Problem 1

Let $0 < p < 1$, repeatedly flp a coin with head probability p. Let X_n be the number of heads on the first n flips.

- Verify that $\{X_n\}$ is a Markov chain, specify the state space, initial probability and transition probability;
- Draw a sketch of the transition graph;
- For $p = \frac{1}{4}$, compute $\mathbb{P}(X_0 = 0, X_1 = 1, X_2 = 1, X_3 = 2)$.

Solution:

Proof.

- $\mathcal{S} = \{0, 1, \cdots, n\}$, initial probability $\nu_0 = 1$ and $\nu_i = 0$, otherwise. The transition probability is

$$p_{ij} = \begin{cases} p, & j = i + 1 \\ 1 - p, & j = i \\ 0, & \text{otherwise} \end{cases}.$$

- Omitted.
- By Markov property,

$$\begin{aligned} & \mathbb{P}(X_0 = 0, X_1 = 1, X_2 = 1, X_3 = 2) \\ = & \nu_0 \times \mathbb{P}(X_1 = 1 \mid X_0 = 0) \times \mathbb{P}(X_2 = 1 \mid X_1 = 1) \times \mathbb{P}(X_3 = 2 \mid X_2 = 1) \\ = & p^2(1 - p) = \frac{3}{64}. \end{aligned}$$

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Problem 2

Suppose a fair six-sided die is repeatedly rolled at times $0, 1, \cdots$ Let $X_0 = 0$, and for $n \geq 1$ let X_n be the largest value that appears among all of the rolls up to time n.

- Verify that $\{X_n\}$ is a Markov chain, specify the state space, initial probability and transition probability;
- Compute two-step transitions $\{p_{35}^{(2)}\}$.

Solution:

Proof.

- $\mathcal{S} = \{0, 1, 2, \cdots, 6\}$, initial probability $\nu_0 = 1$ and $\nu_i = 0$, otherwise. The transition probability is

$$p_{ij} = \begin{cases} \frac{1}{6}, & j > i \\ \frac{i}{6}, & j = i \\ 0, & \text{otherwise} \end{cases},$$

for $i = 0, 1, \cdots, 6$.

- We have

$$p_{35}^{(2)} = \sum_i p_{3i}p_{i5} = p_{33}p_{35} + p_{34}p_{45} + p_{35}p_{55} = \frac{1}{4}.$$

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Problem 3

Let $\{X(t)\}_{t\geq 0}$ be a continuous-time Markov chain on the state space $\mathcal{S} = \{1,2,3\}$, suppose that as $t \to 0$, the transition probabilities are given by

$$P^{(t)} = \begin{pmatrix} 1-7t & 7t & 0 \\ 0 & 1-3t & 3t \\ t & 2t & 1-3t \end{pmatrix} + o(t),$$

Compute the generator matrix G.

Solution:

Proof. By definition,

$$G = \lim_{t\to 0} \frac{P^{(t)} - I}{t} = \begin{pmatrix} -7 & 7 & 0 \\ 0 & -3 & 3 \\ 1 & 2 & -3 \end{pmatrix}.$$

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