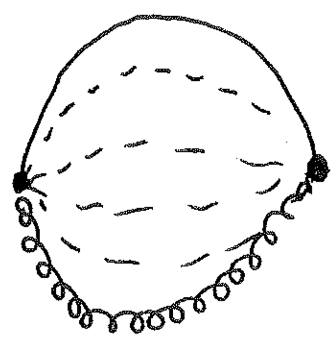
Homotopy

Two functions $f : X \to Y$
$g : X \to Y$

are homotopic if there's a time-varying family $F_t : X \to Y$ so $F_0 = f$ $F_1 = g$.

Ex

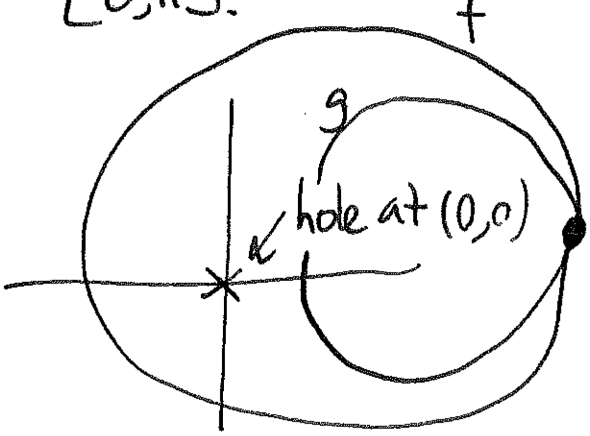


$g, f : [0, \pi] \to \mathbb{R}^2$

$f(x) = (x, \sin x)$

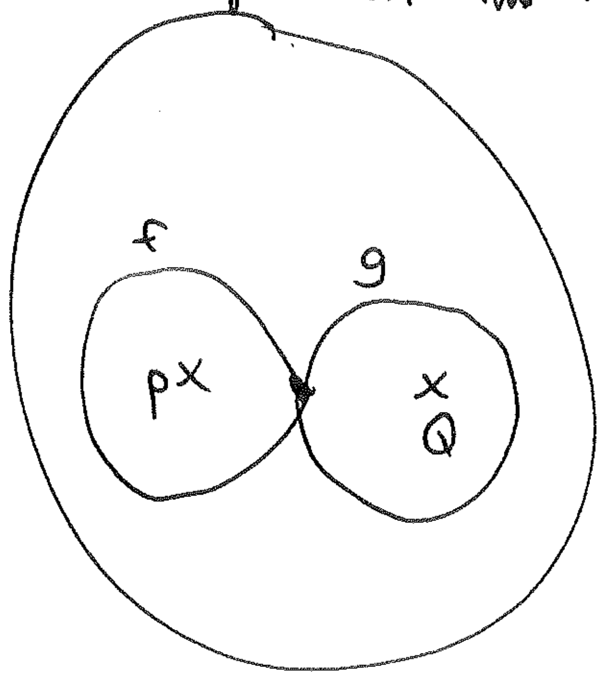
$g(x) = (x, -\sin x)$

Not homotopic use S^1 (circle) as domain instead of $[0, \pi]$.



$f, g : S^1 \to \mathbb{R}^2 \setminus \{(0,0)\}$
punctured plane

plane with two holes:

