Module 4: Metric Spaces and Sequences II

1. Find the closure, interior, and boundary of the following sets using Euclidean distance:

   (i) $\{(x, y) \in \mathbb{R}^2 : y < x^2\} \subseteq \mathbb{R}^2$

   (ii) $[0, 1) \times [0, 1) \subseteq \mathbb{R}^2$

   (iii) $\{0\} \cup \{1/n \colon n \in \mathbb{N}\} \subseteq \mathbb{R}$

2. Prove the following: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space (X, d) that converges to a point $x \in X$. Then x is unique.

3. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}$ such that $x_n \to x$ and and $y_n \to y$, with $\alpha, x, y, \in \mathbb{R}$.

   (i) Show that $\alpha\, x_n \to \alpha\, x$.

   (i) Show that $x_n + y_n \to x + y$.

4. Show that discrete metric spaces (i.e. those with the metric defined as define $d\colon X \times X \to \mathbb{R}$ by $d(x, x) = 0$ and $d(x, y) = 1$ for $x \neq y \in X$) are complete.

5. Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$. Prove that

$$f \text{ is Lipschitz continuous } \Rightarrow f \text{ is uniformly continuous } \Rightarrow f \text{ is continuous.}$$

Provide examples to show that the other directions do not hold.

6. Show that the function $f(x) = \frac{1}{2}\left(x + \frac{5}{x}\right)$ has a unique fixed point on $(0, \infty)$. What is it? (Hint: you will have to restrict the interval.)