DERIVED-EQUIVALENT RATIONAL THREEFOLDS

JOHN LESIEUTRE

ABSTRACT. We describe an infinite set of smooth projective threefolds that have equivalent derived categories but are not isomorphic, contrary to a conjecture of Kawamata. These arise as blow-ups of $\mathbb{P}^3$ at various configurations of 8 points, which are related by Cremona transformations.

1. INTRODUCTION

For a smooth projective variety X, let $D(X) = D^b\mathrm{Coh}(X)$ denote the bounded derived category of coherent sheaves on X. The derived category contains a great deal of information about the variety X: if Y is another variety whose derived category is equivalent to that of X, then X and Y have the same dimension and Kodaira dimension, and, if X is of general type, they are are birational [9]. Extending a result of Bridgeland and Maciocia [4], Kawamata proved that if X is a smooth projective surface, there are only finitely many other smooth projective surfaces Y (up to isomorphism), with $D(X)$ and $D(Y)$ equivalent as triangulated categories. He asked whether this property might hold in all dimensions [9, Conjecture 1.5].

We observe here that there are threefolds for which this is not the case. Let $\mathbf{p}$ denote an ordered 8-tuple of distinct points in $\mathbb{P}^3$, and let $X_{\mathbf{p}}$ be the blow-up of $\mathbb{P}^3$ at the points of $\mathbf{p}$.

Theorem. *There is an infinite set W of configurations of 8 points in $\mathbb{P}^3$ such that if $\mathbf{p}$ and $\mathbf{q}$ are distinct elements of W, then $D(X_{\mathbf{p}}) \cong D(X_{\mathbf{q}})$ but $X_{\mathbf{p}}$ and $X_{\mathbf{q}}$ are not isomorphic.*

The example boils down to three basic observations, made precise in Lemmas 1, 2, and 3.

(1) $X_{\mathbf{p}}$ and $X_{\mathbf{q}}$ are isomorphic if and only if $\mathbf{p}$ and $\mathbf{q}$ coincide, up to permutation and an automorphism of $\mathbb{P}^3$.

(2) If $\mathbf{q}$ can be obtained from $\mathbf{p}$ by a sequence of standard Cremona transformations centered at 4-tuples from among the points of $\mathbf{p}$, then $X_{\mathbf{p}}$ and $X_{\mathbf{q}}$ are connected by a sequence of flops of rational curves with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, and so $D(X_{\mathbf{p}}) \cong D(X_{\mathbf{q}})$.

(3) The orbit of a sufficiently general configuration $\mathbf{p}$ of 8 points under standard Cremona transformations is infinite.

There are several classes of higher-dimensional varieties for which $D(X)$ has been shown to determine the isomorphism class of X, up to finitely many possibilities. These include abelian varieties over $\mathbb{C}$ [14],[7], toric varieties [10], varieties with K_X ample, and Fano varieties [3]. It also known that the number of isomorphism classes of varieties with a given derived category is at most countable [1]. Further discussion of this problem can be found in [7] and [17].

The example is based on the action of Cremona transformations on configurations of points in $\mathbb{P}^3$, as investigated by A. Coble. Most of the results we will need can be found in Dolgachev

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and Ortland's account of Coble's work [6]. We provide self-contained proofs, with references to the more general theory.

It is worth noting that these examples do not pose any problems for the Kawamata-Morrison cone conjecture for klt Calabi-Yau pairs [18]. Through 8 general points in $\mathbb{P}^3$ there is a pencil of quadrics, with base locus a degree 4 curve through the points. If Δ is the sum of two generic quadrics in this pencil, then (X, Δ) is a dlt pair with $K_X + \Delta$ numerically trivial. However, there is no choice of Δ for which $K_X + \Delta$ is numerically trivial and the pair is klt, for the divisor obtained by blowing up the curve in the base locus has log discrepancy 0. If the points are specialized to the base locus of a two-dimensional net of quadrics, the pair (X, Δ) can be made klt, but the configuration is insufficiently general for our construction, and indeed Prendergast-Smith has demonstrated that the cone conjecture holds for this class of examples [16]. Note also that although the blow-up of $\mathbb{P}^3$ at 7 very general points is a weak Fano variety, the blow-up at 8 is not.
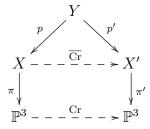
2. The example

