Game theory

(I will follow Math 486 lecture notes from PSU, Christopher Griffin)

Conditional probability

"$\mathbb{P}(A)$" means the probability that A happens.

it's a number between 0 and 1 (inclusive)

"$\cancel{\mathbb{P}(B)}$"

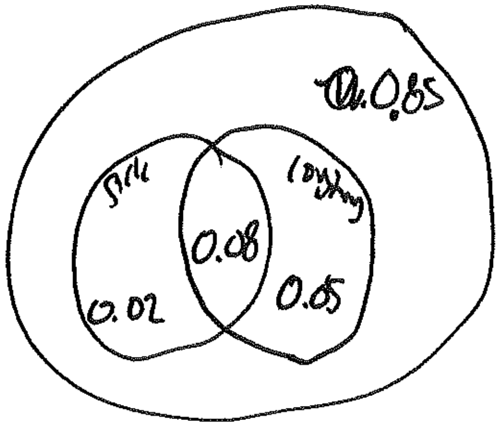
"$\mathbb{P}(B|A)$" is the probability that B happens, given that A happens.

Ex A = person is sick

B = person is coughing

Suppose:

sick + coughing	0.08
sick + not coughing	0.02
not sick + coughing	0.05
not sick + not coughing	0.85
Σ	1



$$\mathbb{P}(A) = 0.08 + 0.02 = 0.10$$

$$\mathbb{P}(B) = 0.08 + 0.05 = 0.13$$

$$\mathbb{P}(\text{sick} \mid \text{coughing}) = \frac{0.08}{0.08 + 0.05} = \frac{8}{13}$$

in general, the rule is:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Challenge: Come up with a formula relating $\mathbb{P}(B \mid A)$ and $\mathbb{P}(A \mid B)$ using that definition?

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \qquad \mathbb{P}(B \mid A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}$$

$$\frac{\mathbb{P}(A \mid B)}{\mathbb{P}(B \mid A)} = \frac{\mathbb{P}(A)}{\mathbb{P}(B)}$$

or

$\boxed{\text{Bayes' rule}}$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\, \mathbb{P}(A)}{\mathbb{P}(B)}$$

The Monty Hall Problem

The host of a game show hides a prize behind one of three doors.

First, you pick a door. (but don't open it)

The host ~~reveals~~ opens one of the other empty doors, showing there's nothing there.

You can stick with your choice, or switch doors.

What should you do?

What's $\mathbb{P}(\text{win} \mid \text{switch})$?

$\mathbb{P}(\text{win} \mid \text{don't switch})$?

P(you picked wrong to start) is $\frac{2}{3}$.

in this case, if you switch, you win.

So if your strategy is to always switch, you'll win $\frac{2}{3}$ of the time!

$$P(\text{win} \mid \text{switch}) = \frac{2}{3}$$

$$P(\text{win} \mid \text{don't switch}) = P\left(\begin{array}{l}\text{you picked right in}\\ \text{the first place}\end{array}\right) = \frac{1}{3}.$$

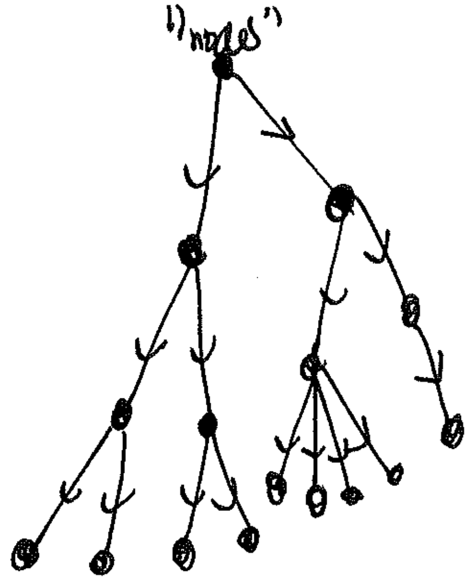
Let's try to define a game using a <u>game tree</u>.

Simplest kind of game: - Complete information

- No chance

We start with a "directed tree" $(\underset{\text{set of vertices}}{V}, \underset{\text{set of edges}}{E})$

no loops



- node for every possible game position

- edge between two nodes if there's a legal move taking you from one to other.

