The Danilov-Khovanskii algorithm for computing Hodge-Deligne polynomials of hypersurfaces in toric varieties

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1 Abstract

In their 1987 paper ([DK]), V. I. Danilov and A. G. Khovanskii described an algorithm for computing the Hodge-Deligne polynomials of generic hypersurfaces — and their complete intersections — in toric varieties. This thesis describes the background knowledge needed to implement this algorithm, a task completed by the author in the Macaulay2 language [1].

2 Introduction

The n-dimensional tori — denoted here as $T^n \cong (\mathbb{C}^*)^n$ — turn out to be useful spaces in which to study hypersurfaces and complete intersections. One motivation is their structure and simplicity: T^n is a commmutative algebraic group with respect to the operation of coordinate-wise multiplication, and the invariant sheaves on this group decompose with respect to the group characters. As a result, cohomology with coefficients in these sheaves can be calculated separately for each character, significantly simplifying computations. This is manifested in an indirect way in the appearance of lattice point enumeration and Ehrhart theory in §3 — two crucial equations used by the algorithm reduce the computation of some coefficients of the Hodge-Deligne polynomial to a calculation of the number of lattice points in a polytope in the character lattice of the torus.

A second motivation for studying tori is that, for algebraic varieties which decompose into a disjoint union of tori, many questions and computations can be reduced to those for the component tori. So in fact answering such questions for tori will answer them for these more complicated varieties. In the present context, this in manifested in the fact that the Hodge-Deligne polynomial behaves "additively" under such decompositions.

In fact, the algorithm can be succinctly, if very imprecisely, described in this context (see Fig. 1 for a visual aid). Given a "generic" hypersurface Z in T^n, one can associate to it a smooth, compact hypersurface $\bar{Z}$, which decomposes into a disjoint union of Z and other, lower-dimensional varieties. By induction on dimension, all the Hodge-Deligne polynomials of the lower-dimensional varieties are assumed to be known; by elementary calculations, the coefficients of "high degree" terms in the Hodge-Deligne polynomial (hereafter "large" Hodge-Deligne numbers) Z are also known; then, via the additivity property, the "large" Hodge-Deligne numbers of $\bar{Z}$ can be computed; Poincaré duality can then be applied to $\bar{Z}$ to get the "small" Hodge-Deligne numbers of $\bar{Z}$, which in turn gives the "small" Hodge-Deligne numbers of Z, again via the additivity property; there are then a handful of "middle" Hodge-Deligne numbers remaining — these are obtained as a linear combination of some of the "small" and "large" Hodge-Deligne numbers and the Euler-Poincaré characteristic of one of the sheaf cohomologies mentioned above.

Once one can compute the Hodge-Deligne polynomial of hypersurfaces in tori, the computation for hypersurfaces in toric varieties reduces to a repeated application of the additivity property. A similar phenomenon occurs for complete intersections of hypersurfaces in toric varieties. In this case, however, one must also take advantage of the behavior of the Hodge-Deligne polynomial with respect to products of varieties. This is discussed very briefly at the end of §4.2, and then again in §7.

[1] A copy of the code can be found at https://github.com/jletai/Danilov-Khovanskii-senior-thesis.

3 Basics: Ehrhart polynomials, series and reciprocity

One of the key steps in the algorithm (see §6.4.4 and Eq. 5) can be reduced to a computation of the number of lattice points contained in a certain polytope related to the hypersurface of interest (polytopes are bounded, convex polyhedra). To simplify the discussion, then, some basic facts and methods about lattice point enumeration are collected below. Most of the results will be stated without proof — the book Beck-Robins ([BR]) contains a nice elementary development of this material and, in particular, proves all of the unproved statements contained in this section (mostly in their Ch. 3).

Let $\Delta \subset \mathbb{R}^d$ be a d-dimensional lattice polytope (i.e. whose vertices lie in $\mathbb{Z}^d$). And define $\ell(\Delta) := \#(\Delta \cap \mathbb{Z}^d)$ to be the number of lattice points contained in Δ, and $\ell^*(\Delta) := \#(\mathring{\Delta} \cap \mathbb{Z}^d)$ to be the corresponding number for the interior of Δ.

Define the **Ehrhart series** of Δ to be

$$Ehr_\Delta(t) := \sum_{n=0}^{\infty} \ell(n\Delta)t^n,$$

and the related sum for interior points:

$$Ehr^*_\Delta(t) := \sum_{n=1}^{\infty} \ell^*(n\Delta)t^n.$$

The following are the relevant fundamental results in the theory of lattice point enumeration for polytopes.

Theorem 1 (Ehrhart, Stanley)**.**

$$Ehr_\Delta(t) := \frac{h_\Delta(t)}{(1-t)^{d+1}} = \frac{1 + h_1^*t + \ldots + h_d^*t^d}{(1-t)^{d+1}}$$

and h_i^* are all non-negative integers.

The degree d polynomial $h_\Delta(t) = 1 + h_1^*t + \ldots + h_d^*t^d$ will be referred to as the **Ehrhart numerator** of Δ.

Corollary 2. *There is a polynomial L_Δ of degree exactly d, such that, for $z \geq 0$,*

$$L_\Delta(z) = \#((z\Delta) \cap \mathbb{Z}^d) = \ell(z\Delta).$$

L_Δ will be referred to as the **Ehrhart polynomial** of Δ.

Theorem 3 (Ehrhart-Macdonald reciprocity)**.** *For a lattice polytope $\Delta \subset \mathbb{R}^d$ as above,*

$$Ehr^*_\Delta(t) = (-1)^{d+1} Ehr_\Delta(\frac{1}{t})$$

One can go back and forth from the Ehrhart numerator, Ehrhart polynomial, and the same for interior points. They all contain the same information.

Corollary 4. *Suppose that*

$$Ehr_\Delta(t) := \frac{h_\Delta(t)}{(1-t)^{d+1}} = \frac{1 + h_1^*t + \ldots + h_d^*t^d}{(1-t)^{d+1}},$$

then the numerator of the interior Ehrhart series is

$$h^*_\Delta(t) = t^{d+1} + h_1^*t^d + \ldots + h_d^*t.$$

Some of the formulae for the Hodge-Deligne numbers of hypersurfaces in toric varieties can be written in terms of the h_i^*. Hence, it will be worthwhile discussing how to compute these numbers for a given polytope. This is done in the lemma and the proof that follows.

Lemma 5. *For a lattice polytope $\Delta \subset \mathbb{R}^d$, the Ehrhart numerator, $h_\Delta(t)$ is determined by the number of lattice points in the first $d+1$ dilations of Δ, beginning with $0 \cdot \Delta$ and ending with $d \cdot \Delta$. Explicitly, one has*

$$1 + h_1^* t + \ldots + h_d^* t^d \equiv (1-t)^{d+1} \sum_{n=0}^{d} \ell(n\Delta) t^n \mod t^{d+1}. \tag{1}$$

Similarly, $h_\Delta^*(t)$, is determined by the number of interior lattice points in $d+1$ dilations of Δ, beginning with $1 \cdot \Delta$ and ending with $(d+1) \cdot \Delta$. Explicitly, one has

$$t^{d+1} + h_1^* t^d + \ldots + h_d^* t \equiv (1-t)^{d+1} \sum_{n=1}^{d+1} \ell^*(n\Delta) t^n \mod t^{d+2}. \tag{2}$$

Proof. To do this, rewrite the equality in Theorem 1 as

$$1 + h_1^* t + \ldots + h_d^* t^d = (1-t)^{d+1} \sum_{n=0}^{\infty} \ell(n\Delta) t^n = \sum_{n=0}^{\infty} a_n t^n,$$

where

$$a_n = \sum_{m=n-(d+1)}^{n} (-1)^{n-m} \binom{d+1}{n-m} \ell(m\Delta).$$

Theorem 1 guarantees that $a_n = 0$ for $n > d$ and that $a_n = h_n^* > 0$ if $n = 0, 1, \ldots, d$. Further, it is clear from the equations for $a_n = h_n^*$ that only the numbers $\ell(k\Delta)$ for $0 \leqslant k \leqslant d$ are required to calculate all the h_n^*. In other words, the number of lattice points in $k\Delta$ for $0 \leqslant k \leqslant d$ determine all of the h_n^*, and hence $h_\Delta(t)$. The congruence modulo t^{d+1} is simply a neat way of eliminating all the residual higher degree terms that would have been cancelled by terms lost in the truncation of the series (the cancellation is guaranteed by Theorem 1).

Likewise, one can write the results of Corollary 4 as

$$t^{d+1} + h_1^* t^d + \ldots + h_d^* t = (1-t)^{d+1} \sum_{n=1}^{\infty} \ell^*(n\Delta) t^n = \sum_{n=0}^{\infty} b_n t^n,$$

where

$$b_n = \sum_{m=n-(d+1)}^{n} (-1)^{n-m} \binom{d+1}{n-m} \ell^*(m\Delta).$$

But now note that $h_n^* = b_{d+1-n}$. Still, it is clear that only the numbers $\ell^*(k\Delta)$ for $1 \leqslant k \leqslant d+1$ are required to calculate all the h_n^*. That is, the number of lattice points interior to $k\Delta$ for $1 \leqslant k \leqslant d+1$ determine all of the h_n^*, and hence $h_\Delta^*(t)$. Again, the congruence modulo t^{d+2} is a neat way to disregard higher order terms that would cancel out with the inclusion of more terms in the series. □

4 Hodge-Deligne numbers

This section recalls the basic features of Hodge structures. Deligne's work on mixed Hodge structures — in the form of the Hodge-Deligne polynomial — will play a central role in the rest of this work.