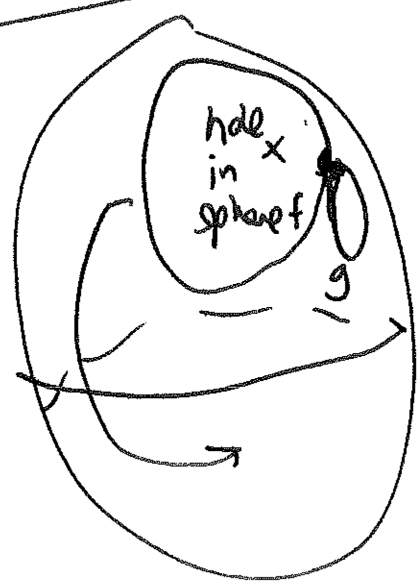


$f, g : S^1 \to \mathbb{R}^2 \setminus \{(P, Q\}\}$

not homotopic

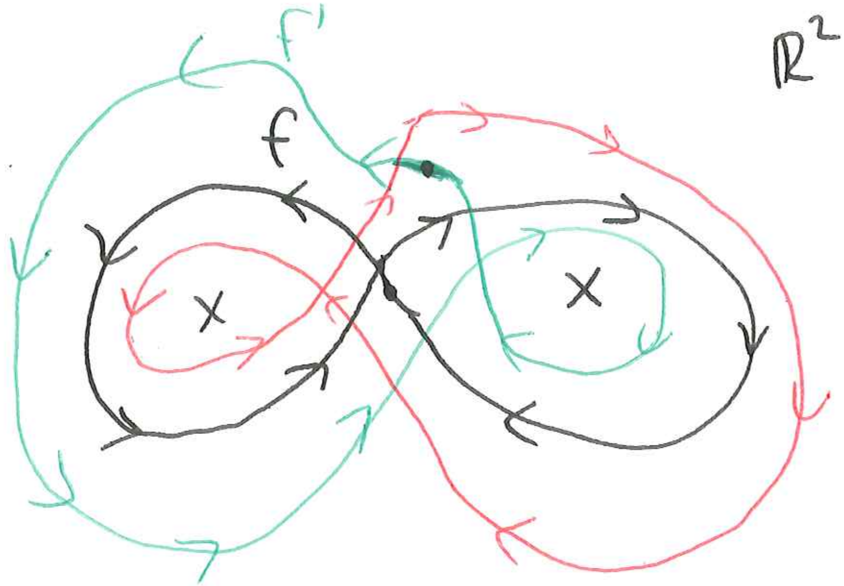
sphere with hole

$f, g : S^1 \to S^2 \setminus P$



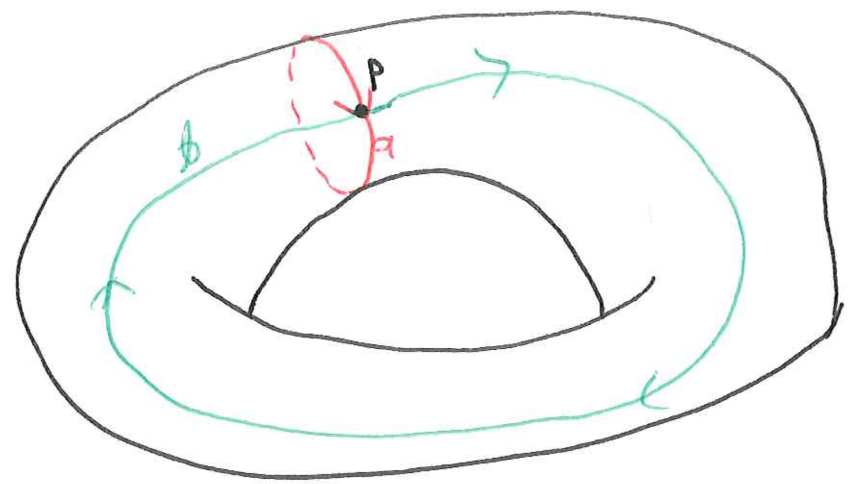
homotopic!

