

# Module 4: Metric Spaces and Sequences II

## Operational math bootcamp



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# Outline

- Sequences
  - Cauchy sequences
  - subsequences



# Sequences

## Definition (Sequence)

Let  $(X, d)$  be a metric space. A *sequence* is an ordered list of points  $x_n$ ,  $n \in \mathbb{N}$ , in  $X$ , denoted  $(x_n)_{n \in \mathbb{N}}$ . We say that a sequence  $(x_n)_{n \in \mathbb{N}}$  *converges* to a point  $x \in X$  if

## Proposition

Let  $(X, d)$  be a metric space, and let  $A \subseteq X$ . Then  $\overline{A}$  is equal to the set of points in  $X$  which are limits of a sequence in  $A$ .

## Proof.



## Proof continued

## Corollary

A set  $F \subseteq X$ , where  $(X, d)$  is a metric space, is closed if and only if every sequence in  $F$  which converges in  $X$  converges to a point in  $F$ .

# Cluster points of a set

## Definition

Let  $(X, d)$  be a metric space and  $A \subseteq X$ . A point  $x \in X$  is a *cluster point* of  $A$  (also called accumulation point) if for every  $\epsilon > 0$ ,  $B_\epsilon(x)$  contains uncountably many points in  $A$ .

## Proposition

*$x \in X$  is a cluster point of  $A \subseteq X$  where  $(X, d)$  is a metric space if and only if there exists a sequence of points  $x_n \in A$ ,  $n \in \mathbb{N}$ , such that  $x_n \rightarrow x$ .*

## Proof.



Combining the previous result with the limit characterization of closure gives the following:

### Corollary

For  $A \subseteq X$ ,  $(X, d)$  a metric space, we have

$$\overline{A} = A \cup \{x \in X : x \text{ is a cluster point of } A\}.$$



# Cauchy sequences

## Definition (Cauchy sequence)

Let  $(X, d)$  be a metric space. A sequence denoted  $(x_n)_{n \in \mathbb{N}} \in X$  is called a *Cauchy sequence* if

## Proposition

Let  $(X, d)$  be a metric space, and let  $(x_n)_{n \in \mathbb{N}}$  be a convergent sequence in  $X$ . Then  $(x_n)_{n \in \mathbb{N}}$  is Cauchy.

## Proof.



## Definition

A metric space where every Cauchy sequence converges (to a point in the space) is called *complete*.

## Proposition

Let  $(X, d)$  be a metric space, and let  $Y \subseteq X$ .

- (i) If  $X$  is complete and if  $Y$  is closed in  $X$ , then  $Y$  is complete.
- (ii) If  $Y$  is complete, then it is closed in  $X$ .

Proof.



# Subsequences

## Definition

Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in a metric space  $(X, d)$ . Let  $(n_k)_{k \in \mathbb{N}}$  be a sequence of natural numbers with  $n_1 < n_2 < \dots$ . The sequence  $(x_{n_k})_{k \in \mathbb{N}}$  is called a *subsequence* of  $(x_n)_{n \in \mathbb{N}}$ . If  $(x_{n_k})_{k \in \mathbb{N}}$  converges to  $x \in X$ , we call  $x$  a *subsequential limit*.

## Example

$$((-1)^n)_{n \in \mathbb{N}}$$

## Proposition

A sequence  $(x_n)_{n \in \mathbb{N}}$  in a metric space  $(X, d)$  converges to  $x \in X$  if and only if every subsequence of  $(x_n)_{n \in \mathbb{N}}$  also converges to  $x$ .

## Proof.



## Proof continued

# Continuity

## Definition

Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces, let  $x_0 \in X$ , and let  $f : X \rightarrow Y$ .  $f$  is *continuous* at  $x_0$  if for every sequence  $(x_n)_{n \in \mathbb{N}}$  in  $X$  that converges to  $x_0$ , we have  $\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$ .

We say that  $f$  is continuous if it is continuous at every point in  $X$ .



## Theorem

Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces, let  $x_0 \in X$ , and let  $f : X \rightarrow Y$ . The following are equivalent:

- (i)  $f$  is continuous at  $x_0$
- (ii) for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $d_Y(f(x), f(x_0)) < \epsilon$  for all  $x \in X$  with  $d_X(x, x_0) < \delta$
- (iii) for each  $\epsilon > 0$ , there is  $\delta > 0$  such that  $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$

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Proof.

(i)  $\Rightarrow$  (ii)



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- (iii) for each  $\epsilon > 0$ , there is  $\delta > 0$  such that  $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$

### Proof continued

(ii)  $\Rightarrow$  (iii)

(iii)  $\Rightarrow$  (i)

## Corollary

*Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces and let  $f : X \rightarrow Y$ . The following are equivalent:*

- (i)  $f$  is continuous*
- (ii) if  $U \subseteq Y$  is open, then  $f^{-1}(U)$  is open*
- (iii) if  $F \subseteq Y$  is closed, then  $f^{-1}(F)$  is closed*

We need the following results about sets and functions:

Let  $X$  and  $Y$  be sets and  $f : X \rightarrow Y$ . Let  $A, B \subseteq Y$ . Then

①  $f^{-1}(A) \subseteq f^{-1}(B)$

②  $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$

Proof.

Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces and let  $f : X \rightarrow Y$ .

(i)  $\Rightarrow$  (ii):



## Proof continued

$$(ii) \Rightarrow (i)$$

$$(ii) \Rightarrow (iii)$$

$$(iii) \Rightarrow (ii)$$









# References

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