Solving a system of linear equations

1. Convert our equations into an augmented matrix

$$\begin{aligned} 3x-2y+z&=7 \\ 2x+y-2z&=3 \end{aligned} \rightsquigarrow \left(\begin{array}{ccc|c} 3 & -2 & 1 & 7 \\ 2 & 1 & -2 & 3 \end{array}\right)$$

2. Do row operations on the matrix to put it into RREF. (row reduced echelon form)
   (this corresponds to eliminating variables)

   a) Add a multiple of a row to another

   b) Multiply a row by a number

   c) Swap two rows.

$$\left(\begin{array}{ccc|c} 3 & -2 & 1 & 7 \\ 2 & 1 & -2 & 3 \end{array}\right) \xrightarrow[R2 \mathrel{+}= -\frac{2}{3}\cdot R1]{} \left(\begin{array}{ccc|c} 3 & -2 & 1 & 7 \\ 0 & 7/3 & -8/3 & -5/3 \end{array}\right)$$

$$\Big\downarrow\ R2 \times= 3$$

$$\left(\begin{array}{ccc|c} 3 & 0 & -9/7 & 39/7 \\ 0 & 7 & -8 & -5 \end{array}\right) \xleftarrow{R1 \mathrel{+}= \frac{2}{7} R2} \left(\begin{array}{ccc|c} 3 & -2 & 1 & 7 \\ 0 & 7 & -8 & -5 \end{array}\right)$$

$$\Big\downarrow\ \begin{array}{l} R1 \times= \frac{1}{3} \\ R2 \times= \frac{1}{7} \end{array}$$

$$\begin{array}{c} \left(\begin{array}{ccc|c} 1 & 0 & -3/7 & 13/7 \\ 0 & 1 & -8/7 & -5/7 \end{array}\right) \\ \begin{array}{cccc} x & y & & \end{array} \end{array}$$

RREF!

$$x - \tfrac{3}{7} z = \tfrac{13}{7}$$
$$y - \tfrac{8}{7} z = -\tfrac{5}{7}$$

3) Read off all solutions from rref.

Identify <u>pivot</u> variables (correspond to leading entries of some row)	Identify <u>free</u> variables (not <u>pivot</u> variables)
x and y	z

Solve for ~~these~~ pivot variables in terms of free variables.

$$\left.\begin{aligned} z &= \text{anything} \\ x &= \tfrac{13}{7} + \tfrac{3}{7} z \\ y &= -\tfrac{5}{7} + \tfrac{8}{7} z \end{aligned}\right] \quad \text{"general solution"}$$

A matrix is in "echelon form" if

1) all rows of 0s at bottom
2) the leading entry is to the right of the leading entry above it
3) all entries <u>below</u> a leading entry are 0

" "row reduced echelon form"

4) every leading entry is 1
5) each leading 1 is only nonzero thing in column.

Find general solution of

$$x+3y+4z=7$$
$$3x+9y+7z=6$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 3 & 9 & 7 & 6 \end{array}\right) \xrightarrow{R2 += (-3)R1} \left(\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{array}\right)$$

$$\Big\downarrow R2 \times= \left(-\tfrac{1}{5}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right) \xleftarrow{R1 += (-4)R2} \left(\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Solution? Pivots: x, z y can be anything.

Free: y Solve for x, z in terms of y.

$$\begin{matrix} x+3y=-5 \\ z=3 \end{matrix} \Longrightarrow \begin{cases} y \text{ is anything} \\ x=-5-3y \\ z=3 \end{cases}$$

e.g. plug in $y=-2$

$\downarrow$

$x=1$
$y=-2$
$z=3$

Balance the reaction:

$$\underline{x_1}\ Na_3PO_4 + \underline{x_2}\ Ba(NO_3)_2 \rightarrow \underline{x_3}\ Ba_3(PO_4)_2 + \underline{\overset{x_4}{\cancel{4}}}\ NaNO_3$$

Fill in the blanks!

Na: $3X_1 = X_4$ $\qquad\qquad x_1 - x_4 = 0$

Ba: $X_2 = \cancel{x_3}\ 3x_3$ $\qquad\qquad x_2 - 3x_3 = 0$

PO_4: $x_1 = 2x_3$ $\qquad\qquad x_1 - 2x_3 = 0$

NO_3: $2x_2 = x_4$ $\qquad\qquad 2x_2 - x_4 = 0$

$$\left(\begin{array}{cccc|c} 3 & 0 & 0 & -1 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 1 & 0 & -2 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 \end{array}\right) \xrightarrow{\text{rref}} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & -1/6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

Pivot: x_1, x_2, x_3

Free: x_4

$x_1 = +1/3\, x_4$

$x_2 = +1/2\, x_4$

$x_3 = +1/6\, x_4$

$x_4 = 6 \qquad x_2 = 3$

$x_1 = 2 \qquad x_3 = 1$

$$2\,Na_3PO_4 + 3\,Ba(NO_3)_2 \rightarrow Ba_3(PO_4)_2 + 6\,NaNO_3.$$

One more:

$$\underline{x_1}\,CH_3NH_2 + \underline{x_2}\,O_2 \longrightarrow \underline{x_3}\,CO_2 + \underline{x_4}\,H_2O + \underline{x_5}\,N_2$$

