Today: More complex calculus.

Is $f(z) = \bar{z}^2$ differentiable?

(z a complex number)

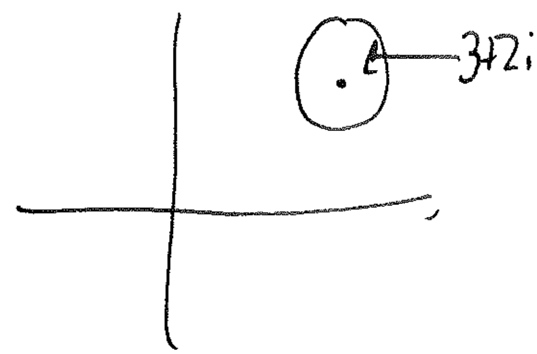
this lets us compute

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = \frac{\pi}{2}$$

not doable with trig etc!

continuous ✓

$$\boxed{\lim_{h \to 0} \frac{f(z+h)-f(z)}{h} \text{ exists}}$$



we have to get the same number no matter which direction h comes from (left, right, positive imaginary direction, diagonally, ...) all have to give same answer.

Let's try:

$f(z) = \bar{z}^2.$ $\quad z = 3+2i.$

Compute

$$\frac{f(z+h)-f(z)}{h}$$

h	$\frac{f(z+h)-f(z)}{h}$
0.1	$6.1-4i$
$0.1i$	$-6+4.1i$
-0.1	$5.9-4i.$
$0.1+0.1i$	$-4.1-6.1i$
$0.02+0.01i$	$0.404-7.222i$

$f(z)=z^2$ $\qquad$ $z=3+2i$

h	$\frac{f(z+h)-f(z)}{h}$
0.1	$6.1+4i$
$0.1i$	$6+4.1i$
$-0.1i$	$6+3.9i$
$0.1+0.1i$	
$0.02+0.01i$	$6.02+4.02i$

$f(z)=z^2$ is differentiable

But $f(z)=\bar{z}^2$ is not.

$f(z) = z^3$

h	$\frac{f(z+h)-f(z)}{h}$
0.1	$+15.91+36.6i$
$0.1i$	$14.39+36.9i$
-0.1	$14.11+35.4i$
$0.1+0.1i$	$15.3+37.52i$
$0.02+0.01i$	$15.12+36.2i$

$3(3+2i)^2 = 15+36i$

differentiable ✓

How to tell from formula if a ~~formula~~ function is differentiable?

u, v real-value

Suppose $f(x+iy) = u(x,y) + i\,v(x,y)$

If $f(z)=z^2$ then

$$f(x+iy) = (x+iy)^2 = x^2 + 2x(iy) + (iy)^2$$

$$= \underbrace{(x^2-y^2)}_{u} + \underbrace{(2xy)}_{v} i$$

If $f(z)=\bar{z}^2$ what are u, v?

$$f(x+iy) = \underbrace{(x^2-y^2)}_{u} + \underbrace{(-2xy)}_{v} i$$

Why was the first differentiable but not the second??

Given $f(x+iy) = u(x,y) + v(x,y)i$ when is it complex-differentiable, in terms of u, v?

What's $\lim_{h \to 0} \frac{f(z+h)-f(z)}{h}$ if h is real?
(in terms of u, v)

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} \frac{[u(x+h,y)+v(x+h,y)i]-[u(x,y)+v(x,y)i]}{h}$$

~~If z+h~~

$$= \lim_{h\to 0} \left[\frac{u(x+h,y)+u(x,y)}{h}\right] + \left[\frac{v(x+h,y)-v(x,y)}{h}\right] i$$

if $z = x+iy$ and h real,

$z+h = (x+h)+iy$

$$= u_x + v_x i.$$

What if h is imaginary? Say $h = ik$ where k real

$z = x+iy$ ~~where~~ then $z+h = x+(y+k)i$

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{k\to 0} \frac{f(z+ik)-f(z)}{ik}$$

$$= \lim_{k\to 0} \frac{[u(x,y+k)+v(x,y+k)i]-[u(x,y)+v(x,y+k)i]}{ik}$$

$$= \lim_{k\to 0} \frac{u(x,y+k)-u(x,y)}{ik} + \frac{v(x,y+k)-v(x,y)}{ik} i.$$

$$= \frac{u_y}{i} + \frac{v_y}{i} i = v_y - i u_y$$

If f is complex-differentiable, those must agree!

$$u_x + v_x i = v_y - i u_y$$

so

$$\boxed{\begin{aligned} u_x &= v_y \\ v_x &= -u_y \end{aligned}}$$

Cauchy-Riemann equations!

Let us check if complex function is differentiable.

