Part 1: Report

Question 1

Since matrix Q (with n = 15) was generated using the rand command, all executions of the code will lead to different matrices, so the code that generated this matrix (lines 5-12) could then generate each of the 100 eQi matrices for part c.

Q

15x15 double

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	-0.0505	0.4097	0.2459	0.1411	-0.0883	-0.0084	0.3375	-0.0041	0.1052	-0.3042	0.3618	0.0978	-0.0564	0.4129	-0.4581
2	-0.0539	-0.1523	-0.2945	0.1864	-0.1743	-0.5374	0.2677	-0.1848	0.0402	0.0955	-0.3712	0.0514	0.4416	0.1577	-0.2372
3	0.3278	0.2917	-0.0499	0.3196	-0.5920	-0.0530	0.0769	0.3145	0.0781	-0.0851	0.0213	-0.2718	0.0406	-0.3529	0.1755
4	-0.3327	0.0259	-0.1674	-0.2809	-0.2014	0.3318	0.1601	0.2459	-0.4069	-0.3495	-0.0251	0.2257	0.4404	-0.0627	0.0882
5	0.1639	-0.3976	0.5373	0.2417	0.0903	-0.0218	0.1465	0.0162	0.0578	-0.3161	-0.1426	0.3887	0.0577	-0.3903	-0.0854
6	-0.3846	-0.0096	-0.2977	0.0894	-0.2333	-0.2335	-0.4471	0.1001	0.1381	-0.1131	0.1384	0.4106	-0.3187	-0.2334	-0.2469
7	0.1353	-0.4162	0.0650	-0.4663	-0.3339	-0.0016	1.3596e-05	0.0242	-0.1187	-0.0964	-0.0150	-0.3947	-0.2010	-0.0412	-0.5033
8	-0.1466	-0.0790	0.0345	-0.3560	-0.0351	-0.4550	0.1966	-0.1532	0.2703	-0.4203	0.2047	-0.1067	-0.0986	0.0036	0.5111
9	0.0635	-0.1500	0.2154	0.1809	0.0788	-0.2815	-0.5757	0.2038	-0.2360	-0.1869	0.2777	-0.2120	0.3876	0.2683	0.0324
10	-0.2256	0.0998	0.0283	0.2852	0.0231	0.0785	-0.1211	-0.1216	-0.1948	-0.4569	-0.5856	-0.2791	-0.3404	0.1821	0.0657
11	-0.4069	0.3345	0.2830	-0.1151	0.3024	-0.3227	0.1077	0.2203	-0.1496	0.1852	-0.0745	-0.2860	0.0537	-0.4358	-0.1913
12	-0.2891	-0.1097	0.3162	-0.1045	-0.2098	0.1190	-0.0663	0.4627	0.5202	0.2091	-0.2926	0.0159	0.0712	0.3215	0.0905
13	-0.1076	-0.3983	-0.2269	0.3084	0.1727	-0.0903	0.3958	0.5015	-0.2260	0.0822	0.2059	-0.0668	-0.3203	0.1437	0.0952
14	-0.2760	-0.2090	-0.2261	0.2603	0.1815	0.3448	0.0209	-0.1475	0.4496	-0.1828	0.2096	-0.4043	0.2702	-0.2205	-0.1504
15	-0.4124	-0.1638	0.3312	0.2293	-0.4365	0.0734	0.0615	-0.4242	-0.2574	0.3293	0.2259	-0.0576	-0.0387	0.0032	0.1752

Table 1. values in randomly generated, orthonormal, nxn matrix Q with n = 15

Question 2

B) The main diagonal of A = 4*J is all 0s, thus the determinant of A = 0, and so it is unsurprising that the eigenvalues of A are also 0. I used the 2-condition number for each of my condition number calculations, and in this case the condition number is ∞, which makes sense, since A is not invertible.

Once the matrix is perturbed by adding the matrices eQi to A the eigenvalues create a plot that forms an ellipse, as seen below, with a couple dozen outliers that do not conform to the pattern but are contained within the ellipse, as well as several values that lie along the x-axis, which represent the real-number values (as complex values are plotted on a 2D plane with x = the real part of the number and y = the imaginary part). It is interesting that both ends of the ellipse along the x-axis the points align quite clearly and seem to radiate outwards, then seem to lose that pattern and cluster along the outline of the ellipse as you approach x = 0. The graph also appears to be symmetrical across the x-axis.

