Banach-Tarski

It's possible to divide up the ball D^3:

$$D^3 = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \quad \text{(disjoint)}$$

Rotate and translate the A_i and reassemble into two D^3's.

$$D^3 = A_1 \cup A_2 \cup A_3 \qquad D^3 = A_4 \cup A_5$$

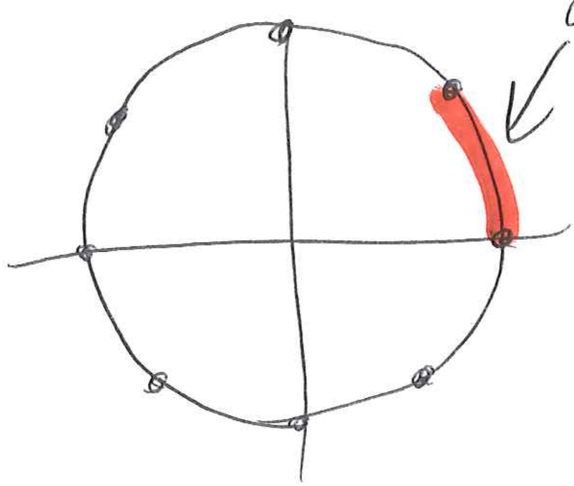
Last time: $G = \left\{ \begin{array}{l} \text{a set of transformations} \\ \text{e.g. all rotations by mult} \end{array} \right\}$
of $\pi/4$ or mults of 1 rad.

(rotations of circle)

orbit representatives

A set of orbit representatives M for G is a collection of points such that every point can be obtained by applying some transformation in G to a (unique) point in M.

$G = \{\text{rotations by } \pi/4\}$

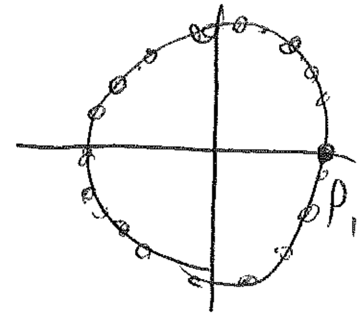


a set of orbit representatives.

$G = \{\text{rotation by } 1 \text{ rad}\}$

can we find a set? of orbit reps?

Algorithm: Pick a point P_1 on circle, and delete its full orbit. $\approx \mathbb{R} \setminus \mathbb{Q}$



Pick a remaining point P_2, delete its full orbit. $\approx \mathbb{R} \setminus \mathbb{Q} \setminus \pi\mathbb{Q}$

Keep going until nothing left. ~~What's left is~~ The set $\{P_i\}$ is a set of orbit reps!

Another way to think about it:

G divides up the circle into an ~~infinite~~ infinite partition by orbits.

each O_α is orbit.

$$\underset{\text{circle}}{S^1} = \bigcup O_\alpha$$

Pick a point P_α in each O_α. That's a set of ORs!

combinations of rotation

To represent an transformation in G, you can use a "word" like:

$$A^{-1}B^{3}ABABA^{-2}B^{-3}A$$

first do A

then B^{-1} 3 times

then A^{-1} 2 times.

$\in S(A^{-1})$

One rule: a word can't AA^{-1}, $A^{-1}A$, BB^{-1}, $B^{-1}B$.

Let $S(A) \subset G = \{$all words starting with $A\}$

$S(A^{-1})$, $S(B)$, $S(B^{-1})$ similar

the empty thing

$$G = S(A) \cup S(A^{-1}) \cup S(B) \cup S(B^{-1}) \cup \{e\}$$

$G = A\,S(A^{-1}) \cup S(A)$

<u>and</u>

$G = B\,S(B^{-1}) \cup S(B)$

First: divide up the sphere S^2:

Let M be a set of orbit representatives for the rotations G on S^2. (Weird set!)
need choice.

$S(A)\,M =$ all points you can get by applying an $S(A)$ transformation to a point of M.

subset of sphere.

