<u>Today</u>: More dynamical systems, more fractals.

<u>HW</u> $r=1$ logistic map

$f(x)=x(1-x)$.

call x_n : so $x_{n+1}=x_n(1-x_n)$

Prove $\lim_{n\to\infty} f^n(x_0)=0$.

Let $y_n = 1/x_n$, Let's prove $y_n \to \infty$

$$y_{n+1}=\frac{1}{x_{n+1}}=\frac{1}{x_n(1-x_n)}=\frac{1}{1/y_n(1-1/y_n)}$$

$$=\frac{1}{\frac{1}{y_n}-\frac{1}{y_n^2}}=\frac{1}{\frac{y_n}{y_n^2}-\frac{1}{y_n^2}}=\frac{1}{\frac{y_n-1}{y_n^2}}$$

$0<x_n<1$

so $y_n>1$

so $y_n-1>0$

$$=\frac{y_n^2}{y_n-1}=\frac{y_n^2-1}{y_n-1}+\frac{1}{y_n-1}=y_n+1+\frac{1}{y_n-1}$$

$$> y_n+1$$

So $\boxed{y_{n+1} > y_n + 1}$

this means $y_n \to \infty$, so $x_n \to 0$.

Julia sets.

If c is a complex number, set $f_c(z) = z^2 + c$.

there exists R so $|f^n(z)| < R$ all n.

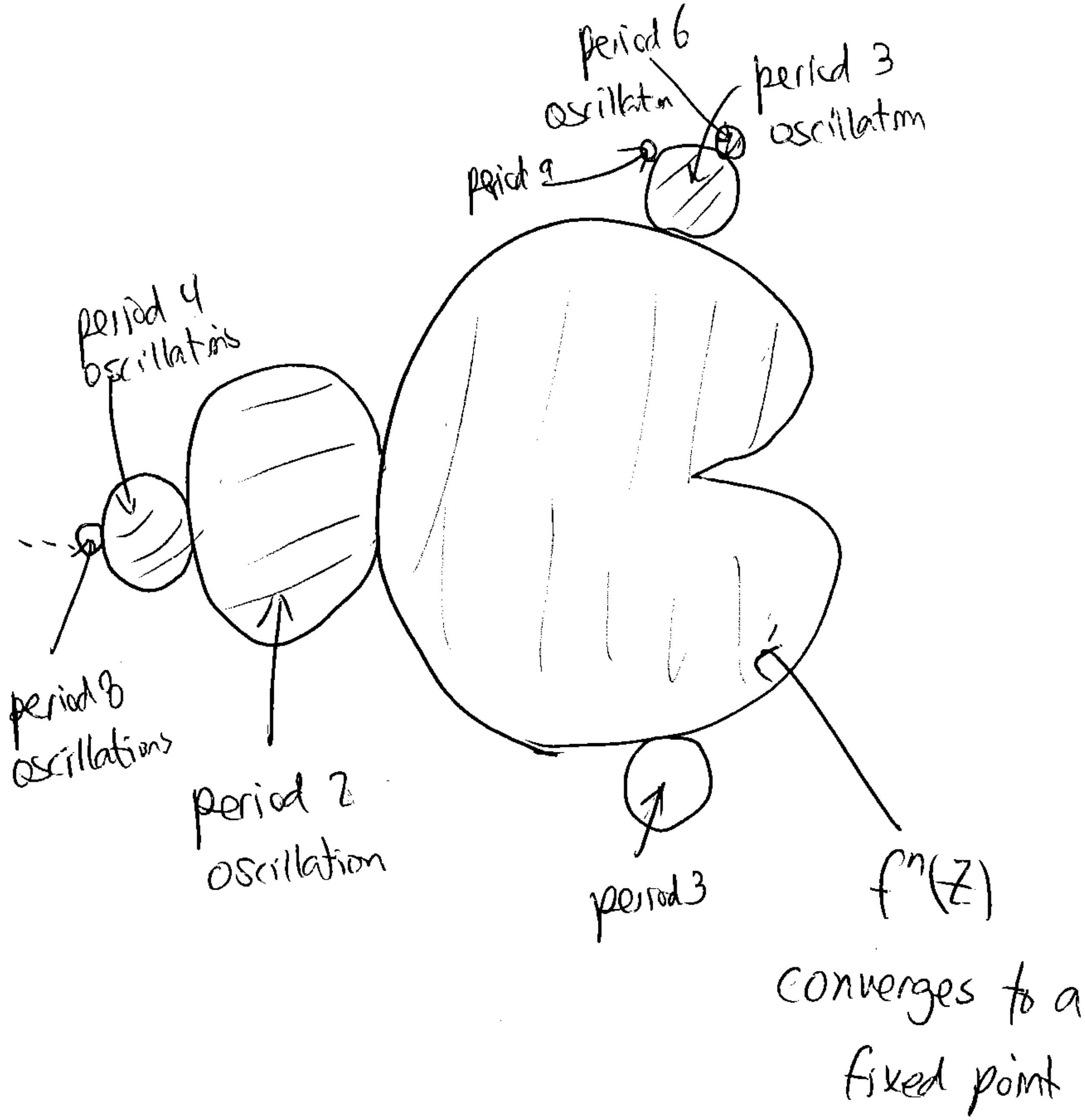
$$J(c) = \{ z : f_c^n(z) \text{ stays bounded as } n \to \infty \}$$

