

Module 4: Metric Spaces and Sequences II

Operational math bootcamp



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Outline

- Sequences
 - Cauchy sequences
 - subsequences
- Continuous functions
 - Contractions
- Equivalence of metrics



Sequences

Definition (Sequence)

Let (X, d) be a metric space. A *sequence* is an ordered list of points x_n , $n \in \mathbb{N}$, in X , denoted $(x_n)_{n \in \mathbb{N}}$. We say that a sequence $(x_n)_{n \in \mathbb{N}}$ *converges* to a point $x \in X$ if

$$\forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N} \quad \text{s.t.} \quad \underline{d(x_n, x) < \varepsilon \text{ for all } n \geq n_\varepsilon}$$

$$\bar{A} := \{x \in X : \forall \varepsilon > 0 \quad \underline{B_\varepsilon(x)} \cap A \neq \emptyset\}$$

Proposition

Let (X, d) be a metric space, and let $A \subseteq X$. Then \bar{A} is equal to the set of points in X which are limits of a sequence in A .

Proof.

\Rightarrow Let $x \in \bar{A}$. Then by definition, for every $\varepsilon > 0$, $B_\varepsilon(x) \cap A \neq \emptyset$. In particular, this is true for $\varepsilon = 1/n, n \in \mathbb{N}$.

For any $n \in \mathbb{N}$, we can choose $x_n \in A$ s.t. $x_n \in B_{1/n}(x)$, which means $d(x, x_n) < 1/n$. Since $1/n \downarrow 0$ monotonically, $x_n \rightarrow x$.



Proof continued

⊆ Let $x \in X$ be the limit of a sequence $(x_n)_{n \in \mathbb{N}}$ in A .
For every $\varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ s.t. $d(x_n, x) < \varepsilon \forall n \geq n_\varepsilon$.
 \Rightarrow For every $\varepsilon > 0$, $\exists x_n \in A$ s.t. $x_n \in B_\varepsilon(x)$.
 $\therefore \forall \varepsilon > 0$, $A \cap B_\varepsilon(x) \neq \emptyset$. We conclude
 $x \in \overline{A}$.

Corollary

A set $F \subseteq X$, where (X, d) is a metric space, is closed if and only if every sequence in F which converges in X converges to a point in F .

$$A \subseteq X \text{ is closed} \iff A = \overline{A}$$

Cluster points of a set

Definition

Let (X, d) be a metric space and $A \subseteq X$. A point $x \in X$ is a *cluster point* of A (also called accumulation point) if for every $\epsilon > 0$, $B_\epsilon(x)$ contains ~~uncountably~~ ^{infinitely} many points in A .

Proposition

$x \in X$ is a cluster point of $A \subseteq X$ where (X, d) is a metric space if and only if there exists a sequence of points $x_n \in A$, $n \in \mathbb{N}$, such that $x_n \rightarrow x$.

Proof.

(\Rightarrow) Suppose \exists sequence $(x_n)_{n \in \mathbb{N}}$ in A s.t. $x_n \rightarrow x$.
Then $\forall \varepsilon > 0$, $B_\varepsilon(x)$ contains infinitely many elements of the sequence x_n . Since each $x_n \in A$, x is a cluster point of A .

(\Leftarrow) Suppose x is a cluster point of A . Then
for any $\varepsilon > 0$, $\exists x_\varepsilon \in A$ s.t. $x_\varepsilon \in B_\varepsilon(x)$.
Take $\varepsilon = 1/n$. $\exists x_n \in A$ s.t. $x_n \in B_{1/n}(x)$ □

By construction, $x_n \rightarrow x$.

Combining the previous result with the limit characterization of closure gives the following:

Corollary

For $A \subseteq X$, (X, d) a metric space, we have

$$\overline{A} = A \cup \{x \in X : x \text{ is a cluster point of } A\}.$$

Cauchy sequences

Definition (Cauchy sequence)

Let (X, d) be a metric space. A sequence denoted $(x_n)_{n \in \mathbb{N}} \in X$ is called a *Cauchy sequence* if

$$\forall \varepsilon > 0 \exists n_\varepsilon \in \mathbb{N} \text{ s.t. } d(x_n, x_m) < \varepsilon \text{ for all } n, m \geq n_\varepsilon$$

Proposition

Let (X, d) be a metric space, and let $(x_n)_{n \in \mathbb{N}}$ be a convergent sequence in X . Then $(x_n)_{n \in \mathbb{N}}$ is Cauchy.

Proof.

Let $\varepsilon > 0$. Let $(x_n)_{n \in \mathbb{N}}$ be a convergent sequence in X . ^{converging to $x \in X$} Then there exists $n_\varepsilon \in \mathbb{N}$ s.t.

$$d(x, x_n) < \varepsilon/2 \quad \forall n \geq n_\varepsilon.$$

Let $n, m \geq n_\varepsilon$, by triangle inequality,

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$\therefore (x_n)_{n \in \mathbb{N}}$ is Cauchy □

Definition

A metric space where every Cauchy sequence converges (to a point in the space) is called *complete*.

