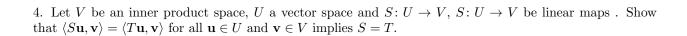
Exercises for Module 8: Linear Algebra II

1. A square matrix is called *nilpotent* if $\exists k \in \mathbb{N}$ such that $A^k = 0$. Show that for a nilpotent matrix A, |A| = 0.

2. A real square matrix Q is called orthogonal if $Q^TQ=I$. Prove that if Q is orthogonal, then $|Q|=\pm 1$.

3. An $n \times n$ matrix is called *antisymmetric* if $A^T = -A$. Prove that if A is antisymmetric and n is odd, then |A| = 0.



5. Let V be an inner product space and $\mathbf{x}_1, \dots, \mathbf{x}_n$ be an orthonormal basis and $\mathbf{y} \in V$. Then, \mathbf{x} has a unique representation $\mathbf{y} = \sum_{i=1}^n \alpha_i \mathbf{x}_i$. Show that $\alpha_i = \langle \mathbf{y}, \mathbf{x}_i \rangle$ for all $i = 1, \dots, n$.

6. Let V be an inner product space and $U \subseteq V$ a subset. Show that U^{\perp} is a subspace of V.

- 7. Let U, V, W be inner product spaces and $S, T \in \mathcal{L}(U, V)$ and $R \in \mathcal{L}(V, W)$. Show that the following holds
 - 1. $(S + \alpha T)^* = S^* + \overline{\alpha} T^*$ for all $\alpha \in \mathbb{F}$
 - 2. $(S^*)^* = S$
 - 3. $(RS)^* = S^*R^*$
 - 4. $I^* = I$, where $I \colon U \to U$ is the identity operator on U