B. For example, projecting from one of the 16 double points of a Kummer Quartic S in $\mathbb{P}^3$ displays $\hat{S}$=S blown up at a double point as a double cover of $\mathbb{P}^2$, branched along 6 lines in general position*, from which Kummer deduced the existence of his famous 16_6 configuration. Projecting a cubic from a point p on the cubic but not on any of the 27 lines, displays the cubic surface (blown up at one point), $\hat{S}$, as a double cover of $\mathbb{P}^2$ with branch locus a smooth quartic curve. The 28 bitangents of the quartic curve are the images of the 27 lines and the exceptional $\mathbb{P}^1$ on $\hat{S}$. In these cases, V was given, and the branch locus B was determined by V.

This can be turned around to a process to construct the "variety of your choice". That is, given a certain divisor B on an N-fold X, if we can construct a branched cover $Y \to X$, with branch locus B, and if we can effectively calculate the invariants of Y, this reduces the problem of constructing N-folds Y with given invariants to the problem of finding the right divisor $B \subset X$, something N-1 dimensional. This is one of the main themes of much recent work applied to the case N=2, of which we mention {Hi1}, {Hö}, {Mi1}, {P} and {So}.

In order to effectively calculate the invariants of Y such as genera, Chern numbers, etc., one must assume the covering $Y \to X$ is Galois. There are principally 4 methods of constructing Galois covers:

* in the combinatorial sense: no q-fold points, q≥3; however tangent to a quadric

A) <u>The root method</u>: Take $B \sim n\bar{B}$, $\mathcal{L} = \mathcal{O}(\bar{B})$ so B is the divisor of a section of $\mathcal{L}^n$. Then, denoting the image of the section by $\sigma(X)$, a cyclic cover is defined by

$$Y := \phi^{-1}(\sigma(X)), \quad \phi: \mathcal{L} \to \mathcal{L}^{\otimes n}.$$

If $\mathcal{L}$ happens to be the trivial bundle, $\bar{B}$ is the divisor of a meromorphic function f on X and Y is the covering defined by the algebraic extension $\mathcal{M}(X)[\sqrt[n]{f}]$, $\mathcal{M}(X)$=function field of X. In this case of course, one may take any $n>1$.

B) <u>The group method</u>: any normal subgroup $N \triangleleft \pi_1(X-B,*)$ defines a Galois cover with branch locus B.

C) <u>Fibre products</u>: If $\pi: Y \to X$ is a Galois cover, and $F \to X$ a fibre space, the fibre product $F^{\sim} = F \times_\pi X$ is again a Galois cover, over F:

$$\begin{array}{ccc} F^{\sim} & \to & F \\ \downarrow & & \downarrow \\ Y & \to & X. \end{array}$$

D) <u>Differential equations</u>: The monodromy of certain differential equations define branched coverings. Classical example is the hypergeometric differential equations, see {DM}.

Assuming one knows the degree of the cover $Y \to X$ -which for type B) and D) can be extremely difficult to determine- it is generally easy to calculate genera and Chern numbers. See, for example, {Hi4}, {Ka}, {P}, {Mi1} and {So}.

Once we have this it is natural to ask what kind of variety one wants to construct. The first systematic utilisation of this approach was to the study of surfaces of general type by Ulf Persson {P}. Using double covers of ruled surfaces, he succeeded in constructing genus 2 fibrations, surfaces of general type with $(c_1^2, c_2) = (a,b)$ for (almost) any $(a,b) \in \mathbb{Z} \times \mathbb{Z}$ with $a \leq 2b$, $5a > b$. The main technical tool was to allow the branch locus to aquire certain singularities. In this guise-coverings branched along a singular divisor as a local phenomenae-the theory of branched covers has a much longer history, going back to the theory of algebroid functions (see Jung {J}, Kähler {Kä}). The two dimensional case was settled by Hirzebruch {Hi5}, and Grauert and Remmert in {GR} studied the analytic covers defined by Behnke-Stein {BS} and Cartan {Ca}, and provided essential technical tools for the study of higher dimensional coverings. Perssons contribution consisted in showing the <u>existence</u> of global branch divisors B with certain prescribed singularities.

Since 1977 when Yau solved Calabi's conjecture, one of the favorite kind of varieties to look for is ball quotients. The Yau inequality, one of the many corollaries of Calabis conjecture {Y}, states that an

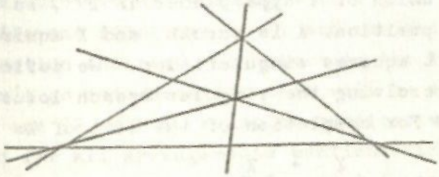
N-dimensional smooth variety Y with ample canonical bundle satisfies

$$(-1)^N c_1^N(Y) \leq (-1)^N \cdot \frac{2(N+1)}{N} \cdot c_1^{N-2}(Y) c_2(Y),$$

with equality holding if and only if Y is a smooth, compact quotient of the complex N-ball. There is also a logarithmic (non-compact) version of this, due to R. Kobayashi {K3}, see 3.3.

The Yau inequality has been used often in past years in the case N=2, that is, for surfaces. In his thesis {Li}, Livné used this to show that certain branched coverings of the elliptic modular surfaces are ball quotients. About the same time, Mostow {Mo1}, who was actually more interested in non-arithmetic lattices in PSU(N,1), used the hypergeometric differential equation to construct directly (bypassing Yau's theorem) 2-dimensional ball quotients as coverings of rational surfaces. Deligne and Mostow {DM} later generalised this method to get examples of (compact or non-compact) ball quotients in all dimensions ≤ 5.

A few years ago Hirzebruch, inspired by Mostows examples, constructed 3 2-dimensional ball quotients as coverings of $\mathbb{P}^2$ branched along arrangements of lines in the plane, defined by algebraic Kummer extensions of the rational function field {Hi1}. Two of these examples, in particular one related to the famous line arrangement :



are identical with two of Mostows examples. (Actually, all of Mostows examples are branched along this line arrangement, with different branching degrees along the different lines. For more details see Höfers thesis {Hö}.) Furthermore, he and Ishida showed that (several) groups of order 25 act freely on this ball quotient, and the quotient is one of the examples constructed by Livné. The 3 arrangements used by Hirzebruch are all related to imprimitive reflection groups. Thus there seemed to be a strong correlation between reflection groups defining line arrangements and discrete subgroups of PSU(2,1). This point of view was intensively studied by M. Yoshida {Yo1},{Yo2}. Last year T. Höfer in his thesis {Hö} showed, by allowing different branching degrees along the different lines of the arrangement (which is different than Hirzebruchs construction), that there are ball quotients covering $\mathbb{P}^2$ and ramifying along the line arrangements defined by all the primitive reflection groups acting on $\mathbb{P}^2$. The theory developed by Höfer actually showed more. In simplified form his main result is

Theorem: Let $Y \xrightarrow{\pi} X$ be a Galois covering of compact, complex surfaces, branched along a normal crossings divisor $L=\Sigma L_i \subset X$. Assume $c_1^2(X)=3c_2(X)$, $c_1(L_i)=\pm 2L_i^2$. Then

$$\left\{\begin{array}{l}Y \text{ is a ball}\\ \text{quotient}\end{array}\right\} \Leftrightarrow \left\{\begin{array}{l}\pi^{-1}(L_i) \text{ are subball quotients, i.e.}\\ \qquad c_1(\pi^{-1}(L_i))=2(\pi^{-1}(L_i))^2\end{array}\right\}$$

This theorem is remarkable in that it reduces the question "Y ball quotient?" to a question involving only curves on Y, and in fact, curves one knows particularly well.

The purpose of the present paper is to generalise Hirzebruch's construction to higher dimensions (Chapter 1) and to prove an N-dimensional anologue of the theorem above (Chapter 2). We know describe the contents in some detail.

In §1 we give the construction of the N-dimensional analogue of the Kummer extensions used in (Hi1), which we call Fermat covers. The Kummer extension $\mathcal{M}(\mathbb{P}^N)[\sqrt[n]{l_2/l_1}, \ldots, \sqrt[n]{l_k/l_1}]$, where the l_i are linear forms defining k hyperplanes in $\mathbb{P}^N$, defines an algebraic variety X. Our viewpoint is to look at X as a complete intersection of k-N-1 Fermat hypersurfaces of degree n in $\mathbb{P}^{k-1}$. X covers $\mathbb{P}^N$ of degree n^{k-1}, ramifying over the union of k hyperplanes in $\mathbb{P}^N$, an arrangement $L \subset \mathbb{P}^N$. If L is in general position, X is smooth, and X aquires singularities as the arrangement L aquires singularities. We define a desingularisation Y of X by resolving the singular branch locus L in the ambient $\mathbb{P}^N$, and taking the Fox completion of the lift of X:

$$\begin{array}{ccc} Y & \to & X \\ \downarrow & & \downarrow \\ \hat{\mathbb{P}}^N & \to & \mathbb{P}^N \end{array}.$$

Of course, in dimensions $N\geq 3$, such a resolution is not unique, but the discusion of the special case N=3 in 2.4. would seem to indicate that Y is a good model of X. Furthermore, this resolution is inductive, in the following sense: If $Y \to \hat{\mathbb{P}}^N$ is the desingularisation of the Fermat cover X as above, then all ramification divisors $R_i \subset Y$ are themselves (products of) desingularisations of Fermat covers $Q_i=Q_i^1\times Q_i^2$,

$$\begin{array}{ccc} R_i=R_i^1\times R_i^2 & \to & Q_i^1\times Q_i^2 \\ \downarrow & & \downarrow \\ \hat{\mathbb{P}}^{m_1}\times\hat{\mathbb{P}}^{m_2} & \to & \mathbb{P}^{m_1}\times\mathbb{P}^{m_2}\end{array}, \quad m_1+m_2=N-1$$

The Y so constructed has a normal crossings ramification divisor $R=\Sigma R_i \subset Y$ and branching degree =n along all R_i. This can be generalised by taking quotients of Y and utilising weighted projective spaces, yielding covers $\overline{Y} \to \hat{\mathbb{P}}^N$ with the same branch locus B but different branching degrees along the different components $B_i \subset B$. This is explained in 1.7. All such quotients yield coverings $\overline{Y} \to \hat{\mathbb{P}}^N$ which have <u>abelian</u> Galois group. Of course some of the most interesting examples might be branched along the same divisor in $\hat{\mathbb{P}}^N$, but with a non-abelian Galois group. Such coverings cannot be constructed by this method.

In §2 we specialise to the case of N=3. We start in 2.1. with an overview of the known interesting arrangements in $\mathbb{P}^3$, and calculate the Chern numbers c_1^3, c_1c_2 and c_3 of Y in dependence on the combinatorial data of the arrangement and the degree n. The resulting formula are atrocious, reflecting the combinatorial complexity of an arrangement of planes in $\mathbb{P}^3$. A more conceptual formula is given in 4.6.1., which the reader may prefer to look at first. Things start to get interesting in 2.3., where we compute under which conditions the canonical bundle is ample. It appears that certain singularities on X "destroy" the ampleness of K_Y. In the next section, where we determine whether Y is a minimal model in the sense of M. Reid, the same singularities appear as the canonical singularities which can occur on X. We get the interesting result that, with some mild restrictions on the arrangement, K_Y is ample $\Leftrightarrow$ X has no canonical singularities. This assures we are working with a good model. In 2.5. we discuss the different fiberings on Y, and use this, together with Viehweg's classification theorems, to determine the Kodaira dimension of Y. Here we don't go into details; this will be done in a forthcoming paper. We end §2 with an examples section, where the Chern numbers are listed for all arrangements mentioned in 2.1.

All of Chapter 1, except sections 2.2. and 2.4. are very elementary and require no special knowledge to follow. The reader will find the calculations of 2.2. quite similar to those used in the proof of Noethers formula in (GH), which is a good reference for this section. The techniques discussed in 2.4. are much more sophisticated; Reid's articles (R1) and (R2) give an attractive introduction.

In Chapter 2 we turn to the question as to when a branched covering $Y \to X$ is (the compactification of) a ball quotient. We get a nice answer if the covering is of type (F), a notion we introduce in 4.1. This is a general enough type of covering to include for example all coverings $Y \to \hat{\mathbb{P}}^N$ branched along (the desingularisation of) an arrangement of hyperplanes in $\mathbb{P}^N$, even those with non-abelian Galois group. The result is:

Theorem: Let $Y \xrightarrow{\pi} X$ be a covering of type (F) of dimension $N\geq 3$, $B=\Sigma B_i \subset X$ the branch divisor, and $\pi^{-1}(B)=\Sigma\pi^{-1}(B_i)$ the reduced ramification locus in Y. Then

$$\left\{\begin{matrix}\text{Y is the compactification}\\ \text{of a ball quotient}\end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix}\pi^{-1}(B_i)\text{ is the compactification of}\\ \text{a subball quotient, all i}\end{matrix}\right\}$$

Subball quotients are defined in §3. This theorem reduces the N-dim. question to an (N-1)-dimensional one about the (reduced) ramification divisors. The corresponding theorem for N=2 was proved in [Hö]; it is remarkable that the proof of this theorem for $N\geq 3$ is <u>much easier</u> than for the case N=2. This theorem has lots of interesting corollaries, a few of which we mention.

Corollary: There are no Fermat covers $Y \to \hat{\mathbb{P}}^N$, $N\geq 3$, such that Y is a compact ball quotient.

A. Sommese has remarked that the proof of the above also yields:

Corollary: There are no Fermat covers $Y \to \hat{\mathbb{P}}^N$, $N\geq 3$, such that T_Y^* is ample.

Using an existence theorem of R. Kobayashi (4.5.) we can show:

Corollary: With the exception of the seven examples of Deligne-Mostow there is precisely one covering of type (F), $Y \to \hat{\mathbb{P}}^3$, branched along the desingularisation of a (known) arrangement of planes in $\mathbb{P}^3$, such that Y is the compactification of a ball quotient. This example ramifies over the arrangement defined by the reflection group G_{576}, number 28 in the S-T list, the symmetry group of the regular 24-cell in $\mathbb{R}^4$.

Of all the reflection groups in dimension 4 (projective: 3) only 4 are the Weyl groups of simple Lie algebras: A_4, B_4, D_4 & F_4. The corresponding arrangements are $A_1(10)=\text{Ceva}^3(1,4)$, $A_1^3(12)$, $A_1^3(16)$ and $A_1^3(24)$, the arrangement defined by G_{576}, the Weyl group of F_4. The seven examples of Deligne-Mostow ramify over $A_1^3(10)$; two of these examples also ramify over $A_1^3(12)$; one example ramifies over $A_1^3(16)$. This, together with the above corollary, yield the fascinating result:

Corollary: All coverings of type (F) $Y \to \hat{\mathbb{P}}^3$ ramifying over an arrangement of planes, such that Y is (the compactification of) a ball quotient, ramify over arrangements defined by the Weyl groups of A_4, B_4, D_4, F_4, and vice versa, each such arrangement is covered by a ball quotient.

This is strikingly different than the 2-dimensional case. The existence of such coverings has the group-theoretic interpretation as the existence of extensions of the Weyl groups $\mathcal{A}_4$, $\mathcal{D}_4$ and $\mathcal{F}_4$ of the Lie algebras A_4, D_4 and F_4:

$$1 \to \Gamma_A \to \tilde{\Gamma}_A \to \mathcal{A}_4 \to 1$$
$$1 \to \Gamma_D \to \tilde{\Gamma}_D \to \mathcal{D}_4 \to 1$$
$$1 \to \Gamma_F \to \tilde{\Gamma}_F \to \mathcal{F}_4 \to 1$$

where Γ_A, Γ_D and Γ_F are discrete subgroups of $PSU(3,1)$ acting properly discontinuously with cocompact quotient on the 3-ball. This result, together with the theorem above, also narrows down the possibilities of such coverings in dimensions > 4 (F_4 is a subgroup of E_6, E_6 of E_7, and E_7 of E_8). We shall consider this question elsewhere.

§3 is a review of mostly well-known results and can be skipped by experts. §4, except part of 4.6. is quite elementary and might be viewed as an extended exercise in the use of the adjunction formula. To keep the presentation brief, in 4.6. we must assume a certain familiarity on the part of the reader with the theory of coverings defined by the hypergeometric differential equation as in [DM]. Furthermore, for the examples we use in a crucial way recent results of R. Kobayashi, [K2], [K3] , which are very deep. We do not give proofs, but if the reader is willing to believe these, the understanding is in no way impaired.

This research was done in the last 18 months during the authors stay at the Max-Planck Institut für Mathematik in Bonn. Many mathematicians have contributed ideas to put this paper in a finished form. In particular I want to thank E. Sato and M. Yoshida of Kyushu University, A. Sommese for helpful comments, help with ugly computations and some moral backing when I needed it. Above all, I express my appreciation for the pleasure of being able to work with F. Hirzebruch, my teacher and the consular of this thesis.

References in the text to §§5-6 of Chapter 3 are references to a forthcoming paper.

§0. Notations and Conventions

0.1. Adjunction formula

Let V be a smooth algebraic variety (or a compact, complex manifold) of dimension N and $W \subset V$ a smooth subvariety (or a submanifold) of dimension M, and let

$$i : W \to V$$

be the inclusion. i induces an exact sequence of locally free sheaves or vector bundles:

$$0 \to T_W \to T_{V|W} \to N_V W \to 0,$$

where $N_V W$ is the normal bundle of W in V. By the multiplicativity of the total Chern class we get the equation in $H^*(W, \mathbb{Z})$:

$$c(T_{V|W}) = c(T_W)c(N_V W).$$

The definition of Chern classes is as in [Hi], and in the algebraic case also as in [Fu]. This is an equation of mixed cohomology classes and sorting out by dimension yields the adjunction formula:

$$c_1(V)_{|W} = c_1(W) + c_1(N_V W)$$

$$c_2(V)_{|W} = c_2(W) + c_1(W)c_1(N_V W) + c_2(N_V W)$$

$$\vdots$$

$$c_M(V)_{|W} = c_M(W) + c_{M-1}(W)c_1(N_V W) + \dots$$

$$+ c_1(W)c_{M-1}(N_V W) + c_M(N_V W),$$

where of course $c_j(N_V W) = 0$ for $j > N-M$.

If $V \supset W$ is a divisor, then $N_V W = \mathcal{O}_V(W)_{|W}$ and $c_1(N_V W)$ is often written $(W)|_W$ or W^2. In this notation, $c_1^j(N_V W) = ((W)_{|W})^j = (W^2)^j = W^{j+1} \in H^*(W)$. Thus $c_1^{N-1}(N_V W) = (W)^N$ is a number, the "self intersection number" of the divisor W. If $V = \mathbb{P}^N$, calculations involving Chern classes are usually done in homology.

0.2. Coverings

Let $f : Y \to X$ be a finite map (a branched covering) of N-dimensional smooth varieties (or compact, complex manifolds). The ramification locus is defined, as a set, as

$$R = \{y \in Y \mid f^* : T_y Y \to T_{f(y)} X \text{ is not iso.}\}.$$

The <u>branch</u> <u>locus</u> $B \subset X$ is defined, as a set, as the image $f(R)$. R can be given a scheme structure by the vanishing of the Jacobian determinant, i.e. the zero scheme of the line bundle $\Lambda^N f^* T_X \otimes \Lambda^N (T_Y^*)$ or the zero scheme of the map

$$\Lambda^N df : \Lambda^N T_Y \to \Lambda^N f^* T_X .$$

This $R \subset Y$ is a non-reduced divisor. Assume $R = \Sigma R_i$ is a normal crossings divisor and that $f: Y \to X$ is Galois. Then each component R_i is of the form

$$R_i = n_i \pi^{-1}(B_i)$$

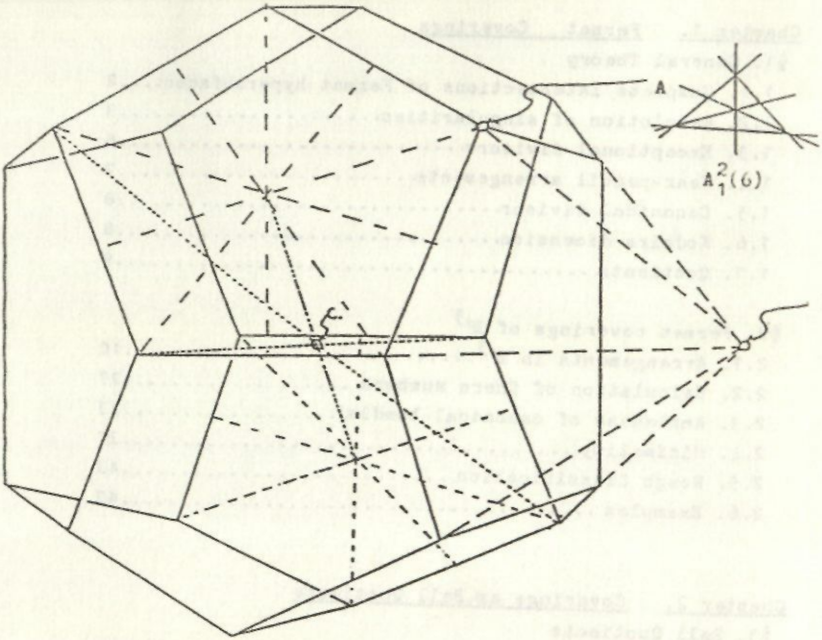
for some component $B_i \subset B$ of the branch locus. The factor n_i is called the <u>branching</u> <u>degree</u> along B_i.

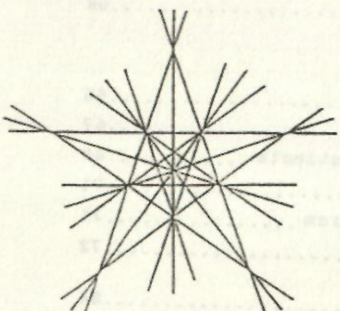
0.3. Notations

$f: Y \to X$ denotes a birational map (Y smooth, X singular) in §1. In §3-§4 $\pi: Y \to X$ usually denotes a branched covering. The context makes clear what is meant. $\hat{\mathbb{P}}^N$ denotes some blow-up of $\mathbb{P}^N$, usually to desingularise a singular divisor. All other notations are introduced where needed. We also mention that when talking about arrangements of hyperplanes , we usually just speak of planes.

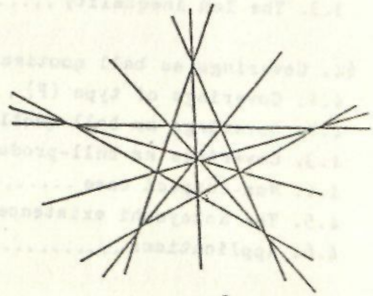
CONTENTS

FERMAT COVERINGS OF PROJECTIVE SPACE





C. $A_1^2(15)$



B. $A_1^2(10)$

The Arrangement $A_1^3(27)$

Chapter 1. Fermat Coverings

In this chapter we introduce the notion of a Fermat covering of $\mathbb{P}^N$, which is a generalisation of a non-singular complete intersection of Fermat hypersurfaces. More precisely, a complete intersection of say k-N-1 Fermat hypersurfaces of degree n in $\mathbb{P}^{k-1}$ can be displayed as a branched cover of $\mathbb{P}^N$, branched along the union of k hyperplanes in $\mathbb{P}^N$ which are in general position, with branching degree n along each plane. The generalisation we study consists of allowing the hyperplanes to no longer be in general position. This is of course the same thing as allowing the complete intersection above to aquire certain very special singularities. Now in general there is no "canonical" minimal resolution in dimesions $N \geq 3$, as in the case of surfaces. However, for the special singularities we consider, there is a sort of "canonical" resolution, one induced by the canonical resolution in the sense of toroidal embeddings, which amounts to blowing up points, curves, etc. in the ambient $\mathbb{P}^{k-1}$ and taking the proper transform of the singular N-fold. We do this in §1 and give some general remarks. In §2 we apply the general theory to the case of algebraic 3-folds, N=3, and in this case study certain questions in more detail.

We start with a self-contained account of the interesting arrangements in $\mathbb{P}^3$ known to us. We then calculate formula for c_1^3, c_1c_2, and c_3 as expressions depending on the combinatorial data of the arrangement and the degree n. It would appear that the formula appearing here are too complicated to be practicably calculated in higher dimensions. However, in dimension 3 we can list the characteristic numbers for all arrangements listed in 2.1 as expressions depending only on n. Finally, we give a criterium for the ampleness of the canonical bundle and consider the rough classification by Kodaira dimension.

§1. General Theory

1.1. Complete intersections of Fermat hypersurfaces

Let $x_o:\ldots:x_{k-1}$ be homogenous coordinates on $\mathbb{P}^{k-1}$, and let $X^N \subset \mathbb{P}^{k-1}$ be the N-dimesional complete intersection (scheme theoretic):

$$X = F_1 \cap \ldots \cap F_{k-1-N},$$

where the F_i are Fermat hypersurfaces defined as the zero loci of the polynomials $F_i(x_o:\ldots:x_{k-1})$:

$$F_1(x_o:\ldots:x_{k-1}) = a_o^1 x_o^n + \ldots + a_{k-1}^1 x_{k-1}^n$$

$$\vdots$$

$$F_{k-1-N}(x_o:\ldots:x_{k-1}) = a_o^{k-1-N} x_o^n + \ldots + a_{k-1}^{k-1-N} x_{k-1}^n \ .$$

If the dimension of X =N, the matrix $A=(a_j^i)_{\substack{i=1,\ldots,k-1-N \\ j=0,\ldots,k-1}}$ will have maximal rank =k-N-1. Assume for the moment the non-degeneracy condition:

(ND) Any (N+1) columns of A form a submatrix of rank N+1

Then X is non-singular. To see this, consider the nth power map:

$$\phi_n : \mathbb{P}^{k-1} \longrightarrow \mathbb{P}^{k-1}$$

$$(x_o:\ldots:x_{k-1}) \longmapsto (x_o^n:\ldots:x_{k-1}^n) = (w_o:\ldots:w_{k-1}).$$

ϕ_n displays $\mathbb{P}^{k-1}$ as an n^{k-1}-sheeted branched cover of itself, branched along the k coordinate axi $w_j=0$, j=0,...,k-1. Notice that the image of the Fermat hypersuface F_i is the zero locus of the linear polynomial

$$F_i(w_o:\ldots:w_{k-1}) = a_o^i w_o + \ldots + a_{k-1}^i w_{k-1},$$

which is a $\mathbb{P}^{k-2}$. It follows that the image of X, $\phi_n(X)$, is a linear subspace $\mathbb{P}^N \subset \mathbb{P}^{k-1}$. ϕ_n thus describes X as a n^{k-1}-sheeted cover of $\mathbb{P}^N$, branched along the k hyperplanes

$$l_j := \{w_j = 0\} \cap \phi_n(X) \subset \mathbb{P}^N.$$

We then have the relations:

$$(*) \qquad \left.\begin{array}{c} a_o^1 l_o + \ldots + a_{k-1}^1 l_{k-1} = 0 \\ \vdots \qquad\qquad \ddots \\ a_o^{k-1-N} l_o + \ldots + a_{k-1}^{k-1-N} l_{k-1} = 0 \end{array}\right\} \text{ on } \phi_n(X).$$

Now suppose some $(k-1-N)\times N$ submatrix of A has rank $N+1$, say the first $N+1$ columns:

$$\det \begin{pmatrix} a_o^1 & \cdots & a_N^1 \\ \vdots & & \vdots \\ a_o^{k-1-N} & \cdots & a_N^{k-1-N} \end{pmatrix} \neq 0$$

Then the first $N+1$ "variables" $l_o,\ldots,l_N$ are projectively independent (not all $N+1$ pass through a point), and we can use $l_o,\ldots l_N$ as projective coordinates on $\phi_n(X)=\mathbb{P}^N$. If the condition (ND) holds, this simply means we can use <u>any</u> $N+1$ of the k planes as coordinates, so that no more than N pass through a point, no more than $(N-1)$ pass through a line, etc. Such an arrangement is said to be in general position.

This shows that X is non-singular. The uniformisation of the covering $X \to \mathbb{P}^N$ is given as follows. Let $p\in\mathbb{P}^1\subset\mathbb{P}^2\subset\ldots\subset\mathbb{P}^N$ be given by

$$p=\{l_{i_1}=l_{i_2}=\ldots=l_{i_N}=0\}$$
$$\mathbb{P}^1=\{l_{i_1}=\ldots=l_{i_{N-1}}=0\},\ l_{i_N}=\text{coordinate on } \mathbb{P}^1, \text{ etc.}$$

Then ϕ_n is given as:

$$\phi_n : X \to \mathbb{P}^N$$

onto nghd of p $\quad (z_1,\ldots,z_N) \to (z_1^n,\ldots,z_N^n)=(l_{i_1},\ldots,l_{i_N})$

onto nghd of $\mathbb{P}^1-\{p\}$ $\quad (z_1,\ldots,z_N) \to (z_1^n,\ldots,z_{N-1}^n,z_N)=(l_{i_1},\ldots,l_{i_{N-1}},l_{i_N})$, etc.

Let $L=\bigcup_j l_j$. We have a stratification of L,

$$L=L^1\supset L^2=\bigcup_{i\neq j}(l_i\cap l_j)\supset L^3\supset\ldots\ldots\supset L^N=\bigcup_{j_1\neq\cdots\neq j_N}(l_{j_1}\cap\ldots l_{j_N}).$$

The number of points, that is the order $|L^N|$, is $\binom{k}{N}$, the number of lines, $|L^{N-1}|=\binom{k}{N-1},\ldots$, the number of $\mathbb{P}^{N-2}$'s in L^2 is $\binom{k}{2}$.

1.2. Resolution of singularities

Now consider the case that (ND) does not hold. In this case X has singular points, curves, etc., lying over the singular parts of the branch locus:

Singularity in X		Singularity in L
sing. pts. of X		pt. where $p>N$ planes meet
sing. curves of X	lie over	line where $q>N-1$ planes meet
⋮		⋮
sing. $N-2$-folds on X		$\mathbb{P}^{N-2}$ where $r>2$ planes meet

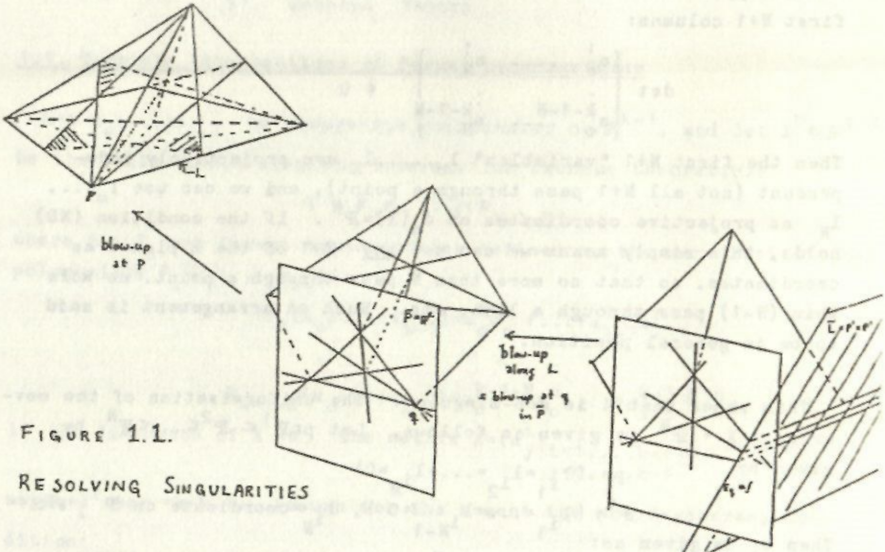


FIGURE 1.1.

