Warm-up to Fourier series

Let's imagine we have an orthonormal basis $e_1, \dots, e_n$ for a vector space V.

Suppose we have a vector v and want to write

$$v = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n. \qquad (c_i \text{ scalars})$$

How to find c_i? If orthonormal, it's easy!

$$\langle v, e_1 \rangle = \langle c_1 e_1 + \cdots + c_n e_n, e_1 \rangle$$

$$= c_1 \langle e_1, e_1 \rangle + c_2 \langle e_2, e_1 \rangle + \cdots c_n \langle e_n, e_1 \rangle.$$

$$= c_1 + 0 + 0 + \cdots + 0 = c_1$$

To get c_j, do $\langle v, e_j \rangle$!

Last time:

$\rightarrow V = \left\{ \begin{matrix} \text{all periodic functions} \\ \text{period } 2\pi \end{matrix} \right\} \qquad \langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} fg \, dx$

An orthonormal basis: $1/\sqrt{2}$, $\cos x$, $\sin x$, $\cos(2x)$, $\sin(2x)$,

$\cos(3x)$, $\sin(3x)$, $\cos(4x)$, $\sin(4x)$, ...

Suppose f is periodic. How to write f as combo of sin & cos?

$$f = \frac{1}{\sqrt{2}} a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx).$$

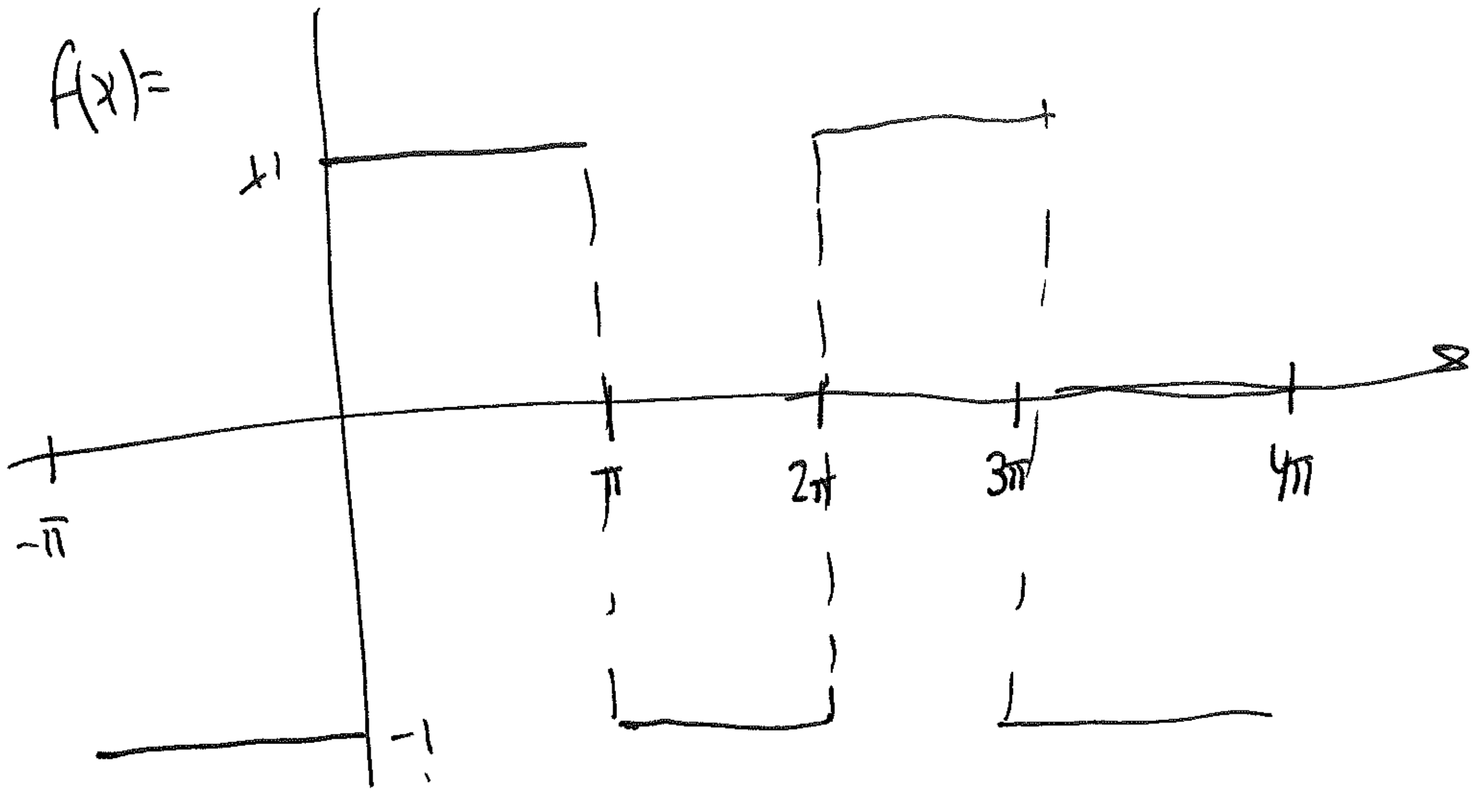
$\cancel{a_0 = \frac{1}{\pi}}$ $\quad a_0 = \left\langle f, \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} f \, dx$

$$a_n = \langle f, \cos(nx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f \cos(nx) \, dx$$

$$b_n = \langle f, \sin(nx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f \sin(nx) \, dx$$

