

Module 4: Statistical inference (I)

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05/10/2022

Outline

This module we will review

- Basics of probability
- Fundamental concepts in inference

Probability distributions

- In statistics, we try to draw conclusions about a larger population from a sample of observations.
- We use mathematical models to capture probabilistic behavior of a population.
- This behavior is modeled using probability distributions.

Density/Distribution functions

Definition (Cumulative Distribution Function)

$$F_X(x) = P(X \leq x) \quad \forall x \in \mathbb{R}$$

Density/Distribution functions (cont'd)

Definition (Probability Mass Function)

For a discrete RV , the probability mass function (PMF) is:

$$f_X(x) = P(X = x) \quad \forall x \in \mathbb{R}$$

Definition (Probability Density Function)

For a continuous RV , the probability density function (PDF) is:

$$f_X(x) = \left. \frac{\partial}{\partial t} F(t) \right|_{t=x}$$

So $F_X(x) = \int_{-\infty}^x f_X(t) dt \forall x \in \mathbb{R}$.

Note that $f_X \geq 0$ for $\forall x$, and thus F_X is an increasing function.

Expectation and Variance

Definition (Expectation)

A measure of central tendency (a weighted average of the values of X)

$$E[X] = \sum_{x \in S} xP(X = x) \text{ for discrete RV taking values from } S$$

$$E[X] = \int_{-\infty}^{\infty} xf_X(x)dx \text{ for continuous RV}$$

Definition (Variance)

A measure of the spread of a distribution

$$\text{Var}(X) = \sum_{x \in S} (x - E[X])^2 P(X = x) \text{ for discrete RV}$$

$$\text{Var}(X) = \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x)dx \text{ for continuous RV}$$

Discrete random variable

A discrete random variable has a countable number of possible values.

Bernoulli and Binomial random variable

- Consider the event of flipping a (possibly unfair) coin.
- $Y \in \{0, 1\}$ represents success and failure.
- Suppose we only flip the coin once,
 - We can express $P(Y = 1) = p$ and $P(Y = 0) = 1 - p$
- Bernoulli distribution

$$P(Y = y) = p^y(1 - p)^{1-y} \quad \text{for } y = 0, 1$$

- If we flip the coin n times,
- Binomial distribution

$$P(Y = y) = \binom{n}{y} p^y(1 - p)^{n-y} \quad \text{for } y = 0, 1, \dots, n$$

Binomial distributions with different values of n and p

If $Y \sim \text{Binomial}(n, p)$, then $E(Y) = np$ and $SD(Y) = \sqrt{np(1-p)}$.

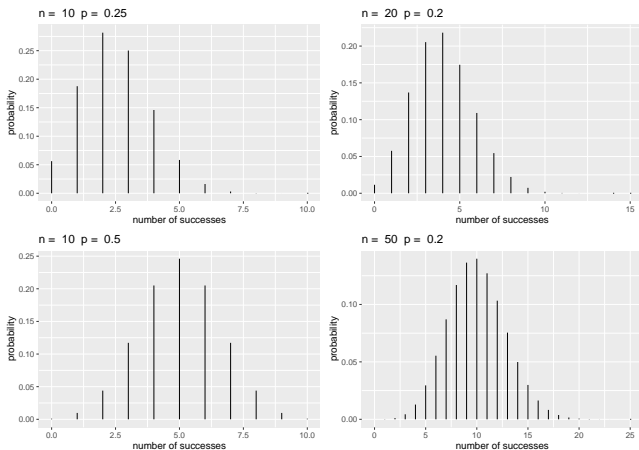


Figure 1: Binomial distributions with different values of n and p .

How to generate in R?

All common distributions have four functions in R:

- Density
`dbinom(x, size, prob)`
- Distribution function
`pbinom(q, size, prob)`
- Quantile function
`qbinom(p, size, prob)`
- Random generation
`rbinom(n, size, prob)`

Not sure? Using `?` with any of the four functions, e.g. `?qbinom`

Example of binomial distribution computing

Question: While taking a multiple choice test, a student encountered 10 problems where she ended up completely guessing, randomly selecting one of the four options. What is the chance that she got exactly 2 of the 10 correct?

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Answer: Knowing that the student randomly selected her answers, we assume she has a 25% chance of a correct response.