4.1 Hodge Structures

Let H be a finite-dimensional vector space over the rationals $\mathbb{Q}$. A **pure Hodge structure of wieght r** on H is a decomposition of the complex vector space

$$H_{\mathbb{C}} = H \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=r} H^{p,q},$$

such that $H^{p,q} = \overline{H^{q,p}}$ (the bar denotes complex conjugation in $H_{\mathbb{C}}$). The dimension of the complex vector space $H^{p,q}$ is called the **Hodge number of type (p, q)** of the structure H:

$$h^{p,q}(H) = dim_{\mathbb{C}}(H^{p,q}).$$

A Hodge structure of weight r on H is equivalent to a **Hodge filtration** F on $H_{\mathbb{C}}$, where

$$F^p = \bigoplus_{s \geqslant p} H^{s,r-s}.$$

This is a descending filtration, and for each integer p, $H_{\mathbb{C}} = F^p \oplus \overline{F^{r-p-1}}$. To obtain a Hodge structure of weight r on $H_{\mathbb{C}}$ given such a filtration F, one need only compute $H^{p,q} = F^p \cap \overline{F^q}$.

One place to find pure Hodge structures is in the cohomologies of compact Kähler manifolds, whose r-dimensional cohomology $H^r(X, \mathbb{C})$ has a pure Hodge structure of weight r. These cohomologies have, in addition to this pure Hodge structure, a second decomposition called the Lefschetz decomposition. Multiplication by the Kähler form gives an operator, L, (the Lefschetz operator) of bi-degree $(1, 1)$ on the space of harmonic forms. Taking its adjoint, Λ, and their commutator, $[L, \Lambda]$, the action of these three operators induces an $\mathfrak{sl}_2(\mathbb{R})$ representation on the space of harmonic forms that descends to the cohomology. The cohomology of the compact Kähler manifold thus decomposes into irreducible representations of this $\mathfrak{sl}_2(\mathbb{R})$ representation. This can, for instance, be used to derive isomorphisms in the cohomologies of a compact Kähler manifold, X, of the form $H^{p,q}(X, \mathbb{C}) \xrightarrow{\sim} H^{n-q,n-p}(X, \mathbb{C})$, for $p + q > n$.

Working with objects with less structure than compact Kähler manifolds motivates a generalization of pure Hodge structures. Given H a vector space over $\mathbb{Q}$, a **mixed Hodge structure** on H consists of the following:

(i) an ascending weight filtration W on H;

(ii) a descending Hodge filtration F on $H_{\mathbb{C}}$;

where the filtration F induces a pure Hodge structure of weight r on the complexification of $Gr^W H = W_r/W_{r-1}$. In particular, for each r there is a decomposition

$$Gr^W H \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=r} H^{p,q}.$$

The dimension of $H^{p,q}$ over $\mathbb{C}$ is again called the **Hodge number of type (p, q)** of the Hodge structure H, and denoted $h^{p,q}(H)$. Mixed Hodge structures will be referred to simply as Hodge structures in the rest of this work.

Next, let H and H' be vector spaces over $\mathbb{Q}$ endowed with Hodge structures. A $\mathbb{Q}$-linear homomorphism $f : H \to H'$ is called compatible with Hodge structures, or a morphism of Hodge structures, if f is compatible with the filtrations F and W: $f(W_r) \subseteq W'_r$ and $f_{\mathbb{C}}(F^p) \subseteq F'^p$, where $f_{\mathbb{C}} : H_{\mathbb{C}} \to H'_{\mathbb{C}}$ is the natural map induced by f taking $h \otimes z \in H_{\mathbb{C}}$ to $f(h) \otimes z \in H'_{\mathbb{C}}$.

Since morphisms of Hodge structures are strictly compatible with the filtrations W and F, it follows that $H^{p,q}$ is exact. That is, if $H' \to H \to H''$ is an exact sequence of Hodge structures, then $H'^{p,q} \to H^{p,q} \to H''^{p,q}$ is exact for all p and q.

Deligne ([D]) showed that for any complex algebraic variety X the cohomology $H^k(X,\mathbb{C})$ carries a Hodge structure. In the case of smooth projective varieties this Hodge structure coincides with the aforementioned pure Hodge structure. These results are also true for cohomology with compact support, $H_c^k(X,\mathbb{C})$, which will be more appropriate to consider in the coming sections. (The $\mathbb{C}$ will sometimes be suppressed in $H_c^k(X,\mathbb{C})$, and $H_c^k(X)$ will be written.)

4.2 Hodge-Deligne theory

This subsection is an overview of some computational aspects of Deligne's mixed Hodge structures theory. For a full development of this material, see [PS]; for some more introductory material, see [H2].

Let X be any quasiprojective variety over $\mathbb{C}$, of dimension n, possibly not irreducible. There is a set of integers $e^{p,q}(X)$, for $0 \leq p \leq n$, and $0 \leq q \leq n$, which satisfy the following properties.

Package these numbers as a polynomial in 2 variables, the **Hodge-Deligne polynomial**:

$$e_X(x,y) := \sum_{p=0}^{n}\sum_{q=0}^{n} e^{p,q}(X)x^p y^q.$$

The coefficients of this polynomial, the $e^{p,q}(X)$'s, will be referred to as the Hodge-Deligne numbers of X. (Note, elsewhere the $h^{p,q}(X)$'s are referred to as the Hodge-Deligne numbers of X.) The $e^{p,q}(X)$'s contain the same information as the Hodge-Deligne polynomial.

Theorem 6 (Properties of Hodge-Deligne numbers)**.** *The following properties hold for all varieties X (closed, open, locally closed, etc.).*

- $e_X(x,y) = e_X(y,x)$, *i.e.* $e^{p,q}(X) = e^{q,p}(X)$, *for all* $0 \leq p,q \leq n$.
- *(Additivity) If X is the disjoint union of Y and Z, then $e_X = e_Y + e_Z$.*
- *(Products) $e_{X\times Y} = e_X e_Y$.*
- *If X is projective and smooth, then*
$$h^{p,q}(X) := h^q(X,\Omega_X^p) = (-1)^{p+q}e^{p,q}(X).$$
- *More generally, $e^{p,q}(X) = \sum_k (-1)^k h^{p,q}(H_c^k(X))$.*

The main goal of the algorithm discussed below is to compute the $e^{p,q}(Z)$'s for Z a hypersurface or complete intersection in a toric variety. A few demonstrative examples of what Hodge-Deligne polynomials look like is therefore in order.

Example 7.

- *If X is a single point, then $e_X(x,y) = 1$.*
- *If $X = \mathbb{P}^1$, then $e_X = 1 + xy$ (direct computation of Hodge numbers is one way to see this).*
- *If $X = \mathbb{C}^1$, then since X is $\mathbb{P}^1$ minus one point, we have $e_{\mathbb{C}^1}(x,y) = xy$, and therefore $e_{\mathbb{C}^n}(x,y) = (xy)^n$.*

- *If X is a smooth, projective, irreducible curve of genus g, then $e_X(x, y) = 1 - g \cdot (x + y) + xy$.*

- *Since $X = \mathbb{C}^*$ is $\mathbb{C}$ minus one point, then $e_{\mathbb{C}^*}(x, y) = xy - 1$, and therefore $e_{(\mathbb{C}^*)^n}(x, y) = (xy - 1)^n$.*

- *For $X = \mathbb{P}^n$, note that X is the disjoint union of $\mathbb{C}^n$ and $\mathbb{P}^{n-1}$. Therefore*

$$e_{\mathbb{P}^n}(x, y) = 1 + xy + (xy)^2 + \ldots + (xy)^n.$$

From the behavior of the Hodge-Deligne polynomials with respect to disjoint unions and products, it is possible deduce their behavior with respect to fiber products. The following corollary to the above theorem makes this explicit.

Corollary 8. *Let $\pi : X \to Y$ be a fiber bundle with fiber F that is locally trivial in the Zariski topology. Then $e_X = e_Y e_F$.*

Proof. Since the fiber bundle is locally trivial in the Zariski topology, there is a finite set of trivializing open sets on Y: $\{\phi_i : \pi^{-1}(U_i) \to U_i \times F\}_{i \in \mathscr{A}}$. Then by an inclusion-exclusion type calculation, one has (denoting $e_Y = e(Y)$, etc., for clarity):

$$e(Y) = \sum_{i_1, i_2, \ldots, i_k} (-1)^k e(U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_k}).$$

The open sets $\{\pi^{-1}(U_i)\}_{i \in \mathscr{A}}$ form a finite cover X, so one likewise has

$$e(X) = \sum_{i_1, i_2, \ldots, i_k} (-1)^k e(\pi^{-1}(U_{i_1}) \cap \pi^{-1}(U_{i_2}) \cap \cdots \cap \pi^{-1}(U_{i_k})).$$

Since

$$\pi^{-1}(U_{i_1}) \cap \pi^{-1}(U_{i_2}) \cap \cdots \cap \pi^{-1}(U_{i_k}) = \pi^{-1}(U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_k}) \cong (U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_k}) \times F,$$

one has

$$e(\pi^{-1}(U_{i_1}) \cap \pi^{-1}(U_{i_2}) \cap \cdots \cap \pi^{-1}(U_{i_k})) = e(U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_k}) e(F).$$

So

$$e(X) = \sum_{i_1, i_2, \ldots, i_k} (-1)^k e(U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_k}) e(F) = e(Y) e(F).$$

□

It will turn out that the computation of Hodge-Deligne polynomials of complete intersections of hypersurfaces can be recast in terms of a similar computation for a related fiber product and hypersurface. As a result, once one can calculate the Hodge-Deligne polynomials of hypersurfaces in toric varieties and of fiber products, one will also be able to calculate the Hodge-Deligne polynomials of complete intersections in toric varieties.

5 Tori and Toric varieties

Although the goal of the following section will be to calculate the Hodge-Deligne numbers of hypersurfaces in tori, extensive use will be made of toric varieties. In particular, smooth compactifications of the hypersurfaces of interest and of the ambient tori will be considered.

This section therefore collects the relevant details of both tori and toric varieties, as well as their constructions.

In this work, an n-dimensional torus will be an algebraic variety T^n isomorphic to $(\mathbb{C}^*)^n$, with T^n inheriting a group structure from the isomorphism. If x_i is the i-th coordinate on $(\mathbb{C}^*)^n$, then for each monomial $x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}$ one can associate $(m_1, m_2, \ldots, m_n) = m \in \mathbb{Z}^n$ — such a monomial will henceforth be abbreviated x^m. Each monomial is a regular function; and each regular function on $(\mathbb{C}^*)^n$ can be uniquely written in the form of a finite linear combination of the monomials x^m for $m \in \mathbb{Z}^n$. That is, the ring of regular functions on $(\mathbb{C}^*)^n$, $\mathbb{C}[(\mathbb{C}^*)^n]$, is isomorphic to the group algebra $\mathbb{C}[\mathbb{Z}^n]$, and $(\mathbb{C}^*)^n = \operatorname{Spec}\mathbb{C}[\mathbb{Z}^n]$ is the spectrum of this algebra.