RESOLVING SINGULARITIES

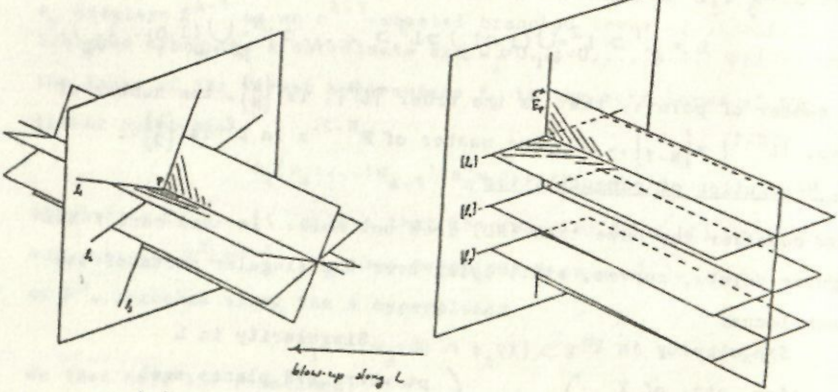


FIGURE 1.2.

A NEAR-PENCIL SINGULAR POINT IN DIMENSION 3

We must differentiate two kinds of singularities.

Definition: a near-pencil singularity is a singular κ-codim. $\xi \subset L^{\kappa}$, $\kappa=3,\ldots,N$, through which r planes pass in one of the following ways:

1) (r-1) meet in a $\mathbb{P}^{\kappa+1}$ through ξ
2) (r-2) meet in a $\mathbb{P}^{\kappa+2}$ through ξ

$\vdots$

N-2-κ) (r-(N-2-κ)) meet in a $\mathbb{P}^{N-2}$ through ξ .

Notice that the condition j) above implies j-1), but not the other way around. Consider, for example, a near-pencil point. Blowing up this point in $\mathbb{P}^N$ induces in the exceptional $\mathbb{P}^{N-1}$ an arrangement of r planes, (r-1) of them passing through a point, (r-2) of them through a line, etc. Such an arrangement (r planes, (r-1) of which pass through a point, (r-2) through a line, etc.) is called a near-pencil arrangement (see 1.4.), which explains the name.

Let us agree to call all other singularities actual; it is they we must resolve. In the process the near-pencil singularities are resolved en passant, so to speak (see Figure2). First, we resolve the branch locus in N-1 steps (see Figure 1):

1) blow up all (actual) sing. pts. $\quad \rho_1 : \hat{\mathbb{P}}^N_1 \longrightarrow \mathbb{P}^N$
2) blow up all (actual) sing. lines $\quad \rho_2 : \hat{\mathbb{P}}^N_2 \longrightarrow \hat{\mathbb{P}}^N_1$

$\vdots$

N-1) blow up all singular $\mathbb{P}^{N-2}$s $\quad \rho_{N-1} : \hat{\mathbb{P}}^N \longrightarrow \hat{\mathbb{P}}^N_{N-2}$

Once this is done we have a new branch locus

$$\hat{\mathbb{P}}^N \supset \tilde{L} = \text{proper transform of } \{L \text{ and exceptional divisors}\}$$

which once again admits a stratification which is uniformisable:

$$\tilde{L} = \tilde{L}^1 \supset \tilde{L}^2 \supset \ldots \supset \tilde{L}^N \ .$$

Let Y be the Fox completion (= the unique completion with a given branching behavior, see [F]) of the lift of X-singX making the following diagramm commute:

$$\begin{array}{ccc} Y & \rightarrow & X \\ \pi \downarrow & & \downarrow \bar{\pi} = \phi_n|_X \\ \hat{\mathbb{P}}^N & \rightarrow & \mathbb{P}^N \ . \end{array}$$

Y is a smooth algebraic manifold with branching degree n along all the k hyperplanes and along all exceptional divisors. The resolution so constructed is easily seen to be equivalent to the following: In the ambient $\mathbb{P}^{k-1}$ blow up all singular points, singular curves, etc., lying on X, and let Y be the proper transform of X in $\hat{\mathbb{P}}^{k-1} = \mathbb{P}^{k-1}$ blown up:

$$\begin{array}{ccc} \hat{\mathbb{P}}^{k-1} & \to & \mathbb{P}^{k-1} \\ \uparrow & & \uparrow \\ Y & \to & X. \end{array}$$

Finally we mention that the singular N-fold X may also be defined in the following manner (which was actually the original definition, see {Hi1}). Let $K(\mathbb{P}^N) = \mathbb{C}({}^{x_1}/_{x_o},\ldots,{}^{x_N}/_{x_o})$ be the rational function field of $\mathbb{P}^N$, and consider the Kummer extension

$$K(\mathbb{P}^N)\{({}^{l_1}/_{l_o})^{1/n},\ldots,({}^{l_k}/_{l_o})^{1/n}\}$$

where l_j=linear form defining the hyperplane l_j (by a slight abuse of notation). This is an abelian extension of degree n^{k-1} with Galois group $(\mathbb{Z}_n)^{k-1}$, and is the function field of X.

1.3. Exceptional divisors

Let p be a point of the arrangement where r planes pass. There will be n^{k-1-r} singular points on X lying over p which are an orbit of the Galois group acting on X. Blowing up p resolves each singular point by an exceptional $P \to \mathbb{P}^{N-1}$ which is a (singular) Fermat covering of one dimension lower corresponding to the induced arrangement. This is an arrangement Λ of r hyperplanes in $\mathbb{P}^{N-1}$. It has singular points where singular lines of the original arrangement L in $\mathbb{P}^N$ pass through p, singular lines where singular planes of L pass through p, etc. Λ is not a near-pencil arrangement, since we did not blow up near-pencil points.

Now consider a singular line l of the arrangement, through which r planes pass. Let σ=#{actual sing. pts. lying on l}. Then for the normal bundle of l in $\hat{\mathbb{P}}^N$, we have

$$N_{\hat{\mathbb{P}}_1^N} l = \mathcal{O}(1-\sigma)\oplus\ldots\oplus\mathcal{O}(1-\sigma).$$

It follows that the exceptional divisor, i.e. the blowup along l, is isomorphic to $\mathbb{P}^1\times\mathbb{P}^{N-2}$. The inverse image of l under the projection $\pi: X \to \mathbb{P}^N$, $\pi^{1}(l)$, consists of n^{k-1-r} distinct curves which are an orbit of the Galois group. The resolving divisors on Y, then, are of the form

$$D \to \mathbb{P}^1\times\mathbb{P}^{N-2}$$

and are in fact a product of Fermat covers in dimensions 1 and N-2, respectively. The arrangement induced in $\mathbb{P}^{N-2}$ is not a near-pencil, since we didn't blow them up. The point arrangement in $\mathbb{P}^1$ consists of s points, where s=#{planes and exceptional $\mathbb{P}^{N-1}$s through which l passes}, the arrangement in $\mathbb{P}^{N-2}$ is r hyperplanes corresponding to the r planes of L through l.

Finally, consider $\eta \subset L^{\kappa}$, a singular κ-codim. plane through which r planes of the arrangement pass. For the normal bundle of η in $\hat{\mathbb{P}}^N_{N-\kappa}$ we have

$$N_{\hat{\mathbb{P}}_{N-\kappa}^N \eta} = \mathcal{O}(1-\sigma) \oplus \ldots \oplus \mathcal{O}(1-\sigma),$$

where $\sigma = \#\{$objects blown up which are contained in $\eta\}$ (since if they are contained in η they are contained in every hyperplane $H \subset \mathbb{P}^N$ containing η). Therefore the exceptional divisor (in $\hat{\mathbb{P}}^N_{N-\kappa}$) is again a product,

$$G = \hat{\mathbb{P}}^{N-\kappa}_{N-\kappa} \times \mathbb{P}^{\kappa-1},$$

and the resolving divisor in Y is a product of Fermat covers:

$$D \to \hat{\mathbb{P}}^{N-\kappa}_{N-\kappa} \times \mathbb{P}^{\kappa-1}.$$

The arrangement induced in $\hat{\mathbb{P}}^{N-\kappa}_{N-\kappa}$ is just the arrangement in $\eta \cong \mathbb{P}^{N-\kappa}$, and the arrangement induced in $\mathbb{P}^{\kappa-1}$ is an arrangement of r planes, corresponding to the r planes of L through η. The arrangement in $\mathbb{P}^{\kappa-1}$ is not a near-pencil, that in $\hat{\mathbb{P}}^{N-\kappa}_{N-\kappa}$, however, may be (if the arrangement L is a near-pencil, for example).

1.4. Near-pencil Arrangements

We now would like to explain briefly why we do not blow up along the near-pencil singularities in the resolution of 1.2.

<u>Definition</u>: a <u>near-pencil</u> <u>arrangement</u> is an arrangement of k planes in $\mathbb{P}^N$, such that one of the following is the case:

1) (k-1) pass through a point
2) (k-2) pass through a line

. . .

. . .

N-1) (k-N+1) pass through a $\mathbb{P}^{N-2}$.

Notice that if k planes pass through a point or (k-1) pass through a line, etc., we cannot do our construction, since no N of the planes can be used as coordinate planes in $\mathbb{P}^N$. The Fermat covers associated to near-pencil arrangements have a simple structure, which we now elucidate. Consider first k planes, of which (k-1) pass through a point p. Let $\hat{\mathbb{P}}^N$ be the blow-up of $\mathbb{P}^N$ at p. This fibres over the exceptional divisor, $\hat{\mathbb{P}}^N \to \mathbb{P}^{N-1}$, the fibre over $x \in \mathbb{P}^{N-1}$ being the unique line l_x in $\mathbb{P}^N$ through p with tangent direction x at p. This fibering induces a fibering of the desingularisation $Y \to S$, with base S=the exceptional divisor resolving p. Since every line of the fibering $\hat{\mathbb{P}}^N \to \mathbb{P}^{N-1}$ meets only 2 branching planes (the exceptional $\mathbb{P}^{N-1}$ and the single plane of the arrangement not passing through p), every inverse image of l_x is still $\mathbb{P}^1$ (via the natural map $[z_o : z_1] \mapsto [z_o^n : z_1^n]$). This displays the fibering $Y \to S$ as a $\mathbb{P}^1$-bundle over the (N-1)-dimensional Fermat cover S. For this reason these arrangements are not particularly interesting. The reader may now check the following:

Arrangement	Fermat Cover
k planes, of which:	
(k-1) pass through a pt.	$\mathbb{P}^1$ - bundle over Y^{N-1}
(k-2) pass through a line	$\mathbb{P}^2$ - bundle over Y^{N-2}
. . .	. . .
k-(N-1) pass through a $\mathbb{P}^{N-2}$	$\mathbb{P}^{N-1}$ -bundle over Y^1

Here Y^i is a Fermat cover of dimension i.

Now consider again the situation of §1.2. Suppose we were to blow up also all near-pencil singularities in our resolution. The resolving exceptional divisors in Y are $\mathbb{P}^i$- bundles over some Y^{N-i}. Since the ruling is in the direction of the blow-up and normal bundles lift naturally, the $\mathbb{P}^i$-fiberings (upstairs) can be blown down again, that is, that which we blew up downstairs can be blown down again upstairs, so we've wasted our time.

1.5. Canonical divisor

Let K_Y, $K_{\hat{\mathbb{P}}^N}$ be the canonical divisors of Y and $\hat{\mathbb{P}}^N$, respectively, let $R \subset Y$ be the ramification divisor and $B \subset \hat{\mathbb{P}}^N$ the branch divisor. Then the following is well-known (where $\{D\}$ = line bundle defined by D):

Lemma: $\{K_Y\} = \pi^*\{K_{\hat{\mathbb{P}}^N}\} \otimes \{R\}$ or $K_Y = \pi^*(K_{\hat{\mathbb{P}}^N}) + R$

Here, R has the scheme structure given as in 0.2. , so in the case of Fermat coverings it is easily seen to take the form:

Lemma: $K_Y = \pi^*(K_{\hat{\mathbb{P}}^N} + \frac{(n-1)}{n}B)$.

This gives us the following useful fact:

Lemma 1.5: For a Fermat cover $Y \longrightarrow \hat{\mathbb{P}}^N$, K_Y can be written (over $\mathbb{Q}$) as a linear sum of branch divisors.

Indeed, write $K_{\mathbb{P}^N}$ as $-(N+1)H = -\frac{1}{2}(l_1 + \ldots + l_{2(N+1)})^*$, take its proper transform in $\hat{\mathbb{P}}^N$, and insert this in the formula above.

*if $k \geq 2(N+1)$, otherwise use some of the l_j more than once

1.6. Kodaira dimension

One of the main conjectures in the classification theory of complex varieties is the

Conjecture $C_{n,m}$: If $W \longrightarrow V$ is a fibre space, V, W, alg. manifolds, n=dimW, m=dimV
Then: $\kappa(W) \geq \kappa(V) + \kappa(F)$, where F is a generic fibre.

The conjecture $C_{n,n-1}$ has been proved by Viehweg {V}, so we may speak of Theorem $C_{n,n-1}$. We can apply this to Fermat coverings. IF the arrangement has at least one singular point, then the desingularisation Y fibres over the exceptional divisor Y^{N-1} . The Kodaira dimensions of both Y^{N-1} and the fibre can be calculated (that of Y^{N-1}

by an inductive process), and theorem $C_{N,N-1}$ gives an estimate on the Kodaira dimension of Y (hopefully equality will hold). We shall do this in detail in dimension 3 (§2.5.).

1.7. Quotients

Suppose we wish to have different branching degrees along the different branching planes. Let $q_o,\dots,q_{k-1}$ be the desired degrees. Set

$$n = \prod_{i=0}^{k-1} q_i\ , \qquad n_i = n/q_i = q_o \dots \hat{q}_i \dots q_{k-1} \qquad (\hat{\ } = \text{delete})$$

$$\mu_p = \text{group of pth roots of unity}$$

$$\tilde{\mu}_n := \mu_{n_o} \times \dots \times \mu_{n_{k-1}} .$$

$\tilde{\mu}_n$ acts on $\mathbb{P}^{k-1}$ in the following manner:

$$\tilde{\mu}_n \times \mathbb{P}^{k-1} \to \mathbb{P}^{k-1}$$

$$((g_o,\dots,g_{k-1}),(x_o,\dots,x_{k-1})) \mapsto (x_o g_o,\dots,x_{k-1} g_{k-1}),$$

$$g_i = \exp\left(\frac{2\pi i b_i}{n_i} i\right).$$

The quotient under this action is the weighted projective space $\mathbb{P}(q_o,\dots,q_{k-1})$. We have a sequence of maps:

$$\mathbb{P}^{k-1} \to \mathbb{P}(q_o,\dots,q_{k-1}) \to \mathbb{P}^{k-1}$$

$$(x_o,\dots,x_{k-1}) \mapsto (x_o^{n_o},\dots,x_{k-1}^{n_{k-1}}) \mapsto ((x_o^{n_o})^{q_o},\dots,(x_{k-1}^{n_{k-1}})^{q_{k-1}})$$
$$= (x_o^{n},\dots,x_{k-1}^{n})$$

There are induced maps

$$\begin{tikzcd}
\mathbb{P}^{k-1} \arrow[r] & \mathbb{P}(q_o,\dots,q_{k-1}) \arrow[r] & \mathbb{P}^{k-1} \\
X \arrow[u, phantom, "\cup"] \arrow[r] & \tilde{X} \arrow[u, phantom, "\cup"] \arrow[r] & \mathbb{P}^N \arrow[u, phantom, "\cup"] \\
Y \arrow[u] \arrow[r] & \overline{Y} \arrow[u] \arrow[r] & \hat{\mathbb{P}}^N \arrow[u] \\
& \tilde{Y} \arrow[u] \arrow[ur, dashed, "\sigma"'] &
\end{tikzcd}$$

$\overline{Y}$ is the quotient of Y under $\tilde{\mu}_n$. Y is smooth, but $\overline{Y}$ may have quotient singularities. $\tilde{Y} \to \overline{Y}$ is a resolution of these singularities (such a resolution is described in {E} or {C}). The map σ displays $\tilde{Y}$ as a branched cover of $\hat{\mathbb{P}}^N$, branched of degree q_i along the l_i, giving the

<u>Theorem 1.7.</u> : Given k hyperplanes (not a "pencil") $l_o,\dots l_{k-1}$ and given $q_o,\dots,q_{k-1}$, there exists a smooth covering $\tilde{Y} \to \hat{\mathbb{P}}^N$, branched of degree q_i along l_i.

§2. Fermat coverings of $\mathbb{P}^3$

2.1. Arrangements in $\mathbb{P}^3$

Consider an arrangement of k planes in $\mathbb{P}^3$, given by linear forms $l_1,\ldots,l_k$. The union L is the zero set of $l_1\cdot\ldots\cdot l_k$. Let $t_q(1)$ denote the number of q-fold lines, i.e. lines through which q planes of the arrangement pass, and t_p the number of p-fold points. The arrangement is in general position if and only if $t_q(1)=0$ for all $q>2$ and $t_p=0$ for all $p>3$. In this case the number of lines is $t_2(1)=\binom{k}{2}$, the number of points is $t_3=\binom{k}{3}$. We shall speak of singular lines and points if $t_q(1)\neq 0$ for some $q>2$ and $t_p\neq 0$ for some $p>3$, respectively. We have the following formula for the number of singular lines:

$$\sum_{q\geq 2} t_q(1)\binom{q}{2} = \binom{k}{2} .$$

If we suppose there are no singular lines, we also have the formula for the number of singular points:

$$\sum_{p\geq 3} t_p\binom{p}{3} = \binom{k}{3} .$$

If we admit both singular lines and singular points, we must consider the data $t_{pq}:=\#$ intersections of a q-fold line with a p-fold point. In this case we get the formula:

$$\sum_{p\geq 3} t_p\binom{p}{3} - \sum_{q\geq 3}\Big\{\sum_p t_{pq}-t_q(1)\Big\}\binom{q}{3} = \binom{k}{3} .$$

Two arrangements will be considered equivalent if they have the same numerical data $t_q(1)$, t_p, t_{pq}, and we speak of the combinatorial type (equivalence class) of an arrangement. We assume from now on that $t_k(1)=t_{k-1}(1)=t_k=0$ (remember these are the cases for which our construction of 1.2. doesn't work).

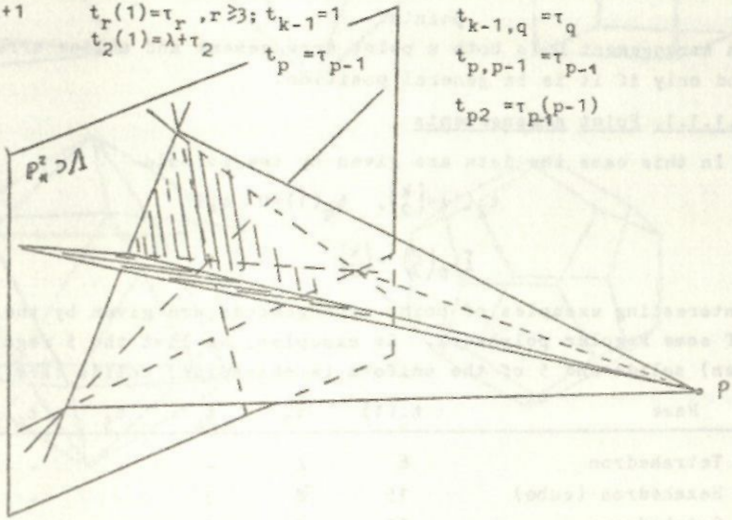
We consider first the <u>join-reducible</u> arrangements (in the language of [GS]) in $\mathbb{P}^3$. These are of two kinds, 1) the join of a point and a line arrangement in $\mathbb{P}^2$, and 2) the join of two point arrangements on two skew lines.

1) Let $\mathbb{P}^2_* \subset \mathbb{P}^3$ be any linear subspace, and let Λ be an arrangement of λ lines in $\mathbb{P}^2_*$. Taking any point p not on the $\mathbb{P}^2_*$, the <u>join</u> L of p and Λ is the arrangement of $\lambda+1$ planes, the λ determined by the lines and the point p, plus the $\mathbb{P}^2_*$. This is of course just the near-pencil (of point type, see 1.4. 1)). Let τ_r=# r-fold points of Λ. The numerical data of L are then

$k=\lambda+1$ $\quad t_r(1)=\tau_r$, $r\geq 3$; $t_{k-1}=1$ $\quad t_{k-1,q}=\tau_q$

$t_2(1)=\lambda+\tau_2$ $\quad t_p=\tau_{p-1}$ $\quad t_{p,p-1}=\tau_{p-1}$

$t_{p2}=\tau_{p-1}(p-1)$



2) Let l, l$'$ be two skew lines in $\mathbb{P}^3$, $p_1,\dots,p_\lambda$, $p_1',\dots,p_{\lambda'}'$ λ and λ' points on l and l$'$ respectively. Take the set of the λ' planes defined by l and p_j' and the λ planes defined by l$'$ and p_j. This gives an arrangement with the data

$$k=\lambda+\lambda' \qquad \left.\begin{array}{ll} t_\lambda(1) = 1 & t_{\lambda+1}=\lambda' \\ t_\lambda(1) = 1 & t_{\lambda'+1}=\lambda \end{array}\right\} \text{near-pencil}$$

$$t_2(1)=\binom{\lambda+\lambda'}{2}-\binom{\lambda}{2}-\binom{\lambda'}{2}=\lambda\cdot\lambda'$$

(the t_{pq} for near-pencil points are automatically $t_{p,p-1}=1$, $t_{p2}=p-1$ for each such point, so we shall omit these data from now on)

An arrangement which is not join reducible, i.e. not of the above types, is called <u>join-irreducible</u>. It is mainly these which are interesting.

2.1.1. Degenerate arrangements

Next we consider arrangements of the following kind, which we call degenerate:

point arrangement : $\Leftrightarrow$ L has only singular points ($t_q(1)=0 \quad q \geq 3$)

line arrangement: $\Leftrightarrow$ L has only singular lines (and of course near-pencil singular points), but no actual singular points.

An arrangement L is both a point arrangement and a line arrangement, if and only if it is in general position.

2.1.1.1. Point arrangements

In this case the data are given by the formula

$$t_2(1)=\binom{k}{2}, \quad t_q(1)=0 \quad q\geq 3$$

$$\sum t_p\binom{p}{3}=\binom{k}{3}$$

Interesting examples of point arrangements are given by the facet planes of some regular polyhedra. As examples, we list the 5 regular (platonian) solids and 5 of the uniform (archimedian) solids (see Fig. 2.2.):

Name	$t_2(1)$	t_3	t_4	t_5	t_6	k
Tetrahedron	6	4	-	-	-	4
Hexahedron (cube)	15	8	3	-	-	6
Octahedron	28	8	12	-	-	8
Dodecahedron	66	40	15	12	-	12
Icosehedron	190	140	90	24	20	20
(3,6,6)	28	32	6			8
(3,8,8)	91	256	27	-	-	14
(4,6,6)	91	256	27	-	-	14
(5,6,6)(soccer ball)	496	3520	120	36	40	32
(3,4,3,4)-cuboctahedron	91	208	39	-	-	14

We discuss another interesting point arrangement later (2.1.3.(i)).

2.1.1.2. Line arrangements

A line arrangement is characterised by the following data:

$$q_1,\dots,q_s,\ \sigma . \quad k=\sum q_i+\sigma, \quad t_2(1)=\binom{k}{2}-\sum\binom{q_i}{2}$$

$$t_3=\binom{k}{3}-\frac{1}{3}\sum q_i(q_i-1)(3k-2q_i-2)$$

Here there are s singular lines, through the i-th one pass q_i planes,

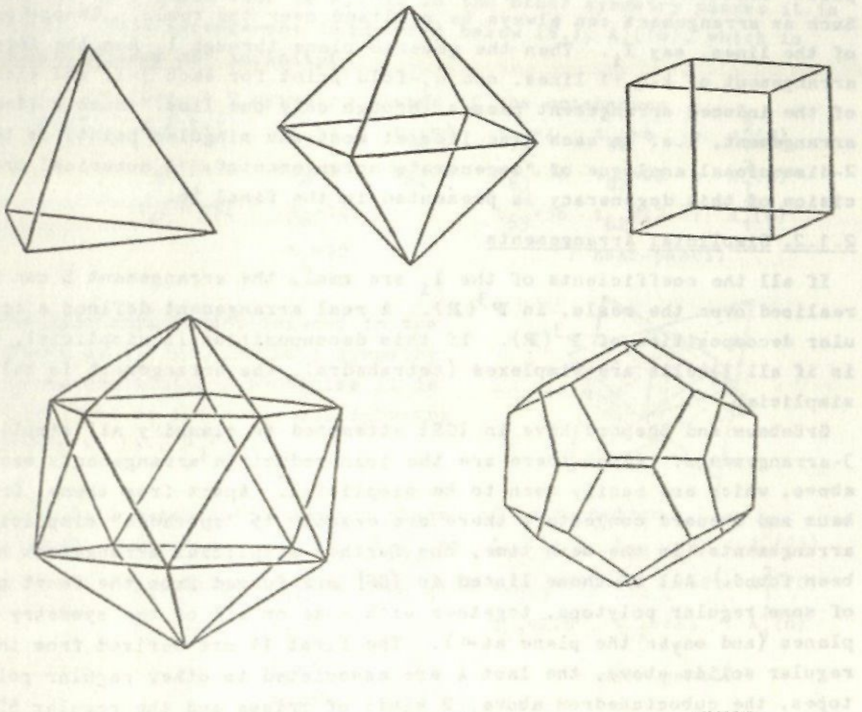


FIG. 2.2.

plus σ additional planes in general position with respect to the others. Such an arrangement can always be realised over the reals. Choose one of the lines, say l_i. Then the generic plane through l_i has the induced arrangement of $k-q_i+1$ lines, one q_j-fold point for each $j=i$, and each line of the induced arrangement passes through only one line. Such a line arrangement, i.e. on each line lies at most one singular point, is the 2-dimensional analogue of "degenerate arrangements". A numerical precision of this degeneracy is presented in the final §6.

2.1.2. Simplicial Arrangements

If all the coefficients of the l_i are real, the arrangement L can be realised over the reals, in $\mathbb{P}^3(\mathbb{R})$. A real arrangement defines a cellular decomposition of $\mathbb{P}^3(\mathbb{R})$. If this decomposition is simplicial, that is if all 3-cells are simplexes (tetrahedra), the arrangement is called simplicial.

Grünbaum and Shepard have in {GS} attempted to classify all simplicial 3-arrangements. First there are the join-reducible arrangements mentioned above, which are easily seen to be simplicial.* Apart from these, Grünbaum and Shepard conjecture there are exactly 15 "sporadic" simplicial arrangements.(In the mean time, one further simplicial arrangement has been found.) All of those listed in {GS} are formed from the facet planes of some regular polytope, together with some or all of the symmetry planes (and maybe the plane at ∞). The first 11 are derived from the regular solids above, the last 4 are associated to other regular polytopes, the cuboctahedron above, 2 kinds of prisma and the regular 60-cell. We shall employ the notation of {GS}, denoting a simplicial N-arrangement of k planes by $A_i^N(k)$, the subscript i denoting which arrangement is meant if there happen to be more that one.

1. $A_1^3(10)$=4 faces and 6 symmetry planes of the tetrahedron

k=10	$t_3(1)=10$	$t_6=5$	$t_{63}=20$	$t_{62}=15$	$A_1^2(6)$
	$t_2(1)=15$	$t_4=10$			near-pencil

This arrangement is depicted in Figure 1.1. The line arrangement induced in each face is also $A_1^2(6)$.

2. $A_1^3(15)$=6 faces & 9 symmetry planes of the cube

k=15	$t_4(1)=3$	$t_9=1$	$t_{94}=3$	$t_{93}=4$	$t_{92}=6$	:	$A_1^2(9)$
	$t_3(1)=19$	$t_8=3$	$t_{84}=3$	$t_{83}=18$	$t_{82}=12$	:	$A_1^2(8)$
	$t_2(1)=30$	$t_6=8$		$t_{63}=32$	$t_{62}=24$	:	$A_1^2(6)$
		$t_5=6$					
		$t_4=18$				:	near-pencil

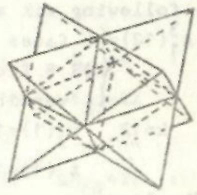
* if the arrangement $\Lambda \subset \mathbb{P}^2_*$ is simplicial.

The line arrangement induced in the faces & 3 symmetry planes parallel to 2 faces each is $A_1^2(9)$; in the other symmetry planes it is $A_1^2(8)$. This arrangement is pictured below (8.), $A_1^3(16)$, which is $A_1^3(15) \cup \{\text{plane at infinity}\}$

3. $A_1(17)$=8 faces & 9 symmetry planes of the octahedron

k=17	$t_4(1)=3$	$t_9=1$	$t_{94}=3$	$t_{93}=4$	$t_{92}=6$	: $A_1^2(9)$
	$t_3(1)=28$	$t_8=6$	$t_{84}=6$	$t_{83}=36$	$t_{82}=24$	: $A_1^2(8)$
	$t_2(1)=34$	$t_6=14$		$t_{63}=56$	$t_{62}=42$	: $A_1^2(6)$
		$t_4=20$				: near-pencil
		$t_3=12$				

The line arrangement induced in the 8 faces is $A_2^2(10)$, in the 3 symmetry planes containing 4 verticies it is $A_1^2(8)$, and in the other six symmetry planes we have $A_3^2(10)$.



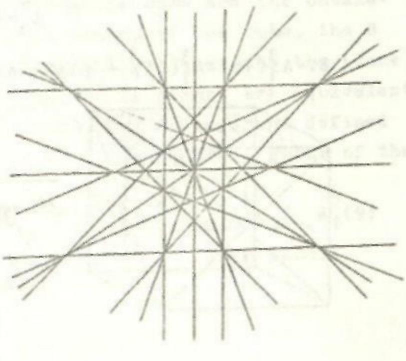
4. $A_1^3(27)$=12 faces & 15 symmetry planes of dodecahedron

k=27	$t_5(1)=6$	$t_{15}=1$	$t_{15,5}=6$	$t_{15,3}=10$	$t_{15,2}=15$	: $A_1^2(15)$
	$t_3(1)=70$	$t_{10}=12$	$t_{10,5}=12$	$t_{10,3}=120$	$t_{10,2}=60$	: $A_1^2(10)$
	$t_2(1)=81$	$t_6=55$		$t_{63}=220$	$t_{62}=165$	: $A_1^2(6)$
		$t_6=12$				: near-pencil
		$t_4=60$				
		$t_3=30$				

This arrangement is depicted on the front cover. The induced line arrangements are $A_4^2(16)$(12 times) and $A_4^2(14)$(15 times).