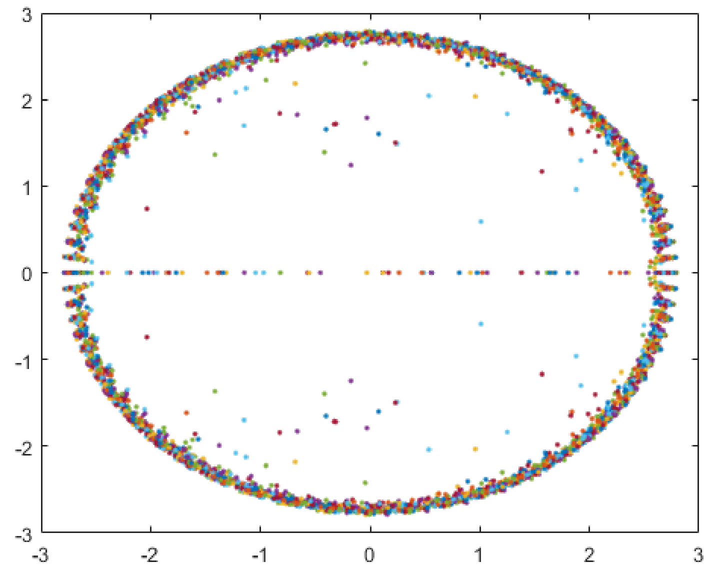

Figure 1. eigenvalues of 4J + eQi for i = 100 perturbation matrices

C) A = 4*J + 4*J^2 still has a main diagonal of 0, and also all eigenvalues are 0, and the 2-condition number is ∞. The eigenvalues of A + eQi are plotted below, and form the shape of a Limaçon of Pascal. Minus some outliers (those within the shape but do not contribute to the shape itself) and the real numbers lying along the x-axis the values that make up the outline of the shape are very concentrated on the left (negative) side of the graph, and seems to become less concentrated (moving from negative x values to approximately x = 4), then more organized into the radiating lines seen in the previous graph. Also similar to the first graph, this seems to be perfectly symmetrical across the x-axis, with each positive y value having a negative counterpart (especially easy to see with the matched outliers).

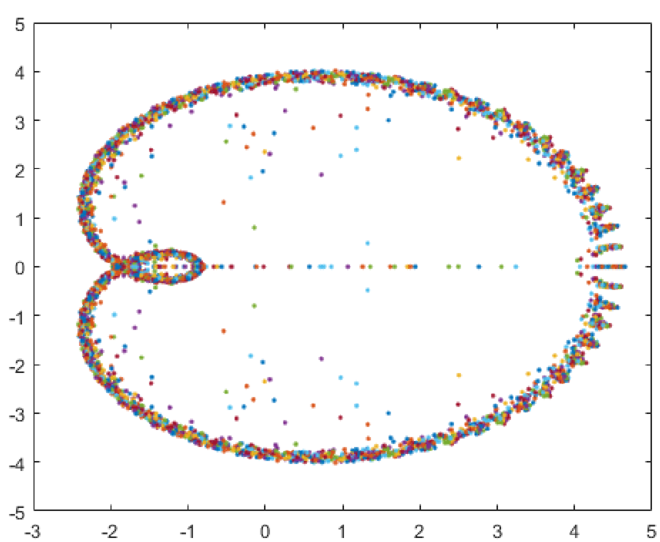

Figure 2. eigenvalues of 4*J + 4*J^2 + eQi for i = 100 perturbation matrices

Question 3

B) Instead of displaying all n eigenvalues here (since n = 50) here are the first 10 eigenvalues. The 2-condition number is ∞, which again makes sense here because there is at least one eigenvalue of 0 (0.000...0 + 0.00...00i = 0), thus the determinant of A is 0 and A is not invertible.

0.000000000000000 + 0.000000000000000i
-0.996210254835969 + 0.000000000000000i
-0.984898468017507 + 0.000000000000000i
-0.966236114702179 + 0.000000000000000i
-0.940506097142895 + 0.000000000000000i
-0.908098456178110 + 0.000000000000000i
-0.869504458610331 + 0.000000000000000i
-0.825309150102121 + 0.000000000000000i
-0.776182486480254 + 0.000000000000000i
-0.722869177888269 + 0.000000000000000i

Table 2. eigenvalues of A = L^-1 * U

As seen on the plot there are a few eigenvalues = 0 for the matrix A, seen at (0,0), many complex values, and many values that lie outside of the interval [0,1]. This is contrary to the assertion that a formula exists in which the exact eigenvalues of A are real and within the interval above. I believe that the discrepancy between my observations and this knowledge is due to computation errors from the calculation using command eig() in MATLAB, which does not calculate symbolic values, and instead results in complex values. These are represented as the vertically stretched ellipse centered at (0,0). Again, as seen in all figures thus far, this graph appears to be symmetric across the x-axis.

