Proof of Boole's inequality using induction

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Boole's inequality states that for a countable set of events, the probability that at least one of them occurs cannot be larger than the sum of the probabilities of the individual events. It is a useful result to obtain an upper bound on the probability of a union of events, which may be difficult to compute, whereas the marginal probabilities are known and can be summed. For an arbitrarily large countable set of events A_i, Boole's inequality can be written as follows:

$$P\Big(\bigcup_{i=1}^{\infty} A_i\Big) \leq \sum_{i=1}^{\infty} P(A_i)$$

The probability on the LHS is the probability of the union of the events, which is equivalent to the probability that at least one of them occurs.

Let us use induction to proove this inequality. We start with the base case, where n = 2, that

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

The above is a consequence of the inclusion-exclusion principle. By Kolmogorov's axioms of probability, all probabilities must be non-negative, so the term $P(A_1 \cap A_2)$ must be greater than or equal to zero. We thus have confirmed our base case for n = 2.

$$P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$$

The case $n = 1$ is trivial, since $P(A_1) = P(A_1)$

For our inductive hypothesis, we assume that the inequality holds true for some n, so that we can eventually show it is true for all n.

$$P\Big(\bigcup_{i=1}^{n} A_i\Big) \leq \sum_{i=1}^{n} P(A_i)$$

We want to show that if the above is true, it will entail that

$$P\Big(\bigcup_{i=1}^{n+1} A_i\Big) \leq \sum_{i=1}^{n+1} P(A_i)$$

We therefore consider the above in the inductive step, and using the result from the inclusion-exclusion principle, we can rewrite the LHS as

$$P\Big(\bigcup_{i=1}^{n+1} A_i\Big) = P\Big(\bigcup_{i=1}^{n} A_i \cup A_{n+1}\Big)$$

$$\iff P\Big(\bigcup_{i=1}^{n} A_i \cup A_{n+1}\Big) = \sum_{i=1}^{n} P(A_i) + P(A_{n+1}) - P\Big(\bigcup_{i=1}^{n} A_i \cap A_{n+1}\Big)$$

The above simply uses $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ to rewrite the LHS. Given the non-negativity of probabilities, for the same reason as in the base case, we know that

$$\iff P\Big(\bigcup_{i=1}^{n} A_i \cup A_{n+1}\Big) \leq \sum_{i=1}^{n} P(A_i) + P(A_{n+1})$$

since $P\Big(\bigcup_{i=1}^{n} A_i \cap A_{n+1}\Big) \geq 0$

The above expression is the desired result, since we can just move back the final A_{n+1} into the union on the LHS, and move $P(A_{n+1})$ into the sum on the RHS.

$$P\Big(\bigcup_{i=1}^{n+1} A_i\Big) \leq \sum_{i=1}^{n+1} P(A_i)$$

Therefore, we have obtained to complete the inductive step, which finishes the proof. The inequality therefore holds for all n. ■