Exercises for Module 5: Topology

1. Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$. Prove that

$$f \text{ is Lipschitz continuous } \Rightarrow f \text{ is uniformly continuous } \Rightarrow f \text{ is continuous.}$$

Provide examples to show that the other directions do not hold.

2. Show that the function $f(x) = \frac{1}{2}\left(x + \frac{5}{x}\right)$ has a unique fixed point on $(0, \infty)$. What is it? (Hint: you will have to restrict the interval.)

3. Prove the following: If two metrics are strongly equivalent then they are equivalent.

4. Let $(x_n)_{n\in N}$ be a sequence in $\mathbb{R}$. Show that $\lim_{n\to\infty} x_n = 0$ if and only if $\limsup_{n\to\infty} |x_n| = 0$.

3. Let $(X, \mathcal{T})$ be a topological space. Prove that $A \subseteq X$ is closed if and only if $\overline{A} = A$.

4. Let $(X, \mathcal{T})$ be a topological space and $\{A_i\}_{i \in I}$ be a collection of subsets of X. Show that

$$\bigcup_{i \in I} \overline{A_i} \subseteq \overline{\bigcup_{i \in I} A_i}.$$

Show that if the collection is finite, the two sets are equal.

5. Let $(X, \mathcal{T})$ be a topological space and $\{A_i\}_{i \in I}$ be a collection of subsets of X. Prove that

$$\overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} \overline{A_i}.$$

Find a counterexample that shows that equality is not necessarily the case.