Module 5: Metric spaces III

Operational math bootcamp

Statistical Sciences
UNIVERSITY OF TORONTO

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Last time

Finished our discussion of open and closed sets:

- Introduced a cluster points of a set:
  $x \in X$ is a *cluster point* of A if for every $\epsilon > 0$, $B_\epsilon(x)$ contains **infinitely** many points in A.

- Sequence characterization of a closed set:
  A set $F \subseteq X$, where (X, d) is a metric space, is closed if and only if every sequence in F which converges in X converges to a point in F.

Last time

Discussed sequences, which includes Cauchy sequences and subsequences:

- Convergent sequence: $x_n \to x \Leftrightarrow \forall \epsilon > 0 \ \exists \, n_\epsilon \in \mathbb{N}$ s.t. $d(x_n, x) < \epsilon$ for all $n \geq n_\epsilon$
- Cauchy sequence: $\forall \epsilon > 0 \ \exists \, n_\epsilon \in \mathbb{N}$ s.t. $d(x_n, x_m) < \epsilon$ for all $n, m \geq n_\epsilon$
- Proved that a convergent sequence is Cauchy
- Discussed complete metric spaces, where all Cauchy sequences converge (like $\mathbb{R}$ with the usual metric, absolute value)
- Proved that a sequence converges to x if and only if all subsequences converge to x

Outline for today

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Continuity

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces, let $x_0 \in X$, and let $f : X \to Y$. f is *continuous* at x_0 if for every sequence $(x_n)_{n\in\mathbb{N}}$ in X that converges to x_0, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

We say that f is continuous if it is continuous at every point in X.

Theorem

Let (X, d_X) and (Y, d_Y) be metric spaces, let $x_0 \in X$, and let $f : X \to Y$. The following are equivalent:

(i) *f is continuous at x_0*

(ii) *for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \epsilon$ for all $x \in X$ with $d_X(x, x_0) < \delta$*

(iii) *for each $\epsilon > 0$, there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$*

(i) f is continuous at x_0
(ii) for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0))) < \epsilon$ for all $x \in X$ with $d_X(x, x_0) < \delta$
(iii) for each $\epsilon > 0$, there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$

Proof. (i) $\Rightarrow$ (ii)

(ii) $\Rightarrow$ (iii)

(iii) $\Rightarrow$ (i)

Corollary

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$. The following are equivalent:

(i) f is continuous

(ii) if $U \subseteq Y$ is open, then $f^{-1}(U)$ is open

(iii) if $F \subseteq Y$ is closed, then $f^{-1}(F)$ is closed

We need the following results about sets and functions:
Let X and Y be sets and $f : X \to Y$. Let $A, B \subseteq Y$. Then

1. $A \subseteq B \implies f^{-1}(A) \subseteq f^{-1}(B)$
2. $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$

Proof.

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$.
(i) $\Rightarrow$ (ii):

$\square$

Proof continued

(ii) $\Rightarrow$ (i)

(ii) $\Rightarrow$ (iii)

(iii) $\Rightarrow$ (ii)

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$.

- f is *uniformly continuous* if for all $\epsilon > 0$, there exists $\delta > 0$ such that for every $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$, we have $d_Y(f(x_1), f(x_2))) < \epsilon$
- f is *Lipschitz continuous* if there exists a $K > 0$ such that for every $x_1, x_2 \in X$ we have $d_Y(f(x_1), f(x_2))) \leq K d_X(x_1, x_2)$

Proposition

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$.

$$f \text{ is Lipschitz continuous} \ \Rightarrow \ \text{f is uniformly continuous} \ \Rightarrow \ \text{f is continuous}$$

Proof is one of your exercises.

Contraction Mapping Theorem

Definition

Let (X, d) be a metric space and let $f : X \to X$. We say that $x^* \in X$ is a *fixed point* of f if $f(x^*) = x^*$.

Definition

Let (X, d) be a metric space and let $f : X \to X$. f is a *contraction* if there exists a constant $k \in [0, 1)$ such that for all $x, y \in X$, $d(f(x), f(y)) \leq kd(x, y)$.

Observe that a function is a contraction if and only if it is Lipschitz continuous with constant $K < 1$.

Theorem (Contraction Mapping Theorem)

Suppose that $f : X \to X$ is a contraction and the metric space X is complete. Then f has a unique fixed point x^*.

