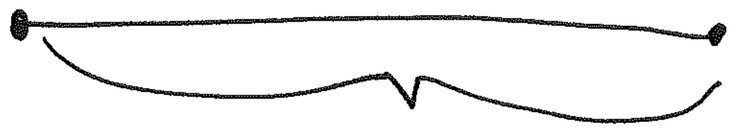
Pluck a string:



L

$u(x,t)$ = displacement of the string at horizontal position x and time t.

Physics: $\cancel{\frac{\partial^2 u}{\partial t^2}}$

T = tension

ρ = density of string

$$\frac{\partial^2 u}{\partial t^2}(x,t) = c^2 \frac{\partial^2 u}{\partial x^2}(x,t) \qquad c = \sqrt{\frac{T}{\rho}}$$

$$u_{tt} = c^2 u_{xx} \qquad u(0,t) = u(L,t) = 0$$

$u_{tt} = c^2 u_{xx}$

Assume $u(x,t) = f(x)\, g(t)$: try to find solutions of this form first.

$f g_{tt} = c^2 f_{xx} g$

f is function of only x,
g is function of only t!

$$\frac{f_{xx}}{f} = \frac{1}{c^2} \frac{g_{tt}}{g}$$

function of only t!

function of only x!

both sides must equal a constant, $-\lambda$.

$f(0) = f(L) = 0$

$$\frac{f_{xx}}{f} = -\lambda$$

$f'' + \lambda f = 0$

$f(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$

$f(0)=0$ says $c_1 + 0 = c_1 = 0$ so $c_1 = 0$

so $f(x) = c_2 \sin(\sqrt{\lambda}\, x)$

$f(L)=0$ says $c_2 \sin(\sqrt{\lambda}\, L) = 0$

this would tell us $c_2 = 0$ so $f(x)=0$ (boring!)

<u>unless</u> $\sin(\sqrt{\lambda}\, L) = 0$ which happens for special λ values.

$\sqrt{\lambda} L = n\pi$

$\lambda L^2 = n^2 \pi^2$

$\lambda = \frac{n^2 \pi^2}{L^2}$.

solution is

$f(x) = c_2 \sin\left(\frac{n\pi}{L} x\right)$.

(for any integer $n \geq 1$)

Now we need to solve

$$\frac{1}{c^2}\frac{g_{tt}}{g} = -\lambda \quad \text{where} \quad \lambda = \frac{n^2\pi^2}{L^2}$$ (for other λ, the f eqn has no sols!)

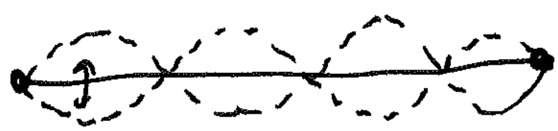
$$g(t) = \cos\left(\frac{n\pi c}{L}t\right)$$

Putting it together:

$$u_n(x,t) = \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi c}{L}t\right).$$

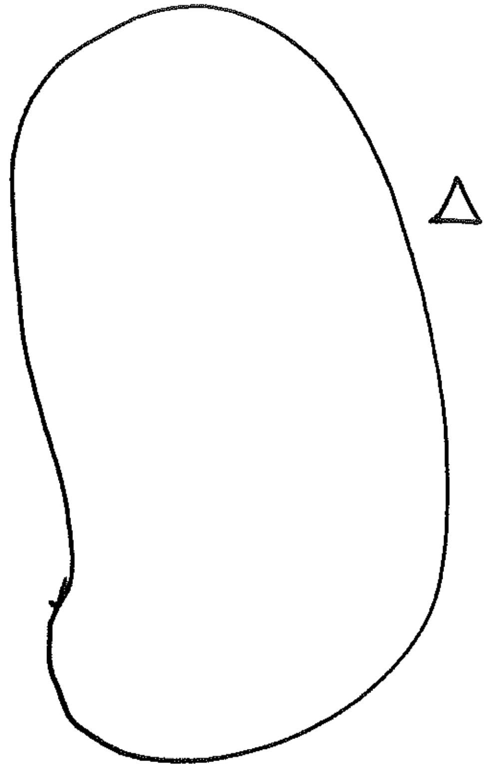
$$c = \sqrt{\frac{\text{Tension}}{\text{density}}}$$

frequency $\frac{nc}{2L}$



The sound of a string will be a combination of $u_n(x,t)$ all added together. What frequencies do we hear?

