Module 9: Linear Algebra III

Operational math bootcamp

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July 26, 2023

Outline

Spectral theory and matrix decompositions:

- Eigenvalues and eigenvectors
- Algebraic and geometric multiplicity of eigenvalues
- Jordan canonical form
- Singular value decomposition
- LU and QR decompositions

Orthogonality and Gram-Schmidt

Definition

Two vectors $\mathbf{x}, \mathbf{y} \in V$ are called *orthogonal* if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, denoted $\mathbf{x} \perp \mathbf{y}$. We call them *orthonormal* if additionally the vectors are normalized, i.e. $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$. A basis $\mathbf{x}_1, \ldots, \mathbf{x}_n$ of V is called *orthonormal basis (ONB)*, if the vectors are pairwise orthogonal and normalized.

Proposition

Let $\mathbf{x}_1, \ldots, \mathbf{x}_k \in V$ be orthonormal. Then the system of vectors is linearly independent.

Proof.

$\square$

Proposition (Orthogonal Decomposition)

Let $\mathbf{x}, \mathbf{y} \in V$ with $\mathbf{y} \neq 0$. Then, there exist $c \in F$ and $\mathbf{z} \in V$ such that $\mathbf{x} = c\mathbf{y} + \mathbf{z}$ with $\mathbf{y} \perp \mathbf{z}$.

Given a basis we can obtain an ONB from it using the Gram-Schmidt algorithm by reiterating the orthogonal decomposition from above.

Proposition (Gram-Schmidt Algorithm)

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in V$ be a system of linearly independent vectors. Define $\mathbf{y}_1 = \mathbf{x}_1 / \|\mathbf{x}_1\|$. For $i = 2, \ldots, n$ define $\mathbf{y}_j$ inductively by

$$\mathbf{y}_i = \frac{\mathbf{x}_i - \sum_{k=1}^{i-1} \langle \mathbf{x}_i, \mathbf{y}_k \rangle \mathbf{y}_k}{\|\mathbf{x}_i - \sum_{k=1}^{i-1} \langle \mathbf{x}_i, \mathbf{y}_k \rangle \mathbf{y}_k\|}.$$

Then the $\mathbf{y}_1, \ldots, \mathbf{y}_n$ are orthonormal and

$$\operatorname{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_n\} = \operatorname{span}\{\mathbf{y}_1, \ldots, \mathbf{y}_n\}.$$

The proof is omitted but can be found in the book.

Recall: connection between matrices and linear maps

Multiplication by a matrix defines a linear map

Let $A \in M_{m\times n}$ be a fixed matrix. Then, we can define a linear map $T_A\colon \mathbb{F}^n \to \mathbb{F}^m$ via $T_A(\mathbf{v}) = A\mathbf{v}$, where we recall matrix vector multiplication $(A\mathbf{v})_i = \sum_{k=1}^n A_{ik}v_k$ for $i = 1, \ldots, m$.

Given a bases for U and V, $T : U \to V$ can be written as a matrix

Let $T \in \mathcal{L}(U, V)$ where U and V are vector spaces. Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ and $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be bases for U and V respectively. The matrix of T with respect to these bases is the $m \times n$ matrix $\mathcal{M}(T)$ with entries A_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$ defined by

$$T\mathbf{u}_k = A_{1k}\mathbf{v}_1 + \cdots + A_{mk}\mathbf{v}_m.$$

Eigenvalues

Definition

Given an operator $A\colon V \to V$ and $\alpha \in \mathbb{F}$, λ is called an *eigenvalue* of A if there exists a non-zero vector $\mathbf{v} \in V \setminus \{\mathbf{0}\}$ such that

$$A\mathbf{v} = \lambda \mathbf{v}.$$

We call such $\mathbf{v}$ an *eigenvector* of A with eigenvalue λ. We call the set of all eigenvalues of A *spectrum* of T and denote it by $\sigma(T)$.

Motivation in terms of linear maps: Let $T\colon V \to V$ be a linear map, where V is a vector space. We would like to describe the action of this linear map in a particularly "nice" way: such that T acts only by scaling, i.e. $T\mathbf{v}_i = \lambda_i \mathbf{v}_i$ where $\lambda_i \in \mathbb{F}$ for $i = 1, \ldots, n$.

