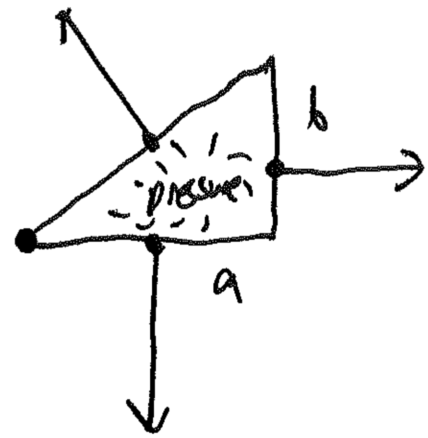
Today: one more day of linear algebra (LU decomposition, invertibility, Markov processes

Next: something fun (requests?)

- Fourier (after a break)
- Game theory?
- Physics proofs?
- Algebraic geometry!



Recap last time: determinants of any size matrix

to find det(M): - use row operation of add $c \times R_i$ to R_j until we get upper triangular. don't change determinant!

- then determinant is product of diagonal entries.

Inverse of an $n \times n$ matrix.

$$\left(\begin{array}{ccc|ccc} 0 & 3 & 2 & 1 & 0 & 0 \\ -1 & 4 & 2 & 0 & 1 & 0 \\ 3 & -4 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\Big\downarrow \text{ rref}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 5 & 2 \\ 0 & 1 & 0 & -5 & 6 & 2 \\ 0 & 0 & 1 & 8 & -9 & -3 \end{array}\right)$$

this is the inverse!

takes n^3 steps.

$$\begin{pmatrix} 0 & 3 & 2 \\ -1 & 4 & 2 \\ 3 & -4 & -1 \end{pmatrix} \xrightarrow{R1 -= R2} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 4 & 2 \\ 3 & -4 & -1 \end{pmatrix} \xrightarrow{R2 += R1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & 2 \\ 3 & -4 & -1 \end{pmatrix}$$

$$\xrightarrow{R3 += (-3) R1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & 2 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow{R3 += (+1/3) R2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & -1/3 \end{pmatrix}$$

upper triangular! det is -1.

Why is this faster? (What is the other way anyway?)

$n \times n$ matrix! (must be square!)

$$NF \begin{pmatrix} 1 & 2 & 3 & 4 & -1 \\ 7 & 1 & 2 & 4 & 1 \\ 3 & 1 & 7 & 0 & 1 \\ 25 & 0 & -6 & 1 & 25 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Look at every possible way to pick n things, one from each row and each column.

There are $n!$ ways to do it.

For each way, multiply circled numbers, add up, half with a $+$, half with a $-$.

Takes $(n+1)!$ steps! Yuck! Row red takes n^3 steps.

LU decomposition

Imagine you want to solve

$$Ax=b_1,\ Ax=b_2, \dots\ Ax=b_6,\ Ax=b_{100},$$

$$Ax=b_{1000000}, \dots$$ many b's, same A, different x for each b.

Given a matrix A, dimensions $m \times n$.

We will find: $A=LU$

where L is an $m \times m$ lower triangular matrix

U is an $m \times n$ ~~upper tria~~ echelon form matrix (upper triangular, basically)

How does this help solve $Ax=b$?

To solve $Ax=b$:

1) Solve $Ly=b$ for y. (easy! L is lower-triangular)
2) Solve $Ux=y$, for the y you just found.

x is solution:

$$Ax=(LU)x=L(Ux)=Ly=X.$$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

The decomposition is:

$$L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$$

$$LU = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \checkmark$$

$Ax = b$ where $b = \begin{pmatrix} 12 \\ 17 \end{pmatrix}$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 7 \\ 3 & -1 & 1 & 0 & -3 \\ 4 & 5 & 6 & 1 & 4 \end{array}\right)$$

$Ly = b$ for y: $\left(\begin{array}{cc|c} 1 & 0 & 12 \\ 3 & 1 & 17 \end{array}\right)$

$x_1 = 12 \quad 3x_1 + x_2 = 17$

$x_2 = 17 - 3x_1 = 17 - 36 = -19$

$y = \begin{pmatrix} 12 \\ -19 \end{pmatrix}$

$Ux = y$ for x $\left(\begin{array}{cc|c} 1 & 2 & 12 \\ 0 & -2 & -19 \end{array}\right)$ $x_2 = \frac{19}{2}$ $x = \begin{pmatrix} -7 \\ 19/2 \end{pmatrix}$

$x_1 = 12 - 2x_2 = -7.$

How to find L&U

0) Use a computer!

if you must:

1) Row reduce A, using only operation $R_i \mathrel{+}= c \cdot R_j$.*

2) Get to echelon form (not reduced)
   (if you must row swap, read about it)

3) LU decomposition records steps of row reduction.

   U = final echelon matrix you got

   L = lower triangular, with L_{ij} = what multiple of row j did you subtract from row i?

   row i
   col j

   diagonal entries = 1

*do in order: first fix first col, then the second, ...

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \xrightarrow{R2 \mathrel{-}= 3R1} \begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 5 & 6 \end{pmatrix} \xrightarrow{R3 \mathrel{-}= 5 \cdot R1} \begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 0 & -4 \end{pmatrix}$$

$$\xrightarrow{R3 \mathrel{-}= 2R2} \begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 0 & 0 \end{pmatrix}$$

that's U!

What about L?

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 2 & 1 \end{pmatrix}$$

L_{21} $\quad$ L_{31} $\quad$ L_{32}

L_{ij} = what multiple of row j did you subtract from row i?

row i
col j

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 5 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

Try one!

$$\begin{pmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{pmatrix}$$

Find $LU = A$

One last topic: (Markov processes)

Suppose that every year:

5% of people in State College move to Bellefonte

95% stay put

3% in Bellefonte move to State College

97% stay put.

Suppose in year 0: 40000 in SC
10000 in Bellefonte.

How many in each place in 1 year?
2 years?
..100 years?
1000000 years?

SC in a year:

$$0.95 \cdot 40000 + 0.03 \cdot 10000$$

Bfte in a year:

$$0.05 \cdot 40000 + 0.97 \cdot 10000$$

$$x_{n+1} = \begin{pmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{pmatrix} x_n$$

columns sum to 1
"Markov matrix"!

stabilizes:

$Ax = x$ ← you eigenvector with eigenvalue 1.

$Ax = Ix$

$(A - I)x = 0.$

$$\left(\begin{array}{cc|c} -0.05 & 0.03 & 0 \\ 0.05 & -0.03 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} -0.05 & 0.03 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$-0.05x + 0.03y = 0 \quad x + y = 50000$

$-5x + 3y = 0 \quad x + y = 50000$

$x = 18750$

$y = 31250$