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R computing:

```
dbinom(2, size = 10, prob = .25)
```

```
## [1] 0.2815676
```

Geometric random variables

- Suppose we are to perform independent, identical Bernoulli trials until the first success.
- If we wish to model Y , the number of failures before the first success
- Geometric distribution

$$P(Y = y) = (1 - p)^y p \quad \text{for } y = 0, 1, \dots, \infty$$

Geometric distributions with $p = 0.3, 0.5$ and 0.7

If $Y \sim \text{Geometric}(p)$, then $E(Y) = \frac{1-p}{p}$ and $SD(Y) = \sqrt{\frac{1-p}{p^2}}$.

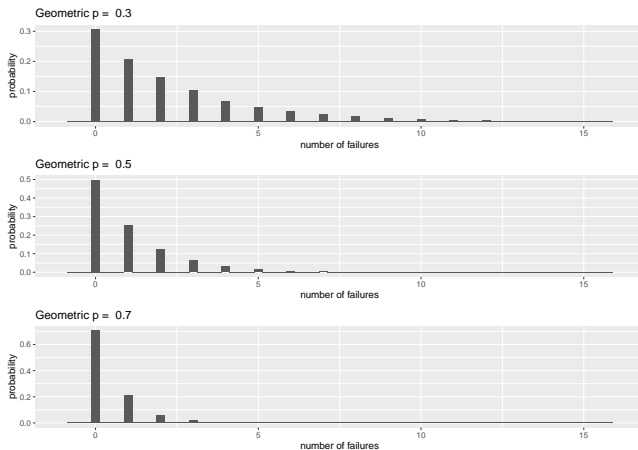


Figure 2: Geometric distributions with $p = 0.3, 0.5$ and 0.7 .

Negative binomial random variable

- If we were to carry out multiple independent and identical Bernoulli trials until the r^{th} success occurs.
- Y , the number of failures before the r^{th} success
- Negative binomial distributions

$$P(Y = y) = \binom{y + r - 1}{r - 1} (1 - p)^y (p)^r \quad \text{for } y = 0, 1, \dots, \infty$$

- When $r = 1$, the geometric distribution is a special case of negative binomial distribution.

Negative binomial distributions with different p and r

If $Y \sim \text{NB}(r, p)$ then $E(Y) = \frac{r(1-p)}{p}$ and $SD(Y) = \sqrt{\frac{r(1-p)}{p^2}}$.

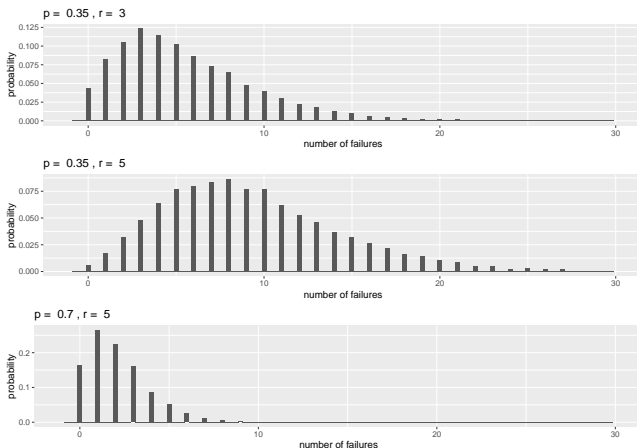


Figure 3: Negative binomial distributions with different values of p and r .

Hypergeometric random variable

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- What if this probability is dynamic?

Hypergeometric random variable

- Bernoulli process assumes the probability of a success remained constant across all trials.
- What if this probability is dynamic?
- Suppose we wanted to select n items **without replacement** from a collection of N objects, m of which are considered successes?
- The probability of selecting a “success” depends on the previous selections.
- Y , the number of successes after n selections
- Hypergeometric random variable

$$P(Y = y) = \frac{\binom{m}{y} \binom{N-m}{n-y}}{\binom{N}{n}} \quad \text{for } y = 0, 1, \dots, \min(m, n).$$

