One more day of pathological functions

Last time:

- limits of functions: $\lim_{x \to a} f(x) = L$ for any $\epsilon > 0$, there exists δ so that if $|x-a|<\delta$, then $|f(x) - L| < \epsilon$.

- continuous functions:

  $\lim_{x \to a} f(x)$ exists at every a value.

Examples from last time	
Raindrop function $f(x)=\begin{cases}\frac{1}{q} & \text{where } x=\frac{p}{q} \text{ if } x\in\mathbb{Q}\\ 0 & \text{if } x\notin\mathbb{Q}.\end{cases}$	continuous at all irrational numbers! but not rational
Conway Base-13 function.	Satisfies IVT despite not continuous. On any interval [a,b) obtains every real value.

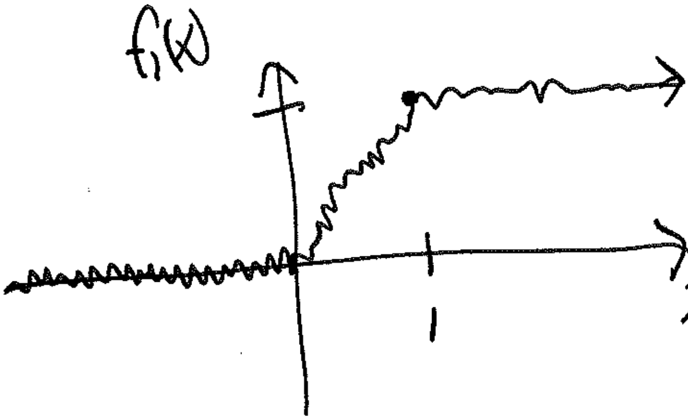
Suppose we have a sequence of continuous functions:

$f_1(x), f_2(x), f_3(x), f_4(x), \ldots$

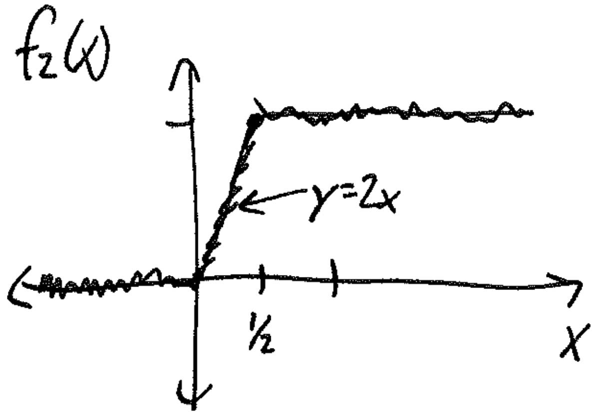
Define $f(x) = \lim_{n\to\infty} f_n(x)$, and assume limit exists for every x.

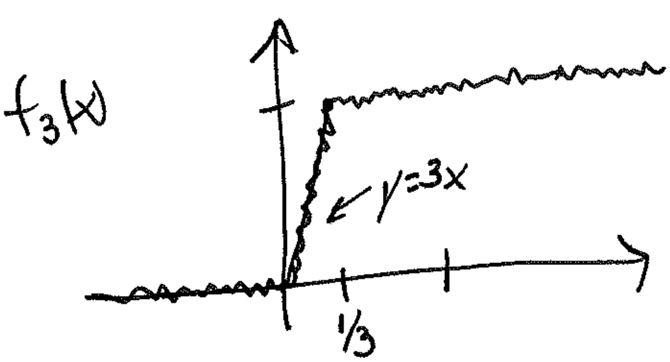
$\boxed{\text{Is } f(x) \text{ continuous?}}$ Maybe, maybe not!