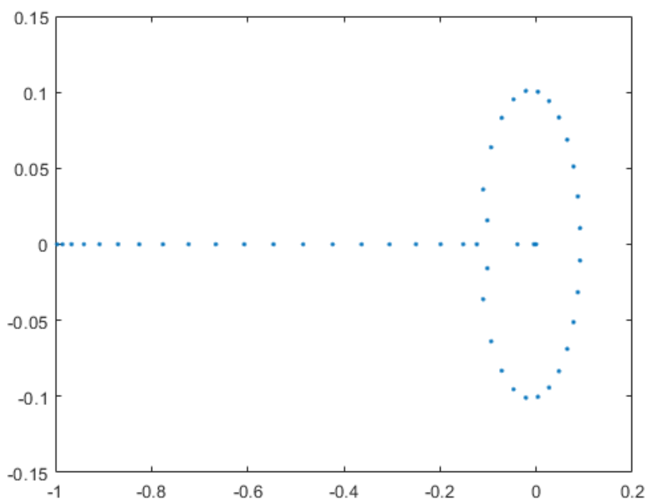

Figure 3. eigenvalues of A = L^-1 * U

C) The graph below, with all 100 perturbations, shows a less perfect ellipse than the unperturbed (above, in b); rather, it is skewed more towards negative x values, creating a teardrop shape. There appear to be many more values concentrated on the x-axis than previous graphs, indicating that there more real number eigenvalues here. As seen in previous graphs many values present as lines radiating from the x-axis on the left side of the ellipse, and are more concentrated at the opposite side along the lines of the shape. Continuing the trend, this graph also is symmetric across the x-axis.

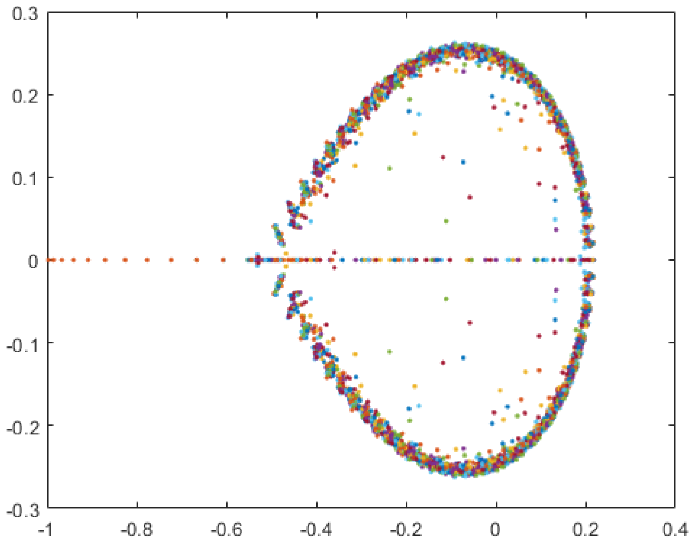

Figure 4. eigenvalues of L^-1 * U + eQi for i = 100 perturbation matrices

Question 4

C) The condition number of A, the companion matrix to the polynomial generated in parts a and b, is extremely high at **1.010603436684278e+21**, showing that A is ill-conditioned. To compare the eigenvalues of A with the roots r I calculated the absolute difference between them, to see if there were any that matched as they mathematically should. There was one; the 35th value calculated had a difference of 0.0000000029, which is less than our epsilon 10^-8 and thus could be reasonably considered to be a match. Otherwise, the difference varied between 3.5918367689 and 0.0043513108. Again, this is likely due to computation error from using eig(A) to calculate the eigenvalues.

