Today:

- Finish defining real numbers straight from axioms.

- Countable vs uncountable

Last time: $\mathbb{N}$ defined using Peano Postulates.

d) There's a set $\mathbb{N}$ with an element 1 and a function $s: \mathbb{N} \to \mathbb{N}$ such that

a) No $n \in \mathbb{N}$ has $s(n) = 1$

b) s is injective

c) (induction, basically)

Then we defined $+$: there's unique binary operation $+$ such that:

a) $n + 1 = s(n)$

b) $n + s(m) = s(n+m)$

Then
(Prove all the basic rules using just these...)

- $a+b=b+a$
- $a+(b+c)=(a+b)+c$

...

We can also define "<":

Say $m<n$ if $m+a=n$ for some $a\in\mathbb{N}$.

(Prove all properties of $<$: $a<b$ and $b<c \Rightarrow a<c$)

Then define multiplication:

There's a unique binary operation $\times : \mathbb{N}\times\mathbb{N}\to\mathbb{N}$

Satisfying

a) $n\times 1 = n$

b) $n\times s(m) = n\times m + n$

Then check distributive law, ...

Now:

Integers

One suggestion:

An integer is an ordered pair of natural numbers $[a,b]$ where $a =$ either 1 or 2 or 3

if $a = 1 \rightsquigarrow$ represents integer 0

$a = 2 \rightsquigarrow$ represents integer b

$a = 3 \rightsquigarrow$ represents integer $-b$

Now we need to define addition for integers:

$$[a,b] + [c,d] = \begin{cases} [2, b+d] & \text{if } a=c=2 \\ [3, b+d] & \text{if } a=c=3 \\ [a,b] & \text{if } c=1 \\ [2, b-d] & \text{if } a=2,\ c=3, \text{ and } b>d \\ \cdots & \end{cases}$$

or combine

$[2,5] + [2,3] = [2,8]$

$[1,3] + [1,4] = [1,7]$

$[2,5] + [1,4] = [2,5]$

$[2,5] + [3,4] = [2,1]$

$[2,5] + [3,6] = [3,1]$

Then you need to check that addition satisfies all the axioms:

$$(x+y)+z = x+(y+z)$$

define multiplication too, check the axioms, ...

(lots of cases!)

There is a little slicker way: let $[\overset{\text{natural}}{a, b}]$ to represent the integer $a-b$.

Downside: $[a,b]$ and $[c,d]$ represent same thing if $a+d=b+c$.

$$[1,4]=[2,5]=[3,6]=\cdots$$

Upside: No cases to define addition now:

$$[a,b]+[c,d]=[a+c,\ b+d]$$

A $\overset{\text{Def}}{}$ rational number is an ordered pair $[a,b]$

of integers. (In our heads: $[a,b]$ represents a/b)

$[a,b]$ and $[c,d]$ represent same rational number

if $ad-bc=0$.

$\frac{a}{b} \stackrel{?}{=} \frac{c}{d}$

$ad=bc$

Then define $+$, $\times$, $<$:

$[a,b]\times[c,d]=[ac,bd]$ $\qquad \frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$

$[a,b]+[c,d]=[ad+bc,bd]$ $\qquad \frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{cb}{bd}$

$=\frac{ad+bc}{bd}$

Then check associativity, distributive, commutative,

work for rational numbers using these rules!

At last: real numbers $\mathbb{R}$

A subset $A \subset \mathbb{Q}$ is called a <u>Dedekind cut</u>

if it has three properties:

a) $A \neq \phi$, $A \neq \mathbb{Q}$
not empty

b) If $x \in A$, and $y \in \mathbb{Q}$ has $y \geq x$, then $y \in A$ too.

c) If $x \in A$, there's some $y \in A$ so $y < x$ and $y \in A$ too.

<u>Examples</u>

positive rational numbers

$A_0 = \{ x \in \mathbb{Q} : x > 0 \}$ is a Dedekind cut

$A_{1/2} = \{ x \in \mathbb{Q} : x > \frac{1}{2} \}$ is a Dedekind cut

$A_q = \{ x \in \mathbb{Q} : x > q \}$ is a Dedekind cut

But

$$A = \{x \in \mathbb{Q} : x > 0 \text{ and } x^2 > 2\}$$ is a Dedekind cut

<u>Def</u> The real numbers $\mathbb{R}$ is the set of all of Dedekind cuts.

What's the Dedekind cut corresponding to $-\sqrt{2}$?

$$A_{-\sqrt{2}} = \{x \in \mathbb{Q} : x^2 < 2 \text{ or } x > 1\}$$

then define $\times$, $+$, $<$ for Dedekind cuts, etc.

Two sets A, B are ~~in bijection~~

the same size if you can find a one-to-one correspondence between elements of A and elements of B.

