MAC 2311 Calculus 1
Hyberbolics & Linear Approximations
Worksheet #6 A

Name: Solutions

+ answer summary

This worksheet is graded on completion. You must complete the entire worksheet to earn any credit. You answers do not have to be correct. Of course, it is better if they are, but the purpose of the this worksheet is the help you master the material. If you aren't sure about the mathematical steps, you can describe the process or support your solution by writing an explanation.

A value of sinh x or cosh x is given. Use the definitions and the identity $\cosh^2 x - \sinh^2 x = 1$ to find the value of the other indicated hyperbolic function.

1) $\sinh x = -\frac{4}{3}$, $\tanh x =$ 1) C

A) $\frac{5}{3}$ B) $-\frac{5}{4}$ C) $\boxed{-\frac{4}{5}}$ D) $\frac{4}{5}$

2) $\sinh x = \frac{8}{15}$, $\operatorname{sech} x =$ 2) A

A) $\boxed{\frac{15}{17}}$ B) $\frac{64}{289}$ C) $\frac{15}{8}$ D) $\frac{17}{15}$

Rewrite the expression in terms of exponentials and simplify the results.

3) $\sinh (4 \ln x)$ 3) A

A) $\boxed{\frac{1}{2}\left(x^4 - \frac{1}{x^4}\right)}$ B) $\frac{1}{2}\left(x^4 + \frac{1}{x^4}\right)$ C) $2\left(x - \frac{1}{x}\right)$ D) $2x$

4) $(\sinh x + \cosh x)^3$ 4) A

A) $\boxed{e^{3x}}$ B) $e^{3x} - e^{-3x}$ C) $\frac{e^{3x}}{4}$ D) e^{x^3}

Find the derivative of y.

5) $y = \cosh x^5$ 5) C

A) $\sinh x^5$ B) $-\sinh x^5$ C) $\boxed{5x^4 \sinh x^5}$ D) $-5x^4 \sinh x^5$

6) $y = \sinh^2 7x$ 6) D

A) $2 \sinh 7x \cosh 7x$
B) $14 \cosh 7x$
C) $2 \cosh 7x$
D) $\boxed{14 \sinh 7x \cosh 7x}$

7) $y = \ln(\sinh 9x)$ 7) B

A) $9 \operatorname{csch} 9x$
B) $9 \coth 9x$
C) $\frac{1}{\sinh 9x}$
D) $\coth 9x$

Find the derivative of y with respect to the appropriate variable.

8) $y = \sinh^{-1}\sqrt{9x}$ 8) A

A) $\frac{9}{2\sqrt{9x}(1 + 9x)}$
B) $\frac{1}{2\sqrt{9x}(1 + 9x)}$
C) $\frac{1}{\sqrt{1 + 9x}}$
D) $\frac{9}{2\sqrt{9x}(9x - 1)}$

9) $y = (\theta^2 + 5\theta)\tanh^{-1}(\theta + 4)$ 9) B

A) $(2\theta + 5)\tanh^{-1}(\theta + 4) - \frac{\theta^2 + 5\theta}{1 + (\theta + 4)^2}$
B) $(2\theta + 5)\tanh^{-1}(\theta + 4) - \frac{\theta}{\theta + 3}$
C) $-\frac{\theta}{\theta + 3}$
D) $(2\theta + 5) - \frac{1}{\theta + 15}$

10) $y = 9\tanh^{-1}(\cos x)$ 10) B

A) $\ln\left(\frac{1}{\sqrt{1 - x^2}}\right)\sin x$
B) $\frac{-9}{\sin x}$
C) $\frac{-9 \sin x}{1 + \cos^2 x}$
D) $\frac{-9}{\cos x}$

11) Verify the identity using the definitions of hyperbolic functions. 11) ______

$$\coth x = \frac{e^{2x} + 1}{e^{2x} - 1}$$

12) Use the fundamental identity $\cosh^2 x - \sinh^2 x = 1$ to verify the identity. 12) ______

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

13) Derive the formula given that $d/dx(\cosh x) = \sinh x$ and $d/dx(\sinh x) = \cosh x$. 13) ______

$$d/dx(\coth x) = -\operatorname{csch}^2 x$$

Find the linearization L(x) of f(x) at x = a.

14) $f(x) = 3x^2 + 4x + 1$, $a = 3$ 14) A

A) $L(x) = 22x - 26$ B) $L(x) = 22x + 28$ C) $L(x) = 14x - 26$ D) $L(x) = 14x + 28$

15) $f(x) = \sqrt{4x + 49}$, $a = 0$ 15) B

A) $L(x) = \frac{4}{7}x - 7$ B) $L(x) = \frac{2}{7}x + 7$ C) $L(x) = \frac{2}{7}x - 7$ D) $L(x) = \frac{4}{7}x + 7$

Express the relationship between a small change in x and the corresponding change in y in the form dy = f′(x) dx.

