Center for Statistics and the Social Sciences Math Camp 2019

Algebra, Functions, & Limits

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Day 1
Outline

- Math notation
- Order of operations
- Rules of exponents, logarithms
- Equation of a line
- Functions, domain, range, examples
- Function transformations
- Continuous and piecewise functions
- Limits

Notation

Real Numbers

- Any number that falls on the continuous line. Often represented by a, b, c, d
- Examples: 2, 3.234, 1/7, $\sqrt{5}$, π
- The set of real numbers is denoted by $\mathbb{R}$. Then $a \in \mathbb{R}$ means a is in the set of real numbers.

Integers

- Any whole number. Often represented by i, j, k, l
- Examples: ...,-3,-2,-1,0,1,2,3,...

Variables

- Can take on different values
- Often represented by x, y, z

Notation

Functions

- Often represented by f, g, h
- Examples: $f(x) = x^2 + 3$, $g(y) = 6y^2 - 2y$, $h(z) = z^3$

Summations

- Often represented by $\sum$ and summed over some integer
- Example:
$$\sum_{i=1}^{3}(i+1)^2 = (1+1)^2 + (2+1)^2 + (3+1)^2 = 2^2 + 3^2 + 4^2 = 29$$

Products

- Often represented by $\prod$ and multiplied over some integer
- Example: $\prod_{k=1}^{3}(y_k + 1)^2 = (y_1 + 1)^2 \times (y_2 + 1)^2 \times (y_3 + 1)^2$

Fractions

Multiplying & Dividing

Fractions are used to describe parts of numbers. They are comprised of two parts:

$$\frac{\texttt{numerator}}{\texttt{denominator}}$$

All numbers can be written as fractions. Examples: $\frac{2}{3}, \frac{16}{4}(=4), \frac{2}{4}=\frac{1}{2}, \frac{8}{1}(=8)$.

Multiplication: Multiply the numerators; multiply the denominators. Examples: $\frac{1}{2} \times \frac{3}{4} = \frac{1 \cdot 3}{2 \cdot 4} = \frac{3}{8}$

Division: Best to change it into a multiplication problem by multiplying the top fraction by the inverse of the bottom fraction. Examples: $\frac{1/2}{7/8} = \frac{1}{2} \times \frac{8}{7} = \frac{1 \cdot 8}{2 \cdot 7} = \frac{8}{14}$.

Simplify: $\frac{8}{14} = \frac{2 \cdot 4}{2 \cdot 7} = \frac{2}{2} \times \frac{4}{7} = 1 \times \frac{4}{7} = \frac{4}{7}$

Fractions

Adding & Subtracting

Adding and subtracting requires that fractions must have the same denominator. If not, we need to find a common denominator (a larger number that has both denominators as factors) and convert the fractions. Then add (or subtract) the two numerators.

Examples: $\frac{1}{7} + \frac{4}{7} = \frac{5}{7}$

$\frac{1}{3} + \frac{1}{4} = \frac{1}{3} \times \frac{4}{4} + \frac{1}{4} \times \frac{3}{3} = \frac{1 \cdot 4}{3 \cdot 4} + \frac{1 \cdot 3}{4 \cdot 3} = \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$

$\frac{17}{20} - \frac{3}{4} = \frac{17}{20} \times \frac{1}{1} - \frac{3}{4} \times \frac{5}{5} = \frac{17 \cdot 1}{20 \cdot 1} - \frac{3 \cdot 5}{4 \cdot 5} = \frac{17}{20} - \frac{15}{20} = \frac{2}{20} = \frac{1}{10}$

Exponents

a^n is 'a to the power of n'. a is multiplied by itself n times. Often a is called the base, n the exponent. Examples:

$$2^3 = 2 \cdot 2 \cdot 2 = 8$$

$$6^4 = 6 \cdot 6 \cdot 6 \cdot 6 = 1296$$

Exponents do not have to be whole numbers. They can be fractions or negative.
Examples:

$$4^{1/2} = \sqrt{4} = 2$$

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

Common Rules

- $a^1 = a$
- $a^k \cdot a^l = a^{k+l}$
- $(a^k)^l = a^{kl}$
- $(ab)^k = a^k \cdot b^k$
- $\left(\frac{a}{b}\right)^k = \left(\frac{a^k}{b^k}\right)$
- $a^{-k} = \frac{1}{a^k}$
- $\frac{a^k}{a^l} = a^{k-l}$
- $a^{1/2} = \sqrt{a}$
- $a^{1/k} = \sqrt[k]{a}$
- $a^0 = 1$

