Module 4: Metric Spaces and Sequences II

Operational math bootcamp

Statistical Sciences
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Outline

- Sequences
  - Cauchy sequences
  - subsequences
- Continuous functions
  - Contractions
- Equivalence of metrics

Sequences

Definition (Sequence)

Let (X, d) be a metric space. A *sequence* is an ordered list of points x_n, $n \in \mathbb{N}$, in X, denoted $(x_n)_{n\in\mathbb{N}}$. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ *converges* to a point $x \in X$ if

$\forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N} \quad$ s.t. $\quad d(x_n, x) < \varepsilon$ for all $n \geq n_\varepsilon$

$\overline{A} := \{x \in X : \forall \varepsilon > 0 \ B_\varepsilon(x) \cap A \neq \emptyset\}$

Proposition

Let (X, d) be a metric space, and let $A \subseteq X$. Then $\overline{A}$ is equal to the set of points in X which are limits of a sequence in A.

Proof.

($\Rightarrow$) Let $x \in \overline{A}$. Then by definition, for every $\varepsilon > 0$, $B_\varepsilon(x) \cap A \neq \emptyset$. In particular, this is true for $\varepsilon = 1/n$, $n \in \mathbb{N}$.

For any $n \in \mathbb{N}$, we can choose $x_n \in A$ s.t. $x_n \in B_{1/n}(x)$, which means $d(x, x_n) < 1/n$.

Since $\frac{1}{n} \downarrow 0$ monotonically, $x_n \to x$.

$\square$

Proof continued

($\Leftarrow$) Let $x \in X$ be the limit of a sequence $(x_n)_{n \in \mathbb{N}}$ in A.

For every $\varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ s.t $d(x_n, x) < \varepsilon$ $\forall n \geq n_\varepsilon$.

$\Rightarrow$ For every $\varepsilon > 0$, $\exists x_n \in A$ s.t $x_n \in B_\varepsilon(x)$.

$\therefore$ $\forall \varepsilon > 0$, $A \cap B_\varepsilon(x) \neq \emptyset$. We conclude $x \in \overline{A}$.

Corollary

A set $F \subseteq X$, where (X, d) is a metric space, is closed if and only if every sequence in F which converges in X converges to a point in F.

$A \subset X$ is closed $\Longleftrightarrow$ $A = \overline{A}$

Cluster points of a set

Definition

Let (X, d) be a metric space and $A \subseteq X$. A point $x \in X$ is a *cluster point* of A (also called accumulation point) if for every $\epsilon > 0$, $B_\epsilon(x)$ contains ~~uncountably~~ infinitely many points in A.

Proposition

$x \in X$ is a cluster point of $A \subseteq X$ where (X, d) is a metric space if and only if there exists a sequence of points $x_n \in A$, $n \in \mathbb{N}$, such that $x_n \to x$.

Proof.

($\Leftarrow$) Suppose $\exists$ sequence $(x_n)_{n \in \mathbb{N}}$ in A s.t. $x_n \to x$. Then $\forall \varepsilon > 0$, $B_\varepsilon(x)$ contains infinitely many elements of the sequence x_n. Since each $x_n \in A$, x is a cluster point of A.

($\Rightarrow$) Suppose x is a cluster point of A. Then for any $\varepsilon > 0$, $\exists x_\varepsilon \in A$ s.t. $x_\varepsilon \in B_\varepsilon(x)$. Take $\varepsilon = 1/n$. $\exists x_n \in A$ s.t. $x_n \in B_{1/n}(x)$. By construction, $x_n \to x$.

$\square$

Combining the previous result with the limit characterization of closure gives the following:

Corollary

For $A \subseteq X$, (X, d) a metric space, we have

$$\overline{A} = A \cup \{x \in X : x \text{ is a cluster point of } A\}.$$

Cauchy sequences

Definition (Cauchy sequence)

Let (X, d) be a metric space. A sequence denoted $(x_n)_{n\in\mathbb{N}} \in X$ is called a *Cauchy sequence* if

$$\forall \varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N} \text{ s.t } d(x_n, x_m) < \varepsilon \text{ for all } n, m \geq n_\varepsilon$$

Proposition

Let (X, d) be a metric space, and let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in X. Then $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

Proof.

