Today:

More differential equations.

Find $y(t)$:

<u>Review:</u> $y'(t) = t\,y(t)$

$$\frac{y'}{y} = t \rightsquigarrow \frac{d}{dt}(\log y) = t$$

$$\log y = t^2/2 + c$$

$$y = e^{t^2/2} \cdot \tilde{c}$$

$\tilde{c} = e^c$; new constant.

$y(t) + \tan(t)\, y' = 1$

"Integrating factor": multiply both sides by $\cos(t)$.

$$\cos t \cdot y + \sin t \cdot y' = \cos t$$

$$\frac{d}{dt}(\sin t \cdot y) = \cos t$$

$$\sin t \cdot y = \sin t + C$$

$$y = 1 + C \cdot \csc t.$$

Another example:

$$2y + ty' = 1$$

$$2ty + t^2 y' = t$$

$$\frac{d}{dt}(t^2 y) = t$$

$$t^2 y = \frac{t^2}{2} + c$$

$$y = \frac{1}{2} + \frac{c}{t^2}.$$

To solve

$$a(t)\, y + b(t)\, y' = c(t)$$

"First-order linear equation"

only $f(t)\cdot y$, $f(t) y'$: no $(y')^2$, $y \cdot y'$, $1/y$

only y', no y'', y''' etc.

general method

$$y + \frac{b(t)}{a(t)} y' = \frac{c(t)}{a(t)}.$$ ← need an integrating factor $p(t)$.

$$p(t) y + p(t) \frac{b(t)}{a(t)} y' = p(t) \frac{c(t)}{a(t)}$$

so
$$p(t) = \frac{d}{dt}\left(p(t) \frac{b(t)}{a(t)}\right) = p'(t) \frac{b(t)}{a(t)} + p(t) \left(\frac{d}{dt} \frac{b(t)}{a(t)}\right).$$

From here, solve for $\frac{p'(t)}{p(t)}$. integrate to get $\log p(t)$, done!

First order linear eqns → solvable.

But for higher order, it's hopeless!

Can't always find a formula for solution.

Classic example: Airy equation $y''=ty$.

Two-dimensional solution space

- One solution can be written as antiderivative of something easy
- Second solution can't be written in a simple way.

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no $(y')^2$ etc

Second-order linear equations with **constant coefficients**

y, y', y''

can't depend on t

homogeneous

$=0$ (every term must depend on y)

$$\boxed{ay''+by'+cy=0}$$

a, b, c are numbers.

Last time:

$my''+by'+ky=0$

mass on a spring with drag.

Example

$$y'' + 5y' + 6y = 0$$

What are all solutions?

Try: Guess $y(t) = e^{rt} + C$

$$r^2 e^{rt} + 5re^{rt} + 6e^{rt} + 6C = 0$$

$\neq 0$ for all t, so $C = 0$.

$$r^2 e^{rt} + 5re^{rt} + 6e^{rt} = 0$$

$$r^2 + 5r + 6 = 0$$

$$(r+2)(r+3) = 0 \quad r = -2 \text{ or } -3.$$

Solutions are e^{-2t}, e^{-3t}.

General solution to equation:

$$y(t) = C_1 e^{-2t} + C_2 e^{-3t}.$$

If we have initial conditions, you can solve for the constants.

$$y(0)=2$$
$$y'(0)=3$$
and $y''+5y'+6y=0$

Solution is $C_1e^{-2t}+C_2e^{-3t}$ $\quad y'(t)=-2C_1e^{-2t}-3C_2e^{-3t}$

plug in 0 for t

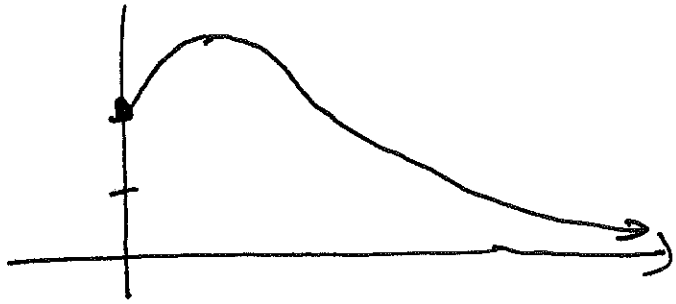
$y(0)=2$ tells us $C_1+C_2=2$

$y'(0)=3$ $\quad -2C_1-3C_2=3$

2 eqns, 2 vars

✓ $C_1=9,\ C_2=-7$

$$\boxed{y(t)=9e^{-2t}-7e^{-3t}}$$



$4y'' - 8y' + 3y = 0$ $\quad y(0) = 2$

$y'(0) = 1/2$

Try: $y(t) = e^{rt}$

what r value will work?

