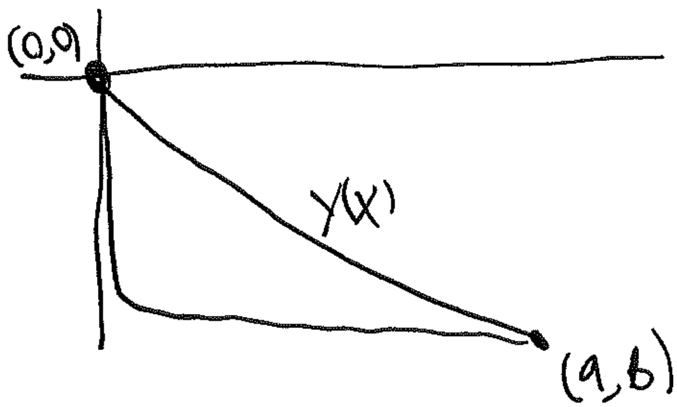
The brachistochrone



last time: the amount of time it takes

is
$$F(y) = \int_0^a \frac{\sqrt{1+y'(x)^2}}{\sqrt{2g\,y(x)}}\,dx$$

want to find y satisfies $y(0)=0$, $y(a)=b$

which minizes F ("functional")

The idea:

To minimize $f(x,y,z)$:

Find a point that make

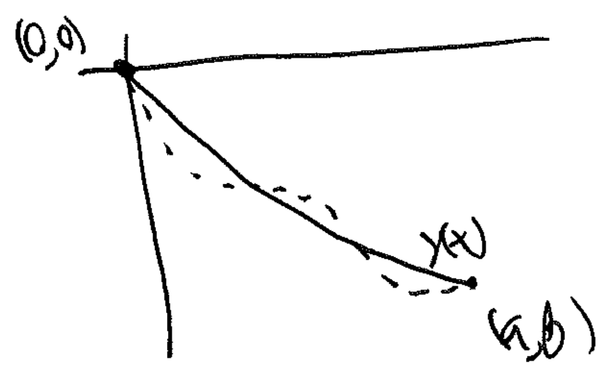
$$\frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = \frac{\partial f}{\partial z}(P) = 0.$$

↑

"moving in x-direction doesn't make things better".

To minimize a functional $F(f)$ ← find f that can't be improved.

We want: no way to perturb f and make F get smaller.



one way to perturb: add $\epsilon \cdot \sin\left(\frac{2\pi}{a}x\right)$

↑

need $\frac{2\pi}{a}x$ to make sure our perturbed fct still goes through (a,b)

look at $y(x) + \epsilon \sin\left(\frac{2\pi}{a}x\right)$ is it better?

We could also try perturbing

$$y(x) + \epsilon \underbrace{x(x-a)}_{0 \text{ at } x=0 \text{ and } x=a}$$

In fact, [CoCalc break]

we found.
Perturbing a line usually makes it better.

We can perturb $y(x)$ not just by $\sin(\frac{2\pi}{a}x)$ or $x(x-a)$,

but by any function v with $v(0)=v(a)=0$

The variation of functional F in direction v is:

$$\delta F(y;v) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(F(y+\epsilon v) - F(y)\right) = \frac{d}{d\epsilon} F(y+\epsilon v)\Big|_{\epsilon=0}$$

If $\delta F(y;v)$ not 0, perturbing y by v improves it!

If a function $y(x)$ is minimizer of F,

then

$$\delta F(y;v) = 0 \text{ for } \underline{\text{all}}\ v \text{ with } v(0)=0 \text{ and } v(a)=0.$$

(But how to find y?)
(It's hard in general!)

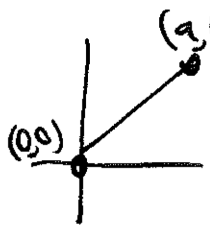
<u>Euler-Lagrange equations.</u>

Suppose that the functional F looks like

$$F(y) = \int_0^a f(x, y, y')\,dx. \quad \text{(common in physics!)}$$

e.g. for brachistochrone: $f(x,y,y')$

$$F(y) = \int_0^a \frac{\sqrt{1+y'(x)^2}}{\sqrt{2g\,y(x)}}\,dx$$



for Shortest path $\quad F(y) = \int_0^a \sqrt{1+y'(x)^2}\,dx$

Suppose

$$F(y) = \int_0^a f(x, y, y')\,dx$$

$$\delta F(y;v) = \frac{d}{d\epsilon}\Big(F(y+\epsilon v)\Big)\Big|_{\epsilon=0}$$

$$= \frac{d}{d\epsilon}\int_0^a f(x,\ y(x)+\epsilon v(x),\ y'(x)+\epsilon v'(x))\,dx \ \Big|_{\epsilon=0}$$

$$= \int_0^a \frac{d}{d\epsilon} f(x,\ y(x)+\epsilon v(x),\ y'(x)+\epsilon v'(x))\,dx \ \Big|_{\epsilon=0}$$

$$= \int_0^a \frac{\partial}{\partial y} f(x, y(x), y'(x))\, v(x)\,dx + \int_0^a \frac{\partial}{\partial y'} f(x, y(x), y'(x))\, v'(x)\,dx$$

$$= \int_0^a \frac{\partial}{\partial y} f(x, y(x), y'(x))\, v(x)\,dx - \int_0^a \frac{d}{dx}\frac{\partial}{\partial y'} f(x, y(x), y'(x))\, v(x)\,dx$$

$$= \int_0^a v(x)\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right)dx$$

If $y(x)$ is optimal, then this integral must be 0

for <u>any</u> v. $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}$ must be 0!