$\mathbb{R}^2 \setminus$ two holes



torus

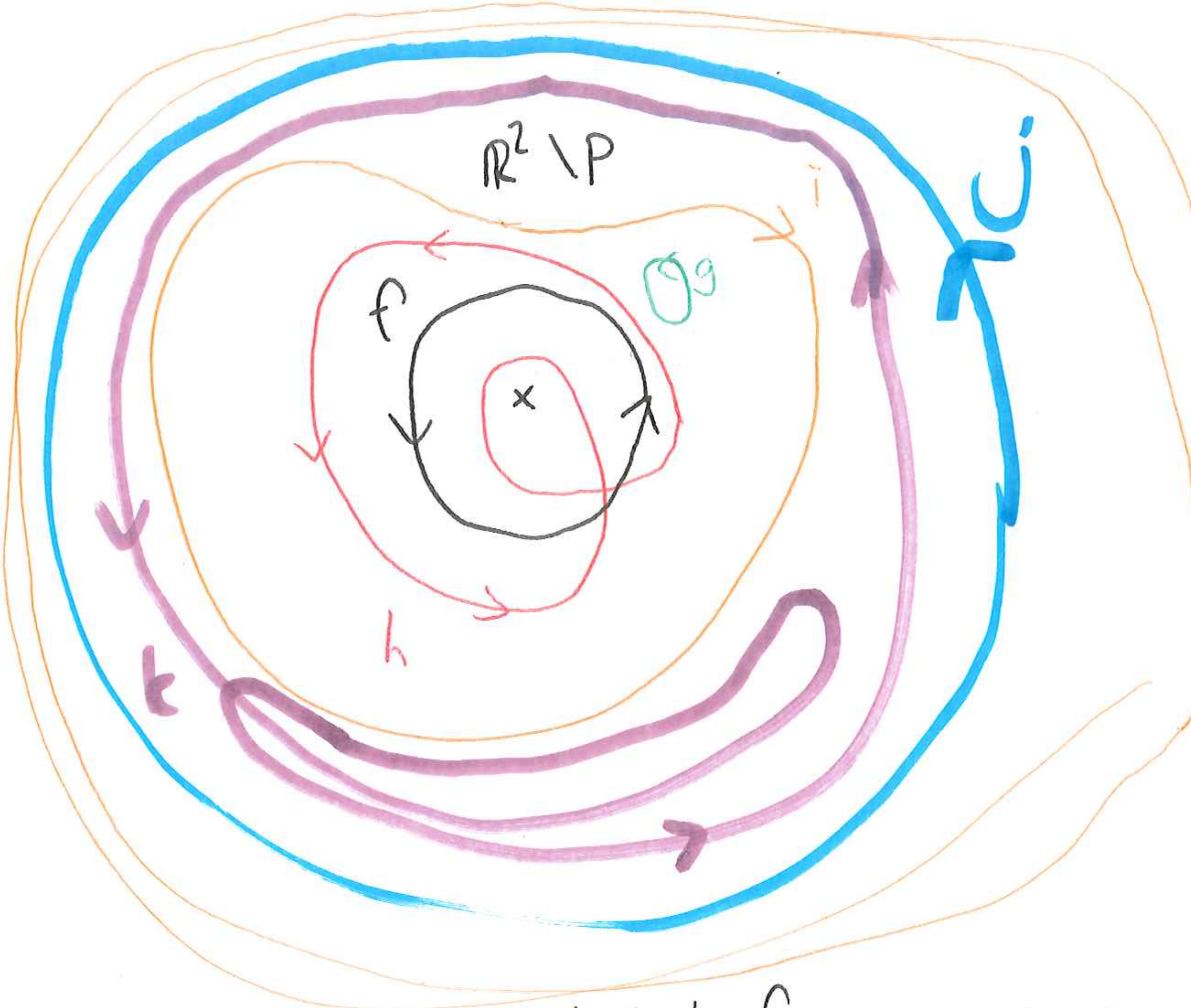
$a, b : S^1 \to T^2$



f = start at p, follow a, then follow b.

g = start at p, follow b ~~a~~, then follow ~~b~~ a.

f, g homotopic! (see video)



L= go around counter clockwise twice, then ~~counterclockwise~~ clockwise once.

j homotopic to f i.e. $j \simeq f$

$j \not\simeq f$

for a loop $S^1 \to \mathbb{R}^2 \setminus P$

the <u>degree</u> is the number of times it goes around the hole (negative if you go opposite direction)

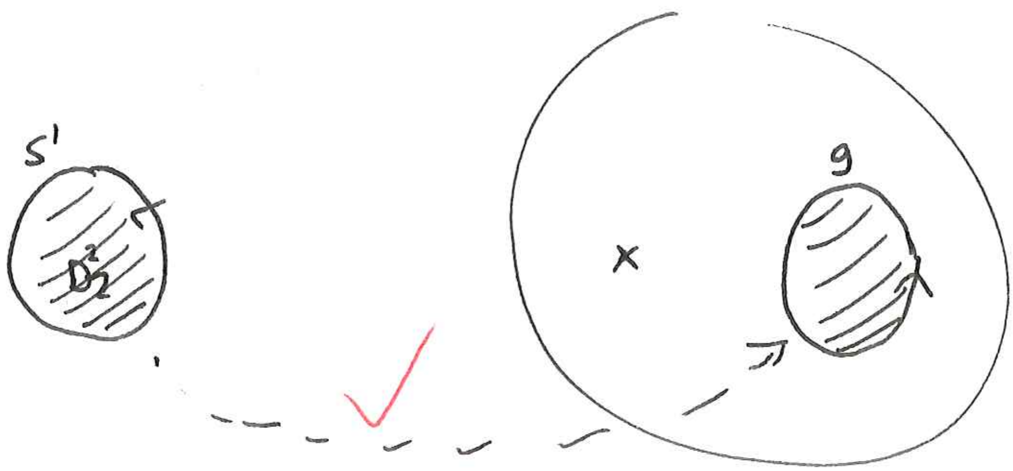
<u>Theorem:</u> Two loops are homotopic if and only if they have the same degree.

