Recap:

Vector space: a set of things where it makes sense to add, multiply by numbers.
(ex: $\mathbb{R}^n$, functions)

subspace: a subset of a vector space closed under addition and scaler mult.
(ex: $\{(x,x,x)\} \subset \mathbb{R}^3$, functions with period π.

inner product: a rule for combining two vectors to get a number.

ex: In $\mathbb{R}^n$, $\langle v,w \rangle$ = dot product

In $C^0([a,b])$ continuous fcts $\langle f,g \rangle = \int_a^b fg \, dx$

Idea: this integral rule has all the same properties as dot product!

e.g. in $\mathbb{R}^n$ $\langle v,v \rangle \geq 0$

in $C^0([a,b])$ $\langle f,f \rangle = \int_a^b f^2 \, dx \geq 0$

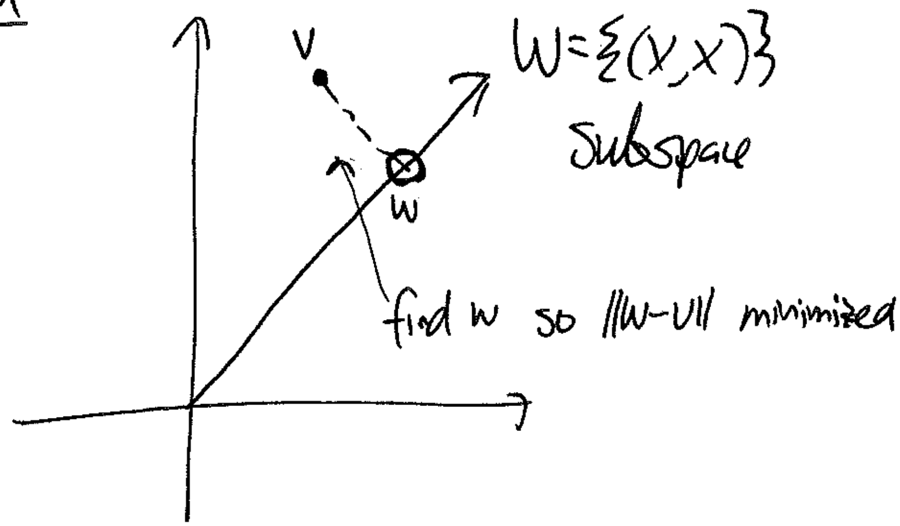
Where we're going.

Given a ~~vector~~ inner product space V and a subspace $W \subset V$.

~~there's~~ Suppose we have a vector $v \in V$ not in W.

We'll find a formula for the point in W "closest" to v.

<u>ex</u> $V = \mathbb{R}^2$



<u>ex</u> $V = C^0([-\pi,\pi])$ continuous functions

$W =$ polynomials of degree ≤ 5.

$v \in V$ could be $\cos(x)$. Formula will find the polynomial p of degree ≤ 5 "closest" to $\cos(x)$

"closest" means best approximation:

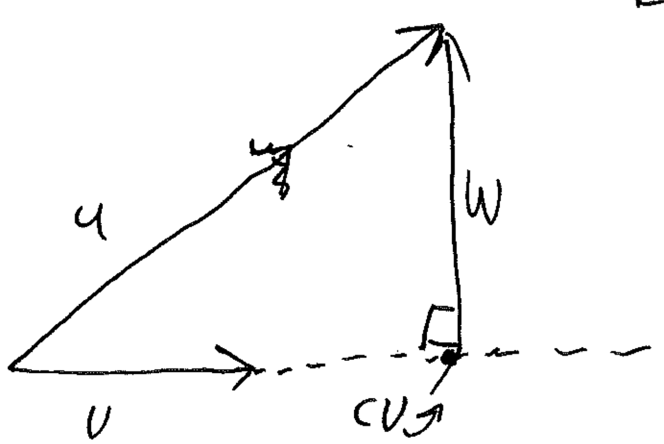
$$\int_{-\pi}^{\pi} (\cos(x) - p(x))^2 \, dx \quad \text{is minimized.}$$

means norm of error $\cos(x) - p$ is minimized: $\displaystyle\int_a^b (\cos(x) - p(x))^2 \, dx$

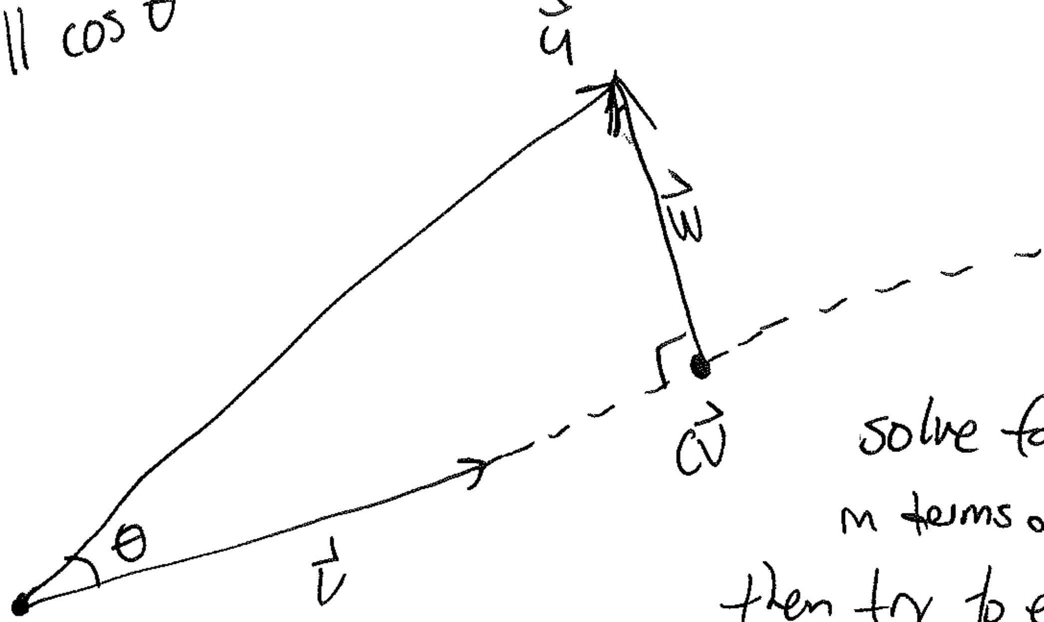
<u>Warm-up:</u> Given vectors u & v, how can we write u as $cv + w$ ← orthogonal to v

