More quiz

More projection formula

Find the point of the plane $x+2y+3z=0$ $\Big]$ subspace W

closest to $v=(2,4,-3)$.

Steps.

1) Find a basis (maybe not orthonormal)

2) Make it orthonormal* (Gram-Schmidt process)

   e_1, e_2

3) Find closest point: use projection

$$\hat{v} = \langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2$$

Gram-Schmidt

Given $v_1, v_2, \ldots, v_n$ a basis.

Set $e_1 = v_1 / \|v_1\| = v_1 / \sqrt{\langle v_1, v_1 \rangle}$ (make first vector length 1)

$$f_2 = v_2 - \underbrace{\langle v_2, e_1 \rangle e_1}_{\text{part of } v_2 \text{ parallel to } e_1}$$ (make 2^{nd} perp. first)

$$e_2 = f_2 / \|f_2\|$$ (make 2^{nd} length 1)

$$f_3 = v_3 - \langle v_3, e_1 \rangle e_1 - \langle v_3, e_2 \rangle e_2$$

$$e_3 = f_3 / \|f_3\|$$

$$\ldots$$

Basis: $v_1=(1,1,-1)$
$v_2=(-2,1,0)$.

$$e_1 = v_1/\|v_1\| = \left(1/\sqrt{3},\ 1/\sqrt{3},\ -1/\sqrt{3}\right)$$

$$f_2 = v_2 - \langle v_2, e_1\rangle e_1$$

$$= (-2,1,0) - \left\langle (-2,1,0),\ \left(1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3}\right)\right\rangle \left(1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3}\right)$$

$$= (-2,1,0) + 1/\sqrt{3}\left(1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3}\right)$$

$$= (-2,1,0) + \left(1/3, 1/3, -1/3\right) = \left(-5/3,\ 4/3,\ -1/3\right)$$

$$e_2 = f_2/\|f_2\| \qquad \|f_2\| = \sqrt{\left(-5/3\right)^2 + \left(4/3\right)^2 + \left(-1/3\right)^2}$$

$$= \frac{\sqrt{42}}{3}$$

$$= \left(-5/\sqrt{42},\ 4/\sqrt{42},\ -1/\sqrt{42}\right)$$

$v = (2, 4, -3)$

Projection

$\hat{v} = \langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2 = \cdots$

$= \left(\frac{27}{14}, \frac{27}{7}, -\frac{45}{14}\right)$

$= \cancel{(3\sqrt{3})}$

Function example

Approximate $\cos(x)$ by a quadratic function on $[-\pi, +\pi]$.

1) Find a basis for quadratic functions: $1, x, x^2$
2) Orthonormalize it

   Gram-Schmidt, $\langle f, g \rangle = \int_{-\pi}^{\pi} fg \, dx$
3) $\langle \cos(x), e_1 \rangle e_1 + \langle \cos(x), e_2 \rangle e_2 + \langle \cos(x), e_3 \rangle e_3$

$v_1 = 1 \qquad v_2 = x \qquad v_3 = x^2$

$$e_1 = v_1 / \|v_1\| \qquad \|v_1\|^2 = \int_{-\pi}^{\pi} 1 \cdot 1 \, dx = 2\pi$$

$$\|v_1\| = \sqrt{2\pi}$$

$$e_2 = \frac{\frac{3}{2}\sqrt{\frac{2}{3}}\, x}{\pi^{3/2}} = \sqrt{\frac{\frac{9}{4} \cdot \frac{2}{3}}{\pi^3}}\, x = \sqrt{\frac{3}{2\pi^3}}\, x$$

$$e_1 = \frac{1}{\sqrt{2\pi}}$$

$$f_2 = v_2 - \langle v_2, e_1 \rangle e_1 \qquad \langle v_2, e_1 \rangle = \int_{-\pi}^{\pi} x \cdot \frac{1}{\sqrt{2\pi}} \, dx = 0$$

$$e_2 = f_2 / \|f_2\| \qquad \|f_2\|^2 = \int_{-\pi}^{\pi} x^2 \, dx = \frac{2\pi^3}{3}$$

so $e_2 = \sqrt{\frac{3}{2\pi^3}}\, x$.

computer says

$$e_3 = -\frac{\frac{5}{4}\sqrt{\frac{2}{5}}}{\pi^{5/2}}(\pi^2 - 3x^2) = \sqrt{\frac{5}{8\pi^5}}\,(3x^2 - \pi^2)$$

$$f_3 = v_3 - \langle v_3, e_1 \rangle e_1 - \langle v_3, e_2 \rangle e_2$$

$$\langle v_3, e_1 \rangle = \int_{-\pi}^{\pi} x^2 \frac{1}{\sqrt{2\pi}}\, dx = \frac{1}{\sqrt{2\pi}} \frac{2\pi^3}{3} \cancel{\sqrt{\tfrac{4}{18}}\,\pi^{5/2}}$$

$$\langle v_3, e_1 \rangle e_1 = \frac{1}{\sqrt{2\pi}} \frac{2\pi^3}{3} \frac{1}{\sqrt{2\pi}} = \frac{\pi^2}{3}$$

