Statistical Sciences

DoSS Summer Bootcamp Probability Module 1

Ichiro Hashimoto

University of Toronto

July 8, 2024

Roadmap

A bridge connecting undergraduate probability and graduate probability

Undergraduate-level probability

- Concrete;
- Examples and scenarios;
- Rely on computation...

Roadmap

A bridge connecting undergraduate probability and graduate probability

Undergraduate-level probability

- Concrete;
- Examples and scenarios;
- Rely on computation...

Graduate-level probability

- Abstract (measure theory);
- Laws and properties;
- Rely on construction and inference...

Roadmap

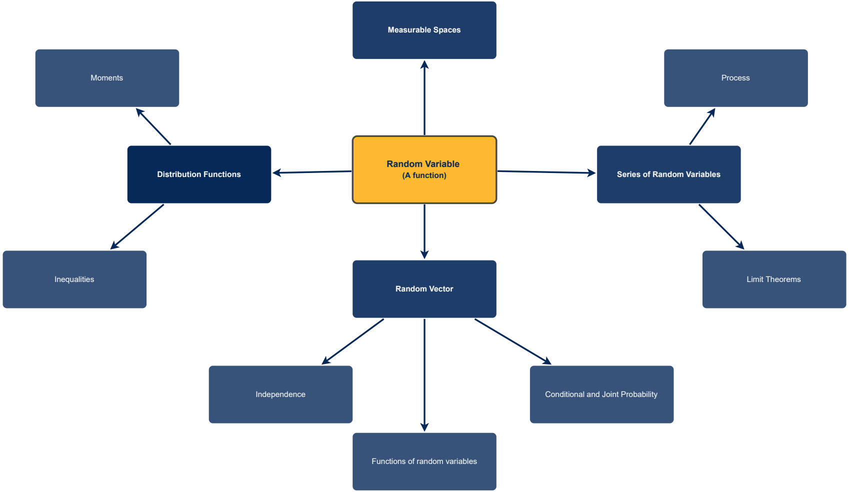

Figure: Roadmap

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Outline

- Measurable spaces
  - Sample Space
  - σ-algebra
- Probability measures
  - Measures on σ-field
  - Basic results
- Conditional probability
  - Bayes' rule
  - Law of total probability

Today

Module 2.

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Measurable spaces

Sample Space

The sample space Ω is the set of all possible outcomes of an experiment.

Examples:

- Toss a coin: $\{H, T\}$ $= \Omega$
- Roll a die: $\{1, 2, 3, 4, 5, 6\}$ $= \Omega$

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Measurable spaces

Sample Space

The sample space Ω is the set of all possible outcomes of an experiment.

Examples:

- Toss a coin: $\{H, T\}$
- Roll a die: $\{1, 2, 3, 4, 5, 6\}$

Event

An event is a collection of possible outcomes (subset of the sample space).

Examples:

- Get head when tossing a coin: $\{H\}$ $\subset \{H, T\} = \Omega$
- Get an even number when rolling a die: $\{2, 4, 6\}$ $\subset \{1,2,3,4,5,6\} = \Omega$

ex1) Tossing a coin twice

$$\Omega = \{HH, HT, TH, TT\} \longrightarrow \text{discrete.}$$

$$\mathbb{P}(HH) = \mathbb{P}(HT) = \mathbb{P}(TH) = \mathbb{P}(TT) = \frac{1}{4}$$

Let X = the number of H

$$\mathbb{P}(X=0) = \mathbb{P}(X=2) = 1/4$$

$$\mathbb{P}(X=1) = 1/2$$

$$\mathbb{P}(X=0) + \mathbb{P}(X=1) + \mathbb{P}(X=2) = 1$$

$$\mathbb{E}X = \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 = 1$$

ex2) Let $X \sim N(\mu, \sigma^2)$ gaussian

continuous

Density $p(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(- \frac{(x-\mu)^2}{2\sigma^2} \right)$

$$\int_{-\infty}^{\infty} p(x)\,dx = 1$$

$$\mathbb{E}X = \int_{-\infty}^{\infty} x\,p(x)\,dx = \mu$$

Discrete case.

assume X only takes integer values for simplicity

$$\mathbb{P}(X \le k) = \sum_{\ell \le k} \mathbb{P}(X = \ell)$$

$$\mathbb{E} X = \sum_{k=-\infty}^{\infty} k \, \mathbb{P}(X = k)$$

Continuity case

$$\mathbb{P}(X \le x) = \int_{-\infty}^{x} p(z) \, dz$$

$$\mathbb{E} X = \int_{-\infty}^{\infty} x \, p(x) \, dx$$

Question Is there any way to explain them in a unified way?

