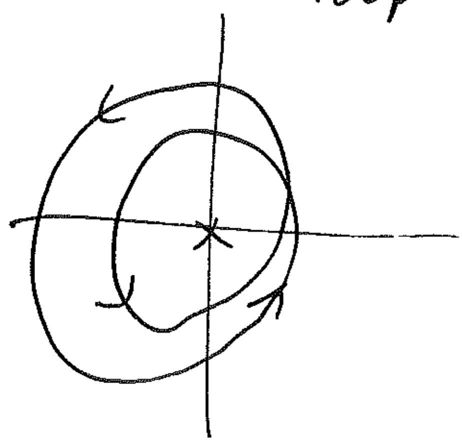
Homotopy & Degree

Reminder:

if you have a function $f: S^1 \to \mathbb{R}^2 \setminus p$

loop in the plane



the degree is $\deg(f)$ is number of times it goes around the hole.

Fact: Two loops f, g are homotopic if and only if they have the same degree.

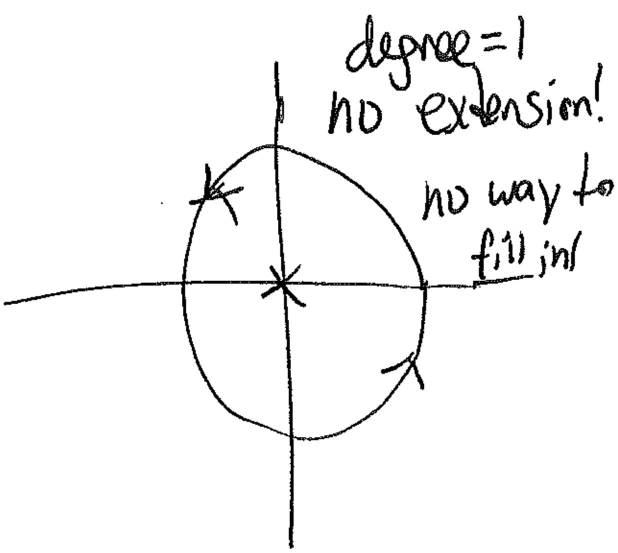
Fact #2:

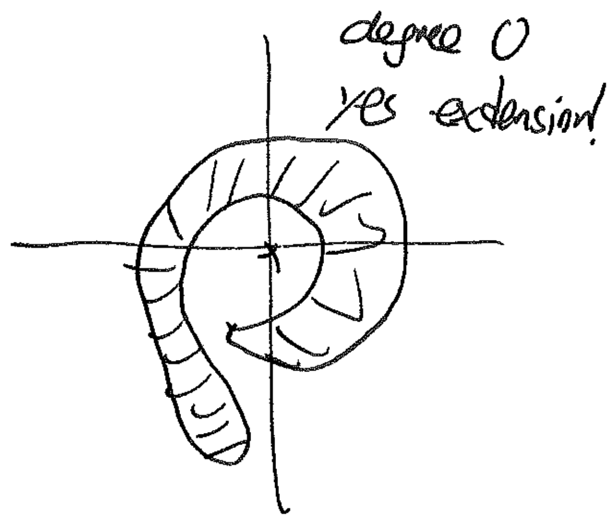
$$f : S^1 \to \mathbb{R}^2 \setminus P$$

can be extended to

$$\tilde{f} : D^2 \to \mathbb{R}^2 \setminus P$$

if and only if degree is 0.





The fundamental theorem of algebra

If $p(x)$ is a nonconstant polynomial with complex coefficients, then $p(x)$ has a (complex) root.

We had polynomial x^2+1, no roots.

So we invented a root: i.

This lets us solve __every__ polynomial. That's crazy!