Hypergeometric distributions with m , N , and n

Y follows a hypergeometric distribution and we define $p = m/N$, then

$$E(Y) = np \text{ and } SD(Y) = \sqrt{np(1-p)\frac{N-n}{N-1}}.$$

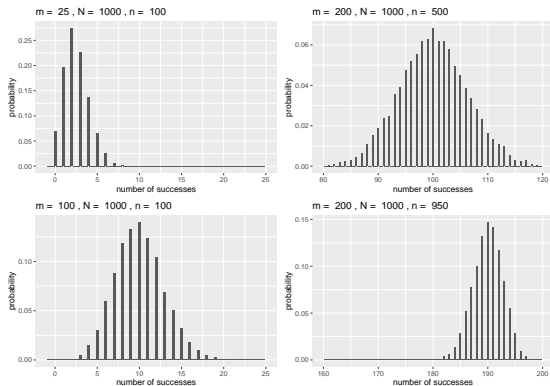


Figure 4: Hypergeometric distributions with different values of m , N , and n

Poisson random variable

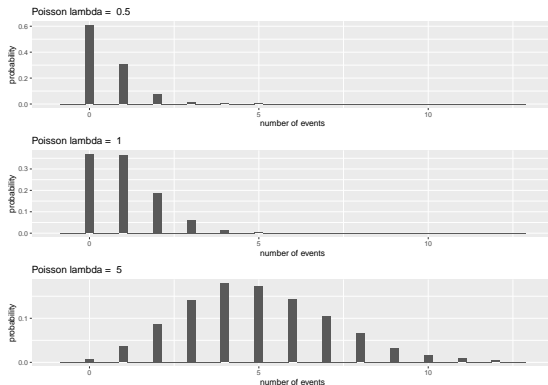
- In a Poisson process, we are counting the number of events per unit of time or space and the number of events depends only on the length or size of the interval.
- Y , the number of events
- Poisson distribution

$$P(Y = y) = \frac{e^{-\lambda} \lambda^y}{y!} \quad \text{for } y = 0, 1, \dots, \infty,$$

where λ is the mean or expected count in the unit of time or space of interest.

Poisson distributions with $\lambda = 0.5, 1$, and 5

$$E(Y) = \lambda \text{ and } SD(Y) = \sqrt{\lambda}$$



Continuous random variable

A continuous random variable can take on an uncountably infinite number of values. Given a pdf $f(y)$,

$$P(a \leq Y \leq b) = \int_a^b f(y)dy$$

Properties:

- $\int_{-\infty}^{\infty} f(y)dy = 1$.
- For any value y , $P(Y = y) = \int_y^y f(y)dy = 0$.
 $P(y < Y) = P(y \leq Y)$.

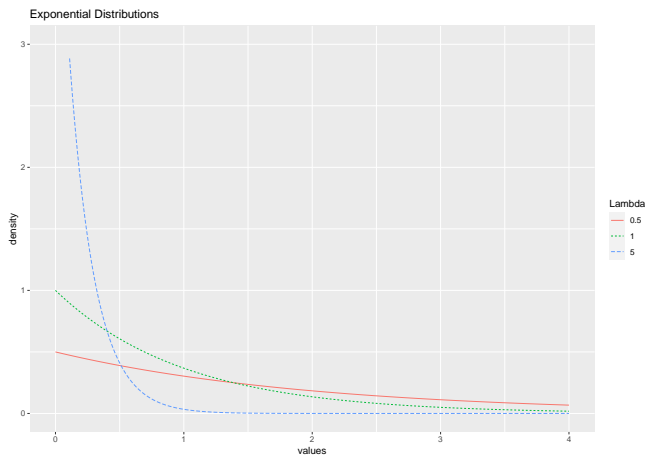
Exponential random variable

- Suppose we have a Poisson process with rate λ
- To model the wait time Y until the first event
- Exponential distribution

$$f(y) = \lambda e^{-\lambda y} \quad \text{for } y > 0,$$

Exponential distributions with $\lambda = 0.5, 1$, and 5

$$E(Y) = 1/\lambda \text{ and } SD(Y) = 1/\lambda$$



Gamma random variable

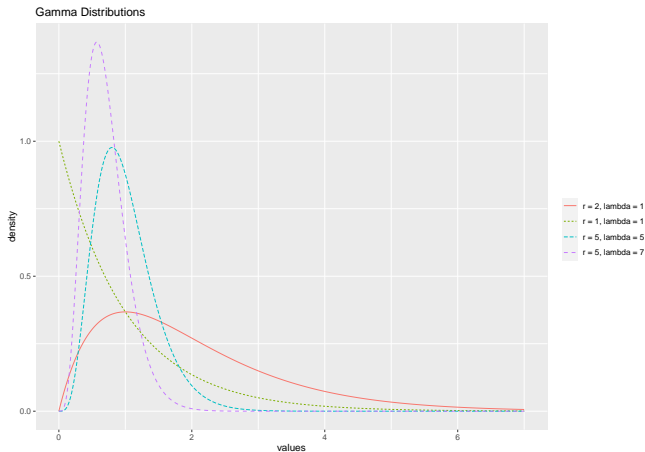
- Consider a Poisson process.
- Y , waiting time before 1 event occurred, follows an exponential distribution.
- Y , waiting time before r events occurred, follows a gamma distribution.