"player vertex assignment": $v: \underset{\text{non-terminal nodes}}{\cancel{V \setminus T}} \to \underset{\text{players}}{P}$

$v: V \setminus \text{terminal} \to P$

<u>every node is assigned to a player.</u> $v: \cancel{V} \to \cancel{A}$ players

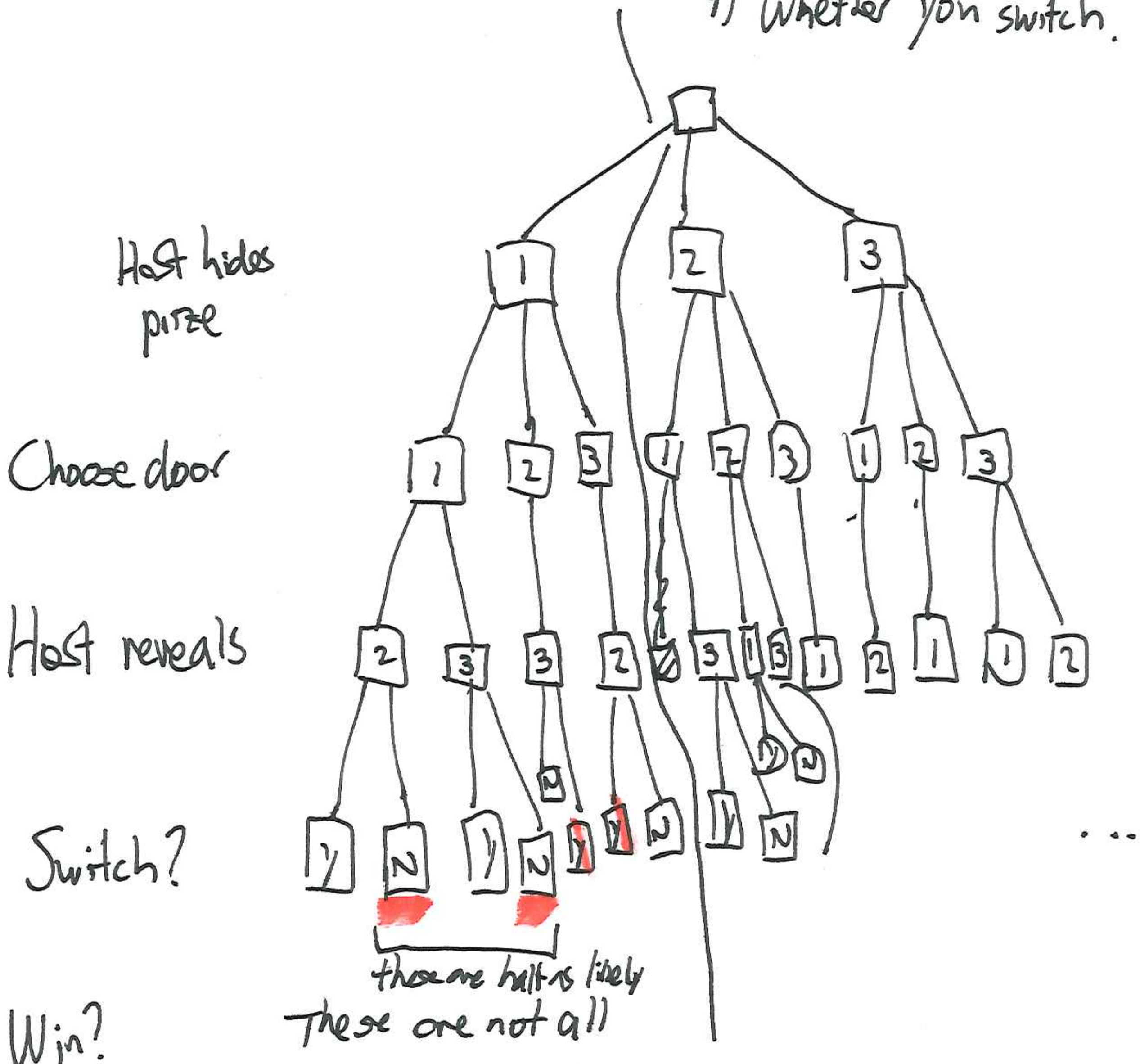
"payoff function"

at every terminal node (final position),

assign a numerical outcome to each player. $F: \{\text{terminal nodes}\} \to \mathbb{R}^N$ ← a vector with an entry for each player

Represent the game as a tree.

1) Which door it's hidden
2) Which door you choose
3) Which door host reveals
4) Whether you switch.