<u>Ex</u> $\mathbb{N}$ (natural) and $\mathbb{Z}$ (integers)

	9	7	5	3	1	2	4	6	8			$\mathbb{N}$
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$			
-5	-4	-3	-2	-1	0	1	2	3	4	5	6	$\mathbb{Z}$

Try it: which of these can you pair off with $\mathbb{N}$

(or with each other)

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $(0,1)$, $[0,1)$, $\mathbb{Z}^2$ (ordered pairs)

$\mathbb{R}^2$, polynomials with integer coefficients, sequences of integers, ..., $\{x \in \mathbb{Q} : 0 < x < 1\} = (0,1)_{\mathbb{Q}}$

I suggest: trying

$\mathbb{R}$ and $(0,1)$

$\mathbb{N}$ and $\mathbb{Z}^2$

$\mathbb{N}$ and $\mathbb{Q}$

$(0,1)$ and $[0,1)$

Sometimes not possible!

$(0,1)\times(0,1)$ vs $(0,1)$

ordered pairs (a,b) with both between 0 & 1.

e.g. $(0.7, 0.4)$

$0.231724152712230 4\ldots$

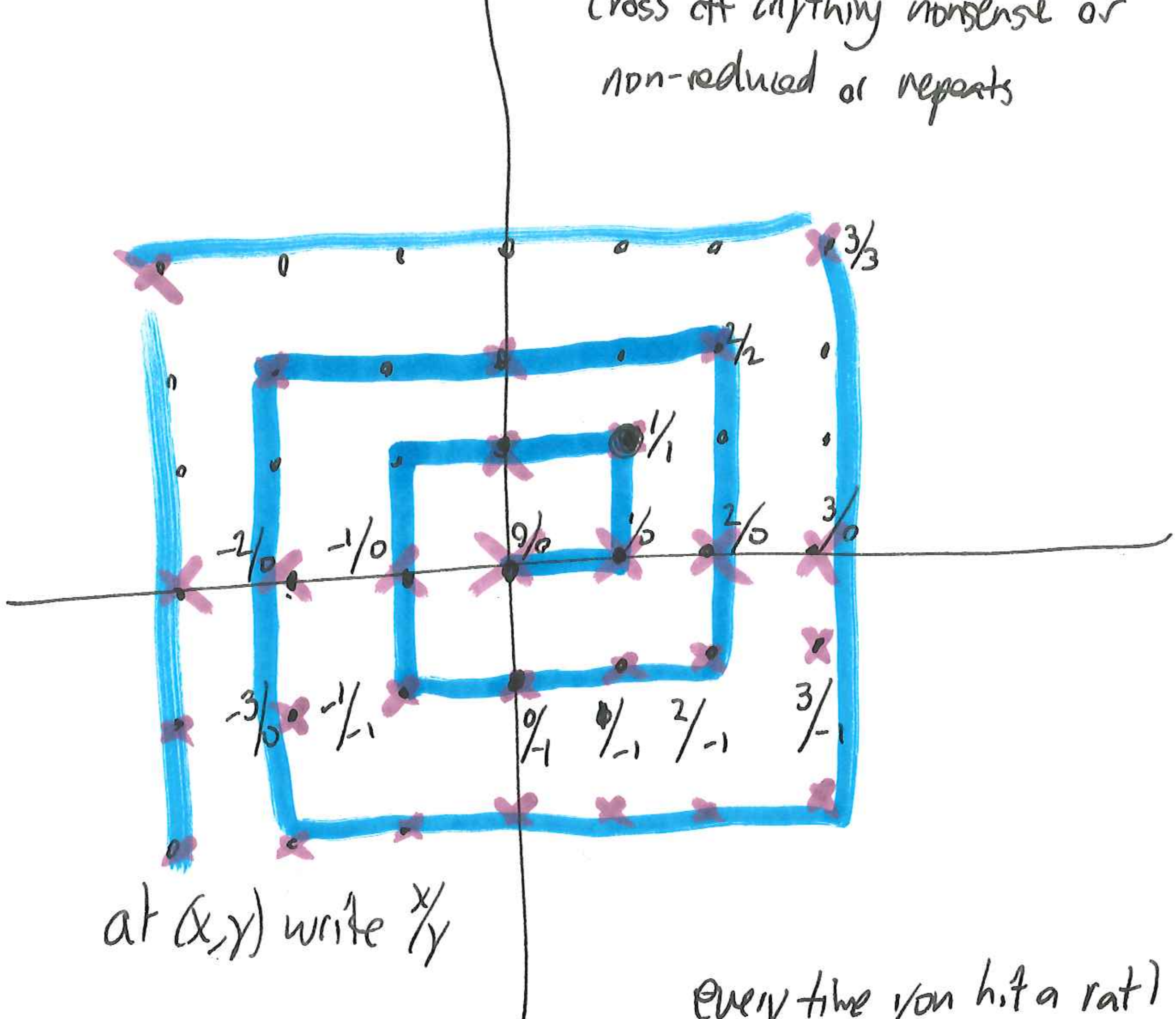
$0.1126675689891 66\ldots$

$\updownarrow$ interlace the digits!

$0.21311276264715\ldots$

$\mathbb{N}$ vs $\mathbb{Q}$

list all ratl numbers in a grid,
cross off anything nonsense or
non-reduced or repeats



at (x,y) write x/y

every time you hit a ratl
that's not crossed out,
add to list

$1/1,\ 0/1,\ -1/1,\ 2/1,\ 1/2$