Logarithms

A logarithm is the power (x) required to raise a base (c) to a given number (a).

$$\log_c(a) = x \Rightarrow c^x = a$$

Examples:

- $2^3 = 8 \Rightarrow \log_2(8) = 3$
- $4^6 = 4096 \Rightarrow \log_4(4096) = 6$
- $9^{1/2} = 3 \Rightarrow \log_9(3) = \frac{1}{2}$

Logarithms

The three most common bases are 2, 10, and $e \approx 2.718$, the natural logarithm. It is often called Euler's number after Leonhard Euler.

Examples:

- $10^2 = 100 \Rightarrow \log_{10}(100) = 2$
- $2^3 = 8 \Rightarrow \log_2(8) = 3$
- $e^2 = 7.3891... \Rightarrow \log(7.3891) = 2$

The natural logarithm ($\log_e$) is the most common; used to model exponential growth (populations, etc). If no base is specified, i.e. $\log(a)$, most often the base is e. Sometimes written as $\ln(a)$.

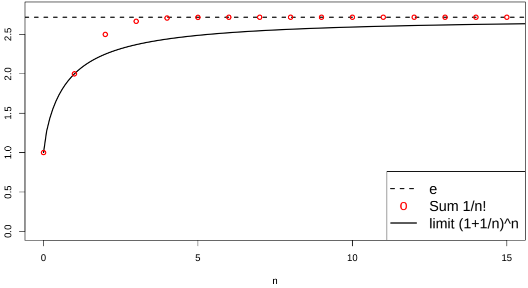
Logarithms

What is e?

The number e is a famous irrational number. The first few digits are $e = 2.718282...$

Two ways to express e:

- $lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$
- $\sum_{n=0}^{\infty}\frac{1}{n!}$



$$\log_c(a \cdot b) = \log_c(a) + \log_c(b)$$

$$\begin{aligned}
x &= \log_c(a \cdot b) \Longleftrightarrow c^x = a \cdot b \\
&\Rightarrow c^{x_1 + x_2} = a \cdot b \text{ where } x_1 + x_2 = x \\
&\Rightarrow c^{x_1} \cdot c^{x_2} = a \cdot b \Rightarrow c^{x_1} = a; c^{x_2} = b \\
&\Rightarrow x_1 = \log_c(a); x_2 = \log_c(b) \\
&\Rightarrow x = x_1 + x_2 \Rightarrow \log_c(a \cdot b) = \log_c(a) + \log_c(b)
\end{aligned}$$

Logarithms

Rules

$\log_c(a^n) = n \cdot \log_c(a)$
For $n = 2$:

$$
\begin{aligned}
x &= \log_c(a^2) \Longleftrightarrow c^x = a^2 \\
&\Rightarrow c^{x_1+x_2} = a \cdot a \text{ where } x_1 + x_2 = x \\
&\Rightarrow c^{x_1} \cdot c^{x_2} = a \cdot a \Rightarrow c^{x_1} = a; c^{x_2} = a \\
&\Rightarrow x_1 = \log_c(a); x_2 = \log_c(a) \\
&\Rightarrow x = x_1 + x_2 \Rightarrow \log_c(a^2) = \log_c(a) + \log_c(a) = 2 \cdot \log_c(a)
\end{aligned}
$$

Logarithms

Rules

$$\log_c\left(\tfrac{a}{b}\right) = \log_c(a) - \log_c(b)$$

$$\begin{aligned}
x &= \log_c\left(\frac{a}{b}\right) \iff c^x = \frac{a}{b} \\
&\Rightarrow c^{x_1+x_2} = \frac{a}{b} \text{ where } x_1 + x_2 = x \\
&\Rightarrow c^{x_1} \cdot c^{x_2} = \frac{a}{b} \Rightarrow c^{x_1} = a; c^{x_2} = \frac{1}{b} = b^{-1} \\
&\Rightarrow x_1 = \log_c(a); x_2 = (-1) \cdot \log_c(b) \\
&\Rightarrow x = x_1 + x_2 \Rightarrow \log_c\left(\frac{a}{b}\right) = \log_c(a) - \log_c(b)
\end{aligned}$$

