

# Module 5: Topology

## Operational math bootcamp



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# Last time

Finished our discussion of open and closed sets:

- Introduced a cluster points of a set:  
 $x \in X$  is a *cluster point* of  $A$  if for every  $\epsilon > 0$ ,  $B_\epsilon(x)$  contains **infinitely** many points in  $A$ .
- Sequence characterization of a closed set:  
A set  $F \subseteq X$ , where  $(X, d)$  is a metric space, is closed if and only if every sequence in  $F$  which converges in  $X$  converges to a point in  $F$ .

# Last time

Discussed sequences, which includes Cauchy sequences and subsequences:

- Convergent sequence:  $x_n \rightarrow x \Leftrightarrow \forall \epsilon > 0 \exists n_\epsilon \in \mathbb{N}$  s.t.  $d(x_n, x) < \epsilon$  for all  $n \geq n_\epsilon$
- Cauchy sequence:  $\forall \epsilon > 0 \exists n_\epsilon \in \mathbb{N}$  s.t.  $d(x_n, x_m) < \epsilon$  for all  $n, m \geq n_\epsilon$
- Proved that a convergent sequence is Cauchy
- Discussed complete metric spaces, where all Cauchy sequences converge (like  $\mathbb{R}$  with the usual metric, absolute value)
- Proved that a sequence converges to  $x$  if and only if all subsequences converge to  $x$

# Last time

Discussed continuous functions:

- Showed that these three definitions of continuous are equivalent metric spaces:  
 $f : X \rightarrow Y$  is *continuous* where  $(X, d_X)$  and  $(Y, d_Y)$  are metric spaces  $\Leftrightarrow$ 
  - if for every  $x_0 \in X$ , for every sequence  $(x_n)_{n \in \mathbb{N}}$  in  $X$  that converges to  $x_0$ , we have  $\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$ .
  - if for every  $x_0 \in X$ , for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $d_Y(f(x), f(x_0)) < \epsilon$  for all  $x \in X$  with  $d_X(x, x_0) < \delta$
  - if  $U \subseteq Y$  is open, then  $f^{-1}(U)$  is open
- Briefly discussed other types of continuity (uniform, Lipschitz) and the Contraction Mapping Theorem

# Outline for today

- Finish metric spaces
  - Equivalent metrics
  - A few extra topics on  $\mathbb{R}$ , including  $\limsup$  and  $\liminf$
- Start topology
  - Basic definitions



# Equivalent metrics

## Definition (Equivalent metrics)

Two metrics  $d_1$  and  $d_2$  on a set  $X$  are *equivalent* if the identity maps from  $(X, d_1)$  to  $(X, d_2)$  and from  $(X, d_2)$  to  $(X, d_1)$  are continuous.

## Proposition

Two metrics  $d_1, d_2$  on a set  $X$  are equivalent if and only if they have the same open sets or the same closed sets.

## Definition

Two metrics  $d_1$  and  $d_2$  on a set  $X$  are *strongly equivalent* if for every  $x, y \in X$ , there exists constants  $\alpha > 0$  and  $\beta > 0$  such

$$\alpha d_1(x, y) \leq d_2(x, y) \leq \beta d_1(x, y).$$

If two metrics are strongly equivalent then they are equivalent. The proof of this is one of the exercises.

## Example

We show that the Euclidean distance (induced by 2-norm) and the metric induced by the  $\infty$ -norm are equivalent on  $\mathbb{R}^n$ .

Can you think of an example that we've seen of a metric that isn't equivalent to the Euclidean metric?





# Right and left continuous

Recall:  $f: \mathbb{R} \rightarrow \mathbb{R}$  is continuous if at  $x_0 \in \mathbb{R}$  if for all  $\epsilon > 0$  there exists a  $\delta > 0$  such that  $|x_0 - y| < \delta$  implies  $|f(x_0) - f(y)| < \epsilon$ .

## Definition

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$ .

- $f$  is *left continuous* at  $x_0 \in \mathbb{R}$  if for all  $\epsilon > 0$  there exists a  $\delta > 0$ , such  $|f(x_0) - f(x)| < \epsilon$  whenever  $x_0 - \delta < x < x_0$ .
- $f$  is *right continuous* at  $x_0 \in \mathbb{R}$  if for all  $\epsilon > 0$  there exists a  $\delta > 0$ , such  $|f(x_0) - f(x)| < \epsilon$  whenever  $x_0 < x < x_0 + \delta$ .