Test: for $f(z) = z^2$, we got

$$f(x+iy) = (x+iy)^2 = \underbrace{(x^2 - y^2)}_{u} + \underbrace{(2xy)}_{v} i$$

$u_x = 2x$ $\quad$ $v_x = 2y$

$v_y = 2x$ $\quad$ $-u_y = 2y$

✓ $\quad$ ✓

but $f(z) = \bar{z}^2$ doesn't satisfy CR eqns!

(10)

Challenge:

Let $u(x,y)=x^2$.

Can you find a $v(x,y)$ so

$$\boxed{u_x = v_y, \quad v_x = -u_y}$$

First eqn says: $v_y = 2x$ $\quad v = 2xy + g(y)$

Second eqn says: $v_x = 0$, impossible!

most $u(x,y)$ aren't the real part of a differentiable

function!! they're rare.

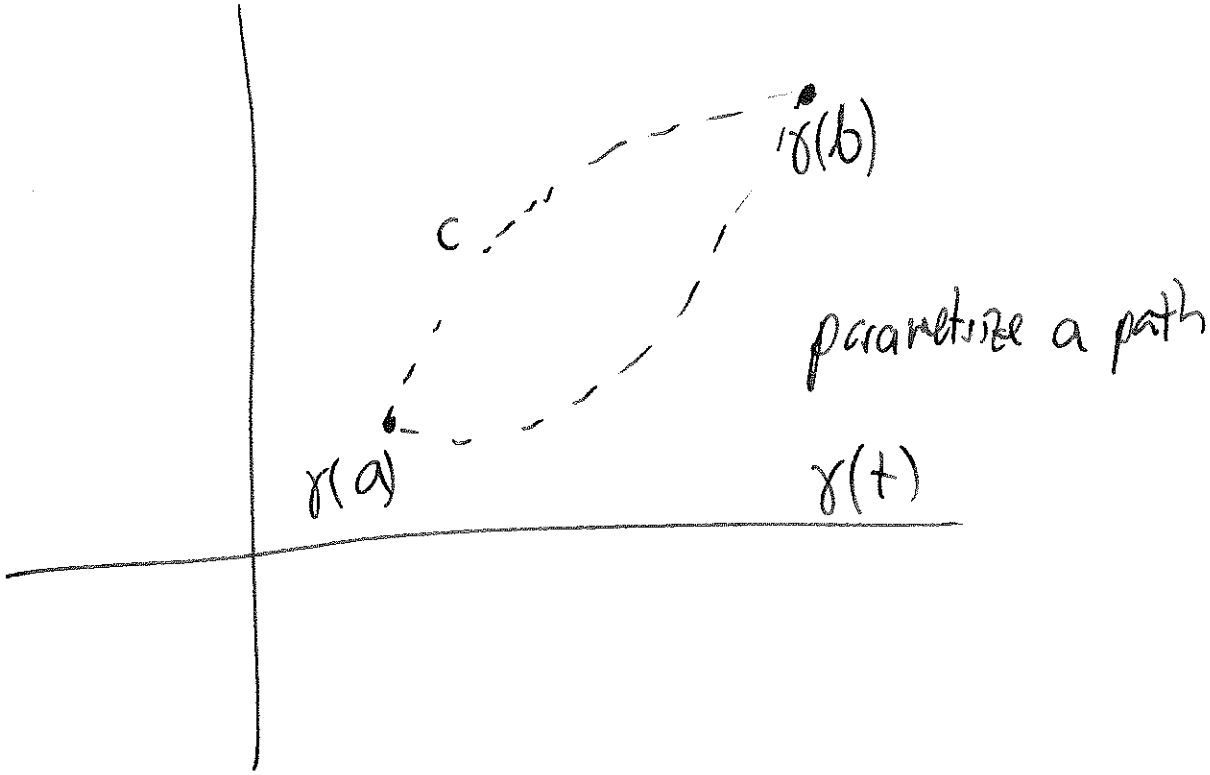
(need $\Delta u = u_{xx} + u_{yy} = 0$)

or v can't exist.

"holomorphic"

"

