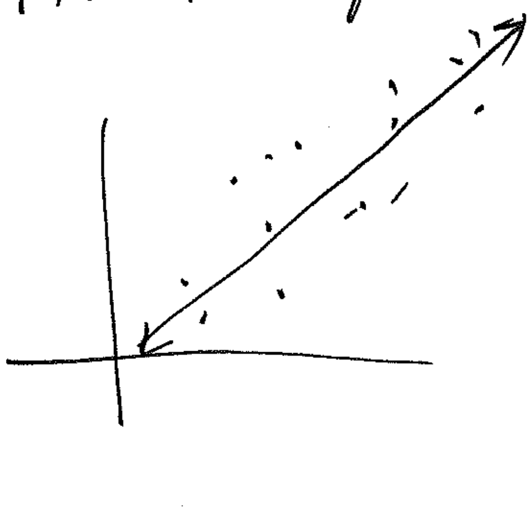
One More Inequality: Jensen's

↳ Introduce abstract algebra.
(Inner product spaces).

A theorem/formula we prove ~~theorem~~
about inner product spaces using only the axioms can be applied to any inner product. The same formula lets us:

→ Find closest point on a plane to a given pt

→  Find best fit line / parabola

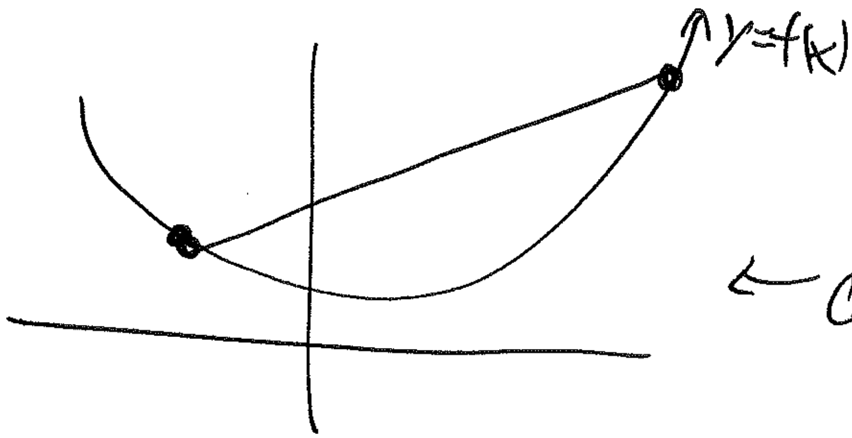
→ Best polynomial approxm. to a non-polynomial fct.

→ Fourier series

$$\rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Jensen's Inequality

A function ~~on~~ is <u>convex</u> on an interval. if a secant ~~line~~ segment always lies above (or on) the graph:

