Mathematics Bootcamp

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Preface

These notes were prepared for the inaugaral Department of Statistical Sciences Graduate Student Bootcamp at the University of Toronto, which is to be held in July 2022.
References are provided for each section. All references are freely available online, though some may require a University of Toronto library log-in to access.

1 Review of proof techniques with examples from algebra and analysis

1.1 Axioms of a field

(A1) *Commutativity in addition:* $x + y = y + x$

(A2) *Commutativity in multiplication:* $x \times y = y \times x$

(B1) *Associativity in addition:* $x + (y + z) = (x + y) + z$

(B2) *Associativity in multiplication:* $x \times (y \times z) = (x \times y) \times z$

(C) *Distributivity:* $x \times (y + z) = x \times y + x \times z$

(D1) *Existence of a neutral element, addition:* There exists a number 0 such that $x + 0 = x$ for every x.

(D2) *Existence of a neutral element, multiplication:* There exists a number 1 such that $x \times 1 = x$ for every x.

(E1) *Existence of an inverse, addition:* For each number x, there exists a number $-x$ such that $x+(-x) = 0$.

(E2) *Existence of an inverse, multiplication:* For each number $x \neq 0$, there exists a number $1/x$ such that $x \times 1/x = 1$.

[**EK:** *This section to be worked on later*]

1.1.1 Exercises

1. For any $a, b \neq 0$, $1/(ab) = 1/a \times 1/b$
2. For $a > 0$, $1/(-a) = -1/a$.
3. For $a, b \neq 0$, $1/(a/b) = b/a$

2 Linear Algebra

2.1 Vector spaces

2.1.1 Axioms of a vector space

Let V be a set and let $\mathbb{F}$ be a field.

Definition 2.1. *We call V a* ***vector space*** *if the following hold:*
Addition:

(A) Commutativity in addition: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ *for all* $\mathbf{u}, \mathbf{v} \in V$

(B) Associativity in addition: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ *for all* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$

(C) Existence of a neutral element, addition: There exists a vector $\mathbf{0}$ *such that for any* $\mathbf{v} \in V$, $\mathbf{0} + \mathbf{v} = \mathbf{v}$

(D) Additive inverse: For every $\mathbf{v} \in V$*, there exists another vector, which we denote* $-\mathbf{v}$*, such that* $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

Multiplication by a scalar:

(E) Existence of a neutral element, multiplication: For any $\mathbf{v} \in V$, $1 \times \mathbf{v} = \mathbf{v}$

(F) Associativity in multiplication: Let $\alpha, \beta \in \mathbb{F}$*. For any* $\mathbf{v} \in V$, $(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$

Associativity:

(G) Let $\alpha \in \mathbb{F}, \mathbf{u}, \mathbf{v} \in V$. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \beta\mathbf{v}$.

(H) Let $\alpha, \beta \in \mathbb{F}, \mathbf{v} \in V$. $(\alpha+\beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$.

Elements of the vector space are called vectors.
Most often we will assume $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$.
Examples of vector spaces: $\mathbb{R}^n$. $\mathbb{C}^n$, $M_{m\times n}$ (matrices of size $m \times n$), $\mathbb{P}_n$ (polynomials of degree n, $p(x) = a_0 + a_1 x + \ldots + a_n x^n$).

Lemma 2.2. *For every $\mathbf{v} \in V$, we have $-\mathbf{v} = (-1) \times \mathbf{v}$.*

Proof. Our goal is to show that $(-1) \times \mathbf{v}$ is the additive inverse of $\mathbf{v}$. We show this as follows:

$$\mathbf{v} + (-1) \times \mathbf{v} = \mathbf{v} \times (1 + (-1)) = \mathbf{v} \times 0 = 0$$

The last step uses Exercise 2.8. [**EK:** *Do by hand in class*] □

2.1.2 Subspaces

Definition 2.3. *A subset U of V is called a* ***subspace*** *of of V if U is also a vector space (using the same addition and scalar multiplication as on V).*

Proposition 2.4. *A subset U of V is a subspace of V if and only if U satisfies the following three conditions:*

1. *$\mathbf{0} \in U$*
2. *Closed under addition: $u, w \in U$ implies $\mathbf{u} + \mathbf{v} \in U$*
3. *Closed under scalar multiplication: $\alpha \in \mathbb{F}$ and $u \in U$ implies $\alpha\mathbf{u} \in U$*

Proof. $\Rightarrow$ If U is a subspace of V, then U satisfies these 3 properties by Definition 2.1.
$\Leftarrow$ Suppose U satisfies the given 3 conditions. Then for any $\mathbf{v} \in U$, there must exist $-\mathbf{v} \in U$ by property 3, since $-\mathbf{v} = (-1) \times \mathbf{v}$ by Lemma 2.2 (property D). Property 1 assures property C. Properties 2 and 3, and the fact that $U \subset V$, assure the remaining properties hold.

