Adv Topics 2 (Lesieutre)
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Problem 1. *For each of the following matrices T, choose a couple sample vectors $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and compute $T\mathbf{v}$. What does the matrix do to a vector, geometrically? What does it do to the unit square?*

a) $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$

This increases the x by a factor of 3 and the y be a factor of 2.

b) $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

This is a "shear transformation". It fixes the y coordinate, and points are move either left or right depending on what the y coordinate is. Points above the plane have the x coordinate increase, and move to the right (the higher up they are, the further they move). For points below the plane, it's the reverse.

c) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

This switches the x and y coordinates. That's equivalent to a reflection about the line $y = x$.

d) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

This sends (x, y) to $(x, -y)$. It's a reflection about the x-axis.

e) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

This sends (x, y) to $(x, 0)$. It keeps the same x coordinate, but sets y to 0. This could be called projection onto the x-axis.

f) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

This sends (x, y, z) to $(x, y, 0)$. It's similar to the above; projection onto the xy-plane.

g) $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

This projects onto the xy-plane, and then rotates by 90°.

h) $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

The geometry of this one is a bit obscure; it's on the list because it's the key to finding a formula for the Fibonacci numbers later on.

Problem 2. *For each of the linear maps described, write down the matrix for the corresponding transformation.*

a) *Reflection about the y-axis.*

This one we can use:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

b) *Rotation by $45°$ clockwise. What about other angles θ?*

A good way to do this is to find the image of the two vectors $(1,0)$ and $(0,1)$. Use the first of these as the left column, and the second as the right column. For the rotation, a bit of trigonometry yields:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

In the case $\theta = 45°$, this is just:

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

(If you see a rotation matrix in another source, it's probably more common to give the counterclockwise version. You can just plug in $-\theta$ to get that here. The only change is to move the $-$ sign from the bottom left to the top right).

c) *In 3D: rotation by an angle θ around the z-axis.*

We want to change x and y as above, while leaving z the same. This is achieved by the following matrix:

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 3. *Let A and B be the first two matrices above, and let f and g be the corresponding linear maps.*

a) *Compute the composition $g \circ f$, using the formulas for the functions. What is the matrix for the composition?*

The functions are $f(x,y) = (3x, 2y)$ and $g(x,y) = (x+3y, y)$. So the composition is:

$$(g \circ f)(x,y) = g(f(x,y)) = g(3x, 2y) = (3x + 6y, 2y).$$

The matrix for this is

$$\begin{pmatrix} 3 & 6 \\ 0 & 2 \end{pmatrix}.$$

b) *Compute the product of the matrices BA. Notice anything?*

If we compute the matrix product directly we get:

$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 6 & 2 \end{pmatrix}$$

I do notice something! We got the same matrix!

c) *Compute $f \circ g$ and AB. Do these match your earlier answers?*

Indeed, the same thing happens this way.

Problem 4. *Suppose that a parallelogram has vertices at $(0, 0)$, (a, c), and (b, d). What is its area? (Try to do this using basic geometry. To make life easy, you can assume (a, c) is in the first quadrant and (b, d) is in the second.)*

Here's how I'm looking at it. The fourth vertex is at $(a + b, c + d)$. There's an enclosing rectangle with vertices at $(b, 0)$, $(b, c + d)$, $(a, c + d)$, and $(a, 0)$. It has area $(a - b)(c + d)$. To get the area of the parallelogram, I can subtract off the areas of four evident triangles:

$$\begin{aligned} A &= (a - b)(c + d) - 2 \cdot (ac)/2 - 2 \cdot (-bd/2) \\ &= (ac - bd) + (ad) - (bc) - (ac) + (bd) = ad - bc. \end{aligned}$$

Problem 5. *One property of a linear map is that it rescales all areas by the same scaling factor.*

a) *For each of the 2×2 maps in the first problem, what is the scaling factor?*

b) *The determinant of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $ad - bc$. Compute the determinants of the 2×2 matrices.*

These two problems have the same answers, so I will just write it once. The scaling factors/determinants are, respectively, 6, 1, -1, -1, 0, X, X, -1. (When a map is between spaces of different dimensions, it doesn't really have a scaling factor; those are the X's. A negative factor means that it scales by that amount, but reverses orientation (i.e. flips images backwards).

c) *Can you guess a formula for the determinant of the product of two matrices?*

Based on the interpretation of determinant as scaling factor and multiplication as composition of functions, we can see that

$$\det(AB) = \det(A)\det(B).$$

Problem 6. *What is the determinant of the* 3×3 *matrix*

$$\begin{pmatrix} 4 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 2 & 2 \end{pmatrix}?$$

Can you guess what the geometric meaning of this might be? (We'll learn another way to compute determinants later on.)