$$f(y) = \frac{\lambda^r}{\Gamma(r)} y^{r-1} e^{-\lambda y} \quad \text{for } y > 0$$

- When $r = 1$, the exponential distribution is a special case of gamma distribution.

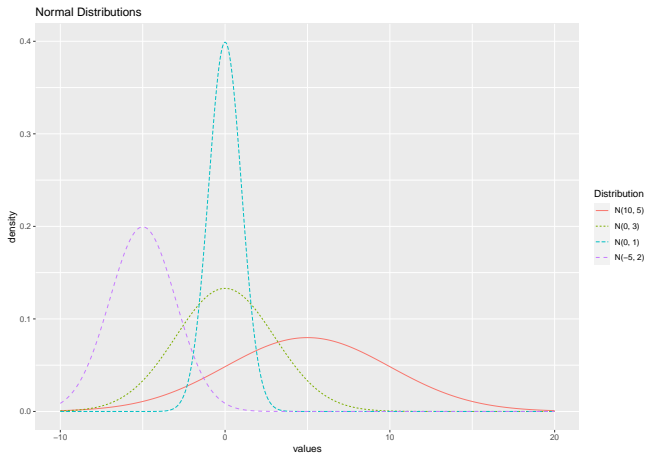
Gamma distributions with different values of r and λ

If $Y \sim \text{Gamma}(r, \lambda)$ then $E(Y) = r/\lambda$ and $SD(Y) = \sqrt{r/\lambda^2}$.



Normal random variable

$Y \in N(\mu, \sigma^2)$, $E(Y) = \mu$ and $SD(Y) = \sigma$.



Beta random variable

We often use beta random variables to model distributions of probabilities bounded below by 0 and above by 1.

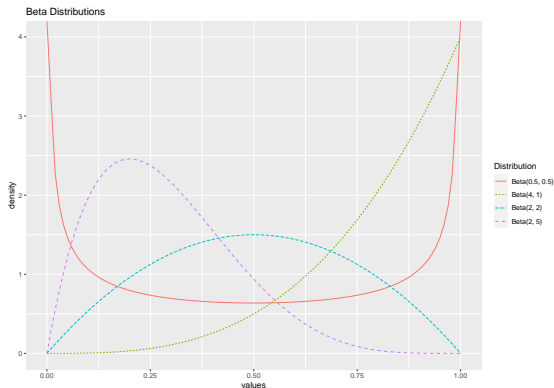
$$f(y) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1} (1-y)^{\beta-1} \quad \text{for } 0 < y < 1$$

- If $\alpha = \beta = 1$, it follows a uniform distribution,

$$\begin{aligned} f(y) &= \frac{\Gamma(1)}{\Gamma(1)\Gamma(1)} y^0 (1-y)^0 \\ &= 1 \quad \text{for } 0 < y < 1. \end{aligned}$$

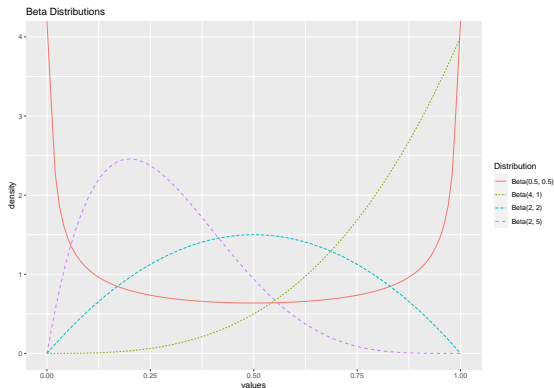
Beta distributions with different values of α and β

$Y \sim \text{Beta}(\alpha, \beta)$, then $E(Y) = \alpha/(\alpha + \beta)$ and $SD(Y) = \sqrt{\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}$.



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Note that when $\alpha = \beta$, distributions are symmetric. The distribution is left-skewed when $\alpha > \beta$ and right-skewed when $\beta > \alpha$.

