

Module 4: Metric Spaces and Sequences II

Operational math bootcamp



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Outline

- Sequences
 - Cauchy sequences
 - subsequences

Sequences

Definition (Sequence)

Let (X, d) be a metric space. A *sequence* is an ordered list of points x_n , $n \in \mathbb{N}$, in X , denoted $(x_n)_{n \in \mathbb{N}}$. We say that a sequence $(x_n)_{n \in \mathbb{N}}$ *converges* to a point $x \in X$ if

Proposition

Let (X, d) be a metric space, and let $A \subseteq X$. Then \overline{A} is equal to the set of points in X which are limits of a sequence in A .

Proof.



Proof continued

Corollary

A set $F \subseteq X$, where (X, d) is a metric space, is closed if and only if every sequence in F which converges in X converges to a point in F .

Cluster points of a set

Definition

Let (X, d) be a metric space and $A \subseteq X$. A point $x \in X$ is a *cluster point* of A (also called accumulation point) if for every $\epsilon > 0$, $B_\epsilon(x)$ contains uncountably many points in A .

Proposition

$x \in X$ is a cluster point of $A \subseteq X$ where (X, d) is a metric space if and only if there exists a sequence of points $x_n \in A$, $n \in \mathbb{N}$, such that $x_n \rightarrow x$.

Proof.



Combining the previous result with the limit characterization of closure gives the following:

Corollary

For $A \subseteq X$, (X, d) a metric space, we have

$$\overline{A} = A \cup \{x \in X : x \text{ is a cluster point of } A\}.$$

Cauchy sequences

Definition (Cauchy sequence)

Let (X, d) be a metric space. A sequence denoted $(x_n)_{n \in \mathbb{N}} \in X$ is called a *Cauchy sequence* if

Proposition

Let (X, d) be a metric space, and let $(x_n)_{n \in \mathbb{N}}$ be a convergent sequence in X . Then $(x_n)_{n \in \mathbb{N}}$ is Cauchy.

Proof.



Definition

A metric space where every Cauchy sequence converges (to a point in the space) is called *complete*.

Proposition

Let (X, d) be a metric space, and let $Y \subseteq X$.

- (i) If X is complete and if Y is closed in X , then Y is complete.
- (ii) If Y is complete, then it is closed in X .

Proof.



Subsequences

Definition

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in a metric space (X, d) . Let $(n_k)_{k \in \mathbb{N}}$ be a sequence of natural numbers with $n_1 < n_2 < \dots$. The sequence $(x_{n_k})_{k \in \mathbb{N}}$ is called a *subsequence* of $(x_n)_{n \in \mathbb{N}}$. If $(x_{n_k})_{k \in \mathbb{N}}$ converges to $x \in X$, we call x a *subsequential limit*.

Example

$$((-1)^n)_{n \in \mathbb{N}}$$

Proposition

A sequence $(x_n)_{n \in \mathbb{N}}$ in a metric space (X, d) converges to $x \in X$ if and only if every subsequence of $(x_n)_{n \in \mathbb{N}}$ also converges to x .

Proof.



Proof continued

Continuity

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces, let $x_0 \in X$, and let $f : X \rightarrow Y$. f is *continuous* at x_0 if for every sequence $(x_n)_{n \in \mathbb{N}}$ in X that converges to x_0 , we have $\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$.

We say that f is continuous if it is continuous at every point in X .

Theorem

Let (X, d_X) and (Y, d_Y) be metric spaces, let $x_0 \in X$, and let $f : X \rightarrow Y$. The following are equivalent:

- (i) f is continuous at x_0
- (ii) for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \epsilon$ for all $x \in X$ with $d_X(x, x_0) < \delta$
- (iii) for each $\epsilon > 0$, there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$

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Proof.

(i) \Rightarrow (ii)



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- (iii) for each $\epsilon > 0$, there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$

Proof continued

(ii) \Rightarrow (iii)

(iii) \Rightarrow (i)

Corollary

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \rightarrow Y$. The following are equivalent:

- (i) f is continuous
- (ii) if $U \subseteq Y$ is open, then $f^{-1}(U)$ is open
- (iii) if $F \subseteq Y$ is closed, then $f^{-1}(F)$ is closed

We need the following results about sets and functions:

Let X and Y be sets and $f : X \rightarrow Y$. Let $A, B \subseteq Y$. Then

① $f^{-1}(A) \subseteq f^{-1}(B)$

② $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$

Proof.

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \rightarrow Y$.

(i) \Rightarrow (ii):



Proof continued

$$(ii) \Rightarrow (i)$$

$$(ii) \Rightarrow (iii)$$

$$(iii) \Rightarrow (ii)$$

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \rightarrow Y$.

- f is *uniformly continuous* if for all $\epsilon > 0$, there exists $\delta > 0$ such that for every $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$, we have $d_Y(f(x_1), f(x_2)) < \epsilon$
- f is *Lipschitz continuous* if there exists a $K > 0$ such that for every $x_1, x_2 \in X$ we have $d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2)$

Proposition

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \rightarrow Y$.

f is Lipschitz continuous $\Rightarrow f$ is uniformly continuous $\Rightarrow f$ is continuous

Contraction Mapping Theorem

Definition

Let (X, d) be a metric space and let $f : X \rightarrow X$. We say that $x^* \in X$ is a *fixed point* of f if $f(x^*) = x^*$.

Definition

Let (X, d) be a metric space and let $f : X \rightarrow X$. f is a *contraction* if there exists a constant $k \in [0, 1)$ such that for all $x, y \in X$, $d(f(x), f(y)) \leq kd(x, y)$.

Observe that a function is a contraction if and only if it is Lipschitz continuous with constant $K < 1$.

Theorem (Contraction Mapping Theorem)

Suppose that $f : X \rightarrow X$ is a contraction and the metric space X is complete. Then f has a unique fixed point x^ .*

Example

Let $f : [-\frac{1}{3}, \frac{1}{3}] \rightarrow [-\frac{1}{3}, \frac{1}{3}]$ be defined by the mapping $x \mapsto x^2$. Assume we use the standard Euclidean metric, $d(x, y) = |x - y|$. f has a unique fixed point because

Equivalent metrics

Definition (Equivalent metrics)

Two metrics d_1 and d_2 on a set X are *equivalent* if the identity maps from (X, d_1) to (X, d_2) and from (X, d_2) to (X, d_1) are continuous.

Proposition

Two metrics d_1, d_2 on a set X are equivalent if and only if they have the same open sets or the same closed sets.

Definition

Two metrics d_1 and d_2 on a set X are *strongly equivalent* if for every $x, y \in X$, there exists constants $\alpha > 0$ and $\beta > 0$ such

$$\alpha d_1(x, y) \leq d_2(x, y) \leq \beta d_1(x, y).$$

If two metrics are strongly equivalent then they are equivalent. The proof of this is one of the exercises.

Example

We show that the Euclidean distance (induced by 2-norm) and the metric induced by the ∞ -norm are equivalent on \mathbb{R}^n .

References

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