Example

Let $f : \left[-\frac{1}{3}, \frac{1}{3}\right] \to \left[-\frac{1}{3}, \frac{1}{3}\right]$ be defined by the mapping $x \mapsto x^2$. Assume we use the standard Euclidean metric, $d(x, y) = |x - y|$. f has a unique fixed point because

Equivalent metrics

Definition (Equivalent metrics)

Two metrics d_1 and d_2 on a set X are *equivalent* if the identity maps from (X, d_1) to (X, d_2) and from (X, d_2) to (X, d_1) are continuous.

Proposition

Two metrics d_1, d_2 on a set X are equivalent if and only if they have the same open sets or the same closed sets.

Definition

Two metrics d_1 and d_2 on a set X are *strongly equivalent* if for every $x, y \in X$, there exists constants $\alpha > 0$ and $\beta > 0$ such

$$\alpha d_1(x, y) \leq d_2(x, y) \leq \beta d_1(x, y).$$

If two metrics are strongly equivalent then they are equivalent. The proof of this is one of the exercises.

Example

We show that the Euclidean distance (induced by 2-norm) and the metric induced by the ∞-norm are equivalent on $\mathbb{R}^n$.

Can you think of an example that we've seen of a metric that isn't equivalent to the Euclidean metric?

Right and left continuous

Recall: $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|x_0 - y| < \delta$ implies $|f(x_0) - f(y)| < \epsilon$.

Definition

Let $f : \mathbb{R} \to \mathbb{R}$.

- f is *left continuous* at $x_0 \in \mathbb{R}$ if for all $\epsilon > 0$ there exists a $\delta > 0$, such $|f(x_0) - f(x)| < \epsilon$ whenever $x_0 - \delta < x < x_0$.
- f is *right continuous* at $x_0 \in \mathbb{R}$ if for all $\epsilon > 0$ there exists a $\delta > 0$, such $|f(x_0) - f(x)| < \epsilon$ whenever $x_0 < x < x_0 + \delta$.

We say that f is left continuous if it is left continuous at all points in the domain, and similar for right continuous.

Proposition

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if it is left and right continuous.

Proof.

$\square$

Proof continued

Bounded sequences and monotone convergence

Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. We call $(x_n)_{n\in\mathbb{N}}$ *bounded* if there exists an $M > 0$ such that $|x_n| < M$ for all $n \in \mathbb{N}$.

Theorem (Monotone convergence theorem)

(i) *Suppose $(x_n)_{n\in\mathbb{N}}$ is an increasing sequence, i.e. $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$, and that it is bounded (above). Then the sequence converges. Furthermore, $\lim_{n\to\infty} x_n = \sup_{n\in\mathbb{N}} x_n$, where $\sup_{n\in\mathbb{N}} x_n := \sup\{x_n : n \in \mathbb{N}\}$.*

(ii) *Suppose $(x_n)_{n\in\mathbb{N}}$ is a decreasing sequence, i.e. $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$, which is bounded (below). Then the sequence converges and $\lim_{n\to\infty} x_n = \inf_{n\in\mathbb{N}} x_n := \inf\{x_n : n \in \mathbb{N}\}$.*

Convention: $\sup A = \infty$ if $A \subseteq \mathbb{R}$ is not bounded above and $\inf A = -\infty$ if A is not bounded below.

Lemma

If $A \subseteq B \subseteq \mathbb{R}$ is non-empty, then $\inf A \leq \sup A$, $\sup A \leq \sup B$, *and* $\inf A \geq \inf B$.

The proof of this follows from the definition of greatest lower and least upper bound.

Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. We define the *limit superior* of $(x_n)_{n\in\mathbb{N}}$ as

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \sup_{k\geq n} x_k.$$

Similarly we define the *limit inferior* of $(x_n)_{n\in\mathbb{N}}$ as

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} \inf_{k\geq n} x_k.$$

If the sequence $(x_n)_{n\in\mathbb{N}}$ is not bounded above, then $\limsup_{n\to\infty} x_n = \infty$. Similarly, if the sequence $(x_n)_{n\in\mathbb{N}}$ is not bounded below, then $\liminf_{n\to\infty} x_n = -\infty$.

Proposition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$.

- The sequence of suprema, $s_n = \sup_{k\geq n} x_k$, is decreasing and the sequence of infima, $i_n = \inf_{k\geq n} x_k$, is increasing.
- The limit superior and the limit inferior of a bounded sequence always exist and are finite.

Proof.

The first part is true by the previous Lemma. The second follows by the Monotone Convergence Theorem. $\square$

Theorem

Let $(x_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathbb{R}$. *Then the sequence converges to* $x \in \mathbb{R}$ *if and only if* $\limsup_{n\to\infty} x_n = x = \liminf_{n\to\infty} x_n$.