C: $x_1 = x_3$	$x_1 - x_3 = 0$
H: $5x_1 = 2x_4$	$5x_1 - 2x_4 = 0$
N: $x_1 = 2x_5$	$x_1 - 2x_5 = 0$
O: $2x_2 = 2x_3 + x_4$	$2x_2 - 2x_3 - x_4 = 0$

$$\left(\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 0 & 0 \\ 5 & 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 & 0 \end{array}\right) \xrightarrow{\text{rref}} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & -9/2 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -5 & 0 \end{array}\right) \end{array}$$

x_5 free!

$$x_1 = 2x_5 \qquad\qquad x_1 = 4$$
$$x_2 = 9/2\,x_5 \xrightarrow{\text{use } 2} x_2 = 9$$
$$x_3 = 2x_5 \qquad\qquad x_3 = 4$$
$$x_4 = 5x_5 \qquad\qquad x_4 = 10$$

What happens to systems with no solution?

$$x + y + z = 2$$
$$2x + 2y + 2z = 5$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right). \quad \text{rref } \checkmark$$

How many solutions to a linear system of eqns?

- Write down matrix, compute rref.

- If there's a row $(0\ 0\ 0\ 0 \mid b)$ where $b \neq 0$, no solutions!

- If there's a free variable (but no $(0\ 0\ 0 \mid b)$ row), then infinitely many solutions.

- If there's no free variable and no $(0\ 0\ 0 \mid b)$ row, then one solution.

Also notice:

If you have more variables than equations, and there is a solution, then there are infinitely many solutions.

Why? e.g. 4 variables, 2 eqns.

There is at most one pivot variable per equation! (At most 2 in this case)

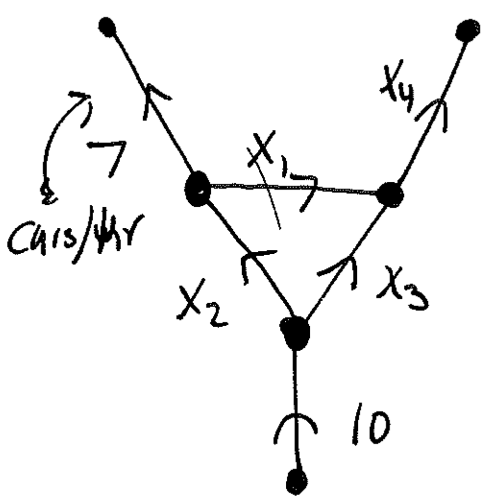
So there must be free variable.

Applications to networks

A "network" is:

- a set of "nodes"
- a set of "branches" connecting nodes

Ex network of roads: nodes are intersections, branches are roads (one-way) connecting the intersections.



to get linear equations:

1) at any intersections, (cars in) = (cars out)

2) (total cars in) = (total cars out)

① $x_2 + x_3 = 10$

② $x_2 = 7 + x_1$

③ $x_1 + x_3 = x_4$

④ $10 = 7 + x_4$

Augmented matrix:

$$\left(\begin{array}{cccc|c} -1 & 1 & 0 & 0 & 7 \\ 0 & 1 & 1 & 0 & 10 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

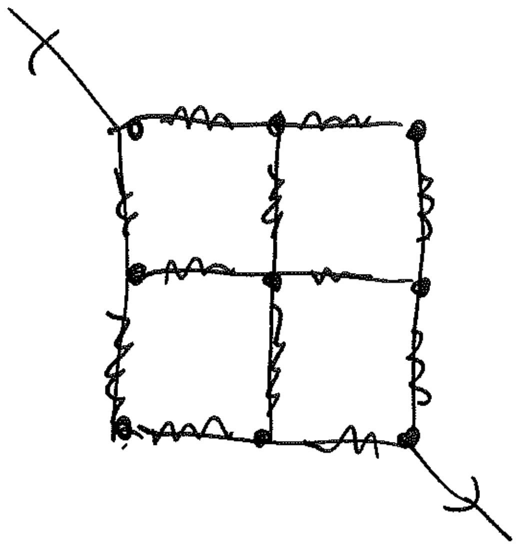
$$\Big\downarrow \text{rref}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Multiple solutions! We need one more equation. Send out somebody else to watch the road, maybe measure value of x_1.

This works for circuits too!

use Kirkhoff's junction rule at each point)



equivalent resistance?

To invert this

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 3 & -4 & 0 & 1 & 0 \\ 7 & 5 & -12 & 0 & 0 & 1 \end{array}\right)$$

$\Big\downarrow$ rref

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \text{inverse of the matrix you started with} \\ 0 & 0 & 1 & \end{array}\right)$$