UNIVERSITY OF TORONTO | Statistical Sciences

DoSS Summer Bootcamp Probability Module 1

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Roadmap

A bridge connecting undergraduate probability and graduate probability

Undergraduate-level probability

- Concrete;
- Examples and scenarios;
- Rely on computation...

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A bridge connecting undergraduate probability and graduate probability

Undergraduate-level probability

- Concrete;
- Examples and scenarios;
- Rely on computation...

Graduate-level probability

- Abstract (measure theory);
- Laws and properties;
- Rely on construction and inference...

Roadmap

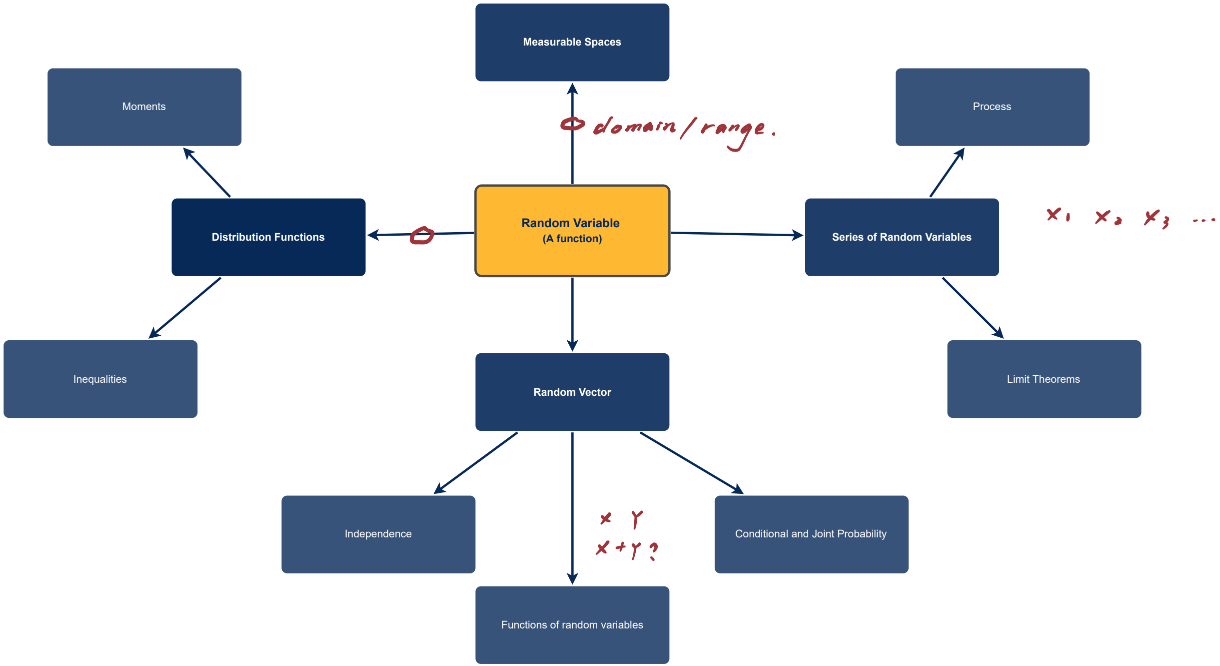

Figure: Roadmap

Outline

- Measurable spaces
  - Sample Space
  - σ-algebra
- Probability measures
  - Measures on σ-field
  - Basic results
- Conditional probability
  - Bayes' rule
  - Law of total probability

Measurable spaces

Sample Space

The sample space Ω is the set of all possible outcomes of an experiment.

Examples:

- Toss a coin: $\{H, T\}$
- Roll a die: $\{1, 2, 3, 4, 5, 6\}$

Measurable spaces

Sample Space

The sample space Ω is the set of all possible outcomes of an experiment.

Examples:

- Toss a coin: $\{H, T\}$
- Roll a die: $\{1, 2, 3, 4, 5, 6\}$

Event

An event is a collection of possible outcomes (subset of the sample space).

$A = \{2, 4\} \subseteq B = \{2, 4, 6\}$

$A = \{1, 3, 5\}$. $B = \{2, 4, 6\}$.