For the rest of this section, $\mathbf{p}$ denotes an ordered 8-tuple of distinct points in $\mathbb{P}^3$, with $\pi_{\mathbf{p}} : X_{\mathbf{p}} \to \mathbb{P}^3$ the blow-up of the points of $\mathbf{p}$. Write E_i for the exceptional divisors of $\pi_{\mathbf{p}}$ and H for the pullback to $X_{\mathbf{p}}$ of the hyperplane class on $\mathbb{P}^3$.

Lemma 1 ([6], Ch. V.1). *The blow-ups $X_{\mathbf{p}}$ and $X_{\mathbf{q}}$ are isomorphic if and only if $\mathbf{p}$ and $\mathbf{q}$ coincide, up to an automorphism of $\mathbb{P}^3$ and permutation of the points.*

Proof. Suppose that $\phi : X_{\mathbf{p}} \to X_{\mathbf{q}}$ is an isomorphism. The restriction $\mathbb{P}^3 \setminus \mathbf{p} \cong X_{\mathbf{p}} \setminus \bigcup E_i \to X_{\mathbf{q}} \to \mathbb{P}^3$ defines a rational map $\psi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$, with indeterminacy locus contained in $\mathbf{p}$ and hence 0-dimensional. Its inverse is likewise regular outside a 0-dimensional set, contained in $\mathbf{q}$. But any rational map $\psi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ for which ψ and ψ^{-1} both have 0-dimensional indeterminacy sets is in fact an automorphism (e.g. by Theorem 1.1 of [2]). Now, $\psi \circ \pi_{\mathbf{p}}$ contracts E_i to a point, and so $\pi_{\mathbf{q}}$ must contract $\phi(E_i)$ to a point. Consequently $\psi(\mathbf{p}) = \mathbf{q}$, so the configurations differ by a permutation and automorphism, and ϕ identifies the exceptional divisors. $\square$

The basic ingredient in constructing other blow-ups which are derived-equivalent to a given one is the action of the standard Cremona transformation $\mathrm{Cr} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$, defined by $[X_0 : X_1 : X_2 : X_3] \mapsto [X_0^{-1} : X_1^{-1} : X_2^{-1} : X_3^{-1}]$. A resolution of this rational map is as follows.

$$
\begin{array}{ccc}
 & Y & \\
{\scriptstyle p}\swarrow & & \searrow{\scriptstyle p'} \\
X \dashrightarrow^{\overline{\mathrm{Cr}}} & & X' \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\
\mathbb{P}^3 \dashrightarrow^{\mathrm{Cr}} & & \mathbb{P}^3
\end{array}
$$

Here π blows up the four standard coordinate points. The strict transforms on X of the six lines ℓ_{ij} between two of these points are smooth rational curves with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. These are flopped by $\overline{\mathrm{Cr}}$; p blows up these curves to divisors isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, which are then contracted along the other ruling by p'. The indeterminacy locus of $\overline{\mathrm{Cr}} : X \dashrightarrow X'$ is the union of these curves. The map π' then blows down the strict transforms of the four planes through three of the four original points.

Suppose that $\mathbf{p}$ is a configuration of 8 points (regarded now as a point on the configuration space $(\mathbb{P}^3)^8 /\!\!/ \mathrm{PGL}(4)$), and four points are chosen from among $\mathbf{p}$ satisfying the following condition:

($*$) No other point of $\mathbf{p}$ lies on any plane defined by three of the four chosen points.