Finding eigenvalues

- Rewrite $A\mathbf{v} = \lambda\mathbf{v}$ as
- Thus, if λ is an eigenvalue, we can find the corresponding eigenvectors by finding the null space of $A - \lambda I$.
- The subspace null$(A - \lambda I)$ is called the *eigenspace*
- To find the eigenvalues of A, one must find the scalars λ such that null$(A - \lambda I)$ contains non-trivial vectors (i.e. not $\mathbf{0}$)
- Recall: We saw that $T \in \mathcal{L}(U, v)$ is injective if and only if null $T = \{\mathbf{0}\}$.
- Thus λ is an eigenvector if and only if $A - \lambda I$ is not invertible.
- Recall: $|A| \neq 0$ if and only if A is invertible.
- Thus λ is an eigenvector if and only if

Theorem

The following are equivalent

1. $\lambda \in \mathbb{F}$ *is an eigenvalue of* A,
2. $(A - \lambda I)\mathbf{v} = 0$ *has a non-trivial solution,*
3. $|A - \lambda I| = 0$.

Characteristic polynomial

Definition

If A is an $n \times n$ matrix, $p_A(\lambda) = |A - \lambda I|$ is a polynomial of degree n called the *characteristic polynomial* of A.

To find the eigenvectors of A, one needs to find the roots of the characteristic polynomial.

Example

Find the eigenvalues of

$$\begin{bmatrix} 4 & -2 \\ 5 & -3 \end{bmatrix}.$$

Multiplicity

Definition

The multiplicity of the root λ in the characteristic polynomial is called the *algebraic multiplicity* of the eigenvalue λ. The dimension of the eigenspace $\text{null}(A - \lambda I)$ is called the *geometric multiplicity* of the eigenvalue λ.

Definition (Similar matrices)

Square matrices A and B are called *similar* if there exists an invertible matrix S such that

$$A = SBS^{-1}.$$

Similar matrices have the same characteristic polynomials and hence the same eigenvalues (see exercise).

Theorem

Suppose A is a square matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be eigenvectors corresponding to these eigenvalues. Then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

Proof

Proof continued

Corollary

If a $A \in M_n(\mathbb{C})$ *has n distinct eigenvalues, then* A *is diagonalizable. That is there exists an invertible matrix* $S \in M_n(\mathbb{C})$ *such that* $A = SDS^{-1}$*, where* D *is a diagonal matrix with the eigenvalues of* A *in the diagonal.*

Theorem

Let $A : V \to V$ be an operator with n eigenvalues. A is diagonalizable if and only if for each eigenvalue λ, the geometric multiplicity of λ and the algebraic multiplicity of λ are the same.

Example: a diagonalizable matrix

$$\begin{bmatrix} 1 & 2 \\ 8 & 1 \end{bmatrix} \text{ is diagonalizable.}$$

.

Example continued

Example: a matrix that is not diagonalizable

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ is } \textit{not} \text{ diagonalizable.}$$

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Theorem

Let $A \in M_n(\mathbb{R})$ be a symmetric matrix. Then, there exists an orthogonal matrix $O \in M_n(\mathbb{R})$ such that $A = ODO^T$, where D is a diagonal matrix with the eigenvalues of A in the diagonal. Furthermore, all eigenvalues of A are real.

We can also state this for $M_n(\mathbb{C})$:
Let $A \in M_n(\mathbb{C})$ be a Hermitian matrix. Then, there exists a unitary matrix $U \in M_n(\mathbb{C})$ such that $A = UDU^*$, where D is a diagonal matrix with the eigenvalues of A in the diagonal. Furthermore, all eigenvalues of A are real.

Block matrices

Definition

A block matrix is a matrix that can be broken into sections called blocks, which are smaller matrices.

Example

$$\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

Definition

A square matrix is called *block diagonal* if it can be written as a block matrix where the main-diagonal blocks are all square matrices and the off-diagonal blocks are all zero.

Example

The matrix

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

is block diagonal.

Definition

A vector $\mathbf{v}$ is called a *generalized eigenvector* of A corresponding to an eigenvalue λ if there exists $k \geq 1$ such that

$$(A - \lambda I)^k \mathbf{v} = 0.$$

The set of generalized eigenvectors of an eigenvalue λ (plus $\mathbf{0}$) is called the *generalized eigenspace* of λ.

Proposition

The algebraic multiplicity of an eigenvalue λ is the same as the dimension of the corresponding generalized eigenspace.