□

This characterisation allows us to easily show that the intersection of subspaces is again a subspace.

Proposition 2.5. *Let V be a vector space and let $U_1, U_2 \subseteq V$ be subspaces. Then $U_1 \cap U_2$ is also a subspace of V.*

Proof. We use the characterization in Proposition 2.4. First, since $\mathbf{0} \in U_1$ and $\mathbf{0} \in U_2$, we have $\mathbf{0} \in U_1 \cap U_2$. Second, for $\mathbf{u}, \mathbf{v} \in U_1 \cap U_2$, since in particular $\mathbf{u}, \mathbf{v} \in U_1$ and $\mathbf{u}, \mathbf{v} \in U_2$ and U_1, U_2 are subspaces, $\mathbf{u} + \mathbf{v} \in U_1$ and $\mathbf{u} + \mathbf{v} \in U_2$. Thus, $\mathbf{u} + \mathbf{v} \in U_1 \cap U_2$. Similarly, one shows $\alpha\mathbf{u} \in U_1 \cap U_2$ for $\alpha \in \mathbb{F}$. □

On the contrary the union of two subspaces is not a subspace in general (see Exercise 2.12). However, the next definition introduces the smallest subspace containing the union.

Definition 2.6. *Suppose $U_1, ..., U_m$ are subsets of V. The sum of $U_1, ..., U_m$, denoted $U_1 + ... + U_m$, is the set of all possible sums of elements of $U_1, ..., U_m$. More precisely,*

$$U_1 + ... + U_m = \{\mathbf{u}_1 + ... + \mathbf{u}_m : \mathbf{u}_1 \in U_1, ..., \mathbf{u}_m \in U_m\}$$

Proposition 2.7. *Suppose $U_1, ..., U_m$ are subspaces of V. Then $U_1 + ... + U_m$ is the smallest subspace of V containing $U_1, ..., U_m$.*

2.1.3 Exercises

Exercise 2.8 (1.1.7 in [1]). Show that $0\mathbf{v} = \mathbf{0}$ for $\mathbf{v} \in V$.

Exercise 2.9 (1.B.1 in [2]). Show that $-(-v) = v$ for $\mathbf{v} \in V$.

Exercise 2.10 (1.B.2 in [2]). Suppose that $\alpha \in \mathbb{F}, \mathbf{v} \in V$, and $\alpha\mathbf{v} = 0$. Prove that $a = 0$ or $v = 0$.

Exercise 2.11 (1.B.4 in [2]). Why is the empty space not a vector space?

Exercise 2.12 (7.4.1 in [1]). Let U_1 and U_2 be subspaces of a vector space V. Prove that $U_1 \cup U_2$ is a subspace of V if and only if $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$.

2.2 Linear (in)dependence and bases

Definition 2.13. *A linear combination of vectors* $\mathbf{v}_1, ..., \mathbf{v}_n$ *of vectors in* V *is a vector of the form*

$$\alpha_1 \mathbf{v}_1 + ... + \alpha_n \mathbf{v}_n = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k$$

where $\alpha_1, ..., \alpha_m \in \mathbb{F}$.

Definition 2.14. *The set of all linear combinations of a list of vectors* $v_1, ..., v_m$ *in* V *is called the* ***span*** *of* $v_1, ..., v_m$*, denoted span*$\{\mathbf{v}_1, ..., \mathbf{v}_n\}$*. In other words,*

$$\mathit{span}\{\mathbf{v}_1, ..., \mathbf{v}_n\} = \{\alpha_1 \mathbf{v}_1 + ... + \alpha_m \mathbf{v}_n : \alpha_1, ..., \alpha_n \in \mathbb{F}\}$$

The span of the empty list is defined to be $\{\mathbf{0}\}$.

Definition 2.15. *A system of vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *is called a basis (for the vector space* V *) if any vector* $\mathbf{v} \in V$ *admits a unique representation as a linear combination*

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \ldots + \alpha_n \mathbf{v}_n = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k$$

In undergrad, you likely thought about this as: the equation $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \ldots + \alpha_n \mathbf{v}_n$, where the x_i are unknown, has a unique solution.
Example of bases:
For $\mathbb{R}^n$: $e_1 = (1, 0, \ldots, 0),\ e_2 = (0, 1, 0, \ldots, 0),\ \ldots,\ e_n = (0, \ldots, 0, 1)$
For $\mathbb{P}^n:\ 1, x, x^2, \ldots, x^n$

Definition 2.16. *The linear combination* $\alpha_1 \mathbf{v}_1 + ... + \alpha_n \mathbf{v}_n$ *is called trivial if* $\alpha_k = 0$ *for every* k.

