RESEARCH STATEMENT: BIRATIONAL GEOMETRY AND ALGEBRAIC DYNAMICS

1. Introduction

My research interests lie at the interface between algebraic geometry and complex dynamics, with an emphasis on the rich interplay between the two. Much of my previous work fits into two main threads: first, applications of dynamical ideas to questions in algebraic geometry, mostly as a source of counterexamples; and second, applications of algebraic geometry to questions about automorphism groups of algebraic varieties and dynamical properties of automorphisms. This has allowed me to prove results in both areas. For example:

(1) I constructed the first example of a variety with infinitely many Fourier–Mukai partners, answering a question about derived categories using dynamical methods [28];
(2) I gave strong constraints on the geometry of smooth three-dimensional varieties admitting positive entropy automorphisms [30];
(3) I found the first example of a projective variety with discrete but not finitely-generated automorphism group [31].

An algebraic variety is the set of solutions to a system of polynomial equations, and an automorphism of a variety is a tuple of rational maps that map the solution set to itself. For example, consider the elliptic curve E defined by $y^2 + y = x^3 + x$. If (x, y) is a point on E, then so is

$$\phi(x,y) = \left(\frac{y^2 - x^3}{x^2}, \frac{x^3 y - x^3 - y^3}{x^3} \right).$$

(This is the map given by translation by $(0, 0)$ in the group law on E.)

Automorphisms of algebraic varieties arise in many contexts. For instance, iteration of the automorphism $\phi : E \to E$ makes it possible to find infinitely many rational solutions to $y^2 + y = x^3 + x$. Starting with $p = (3, 5)$, the points $\phi^n(p)$ provide infinitely many distinct solutions. Automorphism groups also bear on more sophisticated questions in number theory: for example, the $\bar{k}$-twists of a k-variety are classified by the group $H^1(\mathrm{Gal}(\bar{k}/k), \mathrm{Aut}(X_{\bar{k}}))$.

A variety defined over the complex numbers can also be regarded as a complex manifold, and its automorphisms viewed from the perspective of complex dynamics. The map ϕ is simply a translation on a 2-torus, but other examples of algebraic automorphisms are more dynamically interesting. McMullen showed, for example, that there exist rational surfaces whose algebraic automorphisms exhibit Siegel disks [36].

Given an algebraic variety, there are many natural questions to ask about its automorphisms. What is the group of automorphisms? What are the dynamical properties of these automorphisms, and what features of the geometry of X can be revealed by the dynamics? Conversely, suppose that a variety admits automorphisms with certain properties: for example, that the group of automorphisms is very large, or there exists an automorphism with complicated dynamical behavior. What does this imply about the geometry of the variety X? Is it possible to classify the X admitting automorphisms with certain properties?

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2. Overview of results

2.1. **Applications of algebraic dynamics to birational geometry.** The last three decades have seen tremendous progress in the classification of varieties in dimension three and above, chiefly through the methods of the minimal model program (MMP) [5]. However, it is often difficult to find explicit examples of certain pathological phenomena in these high-dimensional settings: most varieties are either too simple to be interesting, or completely intractable from a computational point of view. Varieties admitting dynamically interesting automorphisms strike a balance between these extremes and provide a rich source of examples. A major theme of my work is that these varieties are complicated enough to exhibit various unexpected geometric behaviors, but also simple enough that many computations can be made explicitly.

This outlook resulted in a sequence of papers settling a number of questions and conjectures in birational geometry, based on the explicit construction of counterexamples by essentially dynamical methods. In some instances, these phenomena were expected to be possible but no explicit examples were known; in others, they were conjectured to be impossible. Often it is not clear what possibilities to expect in high dimensions, and counterexamples play an important role in delineating the possible geometric behavior of higher-dimensional varieties.

The following result illustrates of the most naïve form of this strategy. It was conjectured that a non-uniruled projective variety could have only finitely many K_X-negative extremal rays on the cone of curves – i.e. that such a variety can contain only a finite set of curves with a certain property. To give a disproof of this conjecture, it is sufficient to find a variety with a large group of automorphisms and just a single such curve: the orbit of this curve under the action of the automorphism group then yields an infinite set. An explicit example on a partial resolution of a torus quotient disproves the conjecture.

Theorem 1 ([30], answering [23, Problem 4-2-5], [24, III.1.2.5.1]). There exists a non-uniruled variety X with infinitely many K_X-negative extremal rays on $\overline{\mathrm{NE}}(X)$.