Observation

If $A \cap B = \phi$, then $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

For a discrete case. $\{X = k\}$ are disjoint.

$$1 = \sum_{k=-\infty}^{\infty} \mathbb{P}(X = k)$$

countable sum

But for continuous case,

$$\mathbb{P}(X = x) = 0 \text{ for any } x$$

uncountable sum

Therefore.

$$1 = \sum_{x \in \mathbb{R}} \mathbb{P}(X = x) = \sum_{x \in \mathbb{R}} 0 = 0$$

contradiction?

$\Longrightarrow$ uncountable sum is problematic.

$\Longrightarrow$ let's focus on countable sum

Measurable spaces

σ-algebra

A σ-algebra (σ-field) $\mathcal{F}$ on Ω is a non-empty collection of subsets of Ω such that

(i) • If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$, $\rightarrow$ complement is also in $\mathcal{F}$

(ii) • If $A_1, A_2, \cdots \in \mathcal{F}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$. $\rightarrow$ Countable union of $\mathcal{F}$ subsets of $\mathcal{F}$ is also in $\mathcal{F}$.

Remark: $\varnothing, \Omega \in \mathcal{F}$

(Pf) Let $A \in \mathcal{F}$.

By (i), $A^c \in \mathcal{F}$.

By (ii) $\underbrace{A \cup A^c}_{= \Omega} \in \mathcal{F}$ so $\Omega \in \mathcal{F}$.

By (i) again, $\Omega^c \in \mathcal{F}$. So, $\phi \in \mathcal{F}$ //.

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Construction of Probability Theory

Outline.

1) Define the collection of subsets of Ω, $\mathcal{F}$ (σ-algebra)

   on which we can define "Probability measure".

2) Define probability measure as a function

   $$\mathbb{P} : \mathcal{F} \longrightarrow [0, 1]$$

   which has "countable additivity".

3) $(\Omega, \mathcal{F}, \mathbb{P})$ is called "Probability triple".

   Ω: sample space, $\mathcal{F}$: σ-algebra, $\mathbb{P}$: probability measure

Probability measures

Measures on σ-field

A function $\mu : \mathcal{F} \to R^+ \cup \{+\infty\}$ is called a measure if

- $\mu(\varnothing) = 0$,
- If $A_1, A_2, \cdots \in \mathcal{F}$ and $A_i \cap A_j = \varnothing$, then $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

If $\mu(\Omega) = 1$, then μ is called a probability measure.

countable additivity

Probability measures

Measures on σ-field

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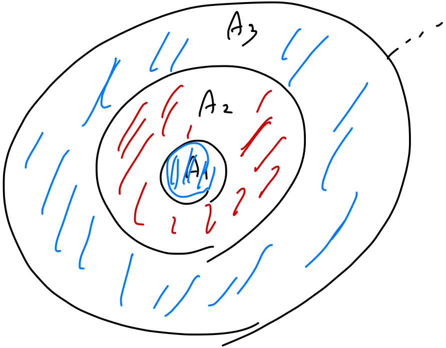
If $\mu(\Omega) = 1$, then μ is called a probability measure.

Properties:

- Monotonicity: $A \subseteq B \quad \Rightarrow \quad \mu(A) \leq \mu(B)$
- Subadditivity: $A \subseteq \cup_{i=1}^{\infty} A_i \quad \Rightarrow \quad \mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$
- Continuity from below: $A_i \nearrow A \quad \Rightarrow \quad \mu(A_i) \nearrow \mu(A)$
- Continuity from above: $A_i \searrow A$ and $\mu(A_1) < \infty \quad \Rightarrow \quad \mu(A_i) \searrow \mu(A)$

Probability measures

Proof of continuity from below:



Let $A_i \in \mathcal{F}$, $A_1 \subset A_2 \subset \cdots\cdots$

$\bigcup_{i=1}^{\infty} A_i = A$.

Let $B_i = A_i \setminus A_{i-1}$, $i \geq 2$.

$B_1 = A_1$

Then B_i are disjoint.

$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i = A$

By countable additivity $\mu(A) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i)$

Note that $\underbrace{A_i = A_{i-1} \cup B_i}_{\text{disjoint union.}}$ implies

$$\mu(A_i) = \mu(A_{i-1}) + \mu(B_i) \text{ by countable additivity.}$$

Therefore $\mu(B_i) = \mu(A_i) - \mu(A_{i-1})$

Thus $$\sum_{i=1}^{n} \mu(B_i) = \mu(B_1) + \sum_{i=2}^{n} \left(\mu(A_i) - \mu(A_{i-1}) \right)$$

$$= \mu(A_1) + \mu(A_n) - \mu(A_1)$$

$$= \mu(A_n)$$

So, $$\sum_{i=1}^{\infty} \mu(B_i) = \lim_{n\to\infty} \mu(A_n)$$

Thus, $$\mu(A) = \lim_{n\to\infty} \mu(A_n)$$

Probability measures

$A_i \searrow A$

Proof of continuity from above:

$\mu(A_1) < \infty$, $A_1 \supset A_2 \supset A_3 \supset \cdots$, $\bigcap_{i=1}^{\infty} A_i = A$

$B_i = A_1 - A_i$

Then $B_1 \subset B_2 \subset B_3 \subset \cdots$, $\bigcup_{i=1}^{\infty} B_i = A_1 \setminus A$

By the continuity from below,

$$\lim_{n\to\infty} \mu(B_n) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu(A_1 \setminus A)$$

$$= \mu(A_1) - \mu(A)$$

Remark: $\mu(A_i) < \infty$ is vital.