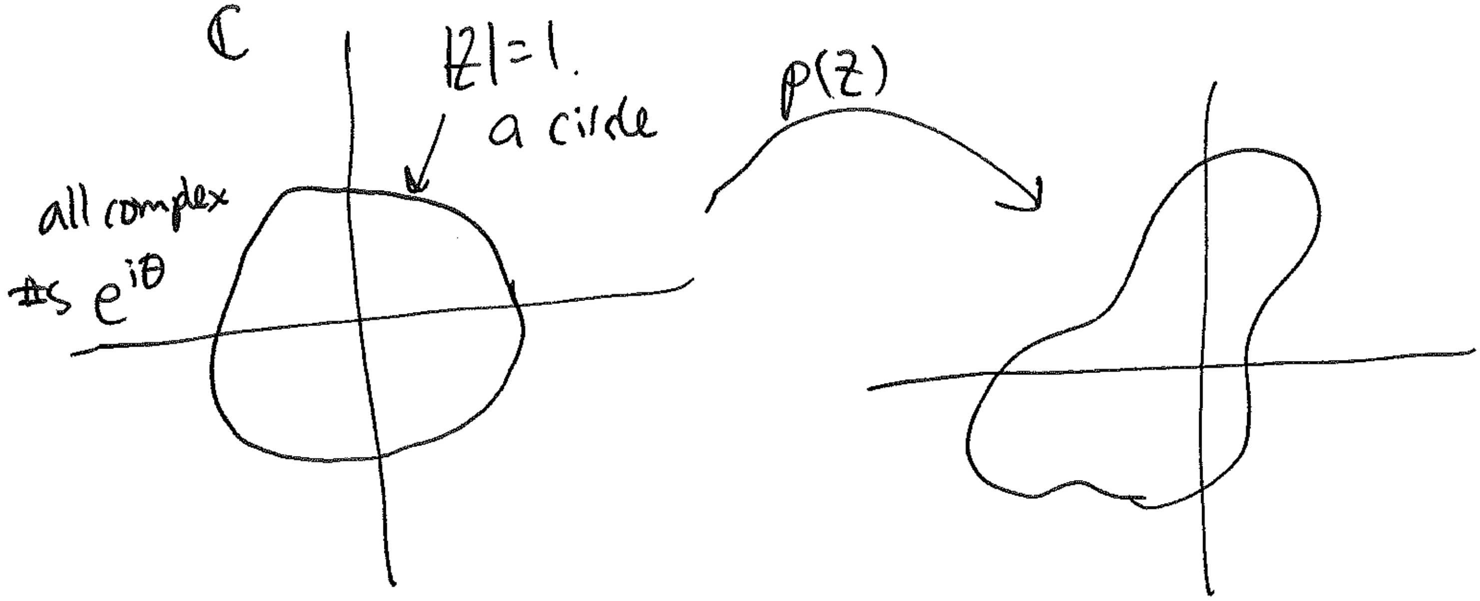
Just adding $\sqrt{2}$ to $\mathbb{Q}$ (for example) still leaves many unsolvable things

Idea:

z = complex variable

Any time you have a polynomial $p(z)$

I can turn it into a loop:



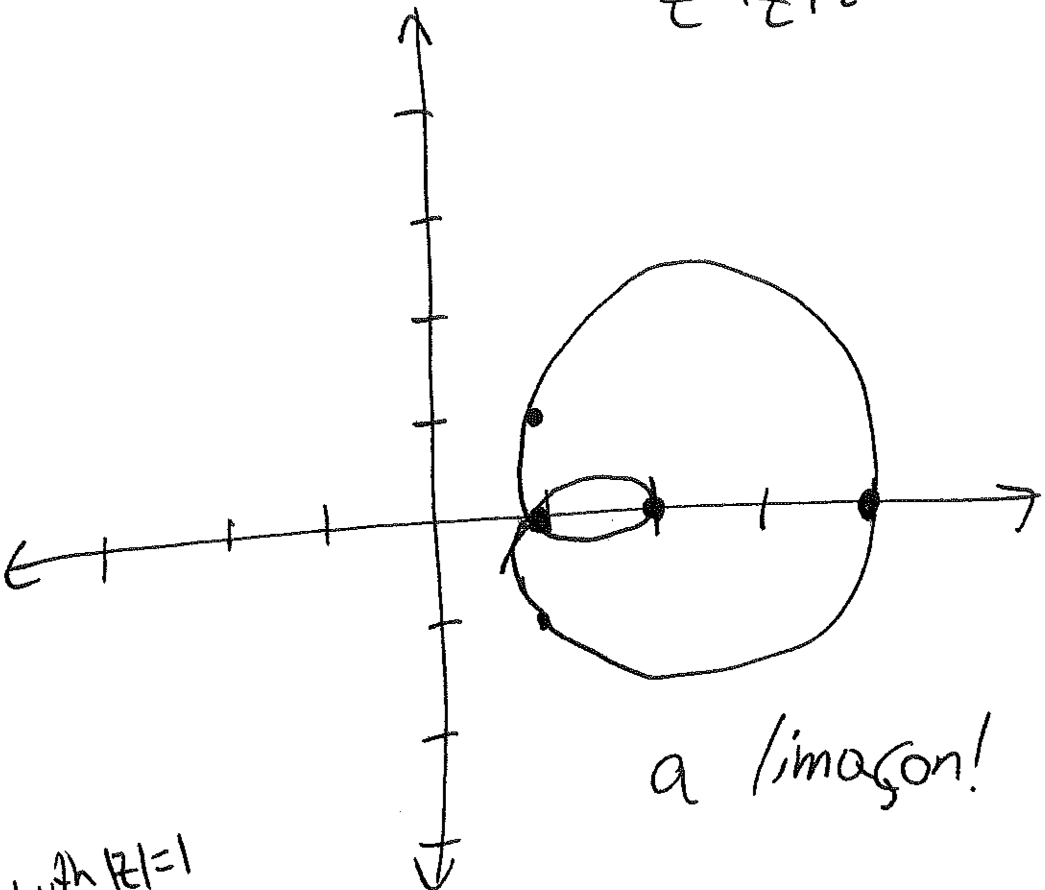
Q What's the loop corresponding

to

$$p(z)=z^2+z+2$$

?

z^2+z+2



with $|z|=1$

z	$p(z)$
1	4
-1	2
i	$1+i$
$-i$	$1-i$
$e^{i\cdot 120°}$	1

(tried lots on computer)

degree of path changes for different quadratics.

e.g shift limaçon left/right by adding constants.

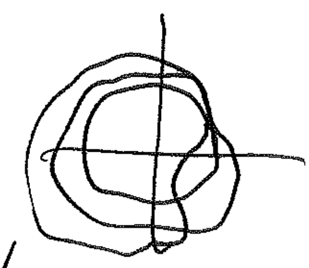
$(e^{i\theta})^3 = e^{3i\theta}$

What about z^3? $\Vdash\!\longrightarrow$ degree 3

goes around 3 times.

What if we look at a polynomial like

$$z^3 + 0.1z^2 - 0.2z + 0.01$$



$\Vdash\!\longrightarrow$ just adds a little wiggle to z^3 loop, but degree still 3.

Conclusion:

$$\text{For } z^n + \overbrace{(\quad)z^{n-1} + \cdots + \cdots}^{\text{all the coefficients very small}}$$

we get a loop of degree n,

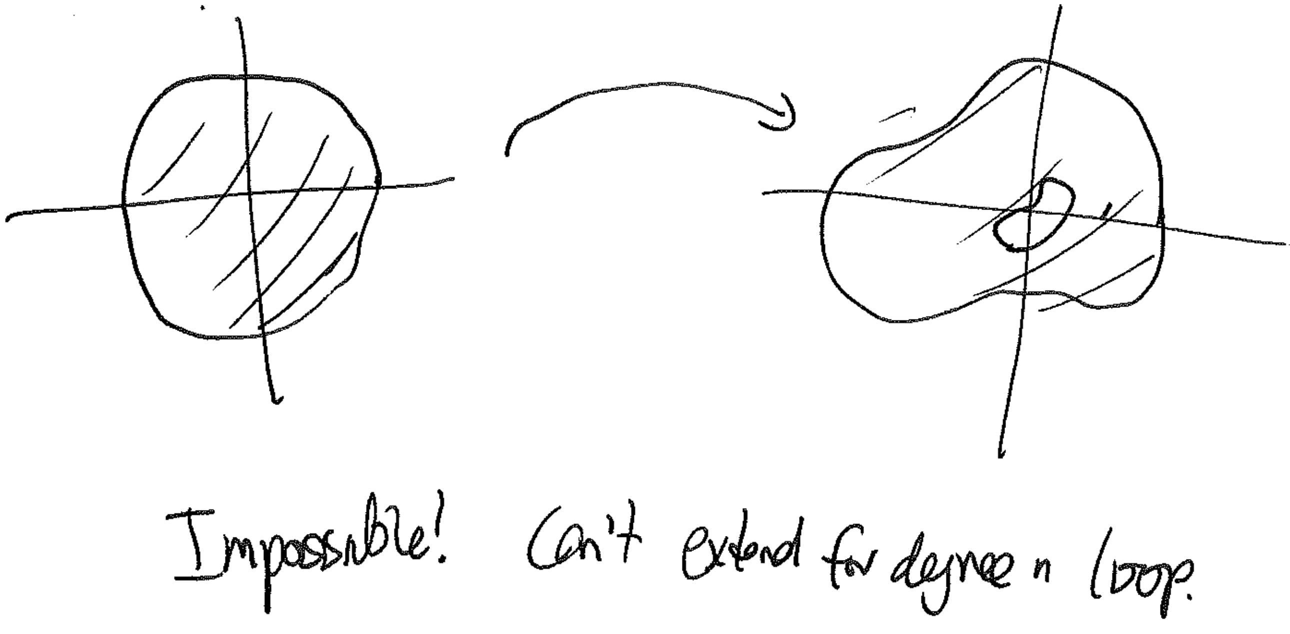
since it's just a wiggle of the z^n loop.