Let $\varepsilon > 0$. Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in X converging to $x \in X$. Then there exists $n_\varepsilon \in \mathbb{N}$ s.t. $d(x, x_n) < \varepsilon/2 \ \forall n \geq n_\varepsilon$.

Let $n, m \geq n_\varepsilon$, by triangle inequality,

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$\therefore (x_n)_{n\in\mathbb{N}}$ is Cauchy $\square$

Definition

A metric space where every Cauchy sequence converges (to a point in the space) is called *complete*.

$\mathbb{R}$, $\mathbb{R}^n$ with usual metrics, are complete

Proposition

Let (X, d) be a metric space, and let $Y \subseteq X$.

(i) If X is complete and if Y is closed in X, then Y is complete.

(ii) If Y is complete, then it is closed in X.

Proof.

(i) Let X be a complete metric space. Let $Y \subseteq X$ be closed. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in Y. Since $Y \subseteq X$, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X.
$\therefore (x_n)_{n\in\mathbb{N}}$ converges to $x \in X$ since X is complete. Since Y is closed, we must have $x \in Y$.
$\therefore Y$ is complete

(ii) (X, d) metric space, $Y \subseteq X$ is complete.
Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in Y that converges to $y \in X$. $(y_n)_{n\in\mathbb{N}}$ is Cauchy in X (and also in Y). Since Y is complete, $(y_n)_{n\in\mathbb{N}}$ converges to $y' \in Y$. Since sequences in metric spaces converge to unique point, $y = y'$. $\therefore Y$ is closed. $\square$

Subsequences

Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space (X, d). Let $(n_k)_{k\in\mathbb{N}}$ be a sequence of natural numbers with $n_1 < n_2 < \cdots$. The sequence $(x_{n_k})_{k\in\mathbb{N}}$ is called a *subsequence* of $(x_n)_{n\in\mathbb{N}}$. If $(x_{n_k})_{k\in\mathbb{N}}$ converges to $x \in X$, we call x a *subsequential limit*.

Example

$((-1)^n)_{n\in\mathbb{N}} = \{-1, 1, -1, 1, \cdots\}$

This sequence diverges. The subsequences $((-1)^{2n})_{n\in\mathbb{N}}$ $((-1)^{2n-1})_{n\in\mathbb{N}}$ converge to 1 and -1, respectively.

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Proposition

A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X,d) converges to $x \in X$ if and only if every subsequence of $(x_n)_{n\in\mathbb{N}}$ also converges to x.

Proof.

$\Leftarrow$ Suppose every subsequence of $(x_n)_{n\in\mathbb{N}}$ converges to $x \in X$. Since $(x_n)_{n\in\mathbb{N}}$ is a subsequence of itself, it must converge to x.

$\square$

Proof continued

($\Rightarrow$) Suppose $(x_n)_{n\in\mathbb{N}}$ converges to $x \in X$.
Let $(x_{n_k})_{k\in\mathbb{N}}$ be an arbitrary subsequence of $(x_n)_{n\in\mathbb{N}}$. Let $\varepsilon > 0$ be arbitrary. $\exists\, n_\varepsilon \in \mathbb{N}$ s.t $d(x_n, x) < \varepsilon$ $\forall n \geq n_\varepsilon$. Choose k_ε such that $n_{k_\varepsilon} \geq n_\varepsilon$ (this must exist since $(n_k)_{k\in\mathbb{N}}$ is strictly increasing).
Then $\forall k \geq k_\varepsilon$, $d(x_{n_k}, x) < \varepsilon$. $\therefore x_{n_k} \to x$

Continuity

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces, let $x_0 \in X$, and let $f : X \to Y$. f is *continuous* at x_0 if for every sequence $(x_n)_{n\in\mathbb{N}}$ in X that converges to x_0, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

We say that f is continuous if it is continuous at every point in X.