If X is a smooth projective variety, the bounded derived category of coherent sheaves on X is an invariant that contains a great deal of information about X. It was conjectured by Kawamata that in fact the derived category of a smooth projective variety X determines the isomorphism type of X, up to finitely many possibilities, and this conjecture was proved for various classes of X in [21, 40, 17, 8, 22]. However, again based on the dynamics of iterated rational maps, I gave a counterexample to this conjecture.

Theorem 2 ([28], answering [21, Conjecture 1.5]). There exists a smooth projective variety X with infinitely many non-isomorphic Fourier–Mukai partners.

Another project, joint with J.C. Ottem, gives a counterexample in a similar vein to the first one. In dimension 2, the set of curves disjoint from a given codimension 1 subvariety is either finite or uncountable (with the latter occuring when such a curve moves in a positive-dimensional family) [43]. We showed that this dichotomy breaks down in higher dimensions, again through the use of iterated rational maps:

Theorem 3 ([33], answering [44, pg. 4]). There exists a smooth projective variety X and an effective divisor D on X for which the set of algebraic curves $C \subset X$ which are disjoint from D is countably infinite.

Given an automorphism (or more generally a pseudoautomorphism, i.e. a rational map such that neither ϕ nor ϕ^{-1} contracts any divisors) $\phi : X \to X$ of positive entropy, one may consider the leading eigenvector of $\phi^* : N^1(X) \to N^1(X)$, viewed as an $\mathbb{R}$-divisor on X. These divisors turn out to have very complicated behavior from the point of view of birational geometry. For example, it is possible that such a divisor has negative intersection with a Zariski dense set of curves on X. This implies that its diminished base locus, an invariant which appears regularly in connection with the MMP, is not a Zariski closed set.

Theorem 4 ([27], answering [16, Remark 1.13], [6, Question 1.4]). There exists a smooth projective variety X and a pseudoeffective divisor D on X for which the diminished base locus $\mathbf{B}_-(D)$ is not a Zariski closed set.

In a similar vein, one can find an example of a divisor which is nef (i.e. has non-negative intersection with every curve), but whose deformations in a family fail to be nef for countably many values of the parameter: thus nefness is not an open condition.

Theorem 5 ([27], answering [26, Remark 1.4.15]). There exists a one-dimensional family of varieties X_t and a family of divisors D_t so that D_t is nef on X_t for very general t, but fails to be nef countably many times.

These eigenvectors also give an example of a divisor for which a certain notion of multiplicity along a subvariety is infinite, another surprising behavior from the point of view of the MMP, where this was not expected.

Theorem 6 ([29], answering [37, pg. 33]). There exists a family $\pi : X \to S$ and a divisor D on X for which the relative asymptotic multiplicity $\sigma_\Gamma(D; X/S)$ is infinite.

2.2. **Applications of birational geometry to algebraic dynamics.** One measure of the richness of the dynamics of an automorphism is the topological entropy. If $\phi : X \to X$ is an endomorphism of a compact metric space, the topological entropy $h(\phi)$ is a measure of the complexity of the dynamics of ϕ: roughly, it detects the tendency of ϕ to separate general points of X. Although in general the entropy can be hard to compute, a famous result of Gromov and Yomdin asserts that if X is an algebraic variety, a holomorphic automorphism $\phi : X \to X$ has positive entropy if and only if the pullback map $\phi^* : H^{1,1}(X; \mathbb{R}) \to H^{1,1}(X; \mathbb{R})$ has a real eigenvalue greater than 1 [20, 45]. This makes it possible to address many questions about the dynamics of holomorphic automorphisms of algebraic varieties by means of purely algebraic techniques.

The classification of algebraic surfaces admitting positive entropy automorphisms is more or less understood by work of Cantat [10], with some important examples due to Bedford–Kim and McMullen [3, 36]. In contrast, many basic questions about holomorphic automorphisms of varieties in higher-dimensional settings remain wide open [38]. The construction and classification of three-dimensional projective varieties admitting automorphisms and birational automomorphisms of positive entropy has recently been the subject of increasing attention among researchers in algebraic dynamics ([41, 2, 7, 39, 1]).