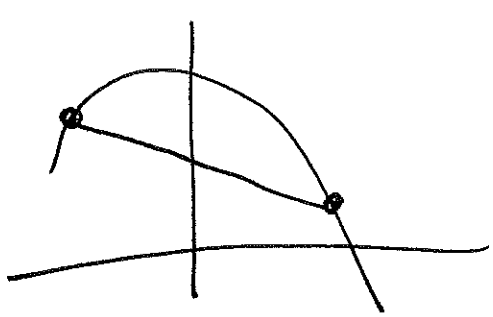


← convex

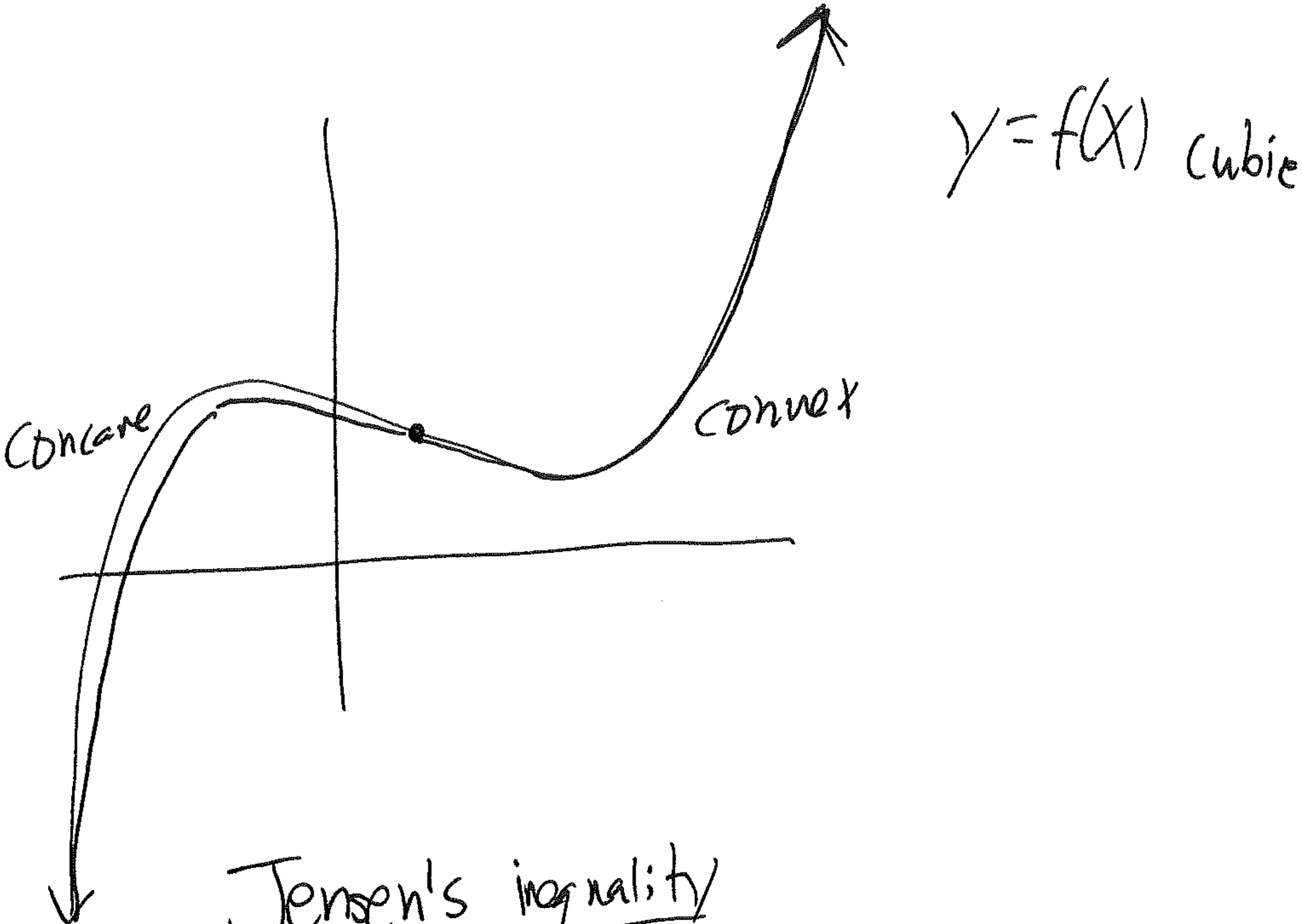
If $f'' \geq 0$, it's convex.

But f can be convex even if not differentiable! $|x|$

A <u>concave</u> fct has secant segments below or on the graph.



f convex $\Longleftrightarrow$ $-f$ concave



Jensen's inequality

Version 1: Suppose f convex on interval (a, b).

Then for $x_1, \ldots, x_n \in (a,b)$ we have

$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq \frac{f(x_1) + \cdots + f(x_n)}{n}.$$

If f is concave, reverse the inequality!

e.g. $f(x)=x^2$.