Condition ($*$) implies that the Cremona transformation centered at the four given points is defined, as these are not coplanar. It also guarantees that no point of $\mathbf{p}$ is on one of the contracted divisors or one of the lines in the indeterminacy locus. We can define a new configuration $\mathbf{q}$ by making a Cremona transformation centered at the four chosen points, and moving the remaining four points under that transformation: if p_j is one of the chosen points, then q_j is defined as the image of the plane through the other three points (which is contracted by π'), while if p_j is not a chosen point, then q_j is just the image of p_j under the Cremona transformation. The configuration $\mathbf{q}$ is defined up to choice of coordinates, and no two points of $\mathbf{q}$ are infinitely near, since no point of $\mathbf{p}$ is on one of the contracted divisors. After blowing up the points of $\mathbf{p}$ and $\mathbf{q}$, there is a rational map $\overline{\mathrm{Cr}} : X_{\mathbf{p}} \dashrightarrow X_{\mathbf{q}}$ which flops six rational curves of normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

We will say that $\mathbf{p}$ and $\mathbf{q}$ are *Cremona equivalent* if there exists a sequence of Cremona transformations centered at 4-tuples from among the points which sends the configuration $\mathbf{p}$ to $\mathbf{q}$, and satisfies condition ($*$) at each step. The *Cremona orbit* of $\mathbf{p}$ is the set of all configurations that are Cremona equivalent to $\mathbf{p}$. If $\mathbf{p}$ is a very general configuration of points, then any sequence of Cremona transformations will automatically satisfy ($*$); however, we prefer not to make any blanket generality assumption on $\mathbf{p}$ at this stage, as it will be useful to consider 8-tuples in slightly special configurations.

Lemma 2. *If $\mathbf{p}$ and $\mathbf{q}$ are Cremona equivalent, then $D(X_{\mathbf{p}}) \cong D(X_{\mathbf{q}})$.*

Proof. Assumption ($*$) on the Cremona transformations implies that there is a sequence of rational maps $X_{\mathbf{p}} = X_{\mathbf{p}_0} \dashrightarrow X_{\mathbf{p}_1} \dashrightarrow \cdots \dashrightarrow X_{\mathbf{p}_n} = X_{\mathbf{q}}$, where each $X_{\mathbf{p}_j} \dashrightarrow X_{\mathbf{p}_{j+1}}$ flops six curves. The lemma is then a consequence of a fundamental result of Bondal and Orlov [3, Theorem 4.3]: if X and X^+ are threefolds, and $\phi : X \dashrightarrow X^+$ is the flop of a rational curve with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, then $D(X) \cong D(X^+)$. $\qquad\square$

There is a subtlety here in that each of these rational maps flops six disjoint curves, while the theorem of Bondal and Orlov is usually stated for the flop of a single curve. Each map can be factored into a sequence of six disjoint flops, but the intermediate varieties encountered are no longer projective. However, the needed result is valid without assuming X and X^+ are projective [8, Remark 11.24ii].

Lemma 3 (cf. [6], Ch. VI). *A very general configuration $\mathbf{p}$ of 8 points has infinite Cremona orbit.*

Proof. We will define a sequence of configurations in which $\mathbf{p}_{n+1}$ is obtained from $\mathbf{p}_n$ by a Cremona transformation centered at four points, satisfying condition ($*$), and such that the $\mathbf{p}_n$ are all distinct. Let $C = C_0$ be a smooth genus 1 curve in $\mathbb{P}^3$ obtained as the complete intersection of two smooth quadrics. Choose p_5, p_6, p_7, and p_8 to be the four points of intersection of some generic hyperplane H with C, and then choose p_1, p_2, p_3, and p_4 to be very general points of C; this guarantees that if $4d - \sum_{i=1}^{8} m_i = 0$, the class $dH|_C - \sum_{i=1}^{8} m_i p_i \in \mathrm{Pic}^0(C)$ is nonzero unless $m_1 = \cdots = m_4 = 0$ (cf. [11, Lemma 2.4]).