c is scalar, so cv parallel to v

[objects sliding down a ramp problem]



$$v \cdot w = \|v\| \, \|w\| \cos\theta$$



solve for $c, \vec{w}$ in terms of θ. then try to express it using only inner product!

$$\cos\theta = \frac{c\|v\|}{\|u\|}$$

$$\text{so} \quad c = \frac{\|u\|}{\|v\|}\cos\theta = \frac{\|u\|}{\|v\|}\,\frac{u \cdot v}{\|u\|\,\|v\|} = \frac{u \cdot v}{\|v\|^2} = \frac{u \cdot v}{v \cdot v}$$

$$\vec{w} = \vec{u} - c\vec{v}.$$

Thm Suppose u, v are vectors in an inner product space. Then we can always write

$$u = cv + w$$

with c a scalar, and w orthogonal to v ($\langle v, w\rangle = 0$)

Use formula

$$c = \frac{\langle u, v\rangle}{\langle v, v\rangle}, \quad w = u - cv.$$

Pf. We need to check that if we use that c value,

~~then $\langle v, w\rangle =$~~

then $\langle v, w\rangle = 0$.

$$\langle v, w\rangle = \langle v, u - cv\rangle = \cancel{\left\langle v, u - \frac{\langle u, v\rangle}{\langle v, v\rangle} v\right\rangle}$$

now use inner product axioms!

$$= \langle v, u\rangle - \langle v, cv\rangle$$

$$= \langle v, u\rangle - c\langle v, v\rangle = \langle v, u\rangle - \frac{\langle u, v\rangle}{\langle v, v\rangle}\langle v, v\rangle$$

$$= \langle v, u\rangle - \langle u, v\rangle = \langle v, u\rangle - \langle v, u\rangle = 0.$$

Take $V = C^0([0,1])$

Let $v = x^2$ and $u = x$. Write $u = cv + w$, where c is a scaler and w is orthogonal to v.

$$c = \frac{\langle u, v\rangle}{\langle v, v\rangle} = \frac{\int_0^1 x^3\,dx}{\int_0^1 x^4\,dx} = \frac{1/4}{1/5} = \frac{5}{4}$$

$$w = u - cv = x - \frac{5}{4}x^2$$

This w is orthogonal to $v = x^2$!

$$\int_0^1 \overset{v}{x^2}\overset{w}{\left(x - \frac{5}{4}x^2\right)}\,dx = 0$$

it works ✓

If $W \subset V$ is a subspace.

a <u>basis</u> for W is a collection of vectors $w_1, \dots, w_r$

such that: 1) Anything in W can be written as a combination.

$$a_1 w_1 + a_2 w_2 + \cdots a_r w_r$$

2) No redundancies: can't write

$$w_i = (\text{combination of the others}).$$

<u>Ex</u> $V = \mathbb{R}^3$, $W = xy\text{-plane} = \{(x, y, 0)\}$.

Basis for W: $w_1 = (1,0,0)$
$w_2 = (0,1,0)$

eg. $(7,-3,0) = 7(1,0,0) + (-3)(0,1,0)$.

$V = \mathbb{R}^3$, $W = \{(x, y, z) : x + y + z = 0\}$

Basis: $w_1 = (1, 0, -1)$

$w_2 = (1, -1, 0)$

Check: can we get

$(10, -7, -3)$

$$3w_1 + 7w_2$$
$$\parallel$$
$$3(1, 0, -1) + 7(1, -1, 0)$$
$$\parallel$$
$$(3, 0, -3) + (7, -7, 0)$$
$$\parallel$$
$$(10, -7, -3).$$

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or $w_1 = (0, 1, -1)$

$w_2 = (-1, 1, 0)$

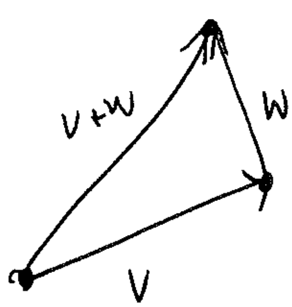
Can we get?

$(10, -7, -3) \leftarrow 3w_1 + (-10)w_2$

Good news: just like orthogonal decomposition formula, many geometric facts about vectors carry over to any inner product space:

1) Pythagorean: if $\langle v,w\rangle = 0$ then

$$\|v+w\|^2 = \|v\|^2 + \|w\|^2$$



2) Cauchy-Schwartz ineq:

For any v, w:

abs val

$$|\langle v,w\rangle| \leq \|v\| \, \|w\|$$

(proof: "boring": try it!)

3) Triangle ineq:

$$\|v\| + \|w\| \geq \|v+w\|$$