Module 1: Proofs

Operational math bootcamp

Statistical Sciences
UNIVERSITY OF TORONTO

Emma Kroell

University of Toronto

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Outline

- Logic
- Review of Proof Techniques
- Introduction to Set Theory

Propositional logic

Propositions are statements that could be true or false. They have a corresponding **truth value**.

ex. "n is odd" and "n is divisible by 2" are propositions . Let's call them P and Q. Whether they are true or not depends on what n is.

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We can combine statements:

- $P \wedge Q$ is the statement "n is odd and n is divisible by 2".
- $P \vee Q$ is the statement "n is odd or n is divisible by 2". We always assume the inclusive or unless specifically stated otherwise.

Examples

Symbol	Meaning
Capital letters	propositions
$\implies$	implies
$\wedge$	and
$\vee$	inclusive or
$\neg$	not

- If it's not raining, I won't bring my umbrella.
- I'm a banana or Toronto is in Canada.
- If I pass this exam, I'll be both happy and surprised.

Truth values

Example

If it is snowing, then it is cold out.
It is snowing.
Therefore, it is cold out.

Write this using propositional logic:

Truth values

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Write this using propositional logic:

$$P \implies Q$$
$$P$$
$$\text{Conclusion: } Q$$

How do we know if this statement is true or not?

Truth table

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$$P \implies Q$$

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

Logical equivalence

$$P \implies Q$$

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$$\neg P \vee Q$$

P	Q	$\neg P$	$\neg P \vee Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Logical equivalence

$$P \implies Q$$

P	Q	$P \implies Q$
T	T	T
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$$\neg P \vee Q$$

P	Q	$\neg P$	$\neg P \vee Q$
T	T	F	T
T	F	F	F
F	T	T	T
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What is $\neg(P \implies Q)$?

Types of proof

- Direct
- Contradiction
- Contrapositive
- Induction

Direct Proof

Approach: Use the definition and known results.

Example

Claim

The product of an even number with another integer is even.

Approach: use the definition of even.

Direct Proof

Claim

The product of an even number with another integer is even.

Definition

We say that an integer n is **even** if there exists another integer j such that $n = 2j$.
We say that an integer n is **odd** if there exists another integer j such that $n = 2j + 1$.

Proof.

Let $n, m \in \mathbb{Z}$, with n even. By definition, there $\exists\ j \in \mathbb{Z}$ such that $n = 2j$. Then

$$nm = (2j)m = 2(jm)$$

Therefore nm is even by definition. $\square$

Claim

If an integer squared is even, then the integer is itself even.

How would you approach this proof?

Proof by contrapositive

$$P \implies Q$$

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

$$\neg P \implies \neg Q$$

P	Q	$\neg P$	$\neg Q$	$\neg Q \implies \neg P$
T	T	F	F	
T	F	F	T	
F	T	T	F	
F	F	T	T	

Proof by contrapositive

$$P \implies Q$$

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

$$\neg Q \implies \neg P$$

P	Q	$\neg P$	$\neg Q$	$\neg Q \implies \neg P$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Proof by contrapositive

Claim

If an integer squared is even, then the integer is itself even.

Proof.

We prove the contrapositive. Let n be odd. Then there exists $k \in \mathbb{Z}$ such that $n = 2k + 1$. We compute

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Thus n^2 is odd.

$\square$

Proof by contradiction

Claim

The sum of a rational number and an irrational number is irrational.

Proof.

Let $q \in \mathbb{Q}$ and $r \in \mathbb{R} \setminus \mathbb{Q}$. Suppose in order to derive a contradiction that their sum is rational, i.e. $r + q = s$ where $s \in \mathbb{Q}$. But then $r = s - q \in \mathbb{Q}$. Contradiction. $\square$

Summary

In sum, to prove $P \implies Q$:

Direct proof:	assume P, prove Q
Proof by contrapositive:	assume $\neg Q$, prove $\neg P$
Proof by contradiction:	assume $P \wedge \neg Q$ and derive something that is impossible

Induction

Well-ordering principle for $\mathbb{N}$

Every nonempty set of natural numbers has a least element.