Logarithms

Examples

- $\log_2(8 \cdot 4) = \log_2(8) + \log_2(4) = 3 + 2 = 5$
- $\log_{10}\left(\frac{1000}{10}\right) = \log_{10}(1000) - \log_{10}(10) = 3 - 1 = 2$
- $\log_4(6^4) = 4 \cdot \log_4(6)$
- $\log(x^3) = 3 \cdot \log(x)$

Order of Operations

Please **E**xcuse **M**y **D**ear **A**unt **S**ally

- **P**arentheses
- **E**xponents
- **M**ultiplication
- **D**ivision
- **A**ddition
- **S**ubtraction

Order of Operations

Examples

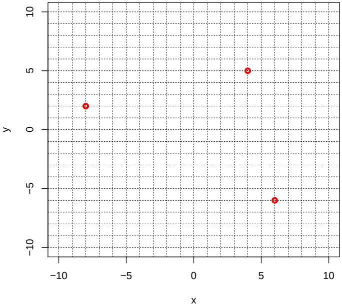
When looking at an expression, work from the left to right following **PEMDAS**. Note: multiplication and division are interchangeable; addition and subtraction are interchangeable.

- $\left((1+2)^3\right)^2 = (3^3)^2 = 27^2 = 729$
- $4^3 \cdot 3^2 - 10 + 27/3 = 64 \cdot 9 - 10 + 9 = 576 - 10 + 9 = 575$
- $(x+x)^2 - 2x + 3 = (2x)^2 - 2x + 3 = 4x^2 - 2x + 3$

Coordinate plane

- The collection of all points (x, y), such that $x \in (-\infty, \infty)$ and $y \in (-\infty, \infty)$.
- **Coordinates** (x, y) provide an “address” for a point in $\mathbb{R}^2$.
- The point (0,0)is where the x and y axes intersect and is called the **origin**.
- Other names: Cartesian plane, two-dimensional (2-D) space, $\mathbb{R}^2$

Examples: (-8,2),(4,5),(6,-6)



Equation of a Line

Linear Equations

If we have two pairs of points $(x_1, y_1), (x_2, y_2)$, we can find a line between the two points.

A common equation for a line is:

$$y = mx + b$$

where m is the **slope** and b is the **y-intercept**. A line is also a way to define a variable y in terms of another variable x.

Another common form (often used in the regression setting) is

$$y = \beta_0 + \beta_1 x$$

, where β_0 is the **y-intercept** and β_1 is the **slope**.

Slopes

The **slope** is the ratio of the difference in the y-values to the difference in the two x-values for any two points on a line. Commonly referred to as **rise** over **run**.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- m measures of the steepness of a line, e.g. how high does the line "rise" in "y-land" when we move one unit to the "right" (toward ∞) in "x"-land.
- The sign of m indicates whether we're going "uphill" $(+)$ or "downhill" (-) when we move to the "right" in "x"-land.

Intercepts

The **intercept**, often denoted b, is the value of y when $x = 0$.

- i.e. every line (that isn't a vertical line) has a point $(0, b)$.
- the vertical height where the line crosses the y-axis.

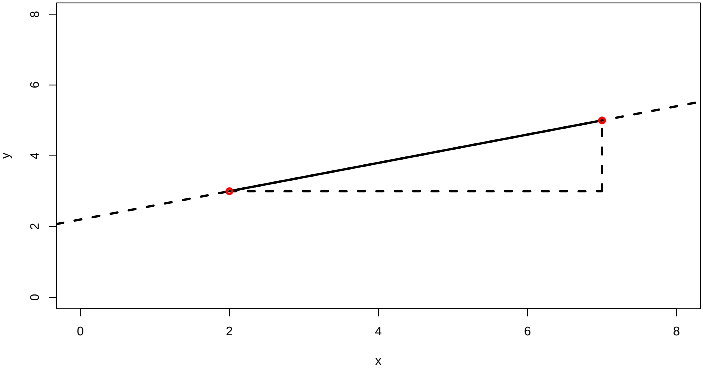
Find the intercept by plugging in one point on the line and the slope into the equation and then solving for the intercept.

$$y_1 = m \cdot x_1 + b \Rightarrow b = y_1 - m \cdot x_1$$

In a simple linear regression setting β_0 can be interpreted as the average value of a dependent variable, y, when the dependent variable x is equal to 0, *if* 0 is a observed or sensible value of your independent variable.