In addition to being regular functions, the monomials above are also group homomorphisms $(\mathbb{C}^*)^n \to \mathbb{C}^*$. It can be shown that all characters of $(\mathbb{C}^*)^n$ are of this form ([H1]). So the characters of $(\mathbb{C}^*)^n$ from a free abelian group of rank n isomorphic to $\mathbb{Z}^n$. For an arbitrary torus, T^n, the group-structure-preserving isomorphism allows one to transfer these notions in a straightforward manner: characters of T^n are morphisms $\chi : T \to \mathbb{C}^*$ that are also group homomorphisms; and they form a free abelian group of rank n. Since there is no natural basis for the group of characters of T^n, it is standard to use the symbol M instead of $\mathbb{Z}^n$. M is referred to as the **character lattice** of T^n. One has $T^n = \operatorname{Spec}\mathbb{C}[M]$.

Each element $f \in \mathbb{C}[M]$, viewed as a map $f : T^n \to \mathbb{C}$, is referred to as a **regular function** or as a **Laurent polynomial**. If $f = \sum_{m\in M} a_m x^m$ (only finitely many a_m are non-zero), then the *support* of f is defined to be the set $Supp(f) = \{m \in M \mid a_m \neq 0\}$.

The convex hull of a set, S, of elements in a real vector space, V, is the set of all convex combinations of points in S:

$$\operatorname{conv}(S) = \{\sum_{p\in S} r_p \cdot p \mid r_p \in \mathbb{R}, \sum_{p\in S} r_p = 1\}.$$

The convex hull of $Supp(f)$ in the real vector space $M_{\mathbb{R}} = M \otimes \mathbb{R}$ is called the **Newton polytope** of f and denoted $\Delta(f)$ — $\Delta(f) = \operatorname{conv}(Supp(f))$.

There is a second set of objects that is in a sense "dual" to the characters and character lattice of a torus. A one-parameter subgroup of T^n is a morphism $\lambda^u : \mathbb{C}^* \to T^n$ that is a group homomorphism. The set of one-parameter subgroups is also a free abelian group (i.e. a lattice) of rank n, and is denoted by N. There is a natural bilinear pairing $\langle \bullet, \bullet \rangle : M \times N \to \mathbb{Z}$ defined as follows: the composition $\chi^m \circ \lambda^n : \mathbb{C}^* \to \mathbb{C}^*$ is a character of $\mathbb{C}^*$ and so is of the form, $t \mapsto t^\ell$ — one then defines $\langle m, u \rangle = \ell$.

The close relation between M and N will make it possible to translate statements relating to M into statements relating to N, and vice versa. Hence, while most of this work will focus on polytopes in the M lattice, one could also consider working with the corresponding "normal fans" of these polytopes in the N lattice.

A cone in a vector space, V, generated by a set, S, of elements in V, is the set

$$\operatorname{Cone}(S) = \{\sum_{v\in S} r_v \cdot v \mid 0 \leqslant r_v \in \mathbb{R}\}.$$

Now, let Δ be an n-dimensional polytope in $M_{\mathbb{R}}$ whose vertices lie in M (i.e. Δ is a lattice polytope). For a face, Γ, in Δ, inclusion will be denoted by $\Gamma \leqslant \Delta$. To each face Γ of Δ one can associate a cone: $\operatorname{Cone}(\Delta, \Gamma) = \{r \cdot (p - q) \mid p \in \Delta, q \in \mathring{\Gamma}, 0 \leqslant r \in \mathbb{R}\}$, which is independent of the choice of q in the interior of Γ. One can further consider the semi-group algebras $\mathbb{C}[\operatorname{Cone}(\Delta, \Gamma) \cap M]$. The toric variety obtained by gluing together the charts $X_{\operatorname{Cone}(\Delta,\Gamma)} = \operatorname{Spec}\mathbb{C}[\operatorname{Cone}(\Delta, \Gamma) \cap M]$ will be denoted $\mathbb{P}_\Delta$. (The use of the symbol $\mathbb{P}_\Delta$ is used in order to emphasize the analogy between these toric varieties and projective space. The details of this construction can be found, for example, in the first two chapters of [CLS].) Since it suffices to work with the charts corresponding to the vertices, $P \leqslant \Delta$, those charts will be abbreviated U_P. Now, $\mathbb{P}_\Delta$ is a compactification of the torus $T_\Delta = X_{\mathrm{Cone}(\Delta,\Delta)}$, whose dimension is equal to n: $n = dim(\mathbb{P}_\Delta) = dim(\Delta) = dim(T_\Delta)$. And for each face Γ one has a closed subvariety in $\mathbb{P}_\Delta$ isomorphic to $\mathbb{P}_\Gamma$, which will also be denoted by the $\mathbb{P}_\Gamma$. Likewise, the image in $\mathbb{P}_\Delta$ of $T_\Gamma \subset \mathbb{P}_\Gamma$ under the aforementioned isomorphism will also be denoted T_Γ. If Γ and Γ' are two faces of Δ, then $\mathbb{P}_\Gamma \cap \mathbb{P}'_\Gamma = \mathbb{P}_{\Gamma\cap\Gamma'}$.

The action of the torus T_Δ, which is dense in $\mathbb{P}_\Delta$, extends to an action on the whole of $\mathbb{P}_\Delta$. Moreover, the T_Γ, for $\Gamma \leqslant \Delta$, are exactly the orbits of this action. They are thus disjoint, and so $\mathbb{P}_\Delta$ decomposes into a disjoint union of tori. This decomposition is termed a *stratification* of $\mathbb{P}_\Delta$, and the T_Γ referred to as the *strata* of the toric variety $\mathbb{P}_\Delta$. This stratification will play an important role in the computation of the Hodge-Deligne polynomial for hypersurfaces in toric varieties (see §7).

Another feature of $\mathbb{P}_\Delta$ that is determined by the structure of Δ is its smoothness or lack thereof.

(Note: the terminology and line of reasoning used in the next few paragraphs follows that used in [DK]. In contemporary literature, "prime polytope" would translate as "simple polytope". But more importantly, it would correspond to a normal fan that is *simplicial* — meaning that all cones are generated by rays that are linearly independent over $\mathbb{R}$. Likewise, "prime with respect to M" corresponds to a *smooth* normal fan — a fan in which all cones are generated by rays that form a subset of a basis [over $\mathbb{Z}$] of N.)

If P is a vertex of Δ, then Δ is **prime at** P (**prime with respect to** M **at** P) if $\mathrm{Cone}(\Delta, P)$ is generated by a basis of $M_\mathbb{R}$ (by a basis of the lattice M). Δ is **prime** (**prime with respect to** M) if it is prime (prime with respect to M) at all of its vertices. The following result on $\mathbb{P}_\Delta$ for Δ prime will be useful when searching for a smooth compactification of T^n.

Proposition 9. *If Δ is prime with respect to M, then $\mathbb{P}_\Delta$ is a smooth variety.*

Proof. If Δ is prime at P with respect to M, then $\mathrm{Cone}(\Delta, P)$ is generated by a basis of M for all vertices P. One then has a natural isomorphism $M \xrightarrow{\sim} \mathbb{Z}^n$ sending the basis of $\mathrm{Cone}(\Delta, P)$ to the standard basis $\{x_1, x_2, \ldots, x_n\}$ of $\mathbb{Z}^n$. This gives that $\mathrm{Cone}(\Delta, P) \cap M \cong \mathrm{Cone}(\{x_1, x_2, \ldots, x_n\}) \cap \mathbb{Z}^n$, which in turn implies that $\mathbb{C}[\mathrm{Cone}(\Delta, P) \cap M] \cong \mathbb{C}[x_1, x_2, \ldots, x_n]$. Hence $U_p = \mathrm{Spec}\,\mathbb{C}[\mathrm{Cone}(\Delta, P) \cap M]$ is isomorphic to affine n-space $\mathbb{C}^n = \mathrm{Spec}\,\mathbb{C}[x_1, x_2, \ldots, x_n]$, which is smooth. Since smoothness is a local notion, and the charts U_P cover $\mathbb{P}_\Delta$, $\mathbb{P}_\Delta$ is a smooth variety. □

(As alluded to above, the corresponding requirement for smoothness of $\mathbb{P}_\Delta$ in terms of the normal fan of Δ is that each cone in the fan is such that its minimal generators form a subset of a basis of N.)

In some cases, the Newton polytope of a Laurent polynomial, $\Delta(f)$ will not be prime with respect to M. This motivates searching for some other, related polytope, Δ', that is prime with respect to M, such that their corresponding toric varieties, $\mathbb{P}_\Delta$ and $\mathbb{P}'_\Delta$, can also be related in a simple way. So, suppose Δ and Δ' are two polytopes in $M_\mathbb{R}$. Then Δ' **majorizes** Δ if there exists a map $\alpha : som(\Delta') \to som(\Delta)$ between the sets of vertices of Δ' and Δ such that $\mathrm{Cone}(\Delta, \alpha(P')) \subseteq \mathrm{Cone}(\Delta', P')$ for all $P' \in som(\Delta')$. If such a map exists, then it is unique. Further, one may extend α to the set of all faces by defining $\alpha(\Gamma')$, $\Gamma' \leqslant \Delta'$, to be the face with vertex set $\{\alpha(P') \,|\, P' \in som(\Gamma')\}$. When Δ' majorizes Δ, there is a natural homomorphism of semi-group algebras $\mathbb{C}[\mathrm{Cone}(\Delta, \alpha(P') \cap M] \to \mathbb{C}[\mathrm{Cone}(\Delta', P') \cap M]$. The corresponding morphism of algebraic varieties, $\rho_{\Delta',\Delta} : \mathbb{P}_{\Delta'} \to \mathbb{P}_\Delta$, then maps the charts $U_{P'}$

to $U_{\alpha(P')}$. This morphism can be viewed as partially resolving the singularities of $\mathbb{P}_\Delta$. For each Δ, there exists a prime Δ' which majorizes Δ.