16) $y = 5x^2 - 2x - 7$ 16) C

A) $dy = 10x - 7\ dx$ B) $dy = 10x\ dx$
C) $dy = (10x - 2)\ dx$ D) $dy = 10x - 4\ dx$

17) $y = x\sqrt{5x + 8}$ 17) A

A) $dy = \dfrac{15x + 16}{2\sqrt{5x + 8}}\ dx$ B) $dy = \dfrac{15x - 16}{\sqrt{5x + 8}}\ dx$

C) $dy = \dfrac{15x + 16}{\sqrt{5x + 8}}\ dx$ D) $dy = \dfrac{15x - 16}{2\sqrt{5x + 8}}\ dx$

Use the differential to approximate the quantity to four decimal places.

18) ln 1.07 18) C

A) 0.0677 B) −0.0726 C) 0.0700 D) −0.0700

19) $e^{0.49}$ (e raised to the power 0.49) 19) D

A) 1.4900 B) 0.6,126 C) .5100 D) 1.6323

20) $\sqrt{101}$ 20) A

A) 10.0500 B) 10.1000 C) 9.9500 D) 11.0000

Answer Key
Testname: MAC 2311 CANVAS WS #6 -- HYBERBOLICS & LINEAR APPROXIMATIONS

1) C
2) A
3) A
4) A
5) C
6) D
7) B
8) A
9) B
10) B

11) $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}} \cdot \dfrac{e^x}{e^x} = \dfrac{e^{2x} + 1}{e^{2x} - 1}$

12) $\dfrac{\cosh^2 x}{\cosh^2 x} - \dfrac{\sinh^2 x}{\cosh^2 x} = \dfrac{1}{\cosh^2 x} \rightarrow 1 - \tanh^2 x = \operatorname{sech}^2 x$

13) $\dfrac{d}{dx}(\coth x) = \dfrac{d}{dx}\left(\dfrac{\cosh x}{\sinh x}\right)$

$= \dfrac{\sinh x(\sinh x) - \cosh x(\cosh x)}{\sinh^2 x}$

$= \dfrac{-1}{\sinh^2 x}$

$= -\operatorname{csch}^2 x$

14) A
15) B
16) C
17) A
18) C
19) D
20) A

MAC 2311
Hyperbolics & Linear Approx.
worksheet #6A

Solutions

① $\boxed{\sinh(x) = -\frac{4}{3}}$

our goal: $\tanh(x)$

$\cosh^2(x) - \sinh^2(x) = 1$

Identity

$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$

$\cosh^2(x) - \left(-\frac{4}{3}\right)^2 = 1$

$\cosh^2(x) - \frac{16}{9} = 1$

$\cosh^2(x) = 1 + \frac{16}{9} = \frac{25}{9}$

$\tanh(x) = \frac{-\frac{4}{3}}{\frac{5}{3}}$

$\boxed{\cosh(x) = \frac{5}{3}}$

C

$\boxed{\frac{-4}{5} = \tanh(x)} = \frac{-4}{3} \cdot \frac{3}{5}$

② $\boxed{\sinh(x) = \frac{8}{15}}$

Goal: $\text{sech}(x) = \frac{1}{\cosh(x)}$

$\cosh^2(x) - \sinh^2(x) = 1$

$\cosh(x) = \sqrt{\frac{289}{225}} = \frac{17}{15}$

$\cosh^2(x) - \left(\frac{8}{15}\right)^2 = 1$

$\Longrightarrow \boxed{\text{sech}(x) = \frac{15}{17}}$

$\cosh^2(x) = 1 + \left(\frac{8}{15}\right)^2$

A

$\cosh^2(x) = \frac{225}{225} + \frac{64}{225}$

$\cosh^2(x) = \frac{289}{225}$

3 $\boxed{\sinh(4\ln x)}$

Defn $\sinh(x) = \dfrac{e^{x} - e^{-x}}{2}$

$$\sinh(4\ln x) = \frac{e^{4\ln x} - e^{-4\ln x}}{2}$$

A

$$= \frac{e^{\ln x^4} - e^{\ln x^{-4}}}{2} = \frac{x^4 - x^{-4}}{2} = \boxed{\frac{1}{2}\left(x^4 - \frac{1}{x^4}\right)}$$

