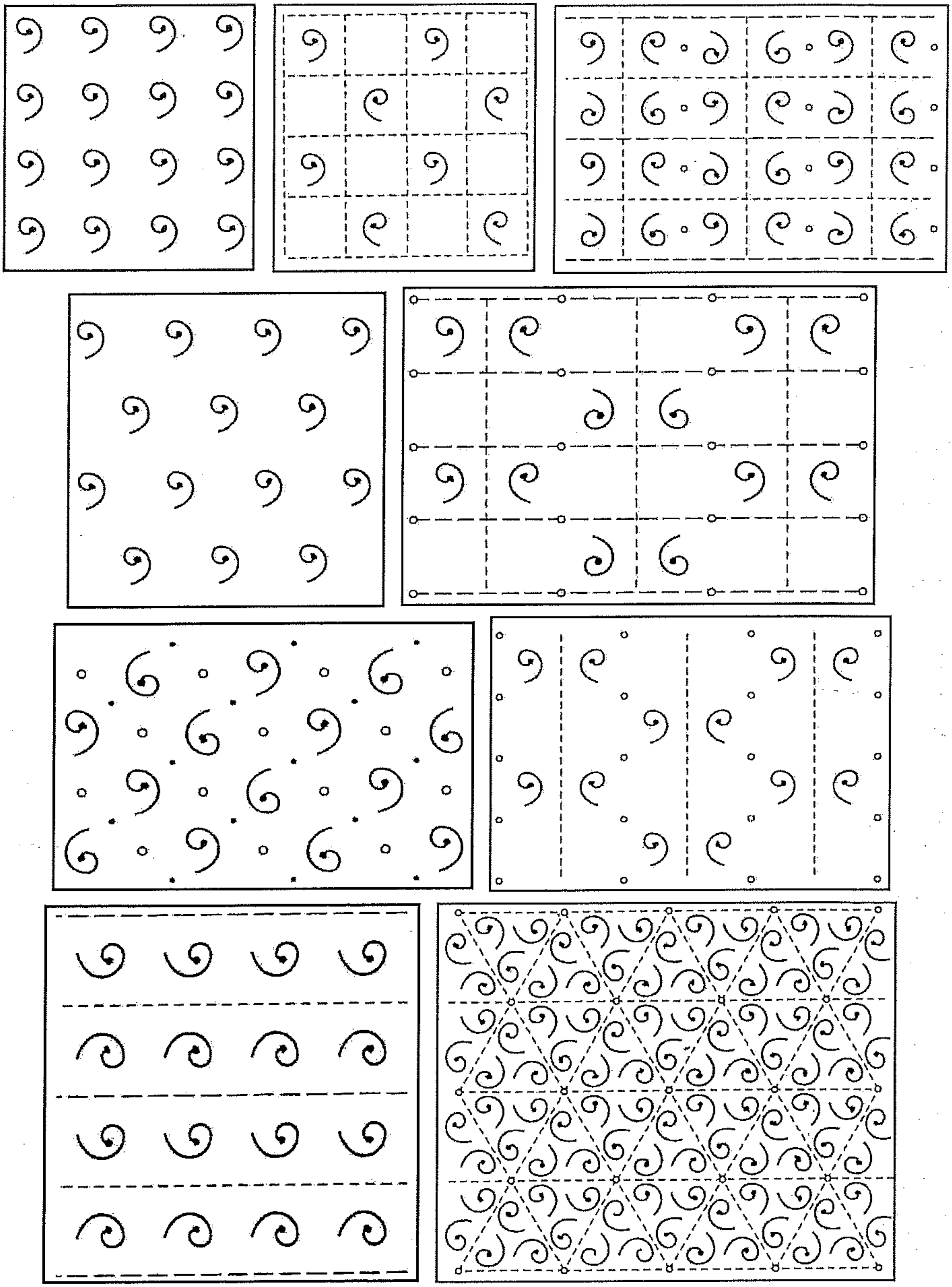
The Seventeen Wallpaper Patterns



From 'Sacred Geometry' by Miranda Lundy, Wooden Books

Wallpaper groups

Last time: what are the possible patterns for repeating wallpaper?

Group theory

A <u>group</u> is: a set G, with a binary operator $\cdot : G \times G \to G$

Given any two $g, h \in G$, you can form $g \cdot h \in G$.

Three requirements:

- Associative: $(g \cdot h) \cdot i = g \cdot (h \cdot i)$
- <u>Identity</u>: there's an $e \in G$ so $e \cdot g = g \cdot e = g$ for any $g \in G$.
- <u>Inverse</u>: Given any g, can find g^{-1} so $g \cdot g^{-1} = e$, $g^{-1} \cdot g = e$

Basic examples:

- G = integers, operation = addition (but not mult!)

- G = nonzero real numbers, operation = multiplication

- Points on an elliptic curve, with the complicated group law

- "Euclidean motions" / "isometries of $\mathbb{R}^2$".

an element is a function

$T : \mathbb{R}^2 \to \mathbb{R}^2$

such that ~~T(x,y)~~

distance

$$d(T(a,b), T(c,d)) = d((a,b),(c,d))$$

If (a,b), (c,d) any two points

Examples:

- Rotation by origin by θ
- Translation by (a, b)
- Reflection over line.

those are all the basic

What's the operation $\bullet$?