complex differentiable



Contour integrals

parametrize a path

$$\int_C f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

complex function

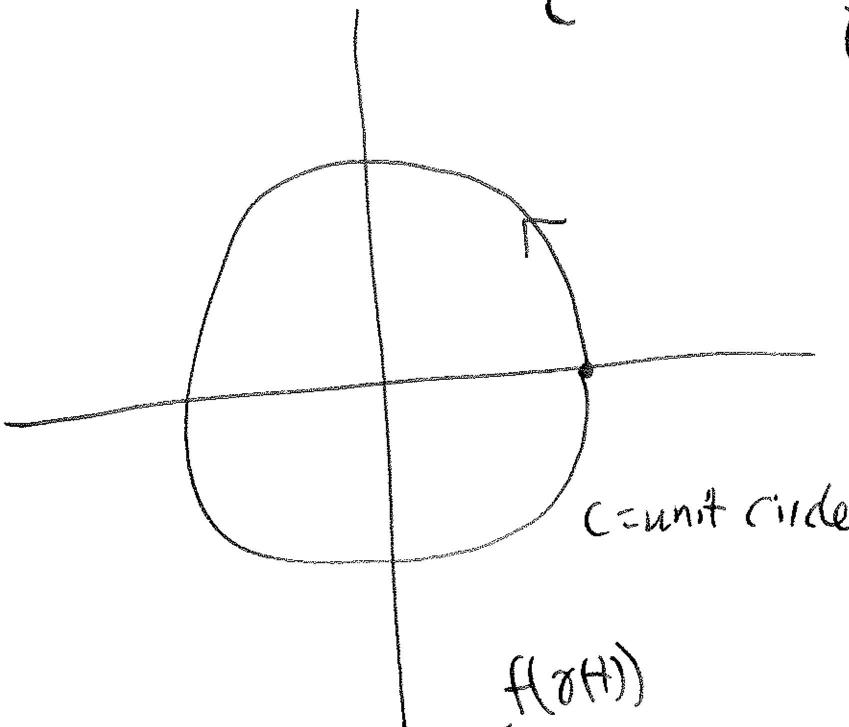
$f(z) = \frac{1}{z}$

$\gamma(t) = \cos(t) + i\sin(t) = \text{cis}\, t$

$= e^{it} \qquad 0 \le t \le 2\pi.$

$\oint_C \frac{1}{z}\,dz$

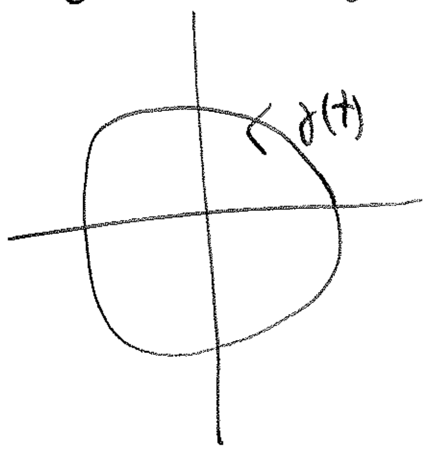
$\gamma'(t) = ie^{it}$



$f(\gamma(t))$

$$\boxed{\oint \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{(e^{it})}(ie^{it})\,dt = 2\pi i}$$

$$\oint_C z\,dz = \int_0^{2\pi} (e^{it})(ie^{it})\,dt = \int_0^{2\pi} ie^{2it}\,dt = \frac{i}{2i} e^{2it}\,dt = \frac{1}{2} - \frac{1}{2} = 0$$



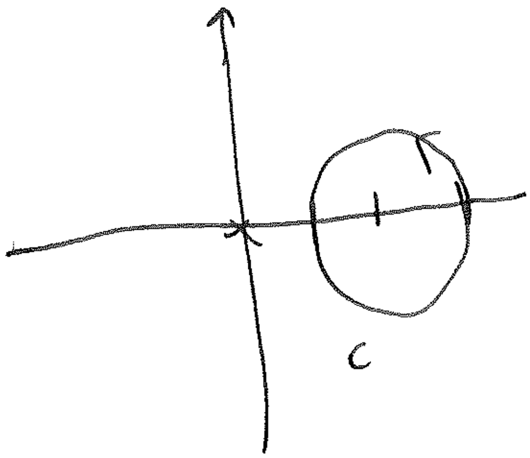
$$\oint_C z^2\,dz = \int_0^{2\pi} (e^{it})^2 (ie^{it})\,dt = 0$$

$$\oint_C z^a\,dz = \int_0^{2\pi} (e^{it})^a (ie^{it})\,dt = i\int_0^{2\pi} e^{(a+1)it}\,dt$$

a integer

$= 0$ if $a \neq -1$

or $2\pi i$ if $a = -1$.



$$\gamma(t) = (\cos(t) + 2) + i \sin t$$
$$= 2 + e^{it}$$
$$\gamma'(t) = ie^{it}$$

$$\oint_c \frac{1}{z}\,dz =$$

$$\int_0^{2\pi} \frac{1}{2+e^{it}}\, ie^{it}\,dt$$

$$= \int_0^{2\pi} \frac{1}{2+e^{it}} \frac{2+e^{-it}}{2+e^{-it}}\, ie^{it}\,dt = \dots = 0.$$

In fact, $\oint_c z^a\,dz = 0$ for any integer a whatsoever!

Next time: If $f(z)$ is holomorphic at every pt inside loop: $\boxed{\oint_c f(z)\,dz = 0}$