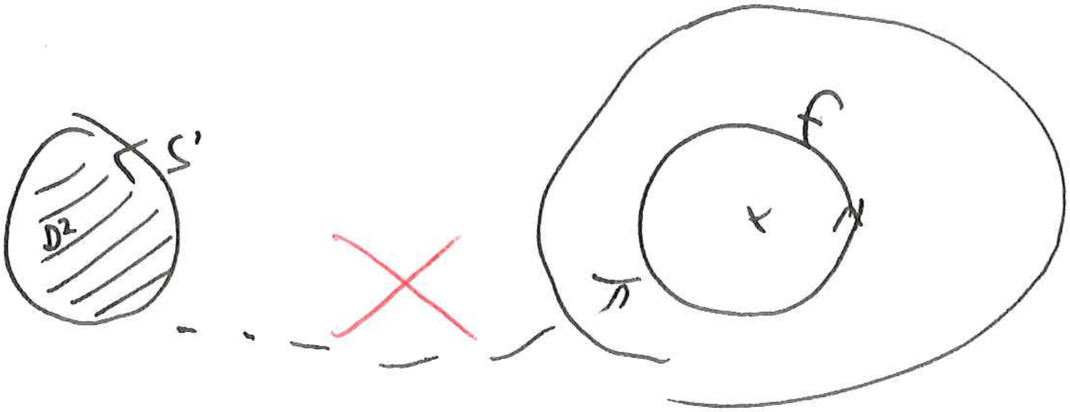
Let's say we have a loop

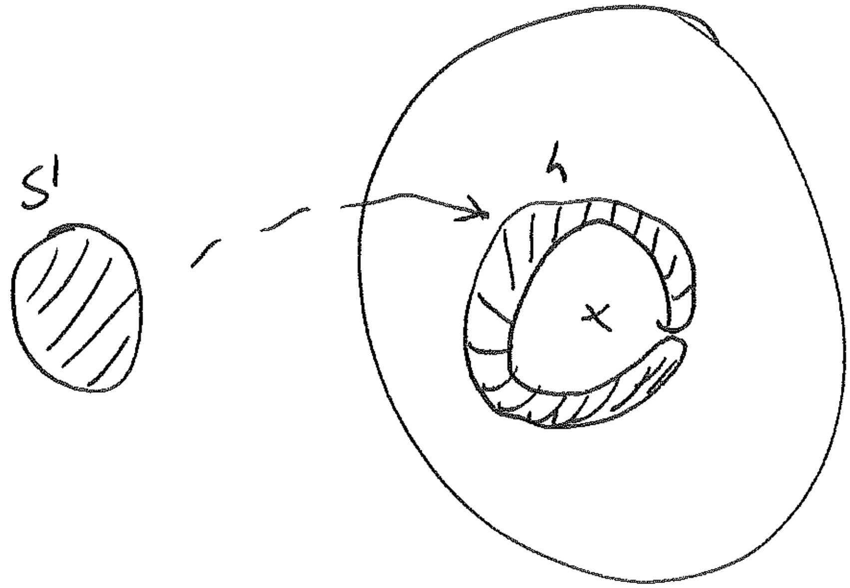
$$\underset{\text{unit circle}}{S^1} \longrightarrow \underset{\text{punctured plane}}{\mathbb{R}^2 \setminus P}$$

