Name: ______________________________

Math 436, Midterm 2
Spring 2021, John Lesieutre

- You have fifty minutes to complete the exam.
- Exam should be submitted on Gradescope once completed. (Scanning and uploading time do not count towards the fifty minutes.)
- You may consult your notes, the course materials on my website, and the textbook.
- Collaboration and all other references are not allowed.
- Although you can write your answers on a copy of the exam, it is not required.
- Justifications or proofs are required for all problems except where indicated otherwise.
- Please either sign below the integrity statement below or copy out this statement at the beginning of your exam.
- I am generous with partial credit! Try not to leave anything blank.
- Good luck!

I affirm that I have complied with all the exam requirements. I have completed the exam within the allotted time and have not consulted any disallowed references.

Signature: ______________________________

Problem	Score	Possible
1		20
2		20
3		20
4		20
5		20
Σ		100

Problem 1. Consider the map $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ given by $T(f) = f + 2f'$.

a) (10 points) Give a basis for this space, and compute the matrix for T with respect to your basis.

b) (10 points) Compute $\mathrm{rref}(M)$ for your answer to (a). Is the map invertible? Justify your answer.

Problem 2. Let $V = \mathcal{P}_3(\mathbb{R})$ and $W = \mathcal{P}_1(\mathbb{R})$. Consider the bases $x^3, x^2, x, 1$ for V and $x, 1$ for W. Suppose that $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_1(\mathbb{R})$ is a linear map whose matrix with respect to these bases is given by:

$$M = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 6 & 4 & -2 \end{pmatrix}.$$

This has

$$\operatorname{rref}(M) = \begin{pmatrix} 1 & 0 & -4 & 2 \\ 0 & 1 & 2 & -1 \end{pmatrix}$$

a) (10 points) Find bases for the nullspace and the range of T.

b) (10 points) Is the vector $3x - 2$ in the range? Justify your answer.

Problem 3. a) (10 points) Give a basis for the product space $\mathbb{R}^{2,2} \times \mathbb{R}^2$. (Here $\mathbb{R}^{2,2}$ denotes the vector space of 2×2 matrices.) What is the dimension of $\mathbb{R}^{2,2} \times \mathbb{R}^2$?

b) (10 points) Find an invertible map $T : \mathbb{R}^{2,2} \times \mathbb{R}^2 \to \mathcal{P}_5(\mathbb{R})$. (You just need to clearly state what your map is; no proof necessary. But if you're not confident, a brief explanation will help with partial credit.)

Problem 4. a) (10 points) Suppose that V is a vector space. Define the dual space of V (a definition for any vector space). Give an example of a nonzero element of the dual space $(\mathbb{R}^3)'$.

b) (10 points) Consider the elements λ, μ, ν of $(\mathcal{P}_2(\mathbb{R}))'$ defined by

$$\begin{aligned} \lambda(f) &= \int_{-1}^{1} f\, dx \\ \mu(f) &= f(0) \\ \nu(f) &= f''(0) \end{aligned}$$

Prove that $3\lambda - 6\mu - \nu = 0$ in $(\mathcal{P}_2(\mathbb{R}))'$.

Problem 5. Suppose that $T : V \to V$ is a linear map.

a) (10 points) Suppose that $U \subset V$ is an invariant subspace for T. Prove that U is also an invariant subspace for the composition $T^2 = T \circ T$.

b) (10 points) Let $\mathbb{R}^\infty$ denote the set of infinite sequences of real numbers. Consider the double-left-shift map $L^2 : \mathbb{R}^\infty \to \mathbb{R}^\infty$ given by

$$L^2(a_0, a_1, a_2, a_3, a_4, \ldots) = (a_2, a_3, a_4, \ldots)$$

Find an eigenvector of this map and the corresponding eigenvalue.