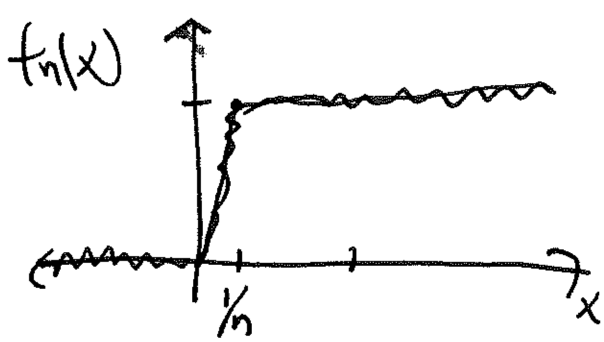
$f_1(x)$



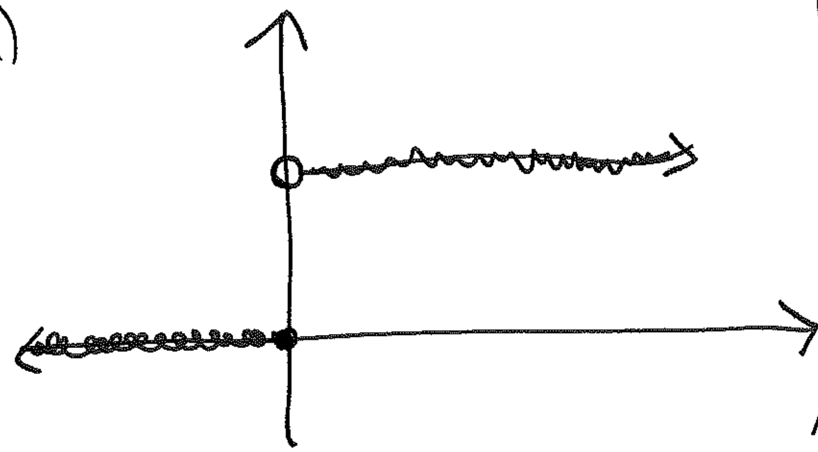
$f_2(x)$







$$f(x) = \lim_{n\to\infty} f_n(x)$$



Even though each $f_n(x)$ is continuous, the limit is not!

$f_1\left(\frac{1}{4}\right) = \frac{1}{4}$, $f_2\left(\frac{1}{4}\right) = \frac{1}{2}$, $f_3\left(\frac{1}{4}\right) = \frac{3}{4}$, $f_4\left(\frac{1}{4}\right) = 1$,

$f_5\left(\frac{1}{4}\right) = 1$, $f_6\left(\frac{1}{4}\right) = 1$, $f_7\left(\frac{1}{4}\right) = 1$

$$\lim_{n\to\infty} f_n\left(\tfrac{1}{4}\right) = 1.$$

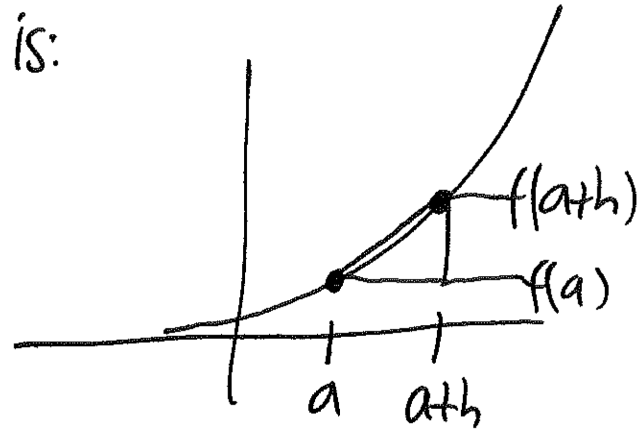
$$\lim_{n\to\infty} f_n(0) = 0$$

$$\lim_{n\to\infty} f_n\left(\tfrac{1}{100}\right) = 1$$

Derivatives

Suppose $f(x)$ is a function. ~~These~~ The derivative of $f(x)$ at the value $x=a$ is:

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$



<u>Example</u> Compute the derivative of $f(x)=x^2$ at ~~a~~ $x=2$. Use the definition, but you can assume basic facts about limits.

$$\lim_{h\to 0} \frac{(2+h)^2 - 2^2}{h} = \frac{(4+4h+h^2)-4}{h} = \frac{4h+h^2}{h} = 4+h$$

$f(a+h)$ $f(a)$

$\cancel{\lim_{h\to 0} 4+\frac{4}{h} = 4}$. $\lim_{h\to 0} 4+h = 4.$

The devil's staircase

Domain is $0 \le x \le 1$

Define $f(x)$ as follows:

1) Write x in base 3
2) If there's a "1" in the expansion, turn every digit after the 1 into 0.
3) Turn all the 2's into 1's.
4) Interpret the result as a binary number.

