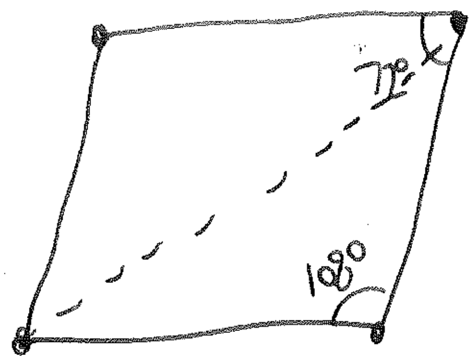
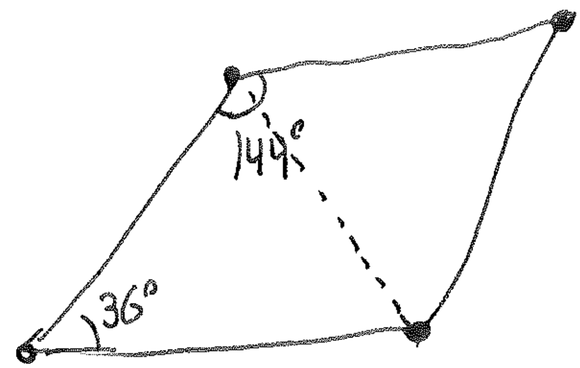
# Penrose tiles

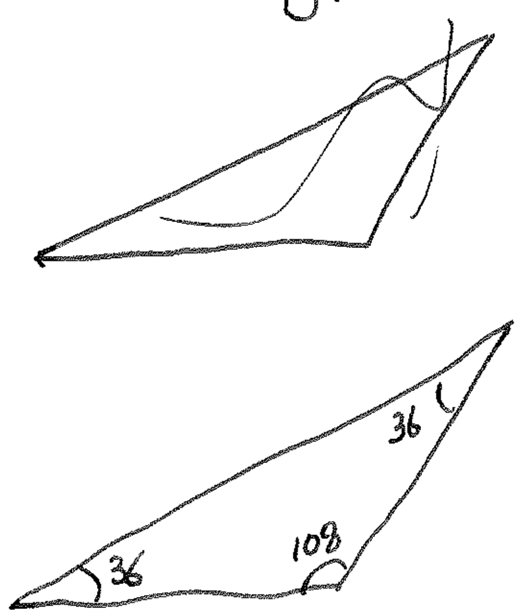


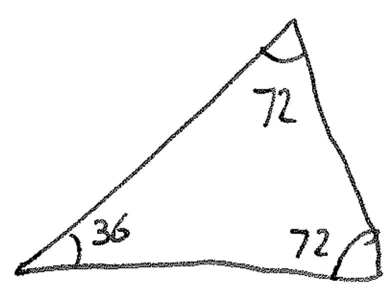

- Why can we tile the plane?
- How do we know it's non-repeating?

Two half-rhombuses:

B:

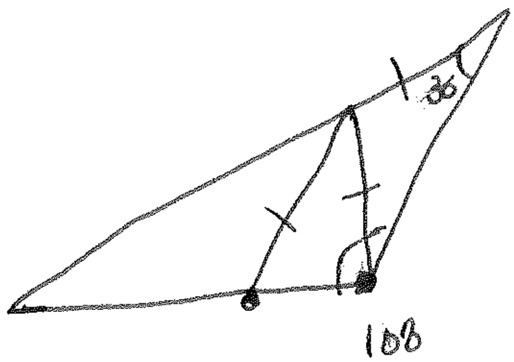


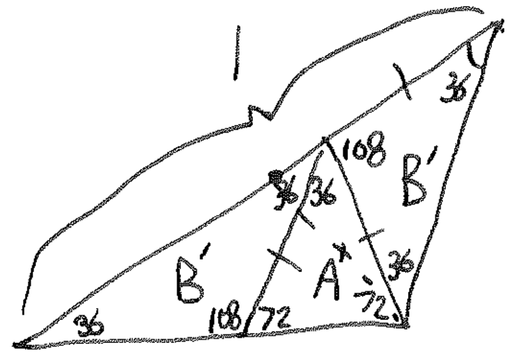

A:



## Subdivision:

B 108-36-36




$$\cos 36^\circ = \frac{1/2}{x}$$

$\varphi - 1$
$=$
$0.618\ldots$
$=$

Geometry:

$$\cos 36^\circ = \frac{1+\sqrt{5}}{4} \qquad x = \frac{1}{2}\sec 36^\circ = \frac{1}{2}\left(\frac{4}{1+\sqrt{5}}\right) = \frac{2}{1+\sqrt{5}} = \boxed{\frac{\sqrt{5}-1}{2}}$$

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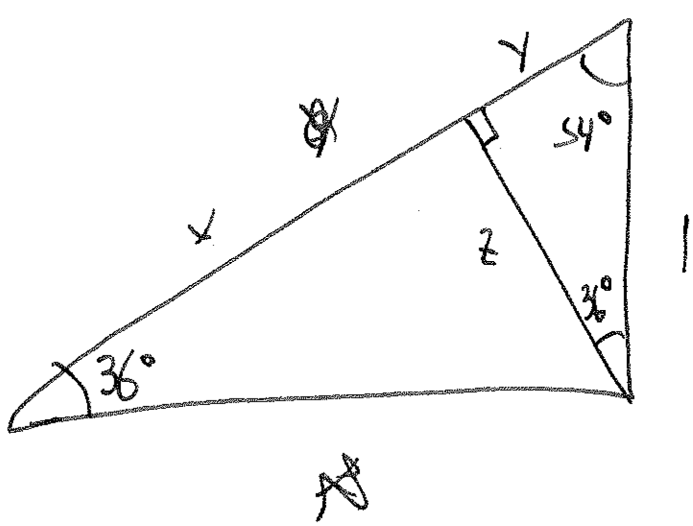



$$\sin 36^\circ = \frac{1}{x+y} = \frac{y}{1}$$

(found a better method at the board)

Let $r_n = \frac{A_n}{B_n}$

$$r_{n+1} = \frac{A_{n+1}}{B_{n+1}} = \frac{A_n + B_n}{A_n + 2B_n} = \frac{r_n B_n + B_n}{r_n B_n + 2B_n} = \frac{r_n + 1}{r_n + 2}.$$

The stable r-value has

$$r = \frac{r+1}{r+2}$$

$$r^2 + 2r = r + 1 \qquad r = \frac{-1 + \sqrt{1-(-4)}}{2} = \frac{\sqrt{5}-1}{2} = \varphi - 1$$

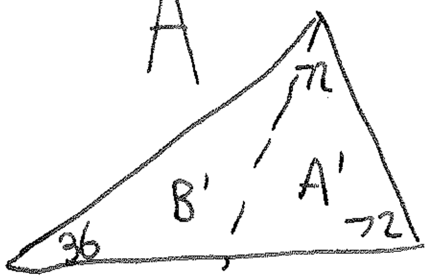
$$r^2 + r - 1 = 0$$

So ratio of A-tiles to B-tiles tends to $\frac{\sqrt{5}-1}{2}$.

More B than A by factor of $1/r = \varphi$!

Irrational, so non-repeating!

A




also scaled by $\varphi - 1$!

let $A_n$ = # A-triangles at step $n$

$B_n$ = # B-triangles at step $n$

after subdividing:

$$A_{n+1} = A_n + B_n$$

$$B_{n+1} = A_n + 2B_n$$

An eigenvector of $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ is a $v = \begin{pmatrix} x \\ y \end{pmatrix}$

such that $Av = \lambda v$ for some scalar $\lambda$.

Here: $v_1 = \begin{pmatrix} \frac{-\sqrt{5}-1}{2} \\ 1 \end{pmatrix}$, $\lambda_1 = \frac{3-\sqrt{5}}{2}$

$v_2 = \begin{pmatrix} \frac{\sqrt{5}-1}{2} \\ 1 \end{pmatrix}$, $\lambda_2 = \frac{3+\sqrt{5}}{2}$

the matrix

To do $A^n \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}$, write $\begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = \overbrace{c_1 v_1 + c_2 v_2}^{\text{eigenvectors,}}$

and then $A^n(c_1 v_1 + c_2 v_2) = c_1 A^n v_1 + c_2 A^n v_2$

$= c_1 \lambda_1^n v_1 + c_2 \lambda_2^n v_2$.

How to analyze

$$A_{n+1} = A_n + B_n$$
$$B_{n+1} = A_n + 2B_n$$

directly?

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} A_{n-2} \\ B_{n-2} \end{pmatrix} \cdots$$

In general:

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \begin{pmatrix} A_0 \\ B_0 \end{pmatrix}$$

need to compute matrix powers!