Problem 7. *One application of this is in the formula for change of variables in multiple integrals. Suppose you want to integrate a function* $f(x, y)$ *over a non-rectangular region* S*. Maybe you can parametrize the region by* $x(s, t)$ *and* $y(s, t)$ *where* $a \leq s \leq b$ *and* $c \leq t \leq d$*.*

a) *One case of this would be integrating over a circle. How would you parametrize a circle, using the description above?*

$x(s, t) = s \cos t$ and $y(s, t) = s \sin t$. (In this case, we're thinking of them as being r and θ in polar, but the method works for other shapes too.)

b) *Imagine the rectangle* $a \leq s \leq b$ *and* $c \leq t \leq d$ *is covered with a "mesh" of rectangles of width* ds *and height* dt*. Applying* $x(s, t)$ *and* $y(s, t)$*, we get a mesh covering* S*, but the elements won't be rectangles anymore.*

Imagine a small rectangle based at (s_0, t_0) *with sides of length* ds *and* dt*, parallel to the axes. When you apply* $x(-, -)$ *and* $y(-, -)$*, what happens to this small rectangle? Imagine* ds *and* dt *are so small that the map looks locally linear there. What is the area of the image?*

Well, our original rectangle had four vertices:

$$\begin{aligned} &(s_0, t_0) \\ &(s_0 + ds, t_0) \\ &(s_0, t_0 + dt) \\ &(s_0 + ds, t_0 + dt). \end{aligned}$$

When we apply the map, we will get the four vertices:

$$\begin{aligned} &= (x(s_0, t_0), y(s_0, t_0)) \\ (x(s_0 + ds, t_0), y(s_0 + ds, t_0)) &\approx (x(s_0, t_0), y(s_0, t_0)) + (x_s(s_0, t_0)\, ds, y_s(s_0, t_0)\, ds) \\ (x(s_0, t_0 + dt), y(s_0, t_0 + dt)) &\approx (x(s_0, t_0), y(s_0, t_0)) + (x_t(s_0, t_0)\, dt, y_t(s_0, t_0)\, dt) \\ (x(s_0 + ds, t_0 + dt), y(s_0 + ds, t_0 + dt)) &\approx (x(s_0, t_0), y(s_0, t_0)) + (x_s(s_0, t_0)\, ds, y_s(s_0, t_0)\, dt) \\ &\quad + x_t(s_0, t_0)\, dt, y_t(s_0, t_0)\, dt) \end{aligned}$$

These are the four vertices of a parallelogram. To find the area, notice that it's obtained by applying the matrix:

$$\begin{pmatrix} x_s(s_0, t_0) & x_t(s_0, t_0) \\ y_s(s_0, t_0) & y_t(s_0, t_0) \end{pmatrix}$$

to the vector

$$\begin{pmatrix} ds \\ dt \end{pmatrix}$$

The area is $(x_s y_t - x_t y_s)(s_0, t_0)$ times the original area $ds\,dt$. You might recognize this as the "Jacobian determinant" from the change-of-variables formula.

c) *How would you compute an integral over your area, using this formula?*
Thinking in terms of Riemann sums, I get:

$$\iint_S f(x,y)\,dx\,dy = \iint_{\substack{a\le s\le b\\ c\le t\le d}} f(x(s,t), y(s,t))(x_s y_t - x_t y_s)(s,t)\,ds\,dt$$

d) *I always find this formula confusing. Use it to compute the area of a circle to make sure everything works.*
We want to integrate the function 1 over the circle we parametrized in part (a).

$$\begin{aligned} x_s &= \cos t \\ x_t &= -s\sin t \\ y_s &= \sin t \\ y_t &= s\cos t \end{aligned}$$

So $x_s y_t - x_t y_s = (\cos t)(s\cos t) - (-s\sin t)(\sin t) = s$. Our "area form" at the end is $s\,ds\,dt$. Now we need to integrate:

$$\int_{x=0}^{1}\int_{t=0}^{2\pi} 1\cdot s\,ds\,dt = \pi.$$

Problem 8. *Let's do another integral over a strange region.*

a) *Consider the area S between $y = 0$ and $y = x^2$ satisfying $0 \le x \le 2$. How could you parametrize this region by functions $x(s,t)$ and $y(s,t)$ where s, t range over a rectangular region?*
Take $x(s,t) = s$ and $y(s,t) = s^2 t$, where $0 \le s \le 2$ and $0 \le t \le 1$.

b) *Compute*

$$\iint_S xy\,dA$$

using change of variables.
First we need the Jacobian:

$$x_s y_t - x_t y_s = (1)(s^2) - (0)(2st) = s^2$$

According to our formula, we get

$$\begin{aligned}
\iint_S xy\,dA &= \int_{s=0}^{2}\int_{t=0}^{1}(s\cdot s^2t)s^2\,dt\,ds \\
&= \int_{s=0}^{2}\int_{t=0}^{1} s^5t\,dt\,ds = \cdots \\
&= \frac{16}{3}
\end{aligned}$$

c) *Compute*

$$\iint_S xy\,dA$$

directly, by choosing suitable bounds for the intgral.