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Note that $\mu(B_n) = \mu(A_1 \setminus A_n) = \mu(A_1) - \mu(A_n)$

Thus $\lim_{n\to\infty} \left(\mu(A_1) - \mu(A_n)\right) = \mu(A_1) - \mu(A)$ $\quad \mu(A_1) < \infty$

$\therefore \quad \lim_{n\to\infty} \mu(A_n) = \mu(A)$

Summary $(\Omega, \mathcal{F}, \mathbb{P})$ probability triple.

"Countable additivity" is the key.

Question How can $(\Omega, \mathcal{F}, \mathbb{P})$ provides unified theory?

Observation

$X : \Omega \to \mathbb{R}$ random variable.

$$\Omega = \{ X \in \mathbb{R} \}$$

$$= \bigcup_{i=-\infty}^{\infty} \{ X \in [i, i+1) \}$$

By countable additivity

$$1 = \mathbb{P}(\Omega) = \sum_{i=-\infty}^{\infty} \mathbb{P}(X \in [i, i+1))$$

$$\Omega = \bigcup_{i=-\infty}^{\infty} \left\{ X \in \left[\frac{i}{n}, \frac{i+1}{n}\right) \right\}$$

becomes finer as $n \nearrow \infty$

$$1 = \mathbb{P}(\Omega) = \sum_{i=-\infty}^{\infty} \mathbb{P}\left(X \in \left[\frac{i}{n}, \frac{i+1}{n}\right)\right)$$

Approximation of Expectation

$$\mathbb{E}X \approx \underbrace{\sum_{i=-\infty}^{\infty} \frac{i}{n} \cdot \mathbb{P}\left(X \in \left[\frac{i}{n}, \frac{i+1}{n}\right)\right)}_{\text{should become more precise as } n \to \infty}$$

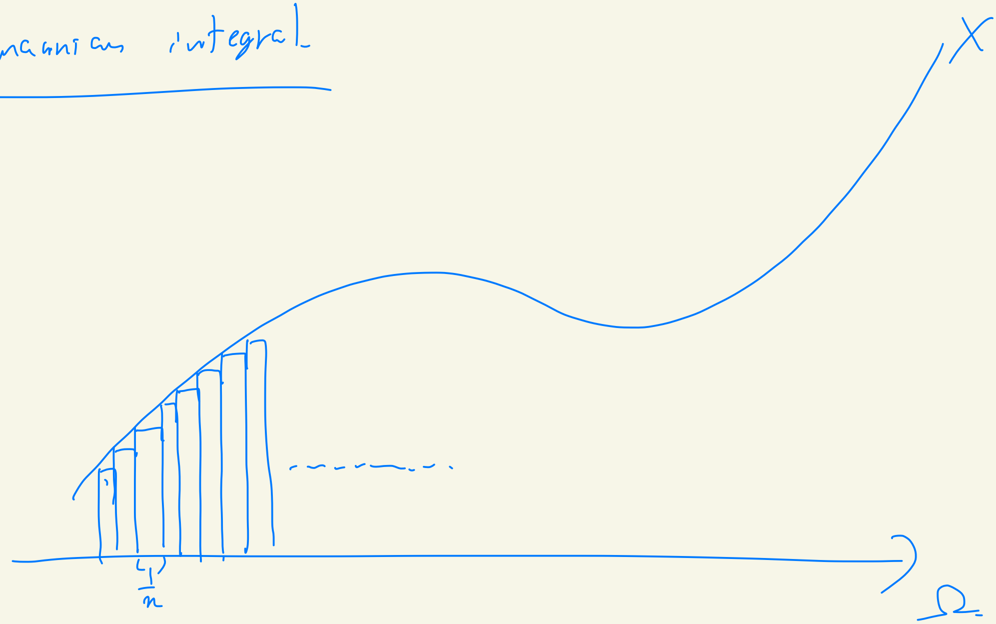
We can use this observation to define $\mathbb{E}X$

$$\mathbb{E}X = \lim_{n \to \infty} \sum_{i=-\infty}^{\infty} \frac{i}{n} \mathbb{P}\left(X \in \left[\frac{i}{n}, \frac{i+1}{n}\right)\right)$$

