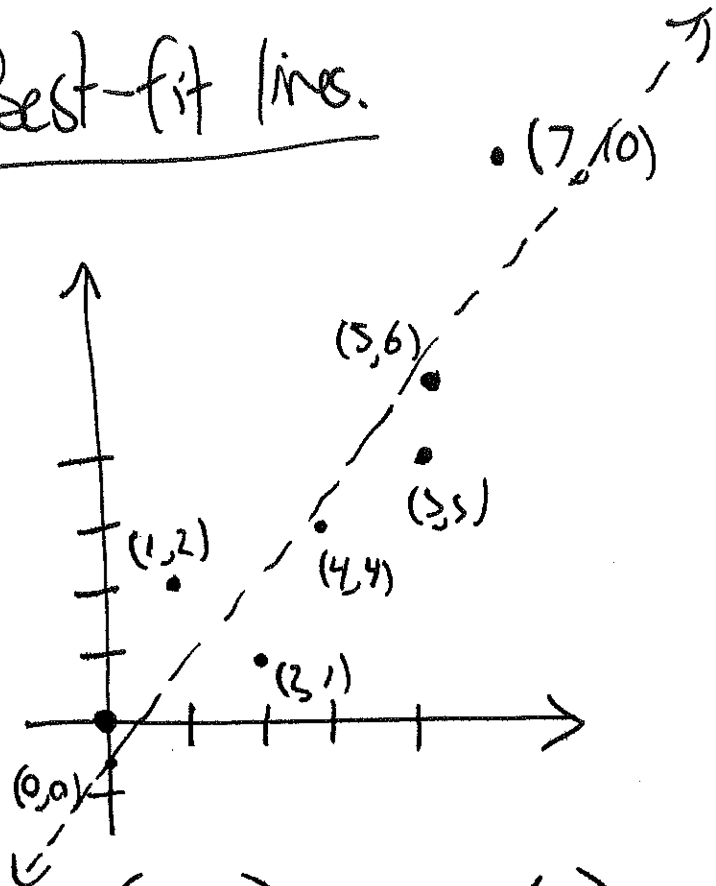
Today: More of the same + Fourier

(experiment with G-S software).

Best-fit lines.



Strategy: $y = mx + b$

To satisfy all equations:

$0 = m0 + b$ $\quad$ $5 = m5 + b$

$2 = m1 + b$ $\quad$ $6 = m5 + b$

$1 = m2 + b$ $\quad$ $10 = m7 + b$

$4 = m4 + b$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \\ 5 & 1 \\ 7 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 4 \\ 5 \\ 6 \\ 10 \end{pmatrix}$$

"$Ax = y$"

try to find closest thing to this so there is a solution

Use Gram-Schmidt!

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \\ 5 \\ 5 \\ 7 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{project} \begin{pmatrix} 0 \\ 2 \\ 1 \\ 4 \\ 5 \\ 6 \\ 10 \end{pmatrix}$$

computer says projection is $\begin{pmatrix} -5/11 \\ 223/264 \\ \vdots \\ 2201/264 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 5 & 1 \\ 5 & 1 \\ 7 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -5/11 \\ 223/264 \\ \vdots \\ 2201/264 \end{pmatrix}$$