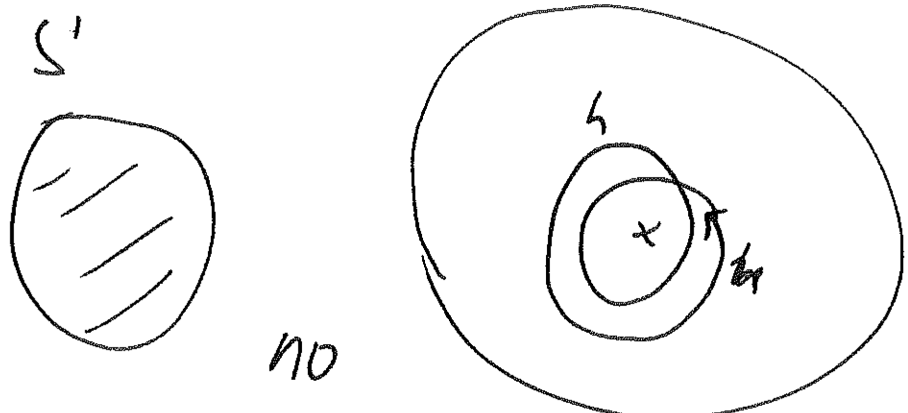
when can we "fill it in" at get a function

$$\underset{\text{unit disk}}{D^2} \longrightarrow \mathbb{R}^2 \setminus P?$$









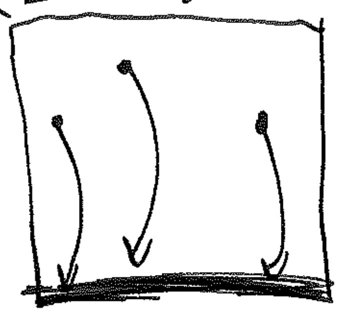
no way to extension

<u>Theorem</u> If $f : S^1 \to \mathbb{R}^2 \setminus p$ has degree 0, it can be extended to D^2. If degree not 0, it can't.

<u>Def</u> Suppose $A \subset X$ is a subset.

A <u>retraction</u> of X to A is a continuous function $f: X \to A$ with $f(a)=a$ for all $a \in A$. "smush X onto A"

X = unit square $\{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$

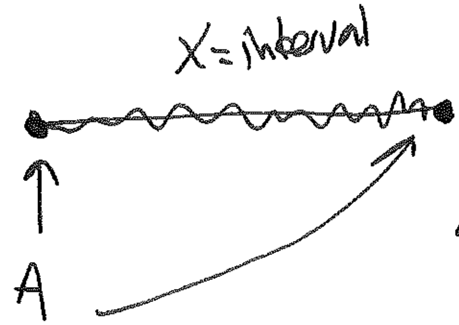


$f(x,y) = (x,0)$

$A = \{(x,0) : 0 \le x \le 1\}$

How about X = annulus.

can we retract it to C_1?



a retraction would be a continuous function

$$f : [0,1] \to \{0,1\}$$

interval   two pt set

$f(a)=a$ means $f(0)=0$, $f(1)=1$

<u>NO</u>

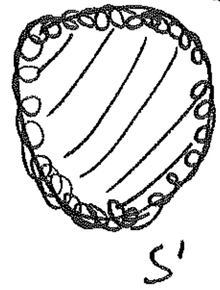
Intermediate value theorem!

Möbius strip onto edge?

<u>no retraction theorem!</u>

No retraction theorem

D^2 can't ~~con~~ retract onto S^1.



Pf. Image you could retract it:

$$f : D^2 \to S^1$$

$f(a) = a$ for in S^1.

that mean you extended the function $f : S^1 \to S^1 \subset \mathbb{R}^2 \setminus P$