The ill-condition of A may explain why the eigenvalues of the perturbation matrices are much noisier (less concentrated) than in the previous graphs:

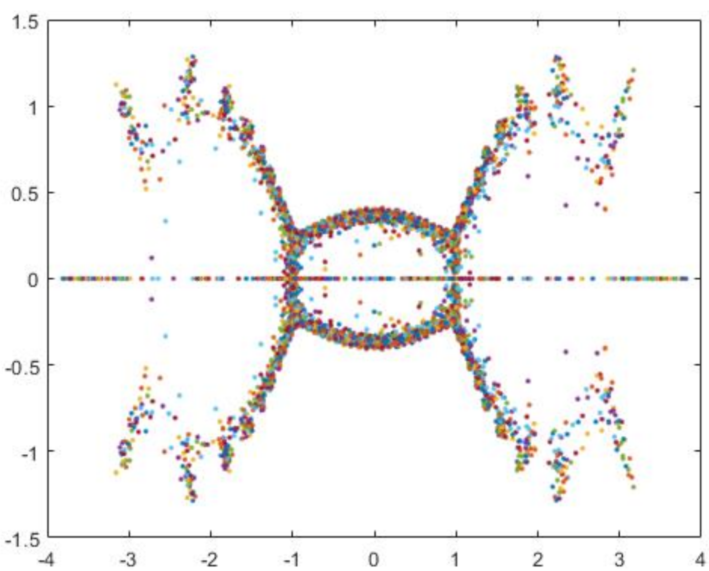

Figure 5. eigenvalues of A + eQi for i = 100 perturbation matrices

This graph has the most real numbers, with a nearly unbroken line across the x-axis. It appears to be symmetric across the x-axis as with the other graphs, and is almost symmetric across the y-axis – at least in the centrally clustered values that create the main shape of the graph.

D) The eigenvalues of B move quite a lot under the perturbations from the matrices, with the values for B + eQi fluctuating between positive and negative values, though not consistently. If one were to take the absolute value of all these eigenvalues the difference between those of B and each of the perturbed matrices appear to be fairly small. This can be seen in the below table, which has the eigenvalues of B, and examples of the 2 perturbed matrices using eQ1 and eQ6:

Q4d eigenvalues of B:	Q4d eigenvalues of B + eQ{1} vs eigenvalues of B + (e(Q{1} + transpose(Q{1}))):		Q4d eigenvalues of B + eQ{6} vs eigenvalues of B + (e(Q{6} + transpose(Q{6}))):	
2.000000000000004	-2.000000006705991	1.999999970855583	-2.000000025159863	-2.000000050319750
1.918367346938788	-1.918367377679508	-2.000000013411992	-1.918367381961474	1.999999936736614
1.836734693877561	1.999999985427775	1.918367356722518	-1.836734702386259	-1.918367416984169
1.755102040816324	-1.836734695403664	-1.918367408420266	1.999999968368299	1.918367360760245
1.673469387755101	1.918367351830642	1.836734667985545	-1.755102038639418	-1.836734710894986
-2.000000000000009	-1.755102067619586	1.755101998664950	-1.673469398031575	-1.755102036462520
1.591836734693876	1.836734680931532	-1.836734696929793	1.918367353849503	-1.673469408308068
1.510204081632652	-1.673469410900402	1.673469325626040	-1.591836764395566	1.836734646845321
1.428571428571430	1.755102019740630	-1.755102094422859	1.836734670361434	1.755101997388929
-1.918367346938773	-1.591836733109738	-1.673469434045712	1.755102019102616	-1.591836794097252
-1.836734693877546	-1.510204082886876	1.591836747624477	-1.510204078661820	1.673469387924211
1.346938775510203	-1.428571445974436	-1.591836731525622	1.673469387839650	-1.510204075690987
-1.755102040816330	1.673469356690571	1.510204083032879	-1.428571439991711	1.591836743344979
1.265306122448982	1.591836741159173	-1.510204084141117	1.591836739019424	-1.428571451412004
1.183673469387757	-1.346938790853739	1.428571411617630	-1.346938807818303	1.510204088940413
-1.673469387755099	1.510204082332768	1.346938726595197	-1.265306118379549	-1.346938840126424
-1.591836734693878	-1.265306140665848	-1.428571463377456	1.510204085286528	1.428571398499020
1.102040816326534	1.428571420094527	1.265306145144266	-1.183673478337753	1.346938752840625
1.020408163265307	1.346938751052699	-1.346938806197286	1.428571413535221	-1.265306114310121
0.938775510204085	-1.183673474579809	1.183673449707903	1.346938764175419	1.265306135932557
-1.510204081632656	1.265306133796617	-1.265306158882714	-1.102040815584538	-1.183673487287752
-1.428571428571432	1.183673459547821	1.102040828577957	1.265306129190762	-1.102040814842552
-1.346938775510199	-1.102040824395653	1.020408165124660	1.183673478485348	1.183673487582953
0.857142857142855	1.102040822452244	-1.183673479771857	-1.020408187418722	-1.020408211572141
-1.265306122448981	1.020408164194982	-1.102040832464783	1.102040825179744	-0.938775484321311
0.775510204081633	-1.020408194325700	-1.020408225386097	1.020408138013683	-0.857142892772077
0.693877551020410	0.938775480569696	-0.938775543509002	-0.938775497262691	1.102040834032962
-1.183673469387756	-0.938775526856539	-0.857142872703024	0.938775496213801	1.020408112762069
-1.102040816326529	-0.857142864922938	0.938775450935315	-0.857142874957461	-0.775510211275203
0.612244897959184	0.857142820806068	0.857142784469279	0.857142847459242	0.938775482223519
0.530612244897958	-0.775510207637659	0.775510150703818	-0.775510207678417	-0.693877625512217
-1.020408163265305	0.775510177392726	-0.775510211193703	0.775510167554366	0.857142837775639
-0.938775510204081	-0.693877575679542	-0.693877600338661	-0.693877588266308	0.775510131027095
0.448979591836735	-0.612244893353227	-0.612244888747270	-0.612244903240535	-0.612244908521886
0.367346938775510	-0.530612247432984	-0.530612249968014	-0.530612261371764	-0.530612277845572
0.285714285714285	0.693877551830256	0.693877552640107	0.693877538861375	0.693877526702347
0.204081632653061	0.612244894130975	0.612244890302770	0.612244881432849	0.612244864906523
0.122448979591837	-0.448979604971500	-0.448979618106279	-0.448979600032980	0.530612220730344
-0.857142857142855	-0.367346948060671	-0.367346957345814	-0.367346950503386	-0.448979608229235
-0.775510204081632	0.530612220951341	0.530612197004724	0.530612232814142	-0.367346962231279
-0.693877551020408	0.448979579871114	0.448979567905502	0.448979581941510	0.448979572046277
-0.612244897959183	-0.285714283105511	-0.285714280496740	-0.285714297576458	-0.285714309438613
0.040816326530612	-0.204081650004453	-0.204081667355853	-0.204081643730936	-0.204081654808812
-0.040816326530612	-0.122448992008050	-0.122449004424250	-0.122449006031244	-0.122449032470654
-0.122448979591837	-0.040816321686885	-0.040816316843165	-0.040816354576087	-0.040816382621558
-0.204081632653061	0.367346928443003	0.367346918110495	0.367346951831208	0.367346964886904
-0.530612244897959	0.040816322481295	0.040816318431972	0.040816311912333	0.040816297294053
-0.448979591836735	0.285714254072676	0.285714222431090	0.285714296382307	0.285714307050324
-0.285714285714286	0.122448980603372	0.122448981614916	0.122448978142836	0.122448976693845
-0.367346938775510	0.204081625214926	0.204081617776769	0.204081629569348	0.204081626485642

Table 3. Comparison of eigenvalues of B and 2 perturbations

In order to compare the eigenvalues of matrices A and B I calculated the absolute difference between them, which gave the following table:

```
Q4d difference between eigenvalues of A vs B
   0.004351310840840
   0.051088574802652
   0.087728921363230
   3.758597517862938
   3.677013711419995
   0.163265340275870
   3.330383730508096
   0.253120927556134
   0.244897627182633
   0.345619139855269
   2.973949623286357
   3.428572888219816
   0.244894108173611
   0.244904798376316
   0.350009368265665
   2.544077678871114
   3.020399389861732
   2.857151559226470
   0.163257980004094
   2.612250705815445
   0.081628184352015
   0.183245420985898
   2.131445737875485
   0.163268416150511
   2.204079835720335
   2.040817147148593
   1.820031942079481
   1.738410098825221
   1.877550730096704
   1.714285793555367
   0.000000016907342
   0.003946066686326
   0.000559753299725
   0.000000002941499
   0.000051708878712
   1.142857142395688
   1.142860694949466
   0.979591836809979
   0.081632479287567
   0.081632653074695
   0.816326535370016
   0.653061224491674
   0.653061224529833
   0.489795918367493
   0.081632653055601
   0.244897959183555
   0.408163265306195
   0.081632653061234
   0.244897959183673
   0.081632653061225
```

Table 4. Difference between the eigenvalues of A and B

There are only 2 eigenvalues in which the difference is close enough to 0 (where the difference is less than our epsilon) that they could be considered to be common between both A and B, otherwise they all differ. Once again, I believe that the reason these eigenvalues are not matching up as they mathematically should is because of the error that is inherent in using Matlab to calculate the values.

