



UNIVERSITY OF  
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# Statistical Sciences

## DoSS Summer Bootcamp Probability Module 7

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# Recap

Learnt in last module:

- Covariance
  - ▷ Covariance as an inner product
  - ▷ Correlation
  - ▷ Cauchy-Schwarz inequality
  - ▷ Uncorrelatedness and Independence
- Concentration
  - ▷ Markov's inequality
  - ▷ Chebyshev's inequality
  - ▷ Chernoff bounds

# Outline

- Stochastic convergence
  - ▷ Convergence in distribution
  - ▷ Convergence in probability
  - ▷ Convergence almost surely
  - ▷ Convergence in  $L^p$
  - ▷ Relationship between convergences

# Stochastic Convergence

## Recall: Convergence

### Convergence of a sequence of numbers

A sequence  $a_1, a_2, \dots$  converges to a limit  $a$  if

$$\lim_{n \rightarrow \infty} a_n = a.$$

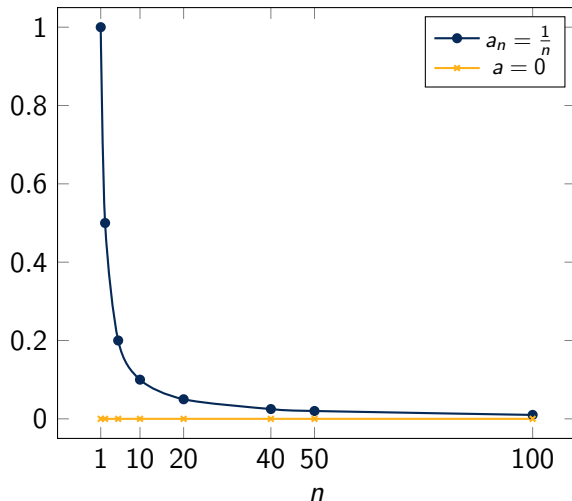
That is, for any  $\epsilon > 0$ , there exists an  $N(\epsilon)$  such that

$$|a_n - a| < \epsilon, \quad \forall n > N(\epsilon).$$

**Example:**  $a_n = \frac{1}{n}$ ,  $\forall \epsilon > 0$ , take  $N(\epsilon) = \lceil \frac{1}{\epsilon} \rceil$ , then for  $n > N(\epsilon)$ ,

$$|a_n - 0| = a_n < \epsilon, \quad \lim_{n \rightarrow \infty} a_n = 0.$$

# Stochastic Convergence



- Capture the property of a series as  $n \rightarrow \infty$ ;
- The limit is something where the series concentrate for large  $n$ ;
- $|a_n - a|$  quantifies the closeness of the series and the limit.

# Stochastic Convergence

## Observation: closeness of random variables

### Sample mean of i.i.d. random variables

For i.i.d. random variables  $X_i, i = 1, \dots, n$  with  $\mathbb{E}(X_i) = \mu$ ,  $\text{Var}(X_i) = \sigma^2$ , then for the sample mean  $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$ ,

$$\mathbb{E}(\bar{X}) = \mu, \quad \text{Var}(\bar{X}) = \frac{\sigma^2}{n}.$$

### Proof:

# Stochastic Convergence

## Example:

Further suppose  $X_i, i = 1, \dots, n$  i.i.d. with distribution  $\mathcal{N}(\mu, \sigma^2)$ , then  $\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$ , so we can draw the probability density plot of  $\bar{X}$ .

Add a .gif showing how the pdf will change as  $n$  increases

# Stochastic Convergence

## Intuition:

- Series of numbers  $a_n \Rightarrow$  Series of random variables  $X_n$ ;
- Limit  $a \Rightarrow$  Limit  $X$ ;
- How to quantify the closeness? ( $|X_n - X|?$ )

## Pointwise convergence / Sure convergence

Suppose random variables  $X_n$  and  $X$  are defined over the same probability space, then we say  $X_n$  converges to  $X$  pointwise if

$$\lim_{n \rightarrow \infty} X_n(\omega) = X(\omega), \quad \forall \omega \in \Omega.$$

## Remark:

Incorporate probability measure in some sense.



# Stochastic Convergence

## Alternatives of describing the closeness:

- Utilize CDF:  $F_{X_n}(x) - F_X(x)$ ;
- Utilize probability of an event:  $\mathbb{P}(|X_n - X| > \epsilon)$ ;
- Utilize the probability over all  $\omega$ :  $\mathbb{P}(\lim_{n \rightarrow \infty} X_n(\omega) = X(\omega))$ ;
- Utilize mean/moments:  $\mathbb{E}|X_n - X|^p$ .

# Stochastic Convergence

## Convergence in distribution

A sequence  $X_1, X_2, \dots$  of real-valued random variables is said to converge in distribution, or converge weakly to a random variable  $X$  if

$$\lim_{n \rightarrow \infty} F_n(x) = F(x),$$

for every number  $x \in \mathbb{R}$  at which  $F(\cdot)$  is continuous. Here,  $F_n(\cdot)$  and  $F(\cdot)$  are the cumulative distribution functions of the random variables  $X_n$  and  $X$ , respectively.

### Notation:

$$X_n \xrightarrow{d} X, \quad X_n \xrightarrow{\mathcal{D}} X, \quad X_n \Rightarrow X.$$

### Remark:

$X_n$  and  $X$  do not need to be defined on the same probability space.