For this one,

$$\begin{aligned}
\iint_S xy\,dA &= \int_{x=0}^{2}\int_{y=0}^{x^2} xy\,dy\,dx \\
&= \int_{x=0}^{2}\frac{1}{2}x^5\,dx \\
&= \frac{16}{3}.
\end{aligned}$$

Phew.

Problem 9. *For each of the maps in the first problem, describe the eigenvectors if you can.*

Problem 10. *The map in part (h) has a bit of a different flavor. I don't think it's especially exciting geometrically, but it has another useful property.*

a) *Compute* $T(0,1)$*,* $T^2(0,1)$*,* $T^3(0,1)$*, … until you find a pattern.*

The pattern is that $T^n(0,1) = (F_n, F_{n+1})$ where F_n is the nth Fibonacci number. So if we can find a formula for the powers, we'll get a formula for Fibonacci numbers.

b) *Can you find the eigenvectors and eigenvalues for this map?*

If a vector (x,y) is an eigenvector with eigenvalue λ, then

$$\begin{aligned}
T(x,y) &= \lambda(x,y) \\
(y, x+y) &= (\lambda x, \lambda y)
\end{aligned}$$

This gives two equations: $y = \lambda x$ and $x + y = \lambda y$. Substituting in the first to the second, we get

$$\begin{aligned} x + \lambda &= \lambda(\lambda x) = \lambda^2 x \\ (\lambda^2 - \lambda - 1)x &= 0 \end{aligned}$$

So either $x = 0$, or $\lambda^2 - \lambda - 1 = 0$. If $x = 0$ then $y = 0$ too, and remember that $(0, 0)$ doesn't count as an eigenvector. So $\lambda^2 - \lambda - 1 = 0$. The roots are given by

$$\begin{aligned} \lambda_1 &= \frac{1 + \sqrt{5}}{2} \\ \lambda_2 &= \frac{1 - \sqrt{5}}{2} \end{aligned}$$

We still need an eigenvector. Remember that if v is an eigenvector, then so is any multiple of v. So there's not going to be a unique solution. We might as well take $x = 1$ and then solve for y. From the first equation, the eigenvectors are $v_1 = (1, \lambda_1)$, with eigenvalue λ_1, and $v_2 = (1, \lambda_2)$ with eigenvalue λ_2.

c) *Can you use this to determine a formula for the pattern you noticed in (a)? (Hint: write* $(1, 0)$ *in terms of the eigenvectors you found in (b).*
Now use your answer to compute $T^n(0, 1)$.
We also need to write $0, 1)$ as a combination of eigenvectors. I notice that $v_1 - v_2 = (0, \lambda_1 - \lambda_2)$, so

$$(0, 1) = \frac{1}{\lambda_1 - \lambda_2} v_1 - \frac{1}{\lambda_1 - \lambda_2} v_2 = \frac{1}{\sqrt{5}} v_1 - \frac{1}{\sqrt{5}} v_2.$$

Then using the fact that we have eigenvectors, we get

$$\begin{aligned} T^n(0, 1) &= \frac{1}{\sqrt{5}} \lambda_1^n v_1 - \frac{1}{\sqrt{5}} \lambda_1^n v_2 \\ &= \frac{1}{\sqrt{5}} \lambda_1^n (1, \lambda_1) - \frac{1}{\sqrt{5}} \lambda_1^n (1, \lambda_2) \\ &= \left(\frac{1}{\sqrt{5}} (\lambda_1^n - \lambda_2^n), \frac{1}{\sqrt{5}} (\lambda_1^{n+1} - \lambda_2^{n+1}) \right). \end{aligned}$$

Since F_n is the first entry of $T^n(0, 1)$, we get

$$F_n = \frac{1}{\sqrt{5}} (\lambda_1^n - \lambda_2^n) = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right).$$

This is the exact formula for Fibonacci numbers that I advertised originally.

Problem 11. *Now we've found the eigenvectors directly for a couple examples. What's the systematic way to do this?*

a) *If* $Tv = \lambda v$*, then* $(T - \lambda I)v = 0$*. This means that* $T - \lambda I$ *sends some vector to* 0*. What does this tell you?*

The determinant is 0.

b) *Can you use this observation to come up with a method for finding eigenvalues? How could you find the eigenvectors once you know the eigenvalues?*

c) *Find the eigenvalues and eigenvectors of these matrices:*

$$\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}.$$

d) *What are the eigenvalues of an upper-triangular matrix?*