Let $\bar{C}$ denote the strict transform of C on $X_{\mathbf{p}_0}$, and say that a prime divisor $D \sim dH - \sum_{i=1}^{8} m_i E_i$ on $X_{\mathbf{p}_0}$ is a *root divisor* if it does not contain $\bar{C}$ and satisfies $4d - \sum_{i=1}^{8} m_i = 0$.

If D is a root divisor, then $D \cdot \bar{C} = 0$ and so D and $\bar{C}$ are disjoint. This implies that $dH|_C - \sum_{i=1}^{8} m_i p_i \in \mathrm{Pic}^0(C)$ is trivial, and so $m_1 = \cdots = m_4 = 0$. If m_5, m_6, m_7, and m_8 are not all equal, then some m_i is greater than d, and the corresponding class cannot be effective. Hence the only root divisor on $X_{\mathbf{p}_0}$ is the strict transform of the plane through the last four points, with numerical class $H - \sum_{i=5}^{8} E_i$.

We now inductively define $\mathbf{p}_{n+1}$ by making Cremona transformation centered at the first four points of $\mathbf{p}_n$, and then cyclically permuting the points so that p_1 comes last. At each step, we show that the first four points of $\mathbf{p}_n$ satisfy condition $(*)$. These transformations induce rational maps $X_{\mathbf{p}_n} \dashrightarrow X_{\mathbf{p}_{n+1}}$; let $\bar{C}_n$ denote the strict transform of $\bar{C}$ on $X_{\mathbf{p}_n}$, and C_n its image in $\mathbb{P}^3$. Since C_0 is the intersection of two quadrics, and Cremona transformations preserve quadrics through the 8 points, each C_n is also an intersection of two quadrics. In particular, no $\bar{C}_n$ can be contained in the indeterminacy locus of $X_{\mathbf{p}_n} \dashrightarrow X_{\mathbf{p}_{n+1}}$.

Suppose that $\mathbf{p}$ is a configuration of points and that $\mathbf{q}$ is the configuration obtained by making a standard Cremona transformation centered at the first four points of $\mathbf{p}$. If D is a divisor in the class $dH - \sum_{i=1}^{8} m_i E_i$ on $X_{\mathbf{p}}$, then its strict transform on $X_{\mathbf{q}}$ has class $d'H' - \sum_{i=1}^{8} m_i E_i'$, where $d' = 3d - \sum_{i=1}^{4} m_i$, $m_i' = 2d + m_i - \sum_{i=1}^{4} m_i$ for $1 \le i \le 4$, and $m_i' = m_i$ for $5 \le i \le 8$ [11]. Write $M : N^1(X_{\mathbf{p}}) \to N^1(X_{\mathbf{q}})$ for the corresponding linear map.

Let $M_\sigma = PM$, where M is as above and P is the permutation matrix which permutes the exceptional divisors by moving the first one to last. If D is a divisor on $X_{\mathbf{p}_i}$, its strict transform on $X_{\mathbf{p}_{i+1}}$ has class $M_\sigma(D)$. The map $M_\sigma^n : N^1(X_{\mathbf{p}_0}) \to N^1(X_{\mathbf{p}_n})$ preserves the effective cones, as well as the property that $4d - \sum_{i=1}^{8} m_i = 0$.

Suppose that $D \sim dH - \sum_{i=1}^{8} m_i E_i$ is a root divisor on $X_{\mathbf{p}_n}$ (i.e. prime, not containing $\bar{C}_n$, and with $4d - \sum_{i=1}^{8} m_i = 0$). Then the strict transform of D on $X_{\mathbf{p}_0}$ is a root divisor as well. Since there is a unique such divisor on $X_{\mathbf{p}_0}$, we conclude that D has numerical class $M_\sigma^n(H - \sum_{i=5}^{8} E_i)$ on $X_{\mathbf{p}_n}$. It is straightforward to check that the classes $M_\sigma^n(H - \sum_{i=5}^{8} E_i)$ are all distinct; the argument is indicated in Lemma 4, which is postponed until the end of this section.