4 $\boxed{(\sinh(x) + \cosh(x))^3}$

$\sinh(x) = \dfrac{e^{x} - e^{-x}}{2}$ $\cosh(x) = \dfrac{e^{x} + e^{-x}}{2}$

$$\left(\frac{e^{x} - e^{-x}}{2} + \frac{e^{x} + e^{-x}}{2}\right)^3 = \left(\frac{2e^{x}}{2}\right)^3 = \boxed{e^{3x}}$$

A

5 $\boxed{y = \cosh(x^5)}$

$$\frac{d}{dx}(\cosh(g(x))) = \sinh(g(x)) \cdot g'(x)$$

$$y' = \sinh(x^5) \cdot 5x^4 = \boxed{5x^4 \sinh(x^5)}$$

C

(6) $\boxed{y = \sinh^2(7x)} = (\sinh(7x))^2$

$$\frac{d}{dx}(\sinh(g(x))) = \cosh(g(x)) \cdot g'(x)$$

$$y' = 2(\sinh(7x))^1 \cdot \underbrace{\frac{d}{dx}(\sinh(7x))}_{\cosh(7x)(7)}$$

$$\boxed{y' = 14\cosh(7x)\sinh(7x)}$$ (D)

(7) $\boxed{y = \ln(\sinh(9x))}$

$$y' = \frac{1}{\sinh(9x)} \cdot \frac{d}{dx}(\sinh(9x))$$

$$y' = \frac{1}{\sinh(9x)} \cdot \cosh(9x) \cdot 9$$

$$y' = \boxed{\frac{9\cosh(9x)}{\sinh(9x)}} = \boxed{9\coth(9x)}$$ (B)

(8) $\boxed{y = \sinh^{-1}(\sqrt{9x})}$

$$\frac{d}{dx}(\sinh^{-1}(g(x))) = \frac{1}{\sqrt{1+(g(x))^2}} \cdot g'(x)$$

$$y' = \frac{1}{\sqrt{1+(\sqrt{9x})^2}} \cdot \frac{d}{dx}(9x)^{1/2}$$

8 $\boxed{\text{continued}}$

$$y' = \frac{1}{\sqrt{1+9x}} \cdot \frac{1}{2}(9x)^{-1/2} \cdot 9$$

$$y' = \frac{1}{\sqrt{1+9x}} \cdot \frac{9}{2} \cdot \frac{1}{\sqrt{9x}}$$

$$y' = \boxed{\frac{9}{2\sqrt{9x}\,\sqrt{1+9x}}} = \boxed{\frac{9}{2\sqrt{9x(1+9x)}}}$$ A

9 $\boxed{y = (\theta^2 + 5\theta)\tanh^{-1}(\theta+4)}$ product rule

$$\frac{d}{dx}\left(\tanh^{-1}(h(x))\right) = \frac{1}{1-(h(x))^2} \cdot h'(x)$$

$f = \theta^2 + 5\theta$ $\quad$ $g = \tanh^{-1}(\theta+4)$

$f' = 2\theta + 5$ $\quad$ $g' = \dfrac{1}{1-(\theta+4)^2} \cdot \underbrace{\dfrac{d}{d\theta}(\theta+4)}_{1}$

$$y' = (2\theta+5)\tanh^{-1}(\theta+4) + \frac{\theta^2+5\theta}{1-(\theta+4)^2}$$

$$y' = (2\theta+5)\tanh^{-1}(\theta+4) + \frac{\theta(\theta+5)}{1-(\theta^2+8\theta+16)}$$

$$y' = (2\theta+5)\tanh^{-1}(\theta+4) + \frac{\theta(\theta+5)}{\underbrace{1-\theta^2-8\theta-16}_{-\theta^2-8\theta-15}}$$

⑨ continued

$$y' = (2\theta+5)\tanh^{-1}(\theta+4) + \frac{\theta\cancel{(\theta+5)}}{-(\theta+3)\cancel{(\theta+5)}}$$

$$\boxed{y' = (2\theta+5)\tanh^{-1}(\theta+4) - \frac{\theta}{\theta+3}}$$ (B)

⑩ $\boxed{y = 9\tanh^{-1}(\cos x)}$

$$\frac{d}{dx}\left(\tanh^{-1}(h(x))\right) = \frac{1}{1-(h(x))^2} \cdot h'(x)$$

$$y' = 9\left(\frac{1}{1-(\cos x)^2}\right) \cdot \underbrace{\frac{d}{dx}(\cos x)}_{-\sin x}$$

$$y' = 9\left(\frac{-\sin x}{1-\cos^2 x}\right) = \frac{-9\sin x}{\sin^2 x} = \boxed{\frac{-9}{\sin x}}$$ (B)

