Problem 1

The Poisson process with intensity λ is an example of CTMC.

- Find $P^{(t)}$;
- Compute the generator matrix G.

Solution:

Proof.
- The state space is $\mathcal{S} = \{0, 1, 2, \cdots\}$, and based on Poisson distribution,

$$P^{(t)} = \begin{bmatrix} e^{-\lambda t} & e^{-\lambda t}(\lambda t) & e^{-\lambda t}(\lambda t)^2/2 & e^{-\lambda t}(\lambda t)^3/6 & \dots \\ 0 & e^{-\lambda t} & e^{-\lambda t}(\lambda t) & e^{-\lambda t}(\lambda t)^2/2 & \dots \\ 0 & 0 & e^{-\lambda t} & e^{-\lambda t}(\lambda t) & \cdots \\ 0 & 0 & 0 & e^{-\lambda t} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

  and further by Taylor expansion,

$$e^{-\lambda t} = 1 - \lambda t + \frac{(\lambda t)^2}{2!} - \frac{(\lambda t)^3}{3!} + \cdots = (1 - \lambda t) + o(t),$$

  we can simplify this as

$$P^{(t)} = \begin{bmatrix} 1-\lambda t & \lambda t & 0 & 0 & \dots \\ 0 & 1-\lambda t & \lambda t & 0 & \dots \\ 0 & 0 & 1-\lambda t & \lambda t & \cdots \\ 0 & 0 & 0 & 1-\lambda t & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} + o(t).$$

- By definition,

$$G = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & \dots \\ 0 & -\lambda & \lambda & 0 & \dots \\ 0 & 0 & -\lambda & \lambda & \cdots \\ 0 & 0 & 0 & -\lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

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Problem 2

If $\{N(t)\}_{t \geq 0}$ is a Poisson process with $\lambda = 3$, compute the probability $\mathbb{P}(N(2) = 4, N(4) = 8)$.

Solution:

Proof. Based on independent increments,

$$\begin{aligned}
\mathbb{P}(N(2) = 4, N(4) = 8) &= \mathbb{P}(N(4) = 8 \mid N(2) = 4) \cdot \mathbb{P}(N(2) = 4) \\
&= \mathbb{P}(N(4) - N(2) = 4 \mid N(2) = 4) \cdot \mathbb{P}(N(2) = 4) \\
&= \mathbb{P}(N(4) - N(2) = 4) \cdot \mathbb{P}(N(2) = 4) \\
&= (\frac{6^4 e^{-6}}{4!})^2.
\end{aligned}$$

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Problem 3

Suppose that undergraduate students and graduate students arrive for office hours according to a Poisson process with rate $\lambda_1 = 5$ and $\lambda_2 = 3$ respectively. What is the expected time until the first student arrives?

Solution:

Proof. By Superposition, the total number of arrivals is a Poisson process with rate $5 + 3 = 8$. So by Poisson-Gamma relationship, the waiting time until the first student arrives follows exponential distribution with parameter $\lambda = 8$. Therefore, the expected time is $\frac{1}{8}$.

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Problem 4

Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion. Show that the followings are Brownian motions.

- $\{Y(t) = B(t + \alpha) - B(\alpha)\}_{t\geq 0}$ for all $\alpha \geq 0$;
- $\{Y(t) = \alpha B(t/\alpha^2)\}_{t\geq 0}$ for all $\alpha \geq 0$.

Solution:

Proof. Omitted.

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