Examples:

- Get head when tossing a coin: $\{H\}$
- Get an even number when rolling a die: $\{2, 4, 6\}$

Measurable spaces

σ-algebra

A σ-algebra (σ-field) $\mathcal{F}$ on Ω is a non-empty collection of subsets of Ω such that

- If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$,
- If $A_1, A_2, \cdots \in \mathcal{F}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Remark: $\varnothing, \Omega \in \mathcal{F}$ $\quad \{\phi, \Omega\}.$ $\quad \{1, 2\}$ $\quad \Omega \subseteq \Omega$

$\exists\, A \in \mathcal{F},\ A^c \in \mathcal{F}$ $\qquad \{\phi, \underbrace{\{1,2\}}_{\Omega}, \{1\}, \{2\}\}$

$A \cup A^c = \Omega \in \mathcal{F}$

$\Omega^c = \phi \in \mathcal{F}$

$\Omega = \{w_1, w_2, \cdots\}.$ power set $2^{\Omega}.$

topological space. $\rightarrow$ Borel. σ-algebra.

$\mathbb{R}.$ $\quad \underbrace{\mathcal{B}(\mathbb{R}), \mathcal{R}}.$

$(\Omega, \mathcal{F}) \leftarrow$ measurable space.

$(\mathbb{R}, \mathcal{R}) \swarrow$

Probability measures

Measures on σ-field

A function $\mu : \mathcal{F} \to \underline{R^+ \cup \{+\infty\}}$ is called a measure if

- $\mu(\varnothing) = 0$,
- If $A_1, A_2, \cdots \in \mathcal{F}$ and $\underline{A_i \cap A_j \overset{i \neq j}{=} \varnothing}$, then $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

If $\underline{\mu(\Omega) = 1}$, then μ is called a probability measure.

} axioms of probability

$(\Omega, \mathcal{F}, \mu) \to$ measure space.

$P. \to$ probability measure space.

Probability measures

Measures on σ-field

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If $\mu(\Omega) = 1$, then μ is called a probability measure.

Properties:

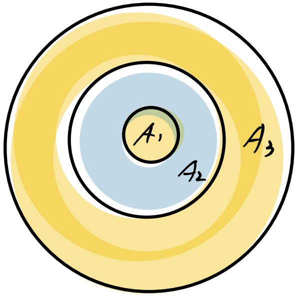
- Monotonicity: $A \subseteq B \quad \Rightarrow \quad \mu(A) \leq \mu(B)$
- Subadditivity: $A \subseteq \cup_{i=1}^{\infty} A_i \quad \Rightarrow \quad \mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$
- Continuity from below: $A_i \nearrow A \quad \Rightarrow \quad \mu(A_i) \nearrow \mu(A)$
- Continuity from above: $A_i \searrow A$ and $\mu(A_i) < \infty \quad \Rightarrow \quad \mu(A_i) \searrow \mu(A)$

$\forall \{x_n\}, \ x_n \to a.$

$\lim_{n \to \infty} f(x_n) = f(a)$

$= f(\lim_{n \to \infty} x_n)$

Probability measures



Proof of continuity from below:

$$A_i \nearrow A \implies \mu(A_i) \nearrow \mu(A).$$

$$A_i \nearrow A \iff A_i \subseteq A_{i+1}, \quad \lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n = A.$$

want to show $\quad \mu(A_i) \nearrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right).$

Let $B_1 = A_1$, $B_2 = A_2 - A_1$, $B_i = A_i - A_{i-1}$

$$= A_2 \cap (A_1)^c.$$

$$B_i \cap B_j = \phi, \quad \bigcup_{i=1}^{n} B_i = A_n, \quad \bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i = A.$$

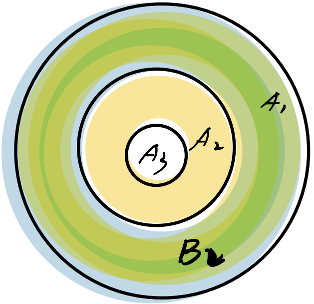
$$\mu(A) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} B_i\right)$$

$$= \lim_{n\to\infty} \mu(A_n)$$

Probability measures

$A_1 = A_i \cup (A_1 - A_i)$

$A_1 = A \cup (A_1 - A)$



Proof of continuity from above:

$A_i \downarrow A$, $\underline{\mu(A_i) < \infty}$. $\Longrightarrow \mu(A_i) \downarrow \mu(A)$.

