Real analysis (the definitions and proofs behind calculus)

What do we mean when we say "The sequence x or $\{x_n\}_{n\geq 1}$ or $\langle x_n \rangle$ or $x_1, x_2, x_3, x_4, \ldots$ has limit L"?

→ The values of the terms are all getting close to L.

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots$ converges to 2? (of course not!)

distance from x_n to L.

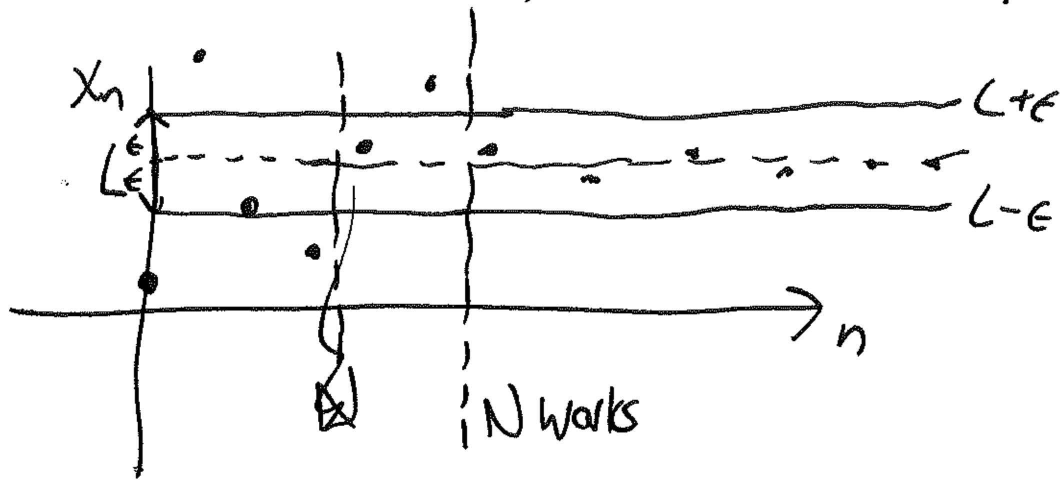
so $|x_n - L|$ approaches 0. ← what does that mean?

get infinitely close...

Does $\frac{1}{2}, \frac{2}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{1}{5}, \frac{2}{5}, \ldots$

approach 0? (Yes!)

Def $x_1, x_2, x_3, x_4, \ldots$ converges to L if for every $\epsilon > 0$, there exists an N (depends on ϵ), so that if $n > N$ then $|x_n - L| < \epsilon$.



Theorem The sequence defined by $x_n = 3 + 2/n$ converges to the limit $L = 3$.

Pf. Suppose that $\epsilon > 0$ is given. Take $N = 2/\epsilon$.

If $n > N$, then $n > 2/\epsilon$ and so $\epsilon > 2/n$.

Then if $n > N$,

$$|x_n - L| = |(3 + 2/n) - 3| = \left|\frac{2}{n}\right| = \frac{2}{n} < \epsilon.$$

Where did $N = 2/\epsilon$ come from?

I wanted $|(3 + 2/n) - 3| < \epsilon$, so how big does n need to be the guarantee this?

Needed $\left|\frac{2}{n}\right| < \epsilon$ so $\frac{2}{n} < \epsilon$ i.e. $\cancel{n > \epsilon/2}$

$\underline{n > 2/\epsilon}$. So I used $N = 2/\epsilon$.

usually do this before sitting down to write proof.

<u>Ex</u> a) $x_n = 2 + 1/\sqrt{n}$

b) $y_n = \dfrac{n+1}{n-1}$

find limit, and prove it using definition.

c) Prove that if $x_n \to L$ and $y_n \to M$ then $x_n + y_n \to L + M$.

a) $X_n = 2 + 1/\sqrt{n}$.

$L = 2$.

Pf. Suppose $\epsilon > 0$. Let $N = 1/\epsilon^2$. If $n > N$, then $n > 1/\epsilon^2$, and so $\epsilon^2 > \frac{1}{n}$, and $\epsilon > \frac{1}{\sqrt{n}}$.

Then $|X_n - L| = \left|\left(2 + \frac{1}{\sqrt{n}}\right) - 2\right| = \left|\frac{1}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}} < \epsilon$.