Principle of mathematical induction

Let n_0 be a non-negative integer. Suppose P is a property such that

1. (base case) $P(n_0)$ is true
2. (induction step) For every integer $k \geq n_0$, if $P(k)$ is true, then $P(k+1)$ is true.

Then $P(n)$ is true for every integer $n \geq n_0$

Note: Principle of strong mathematical induction: For every integer $k \geq n_0$, if $P(n)$ is true for every $n = n_0, \ldots, k$, then $P(k+1)$ is true.

Claim

$n! > 2^n$ if $n \geq 4$.

Proof.

We prove this by induction on n.

Base case: Let $n = 4$. Then $n! = 4! = 24 > 16 = 2^4$.

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Inductive hypothesis: Suppose for some $k \geq 4$, $k! > 2^k$.

Claim

$n! > 2^n$ if $n \geq 4$.

Proof.

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Base case: Let $n = 4$. Then $n! = 4! = 24 > 16 = 2^4$.

Inductive hypothesis: Suppose for some $k \geq 4$, $k! > 2^k$.

Then

$$(k+1)! = (k+1)k! > (k+1)2^k > 2(2^k) = 2^{k+1}.$$

$\square$

Claim

Every integer $n \geq 2$ can be written as the product of primes.

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We prove this by induction on n.
Base case: $n = 2$ is prime.
Inductive hypothesis: Suppose for some $k \geq 2$ that one can write every integer n such that $2 \leq n \leq k$ as a product of primes.

Claim

Every integer $n \geq 2$ can be written as the product of primes.

Proof.

We prove this by induction on n.
Base case: $n = 2$ is prime.
Inductive hypothesis: Suppose for some $k \geq 2$ that one can write every integer n such that $2 \leq n \leq k$ as a product of primes.
We must show that we can write $k + 1$ as a product of primes.
First, if $k + 1$ is prime then we are done.

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Base case: $n = 2$ is prime.
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We must show that we can write $k + 1$ as a product of primes.
First, if $k + 1$ is prime then we are done.
Otherwise, if $k + 1$ is not prime, by definition it can be written as a product of some integers a, b such that $1 < a, b < k + 1$. By the induction hypothesis, a and b can both be written as products of primes, so we are done. $\square$

Introduction to Set Theory

- we define a *set* to be a collection of mathematical objects
- if S is a set and x is one of the objects in the set, we say x is an element of S and denote it by $x \in S$.
- the set of no elements is called empty set and is denoted by $\emptyset$

Definition (Subsets, Union, Intersection)

Let S, T be sets.

- We say that S is a *subset* of T, denoted $S \subseteq T$, if $s \in S$ implies $s \in T$.
- We say that $S = T$ if $S \subseteq T$ and $T \subseteq S$.
- We define the *union* of S and T, denoted $S \cup T$, as all the elements that are in *either* S and T.
- We define the *intersection* of S and T, denoted $S \cap T$, as all the elements that are in *both* S and T.
- We say that S and T are *disjoint* if $S \cap T = \emptyset$.

Some examples

Example

$\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

Example

Let $a < b \cup \{-\infty, \infty\}$.
Open interval: $(a, b) := \{x \in \mathbb{R} : a < x < b\}$
Closed interval: $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$
We can also define half-open intervals.

Example

Let $A = \{x \in \mathbb{N} : 3|x\}$ and $B = \{x \in \mathbb{N} : 6|x\}$ Show that $B \subseteq A$.

Proof.

$\square$

Definition

Let $A, B \subseteq X$. We define the *set-theoretic difference* of A and B, denoted $A \setminus B$ (sometimes $A - B$) as the elements of X that are in A but *not* in B.
The complement of a set $A \subseteq X$ is the set $A^c := X \setminus A$.

Example

Let $X \subseteq \mathbb{R}$ defined as $X = \{x \in \mathbb{R} : 0 < x \leq 40\}$. Then
$X^c = \{x \in \mathbb{R} : x \leq 0 \text{ or } x > 40\}$.

References

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