It is interesting to note that the 35 planes which are the facet and symmetry planes of the icosahedron do not yield a simplicial arrangement. This can be seen by considering the line arrangement induced in the symmetry planes, which is non-simplicial:

non-simplicial arrangement induced in symmetry planes of icosahedron:



This arrangement in nonetheless interesting in its own right, so we list the data, which were provided by P. Orlik. From now on we denote a non-simplicial arrangement of k planes by $a_i^N(k)$.

$a_1^3(35)=20$ faces and 15 symmetry planes of the icosahedron

$k=35$ $t_5(1)=6$ $t_{15}=1$ $t_{15,5}=6$ $t_{15,3}=10$ $t_{15,2}=15$: $A_1^2(15)$

$t_3(1)=130$ $t_{10}=24$ $t_{10,5}=24$ $t_{10,3}=240$ $t_{10,2}=120$: $A_1^2(10)$

$t_2(1)=145$ $t_9=20$ $t_{93}=200$ $t_{92}=120$

$t_6=65$ $t_{63}=260$ $t_{62}=195$: $A_1^2(6)$

$t_5=30$ $t_{53}=60$ $t_{52}=120$ (non-simpl.)

$t_4=260$: near-pencil

$t_3=30$

The following six arrangements are related to the above.

5. $A_1^3(12)=$ { 6 faces of cube & 6 symmetry planes through 2 edges each OR 8 faces of octahedron & 3 symmetry planes containing 4 verticies each + plane at infinity }

$k=12$ $t_3(1)=16$ $t_6=12$ $t_{63}=48$ $t_{62}=36$: $A_1^2(6)$

$t_2(1)=18$ $t_3=12$

The line arrangement $A_1^2(7)$ is induced in each of the 12 planes.

6. $A_1^3(13)=A_1^3(12)$ + plane at infinity

$k=13$ $t_3(1)=19$ $t_7=3$ $t_{73}=18$ $t_{72}=9$: $A_1^2(7)$

$t_2(1)=21$ $t_6=9$ $t_{63}=36$ $t_{62}=27$: $A_1^2(6)$

$t_4=10$: near-pencil

$t_3=6$

7. $A_1^3(14)=A_1^3(12)$ + 2 midplanes of symmetry of the cube

$k=14$ $t_4(1)=1$ $t_8=2$ $t_{84}=2$ $t_{83}=12$ $t_{82}=8$: $A_1^2(8)$

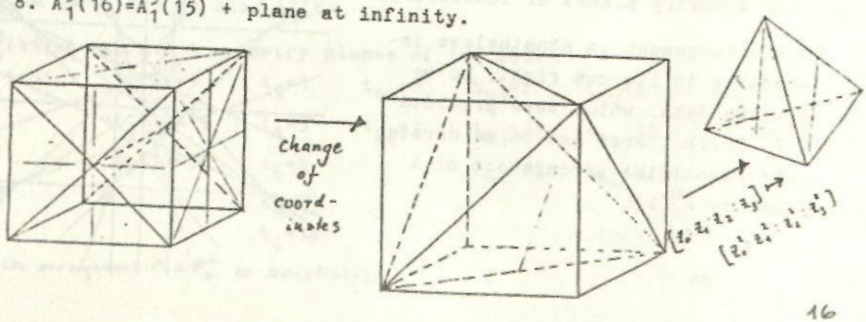
$t_3(1)=20$ $t_7=2$ $t_{73}=12$ $t_{72}=6$: $A_1^2(7)$

$t_2(1)=25$ $t_6=8$ $t_{63}=32$ $t_{62}=24$: $A_1^2(6)$

$t_5=2$, $t_4=16$ } : near-pencil

$t_3=2$

8. $A_1^3(16)=A_1^3(15)$ + plane at infinity.



This arrangement may also be formed (as pictured) by inscribing a tetrahedron in the cube (4+6=10 facet planes) and adding the six symmetry planes of the tetrahedron. Via the map $(z_o:z_1:z_2:z_3) \mapsto (z_o^2:z_1^2:z_2^2:z_3^2)$ $A_1^3(16)$ is mapped onto the arrangement $A_1^3(10)$ (letting the four faces of the tetrahedron be $\{z_o=0\},\ldots,\{z_3=0\}$).

k=16	$t_4(1)=6$	$t_9=4$	$t_{94}=12$	$t_{93}=16$	$t_{92}=24$	:	$A_1^2(9)$
	$t_3(1)=16$	$t_6=8$		$t_{63}=32$	$t_{62}=24$	:	$A_1^2(6)$
	$t_2(1)=36$	$\left.\begin{matrix} t_5=12 \\ t_4=16 \end{matrix}\right\}$				: near-pencil	

9. $A_1^3(18)=A_1^3(17)$ + plane at infinity

k=18	$t_4(1)=$	$t_9=1$	$t_{94}=3$	$t_{93}=4$	$t_{92}=6$	:	$A_1^2(9)$
	$t_3(1)=32$	$t_8=6$	$t_{84}=6$	$t_{83}=36$	$t_{82}=24$	:	$A_1^2(8)$
	$t_2(1)=39$	$t_7=6$		$t_{73}=36$	$t_{72}=24$	:	$A_1^2(7)$
		$t_6=8$		$t_{63}=32$	$t_{62}=24$	:	$A_1^2(6)$
		$\left.\begin{matrix} t_5=3 \\ t_4=36 \end{matrix}\right\}$				: near-pencil	

There are four combinatorial types of line arrangements in the 18 planes; 8 times $A_2^2(10)$, 6 times $A_1^2(11)$, 3 times $A_1^2(9)$ and $A_2^2(13)$ at infinity.

10. $A_1^3(28)=A_1^3(27)$ + plane at infinity

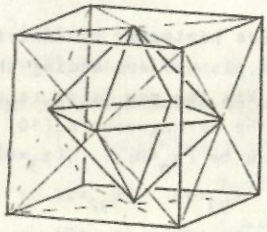
k=28	$t_5(1)=6$	$t_{15}=1$	$t_{15,5}=6$	$t_{15,3}=10$	$t_{15,2}=15$	:	$A_1^2(15)$
	$t_3(1)=76$	$t_{10}=12$	$t_{10,5}=12$	$t_{10,3}=120$	$t_{10,2}=60$	:	$A_1^2(10)$
	$t_2(1)=90$	$t_7=15$		$t_{73}=90$	$t_{72}=45$	:	$A_1^2(7)$
		$t_6=58$		$t_{63}=232$	$t_{62}=174$	:	$A_1^2(6)$
		$t_4=100$			: near-pencil		

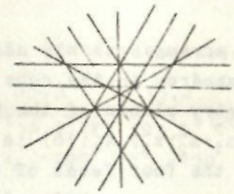
A somewhat more subtile arrangement is

11. $A_1^3(24)$; We quote from {GS}: "Start from the cube and the octahedron inscribed into it, use the 6 facet planes of the cube, the 8 facet planes of the octahedron, the 9 symmetry planes and the plane at infinity." All line arrangements in the 24 planes are equivalent; they are $A_2^2(13)$. This arrangement is also the arrangement defined by the reflection group G_{576} (see 2.1.4.), the symmetry group of the regular 24-cell.

k=24	$t_4(1)=18$	$t_9=24$	$t_{94}=72$	$t_{93}=96$	$t_{92}=144$	:	$A_1^2(9)$
	$t_3(1)=32$	$t_4=96$			: near-pencil		
	$t_2(1)=72$						

$A_1^3(24)$:

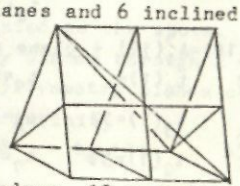




line arrangement $A_2^2(13)$ induced in each plane.

The remaining 4 arrangements are related to other regular polytopes.

12. $A_2^3(15)$=5 faces of prism, 4 symmetry planes and 6 inclined planes.

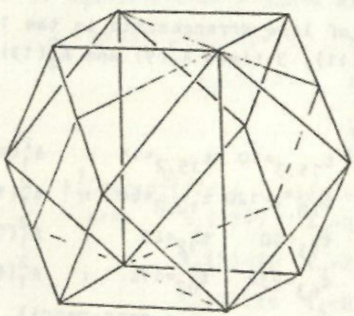


k=15 $t_3(1)=26$ $t_7=9$ $t_{73}=54$ $t_{72}=27$

$t_2(1)=27$ $t_6=6$ $t_{63}=24$ $t_{62}=18$

$t_4=24$ (near-pencil)

13. $A_2^3(28)$=14 facet planes of the cuboctahedron, 13 symmetry planes and the plane at infinity.



k=28 $t_4(1)=16$

$t_3(1)=64$

$t_2(1)=90$

$t_{13}=1$ $t_{134}=9$ $t_{133}=4$ $t_{132}=12$: $A_2^2(13)$

$t_{11}=6$ $t_{114}=24$ $t_{113}=48$ $t_{112}=42$: $A_1^2(11)$

$t_9=3$ $t_{94}=9$ $t_{93}=12$ $t_{92}=18$: $A_1^2(9)$

$t_8=18$ $t_{84}=18$ $t_{83}=108$ $t_{82}=72$: $A_1^2(8)$

$t_6=40$ $t_{63}=160$ $t_{62}=120$: $A_1^2(6)$

$t_5=18$

$t_4=84$: near-pencil

$t_3=24$

The induced arrangements are of 6 types; 6 times $A_4^2(17)$, 4 times $A_1^2(13)$ 6 times $A_2^2(17)$, 8 times $A_3^2(16)$ and 4 times $A_2^2(13)$.

14. $A_1^3(30)$=8 faces of a right prism, 7 symmetry planes, the 12 inclined planes, each through 4 verticies of the prism but not through the center and the plane at infinity.

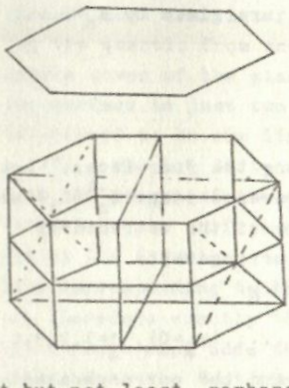
k=30 $t_6(1)=2$ $t_{13}=6$ $t_{13,6}=6$ $t_{13,4}=18$ $t_{13,3}=72$ $t_{13,2}=54$

$t_4(1)=9$ $t_8=18$ $t_{84}=18$ $t_{83}=108$ $t_{82}=72$

$t_3(1)=84$ $t_7=18$ $t_{73}=108$ $t_{72}=54$

$t_2(1)=99$ $t_7=6$: near-pencil

$A_1^3(30)$:



$t_6=36$ $t_{63}=144$ $t_{62}=108$

$t_4=144$: near-pencil

The induced arrangements are of 3 types; $A_4^2(17)$ (18 planes), $A_1^2(13)$ (6 planes) and $A_1^2(19)$ (6 planes).

Last but not least, perhaps the most beautiful arrangement

15. $A_1^3(60)$: This is the arrangement associated to the symmetry group of the regular 120-cell, G_{7200}. It may be generated from the symmetry planes of the regular 120-cell in $\mathbb{R}^4$. All planes are equivalent; the induced arrangement is $A_1^2(31)$.

$k=60$ $t_5(1)=72$

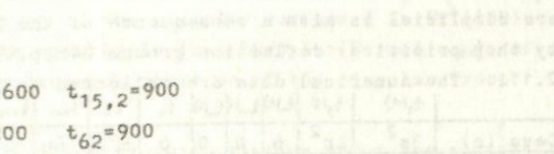
$t_3(1)=200$

$t_2(1)=450$

$t_{15}=60$ $t_{15,5}=360$ $t_{15,3}=600$ $t_{15,2}=900$

$t_6=300$ $t_{63}=1200$ $t_{62}=900$

$\left.\begin{array}{l} t_6=360 \\ t_4=600 \end{array}\right\}$: near-pencil



$A_1^2(31)$

2.1.3. The Ceva Arrangements

Consider the arrangement $A_1^3(10)$ above. Choose coordinates in $\mathbb{P}^3$ such that the four faces of the tetrahedron are given by $z_o=0,\ldots,z_3=0$. There is a natural n-th power map

$$\phi_n:\mathbb{P}^3 \to \mathbb{P}^3$$
$$(z_o,\ldots,z_3)\mapsto(z_o^n,\ldots,z_3^n).$$

This is a covering of degree n^3 branched along the four faces. Let $L_1,\ldots,L_6$ be the six symmetry planes. The inverse image $\phi_n^{-1}(L_i)$ will be a set of n planes passing through the edge of the tetrahedron through which the L_i passes. We define the arrangements

$$\mathrm{Ceva}^3(n) = 6n \text{ planes } \phi_n^{-1}(L_i),\ i=1,\ldots,6$$
$$\mathrm{Ceva}^3(n,j)=\mathrm{Ceva}^3(n) + j \text{ faces } \{z_o=0\},\ldots,\{z_j=0\},\ j=1,2,3,4.$$

The same thing can be done in any dimension and the corresponding arrangements are called $A_j^o(n)=\mathrm{Ceva}^2(n,j)$ $(j=1,2,3)$ in [Hi1] for the case of dimension 2.

$\mathrm{Ceva}^3(2,j)$ can be realised over the reals (since the second root of unity is real), and, as in the case of line arrangements, these are in fact simplicial arrangements, those with names $A_1^3(12)$, $A_1^3(13)$, $A_1^3(14)$ $A_1^3(15)$ and $A_1^3(16)$ in section 2.1.1. The fact that these arrangements are simplicial is also a consequence of the fact that they are defined by the (imprimitive) reflection groups $G(n,p,4)$, $j\equiv\frac{n}{p}\mathrm{mod}(4)$, see section 2.1.4. The numerical data are collected in the following table:

	$t_2(1)$	$t_3(1)$	$t_n(1)$	$t_{n+1}(1)$	$t_{n+2}(1)$	t_4	t_6	t_{n+1}	t_{n+2}	t_{n+3}	t_{3n}	t_{3n+1}	t_{3n+2}	t_{3n+3}
$\mathrm{Ceva}^3(n)$	$3n^2$	$4n^2$	6	0	0	0	n^3	6n	0	0	4	0	0	0
$\mathrm{Ceva}^3(n,1)$	$3n^2+3n$	$4n^2$	3	3	0	n^2	n^3	3n	3n	0	1	3	0	0
$\mathrm{Ceva}^3(n,2)$	$3n^2+6n$	$4n^2$	1	4	1	$2n^2$	n^3	n	4n	n	0	2	2	0
$\mathrm{Ceva}^3(n,3)$	$3n^2+9n$	$4n^2$	0	3	3	$3n^2$	n^3	0	3n	3n	0	0	3	1
$\mathrm{Ceva}^3(n,4)$	$3n^2+12n$	$4n^2$	0	0	6	$4n^2$	n^3	0	0	6n	0	0	0	4

The arrangements induced in the 6-fold points are $\mathrm{Ceva}^2(2)=A_1^2(6)$, those of the $(3n+j)$-fold points are $\mathrm{Ceva}^2(n,j)$. The arrangements induced in the planes are also of appropriate type, for example in $\mathrm{Ceva}^3(n)$ we have $\mathrm{Ceva}^2(n)\cup\{\text{line at infinity}\}$.

2.1.4. Arrangements occuring in classical geometry

(i) Kummer's 16_6 configuration

A quartic hypersurface in $\mathbb{P}^3$ can have no more than 16 nodes, since the dual variety has degree $36-2(\#\text{nodes})$ $(36=d(d-1)^2,\ d=4)$, which must be greater than 2 (a quadric is rational). That such a quartic exists was known very early, (Fresnel's wave surface, Fresnel, Euvres Complete's II, 261), because of its relevance to physics (!). Kummer studied such quartics from a mathematical standpoint (K1). Projecting the quartic from one of its nodes displays the surface S as a double cover of the plane (since each line through the nodes meets the surface in just two other points). The branching locus is easily determined to be six lines in general position* (see for example (GH), p. 774), and the fifteen points of intersection of the six lines are just the images of the other 15 nodes. This means there are 6 planes tangent to the surface passing through each node, and on each of these planes lie 5 further nodes (whose images are the intersections of the line corresponding to the given plane with the 5 other lines). There are therefore exactly 16 planes each passing through 6 of the nodes, and through each node there pass 6 of the 16 planes. This duality arises from the fact that the dual surface S also has degree 4 and 16 nodes.

<u>Remark</u>: Since $S \to \mathbb{P}^2$ is a double cover, it is cyclic, and may be constructed by the "root method", (see for example (P) for double covers), branched over the arrangement of 6 lines in general position. If we take the <u>Fermat</u> <u>Cover</u> over the same branch locus, we get a non-singular complete intersection in $\mathbb{P}^5$, Σ, which may be shown to be the <u>desingularisation</u> of the singular surface S in $\mathbb{P}^3$ ((GH), p.770). Thus we have the remarkable fact that the Fermat cover of this branch locus is the desingularisation of the singular "root" cover, for n=2.

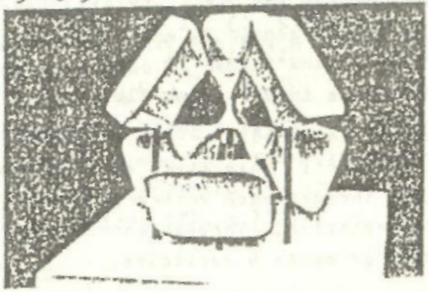
The general equation of a quartic with sixteen nodes has sixteen constants and has the form

$$\sum \lambda_{i_0 i_1 i_2 i_3} x_{i_0} x_{i_1} x_{i_2} x_{i_3} = 0,$$

where $(x_0,\ldots,x_3)$ are homogenous coordinates in $\mathbb{P}^3$. By choosing special values of the constants one gets the following (real) surface:

$$(x_0^2+x_1^2+x_2^2+x_3^2 + 4(x_0x_2+x_1x_3+x_0x_3+x_1x_2+x_0x_1+x_2x_3))^2 = -80x_0x_1x_2x_3$$

which is depicted below.

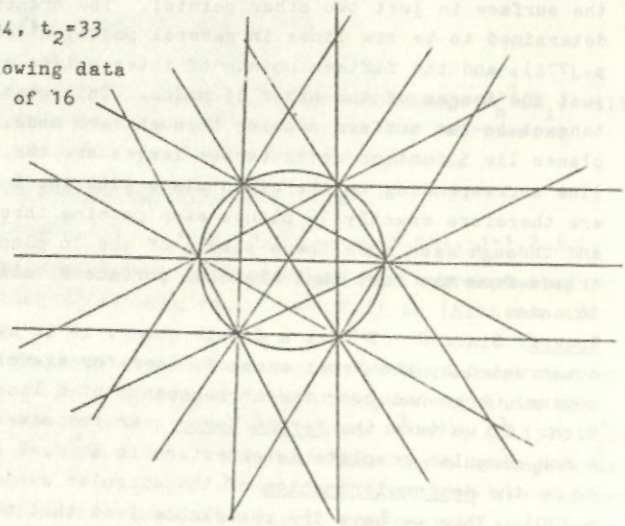


*see footnote on page i

Kummer shows that the six nodes in a plane lie on a conic in that plane. Therefore the arrangement induced in each plane has in general the data: 15 lines, $t_5=6$, $t_2=45$, and if the coefficients are choosen special as above we get the following (real, non-simplicial) arrangement:

15 lines, $t_5=6$, $t_3=4$, $t_2=33$

This gives the following data for our arrangement of 16 planes:



$a_2(16)$ (general)

$k=16$ $t_2(1)=120$

$t_6=16$ $t_{62}=240$

$t_3=240$

$a_3(16)$ (special)

$k=16$ $t_2(1)=120$

$t_6=16$ $t_{62}=240$

$t_4=16$ $t_{42}=96$

$t_3=176$

(ii) Klein's 60_{15} configuration

Let $G(2,4)=Q$ be the Graßmann of lines in $\mathbb{P}^3$, embedded as a smooth quadric hypersurface in $\mathbb{P}^5$ via the Plücker embedding. Let $L=H\cap Q$, H a hyperplane. L is known as a <u>linear</u> <u>line</u> <u>complex</u>, and was the object of study of F. Klein in his thesis {K}, I,II and III (1868-1870). In particular, there are (to use Klein's language) six fundamental linear complexes, corresponding to the coordinate planes $(z_i=0)$, for homogenous coordinates $z_o,\dots,z_5$ on $\mathbb{P}^5$. The intersections $Q\cap H_{i_1}\cap H_{i_2}\cap H_{i_3}\cap H_{i_4}$ $(H_i=(z_i=0))$ consist of 2 points each (Q is a quadric, $H_{i_1}\cap H_{i_2}\cap H_{i_3}\cap H_{i_4}\cong\mathbb{P}^1$) which together determine $2\cdot\binom{6}{4}=30$ lines in $\mathbb{P}^3$, which Klein shows determine the edges of 15 "fundamental tetrahedra", ({K},II,p.60). The faces and verticies are all distinct; this gives an arrangement of 60 planes and 60 verticies; through each vertex pass 15 planes, and each plane contains 15 verticies. Through each of the 30 edges 6 planes pass, and each edge meets 6 verticies.

As is well known ([ST], p. 278), this arrangement is that defined by the unitary reflection group $G_{11,520}$ (see 2.1.4.).

Its numerical data may be read off from the tables of Orlik and Solomon [OS1]:

$a_2^3(60)$

$k=60$	$t_6(1)=30$	$t_{15}=60$	$t_{15,6}=180$	$t_{15,3}=960$	$t_{15,2}=720$
	$t_3(1)=320$	$t_6=480$		$t_{63}=1920$	$t_{62}=1440$
	$t_2(1)=360$	$t_4=960$		(near-pencil)	

(iii) The 45 Tritangents of a smooth cubic surface in $\mathbb{P}^3$

The arrangement we now discuss is even older that the two above. It is very famous how the mathematical world came to know of the 27 lines lying on a smooth cubic surface. Since a line in $\mathbb{P}^3$ meeting a cubic surface in four points must lie completely on that surface, four linear conditions are put on a line to require it to lie on a cubic surface. The Graßmann of lines in $\mathbb{P}^3$, $G(2,4)$ also has dimension 4, which led A. Cayley to conjecture in 1849 in a letter to G. Salmon, that finitely many lines will lie on a cubic surface. G. Salmon replied to him that the finite number is 27.

A few years later in 1854, J. Steiner showed that there are exactly 45 planes meeting the cubic surface in a union of 3 of the 27 lines. He communicated this to L. Schläfli, who in his reply described his famous double-six:

$$N = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{bmatrix}$$

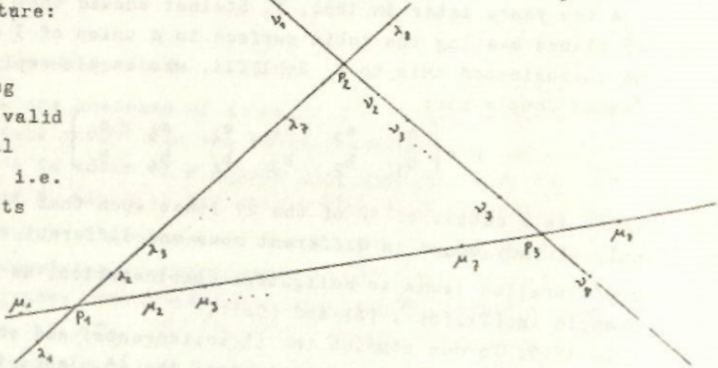
which is a subset of 12 of the 27 lines such that 2 intersect if and only if they occur in different rows and different columns. This configuration leads to delightful combinatorics, as described for example in [T], [B] , [S] and [Co1].

In 1870, Jordan studied the 45 tritangents, and showed that the group acting as even permutations of the 45 planes is a simple group of order 25,920. Around the same time it became known that the simple group of order 25,920 acts on $\mathbb{P}^4$ (not on $\mathbb{P}^3$) as a group of homologies, generated by reflections along a set of 45 primes (hyperplanes) in $\mathbb{P}^4$. In 1905, Coble showed that there is a one-to-one correspondence between the 45 tritangents and the 45 primes in $\mathbb{P}^4$ generating $G_{25,920}$ by reflections (see [T], p.330).

We get a very classical arrangement of 45 planes in $\mathbb{P}^3$. To determine its data, however, we cannot use the look-at-the-tables method, since this arrangement is not defined by a collineation group of $\mathbb{P}^3$. But much of the data is contained in the liturature, especially in Segre's book.* The 27 lines are 5-fold lines of the arrangement, so $t_5(1)=27$. There are in addition 720 double lines, which Segre calls k-lines, so $t_2(1)=720$. This checks in the formula,

$$27\cdot\binom{5}{2} + 720 = \binom{45}{2},$$

so these are all the lines. Each of the 27 lines meets 10 others, so there are $27\cdot 10\cdot 1/2 = 135$ points on the cubic surface where 2 of the 27 lines meet. These are 9-fold points of the arrangement, so $t_9=135$. In each of the 45 planes, 3 of the 5-fold lines and 32 of the double lines lie, and 27 of the 135 points. These 27 points lying in one of the tritangent planes determine dually a line arrangement of 27 lines. We claim this arrangement of 27 lines is $\mathrm{Ceva}^2(8,3)$. We must show this arrangement has $t_{10}=3$, $t_3=64$, $t_2=24$. The three 10-fold points are the duals of the 3 5-fold lines in the tritangent, each containing 10 of the 27 points. Let p_1, p_2 and p_3 be the 3 points in the tritangent where 2 of the 5-fold lines meet, and let λ_i, ν_j, μ_k $i,j,k=1,\dots,8$ be the remaining 24 points. We get the following picture:



* the following discussion is valid for the general cubic surface, i.e. no Eckard points

Choosing any of the λ_i and any of the ν_j, the line joining them contains one of the μ_k, as follows from the addition law on a cubic curve. Therefore there are $8\cdot 8=64$ lines, containing each 3 of the 24 points λ_i, ν_j, μ_k. So the dual arrangement has $t_3=64$. Now through say p_1 and each of the ν_j there is a line containing 2 of the 27 points, so the dual arrangement has $t_2=8\cdot 3 = 24$, as we wanted to show.

For our arrangement of 35 lines in the tritangent we see: the 3 5-fold lines are dual to the 3 "verticies" of $\mathrm{Ceva}^2(8,3)$, and of the 64 triple points, 32 are dual to the remaining 32 lines of the arrangement. Therefore our arrangement in the tritangent plane is dual to the following subset of the points of $\mathrm{Ceva}^2(8,3)$: the 3 verticies, and 4 of the 8 double points lying on each of the 24 lines of $\mathrm{Ceva}^2(8)$. So t_5=#(lines meeting 5 of the 35 points of $\mathrm{Ceva}^2(8,3)$ just mentioned) =24, t_3 =#{lines meeting 3 of the 32 points}=32, $t_2=256+3$, so for our arrangement of 45 planes we get $t_4=32\cdot 45\cdot 1/4 = 360$, $t_3=256\cdot 45\cdot 1/3 = 3840$. Summarising,

$a_1^3(45)$ = 45 tritangents of a smooth cubic surface in $\mathbb{P}^3$

k=45	$t_5(1)=27$	$t_9=135$	$t_{95}=270$	$t_{92}=2160$
	$t_2(1)=720$	$t_4=360$		$t_{42}=2160$
		$t_3=3840$		

2.1.5. Arrangements defined by reflection groups

Let V be a finite dimensional vector space (real or complex). An element $s\in GL(V)$ is a reflection if it has finite order and its fixpoint set is a hyperplane. A finite subgroup $G\subset GL(V)$ is called a reflection group if it is generated by reflections. An (affine) arrangement in V is a finite set $\mathcal{A}$ of hyperplanes, all containing the origin. Such an affine arrangement defines an arrangement in projective space of one dimension less in either one of two ways: If $V=\mathbb{C}^{N+1}$, say, embedd $\mathbb{C}^{N+1}\hookrightarrow\mathbb{P}^{N+1}$. Its complement, the "plane at infinity" is a $\mathbb{P}^N$, and $\mathcal{A}$ induces an arrangement $L\subset\mathbb{P}^N$. We get the same arrangement by blowing up the origin in $\mathbb{C}^{N+1}$ and letting $L\subset\mathbb{P}^N$ (=exceptional divisor) be the arrangement defined as the proper transforms of all the planes of $\mathcal{A}$ through the origin.

Now let $G\subset GL(V)$ be a finite subgroup generated by reflections, and let $\mathcal{A}$ be the set of reflecting hyperplanes $H=\mathrm{Fix}(s)$, $s\in G$ a reflection. We call $\mathcal{A}$ the arrangement defined by G, and will also speak of the arrangement $L\subset\mathbb{P}^N$ defined by the group G', where L is the projective arrangement defined by $\mathcal{A}$ as above, and G' is the associated collineation group, $G'=G/Z$, Z=elements of G represented by scalar matricies ([ST],I,1.).

Now assume $V\cong\mathbb{C}^{N+1}$. G is then called a (finite) unitary reflection group. The "finite" is usually omitted. The unitary reflection groups have been classified by G. Shepard & J. Todd ([ST]). First there are the so called imprimitive groups, which are denoted G(m,p,n),

$m \geq 2$, p divisor of m, $n = \dim V$, which exist in any dimension. The arrangements defined by these groups are the Ceva arrangements above. In addition to these, there are finitely many groups in the dimensions 2-8. For us the groups in dimensions ≥ 4 are interesting. These have the numbers 28-37 in Shepard-Todd's list. In dimension 4 (which define arrangements in $\mathbb{P}^3$), we have 5 groups:

	projective	affine
28.	G'_{576}	G_{1152}
29.	G'_{1920}	G_{7680}
30.	G'_{7200}	G_{14400}
31.	G'_{11520}	$G_{64 \cdot 6!}$
32.	$G'_{25,920}$	$G_{216 \cdot 6!}$

Here the subscript denotes the order of the group.

The arrangements defined by these groups were studied by Orlik-Solomon in {OS1}, {OS2}. Of the above groups 28. and 30. define real arrangements. By a theorem of Coxeter, ({Co2}, Theorem 11.23), such an arrangement is simplicial if G is irreducible. These are just the arrangements $A_1^3(24)$ and $A_1^3(60)$, respectively. $G'_{11,520}$ defines the arrangement $a_2^3(60)$ = Klein's 60_{15} configuration. The group G'_{1920} is a subgroup of G'_{11520} leaving fixed a set of five of the 15 fundamental tetrahedra. It defines an arrangement of 40 planes. $G'_{25,920}$ is a group generated by reflections of order 3, and also defines an arrangement of 40 planes (do not confuse this group with the group of order 25,920 acting on $\mathbb{P}^4$, which is isomorphic to the group of even permutations of the 45 tritangents, which is generated by two-fold reflections)*. We can determine the data from the tables of Orlik-Solomon,

$a_1^3(40)$ = arrangement defined by G'_{1920}

$k=40 \quad t_4(1)=30 \quad t_{12}=20 \quad t_{12,4}=60 \quad t_{12,3}=320$: $\mathrm{Ceva}^2(4)$

$t_3(1)=160 \quad t_9=40 \quad t_{94}=120 \quad t_{93}=160 \quad t_{92}=240$: $A_1^2(9)$

$t_2(1)=120 \quad t_6=160 \quad t_{63}=640 \quad t_{62}=480$: $A_1^2(6)$

$t_4=160$: near-pencil

$a_2^3(40)$ = arrangement defined by the group G'_{25920}

$k=40 \quad t_4(1)=90 \quad t_{12}=40 \quad t_{12,4}=360 \quad t_{12,2}=480 \quad G'_{216}$

$t_2(1)=240 \quad t_5=360$: near-pencil

The arrangement induced in each plane is the 21 lines of the extended Hess-pencil defined by G'_{216}, number 26 in the {ST} list.