If one wishes to study the classification of higher-dimensional varieties that admit dynamically interesting automorphisms and endomorphisms, it is inevitable that one must make use of facts about the classification of higher-dimensional varieties in general. This necessitates the tools of birational geometry in general and the minimal model program in particular. Some fairly coarse classification results are due to Zhang [46], but at present the list of examples of smooth threefolds admitting positive entropy automorphisms is quite short. A basic question is to determine whether the dearth of examples is due to the fact these varieties are indeed very rare, or whether the necessary constructions have simply not been discovered.

The paper [30] is a step in this direction, and marks the beginning of my ongoing work on the classification of varieties admitting positive entropy automorphisms. This paper gives a number of constraints on the geometry of three-dimensional varieties admitting automorphisms of positive entropy, including the following.

Theorem 7 ([30], cf. [38, Question 5.7]). Suppose that X is a smooth three-dimensional variety which does not admit any automorphism of positive entropy, and that Y is a variety constructed as a smooth blow-up of X. Then any positive entropy automorphism of Y admits an equivariant fibration to a two-dimensional variety.

Another recent result of mine demonstrates how one can use geometric techniques to construct a variety whose automorphism group has desired properties. If X is a variety, then the component group $\mathrm{Aut}(X)/\mathrm{Aut}^0(X)$ is a countable group, but almost nothing is known in general about what groups can occur. This is a question of both arithmetic and geometric interest: for example, B. Mazur proved a variety of results about local-to-global properties for rational points on varieties under the hypothesis that $\mathrm{Aut}(X)$ is a finitely generated group [35]. I constructed the first example showing that this hypothesis does not hold for varieties in general:

Theorem 8 ([31], answering [35, 9]). There exists a smooth projective variety over $\mathbb{Q}$ for which $\mathrm{Aut}(X)/\mathrm{Aut}^0(X)$ is not finitely generated.

Although this example does not seem to give a counterexample to the arithmetic questions considered by Mazur, it seems reasonable to hope that similar constructions could reveal new number-theoretic phenomena in the future.

In joint work with Daniel Litt [32], I gave extensions of the some of the methods in [30] to varieties of arbitrary dimension. The results we obtain are less sharp than those in [30], in part due to the absence in high dimensions of many of the very precise classification results provided by the three-dimensional minimal model program. A new ingredient in [32] is the inclusion of p-adic methods. We prove a version of the dynamical Mordell–Lang conjecture, extending results of Bell and collaborators [4], and then show how it can be applied in a geometric context.

Theorem 9 ([32]). Suppose that X is a smooth variety over k, and that $\phi : X \to X$ is an étale endomorphism of X. Let Y and Z be closed subschemes of X. Then the set $A_\phi(Y, Z) = \{n : \phi^n(Y) \subseteq Z\}$ is a union of a finite set and a finite number of arithmetic progressions.

2.3. **Classical questions.** Techniques and constructions from classical algebraic geometry appear consistently in my work. Many of my papers deal with the geometry of blow-ups of projective space and the action of Cremona transformations on configurations of points, a subject was studied by Coble as early as the 1920s [12, 15]. My disproof of the conjecture on finiteness of Fourier–Mukai partners was based on the action of the standard Cremona involution on $\mathbb{P}^3$, a map that Reid has identified has the first example of a flop, tracing its origins to 1837 work of Magnus [42]. Many of my other constructions, such as the variety with non-finitely generated $\mathrm{Aut}(X)$, also have very classical origins.

I have also proved a variety of results with a strongly classical flavor. In collaboration with Izzet Coskun and John Christian Ottem [13], I compute certain cones of effective cycles on blow-ups of projective space at sets of general points. Our motivation was the fact that these cones of cycles have recently been the subject of considerable attention, but few interesting examples have been worked out [18]. The results in our paper can all be understood very concretely as statements about the interpolation of varieties in projective space through prescribed sets of points, and the proofs are based on a variety of degeneration arguments and other classical techniques.

In another ongoing project with J.C. Ottem, I am studying the properties of secant varieties of elliptic normal curves, i.e. genus 1 curves of degree $n+1$ in $\mathbb{P}^n$. We example the singularities of these varieties, and point out their surprising invariance under the action of large groups of birational automorphisms of $\mathbb{P}^n$. These constructions are also connected to questions of current interest, and can be used to exhibit invariant fibrations for a family of pseudo-automorphisms constructed by Perroni and Zhang [41].

In work with Jinhyung Park [34], I determined which blow-ups of products of projective spaces at general points are log Fano varieties, giving a modest extension of work of Castravet and Tevelev [11] identifying which of these blow-ups are Mori Dream Spaces. The arguments combine the results of [11] with a recent characterization of log Fano varieties due to [19].

