Module 3: Metric Spaces and Sequences I

Operational math bootcamp

Statistical Sciences
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Outline

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Definition

Two sets A and B have same cardinality, $|A| = |B|$, if there exists bijection $f : A \to B$.

Example

Which is bigger, $\mathbb{N}$ or $\mathbb{N}_0$?

Cantor-Schröder-Bernstein

Definition

We say that the cardinality of a set A is less than the cardinality of a set B, denoted $|A| \leq |B|$ if there exists an injection $f : A \to B$.

Theorem (Cantor-Bernstein)

Let A, B, be sets. If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Example

$|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$

Proof.

$\square$

Definition

Let A be a set.

1. A is *finite* if there exists an $n \in \mathbb{N}$ and a bijection $f : \{1, \ldots, n\} \to A$
2. A is *countably infinite* if there exists a bijection $f : \mathbb{N} \to A$
3. A is *countable* if it is finite or countably infinite
4. A is *uncountable* otherwise

Example

The rational numbers are countable, and in fact $|\mathbb{Q}| = |\mathbb{N}|$.

Let's look at $\mathbb{Q}^+ := \{x \in \mathbb{Q} : x > 0\}$. The fact that the rationals are countable relies on this famous way of listing the rational numbers:

$$\begin{array}{cccccc} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ 2 & \color{red}{\frac{2}{2}} & \frac{2}{3} & \color{red}{\frac{2}{4}} & \frac{2}{5} & \cdots \\ 3 & \frac{3}{2} & \color{red}{\frac{3}{3}} & \frac{3}{4} & \frac{3}{5} & \cdots \\ 4 & \color{red}{\frac{4}{2}} & \frac{4}{3} & \color{red}{\frac{4}{4}} & \frac{4}{5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Example

This is a map from $\mathbb{N}$ to $\mathbb{Q}^+$. As long as we skip any fraction that is already in our list as we go along, it is injective. Since we can find an injection from $\mathbb{Q}^+$ to $\mathbb{N} \times \mathbb{N}$ (exercise), we have that $|\mathbb{Q}^+| = |\mathbb{N}|$. We can extend this to $\mathbb{Q}$. To do so, let $f : \mathbb{N} \to \mathbb{Q}^+$ be a bijection (which exists by the previous part). Then we can define another bijection $g : \mathbb{N} \to \mathbb{Q}$ by setting $g(1) = 0$ and

$$g(n) = \begin{cases} f(n) & \text{if } n \text{ is even,} \\ -f(n) & \text{if } n \text{ is odd,} \end{cases}$$

for $n > 1$.

Theorem

The cardinality of $\mathbb{N}$ is smaller than that of $(0, 1)$.

Proof.

First, we show that there is an injective map from $\mathbb{N}$ to $(0, 1)$.

Next, we show that there is no surjective map from $\mathbb{N}$ to (0, 1). We use the fact that every number $r \in (0, 1)$ has a binary expansion of the form $r = 0.\sigma_1\sigma_2\sigma_3 \ldots$ where $\sigma_i \in \{0, 1\}$, $i \in \mathbb{N}$. $\square$

Proof.

Now we suppose in order to derive a contradiction that there does exist a surjective map f from $\mathbb{N}$ to $(0, 1)$., i.e. for $n \in \mathbb{N}$ we have $f(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n)\ldots$. This means we can list out the binary expansions, for example like

$$
\begin{aligned}
f(1) =& 0.\textcolor{red}{0}0000000\ldots \\
f(2) =& 0.1\textcolor{red}{1}11111111\ldots \\
f(3) =& 0.01\textcolor{red}{0}1010101\ldots \\
f(4) =& 0.101\textcolor{red}{0}101010\ldots
\end{aligned}
$$

We will construct a number $\tilde{r} \in (0, 1)$ that is not in the image of f. $\square$

Proof.

Define $\tilde{r} = 0.\tilde{\sigma}_1\tilde{\sigma}_2\ldots$, where we define the nth entry of $\tilde{r}$ to be the the opposite of the nth entry of the nth item in our list:

$$\tilde{\sigma}_n = \begin{cases} 1 & \text{if } \sigma_n(n) = 0, \\ 0 & \text{if } \sigma_n(n) = 1. \end{cases}$$

Then $\tilde{r}$ differs from $f(n)$ at least in the nth digit of its binary expansion for all $n \in \mathbb{N}$. Hence, $\tilde{r} \notin f(\mathbb{N})$, which is a contradiction to f being surjective. This technique is often referred to as Cantor's diagonal argument. $\square$

Proposition

(0,1) and $\mathbb{R}$ have the same cardinality.

Proof.

The map $f : \mathbb{R} \to (0, 1)$ defined by $x \mapsto \frac{1}{\pi} \left(\arctan(x) + \frac{\pi}{2}\right)$ is a bijection. $\square$

We have shown that there are different sizes of infinity, as the cardinality of $\mathbb{N}$ is infinite but still smaller than that of $\mathbb{R}$ or $(0, 1)$. In fact, we have

$$|\mathbb{N}| = |\mathbb{N}_0| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}|.$$

Because of this, there are special symbols for these two cardinalities: The cardinality of $\mathbb{N}$ is denoted $\aleph_0$, while the cardinality of $\mathbb{R}$ is denoted $\mathfrak{c}$.

References

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