



UNIVERSITY OF  
TORONTO

# Statistical Sciences

## DoSS Summer Bootcamp Probability Module 8

Miaoshiqi (Shiki) Liu

University of Toronto

June 10, 2022

# Recap

Learnt in last module:

- Stochastic convergence
  - ▷ Convergence in distribution
  - ▷ Convergence in probability
  - ▷ Convergence almost surely
  - ▷ Convergence in  $L^p$
  - ▷ Relationship between convergences

# Outline

- Convergence of functions of random variables
  - ▷ Slutsky's theorem
  - ▷ Continuous mapping theorem
- Laws of large numbers
  - ▷ WLLN
  - ▷ SLLN
  - ▷ Glivenko-Cantelli theorem
- Central limit theorem
  - ▷ Delta method

# Convergence of functions of random variables

**Recall: Stochastic convergence** If  $X_n \rightarrow X$ ,  $Y_n \rightarrow Y$  in some sense, how is the limiting property of  $f(X_n, Y_n)$ ?

## Convergence of functions of random variables (a.s.)

Suppose the probability space is complete, if  $X_n \xrightarrow{a.s.} X$ ,  $Y_n \xrightarrow{a.s.} Y$ , then for any real numbers  $a, b$ ,

- $aX_n + bY_n \xrightarrow{a.s.} aX + bY$ ;
- $X_n Y_n \xrightarrow{a.s.} XY$ .

### Remark:

- Still require all the random variables to be defined on the same probability space

# Convergence of functions of random variables

## Convergence of functions of random variables (probability)

Suppose the probability space is complete, if  $X_n \xrightarrow{P} X$ ,  $Y_n \xrightarrow{P} Y$ , then for any real numbers  $a, b$ ,

- $aX_n + bY_n \xrightarrow{P} aX + bY$ ;
- $X_n Y_n \xrightarrow{P} XY$ .

### Remark:

- Still require all the random variables to be defined on the same probability space

# Convergence of functions of random variables

## Convergence of functions of random variables ( $L^p$ )

Suppose the probability space is complete, if  $X_n \xrightarrow{L^p} X$ ,  $Y_n \xrightarrow{L^p} Y$ , then for any real numbers  $a, b$ ,

- $aX_n + bY_n \xrightarrow{L^p} aX + bY$ ;

### Remark:

- Still require all the random variables to be defined on the same probability space

# Convergence of functions of random variables

**Remark:** Convergence in distribution is different.

## Slutsky's theorem

If  $X_n \xrightarrow{d} X$  and  $Y_n \xrightarrow{P} c$  ( $c$  is a constant), then

- $X_n + Y_n \xrightarrow{d} X + c$ ;
- $X_n Y_n \xrightarrow{d} cX$ ;
- $X_n / Y_n \xrightarrow{d} X / c$ , where  $c \neq 0$ .

**Remark:**

- The theorem remains valid if we replace all the convergence in distribution with convergence in probability.

# Convergence of functions of random variables

**Remark:** The requirement that  $Y_n \xrightarrow{P} c$  ( $c$  is a constant) is necessary.

## Examples:

$X_n \sim \mathcal{N}(0, 1)$ ,  $Y_n = -X_n$ , then

- $X_n \xrightarrow{d} Z \sim \mathcal{N}(0, 1)$ ,  $Y_n \xrightarrow{d} Z \sim \mathcal{N}(0, 1)$ ;
- $X_n + Y_n \xrightarrow{d} 0$ ;
- $X_n Y_n = -X_n^2 \xrightarrow{d} -\chi^2(1)$ ;
- $X_n / Y_n = -1$ .



# Convergence of functions of random variables

## Continuous mapping theorem

Let  $X_n, X$  be random variables, if  $g(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$  satisfies  $\mathbb{P}(X \in D_g) = 0$ , then

- $X_n \xrightarrow{a.s.} X \Rightarrow g(X_n) \xrightarrow{a.s.} g(X) ;$
- $X_n \xrightarrow{P} X \Rightarrow g(X_n) \xrightarrow{P} g(X) ;$
- $X_n \xrightarrow{d} X \Rightarrow g(X_n) \xrightarrow{d} g(X) ;$

where  $D_g$  is the set of discontinuity points of  $g(\cdot)$ .

### Remark:

- If  $g(\cdot)$  is continuous, then ...
- If  $X$  is a continuous random variable, and  $D_g$  only include countably many points, then ...