Supp $x_1=0.3$

$x_2=0.7$

$$f\left(\frac{x_1+x_2}{2}\right)=f(0.5)=0.25$$

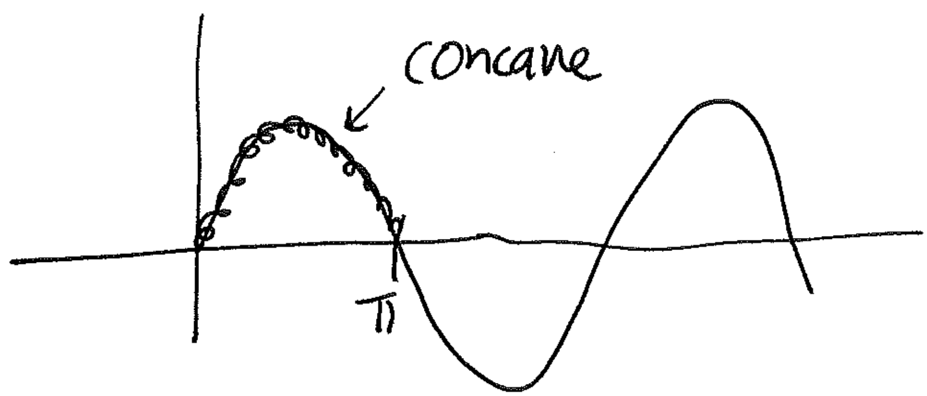
vs $\dfrac{f(x_1)+f(x_2)}{2}=\dfrac{0.09+0.49}{2}=0.29$ ✓

Ex Suppose ABC is a triangle.

Prove

$$\sin(A) + \sin(B) + \sin(C) \leq \frac{3\sqrt{3}}{2}$$

$f(x) = \sin(x)$



Jensen's:

$$\sin\left(\frac{A+B+C}{3}\right) \geq \frac{\sin(A) + \sin(B) + \sin(C)}{3}$$

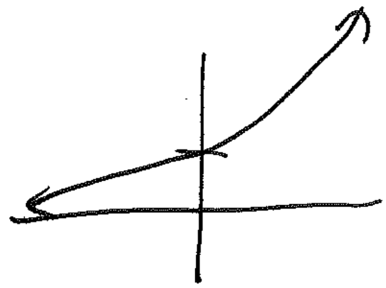
$$\frac{\sin(A) + \sin(B) + \sin(C)}{3} \leq \sin(60^\circ) = \frac{\sqrt{3}}{2}$$

$$\sin(A) + \sin(B) + \sin(C) \leq \frac{3\sqrt{3}}{2}.$$

Suppose $a_1, \ldots, a_n$ are positive.

Let $x_i =$. $\ln(a_i) = x_i$

Consider $f(x) = e^x$. convex.



What does Jensen's ineq tell you?

$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq \frac{f(x_1) + \cdots + f(x_n)}{n}$$

$$e^{(\ln a_1 + \ln a_2 + \cdots + \ln a_n)/n} \leq \frac{e^{\ln a_1} + \cdots + e^{\ln a_n}}{n}$$

$$e^{\ln(a_1 a_2 \cdots a_n)/n} \leq \frac{a_1 + \cdots + a_n}{n}$$

$$(a_1 \cdots a_n)^{1/n} \leq \frac{a_1 + \cdots + a_n}{n}$$

[AM-GM inequality!

Example

Suppose $a, b, c > 0$ and $a+b+c = abc$.

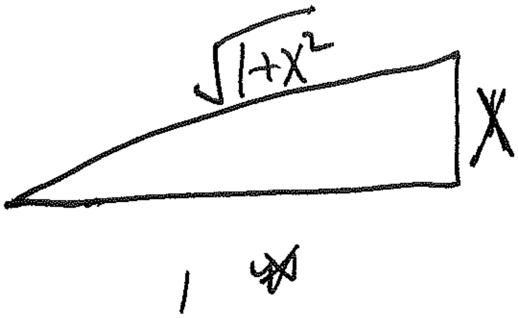
$$\frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \le \frac{3}{2}.$$

Can't simply use Jensen with $f(x) = \frac{1}{\sqrt{1+x^2}}$

because Ƶ

Ƶ: left side $\approx \frac{f(a)+f(b)+f(c)}{3}$

right side: $\frac{f(a+b+c)}{3}$ but I don't know $a+b+c$.