$4r^2 e^{rt} - 8re^{rt} + 3e^{rt} = 0$

$4r^2 - 8r + 3$

$(2r-1)(2r-3) = 0$

$r = 1/2 \quad r = 3/2 \rightsquigarrow e^{t/2}, e^{3t/2}$ solutions.

General sol: $c_1 e^{t/2} + c_2 e^{3t/2}$

Initial conditions: $y(0) = 2$ says $c_1 + c_2 = 2$

$y'(0) = 1/2$ says $1/2 c_1 + 3/2 c_2 = 1/2$

$\rightsquigarrow \quad c_1 = 5/2$

$c_2 = -1/2$

To solve

$$ay'' + by' + cy = 0$$

Guess $y(t) = e^{rt}$

Plug in $\Rightarrow$ $ar^2 + br + c = 0$

solve for r $\Longrightarrow$ two possible values r_1, r_2.

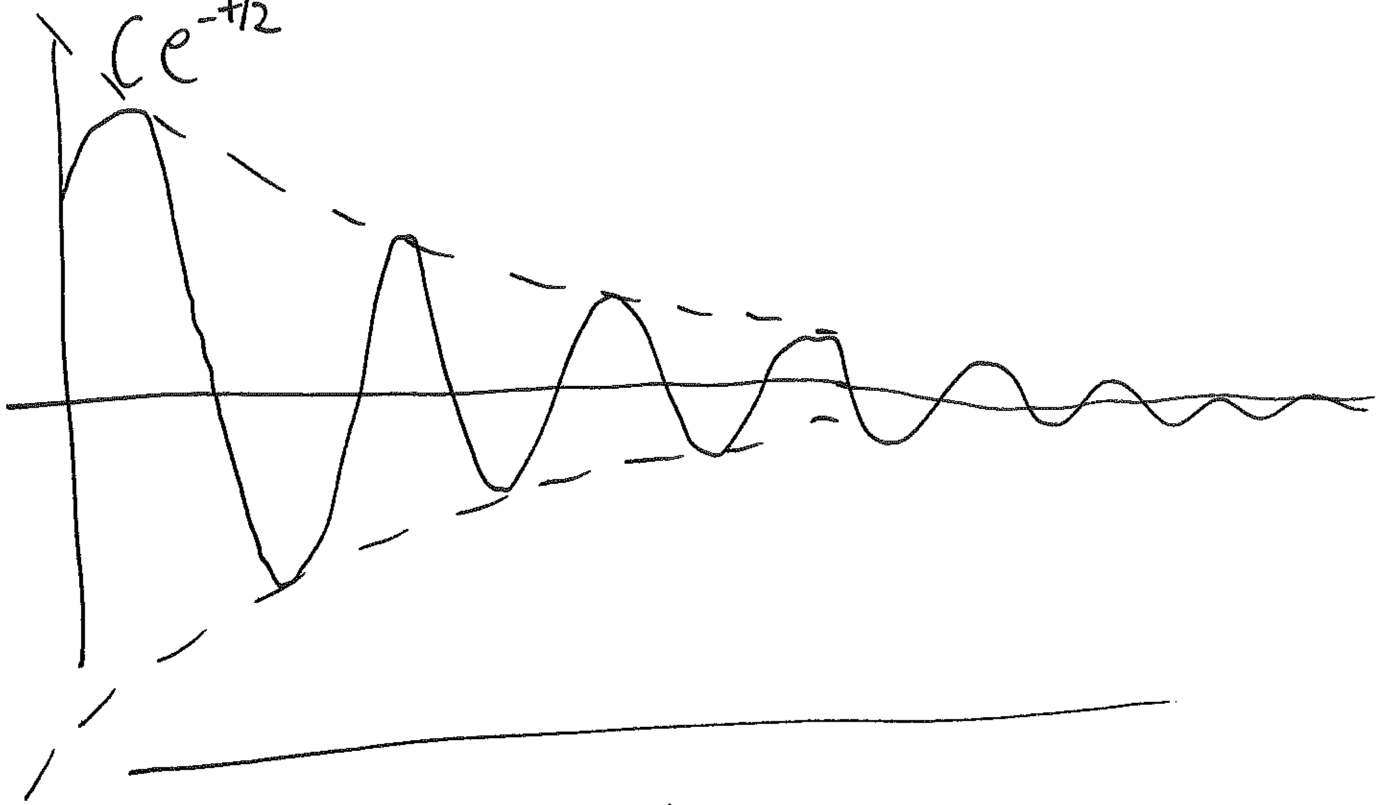
General solution: $c_1 e^{r_1 t} + c_2 e^{r_2 t}$

As long as: - r_1, r_2 distinct

- r_1, r_2 real.

What if they're not?

$\hookrightarrow$ $r_1 = r_2$ or $r_1 = \bar{r}_2$ complex conjugates.