<u>Mandlebrot set</u>

"one piece"

$$M = \{ c : J(c) \text{ is connected} \}$$

$$= \{ c : 0 \text{ has bounded orbit under } f_c \}$$


period 6
oscillation
period 3
oscillation
period 9
period 4
oscillations
period 8
oscillations
period 2
oscillation
period 3
$f^n(z)$
converges to a
fixed point

We've seen some examples of "chaos":

- never goes to ∞, not bounded
- never repeats
- unpredictable: can't guess anything about $f^{1000}(z)$ without calculating it.
  (unlike $r=1/2$ logistic map; then $f^{1000}(z) \approx 0$)
- nearby starting pt have different behavior.
  ("butterfly effect")

How can we define chaos more precisely?

<u>Warm-up</u> How do we define <u>dimension</u>?
(of a fractal, or a non-fractal)

What is dimension of a geometric object?

- How many "independent" directions can we go?

- Number of orthogonal lines you can draw?
  segments.

Another way:

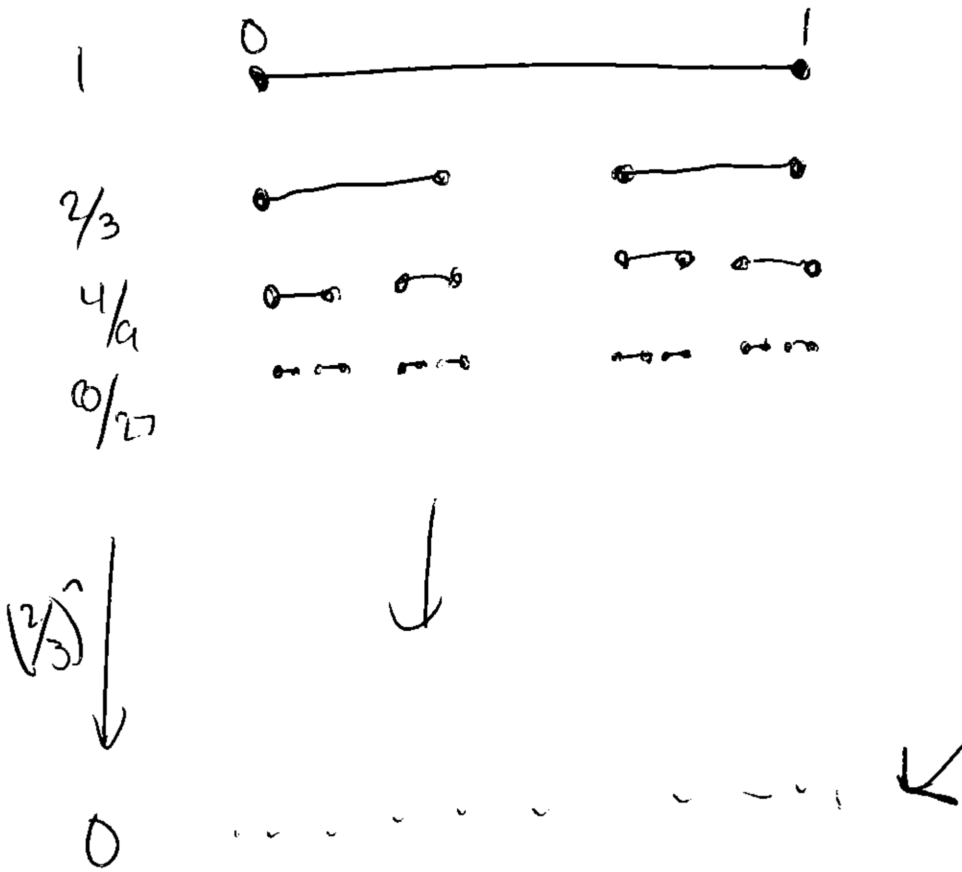
If we double S in every direction, how much does the size $\mu(S)$ change?

