Exercises for Module 6: Topology and Linear Algebra

1. Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$ be dense. Show that if $A \subseteq B \subseteq X$, then B is dense as well.

Since A is dense in X, $\overline{A} = X$.

We need to show $\overline{B} = X$. Since we must have $\overline{B} \subseteq X$, it is enough to show that $\overline{A} = X \subseteq \overline{B}$.

We show $\overline{A} \subseteq \overline{B}$.

Let $A \subseteq B$. Then for any F closed s.t. $B \subseteq F$, we have $A \subseteq B \subseteq F$.

Therefore $\overline{B}$ is a closed set that contains A

$\therefore \overline{A} \subseteq \overline{B}$.

2. Let $(X, \mathcal{T})$ be a Hausdorff topological space. Show that the singleton $\{x\}$ is closed for all $x \in X$. Hint: Show that the complement is open.

Proof. For each $y \in X$, $\exists\, U_y, V_y \in \mathcal{T}$ such that $y \in U_y$, $x \in V_y$ & $U_y \cap V_y = \emptyset$ (since X is Hausdorff).

Then $\bigcup_{y \neq x} U_y = X \setminus \{x\}$, so $\{x\}^c$ is open, and thus $\{x\}$ is closed.

3. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$ be topological spaces and let $f\colon X \to Y$, $g\colon Y \to Z$ be continuous. Show that $g \circ f\colon X \to Z$ is continuous as well.

Proof Let $U \in \mathcal{T}_Z$. We need to show $(f \circ g)^{-1}(U) \in \mathcal{T}_X$.

By definition, $(f \circ g)^{-1}(U) = g^{-1}(f^{-1}(U))$.

Since f is continuous, $f^{-1}(U) \in \mathcal{T}_Y$.

Since g is continuous, $g^{-1}(f^{-1}(U)) \in \mathcal{T}_X$.

4. Let (X, d) be a metric space and $K \subset X$ compact. Show that for all $\epsilon > 0$ there exists $\{x_1, x_2, \ldots, x_n\} \subseteq K$ such that for all $y \in K$ we have $d(y, x_i) < \epsilon$ for some $i = 1, \ldots, n$.

Let $\varepsilon > 0$. The set $\{B_\varepsilon(x)\}_{x \in K}$ is an open cover for K.

Since K is compact, there exists a finite subcover $B_\varepsilon(x_1), \ldots, B_\varepsilon(x_n)$ such that $K \subseteq \bigcup_{i=1}^{n} B_\varepsilon(x_i)$.

Thus for any $y \in K$ $\exists i \in \{1, \ldots, n\}$ such that $y \in B_\varepsilon(x_i)$, which is the required result.

5. Suppose that $\alpha \in \mathbb{F}, \mathbf{v} \in V$, and $\alpha\mathbf{v} = \mathbf{0}$. Prove that $a = 0$ or $v = 0$.

Suppose $\alpha \neq 0$.
Since $\alpha \in \mathbb{F}$, α has a multiplicative inverse, call it α^{-1}.
Then $\alpha\vec{v} = \vec{0} \Rightarrow \alpha^{-1}\alpha\vec{v} = \vec{0} \Rightarrow \vec{v} = \vec{0}$.

Otherwise, if $\alpha = 0$, then $\alpha\vec{v} = 0\vec{v} = \vec{0}$ by lemma from class.

6. Prove the following: Let V be a vector space and let $U_1, U_2 \subseteq V$ be subspaces. Then $U_1 \cap U_2$ is also a subspace of V.

We show that the 3 properties hold.
First, since U_1, U_2 are subspaces, $\vec{0} \in U_1$ & $\vec{0} \in U_2$. Therefore $\vec{0} \in U_1 \cap U_2$.

Second, if $\vec{u}_1, \vec{u}_2 \in U_1 \cap U_2$, then $\vec{u}_1, \vec{u}_2 \in U_1$ & $\vec{u}_1, \vec{u}_2 \in U_2$. Therefore $\vec{u}_1 + \vec{u}_2 \in U_1$ and $\vec{u}_1 + \vec{u}_2 \in U_2$, since U_1, U_2 are subspaces. $\therefore \vec{u}_1 + \vec{u}_2 \in U_1 \cap U_2$.

Finally, let $\alpha \in \mathbb{F}$, $\vec{u} \in U_1 \cap U_2$. Then $\vec{u} \in U_1$ and $\alpha\vec{u} \in U_1$ and similarly $\vec{u} \in U_2$ & $\alpha\vec{u} \in U_2$. $\therefore \alpha\vec{u} \in U_1 \cap U_2$

7. Let U_1 and U_2 be subspaces of a vector space V. Prove that $U_1 \cup U_2$ is a subspace of V if and only if $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$.

$\Rightarrow$ We prove the contrapositive. Suppose $U_1 \not\subseteq U_2$ and $U_2 \not\subseteq U_1$.

Choose $u_1 \in U_1$ s.t. $u_1 \notin U_2$ & $u_2 \in U_2$ s.t. $u_2 \notin U_1$.

Claim: $u_1 + u_2 \notin U_2$ and $u_1 + u_2 \notin U_1$

① Suppose $u_1 + u_2 \in U_2$. Then $u_1 + u_2 - u_1 \in U_2$, which implies $u_2 \in U_2$. $\Rightarrow\Leftarrow$

② Similar argument.

Since $u_1 + u_2 \notin U_1$ & $u_1 + u_2 \notin U_2$, $u_1 + u_2 \notin U_1 \cup U_2$.

Thus $U_1 \cup U_2$ is not a subspace.

$\Leftarrow$ Suppose $U_1 \subseteq U_2$. Then $U_1 \cup U_2 = U_2$ which is a subspace. Similarly, if $U_2 \subseteq U_1$, then $U_1 \cup U_2 = U_1$, which is a subspace.