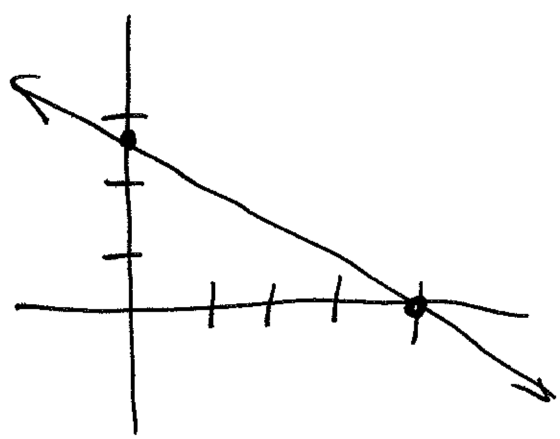
Optimization

- finding critical pts, etc
- Lagrange multipliers
- Linear programming
- Gradient descent (+ Newton's method)

Maximize/minimize $f(x,y)$
subject to $g(x,y)=c$.

occurs where
$Df = \lambda Dg$.

Ex Find the point on $2x+3y=8$
closest to $(0,0)$.

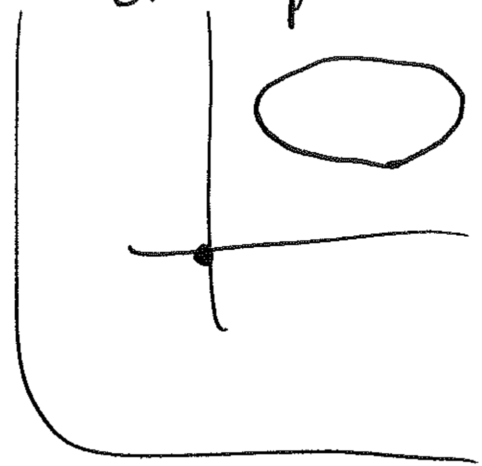


$g(x,y)=2x+3y=8$

~~Max~~ Want to minimize!

$\sqrt{x^2+y^2}$ use $f(x,y)=x^2+y^2$ instead.

can solve similar problems!



$\nabla f = \lambda \nabla g$

$\langle 2x, 2y \rangle = \lambda \langle 2, 3 \rangle$

3 eqns, 3 variables.

$2x = 2\lambda$ $\quad 2x + 3y = 8$

$2y = 3\lambda$

$3y = \frac{9}{2}\lambda$ $\quad 2x + 3y = \frac{13}{2}\lambda = 8$

$\lambda = \frac{16}{13}$

$\boxed{x = \frac{16}{13},\ y = \frac{24}{13}}$

Linear programming

3 variable problem

3 foods available:

	cost	Vit A	Calories
corn	18¢	107	72
milk	23¢	500	121
bread	5¢	0	65

Requirements: ~~Budget: $~~

Vit A: 5000-50000

Calories: 2000-2250

Minimize Budget while satisfying both requirements.

With two variables, we can draw a plot!

With ≥ 3, better to use an algorithm: the simplex method

Use variables c, m, b.

$$107c + 500m + 0b \leq 50000 \qquad c, m, b \geq 0$$

$$107c + 500m + 0b \geq 5000$$

$$72c + 121m + 65b \geq 2000$$

$$72c + 121m + 65 \leq 2250.$$

objective function:

minimize $18c + 23\overset{m}{\cancel{b}} + 5b$

maximize: $-18c - 23m - 5b$

<u>Simplex Method</u>

(we solved it on the computer)

Knapsack problem

Suppose you're packing your backpack.

You have list of possible items, each with a weight + usefulness score.

	Item	Weight	Useful
x_1	Lunch	2	10
x_2	Books	10	1
x_3	Water	1	6
x_4	Computer	5	11
x_5	Spare coat	2	2
x_6	Pencils	0.5	5

all variables must be binary! Either 0 or 1. Simplex alg can handle that

Max weight: 7 lbs

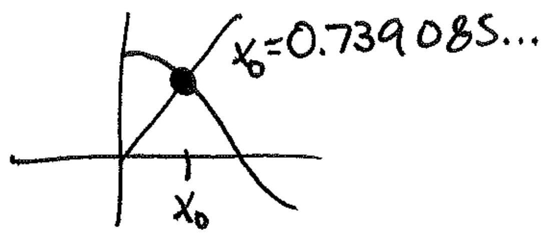
Maximize: $10x_1 + 1x_2 + 6x_3 + 11x_4 + 2x_5 + 5x_6$

Subject to: $2x_1 + 10x_2 + 1x_3 + 5x_4 + 2x_5 + 0.5x_6 \leq 7$

Newton's method (root-finding, not optimization)

Suppose you want to find a root of a function $f(x)$ but can't solve algebraically.