$A_i \downarrow A \iff A_i \supseteq A_{i+1}, \forall i, \bigcap_{i=1}^{\infty} A_i = A$

Let $B_i = A_1 - A_i$, $B_1 = \phi$. $B_2 = A_1 - A_2$, $B_3 = A_1 - A_3$

$B_i \subseteq B_{i+1}$, $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} (A_1 \cap A_i^c)$

$= A_1 \cap \bigcup_{i=1}^{\infty} A_i^c$

$= A_1 \cap (\bigcap_{i=1}^{\infty} A_i)^c$

$= A_1 - A$

By continuity from below

$\mu(B_i) \uparrow \mu(A_1 - A)$

$\mu(A_1 - A_i) \uparrow \mu(A_1 - A)$

$\mu(A_1 - A_i) = \mu(A_1) - \mu(A_i)$

$\mu(A_1 - A) = \mu(A_1) - \mu(A)$

$\lim_{i \to \infty} (\mu(A_1) - \mu(A_i)) = \mu(A_1) - \mu(A)$

$\mu(A_i) < \infty$

$\lim_{i \to \infty} \mu(A_i) = \mu(A)$.

Remark: $\mu(A_i) < \infty$ is vital.

$\mathbb{R}$, $A_i = [i, \infty)$

μ: Lebesgue measure. $\mu((a, b)) = b - a$.

$\mu(A_i) = \infty$ $\qquad \mu(\phi) = 0$.

$A_i \downarrow A = \phi$.

Probability measures

Examples:

$= \bigcup_{i=1}^{\infty} \{\omega_{a_i}\}$.

$\Omega = \{\omega_1, \omega_2, \cdots\}$, $A = \{\omega_{a_1}, \cdots, \omega_{a_i}, \cdots\} \Rightarrow \mu(A) = \sum_{j=1}^{\infty} \mu(\omega_{a_j})$.
Therefore, we only need to define $\mu(\omega_j) = p_j \geq 0$.
If further $\underline{\sum_{i=1}^{\infty} p_j} = 1$, then μ is a probability measure. P.

- Toss a coin: $\{H, T\}$.

  $p(H) = p \in (0,1)$

  $p(T) = 1-p \in (0,1)$.

- Roll a die:

  $\{1, 2, \cdots 6\}$

  $p(\{i\}) = \frac{1}{6}$. $i = 1, \cdots 6$.

Conditional probability

Original problem:

- What is the probability of some event A?
- $P(A)$ is determined by our probability measure.

New problem:

- Given that B happens, what is the probability of some event A?
- $P(A \mid B)$ is the conditional probability of the event A given B.

Conditional probability

Original problem:

- What is the probability of some event A?
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New problem:

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Example:

- Roll a die: $P(\{2\} \mid \text{even number})$

$$P(\{2\} \mid \{2,4,6\}) = \frac{1}{3}.$$

Conditional probability

$$P(\{2\} \mid \{2,4,6\}) = \frac{P(\{2\})}{P(\{2,4,6\})} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}$$

Bayes' rule

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

Remark: Does conditional probability $\underline{P(\cdot \mid B)}$ satisfy the axioms of a probability measure?

Conditional probability

Multiplication rule

$$P(A \cap B) = P(A \mid B)P(B) = P(B \mid A)P(A)$$

Generalization:

Ω

Law of total probability

Let $A_1, A_2, \cdots, A_n$ be a partition of Ω, such that $P(A_i) > 0$, then

$\bigcup_{i=1}^{n} A_i = \Omega.$

$A_i \cap A_j = \phi.$

$$P(B) = \sum_{i=1}^{n} P(A_i)P(B \mid A_i)$$

$\bigcup_{i=1}^{\infty} A_i = \Omega.$

$P(B) = P(B \cap \Omega) = P(B \cap \bigcup_{i=1}^{n} A_i)$

$= \sum_{i=1}^{n} \underbrace{P(B \cap A_i)}$

$P(B) = \sum_{i=1}^{\infty} P(A_i) P(B \mid A_i)$

Problem Set

Problem 1: Prove that for a σ-field $\mathcal{F}$, if $A_1, A_2, \cdots \in \mathcal{F}$, then $\cap_{i=1}^{\infty} A_i \in \mathcal{F}$.

Problem 2: Prove monotonicity and subadditivity of measure μ on σ-field.

Problem 3: (Monty Hall problem) Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?
(Assumptions: the host will not open the door we picked and the host will only open the door which has a goat.)