In terms of the normal fan of Δ, the prime polytope Δ' corresponds to a subdivision, or refinement, of this fan such that all cones in the new fan are simplicial. A toric variety corresponding to a simplicial fan, such as $\mathbb{P}_{\Delta'}$, is called a *simplicial* toric variety. The condition $\mathrm{Cone}(\Delta, \alpha(P')) \subseteq \mathrm{Cone}(\Delta', P')$ implies that there is a map on the normal fans which gives an inclusion of the cones in the normal fan of Δ' into the cones of the normal fan of Δ (i.e. in the opposite direction). In contemporary language, this is equivalent to the condition that the induced map $\rho_{\Delta',\Delta}$ is a *toric morphism* (see [CLS] §3.3). This in particular means that $\rho_{\Delta',\Delta}$ restricts to a group homomorphism from $T_{\Delta'}$ to T_Δ. And the statement that there exists a prime Δ' majorizing Δ is equivalent to the statement that there is a toric morphism from a simplicial toric variety to $\mathbb{P}_\Delta$.

6 Hodge-Deligne numbers of hypersurfaces in tori

The description of the algorithm will begin in earnest in §6.4. In order to make that part of the work as clear and concise as possible, the bulk of the technical details has been placed beforehand. The reader is invited to skip ahead to §6.4 and skim through the algorithm now so that the motivation behind the intervening material is evident. Likewise, the reader is invited to refer to Figure 1 for an overview of the flow of information in the algorithm.

Before continuing to §6.2, it will ease notation to introduce the following variables:

$$e^p(Z) = \sum_q e^{p,q}(Z).$$

6.1 Outline

This subsection collects the results of sections §6.2 and §6.3 in the form that they will be used in the algorithm. Here $\bar{Z}$ is a compactification of $Z = Z_f$ in $\mathbb{P}_{\Delta'}$ for some Δ' majorizing $\Delta(f)$, with $n = dim(Z) = dim(\Delta(f)) = dim(\Delta')$.

The initial step of the algorithm will be a calculation of $e^{p,q}(Z)$ for $p + q > n - 1$. This is done by relating them to those for the ambient torus (see §6.3). For $p + q > n - 1$, one has:

$$e^{p,q}(Z) = e^{p+1,q+1}(T^n) = \begin{cases} (-1)^{n+p+1}\binom{n}{p+1}, & p = q \\ 0, & p \neq q \end{cases}. \tag{3}$$

The next useful equation is a simple restatement of the additivity property of the Hodge-Deligne polynomial (see §4.2). It can be used once all of the numbers $e^{p,q}(\Gamma')$, for $\Gamma' < \Delta'$ have been computed. It will be employed in several different steps of the algorithm, to calculate $e^{p,q}(\bar{Z})$ from $e^{p,q}(Z)$ and vice versa.

$$e^{p,q}(\bar{Z}) = e^{p,q}(Z) + \sum_{\Gamma' < \Delta'} e^{p,q}(\Gamma'). \tag{4}$$

And, lastly, once one has obtained the $e^{p,q}(Z)$ for all $q \neq n - 1 - p$ (i.e. for $p + q > n - 1$ and $p + q < n - 1$), the following equation for $e^p(Z)$ can be used to determine the $e^{p,n-1-p}(Z)$ (see §6.2).

$$e^p(Z) = \sum_q e^{p,q}(Z) = (-1)^{n+1}((-1)^p\binom{n}{p+1} + h^*_{p+1}(\Delta)). \tag{5}$$

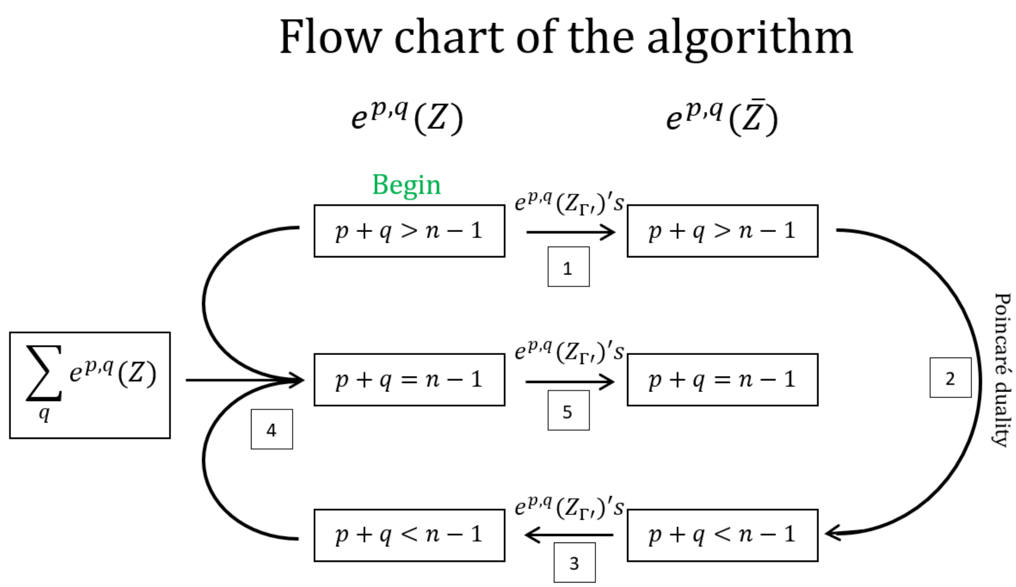

Figure 1: The algorithm begins by using Eq. 3 to obtain $e^{p,q}(Z)$ for $p+q>n-1$. The successive steps are as follows: (1) Compute the $e^{p,q}(Z_{\Gamma'})$'s, for $\Gamma'<\Delta'$, and use them with Eq. 4 to obtain the $e^{p,q}(\bar{Z})$ for $p+q>n-1$. (2) Use Poincaré duality to determine the $e^{p,q}(\bar{Z})$ for $p+q<n-1$ in terms of the $e^{p,q}(Z)$ for $p+q>n-1$. (3) Use the $e^{p,q}(Z_{\Gamma'})$'s with Eq. 4 again to obtain the $e^{p,q}(Z)$ for $p+q<n-1$. (4) For each, p, Eq. 11 together with the $e^{p,q}(Z)$'s already computed determines the $e^{p,q}(Z)$ for $p+q=n-1$. (5) A last application of Eq. 4 then gives $e^{p,q}(\bar{Z})$ for $p+q=n-1$.

6.2 $e^p(Z)$ and the Euler-Poincaré characteristic

Here begins the derivation of the last equation stated in §6.1. This will follow from a sequence of relations between $e^p(Z)$, Euler-Poincaré characteristics of sheaves of germs of differential forms, and the polytope $\Delta = \Delta(f)$.

6.2.1 Relating $e^p(Z)$ to $\chi(\bar{Z}, \Omega^p_{(\bar{Z}, D_Z)})$

The following result, although stated for Z a hypersurface in a torus, holds more generally for any smooth algebraic variety. For the rest of this subsection (§6.2), unless otherwise specified, Δ will be arbitrary and fixed, and Δ' some polytope majorizing Δ. $\bar{Z} = \bar{Z}_{\Delta'} \in \mathbb{P}_{\Delta'}$ will be a smooth compactification of Z such that $\bar{Z} \backslash Z = D_Z$ has transversal intersections in $\bar{Z}$ (i.e. $T_p D_Z = T_p \bar{Z}$ for all points p of self-intersection of D_Z). Similarly, define $D_{\Delta'} = \mathbb{P}_{\Delta'} \backslash T_{\Delta'}$.

Denote the sheaf of germs of regular differential p-forms on $\bar{Z}$ which vanish on D_Z by $\Omega^p_{(\bar{Z}, D_Z)}$: if D_Z decomposes into irreducible components as $D_Z = \bigcup_{i=1}^N D_{Z,i}$, then $\Omega^p_{(\bar{Z}, D_Z)}$ is the kernel of the restriction $\Omega^p_{\bar{Z}} \to \oplus_{i=1}^N \Omega^p_{D_{Z,i}}$. $\Omega^p_{(\mathbb{P}_\Delta, D_\Delta)}$ and $\Omega^p_{(\mathbb{P}_{\Delta'}, D_{\Delta'})}$ are defined similarly. The following proposition and corollary then determine $e^p(Z) = \sum_q e^{p,q}(Z)$ in terms of the Euler-Poincaré characteristic of $\Omega^p_{(\bar{Z}, D_Z)}$. As deriving this relation depends on results too far afield from the task at hand, the proposition and corollary will be stated without proof.

Proposition 10. *There exists a spectral sequence*

$$E_1^{p,q} = H^q(\bar{Z}, \Omega^p_{(\bar{Z}, D_Z)}) \Rightarrow H_c^{p+q}(Z, \mathbb{C}), \tag{6}$$

degenerating at the term E_1 and converging to the Hodge filtration on $H_c^*(Z)$

Corollary 11. *The $e^p(Z)$'s may be expressed as follows.*

$$e^p(Z) = (-1)^p \chi(\bar{Z}, \Omega^p_{(\bar{Z}, D_Z)}) \tag{7}$$

Roughly, the corollary follows from the definition of the $e^{p,q}(Z)$ in terms of the $h^{p,q}(Z)$ and taking the dimension of the objects on either side of the relation given by the proposition.