Theorem

Let (X, d_X) *and* (Y, d_Y) *be metric spaces, let* $x_0 \in X$*, and let* $f : X \to Y$*. The following are equivalent:*

(i) f *is continuous at* x_0

(ii) *for all* $\epsilon > 0$*, there exists* $\delta > 0$ *such that* $d_Y(f(x), f(x_0))) < \epsilon$ *for all* $x \in X$ *with* $d_X(x, x_0) < \delta$

(iii) *for each* $\epsilon > 0$*, there is* $\delta > 0$ *such that* $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$

(i) f is continuous at x_0

(ii) for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \epsilon$ for all $x \in X$ with $d_X(x, x_0) < \delta$

Proof.

(i) $\Rightarrow$ (ii) We prove the contrapositive.

$\exists \varepsilon_0 > 0$ s.t. $\forall \delta > 0$ $\exists x_\delta \in X$ with $d_X(x_\delta, x_0) < \delta$ but $d_Y(f(x_\delta), f(x_0)) \geq \varepsilon_0$ (*)

We need to find a sequence in X that converges to x_0 but the images do not converge.

Let $\delta = \frac{1}{n}$, $n \in \mathbb{N}$. We can pick a sequence x_n using (*) which converges to x_0. For each $n \in \mathbb{N}$,

$$d_Y(f(x_n), f(x_0)) \geq \varepsilon_0.$$

$$\therefore \lim_{n \to \infty} f(x_n) \neq f(x_0).$$

(i) f is continuous at x_0
(ii) for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0))) < \epsilon$ for all $x \in X$ with $d_X(x, x_0) < \delta$
(iii) for each $\epsilon > 0$, there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$

Proof continued

(ii) $\Rightarrow$ (iii) Using the definition of pre-image & open ball

(iii) $\Rightarrow$ (i) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X that converges to x_0. Let $\varepsilon > 0$. By (iii), $\exists\, \delta > 0$ s.t.
$B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0)))$.
$\Rightarrow$ If $x \in B_\delta(x_0)$, then x is such that
$d_Y(f(x_0), f(x)) < \varepsilon$.
Since $(x_n)_{n\in\mathbb{N}}$ converges, $\exists\, N \in \mathbb{N}$ s.t.
$d(x_n, x) < \delta$ for $n \geq N$
So by (iii), $d_Y(f(x_0), f(x_n)) < \varepsilon \;\forall n \geq N$.

$$\therefore \lim_{n\to\infty} f(x_n) = f(x_0).$$

Corollary

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$. The following are equivalent:

(i) f is continuous

(ii) if $U \subseteq Y$ is open, then $f^{-1}(U)$ is open

(iii) if $F \subseteq Y$ is closed, then $f^{-1}(F)$ is closed

We need the following results about sets and functions:
Let X and Y be sets and $f : X \to Y$. Let $A, B \subseteq Y$. Then

1. $A \subseteq B \Rightarrow f^{-1}(A) \subseteq f^{-1}(B)$ (corrected after lecture)
2. $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$

Proof.

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$.
(i) $\Rightarrow$ (ii): Suppose f is continuous (at every point in X) and let $U \subseteq Y$. Let $x \in f^{-1}(U)$. Then $f(x) \in U$.
Since U is open, $\exists\, \varepsilon_0 > 0$ s.t. $B_{\varepsilon_0}(f(x)) \subseteq U$.
By the pr. thm (iii), $\exists\, \delta_0 > 0$ s.t. $B_{\delta_0}(x) \subseteq f^{-1}(B_{\varepsilon_0}(f(x)))$.
Since $B_{\varepsilon_0}(f(x)) \subseteq U$, $f^{-1}(B_{\varepsilon_0}(f(x))) \subseteq f^{-1}(U)$.
Thus for each $x \in f^{-1}(U)$, $\exists\, \delta_0 > 0$ s.t.
$B_{\delta_0}(x) \subseteq f^{-1}(B_{\varepsilon_0}(f(x))) \subseteq f^{-1}(U)$.

