This week: knot theory

What is a knot?

- "A closed loop in $\mathbb{R}^3$
- A function $f : \mathbb{R} \to \mathbb{R}^3$ periodic.

~~$f : \mathbb{R} \to$~~

$0 \le t \le 1$

- $f : [0,1] \to \mathbb{R}^3$

$f(t) = (x(t), y(t), z(t))$

f must be <u>continuous</u> (infinitely differentiable)

(you can draw without lifting pen)

$f(0) = f(1)$

(so it joins up)

otherwise $f(a) \neq f(b)$ for any a and b.

(so it doesn't cross itself)

Definition

A **knot** is an infinitely differentiable function

$f:[0,1]\to\mathbb{R}^3$ given by $f(t)=(x(t), y(t), z(t))$

satisfying:

1) $f(0)=f(1)$

2) $f(a)\neq f(b)$ unless $a=b$ (with exception of $\{0,1\}$).

Two knots are isotopic if:
(think: the same)

Say $f : [0,1] \to \mathbb{R}^3$
$g : [0,1] \to \mathbb{R}^3$

there exists a family of knots

~~$f_s : [0,1] \to$~~ $f_s : [0,1] \to \mathbb{R}^3 \quad (0 \leq s \leq 1)$

~~such~~ smoothly varying as s varies and with

$f_0 = f, \ f_1 = g.$

Question - How can we tell if two knots are isotopic?

- How can we tell if a knot can be untied? (I.e. isotopic to the unknot O)

From playing with knots:

- The trefoil is <u>not</u> isotopic to its reverse.

  overhand knot "chiral" mirror image

- The figure eight knot <u>is</u> isotopic to its reverse.

  "amphichiral"

It's hard to tell whether two knots are isotopic!

It's hard to tell whether a knot is isotopic to the unknot!

Those are the "Perko pair": for 75 years thought to be different knots.

How to prove left-handed trefoil is different from right-handed and both are different from unknot?

Idea: Assign numbers to a knot.

If two knots give different answers, they must be different knots.

→ But: need a number that doesn't depend on the exact way the knot is presented.

|| The <u>crossing number</u> of a knot is the minimum number of times the rope crosses itself in a 2D picture

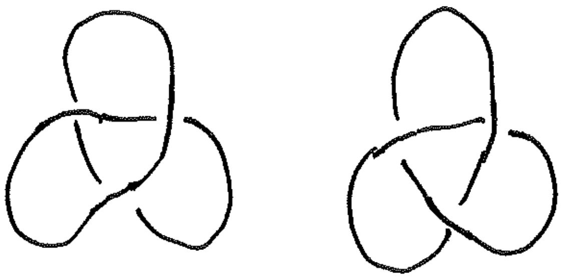
<u>Ex</u> Crossing number of (left- or right-) trefoil is 3.

↳ a) How can you prove it's 3? (⇒ not unknot, or figure eight) which has crossing number 4.

b) This can't distinguish handedness.

"Knot invariants"

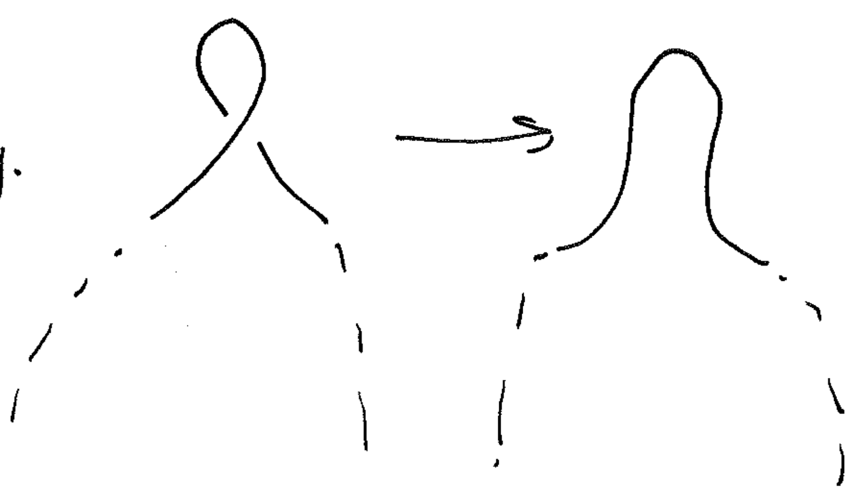
A <u>knot diagram</u> is a 2D picture of a knot, showing which strand goes on top at every crossing:



unknot trefoil

How can we draw isotopies in knot diagrams? What moves can we perform of a knot diagram without changing the knot?