6.2.2 From $\chi(\mathbb{P}_{\Delta'}, \Omega^p_{(\mathbb{P}_{\Delta'}, D_{\Delta'})}(\Delta))$ to $\ell^*(\Delta)$

Any Laurent polynomial $f \in \mathbb{C}[M]$ defines a hypersurface in $T^n = \operatorname{Spec} \mathbb{C}[M]$ — namely, the zero locus of f. Denote this by $Z = Z_f$. Each hypersurface in T^n is of this form. Given Δ, the space of Laurent polynomials whose support lie in Δ will be denoted by $L(\Delta)$; those whose support lie strictly in the interior of Δ will be denoted by $L^*(\Delta)$. One can also associate to Δ and $\mathbb{P}_\Delta$ the ample invertible sheaf $\mathscr{O}_{\mathbb{P}_\Delta}(\Delta)$. $\mathscr{O}_{\mathbb{P}_\Delta}(\Delta)$ is defined as the subsheaf of the sheaf of rational functions on $\mathbb{P}_\Delta$ whose sections over every U_P have the form $x^P f(x)$, where $f(x) \in \mathbb{C}[\operatorname{Cone}(\Delta, P) \cap M]$ is a regular function on U_P and x^P is defined analogously to x^m in §5.

In the case where one wants to consider Δ' majorizing Δ, one instead uses the pullback of $\mathscr{O}_{\mathbb{P}_\Delta}(\Delta)$: $\mathscr{O}_{\mathbb{P}_{\Delta'}}(\Delta) = \rho^*_{\Delta', \Delta}(\mathscr{O}_{\mathbb{P}_\Delta}(\Delta))$. Over the affine chart $U_{P'}$, sections of this sheaf have the form $x^{\alpha(P')} f(x)$, where now $f(x) \in \mathbb{C}[\operatorname{Cone}(\Delta', P') \cap M]$.

The following propositions will use these sheaves to express $\chi(\mathbb{P}_{\Delta'}, \Omega^p_{(\mathbb{P}_{\Delta'}, D_{\Delta'})}(\Delta))$ in terms of $\ell^*(\Delta)$. The reader is referred to [K] for a discussion of the first proposition and to [DK] for the second.

Proposition 12. $H^n(\mathbb{P}_{\Delta'}; \mathscr{O}_{\mathbb{P}_{\Delta'}}(\Delta)) = \begin{cases} L(\Delta), & n = 0 \\ 0, & n > 0 \end{cases}$

Passing from the preceding proposition to the succeeding one makes use of the sheaf $\mathscr{O}_{\mathbb{P}_{\Delta'}}(-D_{\Delta'})$ of germs of functions vanishing on $D_{\Delta'}$ and the isomorphism

$$\Lambda^p(M) \otimes_{\mathbb{Z}} \mathscr{O}_{\mathbb{P}_{\Delta'}}(-D_{\Delta'}) \xrightarrow{\sim} \Omega^p_{(\mathbb{P}_{\Delta'},D_{\Delta'})}$$

given by $m_1 \wedge m_2 \wedge \cdots \wedge m_p \otimes_{\mathbb{Z}} f \mapsto f\frac{dx^{m_1}}{x^{m_1}} \wedge \frac{dx^{m_2}}{x^{m_2}} \wedge \cdots \wedge \frac{dx^{m_p}}{x^{m_p}} = f\frac{dx^m}{x^m}$.

Proposition 13. *Let Δ be an n-dimensional polytope in $M_{\mathbb{R}}$ and let Δ' be a polytope majorizing Δ and prime with respect to M. Then*

$$H^n(\mathbb{P}_{\Delta'}, \Omega^p_{(\mathbb{P}_{\Delta'},D_{\Delta'})}) = \begin{cases} \Lambda^p(M) \otimes L^*(\Delta), & n = 0 \\ 0, & n > 0 \end{cases}.$$

Writing $dim(L^*(\Delta)) = \ell^*(\Delta)$, one then has the following corollary.

Corollary 14. *Let $\Delta(f)$ be an n-dimensional polytope in $M_{\mathbb{R}}$ and let Δ' be a polytope majorizing $\Delta(f)$ and prime with respect to M. Then*

$$\chi(\mathbb{P}_{\Delta'}, \Omega^p_{(\mathbb{P}_{\Delta'},D_{\Delta'})}(\Delta)) = \sum_k (-1)^k dim(H^n(\mathbb{P}_{\Delta'}, \Omega^p_{(\mathbb{P}_{\Delta'},D_{\Delta'})}(\Delta)) = \binom{n}{p}\ell^*(\Delta). \qquad (8)$$

Proof. (Of Prop. 13 $\Longrightarrow$ Cor. 14)

$H^0(\mathbb{P}_{\Delta'}, \Omega^p_{(\mathbb{P}_{\Delta'},D_{\Delta'})}(\Delta))$ is the only non-trivial term, and

$$dim(\Lambda^p(M) \otimes L^*(\Delta)) = dim(\Lambda^p(M))dim(L^*(\Delta)) = \binom{n}{p}\ell^*(\Delta).$$

□

6.2.3 Relating $\chi(\bar{Z}, \Omega^p_{(\bar{Z},D)}(\Delta))$ to $\chi(\mathbb{P}_{\Delta'}, \Omega^p_{(\mathbb{P}_{\Delta'},D_{\Delta'})}(\Delta))$

If $f \in L(\Delta)$, then f can be considered as a global section of $\mathscr{O}_{\mathbb{P}_\Delta}(\Delta)$; the vanishing locus of f is again a hypersurface in $\mathbb{P}_\Delta$, denoted $\bar{Z}_{\Delta(f)}$. If $\Delta'(f) = \Delta'$ majorizes $\Delta(f)$, one can pull f back to $\mathbb{P}_{\Delta'}$ via $\rho_{\Delta',\Delta(f)}$ and hence obtain a hypersurface in $\mathbb{P}_{\Delta'}$, denoted $\bar{Z}_{\Delta'(f)}$. Both $\bar{Z}_{\Delta(f)}$ and $\bar{Z}_{\Delta'(f)}$ are compactifications of Z. And, assuming $dim(\Delta) = n = dim(\Delta')$, one has $Z_{\Delta(f)} = \bar{Z}_{\Delta(f)} \cap T_{\Delta(f)} \cong Z$ and $Z_{\Delta'(f)} = \bar{Z}_{\Delta'(f)} \cap T_{\Delta'(f)} \cong Z$.

A Laurent polynomial $f \in L(\Delta)$ is **non-degenerate with respect to** Δ if the hypersurface $\bar{Z}_{\Delta(f)}$ transversally intersects all strata of $\mathbb{P}_\Delta$. In particular, $\bar{Z}_{\Delta(f)}$ must not intersect $\mathbb{P}_P$ for any vertex $P \leqslant \Delta$. This requires that the coefficient of each monomial corresponding to a vertex of Δ be non-zero. Hence, $\Delta(f) \supseteq L(\Delta)$; and since $\Delta(f) \subseteq L(\Delta)$ by assumption, on in fact has $\Delta(f) = L(\Delta)$. Importantly, given a Δ, there always exist such an $f \in L(\Delta)$ — in fact, any generic choice of coefficients corresponding to the vertices of Δ will give a non-degenerate f (see the proof of Thm. 2 in [K] and the remarks thereafter). If f is non-degenerate with respect to Δ and $\Delta'(f)$ majorizes Δ, then $Z_{\Delta'(f)}$ transversally intersects all strata of $\mathbb{P}_{\Delta'(f)}$ as well. If $\Delta'(f)$ is prime with respect to M, then $\bar{Z}_{\Delta'(f)}$ is a smooth variety.

From now on, f will be a Laurent polynomial non-degenerate with respect to $\Delta = \Delta(f)$, with $\Delta' = \Delta'(f)$ a polytope prime with respect to M that majorizes Δ (if Δ is already prime with respect to M, then one may choose $\Delta' = \Delta$). For clarity, $\bar{Z}_{\Delta'(f)}$ $\mathbb{P}_{\Delta'(f)}$ and $D_{\Delta'(f)}$ will now be henceforth as $\bar{Z}$, $\mathbb{P}$ and D, respectively.

Two relations have been established thus far in §6.2: one between $e^p(Z)$ and the Euler-Poincaré characteristic $\chi(\bar{Z}, \Omega^p_{(\bar{Z},D)}(\Delta))$; and another between $\chi(\mathbb{P}, \Omega^p_{(\mathbb{P},D)}(\Delta))$ and the

$l^*(k\Delta)$. Deriving a relation between $\chi(\bar{Z}, \Omega^p_{(\bar{Z},D)}(\Delta))$ and $\chi(\mathbb{P}, \Omega^p_{(\mathbb{P},D)}(\Delta))$ will therefore establish a relation between $e^p(Z)$ and the $\ell^*(k\Delta)$.

To relate $\chi(\bar{Z}, \Omega^p_{(\bar{Z},D)}(\Delta))$ and $\chi(\mathbb{P}, \Omega^p_{(\mathbb{P},D)}(\Delta))$, one may begin with the following short exact sequence of coherent sheaves:

$$0 \to \Omega^p_{(\bar{Z},D_Z)} \otimes_{\mathscr{O}_{\mathbb{P}}} \mathscr{O}_{\mathbb{P}}(-\bar{Z}) \to \Omega^{p+1}_{(\mathbb{P},D)} \otimes_{\mathscr{O}_{\mathbb{P}}} \mathscr{O}_{\bar{Z}} \to \Omega^{p+1}_{(\bar{Z},D_{\bar{Z}})} \to 0.$$

Since $\bar{Z}$ is defined by a section of the invertible sheaf $\mathscr{O}_{\mathbb{P}}(\Delta)$, it follows that $\mathscr{O}_{\mathbb{P}}(\bar{Z}) \cong \mathscr{O}_{\mathbb{P}}(\Delta)$. One may then take the tensor product of the short exact sequence with the invertible sheaves $\mathscr{O}_{\mathbb{P}}((k+1)\Delta)$, for $k \geqslant 0$, and compute the Euler-Poincaré characteristics:

$$\chi(\bar{Z}, \Omega^p_{(\bar{Z},D_{\bar{Z}})}) = \sum_{k \geqslant 0} (-1)^k \chi(\mathbb{P}, \Omega^{p+k+1}_{(\mathbb{P},D)}((k+1)\Delta) \otimes_{\mathscr{O}_{\mathbb{P}}} \mathscr{O}_{\bar{Z}}).$$

Observing that $\mathscr{O}_{\mathbb{P}}(-\Delta)$ is the sheaf of germs of functions vanishing on $\bar{Z}$, one has a second exact sequence,

$$0 \to \mathscr{O}_{\mathbb{P}}(-\Delta) \to \mathscr{O}_{\mathbb{P}} \to \mathscr{O}_{\bar{Z}}.$$