Theorem (Jordan decomposition theorem)

For any operator A there exists a basis such that A is block diagonal with blocks that have eigenvalues on the diagonal and 1s on the upper off-diagonal. In other words, A can be written in the form

$$A = \begin{bmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_k \end{bmatrix}$$

where the blocks J_i on the main diagonal are Jordan block of the form

$$[\lambda], \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}, \textit{etc.}$$

This form is called Jordan canonical form.

Connection to algebraic and geometric multiplicity:

- The algebraic multiplicity of an eigenvalue λ is the number of times λ appears on the diagonal.
- The geometric multiplicity of λ is the number of Jordan blocks associated with λ.

Why is Jordan form useful?

Singular value decomposition

- $A^T A$ is symmetric
- therefore it is orthogonally diagonalizable and has real eigenvalues
- In fact, the eigenvalues are non-negative (exercise)

Definition

Let A be an $m \times n$ matrix. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A^T A$. Then the *singular values* of A are defined as

$$\sigma_1 = \sqrt{\lambda_1}, \ldots, \sigma_n = \sqrt{\lambda_n}.$$

Theorem (Singular value decomposition)

If A is an $m \times n$ matrix of rank k, then we can write

$$A = U\Sigma V^T$$

where Σ is an $m \times n$ matrix of the form

$$\begin{bmatrix} D_{k\times k} & 0_{k\times(n-k)} \\ 0_{(m-k)\times k} & 0_{(m-k)\times(n-k)} \end{bmatrix},$$

D is a diagonal matrix with the singular values of A, $\sigma_1, \ldots, \sigma_k$, on the diagonal and U and V are both orthogonal matrices (of size $m \times m$ and $n \times n$, respectively).

Uses of SVD:

Differences between JCF and SVD:

LU-decomposition

Definition

The LU-decomposition of a square matrix A is the factorization of A into a lower triangular matrix L and an upper triangular matrix U as follows:

$$A = LU.$$

Why is this useful? Consider the linear system $A\mathbf{x} = \mathbf{b}$

Recall: orthonormal basis

Definition

Two vectors $\mathbf{x}, \mathbf{y} \in V$ are called *orthogonal* if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, denoted $\mathbf{x} \perp \mathbf{y}$. We call them *orthonormal* if additionally the vectors are normalized, i.e. $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$. A basis $\mathbf{x}_1, \ldots, \mathbf{x}_n$ of V is called *orthonormal basis (ONB)*, if the vectors are pairwise orthogonal and normalized.

Starting from a set of linearly independent vectors, we can construct another set of vectors which are orthonormal and span the same space using the **Gram-Schmidt Algorithm**.

QR-decomposition

Definition (QR-decomposition)

The QR-decomposition of an $m \times n$ matrix A with linearly independent column vectors is the factorization of A as follows:

$$A = QR,$$

where Q is an $m \times n$ matrix with orthonormal column vectors and R is an $n \times n$ invertible upper triangular matrix.

One obtains the factorization by applying the Gram-Schmidt algorithm to the columns of A. Let $\mathbf{u}_1, \dots, \mathbf{u}_n$ be the column vectors of A. Let $\mathbf{q}_1, \dots, \mathbf{q}_n$ be the orthonormal vectors obtained by applying Gram Schmidt. Then one can write:

$$\begin{aligned}
\mathbf{u}_1 &= \langle \mathbf{u}_1, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{u}_2, \mathbf{q}_2 \rangle \mathbf{q}_2 + \ldots + \langle \mathbf{u}_n, \mathbf{q}_n \rangle \mathbf{q}_n \\
\mathbf{u}_2 &= \langle \mathbf{u}_2, \mathbf{q}_2 \rangle \mathbf{q}_2 + \ldots + \langle \mathbf{u}_n, \mathbf{q}_n \rangle \mathbf{q}_n \\
&\vdots \\
\mathbf{u}_n &= \langle \mathbf{u}_n, \mathbf{q}_n \rangle \mathbf{q}_n
\end{aligned}$$

Thus the orthonormal vectors obtained using Gram-Schmidt form the columns of Q, while R is the terms needed to go between the columns of A and thsoe of Q, i.e.

$$R = \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{q}_1 \rangle & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \ldots & \langle \mathbf{u}_n, \mathbf{q}_n \rangle \\ 0 & \langle \mathbf{u}_2, \mathbf{q}_2 \rangle & \ldots & \langle \mathbf{u}_n, \mathbf{q}_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \langle \mathbf{u}_n, \mathbf{q}_n \rangle \end{bmatrix}.$$

Why use QR-decomposition?

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