Find the equation of a line using two points

- **Points:** $(2,3),(7,5)$:
- **Slope:** $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{7-2} = \frac{2}{5}$
- **Intercept:** $b = y_1 - mx_1 = 3 - \frac{2}{5} \cdot 2 = 3 - \frac{4}{5} = 11/5$
- **Equation of the line:** $y = \frac{2}{5}x + \frac{11}{5}$



Functions and their Limits

A **function** is a formula or rule of correspondence that maps each element in a set X to an element in set Y.

The **domain** of a function is the set of all possible values that you can plug into the function. The **range** is the set of all possible values that the function $f(x)$ can return.

Examples:
$f(x) = x^2$

- **Domain:** all real numbers $\mathbb{R}$
- **Range:** zero and all positive real numbers, $f(x) \geq 0$

Functions and their Limits

Examples continued

$f(x) = \sqrt{x}$

- **Domain:** zero and all positive real numbers, $x \geq 0$
- **Range:** zero and all positive real numbers, $x \geq 0$

$f(x) = 1/x$

- **Domain:** all real numbers except zero
- **Range:** all real numbers except zero

Solving Linear Equations

Often we would like to find the **root** of a linear equation. This is the value of x that maps $f(x)$ to 0 (where the line crosses the x-axis, or the value of x when $y = 0$).

$$f(x) = mx + b$$

Setting $f(x) = 0$, to find the root we need to solve for x.

$$\begin{aligned} 0 =& mx + b && \text{[subtract } b \text{ from both sides]} \\ -b =& mx && \text{[divide both sides by } m\text{]} \\ \frac{-b}{m} =& x \end{aligned}$$

The value $-b/m$ is the **root** of $f(x) = mx + b$, i.e. most lines (except horizontal lines) have a point $(\frac{-b}{m}, 0)$ on them.

Solving Linear Equations

Why do we do operations on both sides?

On the previous slide, we subtracted b from both sides or added $-b$ to both sides. Why is that okay?

$$
\begin{aligned}
0 &= mx + b \\
\Rightarrow 0 &= mx + b + ({\color{red}b} - {\color{blue}b}) \\
\Rightarrow -{\color{red}b} + 0 &= mx + (b - {\color{blue}b}) \\
\Rightarrow -{\color{red}b} &= mx + 0 \\
\Rightarrow -{\color{red}b} &= mx
\end{aligned}
$$

The number zero is called the **additive identity**. For any number $a \in \mathbb{R}$,

$$a + 0 = a.$$

Solving Linear Equations

Why do we do operations on both sides?

Then, we divided both sides by m or multiplied both sides by $\frac{1}{m}$. Why is that okay?

$$\begin{aligned} -b =& mx \\ \Rightarrow -b =& mx \cdot \frac{1/m}{1/m} \\ \Rightarrow -b \cdot \frac{1}{m} =& mx \cdot \frac{1}{m} \\ \Rightarrow \frac{-b}{m} =& x \end{aligned}$$

The number one is called the **multiplicative identity**. For any number $a \in \mathbb{R}$,

$$a \times 1 = a.$$

Solving Linear Equations

Examples

We may be interested in solving linear equations for values other than zero.

Say you are at the Garage on Capitol Hill (pre-Covid) and you have \$40.00 with you. If shoes are \$7.00 and a lane is \$11.00/hr how long can you bowl?
Let's take x is hours and $f(x)$ total price.

$$f(x) = 7 + 11x$$

How long can you bowl?

$$\begin{aligned} 40 &= 11x + 7 \\ 40 - 7 &= 11x \\ 33 &= 11x \\ 33/11 &= 3 = x \end{aligned}$$

Solving Systems of Linear Equations

We often are interested in finding the **intersection** of two lines or the point (x, y) where two lines cross. This is called solving the system of linear equations.

Suppose we have two equations

$$y = 3 + 0.6x \; y = 8 - 0.8x$$

Since these lie on the same plane (i.e. x and y represent the same dimension in both equations), we now have three different ways to "call" y:

- Given name: y
- Nicknames: $3 + 0.6$, $8 - 0.8x$.