S stretched by factor of 2.

Square: $\mu(2S) = \mu(S) \cdot 2^{2}$ — the dimension

Cube: $\mu(2S) = \mu(S) \cdot 2^{3}$

Cantor set

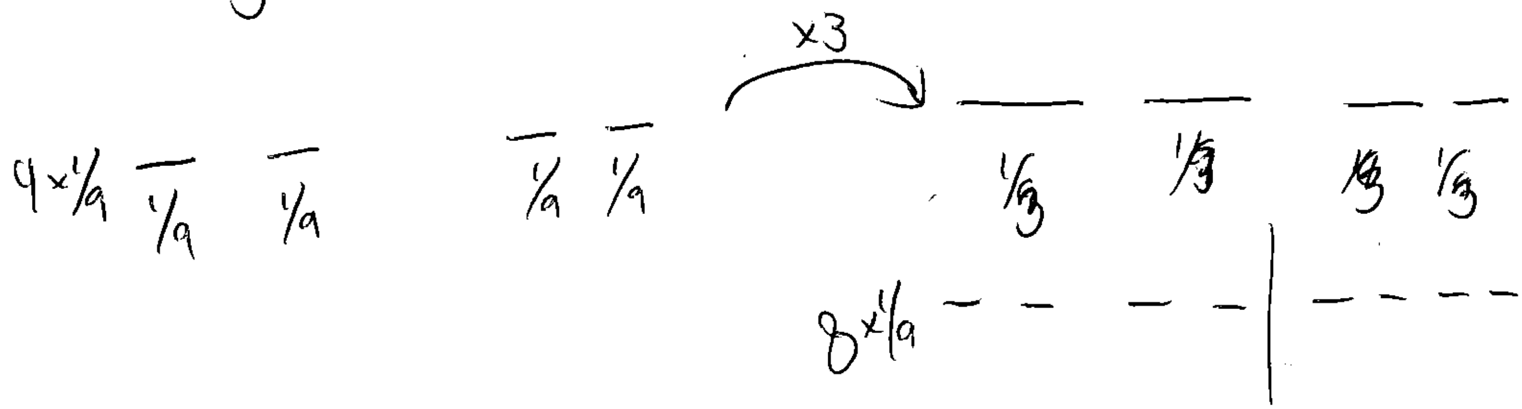


Cantor set is what's left over.

Dimension? If we stretch by a factor of 3, we get:

3 × Cantor set

original



=2 Cantor sets!

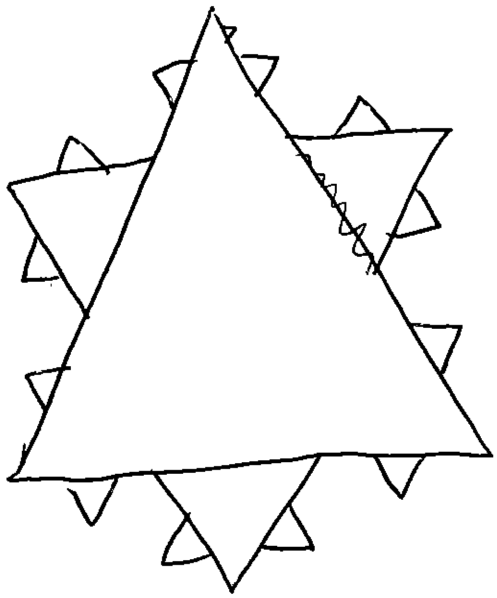
So → $\mu(3\cdot S)=\mu(S)\cdot 2$

but $\mu(3\cdot S)=\mu(S)\cdot 3^d$

$d=\log_3 2 \approx 0.630929\ldots$

"fractal" since not an integer.

Koch snowflake



perimeter of n^{th} iterate:?

