Module 3: Linear Algebra I

Operational math bootcamp

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Outline

- Vector spaces and subspaces
- Linear combinations and bases
- Linear transformations

Definition

We call V a **vector space** if the following hold:

(A) *Commutativity in addition:* $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ for all $\mathbf{u},\mathbf{v}\in V$

(B) *Associativity in addition:* $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$ for all $\mathbf{u},\mathbf{v},\mathbf{w}\in V$

(C) *Existence of a neutral element, addition:* There exists a vector $\mathbf{0}$ such that for any $\mathbf{v}\in V$, $\mathbf{0}+\mathbf{v}=\mathbf{v}$

(D) *Additive inverse:* For every $\mathbf{v}\in V$, there exists another vector, which we denote $-\mathbf{v}$, such that $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$.

(E) *Existence of a neutral element, multiplication:* For any $\mathbf{v}\in V$, $1\times\mathbf{v}=\mathbf{v}$

(F) *Associativity in multiplication:* Let $\alpha,\beta\in\mathbb{F}$. For any $\mathbf{v}\in V$, $(\alpha\beta)\mathbf{v}=\alpha(\beta\mathbf{v})$

(G) Let $\alpha\in\mathbb{F},\mathbf{u},\mathbf{v}\in V$. $\alpha(\mathbf{u}+\mathbf{v})=\alpha\mathbf{u}+\beta\mathbf{v}$.

(H) Let $\alpha,\beta\in\mathbb{F},\mathbf{v}\in V$. $(\alpha+\beta)\mathbf{v}=\alpha\mathbf{v}+\beta\mathbf{v}$.

Definition

A subset U of V is called a **subspace** of of V if U is also a vector space (using the same addition and scalar multiplication as on V).

Proposition

A subset U of V is a subspace of V if and only if U satisfies the following three conditions:

1. $\mathbf{0} \in U$
2. Closed under addition: $u, w \in U$ implies $\mathbf{u} + \mathbf{v} \in U$
3. Closed under scalar multiplication: $\alpha \in \mathbb{F}$ and $u \in U$ implies $\alpha\mathbf{u} \in U$

Proof.

$\Rightarrow$ If U is a subspace of V, then U satisfies these 3 properties by the definition of a vector space.

$\Leftarrow$ Suppose U satisfies the given 3 conditions.

Then for any $\mathbf{v} \in U$, there must exist $-\mathbf{v} \in U$ by property 3, since $-\mathbf{v} = (-1) \times \mathbf{v}$ (exercise). Property 1 assures property C. Properties 2 and 3, and the fact that $U \subset V$, assure the remaining properties hold. $\square$

Linear combinations

Definition

A linear combination of vectors $\mathbf{v}_1, ..., \mathbf{v}_n$ of vectors in V is a vector of the form

$$\alpha_1 \mathbf{v}_1 + ... + \alpha_n \mathbf{v}_n = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k$$

where $\alpha_1, ..., \alpha_m \in \mathbb{F}$.

Span

Definition

The set of all linear combinations of a list of vectors $v_1, ..., v_m$ in V is called the **span** of $v_1, ..., v_m$, denoted $\text{span}\{\mathbf{v}_1, ..., \mathbf{v}_n\}$. In other words,

$$\text{span}\{\mathbf{v}_1, ..., \mathbf{v}_n\} = \{\alpha_1\mathbf{v}_1 + ... + \alpha_m\mathbf{v}_n : \alpha_1, ..., \alpha_n \in \mathbb{F}\}$$

The span of the empty list is defined to be $\{\mathbf{0}\}$.

We say a vector space is *finite dimensional* if it can be spanned by a finite list of vectors; otherwise it is *infinite dimensional*.

Linear independence

Definition

A list of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{F}^n$ is said to be *linearly independent* if

$$0 = \alpha_i \mathbf{v}_i + \ldots + \alpha_n \mathbf{v}_n,$$

where the α_i, $i = 1, \dots, n$ are scalars, admits only the solution $\alpha_1 = \cdots = \alpha_n = 0$.

Otherwise we say the vectors are *linearly dependent*.

Basis

Definition

A system of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is called a basis (for the vector space V) if any vector $\mathbf{v} \in V$ admits a unique representation as a linear combination

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \ldots + \alpha_n \mathbf{v}_n = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k$$

In undergrad, you likely thought about this as: the equation $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \ldots + \alpha_n \mathbf{v}_n$, where the α_i are unknown, has a unique solution.

Claim

All bases of a vector space V have the same length.

Proof.

Definition

The *dimension* of a vector space V, denoted $\dim V$, is the length of any basis of V.

Bases

Example of bases:
For $\mathbb{R}^n$: $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $e_n = (0, \ldots, 0, 1)$
For $\mathbb{P}^n$: $1, x, x^2, \ldots, x^n$

Definition

The linear combination $\alpha_1 \mathbf{v}_1 + ... + \alpha_n \mathbf{v}_n$ is called trivial if $\alpha_k = 0$ for every k.

proposition

A system of vectors $\mathbf{v}_1, \ldots \mathbf{v}_n \in V$ is a basis if and only if it is linearly independent and complete (generating).

Linear transformations

Definition

A **transformation** T from domain X to codomain Y is a rule that assigns an output $y = T(x) \in Y$ to each input $x \in X$

Definition

A transformation from a vector space U to a vector space V is **linear** if

$$T(\alpha \mathbf{u} + \beta \mathbf{v}) = \alpha T(\mathbf{u}) + \beta T(\mathbf{v}) \quad \text{for any } \mathbf{u}, \mathbf{v} \in V,\ \alpha, \beta \in \mathbb{F}$$

Examples

- Differentiation
- Integration
- Rotation of vectors
- Reflection of vectors

Linear transformations

Definition

Let $T : U \to V$ be a linear transformation. We define the following important subspaces:

- *Kernel* or *Null space*:

$$\operatorname{Ker} T = \{\mathbf{u} \in U : T\mathbf{u} = 0\}$$

- *Range*

$$\operatorname{Range} T = \{\mathbf{v} \in V : \exists \mathbf{u} \in U \text{ such that } \mathbf{v} = T\mathbf{u}\}$$

The dimensions of these spaces are often called the following:

- *Nullity*

$$\text{Nullity}(T) = \dim(\text{Ker}(T))$$

- *Rank*

$$\text{Rank}(T) = \dim(\text{Range}(T))$$

Linear transformations

Rank Theorem

For a matrix A or equivalently a linear transformation $A : \mathbb{F}^n \to \mathbb{F}^m$:

$$\text{Rank } A = \text{Rank } A^T$$

Rank Nullity Theorem

Let $T : U \to V$ be a linear transformation, where U and V are finite-dimensional vector spaces. Then

$$\text{Rank}\, T + \text{Nullity}\, T = \dim\ U.$$

Exercises

1. Let U and V be finite-dimensional vector spaces of the same dimension and let $T : U \to V$ be a linear map. Prove that the following are equivalent:
   - T is bijective
   - T is injective
   - T is surjective

References

Axler S. *Linear Algebra Done Right*. 3rd ed. Undergraduate Texts in Mathematics. Springer, 2015. Available from: https://link.springer.com/book/10.1007/978-3-319-11080-6

Treil S. *Linear Algebra Done Wrong*. 2017. Available from: https://www.math.brown.edu/streil/papers/LADW/LADW.html