3. Some future directions

There remain a number of finiteness questions in algebraic geometry that I hope will be amenable to dynamical approaches. Perhaps the most fundamental finiteness question of all is termination of flips, which is closely related to the following question:

Question 1 (Kollár [25, Exercise 82]). Suppose that $\tau : X \dashrightarrow X^+$ the composition of several flips. Can there exist an isomorphism $\phi : X^+ \to X$?

If such an example were to exist, the composition $\phi \circ \tau$ would be a pseudoautomorphism of infinite order, giving rise to a situation amenable to study by dynamical methods. However, a negative answer to Question 1 would follow from any reasonable version of termination of flips. Indeed, if there is a pair of maps τ and ϕ as above, then there exists an infinite sequence of flips given by the maps $(\phi \circ \tau)^{\circ n}$. The majority of the known examples of infinite-order pseudoautomorphisms are on uniruled varieties X for which there does not exist any Δ making (X, Δ) a log canonical pair with $\kappa(K_X + \Delta) \geq 0$, and this is precisely the setting in which the least is known about termination in general. It seems worthwhile to either search for an example of such a flip τ, or to prove that one can not exist, perhaps under additional dynamical hypotheses on the pseudoautomorphism $\phi \circ \tau$.

I also plan to more systematically investigate the connections between dynamical properties of an automorphism, and the numerical invariants of its leading eigenvectors. For example, it is not difficult to show that if $\phi : X \to X$ is a positive entropy automorphism of a threefold, and D is a leading eigenvector of the action of ϕ^* on $N^1(X)$, then $D^2 = 0$ (perhaps after replacing ϕ with ϕ^{-1}). This is rather striking, as nef divisor classes with with self-intersection 0 are unusual in general; such divisors are said to have numerical dimension 1.

Question 2. How do the invariants of the eigenvector D reflect the dynamical properties of the automorphism $\phi : X \to X$? For example, suppose that $\phi : X \dashrightarrow X$ is a pseudoautomorphism, with D a leading eigenvector of $\phi^* : N^1(X) \to N^1(X)$. Under what conditions on the dynamics of ϕ does D admit a Zariski decomposition in the sense of Nakayama?

There are also many open questions about the classification of varieties admitting positive entropy automorphisms. The results of [30] suggest the following conjecture, which although seemingly very strong, is nevertheless consistent with all known examples:

Conjecture 3. Suppose that $\phi : X \to X$ is a positive entropy automorphism of a smooth threefold. Then one of the following holds:

(1) ϕ preserves a fibration $X \to S$ over a smaller-dimensional variety;
(2) there exists a terminal variety Y, birational to X, with $K_Y \equiv 0$ such that ϕ induces an automorphism of Y;
(3) there exists a torus quotient Y, birational to X, such that ϕ induces an automorphism of Y;

I also plan to pursue basic questions about properties of the dynamical degrees of rational maps, a set of invariants which are closely related to the entropy. Given a rational map $\phi : X \dashrightarrow X$, fix an ample divisor H and set

$$\lambda_i(\phi) = \lim_{n\to\infty} ((\phi^n)^*(H^i) \cdot (H^{\dim X - i})^{1/n}.$$

If ϕ is a morphism, then $(\phi^n)^* = (\phi^*)^n$, and $\lambda_i(\phi)$ is given by an eigenvalue of the integer matrix representing $\phi^* : N^i(X) \to N^i(X)$, and so is algebraic. It is not known whether this is the case in general:

Question 4. Does there exist a birational map $\phi : X \dashrightarrow X$ for which some dynamical degree $\lambda_i(\phi)$ is not an algebraic integer?

Computing the dynamical degree of a map given by explicit equations is very hard in practice: the root of the problem is that if ϕ is not a morphism, then the pullback ϕ^* is not functorial, and it can happen that $(\phi^n)^* \neq (\phi^*)^n$. This makes it extremely difficult to control the degrees of the maps $(\phi^n)^*$. This problem arises exactly when some divisor is exactrated by some iterate of ϕ, and then contracted by a subsequent iterate. In dimension 2, a result of Diller and Favre [14] asserts that there exists a birational model of X on which it does hold that $(\phi^n)^* = (\phi^*)^n$; hence either a positive or negative answer to this question will require the methods of high-dimensional geometry.

All of these areas are currently the subject of active research, and I look forward to contributing to these efforts.

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