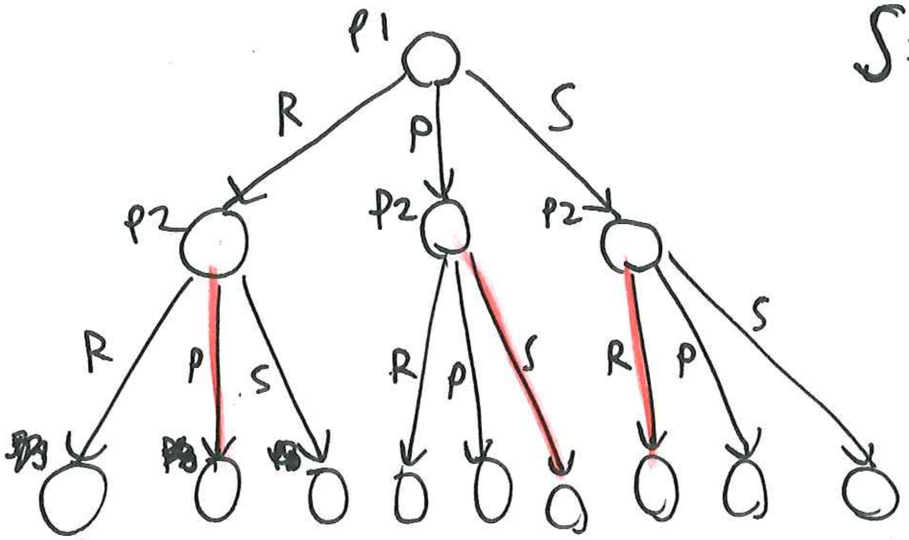


These are not all equally likely!

"move assignment": give a set of "moves" S, and every edge should be assigned to one element of S.

<u>Problem</u> Unfair rock paper scissors

$S = \{R, P, S\}$



$\begin{pmatrix}0\\0\end{pmatrix}$ $\begin{pmatrix}-1\\1\end{pmatrix}$ $\begin{pmatrix}1\\-1\end{pmatrix}$ $\begin{pmatrix}1\\-1\end{pmatrix}$ $\begin{pmatrix}0\\0\end{pmatrix}$ $\begin{pmatrix}-1\\1\end{pmatrix}$ $\begin{pmatrix}-1\\1\end{pmatrix}$ $\begin{pmatrix}1\\-1\end{pmatrix}$ $\begin{pmatrix}0\\0\end{pmatrix}$

✓

everything's defined.

A <u>strategy</u> for Player i in a game like this is: a choice of move at every vertex ~~owned~~ controlled by player.

vertices where it's your turn

$\sigma_i : V_i$:

Imperfect information

Now we want to describe games where the player doesn't have complete information (i.e. player doesn't know which vertex they're at)

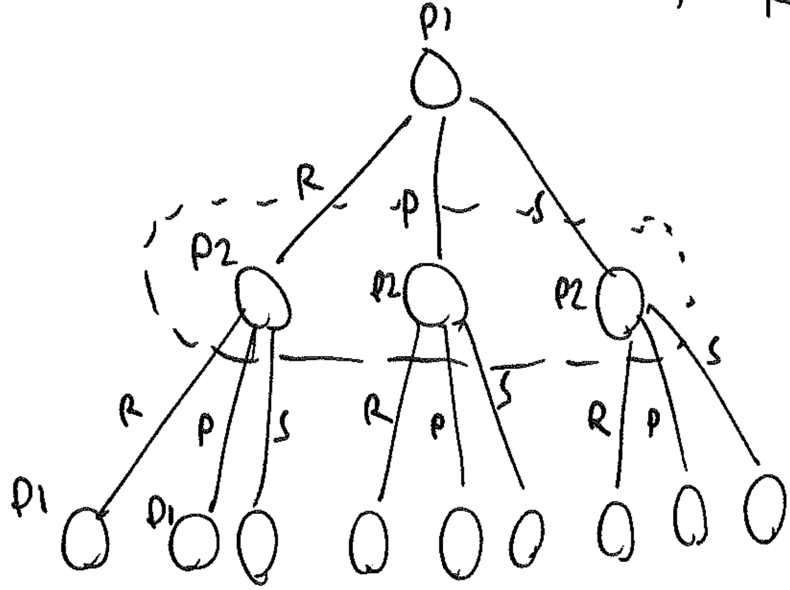
Ex Rock paper scissors, Blackjack, poker, Kriegspiel, ... Monty Hall, ...
(some of these have an element of chance)

How to formalize this?

Divide the vertices into "information sets":

- all vertices in an information set much have same player's turn.
- all vertices in an information set must have the same "move types" available to a player.

Real RPS:



How to incorporate chance into a game?

Add player 0, "Fate"! ~~Now Solve~~

At every "fate" vertex, assign probability to each edge leaving the vertex.