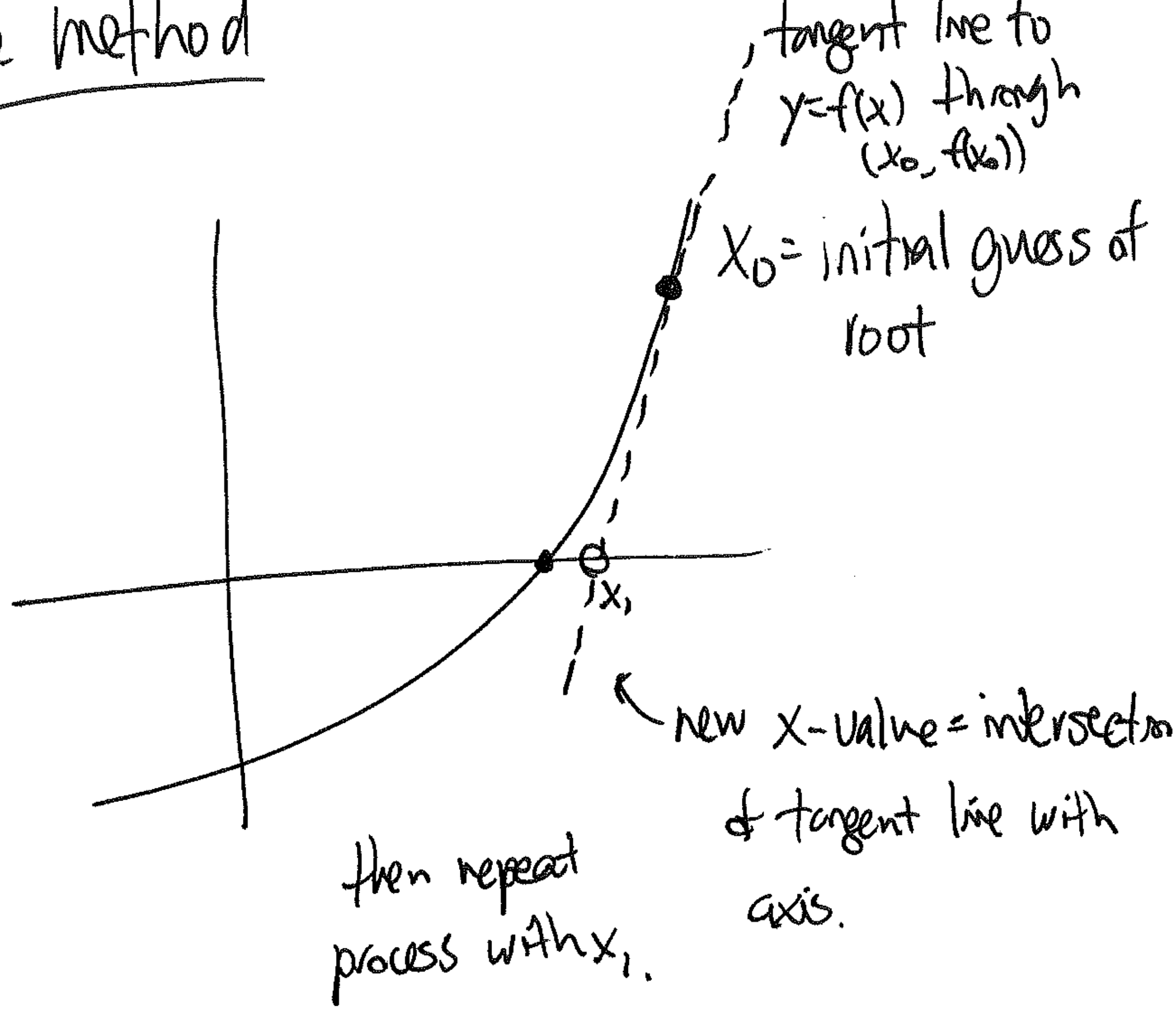
e.g. ~~to~~ $x - \cos x = 0$



Newton's method lets you find a numerically accurate approximation of the root (many digits)

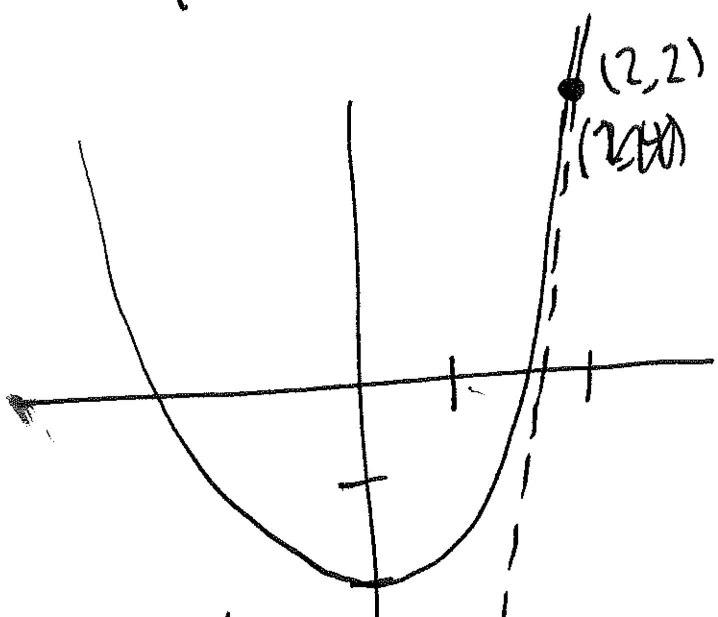
Easier

The method


tangent line to
$y=f(x)$ through
$(x_0, f(x_0))$
x_0 = initial guess of
root
x_1
new x-value = intersection
of tangent line with
axis.
then repeat
process with x_1.