Combining this with the sheaf isomorphism $\Omega^{p+k+1}_{(\mathbb{P},D)}((k+1)\Delta) \otimes_{\mathscr{O}_{\mathbb{P}}} \mathscr{O}_{\mathbb{P}}(-\Delta) \xrightarrow{\sim} \Omega^{p+k+1}_{(\mathbb{P},D)}(k\Delta)$ induced by $f\frac{dx^m}{x^m} \otimes g \mapsto fg\frac{dx^m}{x^m}$, it follows that

$$\begin{aligned}\chi(\mathbb{P}, \Omega^{p+k+1}_{(\mathbb{P},D)}((k+1)\Delta) \otimes_{\mathscr{O}_{\mathbb{P}}} \mathscr{O}_{\bar{Z}}) &= \chi(\mathbb{P}, \Omega^{p+k+1}_{(\mathbb{P},D)}((k+1)\Delta)) - \chi(\mathbb{P}, \Omega^{p+k+1}_{(\mathbb{P},D)}((k+1)\Delta) \otimes_{\mathscr{O}_{\mathbb{P}}} \mathscr{O}_{\mathbb{P}}(-\Delta)) \\ &= \chi(\mathbb{P}, \Omega^{p+k+1}_{(\mathbb{P},D)}((k+1)\Delta)) - \chi(\mathbb{P}, \Omega^{p+k+1}_{(\mathbb{P},D)}(k\Delta)).\end{aligned}$$

So one may rewrite the previous equation purely in terms of the $\chi(\mathbb{P}, \Omega^p_{(\mathbb{P},D)}(k\Delta))$ as follows:

$$\chi(\bar{Z}, \Omega^p_{(\bar{Z},D_{\bar{Z}})}) = \sum_{k \geqslant 0} (-1)^k [\chi(\mathbb{P}, \Omega^{p+k+1}_{(\mathbb{P},D)}((k+1)\Delta)) - \chi(\mathbb{P}, \Omega^{p+k+1}_{(\mathbb{P},D)}(k\Delta))].$$

Using Eq. 8, one thus has $\chi(\mathbb{P}, \Omega^p_{(\mathbb{P},D)}(\Delta))$ in terms of the $\ell^*(k\Delta)$:

$$\chi(\bar{Z}, \Omega^p_{(\bar{X},D)}) = (-1)^{n+1}\binom{n}{p+1} - \sum_{k \geqslant 1} (-1)^k \binom{n+1}{p+k+1} \ell^*(k\Delta). \tag{9}$$

6.2.4 Determining $e^p(Z)$ from $h^*_{p+1}(\Delta)$

Rewriting

$$\sum_{k \geqslant 1} (-1)^k \binom{n+1}{p+k+1} \ell^*(k\Delta) = (-1)^{n+1-(p+1)} \sum_{k \geqslant 1} (-1)^{[n+1-(p+1)]-k} \binom{n+1}{[n+1-(p+1)]-k} \ell^*(k\Delta)$$

and comparing to the equations for the b_n given at the end of Section 3 (though the n there is different), one sees that it is equal to $b_{n+1-(p+1)} = h^*_{p+1}$. Combining this with Eq. 8, one may then express $e^p(Z)$ purely in terms of combinatorial information about Δ:

$$e^p(Z) = \sum_q e^{p,q}(Z) = (-1)^{n+1}((-1)^p \binom{n}{p+1} + h^*_{p+1}(\Delta)).$$

6.3 Setting up the initial step

The theorem stated below will be the key to the initial step of the algorithm. First, recall that a morphism $f : X \to Y$ of varieties of dimension n and m, respectively, induces a homomorphism on homologies: $f_* : H_*(X) \to H_*(Y)$. If X and Y are assumed to be smooth and compact, one also has the Poincaré duality isomorphisms: $D_X : H^*(X) \to H_*(X)$ and $D_Y : H^*(Y) \to H_*(Y)$. One can then define the *Gysin homomorphism* on cohomologies as follows: $f^! = D_Y^{-1} \circ f_* \circ D_X : H^i(X) \to H^{i+m-n}$. In the case of $\bar{Z} \subset \mathbb{P}$, and considering cohomology with compact support, one obtains a map $f^! : H_c^i(\bar{Z}, \mathbb{C}) \to H_c^{i+2}(\mathbb{P}, \mathbb{C})$, since $\bar{Z}$ has real codimension 2 in $\mathbb{P}$. This will induce a homomorphism $H_c^i(Z, \mathbb{C}) \to H_c^{i+2}(T^n, \mathbb{C})$, also called a Gysin homomorphism.

The first step of the algorithm uses the following fact about this induced homomorphism.

Theorem 15. *The Gysin homomorphism $H_c^i(Z, \mathbb{C}) \to H_c^{i+2}(T^n, \mathbb{C})$ is an isomorphism for $i > n-1$. This isomorphism is a morphism of Hodge structures.*

To sketch the proof: One first shows that the Gysin homomorphism $f^! : H_c^i(\bar{Z}, \mathbb{C}) \to H_c^{i+2}(\mathbb{P}, \mathbb{C})$ is an isomorphism for $i > n-1$. This follows because one can fit $f^!$ into the exact sequence

$$H_c^i(\mathbb{P}\backslash\bar{Z}; \mathbb{C}) \to H_c^i(Z; \mathbb{C}) \to H_c^{i+2}(\mathbb{P}; \mathbb{C}) \to H_c^i(\mathbb{P}\backslash\bar{Z}; \mathbb{C}),$$

where the $H_c^i(\mathbb{P}\backslash\bar{Z})$ vanish for $i > n$ since $\mathbb{P}\backslash\bar{Z}$ is affine toroidal. One can then prove that for any open toric subvariety $U \subseteq \mathbb{P}$, the corresponding homomorphism $H_c^i(\bar{Z} \cap U; \mathbb{C}) \to H_c^{i+2}(U; \mathbb{C})$ is also an isomorphism for $i > n-1$. Using $U = T^n$ and $\bar{Z} \cap T^n = Z$, one obtains the statement in the theorem.

The following corollary then determines the $e^{p,q}(Z)$ for $p+q > n-1$. It follows from the fact that the isomorphism in the theorem is compatible with the Hodge structures.

Corollary 16. *For $p+q > n-1$.*

$$e^{p,q}(Z) = e^{p+1,q+1}(T^n) = \begin{cases} (-1)^{n+p+1}\binom{n}{p+1}, & p = q \\ 0, & p \neq q \end{cases}$$

6.4 The algorithm

First, computation of the Hodge-Deligne polynomial for a hypersurface in a torus — Z_f, corresponding to a Laurent polynomial f — can be reduced to the case when $\Delta(f)$ is maximal dimensional. Otherwise, Z_f is the product of a hypersurface in a lower-dimensional torus with a second lower-dimensional torus: $Z_f = Z' \times T'$. The behavior of the Hodge-Deligne polynomial with respect to products then implies $e_Z(x,y) = e_{Z' \times T'}(x,y) = e_{Z'}(x,y)e_{T'}(x,y) = e(Z')(xy-1)^{dim(T')}$. So, if $d' = dim(T')$ one can compute the $e^{p,q}(Z)$ directly from the $e^{p,q}(Z')$ as

$$e^{p,q}(Z) = \sum_{i=0}^{d'} (-1)^i \binom{d'}{i} e^{p-i,q-i}(Z'). \tag{10}$$

The rest of the description of the algorithm therefore assumes that $\Delta(f)$, and hence $\Delta'(f)$, is of maximal dimension.

6.4.1 $e^{p,q}(Z)$ for $p+q > n-1$

As per §6.3, the first step of the algorithm — namely, computing $e^{p,q}(Z)$ for $p+q > n-1$ — is as simple as computing binomial coefficients and taking care to have the correct sign.

With the $e^{p,q}(Z)$ now in hand for $p+q > n-1$, the algorithm will proceed to a recursive step: computing the $e^{p,q}(\Gamma')$ for $\Gamma' < \Delta'$. The recursion terminates when $dim(\Gamma') = 0$ because the hypersurface is assumed to transversally intersect all strata of $\mathbb{P}_\Delta$: this implies that $Z_{\Gamma'} = Z \cap \mathbb{P}_{\Gamma'} = \emptyset$ for all Γ' with $0 = dim(\Gamma') = dim(Z_{\Gamma'})$. Of course, for the recursive step to be valid, it must be true that the remaining steps may be completed with only the information contained in Δ, the $e^{p,q}(Z)$'s for $p+q > n-1$, the $e^{p,q}(Z_{\Gamma'})$'s for all p and q, and the relations described in the previous sections — namely, Eqs. 3, 4, and 5.

The description now proceeds to the second and third steps of the algorithm: computing $e^{p,q}(\bar{Z})$ for $p+q > n-1$ using the additive property of $e_{\bar{Z}}$, and thence $e^{p,q}(\bar{Z})$ for $p+q < n-1$ via Poincaré duality for smooth, compact varieties.

6.4.2 $e^{p,q}(\bar{Z})$ for $p+q > n-1$ and $p+q < n-1$

Since $\bar{Z} = \bigcup_{\Gamma' \leqslant \Delta'} \bar{Z} \cap T_{\Gamma'} = \bigcup_{\Gamma' \leqslant \Delta'} Z_{\Gamma'}$ is a disjoint union, the additivity of the Hodge-Deligne polynomial implies

$$e_{\bar{Z}} = \sum_{\Gamma' \leqslant \Delta'} e_{Z_{\Gamma'}}.$$

One therefore has the following relation among the coefficients (note that $Z_{\Delta'} = Z$):

$$e^{p,q}(\bar{Z}) = e^{p,q}(Z) + \sum_{\Gamma' < \Delta'} e^{p,q}(\Gamma').$$

As noted before, since the $e^{p,q}$'s for a 0-dimensional varieties are known, once it has been shown that Eq.s 4, 3 and 5 plus information contained in Δ determine all of the $e^{p,q}$'s of Z (and $\bar{Z}$) in terms of the $e^{p,q}$'s of the lower-dimensional varieties, it will follow by induction on the dimension of Z that one can compute the $e^{p,q}(Z)$'s (and $e^{p,q}(\bar{Z})$'s) in this way. So for now, one may take the $e^{p,q}(\Gamma')$'s to be given.

