Today: More topology

(But I have to leave extra early to give an exam.)
1:45

Reminder:

X and Y are homeomorphic if

there's a continuous $f: X \to Y$ with continuous

inverse $g: Y \to X$.

("Can stretch one into the other with no gluing/ ripping").

How to know/prove two things are homeomorphic

Just find f & g.

How to prove two things are <u>not</u> homeomorphic:

use a "topological invariant": a property/measure of X that's unaffected by homeomorphism.

- X is <u>simply connected</u> if every loop in X can be shrunk to point

annulus: not simply connected

disk: simply connected

⇒ ~~not simply connected~~ not homeomorphic.

- X is <u>connected</u> if it's just one "piece".

not connected

connected

- the number of edges is a topological invariant

(needs defined)

so cylinder and Möbius strip are not homeomorphic

2 edges — 1 edge

- contractiblity is a topological invariant

a set is ~~contractible~~ simply connected if any loop the set can be shrunk to a point.

contractible	not contractible
$\mathbb{R}$, $\mathbb{R}^2$ — anything convex	annulus
the sphere S^2	Möbius Strip

Examples

- number of holes
- 1-cuttable.

circle

not 1-cuttable

figure-8

1-cuttable

Say X is "1-cuttable" if you can remove a single point and make it not connected.

Möbius strip vs cylinder

- number of edges
- is edge connected

$\mathbb{R}$ vs $\mathbb{R}^2$

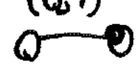
1-cuttable

$\mathbb{R}^2$ vs $\mathbb{R}^3$

$\mathbb{R}^2$: remove pt not simply connected

$\mathbb{R}^3$: remove pt and still simply connected

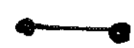
$(0,1)$



no edges

removing any pt makes disconnected

$[0,1]$



2 edges

removing a pt can leave it connected

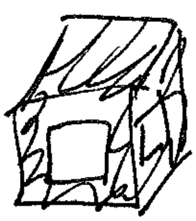
- number of edges is a topological invariant
  ("boundary")
- convex not a topological invt


convex


not

But they are homeomorphic.

- number of holes is topological invt.
  (but hard to define)


cube with holes in horizontal faces


3 holes annulus.

~~Homotopy~~

Homotopy

Two functions $f: X \to Y$

$g: X \to Y$

are **homotopic** if you can smoothly turn one into the other.

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time-dependent

This means you can find a family of functions

$$F_t : X \to Y \quad (0 \le t \le 1)$$

where $F_0 = f$ and $F_1 = g$

You can also think of F as being a single function

ordered pair (x, t)

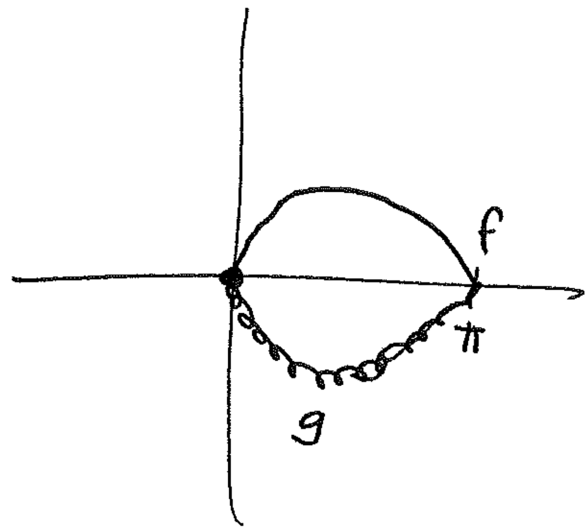
$$F : X \times [0,1] \to Y$$

Ex let $X = [0, \pi]$

$Y = \mathbb{R}^2$

$f(x) = (x, \sin x)$

$g(x) = (x, -\sin x)$



$F_t(x) = (x, (1-2t)\sin x)$

$F_0(x) = (x, \sin x)$

$F_1(x) = (x, -\sin x)$

you could think of this a

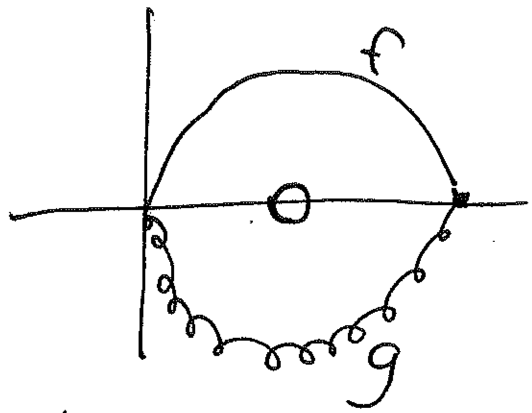
$F(x,t) = (x, (1-2t)\sin x)$

$X = [0, \pi]$

$Y = \mathbb{R}^2 \setminus \{(\frac{\pi}{2}, 0)\}$

$f(x) = (x, \sin x)$

$g(x) = (x, -\sin x)$



Can't use same F_t anymore!

Still homotopic (just move away from hole and do as before)

but finding eqn is hard.

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ex $f: (-\pi/2, \pi/2) \to \mathbb{R}$

$f(x) = \tan(x)$

$g: (-\pi/2, \pi/2) \to \mathbb{R}$

$g(x) = x$

homotopic!

One more:

$$F_t : (-\pi/2, \pi/2) \to \mathbb{R}$$

$F_0 = f$

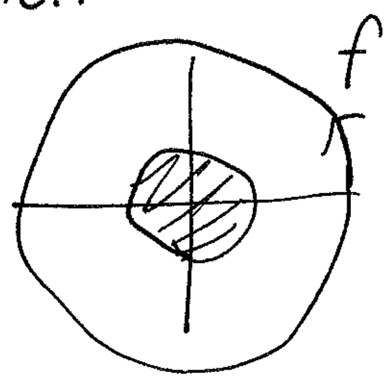
$F_1 = g.$

$$F_t(x) = (1-t)\tan(x) + tx$$

Can two things <u>not</u> be homotopic??

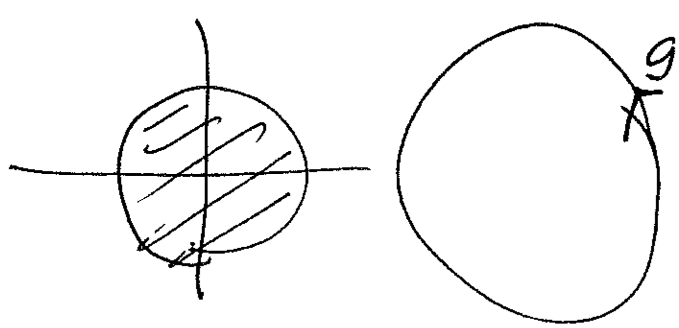
$f : S^1 \to \mathbb{R}^2 \setminus D$

circle — plane minus unit disk



$g : S^1 \to \mathbb{R}^2 \setminus D$

circle



hard to prove!