⑪ $\boxed{\coth(x) = \frac{e^{2x}+1}{e^{2x}-1}}$ verify the identity

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{\frac{e^x+e^{-x}}{2}}{\frac{e^x-e^{-x}}{2}}$$

$$= \frac{e^x+e^{-x}}{2} \cdot \frac{2}{e^x-e^{-x}} = \frac{e^x+e^{-x}}{e^x-e^{-x}}$$

(11) continued

$$\frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^x + \frac{1}{e^x}}{e^x - \frac{1}{e^x}} \left(\frac{\frac{e^x}{1}}{\frac{e^x}{1}} \right) = \boxed{\frac{e^{2x}+1}{e^{2x}-1}}$$

(12) verify the identity

use $\cosh^2 x - \sinh^2 x = 1$

$$1 - \tanh^2(x) = \text{sech}^2 x$$

$$1 - \frac{\sinh^2 x}{\cosh^2 x} = \text{sech}^2 x$$

$$\frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \text{sech}^2 x$$

$$\frac{1}{\cosh^2 x} = \text{sech}^2 x$$

$$\text{sech}^2 x = \text{sech}^2 x \checkmark$$

(13) verify $\frac{d}{dx}\left(\coth(x)\right) = -\text{csch}^2 x$

$\frac{d}{dx}\left(\frac{\cosh(x)}{\sinh(x)}\right)$ quotient rule $\frac{f'g - g'f}{g^2}$

$f = \cosh(x)$ $\quad g = \sinh(x)$

$f' = \sinh(x)$ $\quad g' = \cosh(x)$

$$\frac{d}{dx}\left(\frac{\cosh(x)}{\sinh(x)}\right) = \frac{\sinh^2(x) - \cosh^2(x)}{\sinh^2(x)} = \frac{-1}{\sinh^2 x}$$

$$= \boxed{-\text{csch}^2 x}$$

Find the linearization (tangent line approximation)

$$L(x) = f(a) + f'(a)(x-a)$$

(14) $f(x) = 3x^2 + 4x + 1$ with $a=3$

$f'(x) = 6x + 4$

$f(a) = f(3) = 3(3)^2 + 4(3) + 1$

$= 27 + 12 + 1 = 40$

$f'(a) = f'(3) = 6(3)+4 = 22$

$L(x) = f(a) + f'(a)(x-a)$

$L(x) = 40 + 22(x-3)$

$L(x) = 40 + 22x - 66 \rightarrow \boxed{L(x) = 22x - 26}$ (A)

(15) $f(x) = \sqrt{4x+49} = (4x+49)^{1/2}$ $a=0$

$f'(x) = \frac{1}{2}(4x+49)^{-1/2}(4) = \dfrac{2}{\sqrt{4x+49}}$

$f(0) = (4(0)+49)^{1/2} = 7$

$f'(0) = \dfrac{2}{\sqrt{4(0)+49}} = \dfrac{2}{7}$

$L(x) = f(0) + f'(0)(x-0)$

$L(x) = 7 + \left(\frac{2}{7}\right)(x)$

$\boxed{L(x) = \frac{2}{7}x + 7}$ (B)

$$\boxed{y = f(x) \rightarrow \frac{dy}{dx} = f'(x) \longrightarrow dy = f'(x)\,dx}$$

dx & dy represent small changes in x & y.

16) $y = 5x^2 - 2x - 7$

$$\frac{dy}{dx} = 10x - 2 \longrightarrow \boxed{dy = (10x - 2)dx}$$

17) $y = x\sqrt{5x+8}$ (product rule) $f'g + g'f$

$f = x \quad g = (5x+8)^{1/2}$

$f' = 1 \quad g' = \frac{1}{2}(5x+8)^{-1/2}(5)$

$= \frac{5}{2\sqrt{5x+8}}$

$$\frac{dy}{dx} = (5x+8)^{1/2} + \frac{5x}{2\sqrt{5x+8}}$$

$$\frac{dy}{dx} = \frac{2(5x+8)^{1/2}(5x+8)^{1/2}}{2\sqrt{5x+8}} + \frac{5x}{2\sqrt{5x+8}}$$

$$\frac{dy}{dx} = \frac{2(5x+8) + 5x}{2\sqrt{5x+8}} = \frac{10x + 16 + 5x}{2\sqrt{5x+8}}$$

$$\boxed{dy = \left(\frac{15x + 16}{2\sqrt{5x+8}}\right) dx}$$

Use Differentials for approximations

option #1 Find the linearization

1) Identify the function $f(x)$
2) determine a

option #2 Use $\frac{dy}{dx} = f'(x)$ to see how a small change in x affects y.