We say that  $f$  is left continuous if is left continuous at all points in the domain, and similar for right continuous.

## Proposition

A function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is continuous if and only if it is left and right continuous.

Proof.



## Proof continued

# Bounded sequences and monotone convergence

## Definition

Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$ . We call  $(x_n)_{n \in \mathbb{N}}$  *bounded* if there exists an  $M > 0$  such that  $|x_n| < M$  for all  $n \in \mathbb{N}$ .

## Theorem (Monotone convergence theorem)

- (i) Suppose  $(x_n)_{n \in \mathbb{N}}$  is an increasing sequence, i.e.  $x_n \leq x_{n+1}$  for all  $n \in \mathbb{N}$  and that it is bounded above. Then the sequence converges. Furthermore,  $\lim_{n \rightarrow \infty} x_n = \sup_{n \in \mathbb{N}} x_n$ , where  $\sup_{n \in \mathbb{N}} x_n := \sup\{x_n : n \in \mathbb{N}\}$ .
- (ii)  $(x_n)_{n \in \mathbb{N}}$  is a decreasing sequence, i.e.  $x_n \geq x_{n+1}$  for all  $n \in \mathbb{N}$ , which is bounded below. Then the sequence converges and  $\lim_{n \rightarrow \infty} x_n = \inf_{n \in \mathbb{N}} x_n := \inf\{x_n : n \in \mathbb{N}\}$ .

Convention:  $\sup A = \infty$  if  $A \subseteq \mathbb{R}$  is not bounded above and  $\inf A = -\infty$  if  $A$  is not bounded below.

### Lemma

*If  $A \subseteq B \subseteq \mathbb{R}$  is non-empty, then  $\inf A \leq \sup A$ ,  $\sup A \leq \sup B$ , and  $\inf A \geq \inf B$ .*

The proof of this follows from the definition of greatest lower and least upper bound.

## Definition

Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$ . We define the *limit superior* of  $(x_n)_{n \in \mathbb{N}}$  as

$$\limsup_{n \rightarrow \infty} x_n := \lim_{n \rightarrow \infty} \sup_{k \geq n} x_k.$$

Similarly we define the *limit inferior* of  $(x_n)_{n \in \mathbb{N}}$  as

$$\liminf_{n \rightarrow \infty} x_n := \lim_{n \rightarrow \infty} \inf_{k \geq n} x_k.$$

If the sequence  $(x_n)_{n \in \mathbb{N}}$  is not bounded above, then  $\limsup_{n \rightarrow \infty} x_n = \infty$ . Similarly, if the sequence  $(x_n)_{n \in \mathbb{N}}$  is not bounded below, then  $\liminf_{n \rightarrow \infty} x_n = -\infty$ .

## Proposition

Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$ .

- The sequence of suprema,  $s_n = \sup_{k \geq n} x_k$ , is decreasing and the sequence of infima,  $i_n = \inf_{k \geq n} x_k$ , is increasing.
- The limit superior and the limit inferior of a bounded sequence always exist and are finite.

## Proof.

The first part is true by Lemma 3.45. The second the follows by the Monotone Convergence Theorem. □

## Theorem

Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$ . Then the sequence converges to  $x \in \mathbb{R}$  if and only if  $\limsup_{n \rightarrow \infty} x_n = x = \liminf_{n \rightarrow \infty} x_n$ .

## Proof.





## Proof continued

We can extend this easily to a sequence of functions  $f_n: X \rightarrow \mathbb{R}$  as follows:

Define  $f = \limsup_{n \rightarrow \infty} f_n$  to be the function defined pointwise by  $f(x) = \limsup_{n \rightarrow \infty} (f_n(x))$  and similar for the limit inferior.

There also exists a set theoretic version in terms of unions and intersections which you will encounter in probability.