If y optimizes the functional

$$F(y) = \int_0^a f(x, y, y')\, dx$$

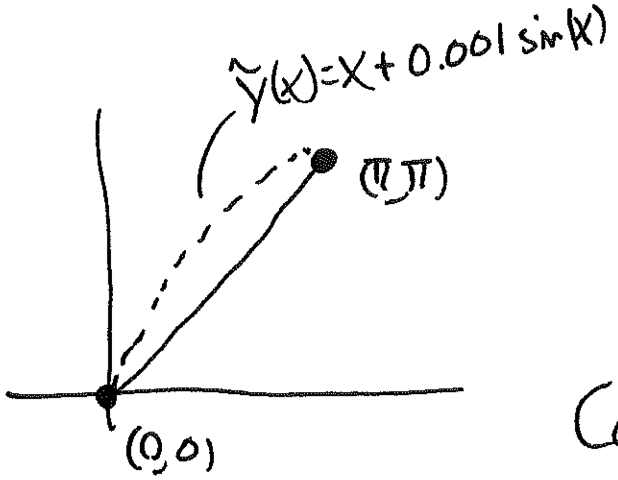
then

$$\frac{\partial}{\partial y} f(x, y, y') = \frac{d}{dx}\frac{\partial}{\partial y'} f(x, y, y').$$

i.e. $$\boxed{\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}.}$$

Euler-Lagrange eqn.

Shortest path.



Consider the initial path

$y(x) = x.$

Could perturbing this by $\sin(x)$ give a shorter path?

Compute

$\delta F(y; \sin x)$ where $f(x, y, y') = \sqrt{1 + y'(x)^2}$

$$\frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+(y')^2}}$$

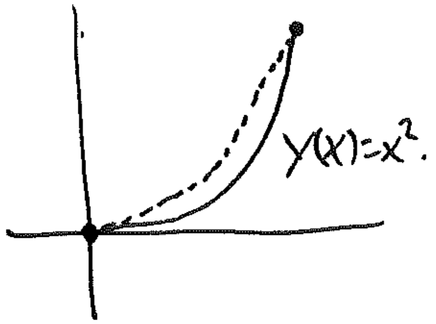
"$y' = z$)

$$\frac{d}{dz}\sqrt{1+z^2} = \frac{2z}{2\sqrt{1+z^2}}$$

$$\frac{d}{dx}\frac{y'}{\sqrt{1+(y')^2}}$$

if $y(x) = x$, this is just $\frac{d}{dx}\frac{1}{\sqrt{2}} = 0.$

so $\delta F(y; \sin x) = \int_0^a \sin x \cdot 0 \, dx = 0$



can we make path shorted by adding $\epsilon \cdot \sin(\pi x)$.

is $\tilde{y}(x) = x^2 + \epsilon \sin(\pi x)$ a shorter or longer path?

we should get:

$\delta F(y; \sin(\pi x)) < 0$ length should get shorter.

$f(x, y, y') = \sqrt{1 + y'(x)^2}$

If $y = x^2$

$\frac{\partial f}{\partial y} = 0 \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+(y')^2}}$

$= \frac{2x}{\sqrt{1+(2x)^2}} = \frac{2x}{\sqrt{1+4x^2}}$

$\frac{d}{dx}\frac{\partial f}{\partial y'} = -\frac{8x^2}{(1+4x^2)^{3/2}} + \frac{2}{(1+4x^2)^{1/2}}$

$\delta F(y; \sin(\pi x)) = \int_0^1 \left(-\frac{8x^2}{(1+4x^2)^{3/2}} + \frac{2}{(1+4x^2)^{1/2}}\right)(\sin \pi x)\, dx$

Shortest path

$$f(x, y', y) = \sqrt{1+y'(x)^2}$$

We want to find y so

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0.$$

$$\frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+(y')^2}}$$

So the shortest path is given by a function $y(x)$ satisfying

$$\frac{d}{dx}\frac{y'(x)}{\sqrt{1+y'(x)^2}} = 0$$

$$(y')^2 = c^2 + (y')^2 c^2$$

$$\frac{y'(x)}{\sqrt{1+y'(x)^2}} = c \qquad \frac{(y')^2}{1+(y')^2} = c^2$$

$$(1-c^2)(y')^2 = c^2$$

$$(y')^2 = \frac{c^2}{1-c^2}$$

$$y = dx + e \text{ is a line} \longleftarrow y' = \sqrt{\frac{c^2}{1-c^2}}$$

Brachistochrone

$$F(y) = \int_0^a \underbrace{\sqrt{\frac{1+(y')^2}{2gy}}}_{f}\, dx .$$

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0}$$

$$f(x, y, y') = \sqrt{\frac{1+(y')^2}{2gy}}$$

$y^{-1/2}$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\sqrt{\frac{1+(y')^2}{2g}}\,\frac{1}{\sqrt{y}} = -\frac{1}{2}\sqrt{\frac{1+(y')^2}{2g}}\,\frac{1}{y^{3/2}}$$

$$\frac{\partial f}{\partial y'} = \frac{\partial}{\partial y}\,\frac{1}{\sqrt{2gy}}\sqrt{1+(y')^2} = \frac{1}{\sqrt{2gy}}\,\frac{y'}{\sqrt{1+(y')^2}}.$$

So

$$-\frac{1}{2}\sqrt{\frac{1+(y')^2}{y^3}} = \frac{d}{dx}\,\frac{y'}{\sqrt{y(1+(y')^2)}}$$

$y(x)$ solving this diff eq is solution to brachistochrone problem!

Answer

Easier to write parametrically

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C \begin{pmatrix} t - \frac{1}{2}\sin(2t) \\ \frac{1}{2} - \frac{1}{2}\cos(2t) \end{pmatrix}$$