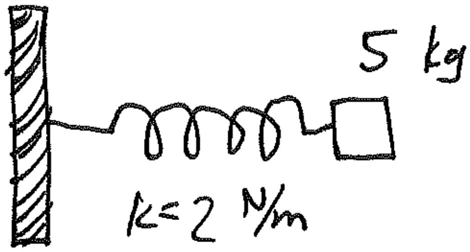
Some differential equations



If we pull it out 3 m and release where will it be in 10 seconds?

Hooke's law: $F = -kx$

$F = ma$

AP Physics:

$\omega = \sqrt{\frac{k}{m}}$

Position at time t: $x(t)$

$m\,x''(t) = -k\,x(t)$ $\leftarrow$ example differential equation.

how to find $x(t)$?

$x''(t) = -\frac{k}{m}x(t)$ ω

$x(t) = \cos\left(\sqrt{\frac{k}{m}}t\right)$

$\sin\left(\sqrt{\frac{k}{m}}t\right)$ work.

Any other solutions? $-\cos\left(\sqrt{\frac{k}{m}}t\right)$, $-\sin\left(\sqrt{\frac{k}{m}}t\right)$

- multiply by constants
- add any two solutions

set of solutions is a vector space!

General solution is:

$$c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right).$$

(every solution is like this for some c_1, c_2).

Another solution is e.g.

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$$5\cos\left(\sqrt{\frac{k}{m}}(t-2)\right)$$

but that's

$$= \underbrace{5}_{\text{some constant}} \cos\left(\sqrt{\frac{k}{m}}t\right) \underbrace{\cos\left(2\sqrt{\frac{k}{m}}\right)} + 5\sin\left(\sqrt{\frac{k}{m}}t\right) \sin\left(2\sqrt{\frac{k}{m}}\right)$$

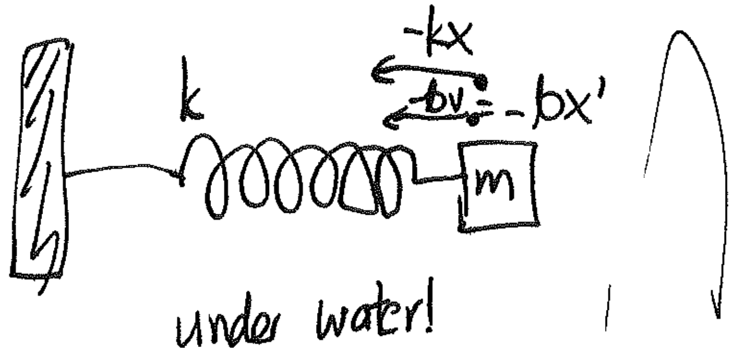
If you have specific initial conditions, like

$x(0)=5$

$x'(0)=2$

these determine a unique solution, because you can solve for c_1, c_2.

What if there's a drag force?



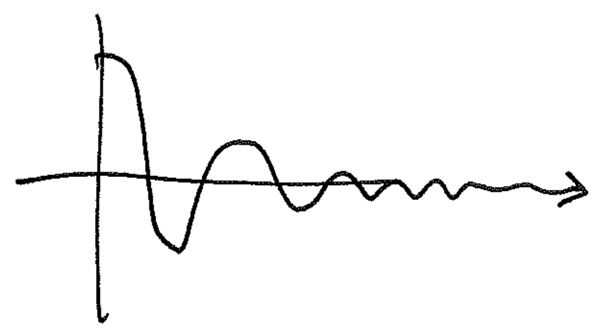
under water!
→ drag force
(proportional to velocity)

$$m x''(t) = -k\,x(t) - b\,x'(t)$$

$$m\,x''(t) + b\,x'(t) + k\,x(t) = 0$$] hard to guess answer!

It ends up being like:

$$e^{-t}\cos(3t)$$



If wall moves back & forth:

$$m\,x''(t) + b\,x'(t) + k\,x(t) = F(t)$$ ← external force.

how to solve?