It's composition: $T \bullet S = T \circ S$

$$\mathbb{R}^2 \xrightarrow{S} \mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$$

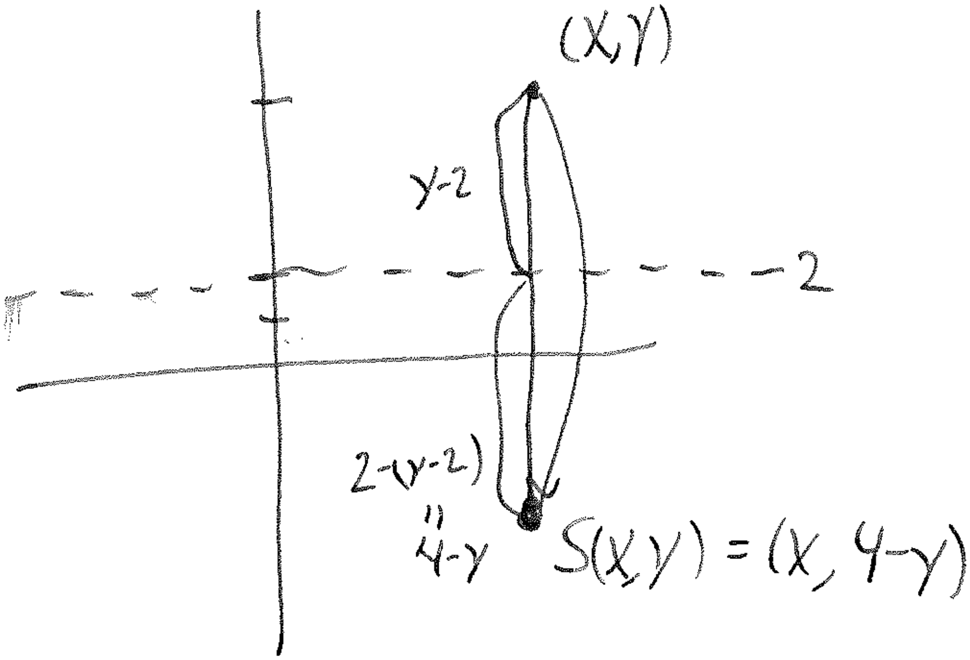
Is it associative?

S = reflect over horizontal line $y=2$

T = shift up 3

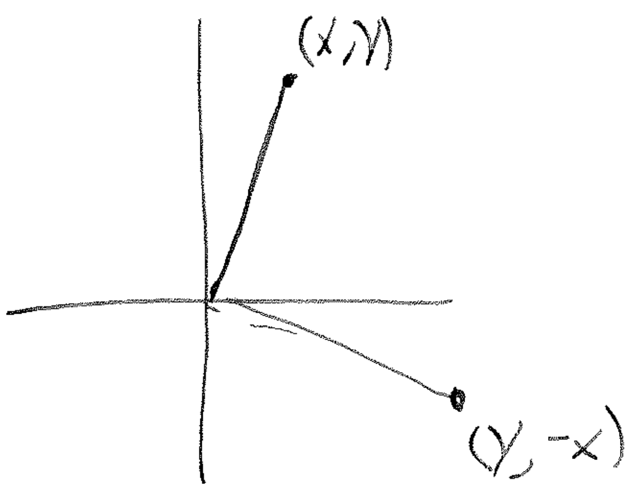
U = rotate $90°$ clockwise.

$S(x,y)=(x,4-y)$



$T(x,y)=(x,y+3)$

$U(x,y)=(y,-x)$



Is $(S \circ T) \circ U = S \circ (T \circ U)$??

What's $(S \circ T)$?

$$(S \circ T)(x,y) = S(T(x,y)) = S((x, y+3)) = (x, 4-(y+3))$$

$$= (x, 1-y)$$

reflection over line $y = 1/2$.

$$((S \circ T) \circ U)(x,y) = (S \circ T)(U(x,y))$$

$$= (S \circ T)(y, -x)$$

$$= (y, 1-(-x)) = (y, x+1)$$

$T \circ u = 1$

$(T \circ u)(x,y) = T(u(x,y)) = T(y,-x) = (y, 3-x)$

$(S \circ (T \circ u))(x,y) = S(y, 3-x) = (y, 4-(3-x))$

$= (y, 1+x)$ ✓

Do they have inverses?

inverse of S: S itself!

inverse of T: shift down 3

inverse of u: rotate 90° counterclockwise.

In fact: every Euclidean motion can be written

in the form $T(x,y) = M\begin{pmatrix} x \\ y \end{pmatrix} + v$

some vector

2×2 matrix preserving distance.

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

e.g. for

$S(x,y) = (x, 4-y)$ how to write in this way?

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} = (x, 4-y)$$

$T(x,y) = (x, y+3)$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

A) If G is a group, a <u>subgroup</u> of G is a subset H such that:

- if $h_1, h_2 \in H$, then $h_1 \cdot h_2 \in H$
- if $h \in H$, $h^{-1} \in H$.

G = Euclidean motions

Some subgroups:

- Translations

In matrix-vector form:

$$T(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

- Rotations around a fixed (P, Q)
- $\{T(x,y) = (x, \pm y + a)\}$

some reflections, some translations.

- Translations by integers

$T(x,y) = \{(x+a, y+b) : a, b \in \mathbb{Z}\}$

If $H \subset$ Euclidean motions.

The point group of H is

represent all elements in matrix-vector form.

$$\cancel{P_H = \{(M,} \quad P_H = \{ M : (M, v) \in H \}$$

for $\{ T(x,y) = (x, \pm y + a) \} = G_2$

$$P_{G_2} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

the translation subgroup of H is:

$$T_H = \{ v : (\underbrace{I_2}_{\text{identity}}, v) \in H \}$$

$$T_{G_2} = \left\{ \begin{pmatrix} 0 \\ a \end{pmatrix} \right\}.$$

A <u>wallpaper group</u> is a subgroup H of Euclidean motions such that:

- The point group P_H has to be finite
- The translation group T_H is given by combinations of just two translations.

theorem: There are only 17 of them!