Under this assumption, Eq. 3 and Eq. 4 above clearly determine $e^{p,q}(\bar{Z})$ for $p+q > n-1$. Further, Poincaré duality for $\bar{Z}$ implies that $e^{n-1-p,n-1-q}(\bar{Z}) = e^{p,q}(\bar{Z})$ is also determined for $p+q > n-1$ — that is, $e^{p,q}(\bar{Z})$ is determined for both $p+q > n-1$ and for $p+q < n-1$.

6.4.3 $e^{p,q}(Z)$ for $p+q < n-1$

Having now determined $e^{p,q}(\bar{Z})$ for $p+q < n-1$, one may now rewrite Eq. 4 as

$$e^{p,q}(Z) = e^{p,q}(\bar{Z}) - \sum_{\Gamma' < \Delta'} e^{p,q}(\Gamma'),$$

from which it immediately follows that the $e^{p,q}(Z)$ are also determined for $p+q < n-1$.

Thus, the $e^{p,q}(Z)$'s (and $e^{p,q}(\bar{Z})$'s) are all determined for $p+q \neq n-1$. So it remains to determine them in the case that $p+q = n-1$. This will be done using Eq. 5 for $e^p(Z)$.

6.4.4 $e^{p,q}(Z)$ (and $e^{p,q}(\bar{Z})$) for $p+q = n-1$

First, observe that the following relation holds:

$$e^{p,n-1-p}(Z) = \sum_q e^{p,q}(Z) - \sum_{q \neq n-1-p} e^{p,q}(Z). \tag{11}$$

Since $q \neq n-1-p \iff p+q \neq n-1$, and the previous two subsections showed that $e^{p,q}(Z)$ are determined for $p+q \neq n-1$, it follows that all of the terms in $\sum_{q \neq n-1-p} e^{p,q}(Z)$ are determined.

Next, recall from §6.2 (i.e. Eq. 5) that

$$e^p(Z) = \sum_q e^{p,q}(Z) = (-1)^{n-1}((-1)^p \binom{n}{p+1} + h^*_{p+1}(\Delta))$$

determines the first term on the right-hand side of Eq. 11 purely in terms of information contained in Δ. Thus, the right-hand side, $e^{p,n-1-p}(Z)$ is determined for every p. Therefore, $e^{p,q}(Z)$ is determined for all p and q.

One last application of Eq. 4 for $p+q = n-1$ then shows that $e^{p,q}(\bar{Z})$ is also determined for all p and q. This completes the algorithm, and the proof that the induction step is valid.

6.5 Examples

For all of the following examples, Δ will be a polytope prime with respect to M (with $\Delta' = \Delta$), f a Laurent polynomial non-degenerate with respect to Δ, and Z its hypersurface in T^n.

Example 17. *(Δ is a point)*
If $M = \mathbb{Z}^0$ and $\Delta = \{0\}$, then the only Laurent polynomials are the constants: $f = a_0$. So the hypersurface Z_f is empty. The $e^{p,q}$'s and $h^{p,q}$'s are then 0 for all p and q.

Example 18. *(Δ is an interval)*
Consider the lattice $M = \mathbb{Z}$ and $\Delta = I = Conv(\{-1,2\}) = [-1,2]$. There are a few ways to compute the $e^{p,q}(Z)$. First, its only faces are $Conv(\{-1\}) = \Gamma_{-1}$ and $Conv(\{2\}) = \Gamma_2$. If the hypersurface of interest, Z, corresponds to a non-degenerate Laurent polynomial, then $Z_{\Gamma_{-1}} = Z \cap T_{\Gamma_{-1}} = \emptyset = Z \cap T_{\Gamma_2} = Z_{\Gamma_2}$. So $\bar{Z} = \bigcup_{\Gamma \leqslant I} Z_\Gamma = Z$. Also, for $p+q > 1-1 = 0$,

$$e^{p,q}(Z) = e^{p+1,q+1}(T^1) = \begin{cases} (-1)^{1+p+1}\binom{1}{p+1}, & p = q \\ 0, & p \neq q \end{cases}.$$

So $e^{p,q} = 0$ when $p+q > 0$.

There only remains $e^{0,0}(Z)$. This is equal to $\sum_q e^{0,q} = (-1)^{1-1}((-1)^0\binom{1}{1} + h^*_{0+1}(I))$, so the problem reduces to finding h^*_1. Now, it is easy to see that $\ell(nI) = 3n+1$, so that the first three terms of $Ehr_I(t)$ are $1+4t$. Using Eq. 1 then gives

$$1 + h^*_1 t \equiv (1-t)^{1+1}(1+4t) \equiv 1 + 2t \mod t^2.$$

So $h^*_1(I) = 2$ and $e^{0,0}(Z) = 1+2 = 3$.

Alternatively, one could simply note that any of the relevant Laurent polynomials are of the form $a_{-1}x^{-1} + a_0 + a_1 x + a_2 x^2$, and that the vanishing locus of such a polynomial in T^1 consists of three points. Using the additive property of the $e_Z(x,y)$ under decomposition and $e_{\{pt\}}(x,y) = 1$ then also yields $e^{0,0}(Z) = 1+1+1 = 3$.

The above argument and computation can be carried out for Δ an arbitrary 1-dimensional lattice polytope: with $h^*_1 = \ell(\Delta) - 2$, $e_Z(x,y) = 1 + h^*_1 = \ell(\Delta) - 1 = \ell^*(\Delta) + 1$. A table of $e^{p,q}(Z) = e^{p,q}(\bar{Z})$ for Δ any interval is given in Table 1.

$p \backslash q$	0	1
0	$\ell(\Delta) - 1$	0
1	0	0

Table 1: $e^{p,q}(Z) = e^{p,q}(\bar{Z})$ for any interval Δ. All other $e^{p,q}(Z) = e^{p,q}(\bar{Z})$ are 0.

Example 19. *(Δ is a square)*

Consider the lattice $M = \mathbb{Z}^2$, the polytope $\Delta = \square = \operatorname{conv}(\{(1,1),(1,-1),(-1,1),(-1,-1)\})$, and an associated hypersurface $Z \subset T^2$. The dimension of M is 2, so for $p+q > 2-1 = 1$, Eq. 3 gives

$$e^{p,q}(Z) = e^{p+1,q+1}(T^2) = \begin{cases} (-1)^{2+p+1}\binom{2}{p+1}, & p = q \\ 0, & p \neq q \end{cases}.$$

$\binom{2}{p+1} = 0$ for $p > 1$, so the only non-zero element is $e^{1,1}(Z) = 1$.

Now, in order to use Eq. 4, we must determine $e^{p,q}(\Gamma)$ for each face $\Gamma < \square$. The above argument for the interval can be used to show that $e^{0,0}(Z_\Gamma) = 2$ and $e^{p,q}(Z_\Gamma) = 0$ when p and q are not both 0. This is true for all of the 1-dimensional faces. Since $\bar{Z}$ "misses the vertices" (i.e. there are no 0-dimensional faces to consider), this accounts for all non-zero $e^{p,q}(\Gamma)$'s, $\Gamma < \square$.

From the preceding paragraph and Eq. 4 it follows that

$$e^{1,1}(\bar{Z}) = e^{1,1}(Z) + \sum_{\Gamma' < \Delta'} e^{p,q}(\Gamma') = 1 + 0 = 1.$$

Poincaré duality then asserts $e^{0,0}(\bar{Z}) = e^{1,1}(\bar{Z}) = 1$, from which is follows that $e^{0,0}(Z) = e^{0,0}(\bar{Z}) - \sum_{\Gamma<\square} e^{0,0}(\Gamma) = 1 - 8 = -7$.

Now, since $e^{0,1}(Z) = e^{1,0}(Z)$, it remains to calculate $e^{1,0}(Z) = \sum_q e^{1,q}(Z) - \sum_{q\neq 0} e^{1,q}(Z)$. The right-most term is known to be just $e^{1,1}(Z) = 1$. By Eq. 5, the first term on the right-hand side of the equation boils down to computing a coefficient of the Ehrhart numerator — namely, $h_2^*(\square)$. Referring back to the discussion of Ehrhart theory, this can be done by counting lattice points in $0 \cdot \square$, $1 \cdot \square$ and $2 \cdot \square$. The answer is easily seen to be $\ell(k \cdot \square) = (2k+1)^2$. As a result, one finds

$$1 + h_1^*(\square)t + h_2^*(\square)t^2 \equiv (1-t)^3 \sum_{k=0}^{2} \ell(k)t^k \equiv 1 + 6x + x^2 \mod t^3,$$

so that $h_2^*(\square) = 1$. From this it follows that $\sum_q e^{1,q}(Z) = (-1)^1((-1)^1\binom{2}{2} - 1) = 0$. So $e^{0,1}(Z) = e^{1,0}(Z) = 0-1 = -1$. And, to wrap things up, $e^{0,1}(\bar{Z}) = e^{1,0}(\bar{Z}) = e^{1,0}(Z)+0 = -1$. We then have Tables 2 and 3 for $e^{p,q}(Z)$ and $e^{p,q}(\bar{Z})$.

$p\backslash q$	0	1
0	-7	-1
1	-1	1

Table 2: $e^{p,q}(Z)$ for $\Delta = \square$

$p\backslash q$	0	1
0	1	-1
1	-1	1

Table 3: $e^{p,q}(\bar{Z})$ for $\Delta = \square$

Since $\bar{Z}$ is compact and smooth, its Hodge numbers can be recovered as $h^{p,q}(\bar{Z}) = (-1)^{p+q}e^{p,q}(\bar{Z})$. In this case, $h^{0,0}(\bar{Z}) = h^{1,0}(\bar{Z}) = h^{0,1}(\bar{Z}) = h^{1,1}(\bar{Z}) = 1$.