\mathbb{R} , \mathbb{R}^n with usual metrics, are complete

Proposition

Let (X, d) be a metric space, and let $Y \subseteq X$.

- (i) If X is complete and if Y is closed in X , then Y is complete.
- (ii) If Y is complete, then it is closed in X .

Proof.

(i) Let X be a complete metric space. Let $Y \subseteq X$ be closed. Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in Y . Since $Y \subseteq X$, $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in X . $\therefore (x_n)_{n \in \mathbb{N}}$ converges to $x \in X$ since X is complete. Since Y is closed, we must have $x \in Y$. $\therefore Y$ is complete.

(ii) (X, d) metric space, $Y \subseteq X$ is complete.

Let $(y_n)_{n \in \mathbb{N}}$ be a sequence in Y that converges to $y \in X$. $(y_n)_{n \in \mathbb{N}}$ is Cauchy in X (and also in Y). Since Y is complete, $(y_n)_{n \in \mathbb{N}}$ converges to $y' \in Y$. Since sequences in metric spaces

converge to unique point, $y = y'$. $\therefore Y$ is closed. □

Subsequences

Definition

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in a metric space (X, d) . Let $(n_k)_{k \in \mathbb{N}}$ be a sequence of natural numbers with $n_1 < n_2 < \dots$. The sequence $(x_{n_k})_{k \in \mathbb{N}}$ is called a *subsequence* of $(x_n)_{n \in \mathbb{N}}$. If $(x_{n_k})_{k \in \mathbb{N}}$ converges to $x \in X$, we call x a *subsequential limit*.

Example

$$((-1)^n)_{n \in \mathbb{N}} = \{-1, 1, -1, 1, \dots\}$$

This sequence diverges. The subsequences $((-1)^{2n})_{n \in \mathbb{N}}$ and $((-1)^{2n-1})_{n \in \mathbb{N}}$ converge to 1 and -1, respectively.

Proposition

A sequence $(x_n)_{n \in \mathbb{N}}$ in a metric space (X, d) converges to $x \in X$ if and only if every subsequence of $(x_n)_{n \in \mathbb{N}}$ also converges to x .

Proof.

(\Rightarrow) Suppose every subsequence of $(x_n)_{n \in \mathbb{N}}$ converges to $x \in X$. Since $(x_n)_{n \in \mathbb{N}}$ is a subsequence of itself, it must converge to x .



Proof continued

(\Rightarrow) Suppose $(x_n)_{n \in \mathbb{N}}$ converges to $x \in X$.
Let $(x_{n_k})_{k \in \mathbb{N}}$ be an arbitrary subsequence of $(x_n)_{n \in \mathbb{N}}$. Let $\varepsilon > 0$ be arbitrary. $\exists n_\varepsilon \in \mathbb{N}$ s.t. $d(x_n, x) < \varepsilon \quad \forall n \geq n_\varepsilon$. Choose k_ε such that $n_{k_\varepsilon} \geq n_\varepsilon$ (this must exist since $(n_k)_{k \in \mathbb{N}}$ is strictly increasing).
Then $\forall k \geq k_\varepsilon$, $d(x_{n_k}, x) < \varepsilon$. $\therefore x_{n_k} \rightarrow x$

Continuity

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces, let $x_0 \in X$, and let $f : X \rightarrow Y$. f is *continuous* at x_0 if for every sequence $(x_n)_{n \in \mathbb{N}}$ in X that converges to x_0 , we have $\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$.

We say that f is continuous if it is continuous at every point in X .

Theorem

Let (X, d_X) and (Y, d_Y) be metric spaces, let $x_0 \in X$, and let $f : X \rightarrow Y$. The following are equivalent:

- (i) f is continuous at x_0
- (ii) for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \epsilon$ for all $x \in X$ with $d_X(x, x_0) < \delta$
- (iii) for each $\epsilon > 0$, there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$

(i) f is continuous at x_0

(ii) for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \epsilon$ for all $x \in X$ with

$d_X(x, x_0) < \delta$

Proof.

(i) \Rightarrow (ii) We prove the contrapositive.

$\exists \epsilon_0 > 0$ s.t. $\forall \delta > 0 \quad \exists x_\delta \in X$ with $d_X(x_\delta, x_0) < \delta$
but $d_Y(f(x_\delta), f(x_0)) \geq \epsilon_0$

We need to find a sequence in X that converges to x_0
but the images do not converge.

Let $\delta = \frac{1}{n}, n \in \mathbb{N}$. We can pick a sequence x_n using
(*) which converges to x_0 . For each $n \in \mathbb{N}$,
 $d_Y(f(x_n), f(x_0)) \geq \epsilon_0$.

$\therefore \lim_{n \rightarrow \infty} f(x_n) \neq f(x_0)$.

- (i) f is continuous at x_0
- (ii) for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \epsilon$ for all $x \in X$ with $d_X(x, x_0) < \delta$
- (iii) for each $\epsilon > 0$, there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$

Proof continued

(ii) \Rightarrow (iii) Using the definition of pre-image & open ball

(iii) \Rightarrow (i) Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X that converges to x_0 . Let $\epsilon > 0$. By (iii), $\exists \delta > 0$ s.t. $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$.