Distributions used in testing

- χ^2 distribution
- t distribution
- F distribution

Some probability distributions in R

Continuous

- Normal (`?rnorm`)
- Uniform (`?runif`)
- Beta (`?rbeta`)
- Chi-sq (`?rchisq`)
- Exponential (`?rexp`)
- t (`?rt`)
- F (`?rf`)
- Logistic (`?rlogis`)
- Lognormal (`?rlnorm`)

Discrete

- Poisson (`?rpois`)
- Binomial (`?rbinom`)
- Geometric (`?rgeom`)
- Negative Binomial (`?rnbinom`)
- Multinomial (`?rmultinom`)

Empirical vs. Theoretical CDF

In statistics, an empirical distribution function is the distribution function associated with the empirical measure of a sample.

- Theoretical CDF

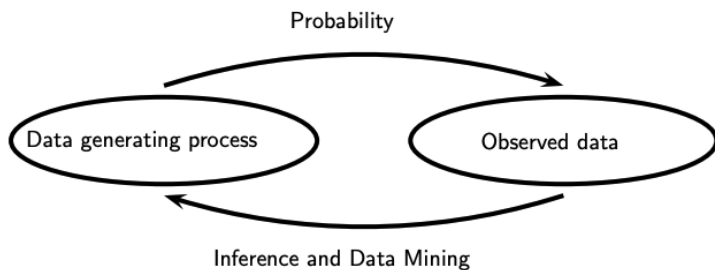
$$F_X(k) = \Pr(X \leq k)$$

- Empirical CDF

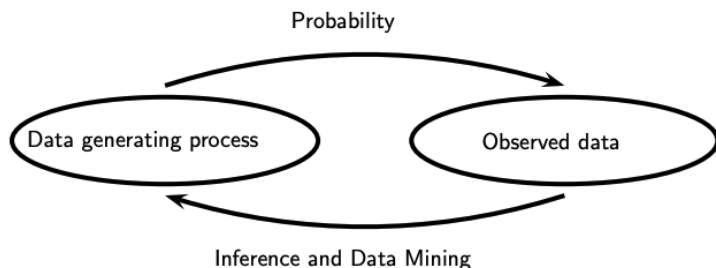
$$\hat{F}_n(k) = \frac{\text{number of elements in the sample} \leq k}{n} = \frac{1}{n} \sum_{i=1}^n I_{X_i \leq k}$$

where X_1, \dots, X_n make up some random sample from the underlying distribution.

Probability and inference



Probability and inference



- Probability: Given a data generating process, what are the properties of the outcomes?
- Statistical inference: Given the outcomes, what can we say about the process that generated the data?

Parametric vs. Nonparametric models

- Statistical model \mathfrak{F} : a set of distributions (or densities or regression functions)

Parametric vs. Nonparametric models

- Statistical model \mathfrak{F} : a set of distributions (or densities or regression functions)
- Parametric model: a set \mathfrak{F} that can be parameterized by a finite number of parameters

$$\mathfrak{F} = \{f(x; \theta) : \theta \in \Theta\}$$

where θ is an unknown parameter (or vector of parameters) that can take values in the parameter space Θ .

- e.g. Normal distribution, a 2-parameter model with density as $f(x; \mu, \sigma)$

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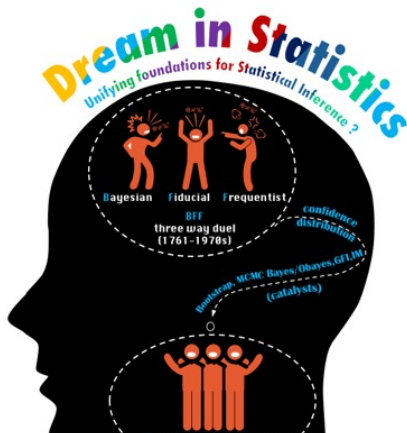
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- e.g. Normal distribution, a 2-parameter model with density as $f(x; \mu, \sigma)$
- Nonparametric model: a set \mathfrak{F} that cannot be parameterized by a finite number of parameters
 - e.g. $\mathfrak{F}_{\text{ALL}} = \{\text{all CDF's}\}$ is nonparametric.