This means

$$3 + 0.6x = 8 - 0.8x.$$

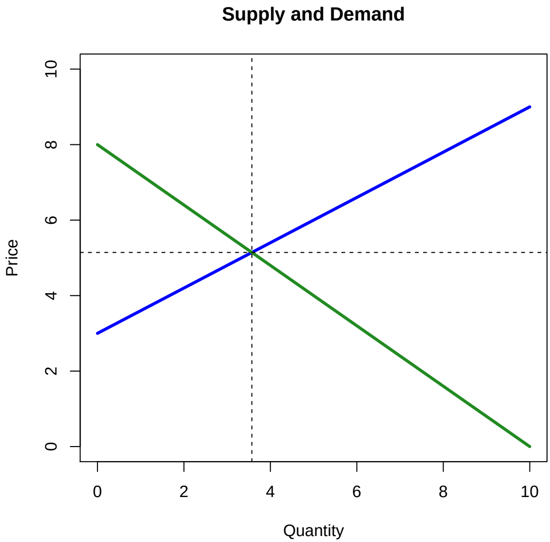
Solving Systems of Linear Equations

We use the fact that we have two different definitions of y to our advantage. Instead of two equations and two unknowns we now have one equation with one unknown!

$$\begin{aligned} 3 + 0.6x &= 8 - 0.8x \\ 3 - 3 + 0.6x + 0.8x &= 8 - 3 - 0.8x + 0.8x \\ 1.4x &= 5 \\ x &= 5/1.4 = 3.571429 \end{aligned}$$

The y-value is found by plugging the found value of x into either original equation: $y = 3 + 0.6(3.571429) = 5.142857$

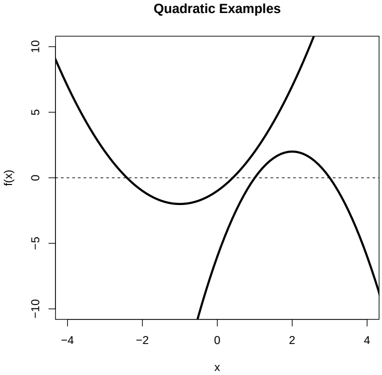
Solving Systems of Linear Equations



Quadratic Equations

Linear functions of x or lines, always take the form $f(x) = mx + b$, where the maximum power of x is 1.

A **quadratic** function has the form $f(x) = ax^2 + bx + c$, where the maximum power x is raised to is 2. Quadratic functions often take the shape of parabolas.



For any quadratic equation $f(x) = ax^2 + bx + c$, we find the **root(s)** (values of x such that $f(x) = 0$, or where the function crosses the x-axis) by using the **quadratic equation**:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \& \quad x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$b^2 - 4ac$ is called the **discriminant**. If the discriminant is

- positive, there will be two roots.
- zero, there will be one root.
- negative, there will be no real roots.

Many quadratic equations can be factored into a more simple form. For example:

$$2x^2 - 6x - 8 = (x - 4)(2x + 2)$$

To see that they are equivalent we can FOIL to multiply the two terms on the right hand side of the equation.

- **F**irst: $x \cdot 2x = 2x^2$
- **O**uter: $x \cdot 2 = 2x$
- **I**nner: $-4 \cdot 2x = -8x$
- **L**ast: $-4 \cdot 2 = -8$

Thus, $(x - 4)(2x + 2) = 2x^2 + 2x - 8x - 8 = 2x^2 - 6x - 8$

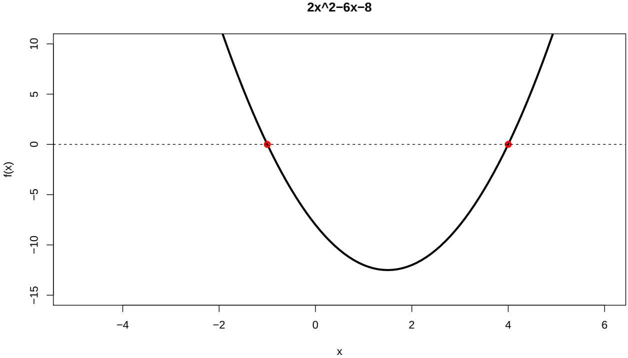
Quadratic Equations

Factoring and FOIL

When your quadratic has been factored you can find the roots by solving each term for zero. For example:

$$2x^2 - 6x - 8 = (x - 4)(2x + 2)$$

has roots when $x - 4 = 0$ and $2x + 2 = 0$. Thus, the roots are found at $x = -1, 4$.



Hunting for the FOIL factors can be tricky! Remember the quadratic equation always works!!