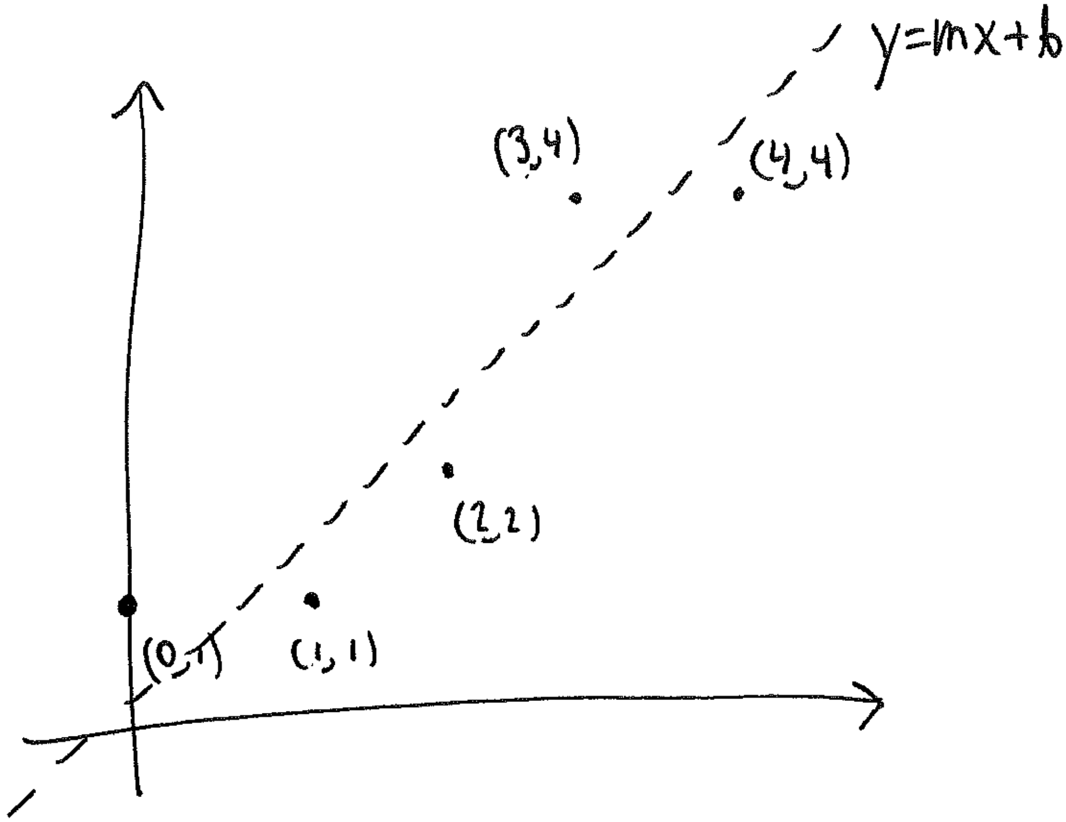
$$\langle v_3, e_2 \rangle = \int_{-\pi}^{\pi} x^2 (\quad x)\, dx = 0$$

$$f_3 = x^2 - \frac{\pi^2}{3}$$

$$\|f_3\|^2 = \int_{-\pi}^{\pi} \left(x^2 - \pi^2/3\right)^2 dx = \dots$$

$$e_3 = f_3 / \|f_3\|$$

Best-fit lines.



Let's solve for m & b to make line go through all points!

$$\begin{aligned} 0m+b&=1 \\ 1m+b&=1 \\ 2m+b&=2 \\ 3m+b&=4 \\ 4m+b&=4 \end{aligned}$$

this has no solutions!

In matrix form:

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \\ 4 \end{pmatrix}$$

x-coords of our pts

all 1's

y-coords of point.

This is an "$Ax=y$" problem: A matrix
y vector
solve for x.

no solutions!

left "
$$\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} m + \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} b$$

the set of y for which this equation has a solution is a plane in 5-D space R^5: it's a subspace!

$Ax=y$ had no solutions.

Let's change y to $\hat{y}$, the closest vector for which there is a solution. That solution $\begin{pmatrix} m \\ b \end{pmatrix}$ gives a line.

Take projection of $\begin{pmatrix} 1 \\ 1 \\ 2 \\ 4 \\ 4 \end{pmatrix}$ onto W.

the set of y for which there is a solution.

Basis for W is $\underbrace{\begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}_{v_2}$

$\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ have a sol?

$\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

for $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, get $\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Want to project $\begin{pmatrix}1\\1\\2\\4\\4\end{pmatrix}$ onto W, which has basis

$$\begin{pmatrix}0\\1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}$$

$$\text{Proj} = \begin{pmatrix}3/5\\3/2\\12/5\\33/10\\21/5\end{pmatrix}.$$

there is a solution here!

if you change y-coords to those, there is a solution.

the distance from to $\begin{pmatrix}1\\1\\2\\4\\4\end{pmatrix}$ is

$$\sqrt{\left(\tfrac{3}{5}-1\right)^2+\left(\tfrac{3}{2}-1\right)^2+\left(\tfrac{12}{5}-2\right)^2+\left(\tfrac{33}{10}-4\right)^2+\left(\tfrac{21}{5}-4\right)^2}$$