Taylor Series

Reminder:

Find a cubic polynomial $g(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

that agrees with $f(x) = \sin x$ "as closely as possible":

1) $g(0) = f(0)$

2) $g'(0) = \cancel{g} f'(0)$ $\quad g'(x) = a_1 + 2a_2x + 3a_3x^2$

   $f'(x) = \cos x$

3) $g''(0) = f''(0)$

4) $g'''(0) = f'''(0)$ $\quad g''(x) = \cancel{a}\ 2a_2 + 6a_3x$

   $f''(x) = -\sin x$

1). gives us $a_0 = 0$ $\quad g'''(x) = 6a_3, \quad f'''(x) = -\cos x$

2) gives $a_1 = 1$

4) gives $a_3 = -\frac{1}{6}$

3) gives $a_2 = 0$

$$\boxed{g(x) = x - \frac{x^3}{6}}$$

Deriving the general formula

We have a function $f(x)$

We want to a power series ("infinite degree polynomial")

that has $g(0), g'(0), g''(0), g'''(0), g^{(4)}(0), \ldots$

all the same as f does.

This should be a good approximation of f, at least near 0.

Let's just solve for it, one coefficient at a time:

Suppose $g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$

We want $f^{(i)}(0) = g^{(i)}(0)$ for every i.

Well, $g^{(4)}(x) = 4 \cdot 3 \cdot 2 \cdot 1 a_4 x^0 + 5 \cdot 4 \cdot 3 \cdot 2 a_5 x^1 + \cdots$

$g^{(i)}(x) = (i!) a_i x^0 +$ (terms with x in them)

$g^{(i)}(0) = (i!) a_i$

We want $g^{(i)}(0) = f^{(i)}(0)$

so $(i!)a_i = f^{(i)}(0)$

$$a_i = \frac{f^{(i)}(0)}{i!}$$

So:

$$g(x) = f(0) + \underbrace{f'(0)\cdot x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \dots}_{\text{Taylor series for } f(x)}$$

$$\boxed{g(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}$$

As you include more and more terms, you get a better and better approximation of your function.

Maclaurin series:

$$g(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Examples

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$* \quad \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 \dots$$

↑ plug in "$-x^2$"

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

$\cos(x) + i\sin(x)$

$\arctan(x)$ / $\tan^{-1}(x)$

$f(x) =$

$1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + 21x^7 + 34x^8 + \dots$

↑

What is it?

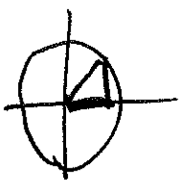
$\tan^{-1}(x)$?

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots$$

$$\tan^{-1}(x) = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

plug in $x=0 \longrightarrow C=0.$

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$



$$= \tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} = \pi/4$$

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \frac{4}{13} - \cdots$$

very slow formula for π!

$\cos(x) + i\sin(x)$

$=$

$$\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) \cancel{+ i\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots\right)}$$
$$+ i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{i}{3!}x^3 + \frac{1}{4!}x^4 + \frac{i}{5!}x^5 - \frac{1}{6!}x^6 - \frac{i}{7!}x^7 + \cdots$$

$$= 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!}$$

$$= e^{ix}$$

$$\boxed{e^{ix} = \cos x + i \sin x}$$

$$f(x) = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + 21x^7 + \cdots$$

"generating function of Fibonacci numbers"

$$x f(x) = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + 21x^8 + \cdots$$

$$f(x) + x f(x) = 1 + 2x + 3x^2 + 5x^3 + 8x^4 + 13x^5 + 21x^6 + \cdots$$

$$= \frac{f(x) - 1}{x}$$

$$f(x) + x f(x) = \frac{f(x) - 1}{x}$$

$$x f(x) + x^2 f(x) = f(x) - 1$$

"generating function for Fibonacci numbers"

$$1 = (1 - x - x^2) f(x)$$

$$\boxed{f(x) = \frac{1}{1 - x - x^2}}$$

Warning! Those series don't converge

for all x values.

ex

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$$

This gives a good approximation of f

only in the range where the series converges!

converges for $-1 < x < 1$.

What about

~~tan(x)=~~ $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$$

Ratio test:

If $a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$
is an infinite series and

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$ exists and is less than 1,

then the series converges.

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$$ for what values of x does this converge?

use ratio:

$$a_n = (-1)^{n+1}\frac{x^n}{n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+2}\frac{x^{n+1}}{n+1}}{(-1)^{n+1}\frac{x^n}{n}}\right| = \frac{n}{n+1}|x|.$$

$$1 - \frac{1}{n+1}$$

$$= \lim_{n\to\infty}\frac{n}{n+1}|x| = |x|.$$

So $|x| < 1$ it converges.

One more use for Taylor series.

Suppose you wanted to solve a differential equation. Often (usually!) you can't really solve for $f(x)$ exactly. But you can find a Taylor series:

$$y'' - x^2 y = e^x \quad (\text{find } y(x))$$

Easier example: $y' = 2y$ (you know the answer: e^{2x})

$y(0) = 1$

Suppose $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots$$

$$2y = 2a_0 + 2a_1 x + 2a_2 x^2 + 2a_3 x^3 + 2a_4 x^4 + \cdots$$

$y(0) = 1 \rightsquigarrow a_0 = 1$

$a_1 = 2a_0$, so $a_1 = 2$.

$2a_2 = 2a_1$, so $a_2 = 2$

$3a_3 = 2a_2$, so $a_3 = 4/3$

$4a_4 = 2a_3$, so $a_4 = 2/3$

$5a_5 = 2a_4$, so $a_5 = 4/15$

$a_0 = 1$

$a_1 = 2$

$a_2 = 2$

$a_3 = 4/3$

$a_4 = 2/3$

$a_5 = 4/15$

$\rightsquigarrow y = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4 + \frac{4}{15}x^5 + \cdots$

(Taylor for e^{2x})