$P_0 = 3$

area enclosed

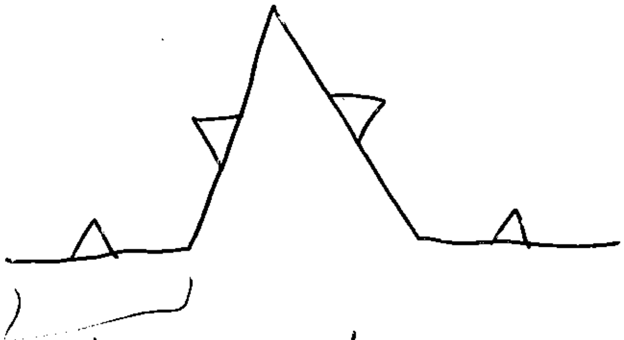
$P_0 = \frac{\sqrt{3}}{4}$

perimeter of $P_n = \left(\frac{4}{3}\right)^n \cdot 3$

perimeter(snowflake) $= \infty$

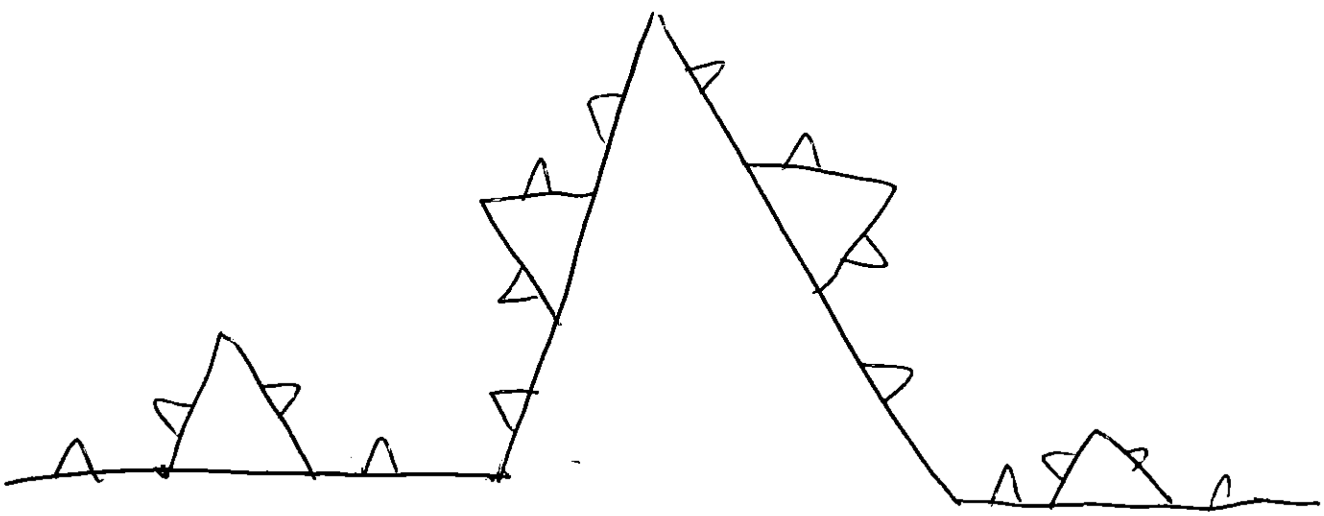
area of P_n = stays finite

dimension = 1? 2? other?