Here's the trick:

Suppose $p(z) = z^n +$ (terms with small coefficients)

has no root: there's no complex number z_0 with $p(z_0) = 0$.

Then we can extend the loop for $p(z)$ to a map from the disk; which misses $(0,0)$ since p had no root.



Impossible! Can't extend for degree n loop unless $n=0$.

→ A polynomial $p(z) = z^n +$ (small stuff)

must have a root!

But every polynomial can be turned into z^n + small stuff.

$$z^3 + 3z^2 - 7z + 4 = 0$$

Let $y = z/1000$.

then

$$(1000y)^3 + 3(1000y)^2 - 7(1000y) + 4 = 0$$

$\div 10^9$

$$y^3 + \underbrace{\frac{3}{1000}y^2 - \frac{7}{1000000}y + \frac{4}{1000000000}}_{\text{all small}} = 0$$

that has a root y_0 by degree,

so our original does too:

$z_0 = 1000 y_0$.

Borsuk-Ulam theorem (run-up to ham sandwich)

Suppose $f: S^2 \to \mathbb{R}^2$ is continuous.

Then there's an x such that $f(x) = f(-x)$

the opposite point on sphere.

Ex.

$$f\begin{pmatrix}\text{a point on} \\ \text{Earth}\end{pmatrix} = \begin{pmatrix}\text{temperature at } x, & \text{humidity at } x\end{pmatrix}$$

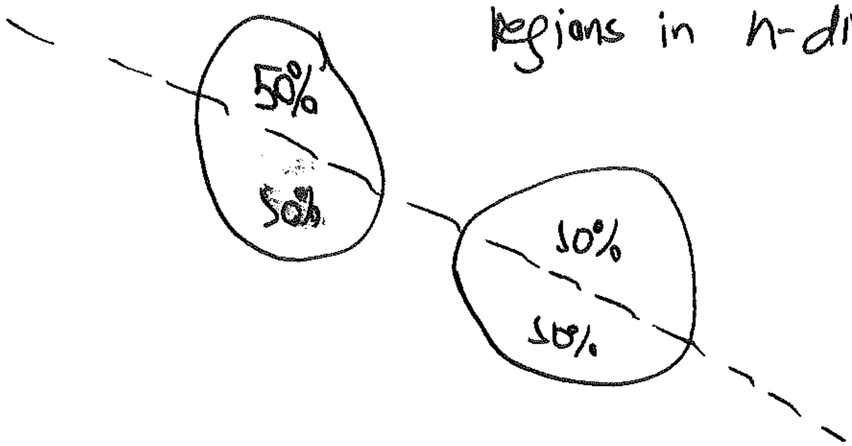
Let's just believe this for now.

(has to do with degree)

Assume B-U theorem

Ham sandwich theorem.