\Rightarrow If $x \in B_\delta(x_0)$, then x is such that $d_Y(f(x_0), f(x)) < \epsilon$.

Since $(x_n)_{n \in \mathbb{N}}$ converges, $\exists N \in \mathbb{N}$ s.t.

$d(x_n, x_0) < \delta$ for $n \geq N$

So by (iii), $d_Y(f(x_0), f(x_n)) < \epsilon \forall n \geq N$.

$$\therefore \lim_{n \rightarrow \infty} f(x_n) = f(x_\infty).$$

Corollary

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \rightarrow Y$. The following are equivalent:

- (i) f is continuous
- (ii) if $U \subseteq Y$ is open, then $f^{-1}(U)$ is open
- (iii) if $F \subseteq Y$ is closed, then $f^{-1}(F)$ is closed

We need the following results about sets and functions:

Let X and Y be sets and $f : X \rightarrow Y$. Let $A, B \subseteq Y$. Then

① $f^{-1}(A) \subseteq f^{-1}(B)$

② $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$

Proof.

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \rightarrow Y$.

(i) \Rightarrow (ii): Suppose f is continuous (at every point in X) and let $U \subseteq Y$. Let $x \in f^{-1}(U)$. Then $f(x) \in U$.

Since U is open, $\exists \varepsilon_0 > 0$ s.t. $B_{\varepsilon_0}(f(x)) \subseteq U$.

By the pr. thm (iii), $\exists \delta_0 > 0$ s.t. $B_{\delta_0}(x) \subseteq f^{-1}(B_{\varepsilon_0}(f(x)))$.

Since $B_{\varepsilon_0}(f(x)) \subseteq U$, $f^{-1}(B_{\varepsilon_0}(f(x))) \subseteq f^{-1}(U)$.

Thus for each $x \in f^{-1}(U)$, $\exists \delta_0 > 0$ s.t.

$$B_{\delta_0}(x) \subseteq f^{-1}(B_{\varepsilon_0}(f(x))) \subseteq f^{-1}(U)$$



$\therefore f^{-1}(U)$ is open.

Proof continued

(ii) \Rightarrow (i) Let's use the def of continuity from pr. thm (iii).
i.e for $x \in X$, for $\varepsilon > 0 \exists \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$

Let $x \in X$ and let $\varepsilon > 0$ be arbitrary.

Since $B_\varepsilon(f(x))$ is open, by (ii), $f^{-1}(B_\varepsilon(f(x)))$
is also open. Since $x \in f^{-1}(B_\varepsilon(f(x)))$, by
def of open set, $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$
(ii) \Rightarrow (iii) We are done.

(ii) \Rightarrow (iii) Let $F \subseteq Y$ be closed.
~~(iii) \Rightarrow (ii)~~
 $\Rightarrow Y \setminus F$ is open

$\Rightarrow f^{-1}(Y \setminus F)$ is open by (ii)
Since $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$, $f^{-1}(F)$
is closed.

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \rightarrow Y$.

- f is *uniformly continuous* if for all $\epsilon > 0$, there exists $\delta > 0$ such that for every $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$, we have $d_Y(f(x_1), f(x_2)) < \epsilon$
- f is *Lipschitz continuous* if there exists a $K > 0$ such that for every $x_1, x_2 \in X$ we have $d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2)$

Proposition

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \rightarrow Y$.

f is Lipschitz continuous $\Rightarrow f$ is uniformly continuous $\Rightarrow f$ is continuous

Proof is one of your exercises.

Contraction Mapping Theorem

Definition

Let (X, d) be a metric space and let $f : X \rightarrow X$. We say that $x^* \in X$ is a **fixed point** of f if $f(x^*) = x^*$.

Definition

Let (X, d) be a metric space and let $f : X \rightarrow X$. f is a **contraction** if there exists a constant $k \in [0, 1)$ such that for all $x, y \in X$, $d(f(x), f(y)) \leq kd(x, y)$.

Observe that a function is a contraction if and only if it is Lipschitz continuous with constant $K < 1$.

Theorem (Contraction Mapping Theorem)

Suppose that $f : X \rightarrow X$ is a contraction and the metric space X is complete. Then f has a unique fixed point x^ .*



Example

Let $f : [-\frac{1}{3}, \frac{1}{3}] \rightarrow [-\frac{1}{3}, \frac{1}{3}]$ be defined by the mapping $x \mapsto x^2$. Assume we use the standard Euclidean metric, $d(x, y) = |x - y|$. f has a unique fixed point because

- $[-\frac{1}{3}, \frac{1}{3}]$ is a complete metric space

- let $x, y \in [-\frac{1}{3}, \frac{1}{3}]$, then

$$|x^2 - y^2| = |x + y| |x - y| \leq \frac{2}{3} |x - y|$$

$\therefore f$ is a contraction with

$$K = 2/3$$

References

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