# Stochastic Convergence

## Example:

Let  $X_n = Z + \frac{1}{n}$ , where  $Z \sim \mathcal{N}(0, 1)$ , then

- $X_n \xrightarrow{d} Z$ ,
- $X_n \xrightarrow{d} -Z$ ,
- $X_n \xrightarrow{d} Y$ ,  $Y \sim \mathcal{N}(0, 1)$ .

## Proof:

# Stochastic Convergence

## Convergence in probability

A sequence  $X_n$  of random variables converges in probability towards the random variable  $X$  if for all  $\epsilon > 0$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P}(|X_n - X| > \epsilon) = 0.$$

**Notation:**  $X_n \xrightarrow{p} X$ ,  $X_n \xrightarrow{P} X$ .

### Remark:

$X_n$  and  $X$  need to be defined on the same probability space.

# Stochastic Convergence

## Examples:

- Let  $X_n = Z + \frac{1}{n}$ , where  $Z \sim \mathcal{N}(0, 1)$ , then  $X_n \xrightarrow{P} Z$ .

**Proof:**

- Let  $X_n = Z + Y_n$ , where  $Z \sim \mathcal{N}(0, 1)$ ,  $\mathbb{E}(|Y_n|) = \frac{1}{n}$ , then  $X_n \xrightarrow{P} Z$ .

**Proof:**

# Stochastic convergence

## Convergence almost surely

A sequence  $X_n$  of random variables converges almost surely or almost everywhere or with probability 1 or strongly towards  $X$  means that

$$\mathbb{P} \left( \lim_{n \rightarrow \infty} X_n = X \right) = \mathbb{P} \left( \omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) = X(\omega) \right) = 1.$$

**Notation:**  $X_n \xrightarrow{a.s.} X$ .

**Remark:**

$X_n$  and  $X$  need to be defined on the same probability space.

# Stochastic convergence

## Examples:

- Let  $X_n = Z + \frac{1}{n}$ , where  $Z \sim \mathcal{N}(0, 1)$ , then  $X_n \xrightarrow{a.s.} Z$ .

**Proof:**

- Let  $X_n = Z + Y_n$ , where  $Z \sim \mathcal{N}(0, 1)$ ,  $\mathbb{E}(|Y_n|) = \frac{1}{n}$ , do we have  $X_n \xrightarrow{a.s.} Z$ ?

**Proof:**

# Stochastic convergence

## Convergence in $L^p$

A sequence  $\{X_n\}$  of random variables converges in  $L_p$  to a random variable  $X$ ,  $p \geq 1$ , if

$$\lim_{n \rightarrow \infty} \mathbb{E}|X_n - X|^p = 0$$

**Notation:**  $X_n \xrightarrow{L^p} X$ .

**Remark:**

$X_n$  and  $X$  need to be defined on the same probability space.



# Stochastic convergence

## Examples:

- Let  $X_n = Z + \frac{1}{n}$ , where  $Z \sim \mathcal{N}(0, 1)$ , then  $X_n \xrightarrow{L^p} Z$ .

**Proof:**

- Let  $X_n = Z + Y_n$ , where  $Z \sim \mathcal{N}(0, 1)$ ,  $\mathbb{E}(|Y_n|^p) = \frac{1}{n}$ , then  $X_n \xrightarrow{L^p} Z$ .

**Proof:**

# Stochastic convergence

Relationship between convergences (on complete probability space):

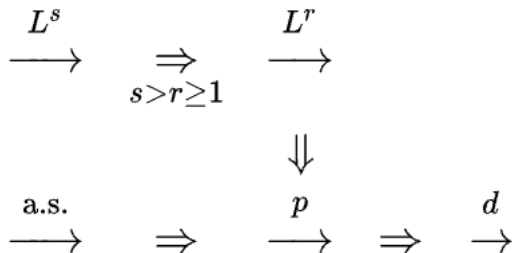


Figure: relationship between convergences

# Stochastic convergence

## Highlights:

- Almost sure convergence implies convergence in probability:

$$X_n \xrightarrow{\text{a.s.}} X \quad \Rightarrow \quad X_n \xrightarrow{P} X;$$

- Convergence in probability implies convergence in distribution:

$$X_n \xrightarrow{P} X \quad \Rightarrow \quad X_n \xrightarrow{d} X;$$

- If  $X_n$  converges in distribution to a constant  $c$ , then  $X_n$  converges in probability to  $c$ :

$$X_n \xrightarrow{d} c \quad \Rightarrow \quad X_n \xrightarrow{P} c, \quad \text{provided } c \text{ is a constant.}$$

# Stochastic convergence

**Proof:**  $X_n \xrightarrow{\text{a.s.}} X \Rightarrow X_n \xrightarrow{P} X.$

# Stochastic convergence

**Proof:**  $X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} X.$

# Stochastic convergence

**Proof:**  $X_n \xrightarrow{d} c \Rightarrow X_n \xrightarrow{P} c$ , provided  $c$  is a constant.

# Problem Set

**Problem 1:** Prove that on a complete probability space, if  $X_n \xrightarrow{L^p} X$ , then  $X_n \xrightarrow{P} X$ .  
(Hint: use Markov's inequality)

**Problem 2:** Let  $X_1, \dots, X_n$  be i.i.d. random variables with  $Bernoulli(p)$  distribution, and  $X \sim Bernoulli(p)$  is defined on the same probability space, independent with  $X_i$ 's. Does  $X_n$  converge in probability to  $X$ ?

**Problem 3:** Give an example where  $X_n$  converges in probability to  $X$ , but not almost surely.