Part 2: Code

Assignment 4 Code

Note: only the last plot is displayed in the published code since all plots are generated in the same section. In order to print each plot individually I commented out all plots but the one I was printing.

```
%--------------------------------Question 1------------------------------
%a)
n = 15;
%random matrix with values in the interval [-1,1]
B = -1 + (1+1)*rand(n);
disp('Q1a B = ');
disp(B);

%b) display random orthonormal matrix Q with n ≥ 10.
[Q, R] = qr(B);
disp('Q1b Q = ');
disp(Q);
```

```
%c)
n = 50;
%e = epsilon = 10^-8
e = 0.0000001;
eQ = cell(10);

for i = 1:100
    A = -1 + (1+1)*rand(n);
    [Q, R] = qr(A);
    eQ{i} = e * Q;
end

%--------------------------------Question 2------------------------------
% a)
v = ones(1,n-1);
J = diag(v,1);

% b)
B = 4*J;
kB = cond(B,2);
eigen_b = eig(B);
disp('Q2b eigenvalues of A = 4*J:');
disp(eigen_b);
disp('Q2b kA:');
disp(kB);
for i = 1:100
    eQi = cell2mat(eQ(i));
    bPerturb = eig(B + eQi);
    plot(bPerturb, '.');
    hold on
end
hold off

% c)
C = (4*J + 4*J^2);
kC = cond(C,2);
eigen_c = eig(C);
disp('Q2c eigenvalues of A = 4*J + 4*J^2:');
disp(eigen_c);
disp('Q2c kA:');
disp(kC);
for i = 1:100
    eQi = cell2mat(eQ(i));
    cPerturb = eig(C + eQi);
    plot(cPerturb, '.');
    hold on
end
hold off
```

```
%--------------------------------Question 3------------------------------
% a)
s = ones(1,n);
s = s *(-2);
S = diag(s);
l = ones(1,n-1);
L = diag(l,-1);
%Matrices
L = S + L;
S = L + J;
U = J;

% b) D = A = L^(-1) * U -> code analyzer recommended L\U > inv(L)*U
D = L\U;
eigen_d = eig(D);
disp('Q3b eigenvalues of A = inv(L)*U:');
disp(eigen_d);
kD = cond(D,2);
disp('Q3b kA:');
disp(kD);
plot(eigen_d, '.');

% c)
for i = 1:100
    eQi = cell2mat(eQ(i));
    dPerturb = eig(D + eQi);
    plot(dPerturb, '.');
    hold on
end
hold off
```

```matlab
%---------------------------------Question 4-------------------------------
% a)
r = linspace(-2,2,n);
p = poly(r);

% b) F = A = companion matrix to p
F = compan(p);

% c)
eigen_f = eig(F);
disp('Q4c Compare roots r with eigenvalues of A');
disp(abs(r' - eigen_f));
kF = cond(F,2);
disp('Q4c kA:');
disp(kF);

for i = 1:100
    eQi = cell2mat(eQ(i));
    fPerturb = eig(F+ eQi);
    plot(fPerturb, '.');
    hold on
end
hold off

% d)
M = 2*rand(n) - 1;
[W,R] = qr(M);
%G = Matrix B as per question
G = W*diag(r)*W';
eigen_g = eig(G);
disp('Q4d eigenvalues of B:');
disp(eigen_g);
gPer_eig = cell(10);
g2Per_eig = cell(10);
%calculate eigenvalues using 10 perturbation matrices
for i = 1:10
    eQi = cell2mat(eQ(i));
    eQiT = eQi';
    gPerturb = eig(G + eQi);
    gPer_eig{i} = gPerturb;
    g2Perturb = eig(G + (eQi + eQiT));
    g2Per_eig{i} = g2Perturb;
    msg = ['Q4d eigenvalues of B + eQ{',num2str(i),'} vs eigenvalues of B + (e(Q{',num2str(i),'} + transpose(Q{',num2str(i),'}))):'];
    disp(msg);
    disp([gPerturb g2Perturb]);
    disp('Q4d difference:');
    disp(abs(eigen_g - gPerturb));
end

%compare eigenvalues of A and B
diff = abs(eigen_f - eigen_g);
disp('Q4d difference between eigenvalues of A vs B');
disp(diff);
```