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b) $y_n = \frac{n+1}{n-1}$ Scratch work: limit is 1. We want $\left|\frac{n+1}{n-1} - 1\right| < \epsilon$; how big does n need to be?

$\frac{n+1}{n-1} - 1 = \frac{n+1}{n-1} - \frac{n-1}{n-1} = \frac{2}{n-1}$, want $< \epsilon$.

$\frac{2}{n-1} < \epsilon$. $\frac{2}{\epsilon} < n-1$, so $n > \boxed{\frac{2}{\epsilon} + 1}$

that's N

c) Suppose x_n converges to L

and y_n converges to M.

Then $x_n + y_n \to L+M$.

<u>Pf.</u> Suppose $\epsilon > 0$. We know there's a value N_1 so that if $n > N_1$, $|x_n - L| < \epsilon/2$. There's another cutoff N_2 so that if $n > N_2$ then $|y_n - M| < \epsilon/2$. Take $N = \max(N_1, N_2)$.

If $n > N$, then both $n > N_1$ and $n > N_2$. This means

$$|(x_n + y_n) - (L+M)| = |(x_n - L) + (y_n - M)| \leq |x_n - L| + |y_n - M|$$

$$< \epsilon/2 + \epsilon/2 = \epsilon.$$

Similar rule for product, quotient, ...

Product formula

Suppose $x_n \to L$, $y_n \to M$. Then $x_n y_n \to LM$.

Pf We'll use some inequality tricks:

$$|x_n y_n - LM| = |(x_n - L)y_n + L(y_n - M)|$$

$$\leq |(x_n - L)y_n| + |L(y_n - M)|$$

want: $< \epsilon$.

$$= |x_n - L|\,|y_n| + |L|\,|y_n - M|$$

Need: $|x_n - L||y_n| < \epsilon/2$ and $|L|\,|y_n - M| < \epsilon/2$.

to get this, pick N_2 so if $n > N_2$ then $|y_n - M| < \frac{\epsilon}{2|L|}$. If $L = 0$ any N_2 works.

y_n converges to M, so $|y_n| \leq |M| + 1$ for $n > N_0$.
(since $|y_n - M| < 1$)

then take N_1 so $|x_n - L| < \frac{\epsilon}{2(|M|+1)}$.

Let $N = \max(N_0, N_1, N_2)$

Limit points

If $x_1, x_2, x_3, x_4, x_5, \ldots$ is a sequence, a <u>subsequence</u> is a new sequence obtained by removing terms.

$1, -1, 1, -1, 1, -1, 1, -1, 1, -1$

sequence has no limit

One subsequence:

$1, 1, 1, 1, 1, 1, \ldots$

but subsequences do!

$-1, -1, -1, -1, -1, \ldots$

(not this one)

$1, 1, -1, -1, 1, 1, -1, -1, 1, 1, \ldots$

A <u>limit point</u> of a sequence is a number R that's a limit of some subsequence.

Q Can you find a sequence with two limit points?

- 10 limit points?
- Infinitely many?
- All positive integers?
- All numbers $0 \leq R \leq 1$?

Integers:

$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots$

k is limit of subsequence $k, k, k, k, k, k, \ldots$

$0 \leq R \leq 1$: almost the "Farey sequence"

$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots$

Why is $\pi/4$ a limit point?

$\pi/4 \approx 0.785398\ldots$ $\quad \frac{7}{10}, \frac{78}{100}, \frac{785}{1000}, \frac{78539}{100000}, \ldots$

converges to $\pi/4$.

<u>Theorem</u> (Bolzano-Weierstrass)

Suppose $x_1, x_2, x_3, \ldots$ is a bounded sequence.

(there are A, B so $A < x_n < B$ for every n).

Then it has a subsequence that converges.

This doesn't work if your sequence isn't bounded:

$1, 2, 3, 4, 5, 6, 7, 8, \ldots$

<u>Series</u> What does it mean that $\sum_{n=1}^{\infty} a_n$ converges?

Define a sequence:

$$S_i = \sum_{n=1}^{i} a_n$$ the "sequence of partial sums"

If sequence S_i converges, we say series converges.

Ex $a_n = \frac{1}{2^n}$ $\quad \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

$\sum_{n=1}^{\infty} a_n$ converges.

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}.$

$S_i = 1 - \frac{1}{2^i}.$

converges to 1 ✓