<u>$S^2 = G\,M$</u> ← for every $P \in S^2$ can find $m \in M$ and a rotation g so $g(m) = P$.

$S^2 = S(A)\,M \cup S(A^{-1})\,M \cup S(B)\,M$
sphere

$\cup\, S(B^{-1}M) \cup M.$ five pieces!

What are all words ~~of the form~~
equivalent to:

$$A\,S(A^{-1})$$

A followed by a word starting A^{-1}

e.g. BAB is $A(A^{-1}BAB)$. that counts:

$$BAB \in A\,S(A^{-1})$$

$A^{-1}B^2A$ is $A(A^{-2}B^2A)$

$ABBA$ is $A(\underbrace{A^{-1}A}_{}BBA)$

not a valid word!

$A\,S(A^{-1}) =$ all words, except those starting with "A"!

$$\overset{F_1}{S^2 = S(A)M} \cup \overset{F_2}{S(A^{-1})M} \cup \overset{F_3}{S(B)M} \cup \overset{F_4}{S(B^{-1})M} \cup \overset{F_5}{M}$$

points obtained by rotating some point of M where last rotation used is an "A".

things in M

What is ~~A~~

what do we get if we rotate every element of F_2 by A?

rotated F_2

$$AF_2 = \underbrace{AS(A^{-1})}M$$

$$AF_2 \cup F_1 = \underbrace{AS(A^{-1})}M \cup \underbrace{S(A)}M$$

$$= (AS(A^{-1}) \cup S(A))M = GM = S^2$$

$$BF_4 \cup F_3 = S^2$$

F_5 is extra! (can be fixed)

Axiom of Choice (an candidate axiom of set theory)

Given any collection of sets U_α, ← indexed by reals, integers, who knows

it's possible to choose a p_α in each U_α.

Other axioms of set theory: you can always take union of two sets, etc.

You can use it or not. Choice is not implied by the other axioms of set theory.

(Theorem: assuming choice won't lead to a contradiction.)

You don't always need this axiom to make a choice.

Bertrand Russell: Imagine each U_α is a pair of shoes.

You can always take p_α = left shoe.

But if each U_α is a pair of socks.

You need choice!

Axiom of Choice

$\Updownarrow$

Well-ordering Principle

$\Updownarrow$

Zorn's Lemma

"obviously true"

"obviously false"

"who knows"

$\Downarrow$ Banach-Tarski

First, divide up the sphere S^2:

lau

To divide up D^3 instead, just use our decomposition of sphere and ~~connect~~ divide up radially. (We'll ignore center.)

Proof for transcendental numbers existing:

- The set of all polynomials with rational coefficients is countable (HW!)
- Each one has only finitely many roots.

$\Rightarrow$ Algebraic numbers are countable.

$\mathbb{R}$ is uncountable, so non-algebraic numbers must exist!

Proof that π is irrational:

Suppose $f(x)$ is any function

Integrate

$$\int f(x) \sin x \, dx$$ by parts twice.

What do you get?

What is

$$\int_0^{\pi} f(x) \sin x \, dx$$

$u = f'(x)$

$$\int f(x) \sin x \, dx = -f(x) \cos x + \int f'(x) \cos x \, dx$$

$u = f(x)$ $\quad$ $v = -\cos x$ $\quad$ $= -f(x) \cos x$

$du = f'(x)\,dx$ $\quad$ $dv = \sin x \, dx$

Actually finding a transcendental number is hard;

π works.

It's easier to prove:

$$0.1\,1\,0\,0\,0\,1\,0\cdots 0\,1$$

$\uparrow$ 1st, $\uparrow$ 2nd, 3, 4, 5, 6, $\uparrow$ 24

every $n!$ place put a 1;

$$L = \sum_{n=1}^{\infty} \frac{1}{10^{n!}}$$

"Liouville number" $\rightarrow$ easier to prove it's transcendental.

$$\int f(x)\sin x = -f(x)\cos x + f'(x)\sin x - \int f''(x)\sin x\,dx$$

We'll use this with $f(x)$ a clever polynomial.