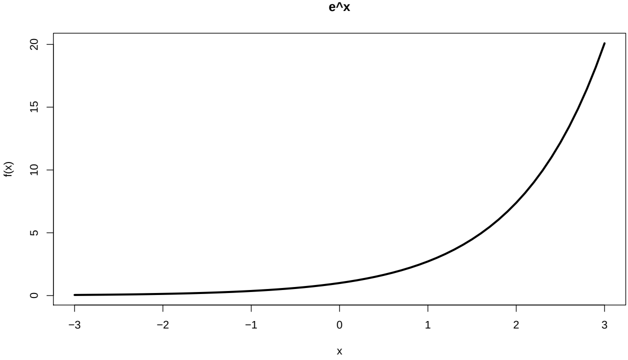
- If $b^2 - 4ac$ is a whole number, a fraction, a squared number, then it can be factored into something simple, if not use the quadratic formula.

Examples:

- $2x^2 + 4x - 16 \Rightarrow b^2 - 4ac = 4^2 - 4 \cdot 2 \cdot (-16) = 144$; 2 roots; factors
- $3x^2 - 2x + 9 \Rightarrow b^2 - 4ac = (-2)^2 - 4 \cdot 3 \cdot 9 = -104$; no real roots

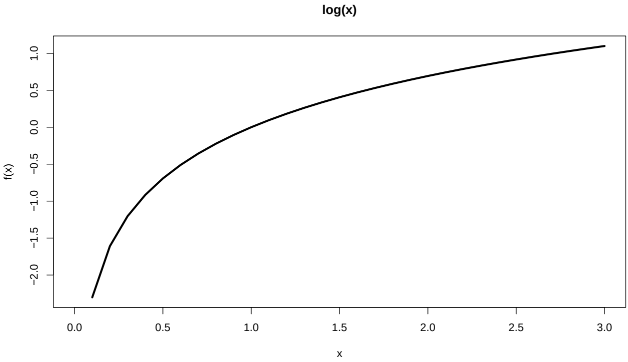
Exponential Functions

Exponential Functions are of the form $f(x) = ae^{bx}$. Often used as a model for population increase where $f(x)$ is the population at time x.



Logarithmic Functions

Logarithmic Functions, $f(x) = c + d \cdot log(x)$, can be used to find the time $f(x)$ necessary to reach a certain population x. It can be thought of as an 'inverse' of the exponential function.



Note: $c = -1/b \cdot log(a)$ and $d = 1/b$ from the previous exponential model.

Continuous & Piecewise Functions

A **continuous** function behaves without break or interruption. If you can follow the ENTIRE graph of a function with your pencil without picking it up, the function is continuous.Examples:

- $f(x) = x^2$
- $f(x) = x + 4$

A **piecewise** functioncan either have 'jumps' in it or can be made up of different functions for different parts of the domain (possible x-values). Example:

- Absolute Value $f(x) = |x|$ can be written as $f(x) = x, x \geq 0$ and $f(x) = -x, x < 0$

Limits

Often we are interested in what a function does as it approaches a certain value. This behavior is called the **limit**.

The limit of $f(x)$ as x approaches a is L:

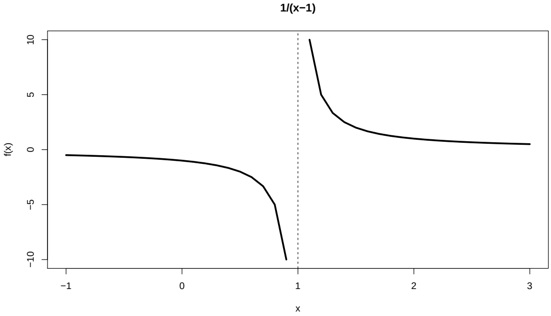
$$lim_{x \to a} f(x) = L$$

It may be that a is not in the domain of $f(x)$ but we can still find the limit by seeing what value $f(x)$ is approaching as x gets very close to a. Examples:

- $lim_{x \to 3} x^2 = 9$ (3 is in the domain)
- $lim_{x \to \infty} (1 + 1/x)^x = e$

Limits

Often limits are different depending on the direction from which you approach a. The limit 'from above' is approaching from the right ($x \downarrow a$) and the limit 'from below' ($x \uparrow a$)is approaching from the left.



If $f(x) = \frac{1}{x-1}$ we have $lim_{x \downarrow 1} \frac{1}{x-1} = \infty$ and $lim_{x \uparrow 1} \frac{1}{x-1} = -\infty$

The End

Questions?