* actually the projective groups are isomorphic, but the reflection groups are not.

2.2. Calculation of the Chern numbers

As described in §1 we construct for each arrangement of planes and each $n \geq 2$ a Fermat cover, a non-singular algebraic 3-fold. In dimension 3 the resolution of singularities of the branch locus takes 2 steps. Let $\hat{\mathbb{P}}_1^3$ denote $\mathbb{P}^3$ blown up at all actual singular points:

$$\rho_1 : \hat{\mathbb{P}}_1^3 \to \mathbb{P}^3,$$

and let $\hat{\mathbb{P}}^3$ denote $\hat{\mathbb{P}}_1^3$ blown up along all singular lines:

$$\rho_2 : \hat{\mathbb{P}}^3 \to \hat{\mathbb{P}}_1^3, \qquad \rho = \rho_1 \circ \rho_2 : \hat{\mathbb{P}}^3 \to \mathbb{P}^3.$$

In each exceptional $P_j = \{p_j \text{ blown-up}\} \cong \mathbb{P}^2$ we have a hyperplane class which we denote by h_j. In each exceptional $L_m = \{l_m \text{ blown-up}\} \cong \mathbb{P}^1 \times \mathbb{P}^1$ we have a fibre class which we denote by f_m. Notice that l_m now refer s to a singular line and not to one of the branching planes as in §1. We gather the neccessary notation:

Object	Notation	Index	
k planes of the arrangement	$H_1,\dots,H_k$	i	$H^2(\hat{\mathbb{P}}^3,\mathbb{Z})$
τ exceptional $\mathbb{P}^2$ s	$P_1,\dots,P_\tau$	j	
σ exceptional $\mathbb{P}^1 \times \mathbb{P}^1$ s	$L_1,\dots,L_\sigma$	m	
$t_2(1)$ n.-s. lines $H_{i_1} \cap H_{i_2}$	$l_1,\dots,l_{t_2(1)}$	i_{12}	$H^4(\hat{\mathbb{P}}^3,\mathbb{Z})$
hyperplane class on P_j	$h_1,\dots,h_\tau$	j	
class of fibre on L	$f_1,\dots,f_\sigma$	m	
#planes through sing. pt. p_j	$\tau(P_j)$		
#planes through sing. line l_m	$\tau(L_m)$		

We use the following coincidence relations:

$$\sigma_{jm} = \begin{cases} 0 & \text{otherwise} \\ 1 & p_j \in l_m \end{cases}$$

$$\hat{\sigma}_{ji_{12}} = \begin{cases} 0 & \text{otherwise} \\ 1 & p_j \in l_{i_{12}} \end{cases} \quad \text{(non-singular line)}$$

$$\sigma_{ji} = \begin{cases} 0 & \text{otherwise} \\ 1 & p_j \in H_i \end{cases} \qquad \sigma_{jmi} = \begin{cases} 0 & \text{otherwise} \\ 1 & \text{if } p_j \in l_m \subset H_i \end{cases}$$

$$\sigma_{mi} = \begin{cases} 0 & \text{otherwise} \\ 1 & l_m \subset H_i \end{cases}$$

$$\tau_{mi} = \begin{cases} 0 & \text{otherwise} \\ 1 & l_m \text{ meets } H_i \text{ at a near-pencil} \end{cases}$$

Let $\{\ \}$ denote proper transform. After blowing up the points we have

$$\text{on } \hat{\mathbb{P}}_1^3 \quad \begin{cases} \{H_i\} = \rho_1^* H_i - \Sigma_j \sigma_{ji} P_j \\ \{l_{i_{12}}\} = \rho_1^* l_{i_{12}} - \Sigma_j \hat{\sigma}_{ji_{12}} h_j \\ \{l_m\} = \rho_1^* l_m - \Sigma_j \sigma_{jm} h_j \end{cases}$$

After blowing up all singular lines we have:

2.2.0. on $\hat{\mathbb{P}}^3$
$$\begin{cases} \{H_i\}=\rho^* H_i-\Sigma_j\sigma_{ji}\{P_j\}-\Sigma_m\sigma_{mi}\{L_m\} \\ \{P_j\}= P_j \text{ blown up at } \Sigma_m\sigma_{jm} \text{ points} \\ \{h_j^i\}=\rho_2^* h_j^i - \Sigma_m\sigma_{jmi}f_m, \quad h_j^i=\{H_i\cap P_j\} \text{, one of the } \tau(P_j) \text{ lines on } P_j \end{cases}$$

2.2.1. <u>First</u> <u>Chern</u> <u>class</u>

Let $\pi:Y\to\hat{\mathbb{P}}^3$ denote the smooth Fermat cover. The first Chern class (= the negative of the canonical divisor) is easy to calculate. We have

$$c_1(Y)=\pi^*\{c_1(\hat{\mathbb{P}}^3)-\frac{n-1}{n}(\sum_{i=1}^{k}\{H_i\}+\sum_{j=1}^{\tau}\{P_j\}+\sum_{m=1}^{\sigma}L_m)\} .$$

Since $c_1(\hat{\mathbb{P}}^3)=4H-2\Sigma_j P_j-\Sigma_m L_m$ (H hyperplane in $\hat{\mathbb{P}}^3$), inserting 2.2.0. we get the formula

2.2.1. $$c_1(Y)=\pi^*\{(4-\frac{n-1}{n}k)H-\sum_j\{2+\frac{n-1}{n}(1-\tau(P_j))\}\{P_j\} -\sum_m\{1+\frac{n-1}{n}(1-\tau(L_m))\}L_m\} .$$

2.2.2. <u>Second</u> <u>Chern</u> <u>class</u>

<u>Lemma 2.2.2.</u>: Y as above, then (modulo torsion)

$$\begin{aligned} c_2(Y)=\pi^*\Big[&(6+\sigma)H^2-\Sigma_{j,m}\sigma_{jm}h_j-\Sigma_m(4-2\Sigma_j\sigma_{jm})f_m \\ &-\frac{n-1}{n}\{(3k+2\sigma)H^2+\Sigma_j\{3-\tau(P_j)-2\Sigma_m\sigma_{jm}\}h_j \\ &\qquad +\Sigma_m\{\tau(L_m)(\Sigma_j\sigma_{jm}-3)+\Sigma_j\sigma_{jm}-\Sigma_i\tau_{mi}\}\} \\ &+(\frac{n-1}{n})^2\{(t_2(1)+\Sigma_m\tau(L_m))H^2+\Sigma_j\{\tau(P_j)-\Sigma_{i_{12}}\hat{\sigma}_{ji_{12}}-\Sigma_m\sigma_{jm}\tau(L_m)\}h_j \\ &\qquad +\Sigma_m\{(1-\tau(L_m))(\Sigma_i\tau_{mi}+\Sigma_j\sigma_{jm})+(\Sigma_j\sigma_{jm}-1)\tau(L_m)\}f_m\}\Big] \end{aligned}$$

proof: The right hand side is a cohomology class in $H^4(Y,\mathbb{Z})$, call it α. It suffices to show: $\alpha_D=c_2(Y)_{|D}$, for all divisors D, that is, α restricted to D is the same as $c_2(Y)$ restricted to D. From adjunction we know
$$c_2(Y)_{|D}=c_2(D)-c_1(D)(D)_{|D}$$
so what we must show is

2.2.2.1. $$\alpha_{|D}=c_2(D)-c_1(D)(D)_{|D} .$$

Furthermore, it sufficies to show 2.2.2.1. only for cohomology class intersections (we are only claiming $\varkappa$ and $c_2(Y)$ are cohomologous). Thus it suffices to check the following cases. Set $D=\pi^*\tilde{D}$, $\tilde{D}\in H^4(\hat{\mathbb{P}}^3,\mathbb{Z})$

1) $\tilde{D}=H=$ hyperplane class in $\hat{\mathbb{P}}^3$ (transversal to branch locus)
2) $\tilde{D}=\frac{1}{n}H_i=$ one of the branching planes
3) $\tilde{D}=\frac{1}{n}P_j=$ one of the exceptional $\mathbb{P}^2$'s
4) $\tilde{D}=\frac{1}{n}L_m=$ one of the exceptional $\mathbb{P}^1\times\mathbb{P}^1$'s .

In cases 1-3 the cover $\pi:D\to\tilde{D}$ is a Fermat cover of surfaces. (The factor $\frac{1}{n}$ is just formal, to keep the calculations strait-we are interested in the <u>reduced</u> ramification divisors). In these cases, the Chern numbers c_1^2 and c_2 are well known (see {Hi1}). In case 4 the cover $D\to\tilde{D}$ is a product of curves (Fermat curves) mapping onto $\mathbb{P}^1\times\mathbb{P}^1$, the branching locus being $\{(b_1)\times\mathbb{P}^1\}\cup\{\mathbb{P}^1\times(b_2)\}$, where b_1, b_2 are the branching divisors on the first and second copies of $\mathbb{P}^1$, respectively. In each case we can calculate $c_2(D)$ and check 2.2.2.1. explicitly. The first 3 cases are easy and may be left to the reader. Case 4 is a little bit trickier, so we do this one. We use adjunction in the following form:

$$(c_2(Y)-c_1(D))_{|D}=c_2(D).$$

Let $\overline{L}_m$ be the reduced ramification divisor lying over L_m. We first calculate $c_1(\overline{L}_m)$:

$$c_1(\overline{L}_m)=\pi^*\{c_1(L_m)-\frac{n-1}{n}(\tau(L_m)b_m+(\Sigma_i\tau_{mi}+\Sigma_j\sigma_{jm})f_m\}\quad b_m,f_m \text{ base \& fibre}$$

$$=\pi^*\{2b_m+2f_m-\frac{n-1}{n}(\tau(L_m)b_m+(\Sigma_i\tau_{mi}+\Sigma_j\sigma_{jm})f_m)\}\ .$$

It is sufficient to consider the components of α which are based on f_m (since $L_m\cdot h_j=L_m\cdot H^2=0$). Set $\alpha=\pi^*\tilde{\alpha}$. We now calculate $(\alpha-c_1(\overline{L}_m))_{|\overline{L}_m}$:

$$n\alpha_{|\pi^*L_m}=\pi^*(\tilde{\alpha}_{|L_m})=\pi^*\{(4-2\Sigma_j\sigma_{jm})-\frac{n-1}{n}(3\tau(L_m)+\Sigma_i\tau_{mi}-\Sigma_j\sigma_{jm})$$

$$+(\frac{n-1}{n})^2(\tau(L_m)+(\tau(L_m)-1)(\Sigma_i\tau_{mi}+\Sigma_j\sigma_{jm}))\}$$

Now $c_1(L_m)\cdot L_m=((c_1\hat{\mathbb{P}}^3)-L_m)_{|L_m})\cdot L_m=(4H-2\Sigma_jP_j-2\Sigma_mL_m)_{|L_m}\cdot L_m=-4+2\Sigma_j\sigma_{jm}-2L_m^3$.

$L_m^3=-2+2\Sigma_j\sigma_{jm}$, so $c_1(L_m)\cdot L_m=-2\Sigma_j\sigma_{jm}$. On the other hand $L_m\cdot b_m=1-\Sigma_j\sigma_{jm}$,

$L_m\cdot f_m=-1$, so we get

$$\pi^*\{\tilde{\alpha}_{|L_m}-\frac{1}{n}(c_1(L_m)-\frac{n-1}{n}(\tau(L_m)b_m+(\Sigma_i\tau_{mi}+\Sigma_j\sigma_{jm})f_m))\cdot L_m\}$$

$$=\pi^*\{\tilde{\alpha}_{|L_m}-(1-\frac{n-1}{n})(\ldots)\cdot L_m\}$$

$$=\pi^*\{4-2\Sigma_j\sigma_{jm}-\frac{n-1}{n}(\tau(L_m)(3-\Sigma_j\sigma_{jm})+\Sigma_i\tau_{mi}-\Sigma_j\sigma_{jm})+(\frac{n-1}{n})^2[\tau(L_m)(1-\Sigma_j\sigma_{jm})$$

$$+(\tau(L_m)-1)(\Sigma_i\tau_{mi}+\Sigma_j\sigma_{jm})\Big] - (1-\tfrac{n-1}{n})\Big[\{-2\Sigma_j\sigma_{jm}-\tfrac{n-1}{n}(\tau(L_m)(1-\Sigma_j\sigma_{jm})$$

$$-(\Sigma_i\tau_{mi}+\Sigma_j\sigma_{jm})))\}\Big]\}$$

$$=\pi^*\{4-\tfrac{n-1}{n}(2\tau(L_m)+2\Sigma_i\tau_{mi}+2\Sigma_j\sigma_{jm})+(\tfrac{n-1}{n})^2(\tau(L_m)(\Sigma_i\tau_{mi}+\Sigma_j\sigma_{jm}))\}$$

$=nc_2(\overline{L}_m)$, as was to be shown (the factor n since π is branched at L_m).

2.2.3. Third Chern class

For $c_3(Y)=e(Y)$, the euler-poincaré characteristic, we use the formula

$$e(Y)=\deg\pi(e(\hat{\mathbb{P}}^3)-\tfrac{n-1}{n}e(B^1)+(\tfrac{n-1}{n})^2e(B^2)-(\tfrac{n-1}{n})^3e(B^3)),$$

where $e(B^i)=\Sigma_\lambda e(B^i_\lambda)$, the B^i_λ being the irreducible components of the codimension i part of the branch locus B^i. $e(B^i)$ is the euler number of a non-singular model (disjoint union of components). (This formula might appear more familiar by multiplying through by n^3 and sorting out by powers of n.) This yields the following formula:

$$c_3(Y)=n^{k-1}\{4+2\tau+2\sigma-\tfrac{n-1}{n}(3k+3\tau+4\sigma+\Sigma_{i,j}\sigma_{ji}+\Sigma_{i,m}\tau_{mi}+\Sigma_{j,m}\sigma_{jm})$$

2.2.3. $$+2(\tfrac{n-1}{n})^2(t_2(1)+\Sigma_j\tau(P_j)+\Sigma_{i,m}\tau_{mi}+\Sigma_{j,m}\sigma_{jm}+\Sigma_m\tau(L_m))$$

$$-(\tfrac{n-1}{n})^3(t_3+\Sigma_{j,m}\sigma_{jm}\tau(L_m)+\Sigma_{j,i_{12}}\sigma_{ji_{12}}+\Sigma_{i,m}\tau_{mi}\tau(L_m))\}.$$

2.2.4. Chern numbers

The third Chern class comes in number form. To calculate the other numbers c_1^3 and c_1c_2 we need the following multiplication tables in $\hat{\mathbb{P}}^3$:*

	H	P_{j_1}	L_{m_1}
H	H^2	0	f_{m_1}
P_{j_2}	0	$-\delta_{j_1j_2}h_{j_1}$	$\sigma_{j_2m_1}f_{m_1}$
L_{m_2}	f_{m_2}	$\sigma_{j_1m_2}f_{m_2}$	$-H^2+(2-2\Sigma_j\sigma_{jm})f_m+\Sigma_j\sigma_{jm}h_j$

	H	P_{j_1}	L_{m_1}
H^2	1	0	0
h_{j_2}	0	$-\delta_{j_1j_2}$	0
f_{m_2}	0	0	$-\delta_{m_1m_2}$

Writing

$$c_1(Y)=\pi^*\{A\cdot H-\Sigma\beta_jP_j-\Sigma_m c_mL_m\},$$ we will have

$$c_1^3(Y)=\deg\pi\{A^3-\Sigma_j\beta_j^3-\Sigma_m c_m^3L_m^3-3\Sigma_m A\cdot c_m^2+3\Sigma_{j,m}\sigma_{jm}\beta_jc_m^2\}$$

yielding the formula

*compare [GH] Chapter on surfaces, section on Noethers formula.

$$c_1^3(Y)=n^{k-1}\{(4-\tfrac{n-1}{n}k)^3 -\textstyle\sum_j\{2+\tfrac{n-1}{n}(1-\tau(P_j)\}^3 -\sum_m\{1+\tfrac{n-1}{n}(1-\tau(L_m))\}^3\cdot$$

$$\{2\Sigma_j\sigma_{jm}-2\}$$

$$-3\textstyle\sum_m(4-\tfrac{n-1}{n}k)\{1+\tfrac{n-1}{n}(1-\tau(L_m))\}^2$$

2.2.4!³

$$+3\textstyle\sum_{j,m}\sigma_{jm}\{2+\tfrac{n-1}{n}(1-\tau(P_j))\}\{1+\tfrac{n-1}{n}(1-\tau(L_m))\}^2\}$$

Finally, the formula for c_1c_2 is

$$c_1(Y)c_2(Y)=n^{k-1}\Big[(4-\tfrac{n-1}{n}k)\{6+\sigma-\tfrac{n-1}{n}(3k+2\sigma)+(\tfrac{n-1}{n})^2(t_2(1)+\Sigma_m\tau(L_m))$$

$$+\textstyle\sum_j\{2+\tfrac{n-1}{n}(1-\tau(P_j))\}\{-\Sigma_m\sigma_{jm}-\tfrac{n-1}{n}(3-\tau(P_j)-2\Sigma_m\sigma_{jm})$$

$$+(\tfrac{n-1}{n})^2\{\tau(P_j)-\Sigma_{i_{12}}\sigma_{ji_{12}}-\Sigma_m\sigma_{jm}\tau(L_m))\}$$

$$+\textstyle\sum_m\{1+\tfrac{n-1}{n}(1-\tau(L_m))\}\{2\Sigma_j\sigma_{jm}-4-\tfrac{n-1}{n}(\tau(L_m)(\Sigma_j\sigma_{jm}-3)+\Sigma_j\sigma_{jm}-\Sigma_i\tau_{mi})\}$$

2.2.4!²

$$+(\tfrac{n-1}{n})^2\{(1-\tau(L_m))(\Sigma_i\tau_{mi}+\Sigma_j\sigma_{jm})+(\Sigma_j\sigma_{jm}-1)\tau(L_m)\}\Big]$$

These formula are quite compact compared with those which come now. We wish to write the formula purely in terms of the combinatorial data of the arrangement as in 2.1. We start with the euler number which is easy,

$$c_3(Y)=n^{k-1}\{4+2\overline{\Sigma}t_p+2\sum_{q\geq 3}t_q(1)-\tfrac{n-1}{n}(3k+\overline{\Sigma}(3+p)t_p+4\sum_{q\geq 3}t_q(1)+\Sigma t_{pq})$$

$$+2(\tfrac{n-1}{n})^2(t_2(1)+\overline{\Sigma}pt_p+\sum_{q\geq 3}qt_q(1)+\Sigma t_{pq})-(\tfrac{n-1}{n})^3(t_3+\Sigma qt_{pq}+\overline{\Sigma}t_{p2})\}.$$

We use the notation $\overline{\Sigma}t_p$, etc. to indicate summation only for actual singular points. This allows us to write $c_3(Y)/n^{k-4}$ as a cubic polynomial,

$$\frac{c_3(Y)}{n^{k-4}}=A_3n^3+B_3n^2+C_3n+D_3,$$

$$A_3=4-3k+2t_2(1)-t_3+2\sum_{q\geq 3}(q-1)t_q(1)+\overline{\Sigma}(p-1)t_p+\sum_{\substack{p\geq 4\\ q\geq 3}}(1-q)t_{pq}-\overline{\Sigma}t_{p2}$$

$$B_3=3k-4t_2(1)+3t_3-4\sum_{q\geq 3}(q-1)t_q(1)-3\overline{\Sigma}(p-1)t_p+3\sum_{\substack{p\geq 4\\ q\geq 3}}(q-1)t_{pq}+3\overline{\Sigma}t_{p2}$$

$$C_3=2t_2(1)-3t_3+2\overline{\Sigma}pt_p+2\sum_{q\geq 3}qt_q(1)-\sum_{\substack{p\geq 4\\ q\geq 3}}(3q-2)t_{pq}-3\overline{\Sigma}t_{p2}$$

$$D_3=t_3+\sum_{\substack{p\geq 4\\ q\geq 3}}qt_{pq}+\overline{\Sigma}t_{p2}$$

This formula now has a form which can be put on a computer.

For $c_1^3(Y)$ we get the formula

$$c_1^3(Y)=n^{k-1}\{(4-\tfrac{n-1}{n}k)^3-\bar{\Sigma}\{2+\tfrac{n-1}{n}(1-p)\}^3t_p-\sum_{q\geq 3}\{1+\tfrac{n-1}{n}(1-q)\}^3(2\bar{\Sigma}t_{pq}-2t_q(1))$$

$$-3\sum_{q\geq 3}(4-\tfrac{n-1}{n}k)\{1+\tfrac{n-1}{n}(1-q)\}^2\cdot t_q(1)$$

$$+3\bar{\Sigma}t_{pq}\{2+\tfrac{n-1}{n}(1-p)\}\{1+\tfrac{n-1}{n}(1-q)\}^2\},$$

where $\bar{\Sigma}t_{pq}$ again indicates summation only over actual singular points. Multiplying through by n^3 we get the following cubic polynomial,

$$\frac{c_1(Y)}{n^{k-4}}=A_{13}n^3+B_{13}n^2+C_{13}n+D_{13},$$

$$A_{13}=\{(4-k)^3-\bar{\Sigma}(3-p)^3t_p-2\sum_{q\geq 3}(2-q)^3(\bar{\Sigma}t_{pq}-t_q(1))-3(4-k)\sum_{q\geq 3}(2-q)^2t_q(1)$$

$$+3\bar{\Sigma}(3-p)(2-q)^2\cdot t_{pq}\}$$

$$B_{13}=\{3k(4-k)^2-3\bar{\Sigma}(3-p)^2(p-1)t_p-3\sum_{q\geq 3}(2-q)^2(q-1)(\bar{\Sigma}t_{pq}-t_q(1))$$

$$-3\sum_{q\geq 3}\{2(4-k)(2-q)(q-1)+k(2-q)^2\}t_q(1)$$

$$+3\bar{\Sigma}\{2(3-p)(2-q)(q-1)+(p-1)(2-q)^2\}t_{pq}\}$$

$$C_{13}=\{3k^2(4-k)-3\bar{\Sigma}(3-p)(p-1)^2t_p-3\sum_{q\geq 3}(2-q)(q-1)^2(\bar{\Sigma}t_{pq}-t_q(1))$$

$$-3\sum_{q\geq 3}\{(4-k)(q-1)^2+2k(2-q)(q-1)\}t_q(1)+3\bar{\Sigma}\{(3-p)(q-1)^2+2(p-1)(2-q)(q-1)\}t_{pq}\}$$

$$D_{13}=\{k^3-\bar{\Sigma}(p-1)^3t_p-\sum_{q\geq 3}(q-1)^3(\bar{\Sigma}t_{pq}-t_q(1))-3k\sum_{q\geq 3}(q-1)^2t_q(1)$$

$$+3\bar{\Sigma}(p-1)(q-1)^2t_{pq}\}$$

The number c_1c_2 is a little bit trickier. The reader may verify

$$c_1(Y)c_2(Y)=n^{k-1}\{(4-\tfrac{n-1}{n}k)(6+\sum_{q\geq 3}t_q(1)-\tfrac{n-1}{n}(3k+2\sum_{q\geq 3}t_q(1))+(\tfrac{n-1}{n})^2(t_2(1)+\sum_{q\geq 3}qt_q(1))\}$$

$$+\bar{\Sigma}\{2+\tfrac{n-1}{n}(1-p)\}\Big\{-\Sigma t_{pq}-\tfrac{n-1}{n}((3-p)t_p-2\sum_{q\geq 3}t_{pq})+(\tfrac{n-1}{n})^2(pt_p-t_{p2}-\sum_{q\geq 3}qt_{pq})\Big\}$$

$$+\sum_{q\geq 3}(1+\tfrac{n-1}{n}(1-q))\Big\{(2\bar{\Sigma}_p t_{pq}-4t_q(1))-\tfrac{n-1}{n}(q(\bar{\Sigma}_p t_{pq}-3t_q(1))+\bar{\Sigma}_p t_{pq}-t_{q+1,q})$$

$$+(\tfrac{n-1}{n})^2\{(1-q)(\sum_{p\geq q}t_{pq})+q(\bar{\Sigma}_p t_{pq}-t_q(1))\}\Big\}\Big].$$

This gives

$$\frac{c_1(Y)c_2(Y)}{n^{k-4}}=A_{12}n^3+B_{12}n^2+C_{12}n+D_{12}$$

$$A_{12}=\{(4-k)(6-3k+t_2(1)+\sum_{q\geq 3}(q-1)t_q(1))+\bar{\Sigma}(3-p)\Big[-t_{p2}+(2p-3)t_p-\sum_{q\geq 3}(q-1)t_{pq}\Big]$$

$$+\sum_{q\geq 3}(2-q)\Big[\bar{\Sigma}_p t_{pq}+t_{q+1,q}+2(q-2)t_q(1)+(1-q)(\sum_{p\geq q}t_{pq})\Big]\}$$

$$B_{12}=\{k(6-3k+t_2(1)+\sum_{q\geq 3}(q-1)t_q(1))+(4-k)(3k-2t_2(1)-2\sum_{q\geq 3}(q-1)t_q(1))$$

$$+\bar{\sum}\Big((3-p)\Big[3(1-p)t_p+2\sum_{q\geq 3}(q-1)t_{pq}+2\Sigma t_{p2}\Big]-(p-1)\Big[-t_{p2}+(2p-3)t_p-\sum_{q\geq 3}(q-1)t_{pq}\Big]\Big)$$

$$+\sum_{q\geq 3}\Big((q-1)\Big[\bar{\sum_p} t_{pq}+t_{q+1,q}+2(q-2)t_q(1)+(1-q)(\sum_{p\geq q}t_{pq})\Big]$$

$$+(2-q)\Big[-qt_q(1)+(1-q)\bar{\sum_p} t_{pq}-t_{q+1,q}-2(1-q)(\sum_{p\geq q}t_{pq})\Big]\Big)\}$$

$$C_{12}=\{k(3k-2t_2(1)-2\sum_{q\geq 3}(q-1)t_q(1))\ +(4-k)(t_2(1)+\sum_{q\geq 3}qt_q(1))$$

$$+\bar{\sum}\Big((p-1)\Big[3(1-p)t_p-2\sum_{q\geq 3}(1-q)t_{pq}+2\Sigma t_{p2}\Big]+(3-p)\Big[pt_p-t_{p2}-\sum_{q\geq 3}qt_{pq}\Big]\Big)$$

$$+\sum_{q\geq 3}\Big((2-q)\Big[(1-q)(\sum_{p\geq q}t_{pq})+q(\bar{\sum_p} t_{pq}-t_q(1))\Big]$$

$$+(q-1)\Big[-qt_q(1)+(1-q)\bar{\sum_p} t_{pq}-t_{q+1,q}-2(1-q)(\sum_{p\geq q}t_{pq})\Big]\Big)\}$$

$$D_{12}=\{k(t_2(1)+\sum_{q\geq 3}qt_q(1))+\bar{\sum}(p-1)\Big[pt_p-t_{p2}-\sum_{q\geq 3}qt_{pq}\Big]$$

$$+\sum_{q\geq 3}(q-1)\Big[(1-q)(\sum_{p\geq q}t_{pq})+q(\bar{\sum_p} t_{pq}-t_q(1))\Big]\}$$

2.3. Ampleness of the canonical bundle

Let K_Y be the canonical bundle on a projective algebraic manifold Y. There is the following criterion for ampleness of K_Y, which was suggested to us by A. Sommese:

Assume: $\left\{ \begin{array}{l} K_Y^N > 0, \quad N=\dim Y, \text{ and for all effective curves } C\subset Y, \\ \qquad\qquad K_Y\cdot C>0 \end{array}\right.$

Then: K_Y is ample.

This follows from a result of Kawamata, which states that the assumptions imply that $K_Y^{\otimes m}$ is generated by its global sections for sufficiently large m, and was first proved for N=3 by P. M. H. Wilson [Wi].

For Fermat covers we have $K_Y=\pi^*(K^Y)$, $K^Y\in \mathrm{Div}(\hat{\mathbb{P}}^N)\otimes\mathbb{Q}$, so it is clearly sufficient to check

for all effective curves $C\subset Y$, $\pi(C)\cdot K^Y>0$.

We now check this condition explicitly for N=3. Let $L\subset\mathbb{P}^3$ be an arrangement and assume:

$k\geq 8$, $t_k(1)=t_{k-1}(1)=t_{k-2}(1)=t_k=t_{k-1}=0$ and L is join-irreducible.

We now write K^Y in a convenient form:

$$K^Y=\sum_{i=1}^{k}\lambda_i(H_i)+\sum_{j=1}^{T}\nu_jP_j+\sum_{m=1}^{Q}\mu_mL_m$$

where $\lambda_i = (\frac{n-1}{n} - \frac{4}{k})$

$$\nu_j = (3-\frac{1}{n} - \frac{4\tau(P_j)}{k})$$

$$\mu_m = (2-\frac{1}{n} - \frac{4\tau(L_m)}{k}).$$

This yields

Lemma 2.3.1.: Assume $\tau(P_j) < \frac{k(3n-1)}{4n}$, $\tau(L_m) < \frac{k(2n-1)}{4n}$ for all singular points p_j and for all singular lines l_m. Then K^Y can be written such that all coefficients ν_j and μ_m are positive. For n=2, k=8, $\lambda_i=0$, otherwise $\lambda_i>0$:

Now recall our curve $C \subset Y$, $\pi(C) \subset \hat{\mathbb{P}}^3$. One of the following must occur:

a) $\pi(C)$ lies in a component of K^Y

or b) it doesn't.

If b) occurs, then by virtue of 2.3.1., $K^Y \cdot \pi(C) > 0$. So assume a), $C \subset D_i \subset K^Y$. Then either

A) $\pi(C)$ is $D_i \cap D_j$ for some j

or B) $\pi(C)$ lies in no other D_j.

This yields conditions on the arrangement for K_Y to be ample. Assume first B)

I) Assume $\pi(C) \subset \{H_i\}$. Then

$$(\pi(C)\cdot H_i)_{\hat{\mathbb{P}}^3} = \pi(C)\cdot h_i > 0 \qquad h_i = \text{hyperplane in } H_i$$

$$(\pi(C)\cdot P_j)_{\hat{\mathbb{P}}^3} = \begin{cases} 1 & \text{if } \pi(C) \text{ passes through } p_j \\ 0 & \text{otherwise} \end{cases}$$

$$(\pi(C)\cdot L_m)_{\hat{\mathbb{P}}^3} = \sigma_{mi}(\pi(C)\cdot h_i) > 0$$

So certainly $K^Y\cdot(C)>0$.