$$f(\theta) = \theta$$

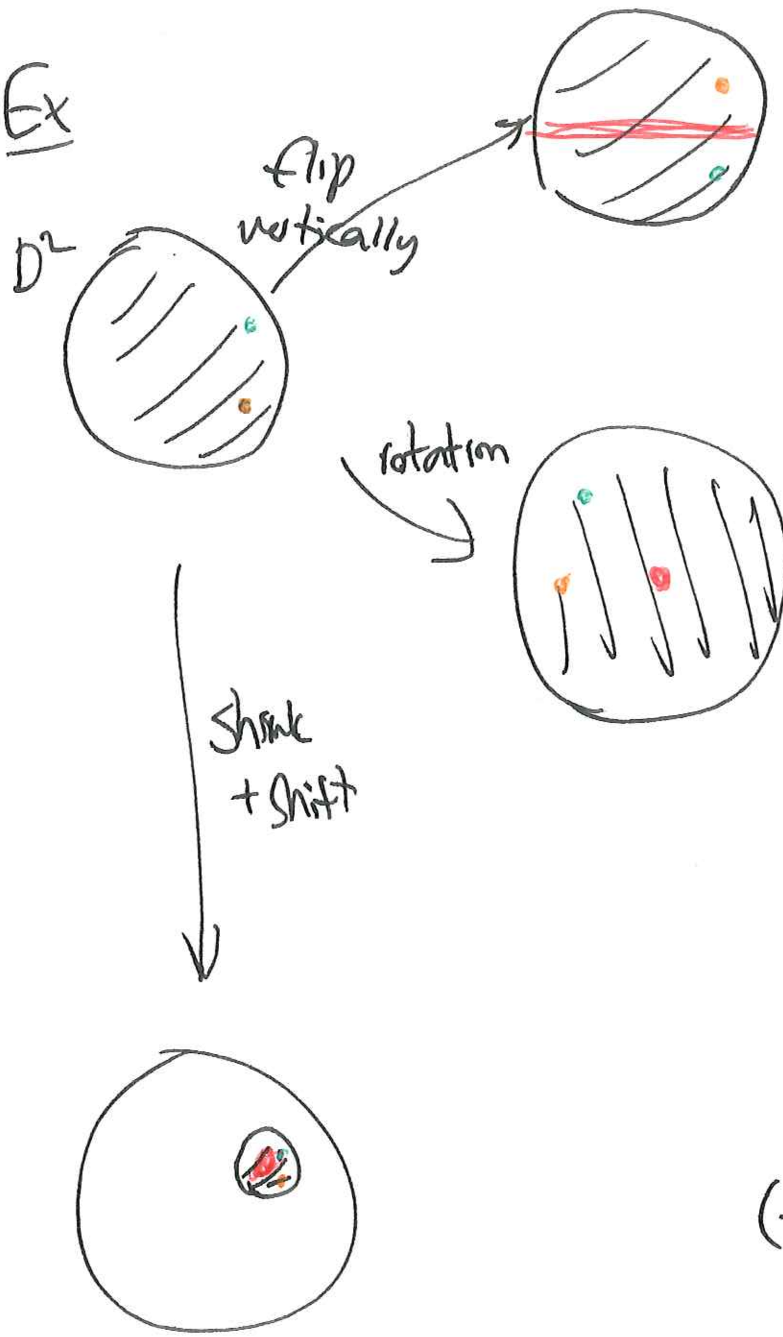
to have domain D^2.

Brouwer fixed point theorem

Any continuous function $f: D^2 \to D^2$ has a fixed pt.

$f(x) = x$

Ex

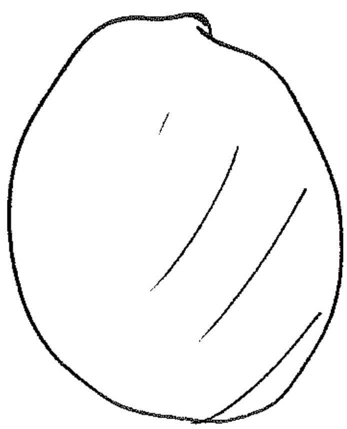


(+ the map thing)

Proof

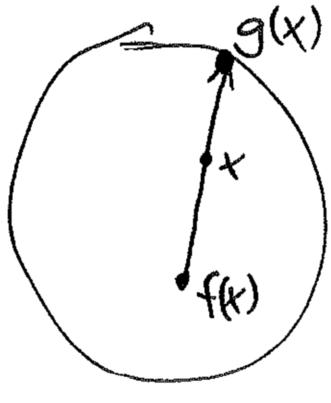
Suppose have continuous function

$f: D^2 \to D^2$ with no fixed points

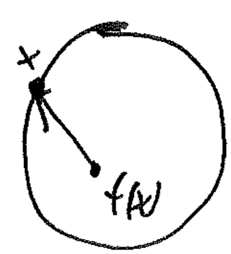


Define $g: D^2 \to S^1$ as follows:

- Given x,
- draw x, $f(x)$ (two different points)
- draw ray ~~fx~~ from $f(x)$ to x, see where hits edge. that's $g(x)$



If x is on edge S^1 already:



$g(x) = x$

This means we have:

$g: D^2 \to S^1$ which is continuous,

and $g(x) = x$ if $x \in S^1$!

This is a retract of D^2 onto S^1; impossible!

<u>Next time(?)</u>

1) Ham sandwich

2) Dirac's belt

3) Fundamental thm of algebra

4) Topological analysis.