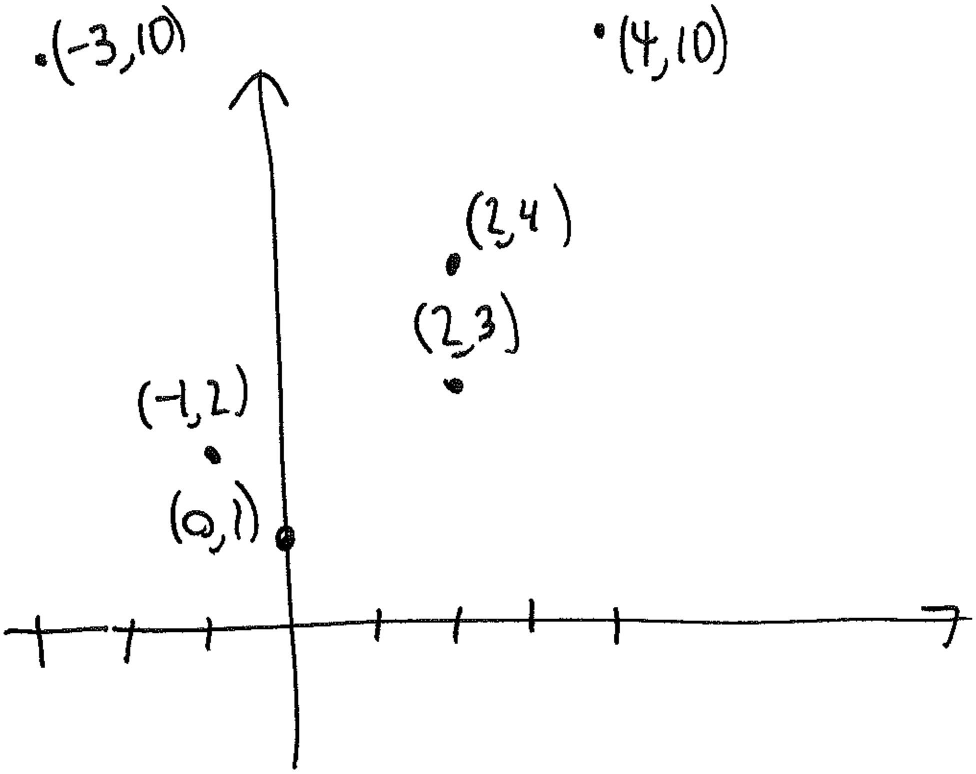
this has a solution!
how to find it?

Eqn 1: $b = -5/11$

Eqn 2: $m + b = 223/264 \Longrightarrow m = 223/264 + 5/11 = 343/264$

1.3
so

Find quadratic through:



$$y = ax^2 + bx + c$$

Dream equations:

$$10 = 9a - 3b + c$$

$$2 = 1a - 1b + c$$

$$1 = 0a + 0b + c$$

$$3 = 4a + 2b + c$$

$$4 = 4a + 2b + c$$

$$10 = 16a + 4b + c$$

$$\begin{pmatrix} 9 & -3 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ 4 & 2 & 1 \\ 4 & 2 & 1 \\ 16 & 4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 10 \\ 2 \\ 1 \\ 3 \\ 4 \\ 10 \end{pmatrix}$$

NO SOLUTIONS!

← replace with projection $\begin{pmatrix} 5663/587 \\ \vdots \\ 12229/1174 \end{pmatrix}$

use Gram-Schmidt on these (computer)

Fourier series

V = all periodic functions with period 2π

A basis* for V is

1
$\cos x$
$\sin x$
$\cos(2x)$
$\sin(2x)$
$\cos(3x)$
$\sin(3x)$
$\cos(4x)$
$\sin(4x)$
$\dots$

Given a periodic function f, ~~how to~~ we can write it in terms of those?

Are those an orthonormal basis?

Use inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, g(x)\, dx$$

Check a few:

$$\langle \cos(x), \cos(x) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 x \, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1+\cos(2x)}{2} \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2} \, dx + \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(2x) \, dx = \frac{1}{\pi}\cdot\frac{1}{2}(2\pi) + \frac{1}{\pi}\left.\frac{\sin(2x)}{2}\right|_{-\pi}^{\pi}$$

✓

$$= 1 + 0 = 1$$

✓ $$\langle \cos(x), \sin(3x) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos x \sin 3x \, dx = 0$$

$$\langle 1, 1 \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} 1 \cdot 1 \, dx = \frac{1}{\pi}(2\pi) = 2.$$

So use $\frac{1}{\sqrt{2}}$ in your basis.

Everything else works!

$\frac{1}{\sqrt{2}}$, $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cos 3x$, $\sin 3x$

is <u>orthonormal</u>.

To write a random function f as a combination of those:

Warm-up: how to write (a, b) as combo of $(1,0)$ and $(0,1)$

$$\underbrace{\big((a,b)\cdot(1,0)\big)}_{1^{st}\text{ coef}}(1,0)+\underbrace{\big((a,b)\cdot(0,1)\big)}_{2^{nd}\text{ coef}}\cdot(0,1)$$

$$f=\left\langle f, \tfrac{1}{\sqrt{2}}\right\rangle \tfrac{1}{\sqrt{2}}+\langle f, \cos x\rangle \cos x+\langle f, \sin x\rangle \sin x$$

$$+\langle f, \cos 2x\rangle \cos 2x, \langle f, \sin 2x\rangle \sin 2x + \cdots$$