II) Assume $\pi(C) \subset P_j$. Then

$$(\pi(C)\cdot P_j)_{\hat{\mathbb{P}}^3} = \pi(C)\cdot(-h_j) < 0$$

$$(\pi(C)\cdot H_i)_{\hat{\mathbb{P}}^3} = \sigma_{ji}(\pi(C)\cdot h_j) > 0$$

$$(\pi(C)\cdot L_m)_{\hat{\mathbb{P}}^3} = \begin{cases} 1 & \text{if } \pi(C) \text{ passes through sing. pt. } L_m\ P_j \\ 0 & \text{otherwise} \end{cases}$$

so in this case we must check

$$\nu_j < \Sigma\sigma_{ji}\lambda_i$$

$$(3-\frac{1}{n} - \frac{4\tau(P_j)}{k}) < \tau(P_j)(\frac{n-1}{n} - \frac{4}{k})$$

$$\frac{3n-1}{n-1} < \tau(P_j)$$

yielding the condition:

$$\begin{cases} n=2, & \text{then } \tau(P_j) > 5 \\ n=3, & \text{then } \tau(P_j) > 4 \end{cases}$$

III) Assume $\pi(C) \subset L_m$. Then

$$(\pi(C)\cdot L_m)_{\hat{\mathbb{P}}^3} = \pi(C)(-b_m + (1-\Sigma\sigma_{jm})f_m)$$

$$(\pi(C)\cdot P_j)_{\hat{\mathbb{P}}^3} = \sigma_{jm}\cdot\pi(C)\cdot f_m$$

$$(\pi(C)\cdot H_i)_{\hat{\mathbb{P}}^3} = \tau_{mi}\cdot\pi(C)\cdot f_m$$

so $\pi(C)\cdot K^Y = \mu_m(-\delta_2 + (1-\Sigma\sigma_{jm})\delta_1) + \delta_1\Sigma\sigma_{jm}\nu_j + \delta_1\Sigma\tau_{mi}\lambda_i + \delta_2\Sigma\sigma_{mi}\lambda_i$

$$= \delta_1(\Sigma\sigma_{jm}\nu_j + \Sigma\tau_{mi}\lambda_i + (1-\Sigma\sigma_{jm})\mu_m) + \delta_2(-\mu_m + \Sigma\sigma_{mi}\lambda_i).$$

(i) Coefficient of $\delta_2 > 0$

$$\tau(L_m) > \frac{\mu_m}{\lambda_i} = \frac{2nk-k-4\tau(L_m)n}{nk-k-4n}$$

yielding the condition

$$\Big\}\ n=2 \quad \tau(L_m) > 3$$

(ii) Coefficient of δ_1 is always positive

Now we come to A)

I) Assume $\pi(C) = H_{i_o} \cap P_j$. Then

$$(\pi(C)\cdot H_{i_o}) = -\pi(C)\cdot P_j = 1$$

$$(\pi(C)\cdot H_{i_k}) = \sigma_{ji_{ok}}$$

$$(\pi(C)\cdot L_m) = \sigma_{jmi_o}.$$

Under these circumstances it is easily verified that $K^Y\cdot\pi(C) > 0$.

II) $\pi(C) = H_{i_1} \cap H_{i_2}$. This case follows immediatly from B) I).

III) $\pi(C) = L_m \cap H_i$, where H_i is one of the $\tau(L_m)$ planes through L_m. This follows from B) I) and III) i).

IV) $\pi(C) = L_m \cap H_i$, where H_i meets L_m at a near pencil point. This follows from B) I) and III) ii).

V) $\pi(C) = P_j \cap L_m$. This follows from B) III) ii) and II).

Putting everything together we get

Theorem 2.3.2.: Assume $-\ \tau(P_j) < \frac{k(3n-1)}{4n}$, $\tau(L_m) < \frac{k(2n-1)}{4n}$

- for $n=2$, $\tau(P_j)>5$, $k>8$, $\tau(L_m)>3$
- for $n=3$ $\tau(P_j)>4$

Then K_Y is ample. If

- $n=2$, $k=8$, $\tau(P_j)=5$ (all j) and $\tau(L_m)=3$ (all m)

Then K_Y is trivial.

Remark: The conditions $\tau(P_j)<k(3n-1)/4n$ and $\tau(L_m)<k(2n-1)/4n$ are not essential. By using a different representation of K^Y it could be replaced by:

The arrangement contains 8 planes,

1) no more than 3 of which pass through a line
2) no more than 5 of which pass through a point.

However, using this representation the calculation gets too ugly, which is why we choose K^Y as we did.

2.4. Minimality

2.4.1. Relatively minimal models

Let X be a complete, non-singular variety.

Definition 2.4.1.: (i) X is a relatively minimal model: $\Leftrightarrow$ any birational map* $f:X \to Y$, where Y is also non-singular, is actually an isomorphism (nothing can be smoothly blown down).

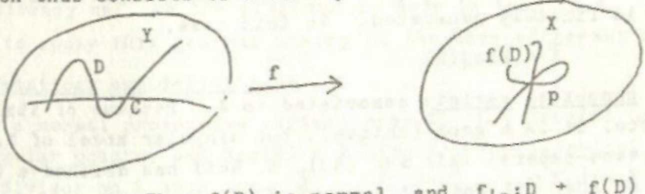
(ii) X is an absolutely minimal model: $\Leftrightarrow$ any birational map $g:Y \to X$ Y non-singular, is actually a morphism (is regular, with no points of indeterminancy).

Recall the situation for surfaces: every smooth surface S has a relatively minimal model S′, and $\rho : S \to S'$ is just the blow-down of finitely many (-1)-curves. The birational equivalence class of S, $\mathcal{C}_S$, contains an absolutely minimal model if and only if S is not birationally ruled. In higher dimensions, there are always relatively minimal models, but no absolutely minimal models in general (compare 2.4.4. & 2.4.5. below).

Now consider 3-folds X. The following is a variant of Zariski′s Main Theorem (in this dimension):

Lemma 2.4.2.: Let $f:Y \to X$ be a birational map of algebraic 3-folds collapsing only a curve $C \subset Y$ to a point $p=f(C) \in X$. Then p is a singular point of X.

proof: Take any ample divisor D on Y, so $D \cdot C > 0$. The set-theoretic intersection thus consists of finitely many points:



Assume X is smooth. Then f(D) is normal, and $f_{|D}: D \to f(D)$ is a birational map with finite fibres, so by Zariski′s Main Theorem is an isomorphism, a contradiction. So X is singular at p.

Because of this, to check that an algebraic 3-fold is relatively minimal it suffices to check that there are no exceptional divisors (blow-ups at non-singular points or curves), and these are all contained in the canonical divisor (considering the behavior of canonical divisors under blow-ups). See [Pi] and [Mi] for more details on this. For Fermat covers, because of our representation of the canonical divisor (2.3.) we only have to check that there are no exceptional divisors in the branch locus. This gives

*everywhere defined

Proposition 2.4.3.: Let Y be the smooth Fermat cover of $\hat{\mathbb{P}}^3$ associated to the arrangement L. If L is join-irreducible (2.1.) then Y is relatively minimal.

However, from the viewpoint of birational geometry, the notion of relative minimality is not at all well behaved, as the following theorems, borrowed from {Ue}, show:

Theorem 2.4.4.: Let X be a relatively minimal Moishezon manifold of dimension ≥ 3. Suppose X contains a rational curve C (which may be singular). Then there exists a relatively minimal model X′ of X, which is not isomorphic to X.

Corollary 2.4.5.: If the relatively minimal Moishezon X contains a ruled surface (which may be singular), the birational equivalence class of X contains continously many distinct minimal models.

In the search for good (unique) minimal models it would be desireable to "blow down" these rational curves, so it will be necessary to consider singular models.

2.4.2. Reid's program for minimal models

Introduction

Let X be a smooth projective variety of general type, and

$$R = \bigoplus_{m\geq 0} H^0(X, mK_X)$$

its canonical ring. This is a birational invariant of X. Assume for the moment that R is finitely generated. In this case,

$$\overline{X} = \mathrm{Proj}(R)$$

is called the canonical variety associated to X. Because of its birational invariance, it is a good (unique), but singular model of X.

In two important papers, {R1} and {R2}, M. Reid has defined a (unique) minimal model of such $\overline{X}$. This is a model X′ in the category of Q-factorial Gorenstein schemes which is a partial resolution

$$\pi : X' \to \overline{X}$$

such that a) π is crepant

b) X′ has only terminal singularities.

Crepant resolutions and terminal sigularities are defined below.

This can be generalised to the case where X is not of general type (or the canonical ring is not finitely generated. At present it isn't known whether there are any such X. If K_X is numerically effective (nef), then by {Ka1}, the canonical ring is finitely generated): (cf.{Ka},{Ka2}, §5)

Let X be a non-singular projective 3-fold, $\kappa(X)\geq 0$ (for simplicity). We look for a minimal model X_m in the category of $\mathbb{Q}$-factorial Gorenstein schemes with only terminal singularities, as follows:

1) We have a series of normal projective 3-folds

$$X = X_o,\ X_1, \ldots, X_m$$

such that X_m has only terminal singularities, and K_{X_m} is nef.

2) for each $i=1,\ldots,m-1$ there is a map ϕ_i such that either

Case a) $\phi_i : X_i \to X_{i+1}$ is a birational map with $\rho(X_i)=\rho(X_{i+1})+1$ ($\rho(X)$=Picard number). ϕ_i is called an <u>elementary</u> <u>contraction</u>.

Case b) $\phi_i : X_i \to X_{i+1}$ is an isomorphism in codim. 1. ϕ_i is called an <u>elementary</u> <u>transformation</u> in this case.

The idea is the following. If K_X is not nef, let $C \subset X$ be a curve such that $K_X \cdot C<0$. Then either

Case α) C moves on a divisor D (we would like to contract this D)

Case β) C doesn't move in a divisor.

Mori pioneered the theory and showed the exsistence of a contraction in Case α), if X is smooth, possibly to a singularity (see [Mo] or [Mi],§2). This was extended to the case where X is singular in [Ka], Theorem 3. This completes the program 2) Case a) by induction. Finally, in [Be], Benveniste showed that in case β), assuming K_{X_i} is integral, it is actually already nef, so the programm is done in this case.

We want to apply this general theory to the case of Fermat covers.

<u>2.4.2.1.</u> <u>Notations</u> <u>and</u> <u>definitions</u>

Let X be a normal projective variety over $\mathbb{C}$. Let Sing(X) be the union of its singular points, and Reg(X) = X-Sing(X). The <u>canonical</u> <u>divisor</u> K_X is a Weil divisor on X such that $\mathcal{O}_{\mathrm{Reg}(X)}(K_X)=\Omega^N_{\mathrm{Reg}(X)}$, N=dim X. The corresponding reflexive sheaf $\mathcal{O}_X(K_X)$ is

$$\omega_X = (\Omega^N_X)^{**} \qquad = \mathcal{O}_X(K_X).$$

One defines
$$\omega_X^{[r]} = (\omega_X^{\otimes r})^{**}.$$

X is called <u>locally</u> <u>$\mathbb{Q}$-factorial</u> if, for any Weil divisor D on X, there exists an integer m such that mD is Cartier. In this case the class

$$\mathrm{cl}(D) \in N(X) = \{\text{Cartier divisors on } X/_{\text{numerical equivalence}}\}\otimes\mathbb{R} \text{ of a Weil}$$

divisor is defined. X is said to have only <u>terminal</u> (resp. <u>canonical</u>) <u>singularities</u>:$\Leftrightarrow$

(1) for some $r\in\mathbb{N}$, rK_X is Cartier

(2) for a resolution of singularities $f:X'\to X$

$$rK_{X'} = f^*(rK_X) + \sum \nu_i E_i$$

such that $\nu_i>0$ (resp. $\nu_i\geq 0$), for all exceptional divisors on X' mapped by f to lower dimensional subvarieties on X.

Let $s\in\omega_X^{[r]}$ be a local base. (2) is equivalent to

(2') for all E_i, $\nu_{E_i}(s)>0$ (resp. $\nu_{E_i}(s)\geq 0$), ν_{E_i}=order along E_i.

If these conditions hold for one resolution f, they hold for any resolution.

Let $f:\overline{X}\to X$ be a birational morphism of (affine or projective) normal varieties, and let $K_{\overline{X}} = f^*K_X + \sum \nu_i E_i$ be the canonical divisor, the E_i's being Weil divisors, $\nu_i\geq 0$.

f is <u>small</u>:$\Leftrightarrow$ $\operatorname{codim}_{\overline{X}} f^{-1}(p) \geq 2$ for all $p\in X$

f is <u>crepant</u>:$\Leftrightarrow$ $\nu_i=0$ all i

f is <u>totally discrepant</u>:$\Leftrightarrow$ $\nu_i>0$ all i.

Notice that small does not imply crepant; if a curve $C\subset X$ is blown up, there is no divisor mapping to a point (the blow up is small), but the exceptional ruled surface may occur with $\nu_i>0$ (not crepant). Neither does crepant imply small; they are logically independent.

Using these notations, terminal singularities are singularities for which every resolution is totally discrepant; the canonical singularities which are <u>not</u> terminal have crepant resolutions. Canonical singularities are always <u>rational</u>, but not all rational singularities are canonical (see [R1], 1.3.). Canonical singularities may be isolated or non-isolated. If they are non-isolated, they are locally of the form $\mathbb{C}\times$(du Val singularity). In addition to the non-isolated canonical singularities, therr are finitely many "dissidents", isolated canonical singularities (loc. cit. 1.14). The terminal singularities are <u>quick</u> (=quotient of isolated compound du Val singularity), and therefore isolated ([R2], 0.6.; see also [Mo1]). Resolving quick singularities with big resolutions introduces $\mathbb{P}^1$'s with $K_X\cdot C<0$* (which correspond to extremal rays in the language of [Mo]), so we don't resolve them. Of course, one could use the small resolutions constructed by Brieskorn and Artin, but ([R2], 0.8. d)) it is not clear that one should make these extra resolutions, since we loose out on both projectivity (mabye) and unicity (certainly).

* C = introduced $\mathbb{P}^1$

2.4.2.2. Canonical and terminal singularities on Fermat 3-folds

Let $X \to \mathbb{P}^3$ be a singular Fermat cover associated to the arrangement L, and Y its desingularisation as in 1.2.

Theorem 2.4.6.: The singularities of X lying over the following singularities of L are canonical:

$$n=2 \begin{cases} \text{3-fold lines} \\ \text{4-fold points} \\ \text{5-fold points} \end{cases}$$

$$n=3 \quad \text{4-fold points}$$

The terminal singularities of X lie over the following singularities of L:

$$n=2 \quad \text{4-fold points.}$$

proof: The 4-fold points induce hypersurface singularities upstairs on X which are analytically equivalent to:

$$x^n + y^n + z^n = w^n, \quad n=2,3$$

so are canonical by [R1], 4.3. (we need $\frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \frac{1}{n} > 1$). The 3-fold lines for n=2 induce (curve)$\times A_1$ upstairs, so are canonical. (Notice that 3-fold points for n=2 yield the only rational singularities on Fermat surfaces.) The 5-fold points for n=2 admit a crepant resolution by rational surfaces ((2,2)-complete intersection (maybe singular) in $\mathbb{P}^4$), so are disidents (if they are isolated, which they needn't be). All other singularities are not rational (being resolved by non-rational surfaces), so certainly not canonical.

For n=2, the 4-fold point downstairs induces a compound A_1 upstairs, the ordinary double point:

$$x^2 + y^2 + z^2 = w^2.$$

These are obviously quick.

We start to draw some conclusions. The following is immediate from the above and 2.3.2.: (under the assumption of the remark following 2.3.2.)

Corollary 2.4.7.: With $Y \to X$ as above, we have

$$\left\{ \begin{array}{c} \text{X has no canonical} \\ \text{singularities} \end{array} \right\} \Longleftrightarrow K_Y \text{ is ample.}$$

We introduce partial resolutions of X:

$$Y \xrightarrow{f} X'' \xrightarrow{g} X' \to X,$$

such that X' (resp. X'') has all (and only) canonical (resp. terminal) singularities of X. Thus f blows down all the resolving $\mathbb{P}^1 \times \mathbb{P}^1$'s (for n=2, 4-fold points) to ordinary double points, otherwise it does nothing. Similarly g blows down (the covers upstairs of):

$$n=2 \begin{cases} \text{3-fold lines} \\ \text{5-fold points} \end{cases}$$

$$n=3 \quad \text{4-fold points.}$$

Corollary 2.4.8.: Notations as above, then X' is the canonical variety of Y.

proof: First notice that 2.4.7., combined with (Ka1), Theorem 1 imply the canonical ring of Y is finitely generated, so the canonical variety is well defined. If K_Y is ample, there are no curves C with $K_Y \cdot C \leq 0$ which could be blown down by the pluricanonical map ϕ_m, so ϕ_m is an embedding, and by 2.4.7. $Y=X'$. If X' has canonical singularities, then by (R1), 1.2.(II), it is the canonical model of a variety of finitely generated general type, i.e. Y.

2.4.2.3. Reid's choice

We now determine Reid's choice of minimal model for Y ((R2), 0.7.).

Theorem 2.4.9.: $g:X'' \to X'$ as above is a crepant resolution and is in fact identical with Reid's crepant resolution (loc.cit. 3.7.). Thus X'' is Reid's choice of minimal model of Y, and $K_{X''}$ is nef.

proof: As already mentioned, that g is crepant is obvious using the description of the canonical divisor given in 2.2. or 2.3. That g is identical with Reid's crepant resolution is obvious by comparing ours (1.2.) with his ((R1), 2.11. and (R2), 3.7.). The invariant k defined in (R1), 2.10 is

$k=0$	$n=2$	3-fold line
		4-fold point
$k=4$	$n=2$	5-fold point
$k=3$	$n=3$	4-fold point

That $K_{X''}$ is nef has already been shown.

Remark: for Fermat covers only the simplest canonical singularities occur. One could get more interesting singularities by taking quotients as in 1.7. It would be an interesting exercise to list all canonical singularities we can get this way.

Summarising, we can state

Theorem 2.4.10.: $Y \to X$ as above, $C \subset Y$ an effective curve.

(i) $K_Y \cdot C = 0 \Leftrightarrow C = \mathbb{P}^1$ and lies on an exceptional divisor arising in the partial resolution $X'' \to X'$ (i.e. resolution of a canonical singularity which is not terminal).

(ii) $K_Y \cdot C < 0 \Leftrightarrow C = \mathbb{P}^1$ and lies on the exceptional divisor arising in the big resolution $Y \to X''$ (i.e. resolution of a terminal singularity).

(iii) $K_Y \cdot C > 0$ otherwise.

This is better than what is expected in general. From the results of [Ka1] it follows that (i) is always true if K_Y is nef (any Y). The "$\Leftarrow$" part of (ii) is also true in general. The other direction is more delicate (see [Be] and the discussion in the introduction of this section), and is (probably) not true in general.

2.5. Rough classification

If the arrangement L is in general position, then X is a non-singular complete intersection for which it is well known, that

k=5,	n<5	$\kappa(X) = -\infty$	(Fano 3-fold)
	n=5	$\kappa(X) = 0$	($K_X = 0$)
	n>5	$\kappa(X) = 3$	(general type)
k=6,	n=2	$\kappa(X) = -\infty$	
	n=3	$\kappa(X) = 0$	
	n>3	$\kappa(X) = 3$	
k=7,	n=2	$\kappa(X) = -\infty$	
	n>2	$\kappa(X) = 3$	
k=8,	n=2	$\kappa(X) = 0$	
	n>2	$\kappa(X) = 3$	
k≥9,	all n	$\kappa(X) = 3$.	

We want to determine the Kodaira dimension for Fermat desingularisations $Y \to X$ of singular X. The answer in this case is much more complex, depending on the type of singularities we consider.

Our basic tool will be the additivity of Kodaira dimensions in fiberings, as well as the other structure theorems in [V1]. We start with a discussion of the fiberings induced on Y by singularities of X.

2.5.1. Fiberings of Y

First consider a singular p_j of the arrangement L. Blowing up at p_j yields an exceptional $\mathbb{P}^2$, and $\hat{\mathbb{P}}_j^3 = \{\mathbb{P}^3$ blown up at $p_j\}$ fibres over the exceptional $\mathbb{P}^2$:

$$\begin{array}{ccc} \hat{\mathbb{P}}_j^3 & \to & \mathbb{P}^2 \\ \cup & & \Psi \\ l_x & \mapsto & x, \end{array} \quad l_x = \text{unique line with tangent } x \text{ at } p_j.$$

We have a commutative diagram ($\hat{\mathbb{P}}^3$ now as in 1.2.; blown up at all actual singularities):

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & \hat{\mathbb{P}}^3 \\ \hat{\rho}\downarrow & & \downarrow\rho \\ \pi_j^{-1}(\mathbb{P}^2) & \xrightarrow[\pi_j]{} & \mathbb{P}^2 \end{array}$$

Corresponding to the composition $\pi_j \circ \hat{\rho} : Y \to \mathbb{P}^2$ we have the Remmert-Stein factorisation $Y \xrightarrow{f} S \xrightarrow{g} \mathbb{P}^2$, S a surface, f has connected fibres and g is finite-to-one. It is easy to see that g is just a Fermat cover (the resolving divisor, see 1.3.), branched along the induced arrangement in $\mathbb{P}^2$. Recall that $\pi^{-1}(\mathbb{P}^2)$ consists of $n^{k-\tau(P_j)-1}$ disjoint components. These are <u>sections</u> of the map f. The generic fibre of f is a covering of $\mathbb{P}^1 = l_x$, branched at $k-\tau(P_j)+1$ points. However, it is important to note that f is in general <u>not flat</u>. Indeed, if l_x is a line in $\mathbb{P}^3$ passing through p_j and another (actual) singular point, then the exceptional divisors (of the other point) lie in the fibres of Y covering l_x. If we wish to have a flat map f, then we only resolve part of the branch locus (so the covering has singularities in the fibres, instead of exceptional divisors), and perform some other resolution of singularities.

Now consider a singular line l_μ of the arrangement. Blowing up along l_μ with exceptional divisor $\mathbb{P}^1 \times \mathbb{P}^1$, $\hat{\mathbb{P}}^3_\mu = \{\mathbb{P}^3$ blown up along $l_\mu\}$ fibres over $\mathbb{P}^1$:

$$\begin{array}{ccc} \hat{\mathbb{P}}^3_\mu & \to & \mathbb{P}^1 \\ \cup & & \in \\ H_t & \longmapsto & t \end{array} \qquad H_t = \text{unique plane through } l_\mu \text{ with tangent } t \text{ there.}$$

Once again, we consider the diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & \mathbb{P}^3 \\ \hat{\rho}\downarrow & & \downarrow \\ \pi_\mu^{-1}(\mathbb{P}^1) & \xrightarrow[\pi_\mu]{} & \mathbb{P}^1 \end{array}$$

and the Remmert-Stein factorisation of $\pi_\mu \circ \hat{\rho}$:

$$Y \xrightarrow{f} C \xrightarrow{g} \mathbb{P}^1.$$

Here f has generic fibres which are Fermat coverings of $\mathbb{P}^2$ $(=H_t)$, branched along $k-\tau(L_\mu)+1$ planes. g is a Fermat cover of $\mathbb{P}^1$ branched at $\tau(L_\mu)$ points. Here we have that f is flat. Once again, the inverse images $\pi^{-1}(\mathbb{P}^1) = \pi^{-1}(f_\mu)$ are sections of f. (f_μ = fibre of L_μ)

<u>2.5.2. Review of Viehwegs results</u>

First we have the complete conjecture C_3 (both C_{31} and C_{32}):

Theorem 2.5.1.: Let V be an algebraic 3-fold , $V \to W$ a fibering with generic fibre F. Then

$$\kappa(V) \geq \kappa(W) + \kappa(F) .$$

proof: {V1}, Sätze III und IV.

This yields the following structure theorem {V1}, Satz I:

Theorem 2.5.2.: Every projective smooth 3fold $\tilde{V}$, which admits the structure of fibre space, has a birational model V, such that

$\kappa(\tilde{V})$	$q(\tilde{V})$	Structure of V		
3		general type		
2		$f:V \to W$	$\dim(F)=1$, $\kappa(F)=0$	
1		$f:V \to W$	$\dim(F)=2$, $\kappa(F)=0$	
0	0	? ? ?		
0	1	the albenese map $\alpha_V: V \to A(V)$ is an étale fibre bundle	$\dim(F)=2$, $\kappa(F)=0$	
0	2	the albenese map $\alpha_V: V \to A(V)$ is an étale fibre bundle	$\dim(F)=1$, $\kappa(F)=0$	
0	3	abelian variety		
$-\infty$	0	? ? ?		
$-\infty$	≥ 1	the Stein factoisation $f:V \to W$ of the albanese map $\alpha_V: V \to A(V)$, W' a desing. of W	$\dim(F)=2$, $\kappa(F)=-\infty$, $q(W')=q(V)$, $\kappa(W') \geq 0$	$\dim\alpha(V)=1$
$-\infty$	≥ 1	the Stein factoisation $f:V \to W$ of the albanese map $\alpha_V: V \to A(V)$, W' a desing. of W	$F=\mathbb{P}^1$, $q(W')=q(V)$, $\kappa(W') \geq 0$	$\dim\alpha(V)=2$

From this we deduce the following sufficient conditions for a Fermat cover $Y \to \hat{\mathbb{P}}^3$ to have $\kappa(Y)=-\infty$, 0, 1 or 2.

Thereom 2.5.3.: I. Let $Y \to S$ be a fibering onto a Fermat surface as above, F the generic fibre. Then

I.1. $\kappa(F)=-\infty$ $\Rightarrow$ $\kappa(Y)=-\infty$

2. $\kappa(F)=0$ a) $\kappa(S)=-\infty$, $\Rightarrow$ $\kappa(Y)=0$

b) $\kappa(S)=2$ $\Rightarrow$ $\kappa(Y)=2$

3. $\kappa(F)=1$, $\kappa(S)=2$ $\Rightarrow$ $\kappa(Y)=3$

II. Let $Y \to C$ be a fibering onto a Fermat curve as above with generic fibre F

II.1. $\kappa(F)=-\infty$ $\Rightarrow$ $\kappa(Y)=-\infty$

2. $\kappa(F)=0$ a) $\kappa(C)=-\infty$ $\Rightarrow$ $\kappa(Y)=0$

b) $\kappa(C)=1$ $\Rightarrow$ $\kappa(Y)=1$

3. $\kappa(F)=2$, $\kappa(C)=1$ $\Rightarrow$ $\kappa(Y)=3$.

2.5.3. Rough classification

We now just list arrangements inducing the conditions I and II in the theorem above. Examples will be studied in Chapter 3, §5. We arrange things by Kodaira dimension.

2.5.3.1. $\kappa(Y)=-\infty$

Arrangements of k planes which have the singularities:

					fibre
I. 1.	(i)	$t_{k-1}=1$	$k\geq 5$	(near pencil)	$\mathbb{P}^1$
	(ii)	$t_{k-2}=1$	$k\geq 6$	$n=2$	$\mathbb{P}^1$
II. 1.	(i)	$t_{k-2}(1)=1$	$k\geq 5$	(near-pencil)	$\mathbb{P}^2$
	(ii)	$t_{k-3}(1)=1$	$k\geq 6$	$n=2$	quadric surface
				$n=3$	cubic surface
	(iii)	$t_{k-4}(1)=1$	$k\geq 7$	$n=2$	(2,2)-complete intersection $\subset \mathbb{P}^4$

2.5.3.2. $\kappa(Y)=0$

First of all, recall the conditions yielding trivial canonical bundle from 2.3. (these have $\kappa(Y)=0$). In addition, arrangements satisfying:

				fibre
I. 2. a)				elliptic curve
	$t_4\geq 1$,	$k=7$,	$n=2$	" "

2.5.3.3. $\kappa(Y)=1$

Arrangements with

II. 2. b)	$t_{k-5}(1)\geq 1$	$k\geq 9$,	$n=2$	K3 surface

2.5.3.4. $\kappa(Y)=2$

Arrangements with

I. 2. b)	$t_{k-2}\geq 1$	$k\geq 9$	$n=3$	elliptic curve
	$t_{k-3}\geq 1$	$k\geq 10$	$n=2$	" "

2.6. Examples

2.6.1. Characteristic exponents

In 2.2. we have calculated the Chern numbers $c_1^3(Y^n)$, $c_1c_2(Y^n)$ and $c_3(Y^n)$ for $Y^n \to \mathbb{P}^3$ a smooth Fermat cover of degree n associated to an arrangement $L \subset \mathbb{P}^3$ as a cubic polynomial in n. These are listed below for all arrangements discussed in 2.1., where we use the notation

$$C13:=c_1^3 \quad , \; C12:=c_1c_2 \quad , \; C3:=c_3 \quad .$$

The Characteristic exponents of the arrangement are the quotients of the leading coefficients A13, A12 and A3. These values are

$$\gamma_1=\frac{A13}{A12} = \lim_{n\to\infty} \frac{c_1(Y^n)}{c_1c_2(Y^n)}$$

$$\gamma_2=\frac{A12}{A3} = \lim_{n\to\infty} \frac{c_1c_2(Y^n)}{c_3(Y^n)}$$

$$\gamma_3=\frac{A13}{A3} = \lim_{n\to\infty} \frac{c_1(Y^n)}{c_3(Y^n)} = \gamma_1\gamma_2.$$

These are listed below, and we point out the following: If $L\subset\mathbb{P}^3$ is a point arrangement (see 2.1.1.), we observe for all known examples

$$\frac{5}{3} \le \gamma_1 < \frac{9}{4} \qquad 2 < \gamma_2 < 4.$$

If on the other hand L is a simplicial arrangement or an arrangement defined by a unitary reflection group, then

$$\frac{9}{4} < \gamma_1 \le 2.5 \qquad 4 < \gamma_2 < 5.$$

This justifies perhaps the name degenerate arrangements used in 2.1.; Fermat covers associated to these arrangements (in all known cases) fulfill

$$c_1(Y^n)/c_1c_2(Y^n) < c_1(\mathbb{P}^2\times\mathbb{P}^1)/c_1c_2(\mathbb{P}^2\times\mathbb{P}^1) = \frac{9}{4}$$

$$c_1(Y^n)/c_3(Y^n) \quad < c_1(\mathbb{P}^2\times\mathbb{P}^1)/c_3(\mathbb{P}^2\times\mathbb{P}^1) = 9$$

for all n>3 (more generally: assuming Y^n is of general type).