Most differential equations can't be solved exactly!
(just like regular equations)

But many common ones are solvable

Let's start with some $\underline{1^{st}\text{ order}}$ diff eqs. has $x'(t)$ but no higher derivatives.

Try it: $$t\,x'(t) + 2\,x(t) = 0$$

$$x'(t) = -\frac{2}{t}\,x(t)$$

Method #1.

$$\frac{x'(t)}{x(t)} = -\frac{2}{t}$$

Guess: $x(t) = Ct^{a}$ try it! see if we can find C, a that work. "ansatz" guess solution.

$$t\,x'(t) + 2\,x(t) = 0$$

$$t(aCt^{a-1}) + 2(Ct^{a}) = 0$$

unless $a=0$

$\Rightarrow aC(t^{a}) = -2Ct^{a}$.

so $a=-2$ and solution is $x(t) = Ct^{-2}$.

Method #2

$$\frac{x'(t)}{x(t)} = -\frac{2}{t}$$

natural log

$$\frac{d}{dt}\left(\log\left(x(t)\right)\right) = -\frac{2}{t}$$

$$\log x(t) = -2\log(t) + C$$

$e^{\sim} \rightarrow$ $$x(t) = e^{-2\log(t) + C} = \tilde{C}\,t^{-2}. \checkmark$$

Try it:

$$t^2 x(t) = 3x'(t)$$

$\downarrow$

$$\frac{d}{dt}\log x(t) = \frac{x'(t)}{x(t)} = \frac{t^2}{3}$$

$$\log x(t) = \frac{t^3}{9} + C$$

$$x(t) = e^{t^3/9}\cdot \tilde{C} \leftarrow e^C.$$

Challenge:

$$t^2 x(t) = 3x'(t) + 2$$

other solutions?

$t^2 e^{-t}$? not quite.

Math How can we ever know there are no other solutions?

$$y' = 0$$

$y(t) = C$ works. are there any other solutions?

If not constant, $y(a) \neq y(b)$ for some a, b.

Mean value theorem: there's a c with $a < c < b$ so

$$y'(c) = \frac{y(b) - y(a)}{b - a} \neq 0.$$

this contradicts $y'(c) = 0$.

$$t\,x'(t) + 2\,x(t) = t^3$$

Multiply by "integrating factor", in this case t.

$$t^2 x'(t) + 2t\ x(t) = t^4$$

$$\frac{d}{dt}\left(t^2 x(t)\right) = t^4$$

product rule!

$$t^2\ x(t) = \frac{t^5}{5} + C$$

$$x(t) = \frac{t^3}{5} + \frac{C}{t^2}$$

$$t^2 x(t) = 3x'(t) + 2$$

$$(t^2)x(t) + (-3)x'(t) = 2$$

want to multiply by $p(t)$ so left side is

chain rule...

what could $p(t)$ be?

$$(t^2 p(t))\, x(t) + (-3p(t))\, x'(t) = 2\, p(t)$$

we need derivative of $-3\,p(t)$ to be $t^2\, p(t)$.

(so we can use product rule)

$$-3\, p'(t) = t^2\, p(t)$$

$$\frac{d}{dt}\log p(t) = \frac{p'(t)}{p(t)} = -\frac{t^2}{3}$$

$$\log(p(t)) = \frac{-t^3}{9} + 0$$

$$p(t) = e^{-t^3/9}.$$

$$\left(t^2 e^{-t^3/9}\right) x(t) + \underbrace{\left(-3e^{-t^3/9}\right)}_{f(t)} x'(t) = 2e^{-t^3/9}$$

$$= f'(t)\, x(t) + f(t)\, x'(t) = 2e^{-t^3/9}$$

$$\frac{d}{dt}\left(-3e^{-t^3/9}\, x(t)\right) = 2e^{-t^3/9}$$

$$-3e^{-t^3/9}\, x(t) = \int 2e^{-t^3/9}\, dt$$

...

$$x(t) = \cdots$$