Example 20. *(Δ is truncated right pyramid)*

Edges	Faces
$a = \mathrm{conv}(\{v_0, v_5\})$	$\alpha = \mathrm{conv}(\{v_0, v_2, v_3, v_5\})$
$b = \mathrm{conv}(\{v_0, v_2\})$	$\beta = \mathrm{conv}(\{v_0, v_1, v_4, v_5\})$
$c = \mathrm{conv}(\{v_3, v_5\})$	$\gamma = \mathrm{conv}(\{v_0, v_1, v_2\})$
$d = \mathrm{conv}(\{v_2, v_3\})$	$\delta = \mathrm{conv}(\{v_3, v_4, v_5\})$
$e = \mathrm{conv}(\{v_0, v_1\})$	$\epsilon = \mathrm{conv}(\{v_1, v_2, v_3, v_4\})$
$f = \mathrm{conv}(\{v_4, v_5\})$	
$g = \mathrm{conv}(\{v_1, v_4\})$	
$h = \mathrm{conv}(\{v_1, v_2\})$	
$i = \mathrm{conv}(\{v_3, v_4\})$	

Table 4: Edges and faces of Δ for Ex. 20.

Consider the lattice $M = \mathbb{Z}^3$ and the polytope $\Delta = \mathrm{conv}(\{v_0 = (-1,-1,-1), v_1 = (4,-1,-1), v_2 = (-1,4,-1), v_3 = (-1,0,-1), v_4 = (0,-1,-1), v_5 = (-1,-1,1)\})$. The edges and faces of Δ are listed in Table 4.

For each edge, $E \in \Delta^{(1)}$, $e^{0,0}(Z_E)$ is $\ell(E) - 1$. These are collected in Table 5. The

Edges	$e^{0,0}$
a	2
b	5
c	1
d	2
e	5
f	1
g	2
h	5
i	1

Table 5: $e^{0,0}$ for each edge in the Δ of Ex. 20.

$e^{p,q}(Z_F)$ for each 2-dimensional face $F \in \Delta^{(2)}$ are likewise compiled in Table 6.

Faces	$e^{1,1}$	$e^{1,0}$	$e^{0,1}$	$e^{0,0}$
α	1	-2	-2	-9
β	1	-2	-2	-9
γ	1	-6	-6	-14
δ	1	0	0	-2
ϵ	1	-2	-2	-9

Table 6: $e^{p,q}$ for each face in the Δ of Ex. 20.

Now, the first $e^{p,q}(Z)$'s to compute are those with $p + q > n - 1 = 2$. By Eq. 3, these are

$$e^{p,q}(Z) = e^{p+1,q+1}(T^2) = \begin{cases} (-1)^{3+p+1}\binom{2}{p+1}, & p = q \\ 0, & p \neq q \end{cases}.$$

Again, $\binom{3}{p+1} = 0$ for $p > 2$, so $e^{2,2}(Z) = 1$ is the only non-zero term while $e^{2,1}(Z) = 0 = e^{1,2}(Z)$.

From this, Eq. 4 implies that $e^{2,2}(\bar{Z}) = e^{2,2}(Z) + 0 = 1$; similarly, $e^{2,1}(\bar{Z}) = 0 = e^{1,2}(\bar{Z})$. And then Poincaré duality implies that $e^{0,0}(\bar{Z}) = e^{2,2}(\bar{Z}) = 1$ and $e^{1,0}(\bar{Z}) = e^{1,2}(\bar{Z}) = 0 =$

$e^{2,1}(\bar{Z}) = e^{0,1}(\bar{Z})$.

Applying Eq. 4 in the other direction then gives

$$e^{0,0}(Z) = e^{0,0}(\bar{Z}) - (\sum_{E\in\Delta^{(1)}} e^{0,0}(E) + \sum_{F\in\Delta^{(2)}} e^{0,0}(F)) = 1 - (24 + (-43)) = 20$$

and

$$e^{1,0}(Z) = e^{1,0}(\bar{Z}) - (\sum_{E\in\Delta^{(1)}} e^{1,0}(E) + \sum_{F\in\Delta^{(2)}} e^{1,0}(F)) = 0 - (0 + (-12)) = 12 = e^{0,1}(Z).$$

It then remains to compute $e^{p,q}(Z)$ for $p+q=2$. For this, one needs $\ell(\Delta) = 34$, $\ell(2\Delta) = 160$, and $\ell(3\Delta) = 441$, from which one can compute

$$(1+\ell(\Delta)t+\ell(2\Delta)+\ell(3\Delta))(1-t)^{3+1} = (1+34t+160t^2+441t^3)(1-t)^4 \equiv 1+30t+30t^2+t^3 \mod t^4.$$

Hence,

$$e^2(Z) = (-1)^2((-1)^2\binom{3}{2+1} + h^*_{2+1}(\Delta)) = 1+1 = 2,$$

$$e^1(Z) = (-1)^2((-1)^1\binom{3}{1+1} + h^*_{1+1}(\Delta)) = -3+30 = 27$$

and

$$e^0(Z) = (-1)^2((-1)^0\binom{3}{0+1} + h^*_{0+1}(\Delta)) = 3+30 = 33.$$

And applying Eq. 11 yields

$$e^{2,0}(Z) = e^2(Z) - \sum_{q\neq 0} e^{2,q}(Z) = 2-1 = 1,$$

$$e^{1,1}(Z) = e^1(Z) - \sum_{q\neq 1} e^{1,q}(Z) = 27-12 = 15$$

and

$$e^{0,2}(Z) = e^0(Z) - \sum_{q\neq 2} e^{2,q}(Z) = 33 - (20+12) = 1.$$

Note that $e^{2,0}(\bar{Z})$ and $e^{0,2}(\bar{Z})$ agree as expected.

One last application of Eq. 4 then yields the final $e^{p,q}(\bar{Z})$: $e^{2,0}(\bar{Z}) = 1 = e^{0,2}(\bar{Z})$ and $e^{1,1}(\bar{Z}) = 15+5 = 20$. The $e^{p,q}$'s of Z and $\bar{Z}$ are recorded in Tables 7 and 8.

$p\backslash q$	0	1	2
0	20	12	1
1	12	15	0
2	1	0	1

Table 7: $e^{p,q}(Z)$ for Δ in Ex. 20.

The Hodge numbers of $\bar{Z}$ can easily be recovered — in this case, signs would only change when $e^{p,q}(Z) = 0$. So $h^{p,q}(\bar{Z}) = e^{p,q}(\bar{Z})$ for all p and q.

$p\backslash q$	0	1	2
0	1	0	1
1	0	20	0
2	1	0	1

Table 8: $e^{p,q}(\bar{Z})$ for Δ in Ex. 20.

7 Hodge-Deligne numbers of hypersurfaces and complete intersections in toric varieties

If Z is not a hypersurface in a torus, but rather a hypersurface in toric variety, X, intersecting all strata of X transversally, then the intersections of Z with the strata of X give a decomposition of Z into a disjoint union of hypersurfaces in tori. I.e. one has $Z = \sum_i Z \cap T_i = \sum_i Z_i$, where the T_i are the strata of X, each of which is isomorphic to a torus of dimension less than or equal to that of X. One can then compute

$$e^{p,q}(Z) = \sum_i e^{p,q}(Z_i).$$

For complete intersections, let $Y = \bigcap_{i=1}^r Z_i$, where the Z_i are the vanishing loci of $f_i = \sum_{m \in M} a_{i,m}\chi^m$, $1 \leqslant i \leqslant r$. To extend the definition of non-degeneracy to the system of equations $\{f_i\}_{i=1}^r$ and their Newton polytopes $\{\Delta(f_i)\}_{i=1}^r$, first make the following definitions.

For each $\xi \in N$ define $\Delta(f_i)^\xi$ to be the face of $\Delta(f_i)$ on which the linear function $m \mapsto \langle m, \xi \rangle$ attains a minimum. Likewise define $f_i^\xi = \sum_{m \in \Delta(f_i)^\xi} a_{i,m}\chi^m$. (Practically, this sets all coefficients of f not on the face $\Delta(f_i)^\xi$ to 0.)

The system of equations $\{f_i\}_{i=1}^r$ is defined to be non-degenerate with respect to the Newton polytopes $\{\Delta(f_i)\}_{i=1}^r$ if, for all solutions z of $0 = f_1^\xi = f_2^\xi = \cdots = f_r^\xi$, the differentials $df_i^\xi(z)$ are linearly independent. Geometrically, if Δ' is a polytope majorizing each of the Δ_i, then the hypersurfaces $\{\bar{Z}_{\Delta'(f_i)}\}_{i=1}^r$ intersect transversally among themselves and are transversal to $D_{\Delta'}$.

Now consider a new function

$$F = 1 - \sum_{i=1}^r \lambda_i f_i,$$

with vanishing locus Z_F. This is a hypersurface in the toric variety $\mathbb{C}^r \times T^n$. The Newton polytope of F, $\Delta(F)$, is $\mathrm{conv}(\{\{0\}, \{x_1\} \times \Delta(f_1), \{x_2\} \times \Delta(f_2), \ldots, \{x_r\} \times \Delta(f_r)\}) \subset \mathbb{R}^r \times M_{\mathbb{R}}$, where $x_1, x_2, \ldots, x_r$ is the natural basis of $\mathbb{Z}^r$. Importantly, F is non-degenerate with respect to $\Delta(F)$.

Restricting the projection $\mathbb{C}^r \times T^n \to T^n$ to Z_F, one obtains a map $\pi : Z_F \to T^n$. Moreover, for $z \in Y$, $\pi^{-1}(z) = \emptyset$, while for $z \in T^n \backslash Y$, $\pi^{-1}(z)$ is an affine linear subspace in $\mathbb{C}^r$. So Z_F is a locally trivial (in the Zariski topology) bundle over $T^n \backslash Y$ with fiber $\mathbb{C}^{r-1}$. By Lemma 8, one then has

$$e(Z_F) = e(T^n \backslash Y)(xy)^{r-1},$$

which implies

$$e^{p,q}(Y) = e^{p,q}(T^n) - e^{p+r-1,q+r-1}(Z_F). \tag{12}$$

Thus the computation boils down to that for Z_F, a hypersurface in the toric variety $\mathbb{C}^r \times T^n$.

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