18) approximate $\ln(1.07)$

option #1 $L(x) \cong f(a) + f'(a)(x-a)$

$f(x) = \ln x$ $\quad f(a) = f(1) = \ln(1) = 0$

$f'(x) = \frac{1}{x}$ $\quad f'(a) = f'(1) = \frac{1}{1} = 1$

$a = 1$ (closest perfect value)

so $f(x) \cong f(a) + f'(a)(x-a)$

$f(x) \cong 0 + (1)(x-1)$

$\boxed{f(x) \cong x-1}$

use this → we need $x = 1.07$

$f(1.07) \cong 1.07 - 1 = \boxed{0.07}$ C

18 $\boxed{\text{option \#2}}$ $\ln(1.07) = \ln(x+dx)$

$\boxed{\begin{array}{c} x = a \\ dx = 0.7 \end{array}}$

$f(x) = \ln x$

$\frac{dy}{dx} = f'(x) = \frac{1}{x}$

$dy = \frac{1}{x} dx \longrightarrow dy = \frac{1}{(1)}(0.7) = 0.7$

So when x increases by 0.7, y increases by 0.7

$\boxed{f(a+dx) \cong f(a) + dy}$ $\boxed{\text{with } dy = f'(a)dx}$

$f(1+0.7) = f(1) + f'(1)(0.7)$ Hence another version of the linearization

$= 0 + (1)(0.7)$

so $f(1.07) = \boxed{\ln(1.07) \cong 0.7}$

19 $\boxed{\text{approximate } e^{0.49}}$

$f(x) = e^x$ $f(0.5) = e^{0.5}$

$f'(x) = e^x$ $f'(0.5) = e^{0.5}$

$\boxed{\text{option } 1}$

$a = 0.50$

$L(x) \cong f(x) = f(a) + f'(a)(x-a)$

$L(x) \cong e^x = e^{0.5} + e^{0.5}(x - 0.5)$

$e^x \cong e^{0.5}(1 + x - 0.5)$

$e^x \cong e^{0.5}(x + 0.5)$

19 Continued

so $e^{x} \approx e^{0.5}(x+0.5)$

$e^{0.49} \approx e^{0.5}(0.49+0.5)$ D

$= e^{0.5}(0.99) \approx \boxed{1.6323}$

option 2
$$\boxed{\begin{gathered} f(a+dx) \approx f(a) + dy \\ \text{or} \\ f(a+dx) \approx f(a) + f'(a)dx \end{gathered}}$$

$e^{0.49} = e^{0.50-0.01}$

$a = 0.5$
$dx = -0.01$

$f(x) = e^{x}$
$f'(x) = e^{x}$

$\boxed{e^{0.49}} \approx f(0.5) + f'(0.5)(-0.01)$

$\approx e^{0.5} + e^{0.5}(-0.01)$

$\approx e^{0.5}(1-0.01) = e^{0.5}(0.99)$

$\approx \boxed{1.6323}$

NOTE: When using differentials for approximations, the values $f(a)$ & $f'(a)$ can usually be determined without a calculator. That is not the case for problem #19.

$e^{0} = 1$ (something we know)

However 0 is not close to 0.49, so using 0 would not be an accurate approximation.

20 $\boxed{\sqrt{101}}$ $f(x) = \sqrt{x}$

option 2

$\sqrt{101} = \sqrt{100+1}$ $\boxed{\begin{array}{l} a = 100 \\ dx = 1 \end{array}}$

$$\boxed{f(a+dx) \cong f(a) + dy \quad \text{with } dy = f'(a)\,dx}$$

$f(x) = \sqrt{x} = x^{1/2}$ $f(100) = \sqrt{100} = 10$

$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ $f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$

$\sqrt{100} = \sqrt{100+1} \cong 10 + \frac{1}{20}(1) \cong \boxed{10.05}$

option 1 Linearizations

$L(x) \cong f(a) + f'(a)(x-a)$

$\sqrt{101} = \sqrt{100+1}$, $100 \downarrow a$

$f(x) = \sqrt{x} = x^{1/2}$ $f(100) = \sqrt{100} = 10$

$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ $f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$

$L(x) = f(100) + f'(100)(x-100)$

$\sqrt{x} = 10 + \frac{1}{20}(x-100) = 10 + \frac{1}{20}x - 5$

$\boxed{\sqrt{x} \cong \frac{x}{20} + 5}$ $\sqrt{101} \cong \frac{101}{20} + 5 =$

$\cong 5.05 + 5$

$\cong \boxed{10.05}$