

## Exercises for Module 8: Linear Algebra II

1. A square matrix is called *nilpotent* if  $\exists k \in \mathbb{N}$  such that  $A^k = 0$ . Show that for a nilpotent matrix  $A$ ,  $|A| = 0$ .
2. A real square matrix  $Q$  is called *orthogonal* if  $Q^T Q = I$ . Prove that if  $Q$  is orthogonal, then  $|Q| = \pm 1$ .
3. An  $n \times n$  matrix is called *antisymmetric* if  $A^T = -A$ . Prove that if  $A$  is antisymmetric and  $n$  is odd, then  $|A| = 0$ .

4. Let  $V$  be an inner product space,  $U$  a vector space and  $S: U \rightarrow V$ ,  $T: U \rightarrow V$  be linear maps. Show that  $\langle S\mathbf{u}, \mathbf{v} \rangle = \langle T\mathbf{u}, \mathbf{v} \rangle$  for all  $\mathbf{u} \in U$  and  $\mathbf{v} \in V$  implies  $S = T$ .

5. Let  $V$  be an inner product space and  $\mathbf{x}_1, \dots, \mathbf{x}_n$  be an orthonormal basis and  $\mathbf{y} \in V$ . Then,  $\mathbf{y}$  has a unique representation  $\mathbf{y} = \sum_{i=1}^n \alpha_i \mathbf{x}_i$ . Show that  $\alpha_i = \langle \mathbf{y}, \mathbf{x}_i \rangle$  for all  $i = 1, \dots, n$ .

6. Let  $V$  be an inner product space and  $U \subseteq V$  a subset. Show that  $U^\perp$  is a subspace of  $V$ .

7. Let  $U, V, W$  be inner product spaces and  $S, T \in \mathcal{L}(U, V)$  and  $R \in \mathcal{L}(V, W)$ . Show that the following holds

1.  $(S + \alpha T)^* = S^* + \overline{\alpha}T^*$  for all  $\alpha \in \mathbb{F}$
2.  $(S^*)^* = S$
3.  $(RS)^* = S^*R^*$
4.  $I^* = I$ , where  $I: U \rightarrow U$  is the identity operator on  $U$