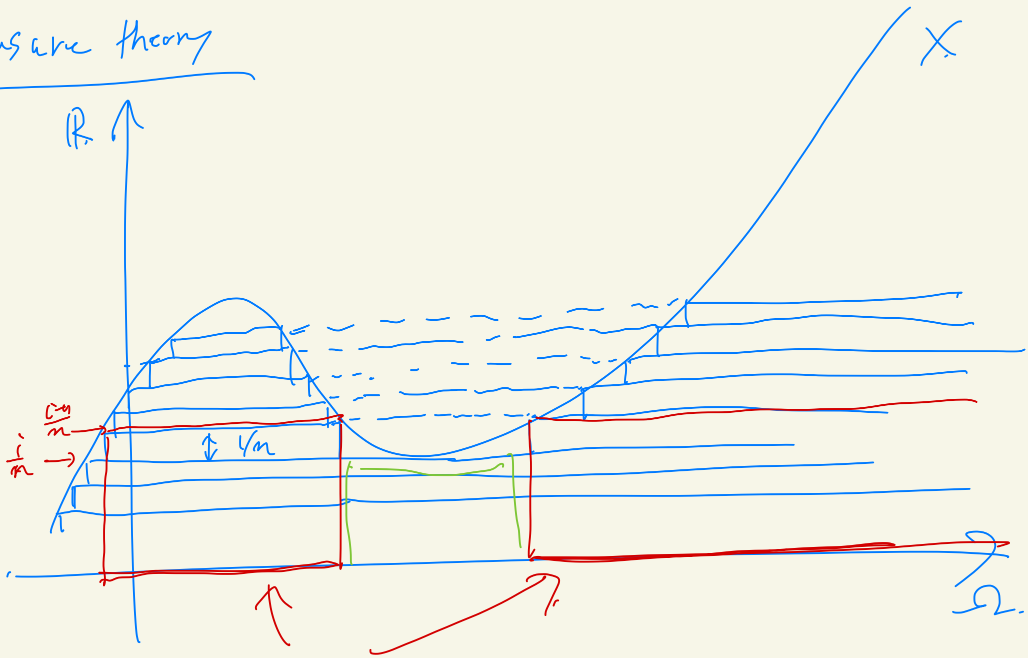
This looks similar to Riemannian integral

Difference between Riemannian integral

Riemannian integral



Measure theory



$\mathbb{P}(X \in [\frac{i}{n}, \frac{i+1}{n}))$

$$\mathbb{E}X = \lim_{n\to\infty} \sum_{i=-\infty}^{\infty} \frac{i}{n} \mathbb{P}\left(X \in \left[\frac{i}{n}, \frac{i+1}{n}\right)\right) = \int_\Omega X \, d\mathbb{P}$$

We can show that

$$EX = \sum_{i=1}^{\infty} k_i \mathbb{P}(X = k_i) \quad \text{discrete case.}$$

$$EX = \int_{-\infty}^{\infty} x p(x) \, dx \quad \text{continuous case.}$$

Probability measures

Examples:

$\Omega = \{\omega_1, \omega_2, \cdots\}$, $A = \{\omega_{a_1}, \cdots, \omega_{a_i}, \cdots\} \Rightarrow \mu(A) = \sum_{j=1}^{\infty} \mu(\omega_{a_j})$.
Therefore, we only need to define $\mu(\omega_j) = p_j \geq 0$.
If further $\sum_{i=1}^{\infty} p_j = 1$, then μ is a probability measure.

- Toss a coin:

- Roll a die:

Conditional probability

Original problem:

- What is the probability of some event A?
- $P(A)$ is determined by our probability measure.

New problem:

- Given that B happens, what is the probability of some event A?
- $P(A \mid B)$ is the conditional probability of the event A given B.

Conditional probability

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New problem:

- Given that B happens, what is the probability of some event A?
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Example:

- Roll a die: $P(\{2\} \mid \text{even number})$

Conditional probability

Bayes' rule

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

Remark: Does conditional probability $P(\cdot \mid B)$ satisfy the axioms of a probability measure?

Conditional probability

Multiplication rule

$$P(A \cap B) = P(A \mid B)P(B) = P(B \mid A)P(A)$$

Generalization:

Law of total probability

Let $A_1, A_2, \cdots, A_n$ be a partition of ω, such that $P(A_i) > 0$, then

$$P(B) = \sum_{i=1}^{n} P(A_i)P(B \mid A_i)$$

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Problem Set

Problem 1: Prove that for a σ-field $\mathcal{F}$, if $A_1, A_2, \cdots \in \mathcal{F}$, then $\cap_{i=1}^{\infty} A_i \in \mathcal{F}$.

Problem 2: Prove monotonicity and subadditivity of measure μ on σ-field.

Problem 3: (Monty Hall problem) Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?
(Assumptions: the host will not open the door we picked and the host will only open the door which has a goat.)