base 3

$$f\left(\frac{1}{2}\right) = f(0.11111111\ldots_3)$$

$$\rightsquigarrow 0.100000000\ldots_2 = \frac{1}{2}$$

$\cancel{f\left(\frac{3}{4}\right)=}$

$$f\left(\frac{4}{5}\right) = f(0.210121012\ldots_3)$$

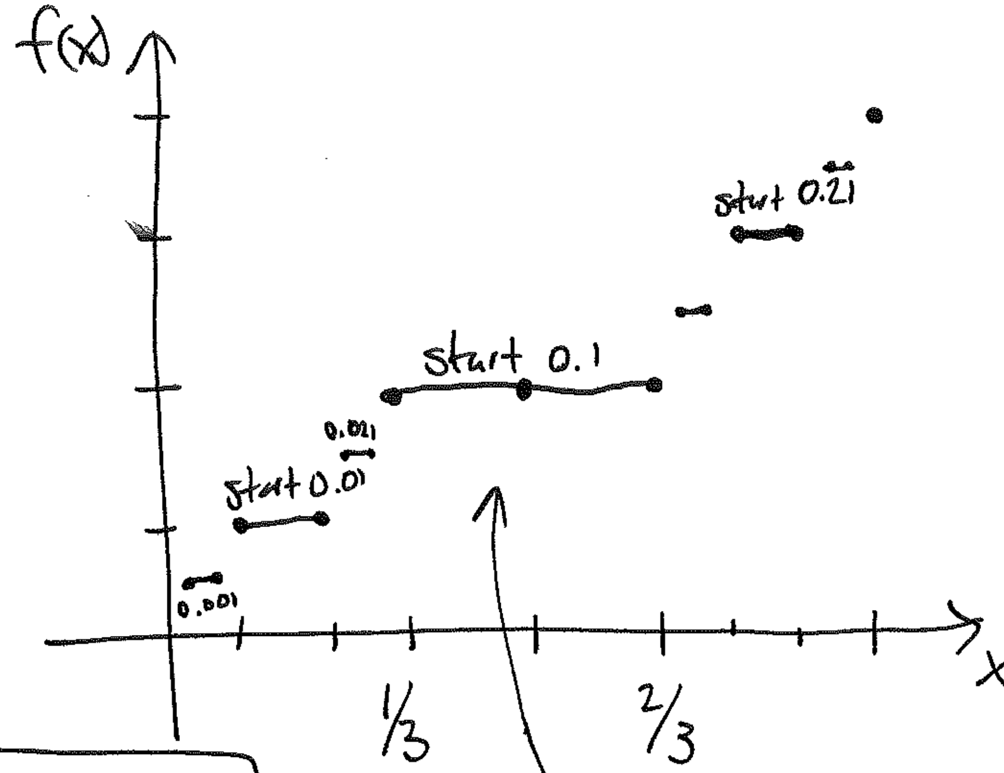
$$\rightsquigarrow 0.21000000$$

$$\rightsquigarrow 0.110000000$$

$$= \frac{3}{4}.$$

Try to plot the function:

(Hint: constant in many places)



3/4 is missing from domain!

continuous?

what number has no 1's?

$0.20202020\ldots_3$

that's

$\frac{2}{3}+\frac{2}{27}+\frac{2}{243}+\cdots$

$$\frac{2/3}{1-1/9}=\frac{\frac{2}{3}}{\frac{8}{9}}=\frac{2}{3}\cdot\frac{9}{8}=\boxed{\frac{3}{4}}$$

anything in this range

starts as $0.1\ldots_3$ in base 3

$\rightsquigarrow f(x)=\frac{1}{2}$.

Value at 3/4 is:

$0.202020\ldots$

$0.10101010\ldots_2$

$$\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\cdots=\frac{1/2}{1-1/4}=\frac{1/2}{3/4}=\boxed{\frac{2}{3}}$$

How much of the domain is "in a stair" so the function is constant nearby?

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \frac{16}{243} + \cdots = \frac{1/3}{1-2/3} = \boxed{1}$$

one big stair — two stairs of size 1/9 — four of size 1/27

Total width is 1!

This function is constant (and has derivative 0) on a bunch of intervals whose length adds up to 1!

It <u>is</u> continuous for all x.

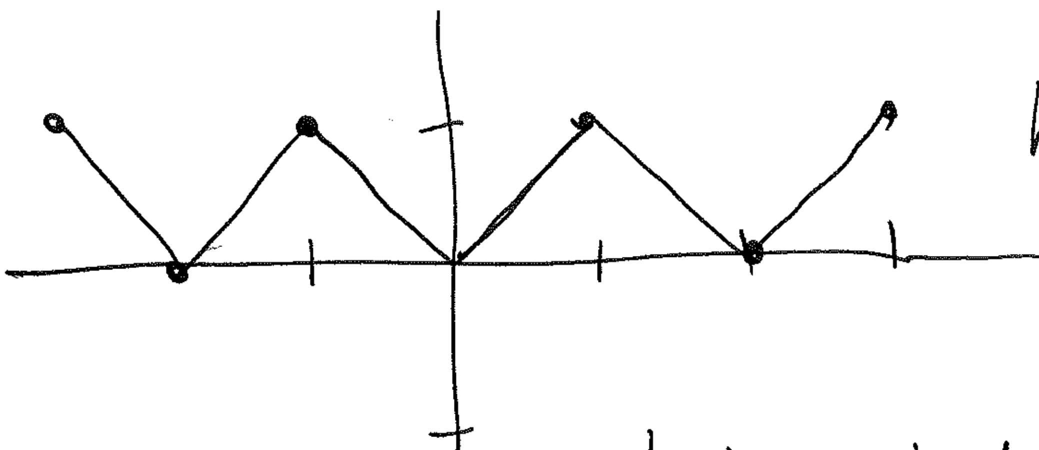
It's not differentiable at points with no 1's in the base-3 expansion (the "Cantor set"). (Otherwise it would violate FTC)

The Weierstrass function.

Let $h(x) = |x|$ if $-1 \leq x \leq 1$

and h has period 2.

HW #2:
Check Staircase
not differentiable at
$x = 2/3$.



h(x)

shrink verically by $1/2^n$

Let

$$h_n(x) = \frac{1}{2^n} h(2^n x).$$ Plot me!

squish horizontally by 2^n

Set. $$g(x) = \sum_{n=0}^{\infty} h_n(x)$$ Definitely converges for any x.

What does this look like?