# Law of large numbers

## Weak Law of Large Numbers (WLLN)

If  $X_1, X_2, \dots, X_n$  are i.i.d. random variables,  $\mu = \mathbb{E}(|X_i|) < \infty$ , then

$$\bar{X} = \frac{\sum_{i=1}^n X_i}{n} \xrightarrow{P} \mu.$$

### Remark:

A more easy-to-prove version is the  $L^2$  weak law, where an additional assumption  $\text{Var}(X_i) < \infty$  is required.

### Sketch of the proof:

# Law of large numbers

## A generalization of the theorem: triangular array

### Triangular array

A triangular array of random variables is a collection  $\{X_{n,k}\}_{1 \leq k \leq n}$ .

$$X_{1,1}$$

$$X_{2,1}, X_{2,2}$$

$$X_{3,1}, X_{3,2}, X_{3,3}$$

$$\vdots$$

$$X_{n,1}, X_{n,2}, \dots, X_{n,n}$$

**Remark:** We can consider the limiting property of the row sum  $S_n$ .

# Law of Large Numbers

## $L^2$ weak law for triangular array

Suppose  $\{X_{n,k}\}$  is a triangular array,  $n = 1, 2, \dots$ ,  $k = 1, 2, \dots, n$ ,  $S_n = \sum_{k=1}^n X_{n,k}$ ,  $\mu_n = \mathbb{E}(S_n)$ , if  $\sigma_n^2/b_n^2 \rightarrow 0$ , where  $\sigma_n^2 = \text{Var}(S_n)$  and  $b_n$  is a sequence of positive real numbers, then

$$\frac{S_n - \mu_n}{b_n} \xrightarrow{P} 0.$$

### Remark:

The  $L^2$  weak law for i.i.d. random variables is a special case of that for triangular array.

# Law of large numbers

**Proof:**

**Remark:**

A more generalized version incorporates truncation, then the second-moment constraint is relieved.

# Law of large numbers

## Strong Law of Large Numbers (SLLN)

Let  $X_1, X_2, \dots$  be an i.i.d. sequence satisfying  $\mathbb{E}(X_i) = \mu$  and  $\mathbb{E}(|X_i|) < \infty$ , then  
$$S_n = \sum_{i=1}^n X_i \xrightarrow{\text{a.s.}} \mu.$$

**Remark:** The proof needs Borel-Cantelli lemma.

## Glivenko-Cantelli theorem

Let  $X_i, i = 1, \dots, n$  i.i.d. with distribution function  $F(\cdot)$ , and let  
$$F_n(x) = \frac{1}{n} \sum_{i=1}^n I(X_i \leq x),$$
 then as  $n \rightarrow \infty$ ,

$$\sup_{x \in \mathbb{R}} |F(x) - F_n(x)| \rightarrow 0.$$

# Law of large numbers

**Proof:**

# Central Limit Theorem

What is the limiting distribution of the sample mean?

## Classic CLT

Suppose  $X_1, \dots, X_n$  is a sequence of i.i.d. random variables with  $\mathbb{E}(X_i) = \mu$ ,  $\text{Var}(X_i) = \sigma^2 < \infty$ , then

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} \mathcal{N}(0, 1).$$

## Remark:

- The proof involves characteristic function.
- A more generalized CLT is referred to as “Lindeberg CLT”.



# Central Limit Theorem

## Example:

Suppose  $X_i \sim \text{Bernoulli}(p)$ , i.i.d., consider  $Z_n = \frac{\sum_{i=1}^n X_i - np}{\sqrt{np(1-p)}}$ , then by CLT,  $Z_n \sim \mathcal{N}(0, 1)$  asymptotically.

Several figures of PMF of  $Z_n$

# Problem Set

**Problem 1:** Prove that on a complete probability space, if  $X_n \xrightarrow{a.s.} X$ ,  $Y_n \xrightarrow{a.s.} Y$ , then  $X_n + Y_n \xrightarrow{a.s.} X + Y$ .

**Problem 2:** Prove that on a complete probability space, if  $X_n \xrightarrow{P} X$ ,  $Y_n \xrightarrow{P} Y$ , then  $X_n + Y_n \xrightarrow{P} X + Y$ .

**Problem 3:** A bank teller serves customers standing in the queue one by one. Suppose that the service time  $X_i$  for customer  $i$  has mean  $\mathbb{E}(X_i) = 2$  (minutes) and  $\text{Var}(X_i) = 1$ . We assume that service times for different bank customers are independent. Let  $Y$  be the total time the bank teller spends serving 50 customers. Find  $\mathbb{P}(90 < Y < 110)$ .