Mass on spring: $$my'' + by' + ky = 0$$

$$mr^2 + br + k = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$ if complex roots:

~~a+bi~~ $a + ci$

real part of root is $a = -\frac{b}{2m}$

$e^{(a+ci)t}$

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$e^{at}\cos(ct)$

Imaginary part: $c = \frac{\sqrt{4mk - b^2}}{2m}$

Solution is: $e^{-b/2m\,t} \cdot \cos\left(\left(\frac{\sqrt{4mk-b^2}}{2m}\right)t\right)$

or sin

$$y'' + y' + 9.25y = 0$$ (this could be a mass on a spring: all coefs +)

Try
$y(t) = e^{rt}$

$$r^2 + r + 9.25 = 0$$

$$\left(r^2 + r + \tfrac{1}{4}\right) + 9 = 0$$

$$\left(r + \tfrac{1}{2}\right)^2 = -9$$

$$r + \tfrac{1}{2} = \pm 3i$$

$$r = -\tfrac{1}{2} \pm 3i$$

Solutions

$$e^{(-\frac{1}{2}+3i)t}, \quad e^{(-\frac{1}{2}-3i)t}.$$

$$\downarrow$$

$$e^{-t/2} e^{3it}$$

$$\shortparallel$$

$$e^{-t/2}(\cos 3t + i \sin 3t)$$

Real part

$$e^{-t/2} \cos 3t$$

Imaginary part

$$e^{-t/2} \sin 3t$$

both solutions.

General sol is

$$c_1 e^{-t/2} \cos 3t + c_2 e^{-t/2} \sin 3t$$

Repeated roots.

$$y'' - 4y' + 4y = 0.$$

Try e^{rt}:

$$r^2 - 4r + 4 = 0$$

$(r-2)(r-2)$. So e^{2t} is a solution.

What's the second solution?

Try $y = (a+bt)e^{rt}$

$$y' = be^{rt} + r(a+bt)e^{rt}$$
$$= (b+ar)e^{rt} + rbte^{rt}$$
$$y'' = r(b+ar)e^{rt} + rb(tre^{rt} + e^{rt})$$
$$= (2br + ar^2)e^{rt} + br^2te^{rt}$$

$y'' - 4y' + 4y = 0$

$0 = e^{rt}(2br + ar^2 - 4b - 4ar + 4a) + te^{rt}(br^2 - 4br + 4b)$

What a, b, r make this 0?

$b(r^2 - 4r + 4) = 0.$

$\rightarrow (r-2)(r-2) = 0$

so $r = 2$. to make te^{rt} term disappear.

Then e^{rt} term is 0 too. As long as $r = 2$, a, b can be anything.

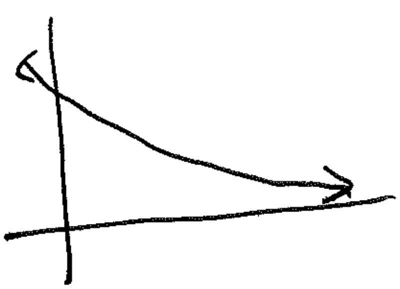
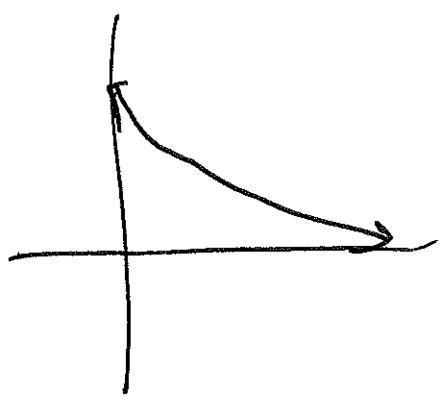
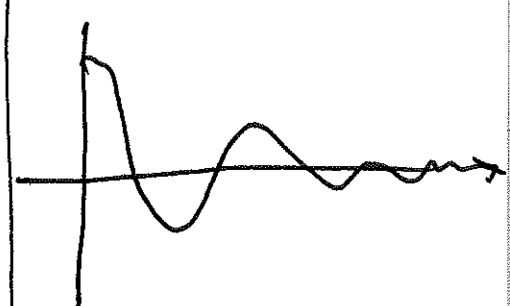
e^{2t} and te^{2t} are our basic solutions.

General solution is

$C_1 e^{2t} + C_2 te^{2t}.$

Mass on a spring.

$my'' + by' + ky = 0.$ Three options

Two distinct real	Double real root	Complex roots.
$b^2 > 4km$	$b^2 = 4km$	$b^2 < 4km$
$c_1 e^{r_1 t} + c_2 e^{r_2 t}$	$c_1 e^{rt} + c_2 t e^{rt}$	$c_1 e^{at}\cos(ct) + c_2 e^{at}\sin(ct)$
 "overdamped"	 "critically damped"	 "underdamped"