Proposition 2.17. *A system of vectors* $\mathbf{v}_1, \ldots \mathbf{v}_n \in V$ *is a basis if and only if it is linearly independent and complete (generating).*

[**EK:** *Proof done by hand*]

2.3 Exercises

From Harvard: Exercise: Suppose v_1, v_2, v_3, v_4 (a) spans V and (b) is linearly independent. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also (a) spans V and (b) is linearly independent.

Exercise: Suppose $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, ..., v_m + w$ is linearly dependent, then $w \in \text{span}(v_1, ..., v_m)$.
Exercise: Suppose that $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Show that $v_1, ..., v_m, w$ is linearly independent if and only if

$$w \notin \text{span}(v_1, ..., v_m)$$

onExercises Exercise: Suppose v_1, v_2, v_3, v_4 (a) spans V and (b) is linearly independent. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also (a) spans V and (b) is linearly independent.
Exercise: Suppose $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, ..., v_m + w$ is linearly dependent, then $w \in \text{span}(v_1, ..., v_m)$.
Exercise: Suppose that $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Show that $v_1, ..., v_m, w$ is linearly independent if and only if

$$w \notin \text{span}(v_1, ..., v_m)$$

[**EK:** *Add a few from books*]

2.4 Linear transformations

Definition 2.18. *A **transformation** T from domain X to codomain Y is a rule that assigns an output $y = T(x) \in Y$ to each input $x \in X$*

Definition 2.19. *A transformation from a vector space U to a vector space V is **linear** if*

$$T(\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha T(\mathbf{u}) + \beta T(\mathbf{v}) \quad \textit{for any } \mathbf{u}, \mathbf{v} \in V,\ \alpha, \beta \in \mathbb{F}$$

Examples: differentiation, rotation of vectors, reflection of vectors
[**EK:** *transpose, adjoint*]

2.5 Determinants

2.6 Spectral theory

Let $T\colon V \to V$ be a linear map, where V is a vector space. We would like to describe the action of this linear map in a particularly nice way. For example, if there exists a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of V such that $T\mathbf{v}_i = \alpha_i \mathbf{v}_i$ where $\alpha_i \in \mathbb{F}$ for $i = 1, \ldots, n$, then T acts on this particular basis merely by scaling the basis vectors. If we look at the matrix of T with respect to this basis, T is a diagonal matrix with α_i in the diagonal. In this section, we will recall when such a basis and other nice descriptions of T exist.

Definition 2.20. *Let V be a vector space. Given a linear map $T\colon V \to V$ and $\alpha \in \mathbb{F}$, α is called an **eigenvalue** of T if there exists a non-zero vector $\mathbf{v} \in V \setminus \{\mathbf{0}\}$ such that $Tv = \alpha v$. We call such $\mathbf{v}$ an **eigenvector** of T with eigenvalue α.*

Note that $Tv = \alpha v$ can be rewritten as $(T - \alpha I)v = 0$. Thus, if α is an eigenvalue, the map $T - \alpha I$ is not invertible, since it must have non-trivial kernel. Using the known characterizations of invertability, this gives the following characterization for eigenvalues.

Theorem 2.21. *Let V be a vector space and $T\colon V \to V$ be a linear map. The following are equivalent*

1. *$\alpha \in \mathbb{F}$ is an eigenvalue of T,*
2. *$(T - \alpha I)\mathbf{x} = 0$ has a non-trivial solution,*
3. *$\det(T - \alpha I) = 0$.*

Note that the last part is to be understood as the determinant of a matrix description of T with respect to a basis of V. We call the polynomial $p_A(\alpha) = \det(T - \alpha I)$ characteristic polynomial of T. Hence, the eigenvalues of T are the zeros of the characteristic polynomial. Recall that basis transformations can be expressed as invertible matrices, thus this is well-defined, since the characteristic polynomial and therefore the eigenvalues are independent of the choice of basis. We will now examine when there exists a basis of eigenvectors.

2.7 Inner product spaces

2.8 Matrix decomposition

2.9 References

The following texts:
Linear Algebra Done Right [2]
Linear Algebra Done Wrong [1]

3 Set theory

4 Metric spaces

5 Topology

6 Functions on $\mathbb{R}$

7 Multivariable calculus

8 Functional analysis

9 Complex analysis and Fourier analysis

References

1. Treil S. Linear Algebra Done Wrong. 2017. Available from: https://www.math.brown.edu/streil/papers/LADW/LADW.html
2. Axler S. Linear Algebra Done Right. 3rd ed. Undergraduate Texts in Mathematics. Springer, 2015. Available from: https://link-springer-com.myaccess.library.utoronto.ca/book/10.1007/978-3-319-11080-6