2.6.2. Lists

2.6.2.1. Point Arrangements

```
C3/N^K-4 =-1 N^3+ 9 N^2+-24 N+ 26
C13/N^K-4 =-5 N^3+ 45 N^2+-135 N+ 135
C12/N^K-4 =-3 N^3+ 24 N^2+-69 N+ 72
```

THE CHARACTERISTIC EXPONENTS ARE:

$a_1^3(6)$ A13/A12= 1.66666666667

Cube A12/A3= 3

A13/A3= 5

$a_1^3(8)$

octahedron

C3/N^K-4 =-8 N^3+ 44 N^2+-88 N+ 80

C13/N^K-4 =-52 N^3+ 276 N^2+-444 N+ 188

C12/N^K-4 =-28 N^3+ 136 N^2+-236 N+ 152

THE CHARACTERISTIC EXPONENTS ARE:

A13/A12= 1.85714285714

A12/A3= 3.5

A13/A3= 6.5

$a_1^3(12)$

dodecahedron

C3/N^K-4 =-57 N^3+ 243 N^2+-378 N+ 250

C13/N^K-4 =-401 N^3+ 1593 N^2+-1899 N+ 555

C12/N^K-4 =-201 N^3+ 774 N^2+-1011 N+ 462

THE CHARACTERISTIC EXPONENTS ARE:

A13/A12= 1.99502487562

A12/A3= 3.52631578947

A13/A3= 7.0350877193

$a_1^3(20)$

icosahedron

C3/N^K-4 =-430 N^3+ 1562 N^2+-2080 N+ 1220

C13/N^K-4 =-3274 N^3+ 10698 N^2+-9966 N+ 1534

C12/N^K-4 =-1582 N^3+ 5128 N^2+-5402 N+ 1880

THE CHARACTERISTIC EXPONENTS ARE:

A13/A12= 2.06953223767

A12/A3= 3.67906976744

A13/A3= 7.61395348837

$a_2^3(8)$

archimedian solid

(3, 6, 6)

C3/N^K-4 =-14 N^3+ 62 N^2+-100 N+ 68

C13/N^K-4 =-58 N^3+ 330 N^2+-606 N+ 350

C12/N^K-4 =-34 N^3+ 172 N^2+-302 N+ 188

THE CHARACTERISTIC EXPONENTS ARE:

A13/A12= 1.70588235294

A12/A3= 2.42857142857

A13/A3= 4.14285714286

$a_2^3(14)$

archimedian solids

(3, 8, 8)

(4, 6, 6)

```
C3/N^K-4 =-193 N^3+ 689 N^2+-856 N+ 418

C13/N^K-4 =-973 N^3+ 3957 N^2+-5151 N+ 2015

C12/N^K-4 =-523 N^3+ 2008 N^2+-2573 N+ 1112

THE CHARACTERISTIC EXPONENTS ARE:

      A13/A12= 1.8604206501

      A12/A3=  2.70984455959

      A13/A3=  5.0414507772
```

$a_1^3(14)$

archimedean solid

(3, 4, 3, 4)

cuboctahedron

```
C3/N^K-4 =-181 N^3+ 653 N^2+-832 N+ 442

C13/N^K-4 =-961 N^3+ 3849 N^2+-4827 N+ 1691

C12/N^K-4 =-511 N^3+ 1936 N^2+-2441 N+ 1040

THE CHARACTERISTIC EXPONENTS ARE:

      A13/A12= 1.88062622309

      A12/A3=  2.82320441989

      A13/A3=  5.30939226519
```

$a_1^{32}(32)$

archimedian solid

(5, 6, 6)

soccer ball

```
C3/N^K-4 =-3396 N^3+ 11000 N^2+-12208 N+ 5000

C13/N^K-4 =-20464 N^3+ 67056 N^2+-70320 N+ 22224

C12/N^K-4 =-10312 N^3+ 33352 N^2+-35648 N+ 12632

THE CHARACTERISTIC EXPONENTS ARE:

      A13/A12= 1.9844840962

      A12/A3=  3.03651354535

      A13/A3=  6.02591283863
```

2.6.2.2. Simplicial Arrangements

$A_1^3(10)$ = Ceva^3(1, 4)

tetrahedron + sym.

```
C3/N^K-4 =-6 N^3+ 40 N^2+-105 N+ 105

C13/N^K-4 =-61 N^3+ 285 N^2+-435 N+ 215

C12/N^K-4 =-26 N^3+ 135 N^2+-250 N+ 165

THE CHARACTERISTIC EXPONENTS ARE:

      A13/A12= 2.34615384615

      A12/A3=  4.33333333333

      A13/A3=  10.1666666667
```

$A_1^3(15) = Ceva^3(2,3)$

Cube + Sym.

```
C3/N^K-4 =-40 N^3+ 196 N^2+-390 N+ 306
C13/N^K-4 =-409 N^3+ 1401 N^2+-1512 N+ 508
C12/N^K-4 =-174 N^3+ 663 N^2+-894 N+ 429
THE CHARACTERISTIC EXPONENTS ARE:
    A13/A12= 2.35057471264
    A12/A3=  4.35
    A13/A3=  10.225
```

$A_1^3(17)$

octahedron + Sym.

```
C3/N^K-4 =-72 N^3+ 324 N^2+-612 N+ 468
C13/N^K-4 =-753 N^3+ 2331 N^2+-2214 N+ 642
C12/N^K-4 =-318 N^3+ 1101 N^2+-1350 N+ 591
THE CHARACTERISTIC EXPONENTS ARE:
    A13/A12= 2.3679245283
    A12/A3=  4.41666666667
    A13/A3=  10.4583333333
```

$A_1^3(27)$

dodecahedron + Sym.

```
C3/N^K-4 =-400 N^3+ 1540 N^2+-2498 N+ 1650
C13/N^K-4 =-4146 N^3+ 11064 N^2+-8586 N+ 1900
C12/N^K-4 =-1756 N^3+ 5228 N^2+-5360 N+ 1912
THE CHARACTERISTIC EXPONENTS ARE:
    A13/A12= 2.36104783599
    A12/A3=  4.39
    A13/A3=  10.365
```

$a_1^3(35)$

icosahedron + Sym.

(non-simplicial)

```
C3/N^K-4 =-1188 N^3+ 4224 N^2+-6329 N+ 3840
C13/N^K-4 =-12008 N^3+ 30174 N^2+-21420 N+ 4038
C12/N^K-4 =-5128 N^3+ 14282 N^2+-13460 N+ 4330
THE CHARACTERISTIC EXPONENTS ARE:
    A13/A12= 2.34165366615
    A12/A3=  4.3164983165
    A13/A3=  10.1077441077
```

$A_4^3(12) = Ceva^3(2)$

C3/N^K-4 =-16 N^3+ 88 N^2+-204 N+ 192

C13/N^K-4 =-172 N^3+ 636 N^2+-756 N+ 292

C12/N^K-4 =-72 N^3+ 300 N^2+-456 N+ 252

THE CHARACTERISTIC EXPONENTS ARE:

A13/A12= 2.38888888889

A12/A3= 4.5

A13/A3= 10.75

$A_4^3(13) = Ceva^3(2,1)$

C3/N^K-4 =-24 N^3+ 124 N^2+-268 N+ 234

C13/N^K-4 =-251 N^3+ 891 N^2+-1002 N+ 356

C12/N^K-4 =-106 N^3+ 421 N^2+-602 N+ 311

THE CHARACTERISTIC EXPONENTS ARE:

A13/A12= 2.3679245283

A12/A3= 4.41666666667

A13/A3= 10.4583333333

$A_1^3(14) = Ceva^3(2,2)$

C3/N^K-4 =-32 N^3+ 160 N^2+-330 N+ 272

C13/N^K-4 =-330 N^3+ 1146 N^2+-1254 N+ 428

C12/N^K-4 =-140 N^3+ 542 N^2+-748 N+ 370

THE CHARACTERISTIC EXPONENTS ARE:

A13/A12= 2.35714285714

A12/A3= 4.375

A13/A3= 10.3125

$A_1^3(16) = Ceva^3(2,4)$

Cube + sym + ∞

C3/N^K-4 =-48 N^3+ 232 N^2+-448 N+ 336

C13/N^K-4 =-488 N^3+ 1656 N^2+-1776 N+ 596

C12/N^K-4 =-208 N^3+ 784 N^2+-1040 N+ 488

THE CHARACTERISTIC EXPONENTS ARE:

A13/A12= 2.34615384615

A12/A3= 4.33333333333

A13/A3= 10.1666666667

$A_1^3(18)$

octahedron + sym + ∞

C3/N^K-4 =-102 N^3+ 434 N^2+-774 N+ 558

C13/N^K-4 =-990 N^3+ 2982 N^2+-2736 N+ 758

C12/N^K-4 =-396 N^3+ 1326 N^2+-1572 N+ 666

THE CHARACTERISTIC EXPONENTS ARE:

A13/A12= 2.5

A12/A3= 3.88235294118

A13/A3= 9.70588235294

$A_1^3(24)$

G_{576} = Weyl group of F_4 = symmetry group of regular 24-cell in $\mathbb{R}^4$

C3/N^K-4 =-240 N^3+ 968 N^2+-1584 N+ 1008

C13/N^K-4 =-2496 N^3+ 6960 N^2+-5904 N+ 1484

C12/N^K-4 =-1056 N^3+ 3288 N^2+-3552 N+ 1344

THE CHARACTERISTIC EXPONENTS ARE:

A13/A12= 2.36363636364

A12/A3= 4.4

A13/A3= 10.4

$A_2^3(15)$

prism + sym

C3/N^K-4 =-48 N^3+ 224 N^2+-441 N+ 351

C13/N^K-4 =-495 N^3+ 1605 N^2+-1611 N+ 497

C12/N^K-4 =-210 N^3+ 759 N^2+-978 N+ 453

THE CHARACTERISTIC EXPONENTS ARE:

A13/A12= 2.35714285714

A12/A3= 4.375

A13/A3= 10.3125

$A_2^2(28)$

cuboctahedron + sym.

C3/N^K-4 =-480 N^3+ 1816 N^2+-2884 N+ 1848

C13/N^K-4 =-4978 N^3+ 13050 N^2+-9936 N+ 2152

C12/N^K-4 =-2108 N^3+ 6166 N^2+-6196 N+ 2162

THE CHARACTERISTIC EXPONENTS ARE:

A13/A12= 2.3614800759

A12/A3= 4.39166666667

A13/A3= 10.3708333333

$A_1^3(28)$ C3/N^K-4 =-516 N^3+ 1924 N^2+-3108 N+ 2040

dodecahedron + C13/N^K-4 =-4954 N^3+ 13398 N^2+-11082 N+ 3074

sym. + ∞ C12/N^K-4 =-2078 N^3+ 5904 N^2+-5596 N+ 1794

THE CHARACTERISTIC EXPONENTS ARE:

A13/A12= 2.38402309913

A12/A3= 4.02713178295

A13/A3= 9.6007751938

C3/N^K-4 =-600 N^3+ 2240 N^2+-3522 N+ 2232

$A_1^3(30)$ C13/N^K-4 =-6198 N^3+ 16086 N^2+-12006 N+ 2504

right prism + sym C12/N^K-4 =-2628 N^3+ 7602 N^2+-7524 N+ 2574

THE CHARACTERISTIC EXPONENTS ARE:

A13/A12= 2.35844748858

A12/A3= 4.38

A13/A3= 10.33

C3/N^K-4 =-5040 N^3+ 17048 N^2+-23340 N+ 12600

$A_1^3(60)$ C13/N^K-4 =-51660 N^3+ 122268 N^2+-82260 N+ 14756

G_{7200} = group of symmetry of regular 120-cell in $\mathbb{R}^4$ C12/N^K-4 =-21960 N^3+ 57804 N^2+-50760 N+ 14940

THE CHARACTERISTIC EXPONENTS ARE:

A13/A12= 2.35245901639

A12/A3= 4.35714285714

A13/A3= 10.25

2.6.2.3. Ceva Arrangements

C3/N^K-4 =-90 N^3+ 396 N^2+-759 N+ 603

C13/N^K-4 =-995 N^3+ 2907 N^2+-2493 N+ 601

$Ceva^3(3)$ C12/N^K-4 =-414 N^3+ 1365 N^2+-1590 N+ 663

THE CHARACTERISTIC EXPONENTS ARE:

A13/A12= 2.40338164251

A12/A3= 4.6

A13/A3= 11.0555555556

$Ceva^3(4)$

```
C3/N^K-4 =-256 N^3+ 1024 N^2+-1784 N+ 1296
C13/N^K-4 =-2796 N^3+ 7500 N^2+-5688 N+ 1132
C12/N^K-4 =-1168 N^3+ 3524 N^2+-3680 N+ 1348
THE CHARACTERISTIC EXPONENTS ARE:
    A13/A12= 2.39383561644
    A12/A3=  4.5625
    A13/A3=  10.921875
```

$Ceva^3(5)$

```
C3/N^K-4 =-550 N^3+ 2080 N^2+-3441 N+ 2385
C13/N^K-4 =-5941 N^3+ 15189 N^2+-10683 N+ 1855
C12/N^K-4 =-2490 N^3+ 7143 N^2+-7002 N+ 2373
THE CHARACTERISTIC EXPONENTS ARE:
    A13/A12= 2.3859437751
    A12/A3=  4.52727272727
    A13/A3=  10.8018181818
```

$Ceva^3(6)$

```
C3/N^K-4 =-1008 N^3+ 3672 N^2+-5880 N+ 3960
C13/N^K-4 =-10796 N^3+ 26748 N^2+-17856 N+ 2788
C12/N^K-4 =-4536 N^3+ 12588 N^2+-11832 N+ 3804
THE CHARACTERISTIC EXPONENTS ARE:
    A13/A12= 2.38007054674
    A12/A3=  4.5
    A13/A3=  10.7103174603
```

2.6.2.4. Arrangements defined in classical geometry

Kummer's 16_6 (general)

```
C3/N^K-4 =-204 N^3+ 768 N^2+-1008 N+ 480
C13/N^K-4 =-1296 N^3+ 4752 N^2+-5616 N+ 2096
C12/N^K-4 =-648 N^3+ 2352 N^2+-2880 N+ 1200
THE CHARACTERISTIC EXPONENTS ARE:
    A13/A12= 2
    A12/A3=  3.17647058824
    A13/A3=  6.35294117647
```

C3/N^K-4 =-188 N^3+ 720 N^2+-976 N+ 512

Kummer's 16_6 C13/N^K-4 =-1280 N^3+ 4608 N^2+-5184 N+ 1664

(special) C12/N^K-4 =-632 N^3+ 2256 N^2+-2704 N+ 1104

THE CHARACTERISTIC EXPONENTS ARE:

A13/A12= 2.0253164557

A12/A3= 3.36170212766

A13/A3= 6.8085106383

C3/N^K-4 =-5376 N^3+ 18080 N^2+-25680 N+ 14760

Kleins 60_{15} C13/N^K-4 =-55776 N^3+ 130080 N^2+-82800 N+ 12980

$= G_{11520}$ C12/N^K-4 =-23616 N^3+ 61440 N^2+-53280 N+ 15480

THE CHARACTERISTIC EXPONENTS ARE:

A13/A12= 2.36178861789

A12/A3= 4.39285714286

A13/A3= 10.375

C3/N^K-4 =-5195 N^3+ 16983 N^2+-19890 N+ 8430

The 45 tritangents C13/N^K-4 =-40490 N^3+ 120258 N^2+-116235 N+ 36261

of a non-singular C12/N^K-4 =-19425 N^3+ 59229 N^2+-60435 N+ 20655

cubic surface THE CHARACTERISTIC EXPONENTS ARE:

(general: no A13/A12= 2.08442728443

Eckhard points). A12/A3= 3.73917228104

A13/A3= 7.79403272377

2.6.2.5. Arrangements defined by reflection groups

C3/N^K-4 =-1536 N^3+ 5440 N^2+-8360 N+ 5280

C13/N^K-4 =-16956 N^3+ 39900 N^2+-25320 N+ 3980

G_{1920} C12/N^K-4 =-7056 N^3+ 18740 N^2+-16800 N+ 5140

(subgroup of G_{11520}) THE CHARACTERISTIC EXPONENTS ARE:

A13/A12= 2.40306122449

A12/A3= 4.59375

A13/A3= 11.0390625

C3/N^K-4 =-1296 N^3+ 4680 N^2+-6480 N+ 3360

C13/N^K-4 =-13176 N^3+ 33480 N^2+-25920 N+ 5900

$G_{25,920}$ C12/N^K-4 =-5616 N^3+ 15840 N^2+-15120 N+ 4920

THE CHARACTERISTIC EXPONENTS ARE:

A13/A12= 2.34615384615

A12/A3= 4.33333333333

A13/A3= 10.1666666667

Chapter 2. Coverings as Ball Quotients

In the last chapter we gave lots of examples of Fermat coverings $Y \to \hat{\mathbb{P}}^3$, and calculated in particular the quotient $Q_1 = c_1^3/c_1c_2$. Yau's Theorem (Theorem 3.3.1. below) states that if $Q_1 = 8/3$, then Y is a ball quotient. Looking through the lists of 2.6. we see there is not a single example of $Q_1 = 8/3$. In fact, we can even prove the following result:

<u>Theorem B</u>: for $N \geq 3$, there are no Fermat coverings

$$Y \to \hat{\mathbb{P}}^N$$

such that Y is a compact ball quotient.

This Theorem will follow from the following more general result, which is the heart of this chapter:

<u>Theorem A</u>: Let $Y \to \hat{\mathbb{P}}^N$ the smooth Fermat cover as in §1. Then:

$$\left\{\begin{matrix} Y \text{ is a compact} \\ \text{ball quotient} \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} \text{all reduced ramification divisors} \\ \overline{B}_i \text{ are subball quotients} \end{matrix}\right\}$$

Subball quotients are defined in 3.2.; they are quotients of suballs. We would now like to indicate roughly how Theorem A implies Theorem B. Set $N=3$ for simplicity, and let $Y \to \hat{\mathbb{P}}^3$ be the smooth Fermat cover constructed in §1 associated with the arrangement $L \subset \mathbb{P}^3$ $(n \geq 2)$. If L has singular lines, their exceptional divisors in the branch locus $B \subset \hat{\mathbb{P}}^3$ are $\mathbb{P}^1 \times \mathbb{P}^1$'s (see 1.3.). The resolving surfaces in the ramification locus of Y are of the type

$$C_1 \times C_2 \to \mathbb{P}^1 \times \mathbb{P}^1,$$

and are not $\mathbb{B}^2$-quotients (they are quotients of $\mathbb{B}^1 \times \mathbb{B}^1$ and have $c_1^2 = 2c_2$). So Y cannot be a ball quotient by Theorem A. Theorem A is a special case of Theorem 4.2.2. below, and its proof is very simple, holding in a much more general context. It holds for all coverings of type (F), a notion we introduce in 4.1.

Theorem A can be generalised in a number of ways. Using induction one can (under mild hypothesis on Y) replace the expression "ramification divisors" on the right hand side by "surfaces in the (N-2) co-dimensional part of the branch locus", as we show in 4.2.3. below.

One gets other generalisations by changing the left-hand side. We consider the following: i) Y is a compact ball-product quotient, and ii) Y is the compactification of a (non-compact) ball quotient. In the last section we give examples. We discuss the 7 examples (2 compact, 5 non-compact) of Deligne-Mostow {DM}, and show that 2 of the non-compact examples can actually be realised as Fermat covers. We also discuss one new 3-dimensional (non-compact) ball quotient, which also yields a new 2-dimensional (non-compact) example, branched over the line arrangement $A_2^2(13)$ in Grünbaums list {Gr}.

§3. Ball Quotients

In this paragraph we develope the necessary facts concerning the Chern numbers of ball quotients and ball-product quotients. First we formulate the Hirzebruch proportionality principle for the case at hand. We then derive a "relative proportionality", which is sort of folk lore. The idea is roughly as follows. Consider a disk $\mathbb{B}^1 \subset \mathbb{B}^2$ sitting inside $\mathbb{B}^2$ as a subball. If Γ acts discretly and freely on $\mathbb{B}^2$ with compact quotient Y, then the image $\tilde{\mathbb{B}}^1 \subset Y$ of $\mathbb{B}^1$ under the action of Γ is easy to spot: the Chern numbers of its tangential and normal bundle have the 2-1 ratio characteristic of a line $l \subset \mathbb{P}^2$, where $c_1(l)=c_1(T_l)=2$, $c_1(N_{\mathbb{P}^2} l)=(l)_{|l}=1$. In the last section the Yau inequality is discussed, which is a sort of converse of Hirzebruch proportionality. There is also an analogous result for non-compact quotients, for which we are indebted to R. Kobayashi.

As much of the material of this paragraph is standard, those familier with Hirzebruch proportionality and Yau´s Theorem might just look at the formula 3.2.4., the form of which is important in the sequel, and then proceed directly to §4.

<u>3.1. Hirzebruch proportionality for ball quotients</u>

<u>3.1.1.</u> <u>Balls and ball-products</u>

The complex N-ball is defined as the set

$$\mathbb{B}^N := \{(x_1,\dots,x_N) \in \mathbb{C}^N \; |\Sigma|x_i|^2 < 1\}$$

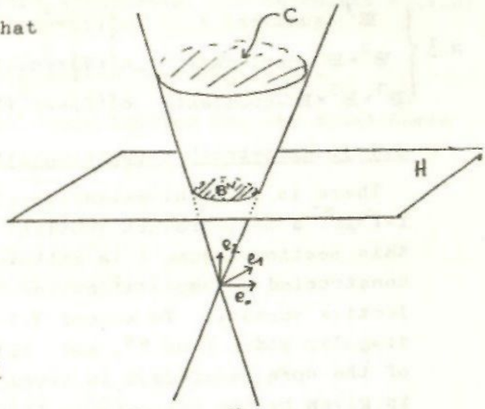
It is the prototype of Stein space and carries a natural Bergmann metric:

$$ds^2 = \sum^N g_{i\bar{j}}\, dx_i\, d\bar{x}_j \, , \qquad g_{i\bar{j}} = \frac{\partial^2 \ln(K_N(x,\bar{x}))}{\partial x_i \partial \bar{x}_j} \, ,$$

where $K_N(x,\bar{x}) = (1-|x|^2)^{-(N+1)}$ is the Bergmann Kernel function.

$\mathbb{B}^N$ is naturally imbedded in projective space of the same dimension as follows: let $\mathbb{C}^{N+1}$ be equipped with a hermitian bilinear form $\langle , \rangle$ with signature $(N,1)$ (N negative eigenvalues). Let $C \subset \mathbb{C}^{N+1}$ be the cone

$$C = \{(x_o,\dots,x_N) \in \mathbb{C}^{N+1} \mid \langle x,x \rangle > 0\}$$



Fix a base $e_o,\dots,e_N$ of $\mathbb{C}^{N+1}$ such that that with respect to $\{e_i\}$ the form $\langle , \rangle$ is described by a diagonal matrix:

$$\begin{pmatrix} -1 & & & & \\ & -1 & & 0 & \\ & & \ddots & & \\ & 0 & & -1 & \\ & & & & 1 \end{pmatrix} \left.\vphantom{\begin{matrix} a\\a\\a\\a \end{matrix}}\right] N$$

Let H be an affine hyperplane in $\mathbb{C}^{N+1}$, perpendicular to the e_N-axis. The intersection $H \cap C$ is a ball $\mathbb{B}^N$, so the image of C under the natural projection $\mathbb{C}^{N+1}-\{0\} \to \mathbb{P}^N$ is a ball $\mathbb{B}^N \subset \mathbb{P}^N$. The embedding is such that every automorphism of $\mathbb{B}^N$ extends to one of $\mathbb{P}^N$. $\mathrm{Aut}(\mathbb{B}^N) = PSU(N,1)$.

A ball-product is the cartesian product of balls,

$$\mathbb{B}^{M^\cdot} = \mathbb{B}^{m_1} \times \dots \times \mathbb{B}^{m_q} ,$$

$M^\cdot = (m_1,\dots,m_q)$, $\dim \mathbb{B}^{M^\cdot} = N = \Sigma m_i$. As above it can be imbedded in the product projective space $\mathbb{P}^{M^\cdot} = \mathbb{P}^{m_1} \times \dots \times \mathbb{P}^{m_q}$. Ocassionaly we shall view the N-ball $\mathbb{B}^N$ as a special case of ball-product.

3.1.2. Hirzebruch proportionality for balls

Let $\Gamma \subset PSU(m_1,1) \times \dots \times PSU(m_q,1)$ be a discrete subgroup acting properly discontinuously and freely on $\mathbb{B}^{M^\cdot}$ with compact quotient $Y = \Gamma \backslash \mathbb{B}^{M^\cdot}$. Y inherits a Kähler structure from the Bergman metric on $\mathbb{B}^{M^\cdot}$ and is in fact a smooth algebraic variety. We refer to Y as a ball-product quotient. The Chern numbers of Y fullfill the following remarkable relations:

Proposition 3.1.2. (Hirzebruch proportionality, [Hi2], Satz3.)

$$c_{\nu_1}(Y) \cdot \dots \cdot c_{\nu_k}(Y) = \frac{c_{\nu_1}(\mathbb{P}^{M^\cdot}) \cdot \dots \cdot c_{\nu_k}(\mathbb{P}^{M^\cdot})}{c_{\mu_1}(\mathbb{P}^{M^\cdot}) \cdot \dots \cdot c_{\mu_\ell}(\mathbb{P}^{M^\cdot})} \, c_{\mu_1}(Y) \cdot \dots \cdot c_{\mu_\ell}(Y)$$

for any two partitions $\{\nu_i\}, \{\mu_i\}, \Sigma \nu_i = \Sigma \mu_i = N = \dim Y$. The Chern numbers of $\mathbb{P}^{M^\cdot}$ are given by

$$c_{\nu_1}(\mathbb{P}^{M^\cdot}) \cdot \dots \cdot c_{\nu_k}(\mathbb{P}^{M^\cdot}) = \sum_{\substack{m_{i_1},\dots,m_{i_k} \\ \forall j \;\; \nu_j \le m_{i_j}}} \binom{m_{i_1}+1}{\nu_1} \cdot \quad \cdot \binom{m_{i_k}+1}{\nu_k}$$

In dimensions 2 and 3 we have

$$N=2\begin{cases} \mathbb{B}^2\text{- quotient :} & c_1^2(Y)=3c_2(Y) \\ \mathbb{B}^1\times\mathbb{B}^1\text{-quotient:} & c_1^2(Y)=2c_2(Y) \end{cases}$$

$$N=3\begin{cases} \mathbb{B}^3\text{-quotient :} & c_1^3(Y)=\frac{8}{3}c_1(Y)c_2(Y) & c_1^3(Y)=16c_3(Y) & c_1(Y)c_2(Y)=6c_3(Y) \\ \mathbb{B}^2\times\mathbb{B}^1\text{-quotient:} & c_1^3(Y)=\frac{9}{4}c_1(Y)c_2(Y) & c_1^3(Y)=9c_3(Y) & c_1(Y)c_2(Y)=4c_3(Y) \\ \mathbb{B}^1\times\mathbb{B}^1\times\mathbb{B}^1\text{-quotient:} & c_1^3(Y)=2c_1(Y)c_2(Y) & c_1^3(Y)=6c_2(Y) & c_1(Y)c_2(Y)=3c_3(Y) \end{cases}$$

3.1.3. Hirzebruch proportionality in the non-compact case

There is a useful extension of 3.1.2. due to Mumford {M}. Consider $Y=\Gamma\backslash\mathbb{B}^{M^\cdot}$ a ball-product quotient which is not compact. For the rest of this section assume Γ is arithmetic. In this case, Baily and Borel {BB} constructed a compactification of Y, Y^*, such that Y^* is a normal projective variety. To extend 3.1.2. we need two things: i) a good non-singular model $\overline{Y}$ of Y^*, and ii) a good definition of the Chern numbers of the open manifold Y in terms of those of $\overline{Y}$ and $D=\overline{Y}-Y$. A good model is given by the compactification $\overline{Y}$ of Y constructed using toroidal embeddings by Ash, Mumford, Rapoport and Tai in {AM}, which they show to be a desingularisation of Y^* (loc. cit. p.254). Mumford then gave in {M} the right definition of "Chern classes for Y". We now describe this briefly.

Let $\Omega^q_{\overline{Y}}$ be the sheaf of holomorphic q-forms on the compact $\overline{Y}$, and suppose D is given in local coordinates $z_1,\dots,z_N$ by $z_1=0=\dots=z_s$. The sheaf of logarithmic q-forms along D $(q\geq s)$ is $\Omega^q_{\overline{Y}}\left\{\frac{dz_1}{z_1}\wedge\dots\wedge\frac{dz_s}{z_s}\wedge dz_{s+1}\wedge\dots\wedge dz_q\right\}$, the sheaf of q-forms with poles of order ≤ 1 along D. For details see {Ii}, ch. 11, or {GH} ch.5. This sheaf is denoted $\Omega^q_{\overline{Y}}(\log D)$. Now define the logarithmic Chern class of $(\overline{Y},D)$ by

$$\overline{c}_i(\overline{Y},D):=(-1)^i c_i(\Omega^1_{\overline{Y}}(\log D)).$$

(The sign is of course due to taking Ω^1, the cotangent bundle, instead of the tangent bundle). These are thought of as the "Chern classes of the non-complete variety Y". Using $\overline{c}_i$, a result anologous to 3.1.2. can be formulated:

Proposition 3.1.3. (Hirzebruch proportionality in the non-compact case, {M}, Theorem 3.2. and Prop. 3.4. a)). Y and $\overline{Y}$ as above

$$\overline{c}_{\nu_1}(\overline{Y},D)\cdot\ldots\cdot\overline{c}_{\nu_k}(\overline{Y},D)=\frac{c_{\nu_1}(\mathbb{P}^{M^\cdot})\cdot\ldots\cdot c_{\nu_k}(\mathbb{P}^{M^\cdot})}{c_{\mu_1}(\mathbb{P}^{M^\cdot})\cdot\ldots\cdot c_{\mu_l}(\mathbb{P}^{M^\cdot})}\overline{c}_{\mu_1}(\overline{Y},D)\cdot\ldots\cdot\overline{c}_{\mu_l}(\overline{Y},D)$$

Actually, the sheaf $\Omega^1_{\bar{Y}}(\log D)$ turns out to be intrinsic to the open manifold Y, and does not depend of the particular desingularisation $\bar{Y}$ of Y^* used. This is proven in {Ii}, Theorem 11.1.

Now assume D consists of disjoint components. To calculate $\bar{c}_j(\bar{Y},D)$ recall the exact sequence

$$0 \to \Omega^1_{\bar{Y}} \to \Omega^1_{\bar{Y}}(\log D) \to j_*\mathcal{O}_D \to 0,$$

where $j:D \hookrightarrow \bar{Y}$ is the inclusion. This implies for the total Chern classes

$$c(\Omega^1_{\bar{Y}}(\log D)) = c(\Omega^1_{\bar{Y}})c(j_*\mathcal{O}_D)$$

$$(1-\bar{c}_1+\ldots\pm\bar{c}_N) = (1-c_1+\ldots\pm c_N)(1+D+D^2+\ldots+D^N)$$

$$\bar{c}_1(\bar{Y},D) = c_1(\bar{Y})-(D)$$

$$\bar{c}_2(\bar{Y},D) = c_2(\bar{Y})-c_1(\bar{Y})(D)+D^2$$

$$\vdots$$

$$\bar{c}_N(\bar{Y},D) = c_N(\bar{Y})-c_{N-1}(\bar{Y})(D)+\ldots\pm D^N$$

Now assume $D\subset\bar{Y}$ is a divisor. Then for the euler number, for example, we see (by applying adjunction)

$$\begin{aligned}\bar{c}_N(\bar{Y},D) &= c_N(\bar{Y}) - \{c_{N-1}(D)+c_{N-2}(D)(D)\} \\ &\quad + \{c_{N-2}(D)+c_{N-3}(D)(D)\}(D) \\ &\quad \vdots \\ &\quad \pm D^N \\ &= c_N(\bar{Y})-c_{N-1}(D),\end{aligned}$$

since everything else cancels. In a similar manner the following is shown:

Proposition 3.1.4. $\bar{Y}$ as above, D= union of disjoint divisors. Then

$$\bar{c}_j(\bar{Y},D) = c_j(\bar{Y})-c_{j-1}(D).$$

In many cases, D is the disjoint union of complex tori of dimension N-1 (this is the natural desingularisation of a parabolic fixed point of Γ on the boarder $\partial\mathbb{B}^N$, see {He}). In this case, combining 3.1.3. and 3.1.4., applying adjunction to D and using the fact that $c_i(D)=0$ (since D consists of tori), we get the following.

