Exercises for Module 5: Topology

1. Prove the following: If two metrics are strongly equivalent then they are equivalent.

Proof. Let X be a set and d_1, d_2 be two metrics on X. Suppose they are strongly equivalent, i.e. for every $x, y \in X$ $\exists\, \alpha, \beta > 0$ s.t.

$$\alpha d_1(x,y) \leq d_2(x,y) \leq \beta d_1(x,y).$$

Let f be the identity map from (X, d_1) to (X, d_2). We show it is continuous using $\varepsilon - \delta$ definition.

Let $\varepsilon > 0$ be arbitrary. Choose $\delta = \varepsilon/\beta$. Then if $d_1(x,y) < \delta = \varepsilon/\beta$, we have

by strong eq.

$d_2(f(x), f(y)) = d_2(x,y) \leq \beta d_1(x,y) < \beta\, \varepsilon/\beta = \varepsilon$, so f is cont.

Similarly, for the id. map from (X, d_2) to (X, d_1): let $\varepsilon > 0$. Choose $\delta = \alpha\varepsilon$. Then $d_1(x,y) \leq \frac{1}{\alpha} d_2(x,y) < \frac{1}{\alpha}\alpha\varepsilon = \varepsilon$, so it is continuous as well.

strong eq.

2. Let $(x_n)_{n\in N}$ be a sequence in $\mathbb{R}$. Show that $\lim_{n\to\infty} x_n = 0$ if and only if $\limsup_{n\to\infty} |x_n| = 0$.

($\Rightarrow$) Suppose $\lim_{n\to\infty} x_n = 0$. Then by the theorem in lecture, $\limsup_{n\to\infty} x_n = 0$. This implies $\limsup_{n\to\infty} |x_n| = 0$.

($\Leftarrow$) Suppose $\limsup_{n\to\infty} |x_n| = 0$.

Since $\limsup_{n\to\infty} |x_n| \geq \liminf_{n\to\infty} |x_n|$ and $|x_n| \geq 0\ \forall n \Rightarrow \liminf_{n\to\infty} |x_n| \geq 0$, we have

$$0 = \limsup_{n\to\infty} |x_n| \geq \liminf_{n\to\infty} |x_n| \geq 0$$

$$\Rightarrow \limsup_{n\to\infty} |x_n| = 0 = \liminf_{n\to\infty} |x_n|$$

$$\Rightarrow \lim_{n\to\infty} |x_n| = 0 \text{ by theorem from class}$$

$$\Rightarrow \lim_{n\to\infty} x_n = 0$$

Lemma $x_n \to 0 \iff |x_n| \to 0$

($\Rightarrow$) Suppose $x_n \to 0$. Then $\forall \varepsilon > 0\ \exists n_\varepsilon \in \mathbb{N}$ s.t. $|x_n - 0| < \varepsilon\ \forall n \geq n_\varepsilon$, i.e. $|x_n| < \varepsilon\ \forall n \geq n_\varepsilon$.
Let $\varepsilon > 0$ arbitrary. Choose $n = n_\varepsilon$, then $\forall n \geq n_\varepsilon$, $||x_n| - 0| = |x_n| < \varepsilon$ as required.

($\Leftarrow$) Similar.

3. Let $(X, \mathcal{T})$ be a topological space. Prove that $A \subseteq X$ is closed if and only if $\overline{A} = A$.

$(\Rightarrow)$ Suppose A is closed.

By definiton $A \subseteq \overline{A}$, so we only need to show $\overline{A} \subseteq A$.

But from class we have that $\overline{A} = \bigcap\{F : F \text{ closed}, A \subseteq F\}$.

Since A is closed, A is in this set, so $\overline{A} = \bigcap\{F : F \text{ closed}, A \subseteq F\} \subseteq A$.

$(\Leftarrow)$ Suppose $\overline{A} = A$. Then $A = \{x \in X : \forall U \in \mathcal{T} \text{ with } x \in U, U \cap A \neq \emptyset\}$.

$\Rightarrow \forall x \in A^c$, $\exists U_x \in \mathcal{T}$ with $x \in U_x$ s.t. $U_x \cap A = \emptyset$, i.e. $U_x \subseteq A^c$

Then $\bigcup_{x \in A^c} U_x \subseteq A^c$ and $A^c \subseteq \bigcup_{x \in A^c} U_x$, so $A^c = \bigcup_{x \in A^c} U_x$.

Since the union of open sets is open, A^c is open, so A is closed.

4. Let $(X, \mathcal{T})$ be a topological space and $\{A_i\}_{i \in I}$ be a collection of subsets of X. Show that

$$\bigcup_{i \in I} \overline{A_i} \subseteq \overline{\bigcup_{i \in I} A_i}.$$

Show that if the collection is finite, the two sets are equal.

First we show $\bigcup_{i \in I} \overline{A_i} \subseteq \overline{\bigcup_{i \in I} A_i}$:

Let $x \in \bigcup_{i \in I} \overline{A_i}$. Then $\exists i \in I$ s.t. $x \in \overline{A_i}$

$\Rightarrow \exists i \in I$ s.t. $\forall U \in \mathcal{T}$ with $x \in U$, $U \cap A_i \neq \emptyset$

$\Rightarrow \forall U \in \mathcal{T}$ with $x \in U$, $U \cap \left(\bigcup_{i \in I} A_i\right) \neq \emptyset$

$\Rightarrow x \in \overline{\bigcup_{i \in I} A_i}$

Next suppose we have a finite collection. We show $\overline{\bigcup_{i=1}^n A_i} \subseteq \bigcup_{i=1}^n \overline{A_i}$.

Note that $\bigcup_{i=1}^n \overline{A_i}$ is closed.

Then $\overline{\bigcup_{i=1}^n A_i} = \bigcap\{F : F \text{ is closed and } \bigcup_{i=1}^n A_i \subseteq F\} \subseteq \bigcup_{i=1}^n \overline{A_i}$

since it is in the set;

5. Let $(X, \mathcal{T})$ be a topological space and $\{A_i\}_{i \in I}$ be a collection of subsets of X. Prove that

$$\overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} \overline{A_i}.$$

Find a counterexample that shows that equality is not necessarily the case.

Since $A_i \subseteq \overline{A_i}$ $\forall i \in I$, $\bigcap_{i \in I} A_i \subseteq \bigcap_{i \in I} \overline{A_i}$, and $\bigcap_{i \in I} \overline{A_i}$ is closed.

Then $\overline{\bigcap_{i \in I} A_i} = \bigcap \{F : F \text{ closed}, \bigcap_{i \in I} A_i \subseteq F\} \subseteq \bigcap_{i \in I} \overline{A_i}$ since it is in the set.

Counterexample that shows $\bigcap_{i \in I} \overline{A_i}$ not necessarily $\overline{\bigcap_{i \in I} A_i}$.

Let $A_1 = [0,1)$, $A_2 = (1,2]$ and the topology be the one induced by $|\cdot|$ metric.

Then $\overline{A_1} = [0,1]$, $\overline{A_2} = [1,2]$, and $\bigcap_{i=1,2} \overline{A_i} = [0,1] \cap [1,2] = \{1\}$

but $\overline{\bigcap_{i=1,2} A_i} = \overline{[0,1) \cap (1,2]} = \overline{\emptyset} = \emptyset$.