$$\underset{\substack{\uparrow \\ u_1}}{\frac{c}{2L}}, \quad \underset{\substack{\uparrow \\ u_2}}{\frac{2c}{2L}}, \quad \underset{\substack{\uparrow \\ u_3}}{\frac{3c}{2L}}$$



drum in shape Δ.

what frequencies show up when we hit it?

(it depends on Δ)

You can set up a function $u(x,y,t)$

$$u_{tt} = c^2(u_{xx} + u_{yy}) = c^2 \Delta u$$

"Laplacian of u"

Wave eqn for drum:

$$u_{tt} = c^2 \Delta u$$

If f is function f where

$$\Delta f = \lambda f$$

multiple of f,

then λ is called "eigenvalue of Laplacian", and it's a frequency of the drum!

The sound of the drum is determined

by the set of eigenvalues of laplacian (depends on Δ)
shape.

Q Can you hear shape of a drum?

((Can two different shapes yield the same λ values?)

A Two drums can produce the same

frequencies!

Random walks

1D.

0

start here

- move left or right with probability $\frac{1}{2}$.
- Q: what's the probability you get back to 0 eventually?
- warm-up: what's the probability you get back to 0 after exactly n steps?

  (what if $n=4$?)

What's the chance you're back after n steps?

If n is odd: 0. You need to have made same number of L and R.

If n is even:

Total number of length n strings with L&R:

2^n

Total number that get you back to 0: need the same number of L & R

$n = 2m$. then

$$\underbrace{LLLL}_{n/2}\ \underbrace{RRRR}_{n/2} \qquad \frac{(2m)!}{m!\,m!} = \binom{2m}{m}$$

The chance we're back is $u_{2m} = \dfrac{\binom{2m}{m}}{2^{2m}}$

Now look at

$$S = \sum_{m=1}^{\infty} \frac{1}{2^{2m}} \binom{2m}{m}$$

this tells us: the expected number of times our random path comes back to 0.

If walk is guaranteed to return: $S = \infty$
sum diverges

it not: S is finite
sum converges.

$$S = \sum_{m=1}^{\infty} \frac{1}{2^{2m}} \binom{2m}{m} = \sum_{m=1}^{\infty} \frac{1}{2^{2m}} \frac{(2m)!}{m!\, m!}$$

Stirling's approximation:

$$n! \approx \sqrt{2\pi n}\, e^{-n} n^n$$

$$\sum_{m=1}^{\infty} \frac{1}{2^{2m}} \frac{(2m)!}{m!\, m!} =$$

$$\frac{1}{2^{2m}} \frac{(2m)!}{m!\, m!} \approx \frac{1}{2^{2m}} \frac{\sqrt{4\pi m}\, e^{-2m} (2m)^{2m}}{\left(\sqrt{2\pi m}\, e^{-m} m^m\right)^2}$$

$$= \cancel{\frac{1}{2^{2m}}} \frac{\sqrt{4\pi m}}{(\sqrt{2\pi m})^2} \cdot \frac{e^{-2m}}{(e^{-m})^2} \frac{1}{2^{2m}} \frac{(2m)^{2m}}{(m^m)^2}$$

$$= \frac{1}{\sqrt{\pi m}} \cdot 1 \cdot 1$$

$$\sum_{m=1}^{\infty} \frac{1}{2^{2m}} \frac{(2m)!}{m!\, m!} \approx \sum_{m=1}^{\infty} \frac{1}{\sqrt{\pi m}}$$

diverges!

random walk comes back!