# Topology

- Let  $X$  be a set. If  $X$  is not a metric space, can we still have open and closed sets?
- One can think of a topology on  $X$  as a specification what subsets of  $X$  are open

## Definition

Let  $\mathcal{T} \subseteq \mathcal{P}(X)$ . We call  $\mathcal{T}$  a *topology* on  $X$  if the following holds:

- (i)  $\emptyset, X \in \mathcal{T}$
- (ii) Let  $A$  be an arbitrary index set. If  $U_\alpha \in \mathcal{T}$  for  $\alpha \in A$ , then  $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$  ( $\mathcal{T}$  is closed under taking arbitrary unions)
- (iii) Let  $n \in \mathbb{N}$ . If  $U_1, \dots, U_n \in \mathcal{T}$ , then  $\bigcap_{i=1}^n U_i \in \mathcal{T}$  ( $\mathcal{T}$  is closed under taking finite intersections)

If  $U \in \mathcal{T}$ , we call  $U$  *open*. We call  $U \subseteq X$  *closed*, if  $U^c \in \mathcal{T}$ . We call  $(X, \mathcal{T})$  a *topological space*.

## Example

For a set  $X$ , the following  $\mathcal{T} \subseteq \mathcal{P}(X)$  are examples of topologies on  $X$ .

- Trivial topology:  $\mathcal{T} = \{\emptyset, X\}$ ,
- Discrete topology:  $\mathcal{T} = \mathcal{P}(X)$ ,
- Let  $X$  be an infinite set. Then,  $\mathcal{T} = \{U \subseteq X : U^c \text{ is finite}\} \cup \{\emptyset\}$  defines a topology on  $X$ .
- Topology induced by a metric: i.e. if  $d$  is a metric on  $X$  we can define

$$\mathcal{T}_d = \{U \subseteq X \mid \forall x \in U \exists \epsilon > 0 \text{ such that } B_\epsilon(x) \subseteq U\}.$$

The discrete topology is also induced by a metric, can you guess which one?

Given a topological space  $(X, \mathcal{T})$  and a subset  $Y \subseteq X$ , we can restrict the topology on  $X$  to  $Y$  which leads to the next definition.

### Definition (Relative topology)

Given a topological space  $(X, \mathcal{T})$  and an arbitrary non-empty subset  $Y \subseteq X$ , we define the relative topology on  $Y$  as follows

$$\mathcal{T}|_Y = \{U \cap Y : U \in \mathcal{T}\}.$$

## Definition

Let  $(X, \mathcal{T})$  be a topological space and let  $A \subseteq X$  be any subset.

- The *interior* of  $A$  is  $\overset{\circ}{A} := \{a \in A : \exists U \in \mathcal{T} \text{ s.t. } U \subseteq A \text{ and } a \in U\}$ .
- The *closure* of  $A$  is  $\overline{A} := \{x \in X : \forall U \in \mathcal{T} \text{ with } x \in U, U \cap A \neq \emptyset\}$ .
- The *boundary* of  $A$  is  $\partial A := \{x \in X : \forall U \in \mathcal{T} \text{ with } x \in U, U \cap A \neq \emptyset \text{ and } U \cap A^c \neq \emptyset\}$ .

- The *interior* of  $A$  is  $\overset{\circ}{A} := \{a \in A : \exists U \in \mathcal{T} \text{ s.t. } U \subseteq A \text{ and } a \in U\}$ .
- The *closure* of  $A$  is  $\overline{A} := \{x \in X : \forall U \in \mathcal{T} \text{ with } x \in U, U \cap A \neq \emptyset\}$ .
- The *boundary* of  $A$  is  $\partial A := \{x \in X : \forall U \in \mathcal{T} \text{ with } x \in U, U \cap A \neq \emptyset \text{ and } U \cap A^c \neq \emptyset\}$ .

## Example

Let  $X = \{a, b, c\}$  and  $\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ . Then

- $\overset{\circ}{\{a\}} =$
- $\overset{\circ}{\{c\}} =$
- $\overline{\{a\}} =$
- $\overline{\{c\}} =$



## Proposition

Let  $(X, \mathcal{T})$  be a topological space and  $A \subseteq X$ . Then,

$$\overline{A} = \bigcap \{F : F \text{ is closed and } A \subseteq F\}.$$

## Proof.



## Proof continued

Similarly, one can show  $\overset{\circ}{A} = \bigcup \{U : U \text{ is open and } U \subseteq A\}$ . Hence, we see that the interior of  $A$  is the largest open set contained in  $A$  and the closure is the smallest closed set that contains  $A$ .

# Next time

- Finish topology
  - Dense subsets
  - Compactness
  - Continuity
- Start linear algebra
  - Vector spaces and subspaces

# References

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