Square wave

$f(x) =$



even × odd

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} f \, dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f \sin(nx) \, dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f \cos(nx) \, dx = 0$$

odd × even

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \overbrace{f \sin(nx)}^{\text{odd} \times \text{odd} = \text{even}}\, dx = \frac{2}{\pi}\int_0^{\pi} f \sin(nx)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin(nx)\, dx$$

$\cancel{\cos(n\pi) =}$

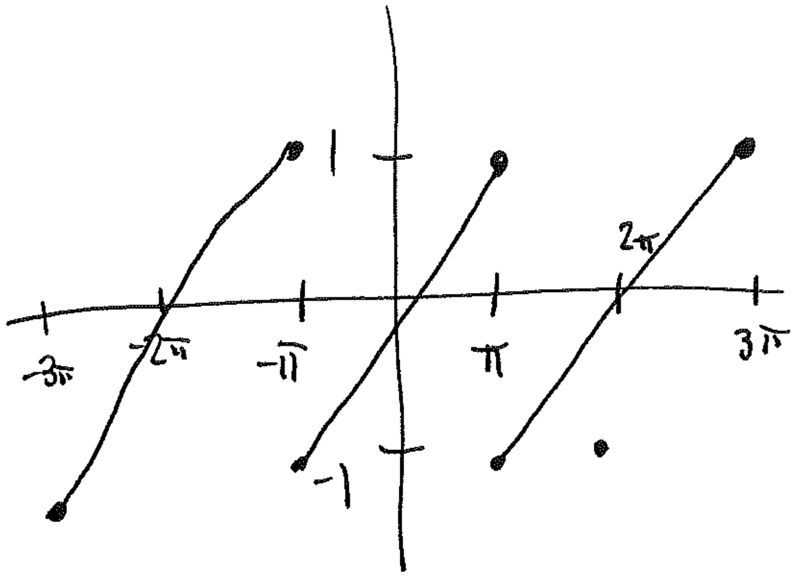
$$\cos(n\pi) = (-1)^n = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$$

$$= \frac{2}{\pi}\left(-\frac{1}{n}\cos(nx) \Big|_0^{\pi} \right)$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{4}{\pi n} & n \text{ odd} \end{cases}$$

$$f(x) = \frac{4}{\pi}\sin(x) + \frac{4}{3\pi}\sin(3x) + \frac{4}{5\pi}\sin(5x) + \frac{4}{7\pi}\sin(7x) + \dots$$

Sawtooth wave:



$$a_0 = \langle f, \tfrac{1}{\sqrt{2}} \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{\sqrt{2}} f\, dx$$

$$a_n = \langle f, \cos(nx) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f\cos(nx)\, dx$$

$$b_n = \langle f, \sin(nx) \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f\sin(nx)\, dx$$

$a_i = 0$ (odd!)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\, dx = \frac{2}{\pi}\int_{0}^{\pi} x\sin(nx)\, dx$$

$$\int x\sin(nx)\, dx = -\frac{x}{n}\cos(nx) + \int \frac{1}{n}\cos(nx)\, dx$$

$u = x \quad v = -\frac{1}{n}\cos(nx)$

$du = dx \quad dv = \sin(nx)\, dx$

$$= -\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)$$

$$= \frac{2}{\pi}\left(\left(-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right)\Big|_0^{\pi}\right) = \left(\right)$$

$$= \frac{2}{\pi}\left(\left(-\frac{\pi}{n}(-1)^n + \frac{0}{n}\cos(0x)\right)\right) = -\frac{2}{n}(-1)^n = \frac{2}{n}$$

$$= \begin{cases} \frac{2}{n} & n \text{ odd} \\ -\frac{2}{n} & n \text{ even.} \end{cases}$$

$$f(x) = 2\sin(x) - \frac{2}{2}\sin(2x) + \frac{2}{3}\sin(3x) - \frac{2}{4}\sin(4x) + \frac{2}{5}\sin(5x)$$

Pythagorean theorem.