$1/3$

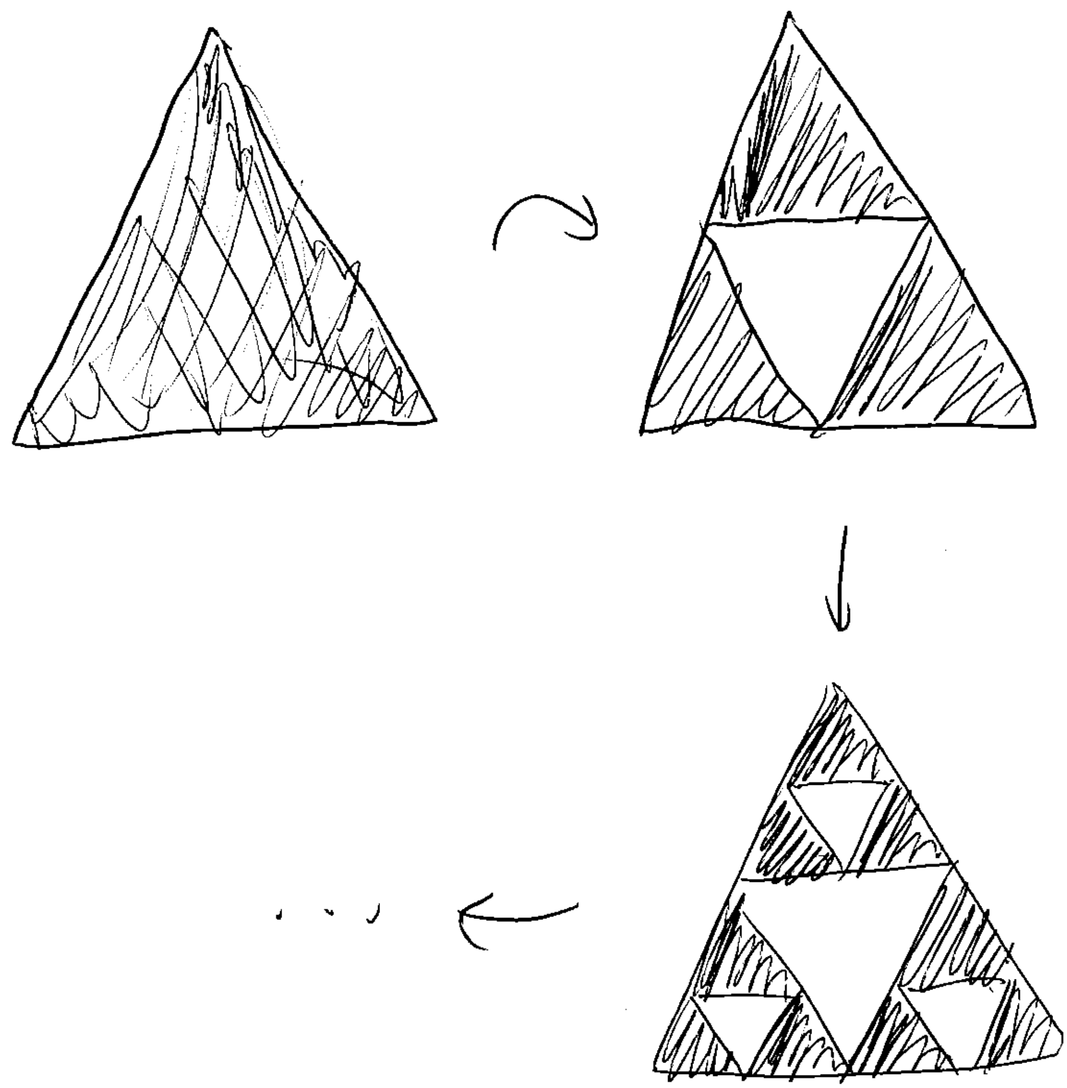
$\downarrow$ scale by 3



$$\mu(3\cdot S) = 4\,\mu(S)$$

but $\mu(3\cdot S) = \mu(S)\cdot 3^d \implies d = \log_3 4 = 1.2618\ldots$

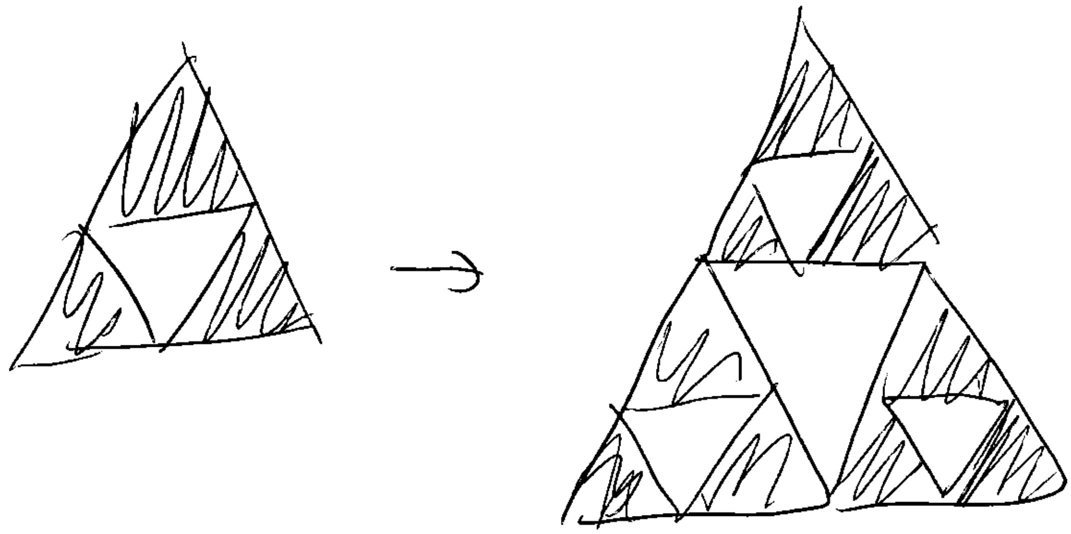
Sierpinski triangle



$\text{area}(P_n) \approx \left(\frac{3}{4}\right)^n$ final area is 0!

Dimension = ?

Stretch by 2 $\rightsquigarrow$ 3 copies of original!



$$\mu(2\cdot S) = 3\,\mu(S)$$

$$\mu(2\cdot S) = \mu(S)\cdot 2^d \qquad d = \log_2 3$$