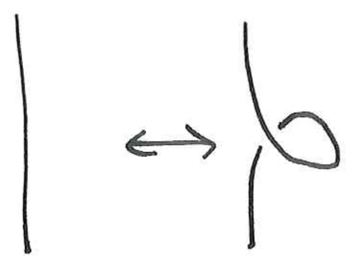
e.g.



Any isotopy can be made as a combination of three basic moves:

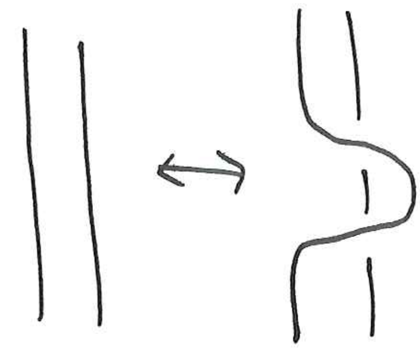
("Reidemeister moves")

1)



untwist

2)



cross over

3)

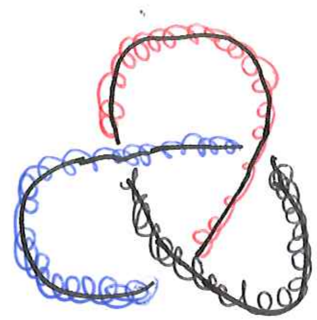
pass over a crossing
(or under)

Every isotopy can be broken down into these steps! (Annoying to prove, though.)

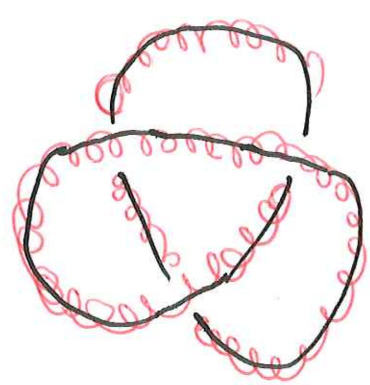
Def A knot is <u>tricolorable</u> if (True/false invariant, not a number

in a knot diagram for the knot,
we can color each ~~sth~~ arc of the knot
with one of three colors, such that:

1) At each crossing, either only one color, or all three colors appear.
2) At least two colors are used.



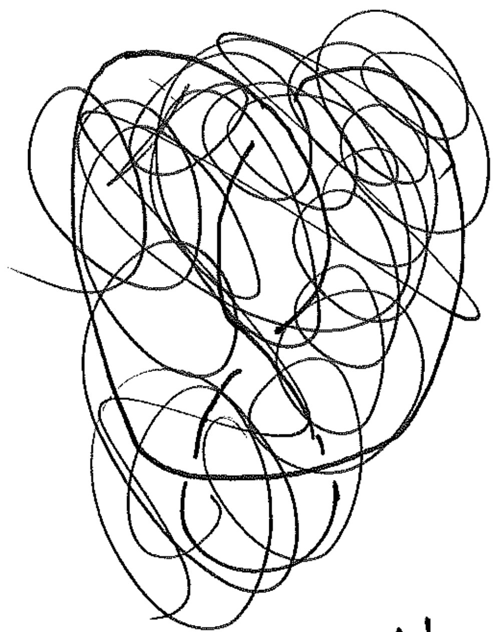
trefoil
tricolorable

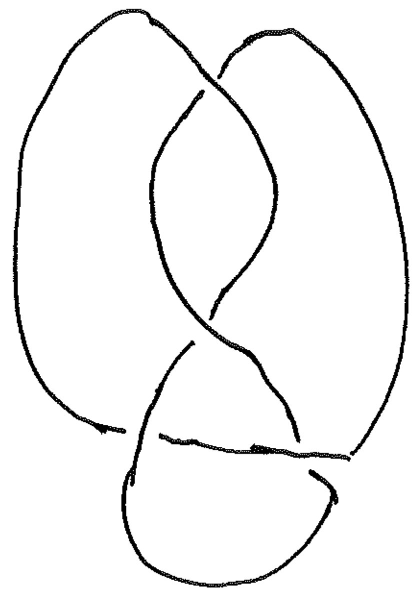


unknot
not tricolorable

(we still need to check that Reidemeister moves don't affect tricolorability!)

Is figure eight tricolorable?





No

→ not the same as trefoil

→ but maybe the same as unknot?