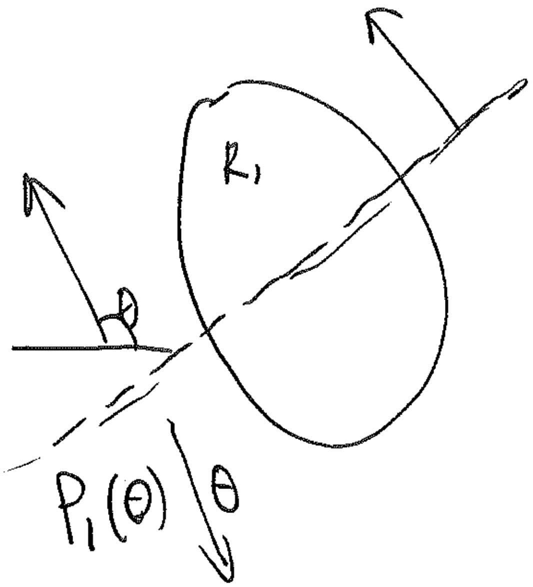
Say we have n ~~B~~ regions in n-dimensional space.



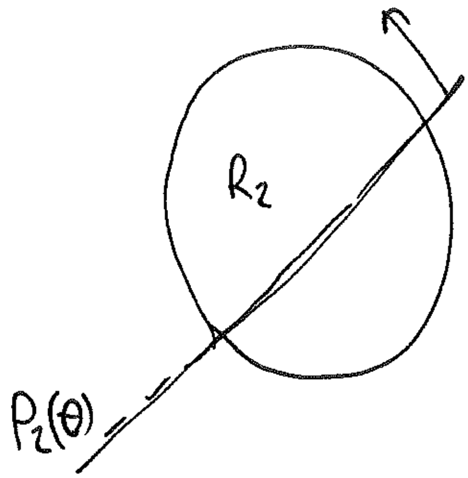
Can make a cut that splits every region in half.

2D proof.

<u>2D proof</u>



for any angle θ, you can find a ~~plane~~ cut $P_1(\theta)$ ~~cutting~~ with normal direction θ which cuts R_1 in half, and a ~~plane~~ cut $P_2(\theta)$ with normal θ which cuts R_2 in half.



goal: Show there's a θ where the two are the same.

$d(\theta) < 0$ here.

parallel planes

Define a function $d(\theta)$ = distance between $P_1(\theta)$ & $P_2(\theta)$

positive if $P_2(\theta)$ is in positive normal direction, from $P_1(\theta)$

negative if $P_2(\theta)$ is in negative direction.

What if we use $-\theta$?

$P_1(\theta) = P_1(-\theta)$

$P_2(\theta) = P_2(-\theta)$

$d(\theta) = -d(-\theta)$

$d: S^1 \to \mathbb{R}$

Sol:

2D Borsuk-Ulam theorem:

If $f: \overset{\text{circle}}{S^1} \to \mathbb{R}$, there's an x so $f(x)=f(-x)$

Applying to our d function from before:

There's a θ so $d(\theta)=d(-\theta)$

But we know: $d(\theta)=-d(-\theta)$.

This means $d(\theta)=0$! So $P_1(\theta)=P_2(\theta)$!

So cut at this angle θ: the two planes agree.

~~For 3D one next~~

For 3D,

given a point on sphere S^2, think of it as giving a normal direction $\hat{n}$.

find cuts $P_1(\hat{n})$, $P_2(\hat{n})$, $P_3(\hat{n})$ cutting regions in half. (want to find $\hat{n}$ that makes them the same.)

Define $d_{12}(\hat{n})=$ signed distance $P_1(\hat{n})$ & $P_2(\hat{n})$

$d_{13}(\hat{n})=$ distance $P_1(\hat{n})$ & $P_3(\hat{n})$.

$\leadsto$ Gives us

$$d: S^2 \to \mathbb{R}^2$$

$$d(\hat{n}) = (d_{12}(\hat{n}), d_{13}(\hat{n}))$$

~~$d(\hat{n}) = -d(-\hat{n})$~~

$d(-\hat{n}) = -d(\hat{n})$.

BU tells us there's an $\hat{n}$ so $d(\hat{n}) = d(-\hat{n})$ as before.

so $d(\hat{n}) = 0$, all three the same.