Proposition 3.1.5. $\bar{Y}$ smooth compactification of a non-compact ball quotient, $D=\bar{Y}-Y$ a disjoint union of complex tori of dimension N-1, then

$$Nc_1^N(\bar{Y})-2(N+1)c_1^{N-2}(\bar{Y})c_2(\bar{Y}) = (-1)^{N-1}N(D)^N.$$

3.2. Subballs

3.2.1. Subballs

A subball $\mathbb{B}^{N-1} \subset \mathbb{B}^N$ is the subset $\mathbb{B}^{N-1} := \{(x_1,\ldots,x_N) \in \mathbb{B}^N \mid l(x)=0,$ l eine linear form$\}$. With respect to the embedding described above it is a hyperplane section of $\mathbb{B}^N$, $\mathbb{B}^{N-1} = \mathbb{P}^{N-1} \cap \mathbb{B}^N \subset \mathbb{P}^N$. Let $\Gamma \subset PSU(N,1)$ be a discrete subgroup acting on $\mathbb{B}^N$ with (compact or non-compact) quotient Y. Let $D \subset Y$ denote the image of a subball $\mathbb{B}^{N-1} \subset \mathbb{B}^{N\,*}$ under the map

$$\begin{array}{ccc} \mathbb{B}^N & \to & \Gamma\backslash\mathbb{B}^N = Y \\ \cup & & \cup \\ \mathbb{B}^{N-1} & \to & D \;. \end{array}$$

We call D a <u>subball quotient</u>. As an (N-1)-dimensional ball quotient, D of course fulfills the proportionality 3.1.2. (or 3.1.3.), but actually we can say more. A relative version of Hirzebruch proportionality is also true. If Y is compact, we get

<u>Lemma 3.2.1.</u>: For the inclusion $D \subset Y$ the same proportionality of the Chern classes holds between D and Y as between $\mathbb{P}^{N-1} \subset \mathbb{P}^N$:

$$c_j(Y)_{|D} = \frac{N+1}{N+1-j}\, c_j(D) \quad \in \quad H^{2j}(D,\mathbb{Q}).$$

Proof: For the Bergmann Kernel function $K_N(x,\bar{x})$ and $K_{N-1}(x,\bar{x})$ on $\mathbb{B}^N$ and $\mathbb{B}^{N-1}$, respectively, we have

$$K_N(x,\bar{x})_{|\mathbb{B}^{N-1}} = (K_{N-1}(x,\bar{x}))^{\frac{N+1}{N}} \;.$$

The Lemma now follows from the way the Chern classes are determined by the elementary symmetric functions of ($-1/2\pi i$ times) the curvature form defined by the metric, which is $\partial\bar{\partial}\ln(K(x,\bar{x}))$. q.e.d.

In the non-compact case, modifying this argument in an obvious way yields a similiar result. However, for technical reasons we need the result one gets by going to Chern <u>numbers</u>:

<u>Lemma 3.2.2.</u>: Let $\bar{Y}$, Y be as in 3.1.3., $T=\bar{Y}-Y$ (=D in 3.1.3.) and D a non-compact suball quotient with compactification $\bar{D}$. Then

a) $\bar{c}_1^{N-1}(\bar{Y},T)|_{\bar{D}} = (\frac{N+1}{N})^{N-1}\bar{c}_1^{N-1}(\bar{D},\bar{D}\cap T)$

b) $\bar{c}_1^{N-3}(\bar{Y},T)\bar{c}_2(\bar{Y},T)_{|\bar{D}} = 2\frac{N+1}{N-1}\cdot(\frac{N+1}{N})^{N-3}\bar{c}_1^{N-3}(\bar{D},\bar{D}\cap T)\bar{c}_2(\bar{D},\bar{D}\cap T)$

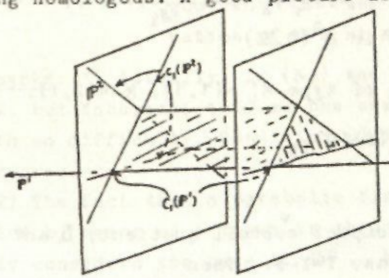
3.2.2. Subball-Products

Consider the ball-product $\mathbb{B}^{m_1} \times \ldots \times \mathbb{B}^{m_q}$. This has several types of

*here we assume Γ acts prop. discont. on $\mathbb{B}^{N-1}$ also subballs. To avoid confusion recall that a submanifold of a ball is a subball if and only if it is totally geodesic with respect to the Bergman metric. Applying this criterium here yields the following characterisation of subball-products. Let $M_i^{\cdot}=(m_1,\ldots,m_i-1,\ldots,m_q)$. The subball $\mathbb{B}^{M_i^{\cdot}}\subset\mathbb{B}^{M^{\cdot}}$ is $\mathbb{B}^{M_i^{\cdot}}=\mathbb{B}^{m_1}\times\ldots\times\mathbb{B}^{m_i-1}\times\ldots\times\mathbb{B}^{m_q}$, and with respect to the imbedding above it is the intersection :

$$\mathbb{B}^{M_i^{\cdot}}=\mathbb{P}^{M_i^{\cdot}}\cap\mathbb{B}^{M^{\cdot}}, \quad \text{where} \quad \mathbb{P}^{M_i^{\cdot}}=\mathbb{P}^{m_1}\times\ldots\times\mathbb{P}^{m_i-1}\times\ldots\times\mathbb{P}^{m_q}$$

As above, let $D\subset Y$ denote the image of $\mathbb{B}^{M_i^{\cdot}}\subset\mathbb{B}^{M^{\cdot}}$ under the natural projection. Assume Y is compact (and smooth of course). We refer to D as a <u>subball-product quotient</u> (sbpq). Of course, the subball $\mathbb{B}^{m_i-1}\subset\mathbb{B}^{m_i}$ sitting in side its "over-ball" $\mathbb{B}^{m_i}$ fulfills the proportionality 3.2.1., but $\mathbb{B}^{M_i^{\cdot}}\subset\mathbb{B}^{M^{\cdot}}$ is more subtile. Notice that $(\mathbb{B}^{M_i^{\cdot}})^{m_i}=0$. To see this, write $\mathbb{B}^{M_i^{\cdot}}=p_i^*(\mathbb{B}^{m_i-1})$, where $p_i:\mathbb{B}^{M^{\cdot}}\to\mathbb{B}^{m_i}$ is the projection. Then $(\mathbb{B}^{M_i^{\cdot}})^{m_i-1}=p_i^*((\mathbb{B}^{m_i-1})^{m_i-1})=p_i^*(pt)$, which is the fibre of p_i. Thus, for example, any subball quotient in $\mathbb{B}^1\times\ldots\times\mathbb{B}^1$ has trivial normal bundle*. Thus in this case the only condition (except of course the "absolute" proportionality $c_1^N(D)=2c_1^{N-2}(D)c_2(D)$) on a sbpq $D\subset Y$ is $(D)^2=0$. Furthermore we cannot expect statements as in 3.2.1. between homology classes, since the product structure prevents both sides from being homologous. A good picture is the following: Let $\mathbb{P}^{M^{\cdot}}=\mathbb{P}^2\times\mathbb{P}^1$.



$$\begin{array}{ccc} \mathbb{P}^2\times\mathbb{P}^1 & \xrightarrow{p_1} & \mathbb{P}^1 \\ \downarrow{\scriptstyle p_2} & & \\ \mathbb{P}^2 & & \end{array}$$

$c_1(\mathbb{P}^{M^{\cdot}})|_{\mathbb{P}^1\times\mathbb{P}^1}=3b+2f$.

$b=p_2^*(pt)$, $f=p_1^*(pt)p_2^*(H)$, the base and fibre of the compact dual of the subball $\mathbb{B}^1\times\mathbb{B}^1\subset\mathbb{B}^2\times\mathbb{B}^1$, and H is a line in $\mathbb{P}^2$. On the other hand, $c_1(\mathbb{P}^1\times\mathbb{P}^1)=2b+2f$, so an equation as in 3.2.1. wouldn't make sense. We have to work with numbers. For example on $\mathbb{P}^2\times\mathbb{P}^1$ we have

$$c_1^2(\mathbb{P}^1\times\mathbb{P}^1)=8$$
$$c_2(\mathbb{P}^1\times\mathbb{P}^1)=4$$
$$c_1(\mathbb{P}^1\times\mathbb{P}^1)(\mathbb{P}^1\times\mathbb{P}^1)^2=2, \quad \left((\mathbb{P}^1\times\mathbb{P}^1)|_{\mathbb{P}^1\times\mathbb{P}^1}\right)^2=0$$

yielding $\quad c_1^2=2c_2, \quad c_2=2c_1(\ )|_{(\ )}, \quad c_1^2=4c_1(\ )|_{(\ )}$

The reader may now check the following (the factors are calculated by general theory).

*excepting of course the diagonal of $\mathbb{B}^1\times\mathbb{B}^1$

Lemma 3.2.3.: If $D \subset Y$ is a subball quotient of type $M_{i_0} = (m_1, \ldots, m_{i_0}-1, \ldots, m_q)$

$N = \Sigma m_i$. Then

i) $(D)^h = 0$, all $h \geq m_{i_0}$

ii) $$c_1^{N-1}(Y)_{|D} = \left(\frac{m_{i_0}+1}{m_{i_0}}\right)^{m_{i_0}-1} \cdot c_1^{N-1}(D)$$

iii) $$c_1^{N-3}(Y)c_2(Y)_{|D} = \left[\frac{(m_{i_0}+1)}{m_{i_0}}\right]^{m_{i_0}-1} \frac{1}{(m_{i_0}-1)(m_{i_0}-2)} \cdot$$

$$\cdot \frac{\left\{\frac{m_{i_0}(m_{i_0}-1)(m_{i_0}-2)}{2(m_{i_0}+1)} + \sum_{j \neq i_0} \frac{m_j^2(m_j-1)}{2(m_j+1)}\{m_j + 2(m_{i_0}+1)\} + \sum_{j \neq k \neq i_0} \frac{m_j m_k}{(m_j+1)(m_k+1)}\right\}}{\left\{\frac{m_{i_0}-1}{2m_{i_0}} + \frac{1}{(m_{i_0}-1)(m_{i_0}-2)}\left[\sum_{i \neq i_0} \frac{m_i}{2(m_i+1)}\{m_i(m_i-1) + 2(m_i+1)(m_{i_0}-2)\} + \sum_{i \neq j \neq i_0} m_i m_j\right]\right\}} \cdot c_1^{N-3}(D)c_2(D)$$

We gather the most important formula in the

Formulae 3.2.4.:

(i) suppose $D \subset Y$ is a suball quotient $(M' = (N))$. Then

a) $c_1(D) = N(D)_{|D} \in H^2(D, \mathbb{Z})$

b) $c_2(D) = \binom{N}{2}\left[(D)_{|D}\right]^2 \in H^4(D, \mathbb{Z})$

(ii) suppose $D \subset Y$ is a sbpq of type $M_i' = (1, \ldots, \underset{i}{0}, \ldots, 1)$, $M' = (1, \ldots, 1)$.

Then $(D)_{|D} \equiv 0 \in$ in $H^2(D, \mathbb{Z})$

(iii) suppose $D \subset Y$ is a sbpq of type $M_1' = (1,1)$, $M' = (2,1)$. Then

a) $c_2(D) = 2c_1(D)(D)_{|D}$

b) $\left[(D)_{|D}\right]^2 = 0$

(iv) suppose $D \subset Y$ is a non-compact subball quotient, $\bar{D}$ and $\bar{Y}$ the compactifications, $T = \bar{Y} - Y$. Then

a) $\bar{c}_1^{N-1}(\bar{Y}, T)_{|D} = \left(\frac{N+1}{N}\right)^{N-1} \cdot \bar{c}_1^{N-1}(\bar{D}, \bar{D} \cap T)$

b) $\bar{c}_1^{N-3}(\bar{Y}, T)\bar{c}_2(\bar{Y}, T)_{|D} = 2\frac{N+1}{N-1} \cdot \left(\frac{N+1}{N}\right)^{N-3} \cdot \bar{c}_1^{N-3}(\bar{D}, \bar{D} \cap T)\bar{c}_2(\bar{D}, \bar{D} \cap T)$

3.3. The Yau inequality

The following theorem, known as the Yau inequality, is one of the many corollaries following from Yau's celebrated solution of Calabi's conjecture [Y]:

Theorem 3.3.1.: M a compact Kähler manifold. If K_M is ample, then

$$(-1)^N c_1^N(M) \le \frac{2(N+1)}{N}(-1)^N c_1^{N-2}(M)c_2(M),$$

with equality holding if and only if M is covered by the complex N-ball.

The reader might consult {KW},I,1.6. for an elementary derivation of this result if M carries a Kähler-Einstein metric. Yau's solution of Calabi's conjecture implies that M carries a Kähler-Einstein metric if K_M is ample. Notice that this result is stronger than the converse of Hirzebruch proportionality; here the proportionality of just two of the Chern numbers is required.

There is also a non-compact version of this, due to R. Kobayashi, which we shall refer to as the Kobayashi-Yau inequality {K3}.

Theorem 3.3.2.: M a compact Kähler manifold, $D \subset M$ a divisor, and assume:

i) D is a disjoint union of complex tori

ii) $K_M + D$ is ample on M-D

Then: $$(-1)^N \bar{c}_1^N(M,D) \le \frac{2(N+1)}{N}(-1)^N \bar{c}_1^{N-2}(M,D)\bar{c}_2(M,D),$$

with equality holding if and only if M-D is a ball quotient $\Gamma \backslash \mathbb{B}^N$ and each torus in D is the compactification of M-D at a parabolic fixed point of Γ on $\partial\mathbb{B}^N$.

Remarks: 1) Actually, in {K3} the inequality is only proven for surfaces, N=2, but Kobayashi told me the same proof goes through in all dimensions with no difficulty. Condition i) is not necessary, but suffices for our purposes.

2) The fact that a parabolic fixed point of Γ on $\partial\mathbb{B}^N$ can be compactified by means of a torus* is shown in {He}. Actually, there he also only considers the case of $\mathbb{B}^2$, but using a local presentation of Γ by means of matricies, it is easy to see that this carries through to any $N \ge 2$.

* generically

§4. Coverings as Ball Quotients

4.1. Coverings of type (F)

Let $\pi : Y \to X$ be a Galois covering of complex manifolds of dimension N, satisfying the following conditions:

(i) The branch locus $B \subset X$ is a divisor with normal crossings, $B = \Sigma B_i$, B_i the irreducible components

(ii) B has a stratification

$$B = B^1 = \bigcup_i B_i \supset B^2 = \bigcup_{i \neq j} \{B_i \cap B_j\} \supset B^3 \supset \ldots \supset B^N.$$

(iii) choose local coordinates $z_1, \ldots, z_N$ such that

$B_i = \{z_i = 0\}$, $B_{i_1} \cap B_{i_2} = \{z_{i_1} = z_{i_2} = 0\}$, etc. Then π is given by

$$(x_1, \ldots, x_N) \mapsto (x_1^{n_{i_1}}, x_2, x_3, \ldots, x_N) = (z_1, \ldots z_N) \text{ on } B_{i_1} - \{B_{i_1} \cap B_{i_j}\} \quad j \neq 1$$

$$(x_1, \ldots, x_N) \mapsto (x_1^{n_{i_1}}, x_2^{n_{i_2}}, x_3, \ldots, x_N) = (z_1, \ldots, z_N) \text{ on}$$

$$B_{i_1} \cap B_{i_2} - \{B_{i_1} \cap B_{i_2} \cap B_{i_j}\} \quad j \neq 1,2$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$(x_1, \ldots, x_N) \mapsto (x_1^{n_{i_1}}, x_2^{n_{i_2}}, \ldots, x_N^{n_{i_N}}) = (z_1, \ldots, z_N) \text{ on } B_{i_1} \cap B_{i_2} \cap \ldots \cap B_{i_N}.$$

n_i = branching degree along B_i

$n_{i_1} n_{i_2}$ = branching degree along $B_{i_1} \cap B_{i_2}$

$$\vdots \qquad \vdots \qquad \vdots$$

(iv) for all components $B_i \subset B^1$, $n_i > 1$

(v) $c_1(Y)$ can be written (over $\mathbb{Q}$) as a linear combination of the ramification divisors $R_i = \pi^*(B_i)$, $c_1(Y) = \sum \lambda_i R_i$, $\lambda_i \in \mathbb{Q}$.

We call $\pi : Y \to X$ a <u>covering of type (F)</u>, if it fulfills (i)-(v) above. Obviously, the Fermat covers of §1 are of type (F), and even in fact the prototype of such coverings. The terminology should suggest a generalisation of Fermat covers.

<u>Remarks on the definition</u>

1) The conditions (i) & (ii) on the branch divisors imply the same thing for the ramification divisors R_i.

2) The condition (iv) is to insure the R_i are pointwise fixed by the Galois group. This is important for subballs.

3) Condition (v) insures we can use induction by adjunction in the proofs of the main theorems of this chapter. It is not as restrictive

as it looks, for example, any covering of projective space does the job.

4) In the sequel we will be dealing with the <u>reduced</u> ramification divisors $\overline{B}_i := \pi^{-1}(B_i)$. (Notice $R_i = \pi^*(B_i) = n_i \pi^{-1}(B_i)$, so one shouldn't apply adjunction to R_i.)

5) We use the notation R^j for the reduced ramification locus in codimension j, $R^j = \pi^{-1}(B^j)$.

<u>4.2. Coverings as ball quotients</u>

4.2.1. Suppose first $\pi : Y \to X$ is a covering of type (F), such that Y is a ball quotient. Then by (iv) of the definition, the reduced ramification divisors $\overline{B}_i$ are pointwise fixed by the Galois group, and therefore are totally geodesic, yielding

<u>Proposition 4.2.1.</u>: If $\pi : Y \to X$ is a covering of type (F) such that Y is a ball quotient, then each $\overline{B}_i$ is a subball quotient (i.e. fulfills 3.1.2. and 3.2.4. (i)).

4.2.2. We now want to show the converse of 4.2.1. Assume for the rest of this section that K_Y is ample and $N = \dim Y \geq 3$. Then, by Yau's theorem, what we must show is $c_1^N(Y) = \frac{2(N+1)}{N} c_1^{N-2}(Y) c_2(Y)$.

<u>Theorem 4.2.2.</u>: Let $\pi : Y \to X$ be a covering of type (F) with K_Y ample. Then:

$$\left\{ \begin{array}{c} Y \text{ is a ball} \\ \text{quotient} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} \text{for all } i,\ \overline{B}_i \text{ is a subball quotient} \\ \text{(fulfills 3.2. . (i) \& 3.1.2.)} \end{array} \right\}$$

prooof: " $\Rightarrow$ " is 4.2.1. " $\Leftarrow$ ": We must show $Nc_1(Y) - 2(N+1)c_1^{N-2}(Y)c_2(Y) = 0$. Write this in the following form:

$$2(N+1)c_1^{N-2}(Y)c_2(Y) - Nc_1^N(Y) = c_1(Y)\left\{2(N+1)c_1^{N-3}(Y)c_2(Y) - Nc_1^{N-1}(Y)\right\}$$
$$= c_1(Y) \cdot \mathcal{D},\ \mathcal{D} \in H^{2N-2}(Y, \mathbb{Z}).$$

By condition (v) of 4.1., it is sufficient to show $\mathcal{D}_{|\overline{B}_i} = 0$ for all $\overline{B}_i$.

But by adjunction applied to $\overline{B}_i \subset Y$,

$$\mathcal{D}_{|\overline{B}_i} = \Big[2(N+1)\{c_2(\overline{B}_i) + c_1(\overline{B}_i)(\overline{B}_i)|_{\overline{B}_i}\} - N\{c_1(\overline{B}_i) + 2c_1(\overline{B}_i)(\overline{B}_i)|_{\overline{B}_i} + (\overline{B}_i)|_{\overline{B}_i}\}\Big] \cdot c_1^{N-3}(Y)|_{\overline{B}_i}$$

$$= \Big[\{2Nc_2(\overline{B}_i) - (N-1)c_1^2(\overline{B}_i)\} + \{Nc_1(\overline{B}_i)(\overline{B}_i)|_{\overline{B}_i} - c_1^2(\overline{B}_i)\} + \{c_2(\overline{B}_i) - \tfrac{N-1}{2}c_1(\overline{B}_i)(\overline{B}_i)|_{\overline{B}_i}\} + \left\{c_2(B_i) - \binom{N}{2}\left((\overline{B}_i)|_{\overline{B}_i}\right)^2\right\}$$

$$+\frac{N-3}{2}\{N(\bar{B}_i)|_{\bar{B}_i} - c_1(\bar{B}_i)(\bar{B}_i)|_{\bar{B}_i}\}\Big]\cdot c_1^{N-3}(Y)|_{\bar{B}_i}$$

$=0$ by 3.2.4. (i).

We now make the further assumptions on $\pi: Y \to X$:

(i) for all components $C \subset R^2$, $N_Y C = \mathcal{L}\oplus\mathcal{L}$ for some line bundle $\mathcal{L}$ on C

(ii) for all components $\xi \subset R^K$, $\xi \to \pi(\xi)$ is also a covering of type (F) (and in particular fulfills 4.1.(v)).

These assumptions on Y assure we can use induction to improve the above to

Corollary 4.2.3.: Y as above, then

$$\left\{\begin{array}{c}\text{Y is a ball}\\ \text{quotient}\end{array}\right\} \Leftrightarrow \left\{\begin{array}{c}\text{All surfaces } S\subset R^{N-2} \text{ are subballs with}\\ \text{respect to all embeddings.}\end{array}\right\}$$

proof: "$\Rightarrow$" is obvious by descending induction applied to 4.2.1. "$\Leftarrow$". Proof by induction on N=dim Y. N=3 is the theorem above, so assume "$\Leftarrow$" for all ramification divisors $\bar{B}_i \subset Y$. They are then ball quotients and fulfill 3.1.2. We must show that they satisfy 3.2.4. (i), since the result then follows by the theorem above. Set $\mathcal{D} = c_1(\bar{B}_i) - N(\bar{B}_i)|_{\bar{B}_i}$. By assumption (ii) above it is sufficient to show $\mathcal{D}|_C = 0$ for all components $C \subset R^2$. But

$$\begin{aligned}\mathcal{D}|_C &= -c_1(C) - (C)|_C + N(C)(\bar{B}_i)|_{\bar{B}_i}\\ &= N((C)(\bar{B}_i)|_{\bar{B}_i} - (C)|_C)\\ &= 0\end{aligned}$$

because of the assumption (i) above. (Here of course $(C)|_C = c_1(N_{\bar{B}_i} C)$.)

Now assume given a covering $\pi: Y \to \hat{\mathbb{P}}^N$ branched only along hyperplanes (and the exceptional divisors introduced in the resolution of singularities as in 1.2.). Then we can go one step further,

Corollary 4.2.4.: Under these assumptions

$$\left\{\begin{array}{c}\text{Y is a ball}\\ \text{quotient}\end{array}\right\} \Leftrightarrow \left\{\begin{array}{c}\text{all curves } C\subset R^{N-1} \text{ satisfy } e(C) = 2C\cdot C\\ \text{with respect to all embeddings}\end{array}\right\}$$

proof: "$\Rightarrow$" follows again from 4.2.1. "$\Leftarrow$" We must show all surfaces fulfill 3.1.2. and 3.2.4.(i). 3.1.2. follows from Höfers theory [Hö], so we only have to show 3.2.4.(i). But this follows by the same argument as in the proof of 4.2.3.

4.3. Coverings as ball-product quotients

The purpose of this section is to extend the results of 4.2. as far as possible to the case of quotients of ball-products $Y=\Gamma\backslash\mathbb{B}^{M^\cdot}$, $M^\cdot=(m_1,\ldots,m_q)$. Of course, we cannot expect nice equivalences like 4.2.2., because we have no anologue of Yau's theorem in this case. Furthermore, there are complications if $m_i \nmid m_j$ for some i,j, due to the fact that then there are more than one type of subball quotient (see 3.2.2.). For this reason it is natural to expect the best results for $M^\cdot=(m,\ldots,m)$. In this case, all subball-products are of the same type. Let us agree to call such ball-products homogenous, i.e. $\mathbb{B}^{M^\cdot}$ is a homogenous ball-product $:\Leftrightarrow$ $M^\cdot=(m,\ldots,m)$.

4.3.1. Just as in 4.2.1. we have the following

Proposition 4.3.1.: If $Y\to X$ is a covering of type (F) such that Y is a ball-product quotient of type $M^\cdot=(m_1,\ldots,m_q)$. Then for all ramification divisors $\overline{B}_i\subset Y$, one of the following hold:

(k) $\left\{\overline{B}_i \text{ is a sbpq of type } M_k^\cdot=(m_1,\ldots,m_k-1,\ldots,m_q)\right\}$

Recall that a sbpq $D\subset Y$ fulfills the numerical conditions 3.2.2.

4.3.2. Now consider a cover $Y\to X$ of type (F), such that all ramification divisors $\overline{B}_i$ fulfill (i) 3.1.2., (ii) 3.2.3. for $M_i^\cdot=(m,\ldots,m-1,\ldots,m)$, that is, all $\overline{B}_i$ are sbpq's of a homogenous ball-product $\mathbb{B}^{M^\cdot}$, $M^\cdot=(m,\ldots,m)$, $m>1$.

Theorem 4.3.2.: Under these assumptions on $\overline{B}_i$, $(N\geq 3)$

$$c_1^N(Y)=\frac{2(N-1)}{((m-1)+2m(m+1)(q-1))}\frac{(m+1)}{m}c_1^{N-2}(Y)c_2(Y),\quad N=mq$$

that is, $c_1^N(Y)$ and $c_1^{N-2}(Y)c_2(Y)$ have the same proportionality as the homogenous product projective space $\mathbb{P}^{M^\cdot}=\mathbb{P}^m\times\ldots\times\mathbb{P}^m$.

Proof: Let α be the factor above. Then

$$c_1^N(Y)-\alpha c_1^{N-2}(Y)c_2(Y)=c_1(Y)\left\{c_1^{N-1}(Y)-\alpha c_1^{N-3}(Y)c_2(Y)\right\}=c_1(Y)\cdot\mathcal{D}$$

and as before it is sufficient to show: $\mathcal{D}|_{\overline{B}_i}=0$ for all $\overline{B}_i\subset Y$. By assumption, the $\overline{B}_i$ satisfy

$$c_1^{N-1}(Y)|_{\overline{B}_i}=\gamma\cdot c_1^{N-1}(\overline{B}_i),$$

$$c_1^{N-3}(Y)c_2(Y)|_{\overline{B}_i}=\beta\cdot c_1^{N-3}(\overline{B}_i)c_2(\overline{B}_i)$$

and $\quad c_1^{N-1}(\bar{B}_i) = \delta c_1^{N-3}(\bar{B}_i)c_2(\bar{B}_i)$,

where the factors are as in 3.1.2. and 3.2.3. We must therefore show

$$\gamma c_1^{N-1}(\bar{B}_i) - \alpha\beta c_1^{N-3}(\bar{B}_i)c_2(\bar{B}_i) = 0$$

or $$(\gamma\delta-\alpha\beta)c_1^{N-3}(\bar{B}_i)c_2(\bar{B}_i) = 0$$

$$\Leftrightarrow \qquad \gamma\delta = \alpha\beta \quad .$$

This formal identity is proven using standard methods.

Much easier is the case $M^* = (1,\dots,1)$:

Theorem 4.3.3.: Suppose for all $\bar{B}_i \subset Y$:

(i) $c_1^{N-1}(\bar{B}_i) = 2c_1^{N-3}(\bar{B}_i)c_2(\bar{B}_i)$

(ii) $(\bar{B}_i)_{|\bar{B}_i} \equiv 0$

Then: $c_1^N(Y) = 2c_1^{N-2}(Y)c_2(Y)$.

Proof: indeed, in this case (notations as above)

$$\mathcal{D}_{|\bar{B}_i} = c_1^{N-1}(\bar{B}_i) - 2c_1^{N-3}(\bar{B}_i)c_2(\bar{B}_i) \ + \ \text{terms containing } (\bar{B}_i)_{|\bar{B}_i}$$

$$= 0 \text{ by assumption.}$$

As the general case is much more complicated, we consider only the simplest case: $\mathbb{B}^2 \times \mathbb{B}^1$ - proportionality. Let $Y \to X$ be a covering of type (F), and assume $\bar{B}_i$ are $\mathbb{B}^2$ - subballs for $i=1,\dots,r$ and the other $\bar{B}_j$ are $\mathbb{B}^1 \times \mathbb{B}^1$- subballs, $j=r+1,\dots,s$. Let

$$c_1(Y) = \sum \lambda_i \bar{B}_i + \sum \mu_j \bar{B}_j$$

be the first Chern class of Y.

Theorem 4.3.4.: Assumptions as above, and also assume:

$$\tfrac{3}{2}\sum \mu_j c_2(\bar{B}_j) = \sum \lambda_i c_1^2(\bar{B}_i) \ .$$

Then: $c_1^3(Y) = \frac{9}{4}c_1(Y)c_2(Y)$.

Proof: As before,

$$4c_1(Y) - 9c_1(Y)c_2(Y) = c_1(Y)(4c_1(Y) - 9c_2(Y))$$
$$= (\Sigma\lambda_i\bar{B}_i + \Sigma\mu_j\bar{B}_j)(4c_1(Y) - 9c_2(Y))$$

using adjunction $$=\sum\lambda_i\left\{3(3c_2(\bar{B}_i)-c_1^2(\bar{B}_i))-c_1^2(\bar{B}_i)\right\}$$
$$+\sum\mu_j\left\{4(2c_2(\bar{B}_j)-c_1^2(\bar{B}_i))+(c_2(\bar{B}_j)-2c_1(\bar{B}_j)(\bar{B}_j)_{|\bar{B}_j})\right.$$
$$\left.+3c_1(\bar{B}_j)(\bar{B}_j)_{|\bar{B}_j}\right\}$$

The theorem follows from this using 3.2.4. iii).

4.4. Non-compact case

We now want to extend the results of 4.2. as far as possible in the case Y is the compactification of a non-compact ball quotient.