Example

Find a root of, $x^2-2=0$



Initial guess: $x_0=2$

Tangent line to x^2-2 at $x=2$?

~~y=2x+3~~ $y=4x-6$

$y=0$ when $x=3/2$

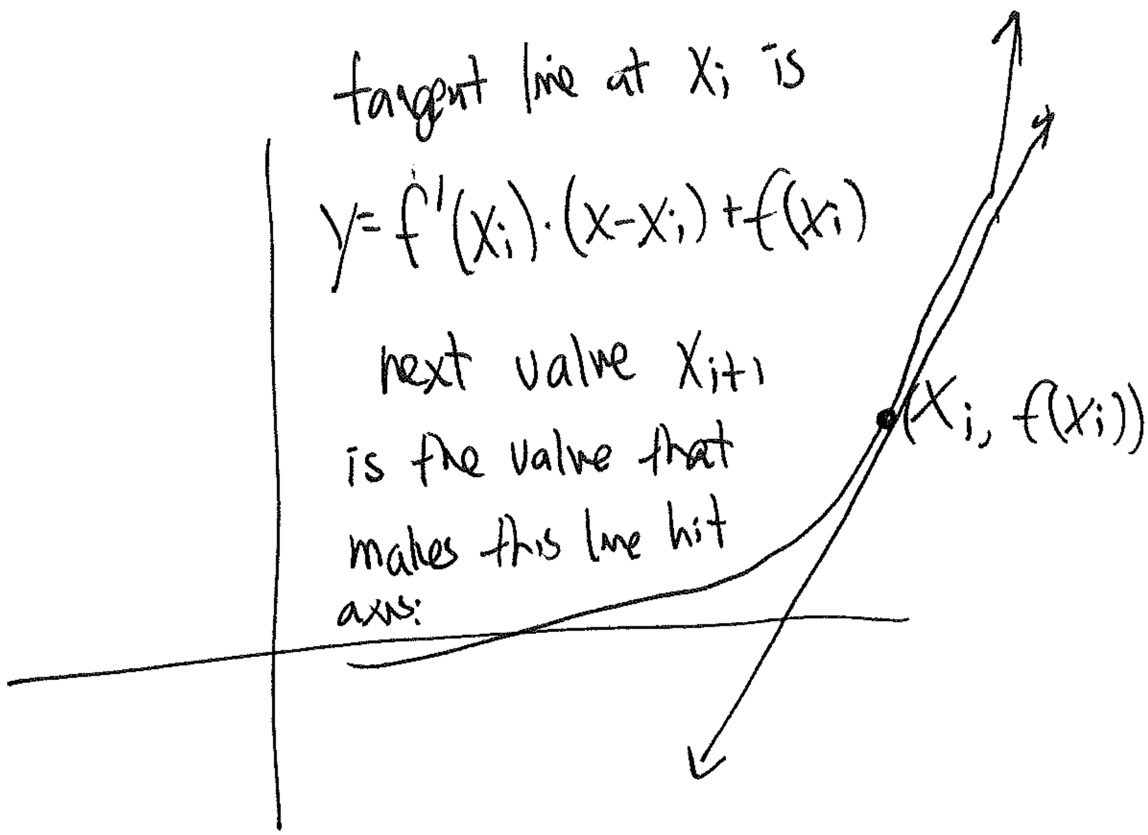
x_0	2	$f(x_0)$
x_1	$3/2$	

i	x_i	$f(x_i)$
0	2	2
1	$3/2$	$1/4$
2	$17/12$	$1/144$

Keep going:

...

General formula



$$f'(x_i)\cdot(x - x_i) + f(x_i) = 0$$

$$x - x_i = -\frac{f(x_i)}{f'(x_i)}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Newton examples

$f(x) = x^2 - 2 \rightsquigarrow$ approximations of $\sqrt{2}$

$2, \ 3/2, \ 17/12, \ 577/408, \ 665857/470832$

or $1,$ $\quad 1/2, \ 9/4, \ 113/72, \ 23137/16272, \ldots$

$f(x) = x - \cos x$

$1.000, \ 0.75036, \ 0.73911,$

$0.7390851333, \ 0.7390851332151 61, \ldots$

Doubles the number of correct digits every iterations!

$f(x) = \sin x \longrightarrow$ approximate π

$3, \ 3.142546\ldots, \ 3.14159265330048, \ldots$

$3.14159265358979, \ldots$