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# CS180 Homework 8

## Question 1

Consider a bipartite graph G with a bipartition of L and R, where L denotes the set of teams (and |L| = 2n), and the set R denotes the set of days (and |R| = 2n – 1). Let an edge (t, d) ∈ G if for every t ∈ L and d ∈ R, where team t won a match on day d. The problem then is equivalent to asking whether the graph contains a matching of size |R|, where for every day we can find a unique team that won on that day. This can be proved using Hall's Marriage Theorem, which is true iff $\forall S \subseteq R, |S| \leq |N(S)|$, where N(S) represents the number of neighbors of set S.

Assume that |N(S)| < |S| and let t ∈ L \ N(S) be a team that didn't win in any day in set S. Since |N(S)| < |S| < 2n = |L|, such a team exists for the days in S. However, on each of the days in S, team t must have played and lost to a different opponent (as it didn't win any games), which means that there exists at least |S| teams that must have won some game, so |S| ≤ |N(S)|, which is a contradiction. Therefore, |S| ≤ |N(S)| $\forall S \subseteq R$; and thus, it is possible to choose one winning team from each team without choosing any team more than once.

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Question 2

For a given graph G = (V, E) and integer k, the Clique Problem is to find whether G contains a clique of size >= k. For a given graph G' = (V', E') and integer k', the Independent Set Problem is to find whether G' contains an Independent Set of size >= k'. To reduce the Clique Problem to an Independent Set problem for a given graph G = (V, E), construct a complementary graph G' = (V', E') such that (1) V = V' (i.e. complement graph has the same vertices as the original graph) and (2) E' is the complement of E (i.e. G' has all the edges not present in G), which can be constructed in polynomial time.

Then, if there is an independent set of size k that exists in the complement graph G', it implies that no two vertices share an edge in G'. This further implies that all of the vertices share an edge with all others in G, which forms a clique. Thus, there exists a clique of size k in G. Conversely, if there is a clique of size k in the graph G, it implies that all vertices share an edge with all others in G. This further implies that no two of these vertices share an edge in G', which forms an independent set. Thus, there exists an independent set of size k in G'.

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Question 3

(a) Set cover reduced to hitting set

The set cover can be reduced to hitting set by checking that for a solution a1, …, ak , every set Bj is hit by at least one of the ai. Given sets S1, …, Sn, and a ground set W = {w1, …, wm} of elements to cover, let a1, …, an represent the sets and let B1, …, Bm represent the elements. We can say that ai hits B jiff set Si contains element wj. It follows that there is a hitting set of size at most k iff there is a set cover of size at most k.

(b) Hitting set reduced to dominating set

(c) Dominating set reduced to set cover

Given a graph G = (V, E), we can construct a set cover (U, S) where U is equal to V, and the family of subsets is S = {S1, …, Sn} such that Sv consists of the vertex v and all vertices adjacent to v in G. Now, if D is a dominating set for G, then C = {Sv, where v ∈ D} is a solution to the set cover problem with |C| = |D|. Conversely, if C = {Sv, where v ∈ D} is a solution to the set cover problem, then D is a dominating set for G with |D| = |C|. Therefore, the size of the dominating set for G is equal to the size of the set cover for (U, S).

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Question 4

(a) Reduction from Hamiltonian Cycle to Hamiltonian Path

Given a graph G for which we need to find if a Hamiltonian Cycle exists, for a single edge e = (u, v) add new vertices u' and v' such that u' is connected only to u and v' is connected only to v to give anew graph G'. G' has a Hamiltonian Path iff G has a Hamiltonian Cycle with the edge e = (u, v). Run the Hamiltonian Path algorithm on each G' for each edge $e \in G$. If all the graphs have no Hamiltonian Path, then G has no Hamiltonian Cycle. On the other hand, if at least one G' has a Hamiltonian Path, then G has a Hamiltonian Cycle which contains the edge e.

(b) Reduction from Hamiltonian Path to Hamiltonian Cycle

Given a graph G which contains a Hamiltonian Cycle, choose an arbitrary node v and split it into two nodes v' and v'' to get a new graph G'. Any Hamiltonian Path in G' must start at v' and end at v''. If G' has a Hamiltonian Path, then the same ordering of nodes (after combining v' and v'' back together to get v) is a Hamiltonian Cycle in G. Similarly, if G has a Hamiltonian Cycle, then the same ordering of nodes is a Hamiltonian Path of G' by splitting v to v' and v''.