□

$\therefore f^{-1}(U)$ is open.

Proof continued

(ii) $\Rightarrow$ (i) Let's use the def of continuity from pr. thm (iii).
i.e for $x \in X$, for $\varepsilon > 0$ $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$

Let $x \in X$ and let $\varepsilon > 0$ be arbitrary.
Since $B_\varepsilon(f(x))$ is open, by (ii), $f^{-1}(B_\varepsilon(f(x)))$ is also open. Since $x \in f^{-1}(B_\varepsilon(f(x)))$, by def of open set, $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$
We are done.

(ii) $\Rightarrow$ (iii)

(ii) $\Rightarrow$ (iii) Let $F \subseteq Y$ be closed.
$\Rightarrow Y \setminus F$ is open
$\Rightarrow f^{-1}(Y \setminus F)$ is open by (ii)
Since $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$, $f^{-1}(F)$ is closed.

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$.

- f is *uniformly continuous* if for all $\epsilon > 0$, there exists $\delta > 0$ such that for every $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$, we have $d_Y(f(x_1), f(x_2))) < \epsilon$
- f is *Lipschitz continuous* if there exists a $K > 0$ such that for every $x_1, x_2 \in X$ we have $d_Y(f(x_1), f(x_2))) \leq K d_X(x_1, x_2)$

Proposition

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$.

$$f \text{ is Lipschitz continuous } \Rightarrow \text{ f is uniformly continuous } \Rightarrow \text{ f is continuous}$$

Proof is one of your exercises.

Contraction Mapping Theorem

Definition

Let (X, d) be a metric space and let $f : X \to X$. We say that $x^* \in X$ is a *fixed point* of f if $f(x^*) = x^*$.

Definition

Let (X, d) be a metric space and let $f : X \to X$. f is a *contraction* if there exists a constant $k \in [0, 1)$ such that for all $x, y \in X$, $d(f(x), f(y)) \leq kd(x, y)$.

Observe that a function is a contraction if and only if it is Lipschitz continuous with constant $K < 1$.

Theorem (Contraction Mapping Theorem)

Suppose that $f : X \to X$ is a contraction and the metric space X is complete. Then f has a unique fixed point x^*.

Example

Let $f : \left[-\frac{1}{3}, \frac{1}{3}\right] \to \left[-\frac{1}{3}, \frac{1}{3}\right]$ be defined by the mapping $x \mapsto x^2$. Assume we use the standard Euclidean metric, $d(x, y) = |x - y|$. f has a unique fixed point because

- $[-\frac{1}{3}, \frac{1}{3}]$ is a complete metric space
- let $x, y \in [-\frac{1}{3}, \frac{1}{3}]$, then

$$|x^2 - y^2| = |x+y||x-y| \leq \frac{2}{3}|x-y|$$

$\therefore f$ is a contraction with $K = 2/3$

Equivalent metrics

Definition (Equivalent metrics)

Two metrics d_1 and d_2 on a set X are *equivalent* if the identity maps from (X, d_1) to (X, d_2) and from (X, d_2) to (X, d_1) are continuous.

Proposition

Two metrics d_1, d_2 on a set X are equivalent if and only if they have the same open sets or the same closed sets.

Definition

Two metrics d_1 and d_2 on a set X are *strongly equivalent* if for every $x, y \in X$, there exists constants $\alpha > 0$ and $\beta > 0$ such

$$\alpha d_1(x, y) \leq d_2(x, y) \leq \beta d_1(x, y).$$

If two metrics are strongly equivalent then they are equivalent. The proof of this is one of the exercises.

Example

We show that the Euclidean distance (induced by 2-norm) and the metric induced by the ∞-norm are equivalent on $\mathbb{R}^n$.

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