Let $a=\tan A$
$b=\tan B$
$c=\tan C$
all between $0, \pi/2$.

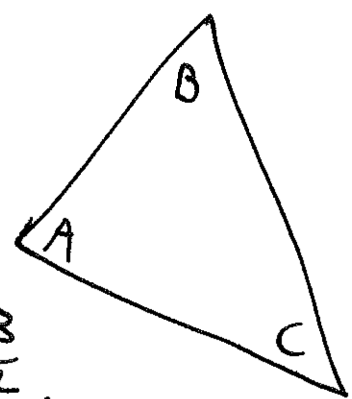
and apply Jensen's inequality to A, B, C instead.

$a+b+c=abc$ tells us what about A, B, C?

$$\tan(A+B+C)=\frac{\overbrace{\tan A+\tan B+\tan C-\tan A\tan B\tan C}^{a+b+c-abc}}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}=0$$

so $\tan(A+B+C)=0$

and so $A+B+C=\pi$



we want $\cos(A)+\cos(B)+\cos(C)\le\frac{3}{2}$.

$$\cos(A)+\cos(B)+\cos(C)\le 3\cos\left(\frac{A+B+C}{3}\right)=3\cos(60^\circ)=\frac{3}{2}.$$

Vector spaces

Suppose V is a set.

An addition rule on V means a rule assigning a sum $u+v \in V$ for any $u \in V$ and $v \in V$.

A scalar multiplication rule on V is a rule assigning a product $av \in V$ for any $v \in V$ and $a \in \mathbb{R}$.
(tells you how to do scalar $\times$ vector)

Def A vector space V is a set with an addition rule and a scalar mult rule that satisfy some axioms:

1) $u+v=v+u$ (addition is commutative)

2) $(u+v)+w=u+(v+w)$ and $(ab)v=a(bv)$

3) There's an identity $0 \in V$ so $v+0=0+v=v$.

4) For any v, there's a w so $v+w=0$

5) $1v=v$

6) $a(u+v)=au+av$, $(a+b)v=av+bv$.

Examples:

- regular old vectors of length n (for any n) [dimension] $(\mathbb{R}^n)$
- $m \times n$ matrices are a vector space. $(M^{m\times n})$
- complex numbers $(\mathbb{C})$
- imaginary numbers (bi) $(\mathbb{R}i)$ incl $0 \cdot i$.
- continuous fcts $f: \mathbb{R} \to \mathbb{R}$ $(C^0(\mathbb{R}))$
- infinite sequences $(a_1, a_2, a_3, a_4, \ldots)$ $(\mathbb{R}^\infty)$

What about:

a) Integers

No: can't multiply by $\frac{1}{2}$

f) Functions with period π

Yes ✓ $\quad f(x+\pi)=f(x)$ any x.

b) Positive real numbers

No: no additive identity 0 / can't mult by -1.

g) Periodic functions

$\sin(x)+\sin(\pi x)$

c) Functions f with $f(1)=1$

No: x^2+1 is one, x^4+1 is one but $x^4+x^2+\underline{\underline{2}}$ has $f(1)=2$.

d) Functions f with $f(1)=0$

Yes ✓

e) Polynomials of any degree

✓

What if we defined addition by $\oplus$

$$(f \oplus g)(x) = f(x) + g(x) - 1.$$

If $f(1)=1$ and $g(1)=1$ then $(f \oplus g)(1) = 1$.

Is functions with $f(1)=1$ a vector space if we use $\oplus$?

Is $a(u \oplus v) = au \oplus av$?

$$a(u+v-1) \stackrel{?}{=} au+av-1$$

$$au+av-a \stackrel{?}{=} au+av-1$$