Frequentist, Bayesian, Fiducial inference (BFF)

- Frequentist: statistical methods with guaranteed frequency behavior
- Bayesian: statistical methods for using data to update beliefs
- Fiducial: statistical methods based on inverse probability without calling on prior probability distributions

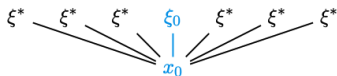


Difference: math details, interpretation, replication

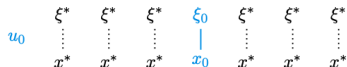
- Frequentist: modeling collection of distributions $\mathcal{P} = \{P_\xi\}_{\xi \in \Xi}$
 - parameter ξ_0 fixed, data x replicated



- Bayesian: modeling one joint distribution $f(x | \xi) \cdot \pi(\xi)$
 - data x_0 fixed, parameter ξ replicated



- Fiducial: modeling data generating algorithm $\mathbf{x} = G(\mathbf{u}, \xi)$
 - data x & parameter ξ linked through DGA, auxiliary variable u replicated



Fundamental concepts in inference

- Point estimation
- Hypothesis testing
- Confidence sets

Point estimation

- Providing a single “best guess” of some quantity of interest
- Notations
 - Parameter θ : fixed, unknown quantity
 - Point estimator $\hat{\theta}$: depends on data, random variable

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Definition (Point estimator)

Let X_1, \dots, X_n be n IID data points from some distribution F . A point estimator $\hat{\theta}_n$ of a parameter θ is some function of X_1, \dots, X_n :

$$\hat{\theta}_n = g(X_1, \dots, X_n)$$

- Properties
 - Unbiasedness
 - Consistency
 - Efficiency

Point estimation (cont'd)

- Bias

$$\text{bias}(\hat{\theta}_n) = \mathbb{E}_{\theta}(\hat{\theta}_n) - \theta$$

- Consistency

$$\hat{\theta}_n \xrightarrow{P} \theta$$

- Standard error

$$\text{se} = \text{se}(\hat{\theta}_n) = \sqrt{\mathbb{V}(\hat{\theta}_n)}$$

- Mean square error

$$\text{MSE} = \mathbb{E}_{\theta}(\hat{\theta}_n - \theta)^2$$

Confidence sets

Definition (Confidence set)

A $1 - \alpha$ confidence interval for a parameter θ is an interval $C_n = (a, b)$ where $a = a(X_1, \dots, X_n)$ and $b = b(X_1, \dots, X_n)$ are functions of the data such that

$$\mathbb{P}_\theta(\theta \in C_n) \geq 1 - \alpha \quad \forall \theta \in \Theta$$

- If θ is a vector, we use **Confidence sets** instead of **Confidence intervals**.
- In Frequentist, θ is fixed while C_n is random.
 - Confidence interval is not a probability statement about θ .
- In Bayesian, θ is random.
 - Bayesian interval refers to degree-of-belief probabilities.

Hypothesis testing

Definition (Hypothesis testing)

Suppose that we partition the parameter space Θ into two disjoint sets Θ_0 and Θ_1 and that we wish to test

$$H_0 : \theta \in \Theta_0 \quad \text{versus} \quad H_1 : \theta \in \Theta_1$$

We call H_0 the null hypothesis and H_1 the alternative hypothesis.

Hypothesis testing (cont'd)

Let X be a random variable, \mathcal{X} be the range of X . We test a hypothesis by finding the rejection region $R \subset \mathcal{X}$,

$$X \in R \implies \text{reject } H_0$$

$$X \notin R \implies \text{retain (do not reject) } H_0$$

Common form of R ,

$$R = \{x : T(x) > c\}$$

where T is a test statistic and c is a critical value.

Hypothesis testing (cont'd)

- Type I error: Rejecting H_0 when H_0 is true
- Type II error: Retaining H_0 when H_1 is true

Definition (Power function)

The power function of a test with rejection region R is defined by

$$\beta(\theta) = \mathbb{P}_\theta(X \in R).$$

The size of a test is defined to be

$$\alpha = \sup_{\theta \in \Theta_0} \beta(\theta).$$

A test is said to have level α if its size is less than or equal to α .

Resources

This tutorial is based on

- Havard Biostatistics Summer Pre Course [link]
- “Beyond Multiple Linear Regression” by Paul Roback and Julie Legler [link]
- “Short course on Generalized Fiducial Inference” by Jan Hannig [link]

More resources: - BFF, Bayesian, Fiducial & Frequentist:

<http://bff-stat.org/about/>