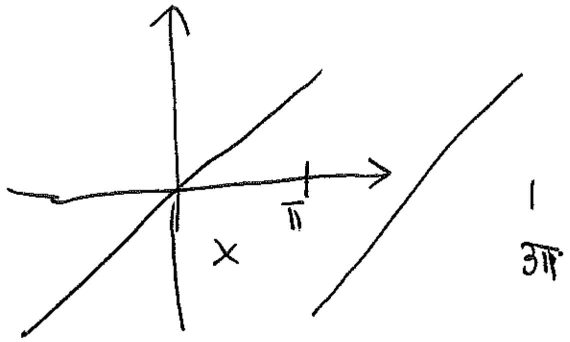
length of $(2,3,4)$? $\sqrt{2^2+3^2+4^2}$.

e_i orthonormal basis.

$$\|c_1e_1 + \cdots + c_ne_n\|^2 = \|c_1e_1\|^2 + \cdots + \|c_ne_n\|^2$$

$$= c_1^2\|e_1\|^2 + \cdots + c_n^2\|e_n\|^2$$

$$= c_1^2 + c_2^2 + c_3^2 + \cdots + c_n^2.$$



Before

$$f(x) = \frac{2}{1}\sin(x) - \frac{2}{2}\sin(2x) + \frac{2}{3}\sin(3x) - \frac{2}{4}\sin(4x) + \frac{2}{5}\sin(5x) - \cdots$$

$$= c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 + \cdots$$

$$\|f\|^2 = c_1^2 + c_2^2 + c_3^2 + c_4^2 + \cdots$$

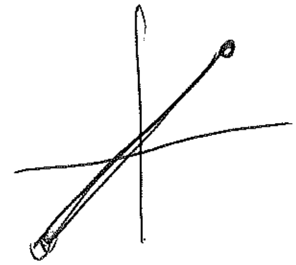
$$= \left(\frac{2}{1}\right)^2 + \left(-\frac{2}{2}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(-\frac{2}{4}\right)^2 + \left(\frac{2}{5}\right)^2 + \cdots$$

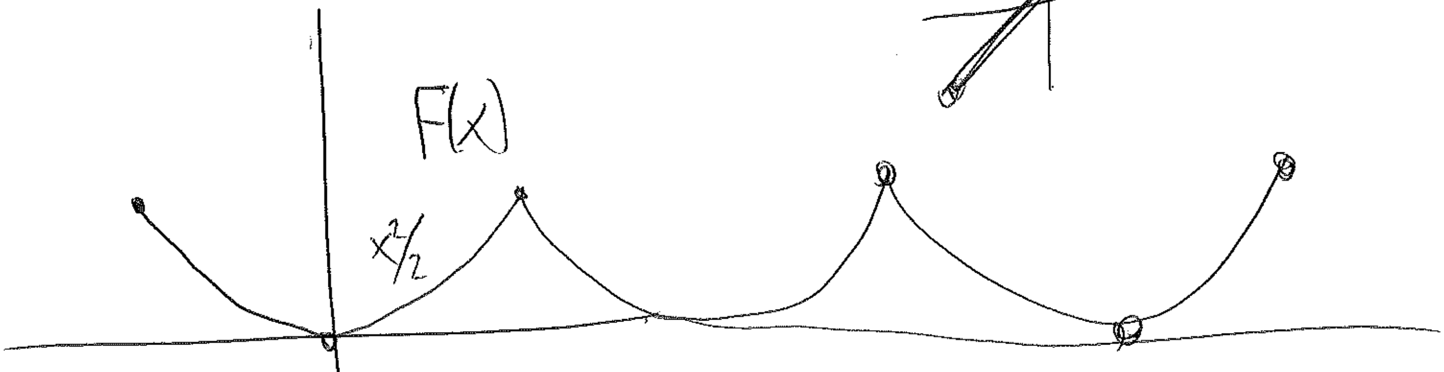
$$= 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots\right)$$

$$\|f\|^2 = \langle f, f \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi}\left(\frac{2\pi^3}{3}\right) = \frac{2\pi^2}{3}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{2\pi^2}{3} \cdot \frac{1}{4} = \frac{\pi^2}{6}.$$

What other sums can we get?





$$f(x) = \frac{2}{1}\sin(x) - \frac{2}{2}\sin(2x) + \frac{2}{3}\sin(3x) - \frac{2}{4}\sin(4x)$$

$$C + F(x) = -\frac{2}{1}\cos(x) + \frac{2}{4}\cos(2x) - \frac{2}{9}\cos(3x) + \frac{2}{16}\cos(4x) + \cdots$$

Pythagorean thm:

$$\left(-\frac{2}{1}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(-\frac{2}{9}\right)^2 + \left(\frac{2}{16}\right)^2 + \left(-\frac{2}{25}\right)^2 .$$

$$\| \; 4\sum_{n=1}^{\infty} \frac{1}{n^4} = \|f\|^2$$

$$\hookrightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$