Theorem 4.4.1.: Let $Y \to X$ be a covering of type (F), $D=\bigcup_i T_i \subset Y$ a divisor which is the disjoint union of complex tori, and suppose K_Y+D is ample on $Y-D$. Then $(N\geq 3)$

$$\left\{\begin{matrix}\text{Y-D is a ball}\\ \text{quotient}\end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix}\text{for all } i, \quad \bar{B}_i-D \text{ is a subball quotient}\\ \text{(fulfills 3.1.3. and 3.2.4.(iv))}\end{matrix}\right\}$$

Proof: "$\Rightarrow$" follows as in 4.2.1. We assume the right hand side and as previously set (this time with $\bar{c}_i$'s)

$$2(N+1)\bar{c}_1^{N-2}(Y,D)\bar{c}_2(Y,D)-N\bar{c}_1^N(Y,D) = \bar{c}_1(Y,D)\cdot\mathcal{D},$$

and it is sufficient to show $\mathcal{D}_{|\bar{B}_i}=0$ for all $\bar{B}_i$.* Because of 3.1.3. and 3.2.4.(iv), the proof goes through as in 4.3.2. (with the easier factors $\alpha=\frac{N+1}{N-1}$ $\beta=\left(\frac{N+1}{N}\right)^{N-3}\frac{2(N+1)}{N-1}$, $\gamma=\left(\frac{N+1}{N}\right)^{N-1}$, $\delta=\frac{N}{N-2}$). q.e.d.

We leave it to the reader to check that 4.2.3. also carries over to the non-compact case.

4.5. The Kobayashi existence theorem

It is in general a difficult question whether a branched covering $Y \to X$ exists, with prescribed branch divisor $B\subset X$ and branch degrees n_i along the components B_i of B. In the most important case, however, there is an existence theorem, due to R. Kobayashi. The theorem in dimension 2 is found in {K2}. To formulate the theorem in higher dimensions, we need the notion of the Chern numbers of the orbifold (X,B). Assume first there exists a covering $\pi:Y \to X$, branched along the B_i with the prescribed branching degrees n_i. Then the Chern numbers of Y are of the form

$$c_1^N(Y) = \deg\pi\cdot\tilde{c}_1^N(X,B)$$
$$c_1^{N-2}(Y)c_2(Y)=\deg\pi\cdot\tilde{c}_1^{N-2}(X,B)\tilde{c}_2(X,B),$$

*by 3.2.2.

where $\tilde{c}_1(X,B)$ is an expression in the Chern classes of X and B. The <u>Chern numbers</u> $\tilde{c}_1^N(X,B)$, $\tilde{c}_1^{N-2}\tilde{c}_2(X,B)$ <u>of the orbifold (X,B)</u> are defined by the right-hand side of the equation. Let $D\subset B$ be the union of all components of B which have $n_i=\infty$ ($\frac{n_i-1}{n_i}=1$).

<u>Theorem 4.5.1.</u>: Let (X,B) be as above, and assume:

1) $\kappa(X,B) = N$

2) $\tilde{c}_1(X,B)=c_1(X) - \Sigma\frac{n_i-1}{n_i} B_i$ is ample away from D.

3) $\tilde{c}_1^N(X,B)=\frac{2(N+1)}{N}\tilde{c}_1^{N-2}\tilde{c}_2(X,B)$

Then: the universal covering of X-D is the ball $\mathbb{B}^N$, and there exists a finite branched covering $Y \to X$ with the prescribed branching degrees n_i along B_i. Y is thus the compactification of a ball quotient.

The proof of this theorem goes through in higher dimensions just as in the case of N=2 (see [K2]).

4.6. Applications

4.6.1. Chern numbers for Fermat covers of $\mathbb{P}^3$

Instead of calculating the Chern numbers c_1^3 and c_1c_2 as in 2.2., we could consider the <u>difference</u> $3c_1^3-8c_1c_2$. Using the methods of 4.2. this can be given a very conceptual form. Write

$$3c_1^3(Y)-8c_1(Y)c_2(Y)=c_1(Y)\{3c_1^2(Y)-8c_2(Y)\}$$

$$=-1\left\{\Sigma\lambda_i\pi^*(H_i)+\Sigma\nu_j\pi^*(P_j)+\Sigma\mu_m\pi^*(L_m)\right\}(3c_1^2(Y)-8c_2(Y)),$$

coefficients λ_i,ν_j,μ_m, as in 2.3. Now apply adjunction. This yields:

$$\frac{1}{n}(3c_1^3(Y)-8c_1(Y)c_2(Y))=-1\Bigg[\sum_{i=1}^{k}\lambda_i\Big\{2(c_1^2(\bar{H}_i)-3c_2(\bar{H}_i)) + (c_1^2(\bar{H}_i)-3c_1(\bar{H}_i)(\bar{H}_i)_{|\bar{H}_i})$$
$$+ (c_1(\bar{H}_i)(\bar{H}_i)_{|\bar{H}_i}-c_2(\bar{H}_i)) + (3\left((\bar{H}_i)_{|\bar{H}_i}\right)^2-c_2(\bar{H}_i))\Big\}$$
$$+\sum_{j=1}^{\tau}\nu_j\Big\{2(c_1^2(\bar{P}_j)-3c_2(\bar{P}_j)) + (c_1^2(\bar{P}_j)-3c_1(\bar{P}_j)(\bar{P}_j)_{|\bar{P}_j})$$
$$+ (c_1(\bar{P}_j)(\bar{P}_j)_{|\bar{P}_j}-c_2(\bar{P}_j)) + (3\left((\bar{P}_j)_{|\bar{P}_j}\right)^2-c_2(\bar{P}_j))\Big\}$$
$$+\sum_{m=1}^{\sigma}\mu_m\Big\{3\left((\bar{L}_m)_{|\bar{L}_m}\right)^2- 2c_1(\bar{L}_m)(\bar{L}_m)_{|\bar{L}_m} -2c_2(\bar{L}_m)\Big\}$$

$$\frac{1}{n}(c_1(Y)-2c_1(Y)c_2(Y))=\sum_{i=1}^{k}\lambda_i\left[\{2c_2(\bar{H}_i)-c_1^2(\bar{H}_i)\}-\{\left((\bar{H}_i)_{|\bar{H}_i}\right)^2\}\right]$$

$$+\sum_{j=1}^{r}\nu_j\left[\{2c_2(\bar{F}_j)-c_1^2(\bar{F}_j)\}-\{\left((\bar{F}_j)_{|\bar{F}_j}\right)^2\}\right]$$

$$-\sum_{m=1}^{\sigma}\mu_m\{(\bar{L}_m)_{|\bar{L}_m}{}^2\}.$$

All brackets $\{\ldots\}$ are defect factors, and vanish if the $\bar{H}_i$ are subball (-product) quotients of the corresponding type. This shows Theorems 4.2.2. and 4.3.3. in this special case purely computationally.

4.G.2. Fermat covers as ball quotients

The purpose of this section is to prove the following

Theorem 4.6.1.: In dimensions $N\geq 3$, there are no Fermat covers

$$Y \to \hat{\mathbb{P}}^N$$

such that Y is a compact, smooth ball quotient.

Proof: As explained in the introduction to this chapter, if there are singular lines l_m with $\tau(L_m)\geq 3$ in an arrangement $L\subset\mathbb{P}^3$, the Fermat cover Y has $\mathbb{B}^1\times\mathbb{B}^1$-quotients in its ramification locus, so by 4.3.2. cannot possibly be a ball quotient. Therefore, we need only consider the following two cases:

a) L has no singular lines

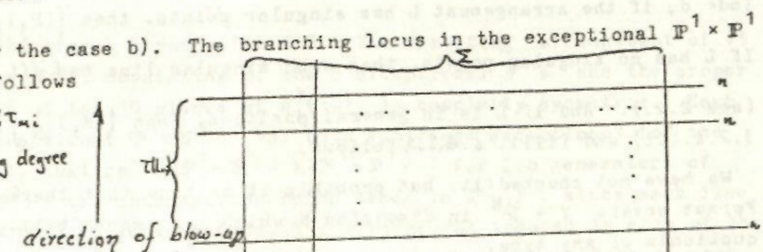
b) The exceptional divisors upstairs, $S=C_1\times C_2\to\mathbb{P}^1\times\mathbb{P}^1=L_m$ has $C_i=\mathbb{P}^1$ for some $i=1,2$, and has $C_i\cdot S=-1$ and can be blown down.

If a) holds, the arrangement in-duced in each of the exceptional $\mathbb{P}^2$'s is in general position (singular points of an arrangement in an exceptional $\mathbb{P}^2$ correspond to singular lines of L through that point), and the covering surfaces are non-singular complete intersections, have therefore $c_1^2\leq 2c_2$, so are not ball quotients. Applying 4.2.2. finishes in this case.

Consider the case b). The branching locus in the exceptional $\mathbb{P}^1\times\mathbb{P}^1$ looks as follows

$\Sigma=\sum_j\sigma_{jm}+\sum_i\tau_{mi}$

n = branching degree



The euler numbers may be calculated by Hurwitz formula,

$$e(C_1)=n^{\tau(L_m)-1}(2-\tau(L_m))+n^{\tau(L_m)-2}\tau(L_m)$$

$$e(C_2)=n^{\Sigma-1}(2-\Sigma)+n^{\Sigma-2}\Sigma$$

so

$$C_1=\mathbb{P}^1 \iff e(C_1)=2 \iff \quad n=2, \quad \tau(L_m)=3$$

$$C_2=\mathbb{P}^1 \iff e(C_2)=2 \iff \begin{cases} n\geq 2 & \Sigma=2 \\ n=2 & \Sigma=3. \end{cases}$$

The self intersections are

$$C_1\cdot C_1 = -n^{\tau(L_m)-2}$$

$$C_2\cdot C_2 = n^{\Sigma-2}(1-\Sigma\sigma_{jm})$$

so for the cases above,

		C^2	
$n=2$	$\tau(L_m)=3$	-2	
$n\geq 2$	$\Sigma=2$	-1	(since $\Sigma\sigma_{jm}=2$ in this case)
$n=2$	$\Sigma=3$	-2	$\Sigma\sigma_{jm}=2$
		0	$\Sigma\sigma_{jm}=1$
		-4	$\Sigma\sigma_{jm}=3$

The only possibility is: $n\geq 2$, $\Sigma=2$. But this is a pathological case. Consider each of the planes passing through the line. The induced arrangements are of type (12), in the language of [Hi1], and after blowing down the exceptional divisor, each of the branch divisors is a product of curves, therefore $\mathbb{B}^1\times\mathbb{B}$-quotient, so applying 4.2.2. once again, 4.6.1. is proved in dimension 3. Applying 4.2.2. one last time proves (inductively) 4.6.1. for $N\geq 3$.

In passing we notice the following almost trivial

Theorem 4.6.1a: There are no Fermat coverings $Y\to\hat{\mathbb{P}}^3$ such that Y is (i) a compact $\mathbb{B}^2\times\mathbb{B}^1$-quotient or (ii) a compact $\mathbb{B}^1\times\mathbb{B}^1\times\mathbb{B}^1$-quotient.

Indeed, if the arrangement L has singular points, then $\left((P_j)_{|P_j}\right)^2\neq 0$. If L has no singular points, then each singular line has $\left((L_m)_{|L_m}\right)^2\neq 0$ (see 2.2.). And if L is in general position, then $\left((H_i)_{|H_i}\right)^2\neq 0$, so by 3.2.4.(ii) and (iii), 4.6.1.a follows.

We have not checked it, but probably it is true that there are no Fermat covers $Y\to\hat{\mathbb{P}}^N$ in dimension N which are smooth ball-product quotients of any type.

A. Sommese has remarked that the proof of the above also yields

Corollary 4.6.3.: There are no Fermat covers $Y \to \hat{\mathbb{P}}^3$ such that Y has ample cotangent bundle.

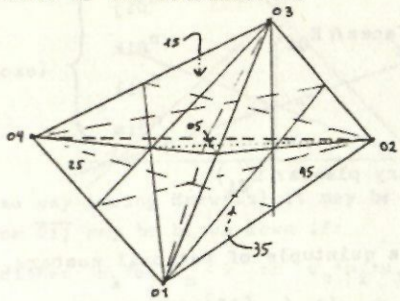
Proof: If Y has ample cotangent bundle, then all divisors must have. But, as mentioned above, if the arrangement has singular lines, there are divisors $\bar{B}_i = C_1 \times C_2$ in the branch locus. Such a product cannot have ample cotangent bundle (since $T^*_{B_i}|C_j = T^*_{C_j} \oplus \mathcal{O}_{C_j}$ holomorphically).

If the arrangement has no singular lines, the induced arrangement of a hyperplane in $\mathbb{P}^3$ intersecting the arrangement is in general position and the divisor in Y covering it cannot have ample cotangent bundle, by Sommese's results ([So]).

4.6.3. Ball quotients covering $\mathbb{P}^3$ and ramifying along an arrangement

Although there are no Fermat covers which are compact ball quotients, there are 2 examples which are compactifications of non-compact quotients. Altogether there are 7 examples due to Deligne-Mostow and one new example we discuss below. Two of Deligne-Mostows examples are compact; the rest are compactifications of ball quotients.

We start with the arrangement $A_1^3(10)$ consisting of the 4 faces and 6 symmetry planes of the tetrahedron:



Resolve the branch locus as in 1.2. The resulting "arrangement of 15 planes in $\hat{\mathbb{P}}^3$", consisting of the 5 exceptional $\mathbb{P}^2$'s and the proper transforms of the 10 planes of $A_1^3(10)$, is completely symmetric. Each of the exceptional $\mathbb{P}^1 \times \mathbb{P}^1$'s has both fiberings exceptional (of the first kind, that is $f \cdot \mathbb{P}^1 \times \mathbb{P}^1 = b \cdot \mathbb{P}^1 \times \mathbb{P}^1 = -1$ for f, b generators of $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$, the intersection being taken in $\hat{\mathbb{P}}^3$), since each line meets 2 actual singular points. The arrangement induced in each of the 15 planes is the arrangement $A_1^2(6)$, and the proper transform of each is a $\mathbb{P}^2$ blown up in 4 points.

We number the 15 planes H_{ij}, $i<j \in \{0,..,5\}$ such that:

$$H_{ij} \cap H_{kl} \neq \phi \Leftrightarrow i \neq j \neq k \neq l.$$

We may assume the exceptional $\mathbb{P}^2$'s are numbered 01,...,05. We denote by 0ij the singular line passing through 0i and 0j. We then have:

$$H_{ij} \cap H_{kl} = 0\lambda\mu \Leftrightarrow (i,j,k,l) \cap (0,\lambda,\mu) = \phi.$$

We want to consider branched coverings $Y \to \hat{\mathbb{P}}^3$, branched along $A_1^3(10)$ with branching degrees n_{ij} along H_{ij} and n_{0ij} along the exceptional divisor 0ij. Such coverings are obviously of type (F), so the theorems of 4.2. apply. The existence of such coverings, in general a highly non-trivial problem, is assured by the Kobayashi existence theorem, provided we are constructing (compactifications of) ball quotients. By 4.4.1., to show that Y is (a compactification of) a ball quotient, we must show:

a) the reduced ramification divisors $\overline{H_{ij}} \to H_{ij}$ are (compactifications of) ball quotients or abelian varieties.

b) all ramification divisors $\overline{0ij} \to 0ij$ blow down (in one direction or the other) or are abelian varieties.

a) Consider for concreteness H_{0i}. The induced branching degrees (for the induced arrangement $A_1^2(6)$ in H_{0i}) are seen to be:

$$\left.\begin{matrix} n_{kl} \\ n_{lm} \\ n_{km} \end{matrix}\right\} (\text{faces} \cap H_{0i}) \qquad \left.\begin{matrix} n_{0ij} \\ n_{0ik} \\ n_{0il} \\ n_{0im} \end{matrix}\right\} (\text{exceptional } 0ij \cap H_{0i})$$

$$\left.\begin{matrix} n_{jk} \\ n_{jl} \\ n_{jm} \end{matrix}\right\} (\text{symmetry planes} \cap H_{0i})$$

In this case there is a quintuple of rational numbers, $\mu'_{0i}, \mu'_j, \mu'_k, \mu'_l, \mu'_m$, $\Sigma\mu'_\lambda = 2$ such that $n_{\lambda_1\lambda_2} = (1-\mu'_{\lambda_1}-\mu'_{\lambda_2})^{-1}$ (this is the theory for the surface case, see [DM], [Hö], or [Hi3]). On the other hand, the branching degree n_{0ij} is given by:

$$2\left(\frac{1}{n_{kl}} + \frac{1}{n_{lm}} + \frac{1}{n_{km}} - 1\right)^{-1} = n_{0ij} = 2\left(\frac{1}{n_{0i}} + \frac{1}{n_{0j}} + \frac{1}{n_{ij}} - 1\right)^{-1}$$

(the first equality is the surface case, the second follows from Hurwitz' formula applied to the exceptional $C_1 \times C_2 \to \mathbb{P}^1 \times \mathbb{P}^1$). The usual convention is to set $n_{oij}=\infty$ if $\frac{1}{n_{oi}}+\frac{1}{n_{oj}}+\frac{1}{n_{ij}}=1$. In this case, the curve covering the exceptional $\mathbb{P}^1=0ij$ (in H_{0i}) is an elliptic curve compactifying a non-compact quotient.

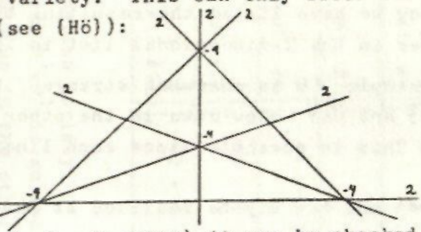
Because of the symmetry of the 15 branching planes, there is a sextuple of rational numbers $\mu_o, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5$ such that $n_{ij}=(1-\mu_i-\mu_j)^{-1}$. Inserting this in the above we get:

(surface$\rightarrow$) $\mu_k'+\mu_l'+\mu_m' = \mu_0+\mu_i+\mu_j$ ($\leftarrow$ in 3-fold)

These equations yield

$$\mu_0+\mu_i = \mu_{0i}'$$
$$\mu_j = \mu_j'$$
$$\mu_k = \mu_k'$$
$$\mu_l = \mu_l'$$
$$\mu_m = \mu_m' \quad .$$

There are 15 such systems of equations relating the μ_i' to the μ_i. Here, once again, if $\mu_i+\mu_j=1$, $n_{ij}=\infty$ and the ramification divisor H_{ij} must be an abelian variety. This can only occur for the following branching degrees (see (Hö)):



b) In the same way (using Hurwitz) it may be checked that the ramification divisor $\overline{0ij}$ may be blown down if:

either $\mu_k+\mu_l+\mu_m<1$ or $\mu_o+\mu_i+\mu_j<1$.

We need ten such triples to get a compact quotient. The ramification divisor $\overline{0ij}$ is an abelian variety if:

$\mu_k+\mu_l+\mu_m=1$ and $\mu_0+\mu_i+\mu_j=1$.

Summing up,

<u>Necessary and sufficient conditions for Y to be the compactification of a ball quotient</u>

A sextuple of rational numbers $\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5$ satisfying:

(i) $\Sigma\mu_i=2$

(ii) $(1-\mu_i-\mu_j)^{-1}\in\mathbb{Z}\cup\{\infty\}$

(iii) for all pairs i,j: $(\mu_i+\mu_j),\mu_k,\mu_l,\mu_m,\mu_n$ satisfy

either a) $\mu_i+\mu_j=1$
or b) $(1-\mu_{\lambda_1}-\mu_{\lambda_2})^{-1}\in\mathbb{Z}$, $\lambda_i\in\{ij,k,l,m,n\}$

(iv) for each pair $i,j\in\{1,\dots 5\}$, $\mu_i,\mu_j,\mu_k,\mu_l,\mu_m,\mu_0$ fulfill

either a) $\mu_i+\mu_j+\mu_0<1$ or $\mu_k+\mu_l+\mu_m<1$
or b) $\mu_i+\mu_j+\mu_0=1$ and $\mu_k+\mu_l+\mu_m=1$.

It can be verified that these are the integrality conditions given in [DM]. There are 7 sextuples listed in [DM], §14, satisfying the above. They are:

1) $\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3}$
2) $\frac{1}{2},\frac{1}{2},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}$
3) $\frac{2}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}$
4) $\frac{1}{2},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},\frac{1}{6}$
5) $\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{3}{8},\frac{1}{8}$
6) $\frac{5}{12},\frac{5}{12},\frac{5}{12},\frac{1}{4},\frac{1}{4},\frac{1}{4}$
7) $\frac{7}{12},\frac{5}{12},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}$

In the table below we have listed the resulting branching degrees, as well as the number in the 2-dimensional list in [DM] for each $\overline{H}_{ij}$.

Remarks: 1) The example #6 is somewhat strange. Here the ramification divisors $\overline{024}$, $\overline{025}$ and $\overline{045}$ blow down in the other direction than that of the blow up. This is possible since each line meets 2 actual singular points.

2) The examples #1 and #4 may be realised as Fermat covers:

#1. Arrangement $A_1^3(10)$, $n=3$

From our formula in §2.2 we have

$$c_1^3(Y)=-172\cdot 3^6,\quad c_1c_2(Y)=-72\cdot 3^6,\quad c_3(Y)=-12\cdot 3^6.$$

Since each line meets 2 actual singular points, the self-intersection is

$$((L_m)|L_m)^2=(2-2\Sigma\sigma_{jm})=-2.$$

The ramification divisors over the exceptional $\mathbb{P}^1\times\mathbb{P}^1$'s are abelian varieties, and letting D denote thier union,

$$c_1c_2(Y)/c_3(Y)=\overline{c}_1\overline{c}_2(Y,D)/\overline{c}_3(Y,D)=6$$

$$3\overline{c}_1^3(Y,D)-8\overline{c}_1(Y,D)\overline{c}_2(Y,D)=3c_1^3(Y)-8c_1c_2(Y)-3(D_{|D})^2$$
$$=60+3\cdot 10\cdot(-2)$$
$$=0,$$

	01	02	03	04	05	12	13	14	15	23	24	25	34	35	45	012	013	014	015	023	024	025	034	035	045
#2	4 #3	4 #3	4 #3	4 #3	∞ AB	2 #2	2 #2	2 #2	4 #3	2 #2	2 #2	4 #3	2 #2	4 #3	4 #3	∞	∞	∞	-4	∞	∞	-4	∞	-4	-4
#3	∞ AB	∞ AB	∞ AB	∞ AB	∞ AB	2 #3	2 #3	2 #3	2 #3	2 #3	2 #3	2 #3	2 #3	2 #3	2 #3	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
#1	3 #1	3 #1	3 #1	3 #1	3 #1	3 #1	3 #1	3 #1	3 #1	3 #1	3 #1	3 #1	3 #1	3 #1	3 #1	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
#4	6 #5	6 #5	6 #5	6 #5	3 #1	3 #8	3 #8	3 #8	2 #7	3 #8	3 #8	2 #7	3 #8	2 #7	2 #7	-6	-6	-6	∞	-6	-6	∞	-6	∞	∞
#5	4 #10	4 #10	4 #10	4 #10	2 #9	4 #10	4 #10	4 #10	2 #9	4 #10	4 #10	2 #9	4 #10	2 #9	2 #9	-8	-8	-8	-8	-8	-8	-8	-8	-8	-8
#6	6 #16	3 #20	6 #16	3 #20	3 #20	3 #20	6 #16	3 #20	3 #20	3 #20	2 #22	2 #22	3 #20	3 #20	2 #22	-12	-4	-12	-12	-12	12 *	12 *	-12	-12	12 *
#7	6 #16	6 #16	6 #16	6 #16	∞ AB	2 #21	2 #21	2 #21	3 #17	2 #21	2 #21	3 #17	2 #21	3 #17	3 #17	-12	-12	-12	-4	-12	-12	-4	-12	-4	-4

AB means the cover of H_{ij} is an abelian variety.

∞ means the cover of an exceptional $\mathbb{P}^1 \times \mathbb{P}^1$ is an abelian variety

* these exceptional divisors blow down in the other direction than that of the blow-up

so by the Kobayashi-Yau inequality, Y is a compactification of a ball quotient.

\# 4: Arrangement $A_1^3(12)$, n=3

In this case we have $((L_m)|_{L_m})^2=-4$,

$$c_1^3(Y)=-896\cdot 3^8;\quad c_1c_2(Y)=-360\cdot 3^8,\quad c_3(Y)=-60\cdot 3^8.$$

Once again the ramification divisors covering the exceptional $\mathbb{P}^1\times\mathbb{P}^1$'s are abelian varieties, and letting D denote thier union,

$$c_1c_2(Y)/c_3(Y) = \bar{c}_1(Y,D)\bar{c}_2(Y,D)/\bar{c}_3(Y,D) = 6$$

$$3\bar{c}_1^3(Y,D)-8\bar{c}_1(Y,D)\bar{c}_2(Y,D) = 192 + 3(16)(-4)=0.$$

In a manner similar to [Hö], 5.2.3., it is easily verified that this is # 4 in the [DM]-list.

The rest of this section is devoted to proving the following:

Theorem 4.6.2.: Besides the seven examples of Deligne-Mostow there is precisely one covering of type (F) branched along one of the arrangements of planes listed in 2.1., $Y \to \hat{\mathbb{P}}^3$, such that Y is a compactification of a ball quotient.

We start with the

Lemma 4.6.3.: The arrangement L such that $Y \to \hat{\mathbb{P}}^3$ branched along L is a compactification of a ball quotient must fulfill:

$t_q(1)=0$ for $q>4$ (assuming each such line meets ≥ 4 other branch divisors)

$t_4(1)\neq 0 \Rightarrow$ all 4-fold lines meet ≤ 4 other branch div.

Proof: reasoning in the same way as in the proof of 4.6.1. the coverings of the excetional $\mathbb{P}^1\times\mathbb{P}^1$'s must be either exceptional and blow down or be abelian varieties. Niether is possible for q-fold lines unless q=3 or 4. To list the possibilities we introduce the following notation: $n_1,\ldots,n_q$ are the branching degrees of the q planes passing through the q-fold line, τ=number of branch divisors the line meets transversally, $m_1,\ldots,m_\tau$ these branching degrees. The possibilities are listed below.

q=3	$\tau\neq 3$: n_1	n_2	n_3	n_{123}	$\tau=3$	m_1	m_2	m_3	n_{123}
	2	2	n	-2n		2	2	n	-2n
	2	3	3	-12		2	3	3	-12
	2	3	4	-24		2	3	4	-24
	2	3	5	-60		2	3	5	-60
	2	3	6	∞		2	3	6	∞
	2	4	4	∞		2	4	4	∞
	3	3	3	∞		3	3	3	∞

q=4		n_1	n_2	n_3	n_4	m_1	m_2	m_3	m_4	n_{1234}
	τ=4	2	2	2	2	2	2	2	2	∞
	τ=3	2	2	2	2	2	3	6		∞
						2	4	4		∞
						3	3	3		∞
			anything			2	2	n		-2n
						2	3	3		-12
						2	3	4		-24
						2	3	5		-60

Applying this Lemma we can eliminate all arrangements of 2.1. except

1. Ceva arrangements
2. $A_1^3(17)$, $A_1^3(18)$, $A_2^3(15)$ and $A_1^3(24)$.

All coverings of type (F) over Ceva arrangements can be derived from the Deligne-Mostow examples, so we need not consider these. We further can exclude $A_1^3(17)$, $A_1^3(18)$ and $A_2^3(15)$ by the following.

<u>Lemma 4.6.4.:</u> Let $A_2(10)$ be the simplicial line arrangement in Grünbaum's list. There is no covering of type (F) over $A_2(10)$ $S \to \mathbb{P}^2$, such that S is a compactification of a ball quotient.

Indeed, assuming this lemma, we may exclude the 3 plane arrangements $A_1^3(17)$, $A_1^3(18)$ and $A_2^3(15)$ by 4.4.1., since in each there are induced line arrangements of type $A_2(10)$.

Proof: We use the methods developed in {Hö}. The matrix Q" ({Hö}, p.47) for the arrangement is

Matrix Q"

$$\begin{array}{cccccccccc}
2 & . & . & 1 & . & . & . & . & . & 1 \\
. & 2 & . & . & 1 & . & . & . & . & 1 \\
. & . & 2 & . & . & 1 & . & . & . & 1 \\
1 & . & . & 2 & . & . & . & . & . & . \\
. & 1 & . & . & 2 & . & . & . & . & . \\
. & . & 1 & . & . & 2 & . & . & . & . \\
. & . & . & . & . & . & 2 & . & . & . \\
. & . & . & . & . & . & . & 2 & . & . \\
. & . & . & . & . & . & . & . & 2 & . \\
1 & 1 & 1 & . & . & . & . & . & . & 2
\end{array}$$

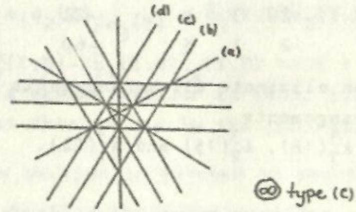
and the eigenvectors are

16	16	16	1	1	1	6	6	6	-15
0	0	0	2	2	2	1	1	1	2,

from which it follows that there are no weights yielding a covering which is a (compactification of a) ball quotient.

We now concentrate on the arrangement $A_1^3(24)$ in 2.1. Recall that the line arrangement induced in each of the 24 planes is $A_2^2(13)$. By 4.4.1. we need a covering of type (F) branched along $A_2^2(13)$ which is a ball quotient. This is provided by

Lemma 4.6.5.: A covering of type (F) of $\mathbb{P}^2$ branched along $A_2^2(13)$ is a compactification of a ball quotient if the branching degrees are as below. The existence of such a covering is provided by Kobayashis theorem 4.5.



Proof: We use (a slight modification of) Höfers formula (Hö), 3.2.3. For the exceptional $\mathbb{P}^1$s we have

4-fold points	2	2	∞	∞	2
	2	4	4	∞	2
3-fold points	2	2	2		-4

Considering the 4 types of lines, the following equations prove the lemma:

(I) type (a) $6\,\frac{1}{2} + \frac{6}{4} - \frac{6}{4} - \frac{6}{2} = 0$

(II) type (b) $6 - 4\,\frac{3}{2} = 0$

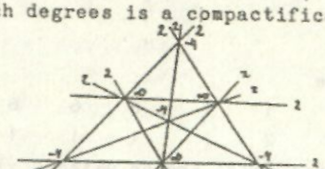
(III) type (c) $4\,\frac{3}{4} + \frac{3}{2} - \frac{9}{2} = 0$

(IV) type (d) same as (I).

The line at infinity is of type (c).

We now consider the induced arrangement in the exceptional $\mathbb{P}^2$s of the 9-fold points of $A_1^3(24)$. The induced arrangement is $A_1^2(9)=\mathrm{Ceva}^2(2,3)$

Lemma 4.6.6.: The covering of type (F) of $A_1(9)$ with the following branch degrees is a compactification of a ball quotient.



Proof: By remark 5.2.3. in (Hö) it is easy to see that this is the ball quotient #3 in the list in (DM).

Using Corollary 4.2.5. (together with the proof of 4.6.5.) and Kobayashi s theorem 4.5.1., the final step in the proof of Theorem 4.6.2. is provided by the following, easy to prove

<u>Lemma 4.6.7.</u>: Consider the arrangement $A_1^3(24)$, and let the branching degrees be as below. Then the branching degrees in the branch divisors are those listed in 4.6.5. and 4.6.6. The covers of the exceptional $\mathbb{P}^1 \times \mathbb{P}^1$'s corresponding to the 4-fold lines are abelian varieties; those corresponding to the 3-fold lines are ruled surfaces which are exceptional and may be blown down.

branching degree	divisor
2	24 planes
2	exceptional $\mathbb{P}^2$'s
-4	3-fold lines
∞	4-fold lines

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