Proof.

Proof continued

We can extend this easily to a sequence of functions $f_n \colon X \to \mathbb{R}$ as follows:

Define $f = \limsup_{n\to\infty} f_n$ to be the function defined pointwise by $f(x) = \limsup_{n\to\infty}(f_n(x))$ and similar for the limit inferior.

There also exists a set theoretic version in terms of unions and intersections which you will encounter in probability.

Topology

- Let X be a set. If X is not a metric space, can we still have open and closed sets?
- One can think of a topology on X as a specification of what subsets of X are open

Definition

Let $\mathcal{T} \subseteq \mathcal{P}(X)$. We call $\mathcal{T}$ a *topology* on X if the following holds:

(i) $\emptyset, X \in \mathcal{T}$

(ii) Let A be an arbitrary index set. If $U_\alpha \in \mathcal{T}$ for $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$ ($\mathcal{T}$ is closed under taking arbitrary unions)

(iii) Let $n \in \mathbb{N}$. If $U_1, \ldots, U_n \in \mathcal{T}$, then $\bigcap_{i=1}^n U_i \in \mathcal{T}$ ($\mathcal{T}$ is closed under taking finite intersections)

If $U \in \mathcal{T}$, we call U *open*. We call $U \subseteq X$ *closed*, if $U^c \in \mathcal{T}$. We call $(X, \mathcal{T})$ a *topological space*.

Example

For a set X, the following $\mathcal{T} \subseteq \mathcal{P}(X)$ are examples of topologies on X.

- Trivial topology: $\mathcal{T} = \{\emptyset, X\}$,
- Discrete topology: $\mathcal{T} = \mathcal{P}(X)$,
- Let X be an infinite set. Then, $\mathcal{T} = \{U \subseteq X \;:\; U^c \text{ is finite}\} \cup \emptyset$ defines a topology on X.
- Topology induced by a metric: i.e. if d is a metric on X we can define

$$\mathcal{T}_d = \{U \subseteq X \mid \forall x \in U \; \exists \epsilon > 0 \text{ such that } B_\epsilon(x) \subseteq U\}.$$

The discrete topology is also induced by a metric, can you guess which one?

Given a topological space $(X, \mathcal{T})$ and a subset $Y \subseteq X$, we can restrict the topology on X to Y which leads to the next definition.

Definition (Relative topology)

Given a topological space $(X, \mathcal{T})$ and an arbitrary non-empty subset $Y \subseteq X$, we define the relative topology on Y as follows

$$\mathcal{T}|_Y = \{U \cap Y : \ U \in \mathcal{T}\}.$$

Definition

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$ be any subset.

- The *interior* of A is $\mathring{A} := \{a \in A \colon \exists U \in \mathcal{T} \text{ s.t. } U \subseteq A \text{ and } a \in U\}$.
- The *closure* of A is $\overline{A} := \{x \in X : \forall U \in \mathcal{T} \text{ with } x \in U, U \cap A \neq \emptyset\}$.
- The *boundary* of A is $\partial A := \{x \in X : \forall U \in \mathcal{T} \text{ with } x \in U,\ U \cap A \neq \emptyset \text{ and } U \cap A^c \neq \emptyset\}$.

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- The *closure* of A is $\overline{A} := \{x \in X \colon \forall U \in \mathcal{T} \text{ with } x \in U, U \cap A \neq \emptyset\}$.
- The *boundary* of A is $\partial A := \{x \in X \colon \forall U \in \mathcal{T} \text{ with } x \in U,\ U \cap A \neq \emptyset \text{ and } U \cap A^c \neq \emptyset\}$.

Example

Let $X = \{a, b, c\}$ and $\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then

- $\mathring{\{a\}} =$
- $\mathring{\{c\}} =$
- $\overline{\{a\}} =$
- $\overline{\{c\}} =$

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Proposition

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then,

$$\overline{A} = \bigcap \{F \colon F \text{ is closed and } A \subseteq F\}.$$

Proof.

$\square$

Proof continued

Similarly, one can show $\mathring{A} = \bigcup\{U : U \text{ is open and } U \subseteq A\}$. Hence, we see that the interior of A is the largest open set contained in A and the closure is the smallest closed set that contains A.

Next time

- Finish topology
    - Dense subsets
    - Compactness
    - Continuity
- Start linear algebra
    - Vector spaces and subspaces

References

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