Exercises for Module 2: Set Theory

1. Is $\mathbb{R} \times \mathbb{R}$ with the ordering $(x_1, y_1) \preceq (x_2, y_2)$ if $x_1 \leq x_2$ a partially ordered set?

No, it is not. Take $(x_1, y_1) = (2, -1)$ and $(x_2, y_2) = (2, 8)$.
Observe that $(2,-1) \preceq (2,8)$ and $(2,8) \preceq (2,-1)$, but $(2,-1) \neq (2,8)$.
The ordering is not anti-symmetric, so it is not a partial order.

2. Let S be a non-empty set. A relation R on S is called an equivalence relation if it is

   (i) Reflexive: $(x, x) \in R$ for all $x \in S$

   (ii) Symmetric: if $(x, y) \in R$ then $(y, x) \in R$ for all $x, y \in S$

   (iii) Transitive: if $(x, y), (y, z) \in R$ then $(x, z) \in R$ for all $x, y, z \in S$

Given $x \in S$ the equivalence class of x (with respect to a given equivalence relation R) is defined to consist of those $y \in S$ for which $(x, y) \in R$. Show that two equivalence classes are either disjoint or identical.

Proof.
Let $x_1, x_2 \in S$ such that $x_1 \neq x_2$. Let E_1 be the equivalence class of x_1 and E_2 be the equivalence class of x_2.
Any two sets are either disjoint or not disjoint. If E_1 and E_2 are disjoint, we are done. So we assume E_1 and E_2 are <u>not</u> disjoint. We must show that they are identical.

To show $E_1 = E_2$, we must show $E_1 \subseteq E_2$ & $E_2 \subseteq E_1$.
Note that since E_1 and E_2 are not disjoint, $\exists y \in S$ such that $y \in E_1$ and $y \in E_2$.
$E_1 \subseteq E_2$: Let $z \in E_1$. Then $(x, z) \in E_1$. Since $(x, y) \in E_1$, by symmetry and transitivity, $(y, z) \in R$. But $(x_2, y) \in R$ since $y \in E_2$.
Therefore by symmetry & transitivity, $(x_2, z) \in R$. Thus $z \in E_2$.
To show $E_2 \subseteq E_1$, repeat with the roles of x_1 & x_2 reversed.
Thus $E_1 = E_2$.

3. Let $(X, \leq)$ be a partially ordered set and $S \subseteq X$ be bounded. Show that the infimum and supremum of S are unique. (if they exist).

Proof. Let $(X, \leq)$ be a partially ordered set and $S \subseteq X$ bounded.
Suppose that S has 2 suprema; call them r_1 and r_2.
We have $r_1, r_2 \in X$ (not necessarily in S).
By the definition of supremum, since r_1 is the sup and r_2 and r_1 is another upper bound, $r_1 \leq r_2$.
But since r_2 is a sup & r_1 is another upper bound, we have $r_2 \leq r_1$.
Since $(X, \leq)$ is partially ordered, by anti-symmetry, we have that $r_1 = r_2$.
Thus if S has a sup, it is unique.

The proof for the inf is similar.

4. Let $S, T \subseteq \mathbb{R}$ and suppose both are bounded above. Define $S + T = \{s + t : s \in S, t \in T\}$. Show that $S + T$ is bounded above and $\sup(S + T) = \sup S + \sup T$.

Proof Since both $S, T \subseteq \mathbb{R}$ are bounded above, they both have a supremum.
Let $x = \sup S$ and $y = \sup Y$. By definition, $s \leq x \; \forall s \in S$ and $t \leq y \; \forall t \in T$.
Therefore $s + t \leq x + y \; \forall s \in S, \forall t \in T$, so $S + T$ is bounded above by $x + y$.
We claim that $\sup(S+T) = x + y$.
We use the characterization of sup from Prop 2.22. We have already shown that $x+y$ is an upper bound for $S+T$, so it remains to show that $\forall \varepsilon > 0 \; \exists s+t \in S+T$ s.t. $x + y - \varepsilon < s + t$.

Let $\varepsilon > 0$ be arbitrary.
Since $x = \sup S$, by Prop 2.22, $\exists s \in S$ s.t $x - \varepsilon/2 < s$. (1)
Similarly, since $y = \sup T$, $\exists t \in T$ s.t $y - \varepsilon/2 < t$. (2)
$\therefore \exists s \in S, \exists t \in T$ s.t. $x + y - \varepsilon < s + t$ (add (1) & (2))
Thus $\sup(S+T) = \sup(S) + \sup(T)$.

5. Let $f : X \to Y$ be defined by the map $x \mapsto \sin(x)$. For what choices of X and Y is f injective, surjective, bijective, or neither?

$X, Y \subseteq \mathbb{R}$

injective: $X = [0, 2\pi)$, $Y = \mathbb{R}$

surjective: $X = \mathbb{R}$, $Y = [-1, 1]$

bijective: $X = [0, 2\pi)$, $Y = [-1, 1]$

neither: $X = Y = \mathbb{R}$

(solution not unique)

6. Show that for sets $A, B \subseteq X$ and $f : X \to Y$, $f(A \cap B) \subseteq f(A) \cap f(B)$.

Proof Let $A, B \subseteq X$, $f : X \to Y$.
Let $y \in f(A \cap B)$. Then by definition, $\exists x \in A \cap B$ such that $f(x) = y$. Since $x \in A$, this means $y \in f(A)$ by definition. Since also $x \in B$, this means $y \in f(B)$.

$\therefore y \in f(A) \cap f(B)$

7. Let $f : X \to Y$ and $B \subseteq Y$. Prove that $f(f^{-1}(B)) \subseteq B$, with equality iff f is surjective.

<u>Proof.</u> Let $f: X \to Y$, $B \subseteq Y$.

First we show $f(f^{-1}(B)) \subseteq B$ for any $f: X \to Y$.

Let $y \in f(f^{-1}(B))$. Then $\exists x \in f^{-1}(B)$ s.t. $y = f(x)$.

Since $x \in f^{-1}(B)$, $f(x) \in B$. Thus $y = f(x) \in B$.

Next, suppose that f is surjective. We show $B \subseteq f(f^{-1}(B))$.

Let $y \in B$. Since f is surjective, $\exists x \in X$ s.t. $f(x) = y$.

Since $y \in B$, $x \in f^{-1}(B)$. Thus $y \in f(f^{-1}(B))$.

Finally, we show $f(f^{-1}(B)) = B \Rightarrow f$ is surjective.

We show the contrapositive:

Suppose $f: X \to Y$ is not surjective. Then $\exists y \in Y$ such that $\forall x \in X$, $f(x) \neq y$, i.e. $y \notin f(X)$. However, since $f^{-1}(Y) = X$, we have $y \notin f(f^{-1}(Y))$.

Thus $\exists B \subseteq Y$ (in this case Y itself), such that $Y \nsubseteq f(f^{-1}(Y))$.

8. Prove that $f(\cup_{i\in I} A_i) = \cup_{i\in I} f(A_i)$, $A_i \subseteq Y$ $\forall i \in I$, $f: X \to Y$

<u>Proof.</u>

Let $y \in f(\bigcup_{i\in I} A_i)$.

$\Leftrightarrow \exists x \in \bigcup_{i \in I} A_i$ s.t. $f(x) = y$

$\Leftrightarrow \exists i^* \in I$ s.t. $x \in A_{i^*}$ and $f(x) = y$

$\Leftrightarrow \exists i^* \in I$ s.t. $y \in f(A_{i^*})$

$\Leftrightarrow y \in \bigcup_{i \in I} f(A_i)$