It follows that condition $(*)$ holds for the configuration $\mathbf{p}_n$: if any four points p_{j_1}, ..., p_{j_4} of $\mathbf{p}_n$ were coplanar, then $H - \sum_{i=1}^{4} E_{j_i}$ would be a root divisor on $X_{\mathbf{p}_n}$, which is possible only if $n = 0$ and the points in question are p_5, p_6, p_7, and p_8. In particular, the Cremona transformation defining $\mathbf{p}_{n+1}$ is well-defined for every n, and this gives an infinite sequence of configurations connected by Cremona transformations, all satisfying $(*)$. Since the degrees of the classes $M_\sigma^n(H - \sum_{i=5}^{8} E_i)$ grow unboundedly, there are infinitely many distinct configurations among the $\mathbf{p}_n$, even up to permutation of the points. As the Cremona orbit is infinite for the special configuration $\mathbf{p}_0$, it is also infinite for very general configurations. $\quad\square$

Proof of Theorem. Let $\mathbf{p}$ be a very general 8-tuple of points in $\mathbb{P}^3$, and let W be the Cremona orbit of $\mathbf{p}$. By Lemma 3, W contains infinitely many distinct configurations of points, even up to permutations. Lemma 2 then shows that the blow-ups $X_{\mathbf{q}}$ for $\mathbf{q} \in W$ are all derived-equivalent, but by Lemma 1 no two are isomorphic. $\quad\square$

Remarks. In fact this construction can easily be generalized to configurations of $k \ge 8$ points. Permutations of the points, together with the Cremona transformation at the first four, generate the action of a Coxeter group of type $T_{2,4,k-4}$ on the configuration space $(\mathbb{P}^3)^k /\!/ \mathrm{PGL}(4)$ by birational maps. This group is infinite as soon as $k \ge 8$, and sufficiently general configurations have infinite orbits. This is explored in detail in [6, Ch. VI]. The

construction in Lemma 3 simply follows the orbit of a single special configuration under the iteration of a Coxeter element in this group.

The property of having only a finite orbit under Cremona transformations is quite special: a complete classification of such configurations in $\mathbb{P}^2$ has been given by Dolgachev and Cantat [5]. The case in which $\mathbf{p}$ is a configuration which is Cremona equivalent to itself under some nontrivial sequence of Cremona transformations is also quite interesting; in this case there is a pseudoautomorphism $\phi : X_{\mathbf{p}} \dashrightarrow X_{\mathbf{p}}$ [15], inducing an autoequivalence $D(X_{\mathbf{p}}) \to D(X_{\mathbf{p}})$.

Lemma 4. *For any m and n, the classes $M_\sigma^m(H - \sum_{i=5}^8 E_i)$ and $M_\sigma^n(H - \sum_{i=5}^8 E_i)$ are distinct.*

Proof. Explicitly, the matrix M_σ is given with respect to the basis H, E_i as

$$M_\sigma = \begin{pmatrix} 3 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ -2 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -2 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $M_\sigma = SJS^{-1}$ be the Jordan decomposition. A computation shows that J has a 3×3 Jordan block associated to the eigenvalue 1. One can compute the coefficients of $H - \sum_{i=5}^8 E_i$ in the Jordan basis as the entries of $S^{-1}\left(H - \sum_{i=5}^8 E_i\right)$, and observe $H - \sum_{i=5}^8 E_i$ has nonzero coefficients for two of the generalized eigenvectors in the nontrivial Jordan block. It follows that the powers $M_\sigma^n(H - \sum_{i=5}^8 E_i)$ are all distinct. $\square$

For a more enlightened perspective on this calculation from the point of view of Coxeter groups, we refer to [15, §2] and [12, Thm. 2.2].

Acknowledgements

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Department of Mathematics, MIT, 77 Massachusetts Avenue, Cambridge, MA 02139, USA
E-mail address: johnl@math.mit.edu