

# Module 2: Set Theory

## Operational math bootcamp



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# Outline

- Review of basic set theory
- Ordered Sets
- Functions

# Introduction to Set Theory

- We define a *set* to be a collection of mathematical objects.
- If  $S$  is a set and  $x$  is one of the objects in the set, we say  $x$  is an element of  $S$  and denote it by  $x \in S$ .
- The set of no elements is called empty set and is denoted by  $\emptyset$ .

## Definition (Subsets, Union, Intersection)

Let  $S, T$  be sets.

- We say that  $S$  is a *subset* of  $T$ , denoted  $S \subseteq T$ , if  $s \in S$  implies  $s \in T$ .
- We say that  $S = T$  if  $S \subseteq T$  and  $T \subseteq S$ .
- We define the *union* of  $S$  and  $T$ , denoted  $S \cup T$ , as all the elements that are in *either*  $S$  or  $T$ .
- We define the *intersection* of  $S$  and  $T$ , denoted  $S \cap T$ , as all the elements that are in *both*  $S$  and  $T$ .
- We say that  $S$  and  $T$  are *disjoint* if  $S \cap T = \emptyset$ .

# Some examples

## Example

$$\mathbb{N} \subseteq \mathbb{N}_0 \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$$

## Example

Let  $a, b \in \mathbb{R}$  such that  $a < b$ .

Open interval:  $(a, b) := \{x \in \mathbb{R} : a < x < b\}$  ( $a, b$  may be  $-\infty$  or  $+\infty$ )

Closed interval:  $[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$

We can also define half-open intervals.

## Example

Let  $A = \{x \in \mathbb{N} : 3|x\}$  and  $B = \{x \in \mathbb{N} : 6|x\}$  Show that  $B \subseteq A$ .

*Proof.*

# Difference of sets

## Definition

Let  $A, B \subseteq X$ . We define the *set-theoretic difference* of  $A$  and  $B$ , denoted  $A \setminus B$  (sometimes  $A - B$ ) as the elements of  $X$  that are in  $A$  but *not* in  $B$ .

The complement of a set  $A \subseteq X$  is the set  $A^c := X \setminus A$ .

## Example

Let  $X \subseteq \mathbb{R}$  be defined as  $X = \{x \in \mathbb{R} : 0 < x \leq 40\} = (0, 40]$ . Then  $X^c$

Recall that for sets  $S, T$ :

- the *union* of  $S$  and  $T$ , denoted  $S \cup T$ , is all the elements that are in *either*  $S$  and  $T$
- and the *intersection* of  $S$  and  $T$ , denoted  $S \cap T$ , is all the elements that are in *both*  $S$  and  $T$ .

We extend the definition of union and intersection to an arbitrary family of sets as follows:

### Definition

Let  $S_\alpha$ ,  $\alpha \in A$ , be a family of sets.  $A$  is called the *index set*. We define

$$\bigcup_{\alpha \in A} S_\alpha := \{x : \exists \alpha \text{ such that } x \in S_\alpha\},$$

$$\bigcap_{\alpha \in A} S_\alpha := \{x : x \in S_\alpha \text{ for all } \alpha \in A\}.$$



## Example

$$\bigcup_{n=1}^{\infty} [-n, n] =$$

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) =$$

## Theorem (De Morgan's Laws)

Let  $\{S_\alpha\}_{\alpha \in A}$  be an arbitrary collection of sets. Then

$$\left( \bigcup_{\alpha \in A} S_\alpha \right)^c = \bigcap_{\alpha \in A} S_\alpha^c \quad \text{and} \quad \left( \bigcap_{\alpha \in A} S_\alpha \right)^c = \bigcup_{\alpha \in A} S_\alpha^c$$

*Proof.*

Since a set is itself a mathematical object, a set can itself contain sets.

### Definition

The power set  $\mathcal{P}(S)$  of a set  $S$  is the set of all subsets of  $S$ .

### Example

Let  $S = \{a, b, c\}$ .

Then  $\mathcal{P}(S) =$

Another way of building a new set from two old ones is the Cartesian product of two sets.

### Definition

Let  $S, T$  be sets. The *Cartesian product*  $S \times T$  is defined as the set of tuples with elements from  $S, T$ , i.e

$$S \times T = \{(s, t) : s \in S \text{ and } t \in T\}.$$

This can also be extended inductively to a finite family of sets.

# Ordered set

## Definition

A *relation*  $R$  on a set  $X$  is a subset of  $X \times X$ . A relation  $\leq$  is called a *partial order* on  $X$  if it satisfies

- ① reflexivity:
- ② transitivity:
- ③ anti-symmetry:

The pair  $(X, \leq)$  is called a *partially ordered set*.

A *chain* or *totally ordered set*  $C \subseteq X$  is a subset with the property  $x \leq y$  or  $y \leq x$  for any  $x, y \in C$ .

### Example

The real numbers with the usual ordering,  $(\mathbb{R}, \leq)$  are totally ordered.

### Example

The power set of a set  $X$  with the ordering given by subsets,  $(\mathcal{P}(X), \subseteq)$  is partially ordered set.

## Example

Let  $X = \{a, b, c, d\}$ . What is  $\mathcal{P}(X)$ ? Find a chain in  $\mathcal{P}(X)$ .

$$\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{c, d\}, \{b, d\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{b, c, d\}, \{a, b, d\}, \{a, c, d\}, X\}$$

## Example

Consider the set  $C([0, 1], \mathbb{R}) := \{f : [0, 1] \rightarrow \mathbb{R} : f \text{ is continuous}\}$ .

For two functions  $f, g \in C([0, 1], \mathbb{R})$ , we define the ordering as  $f \leq g$  if  $f(x) \leq g(x)$  for  $x \in [0, 1]$ . Then  $(C([0, 1], \mathbb{R}), \leq)$  is a partially ordered set.

Can you think of a chain that is a subset of  $(C([0, 1], \mathbb{R}))$ ?



## Definition

A non-empty partially ordered set  $(X, \leq)$  is *well-ordered* if every non-empty subset  $A \subseteq X$  has a minimum element.

Example:

$(\mathbb{N}, \leq)$  is...

$(\mathbb{R}, \leq)$  is...

## Definition

Let  $(X, \leq)$  be a partially ordered set and  $S \subseteq X$ .

Then  $x \in X$  is an *upper bound* for  $S$  if for all  $s \in S$  we have  $s \leq x$ .

Similarly  $y \in X$  is a *lower bound* for  $S$  if for all  $s \in S$ ,  $y \leq s$ .

If there exists an upper bound for  $S$ , we call  $S$  *bounded above* and if there exists a lower bound for  $S$ , we call  $S$  *bounded below*. If  $S$  is bounded above and bounded below, we say  $S$  is *bounded*.

We can also ask if there exists a least upper bound or a greatest lower bound.

## Definition

Let  $(X, \leq)$  be a partially ordered set and  $S \subseteq X$ .

We call  $x \in X$  *least upper bound* or *supremum*, denoted  $x = \sup S$ , if  $x$  is an upper bound and for any other upper bound  $y \in X$  of  $S$  we have  $x \leq y$ .

Likewise,  $x \in X$  is the *greatest lower bound* or *infimum* for  $S$ , denoted  $x = \inf S$ , if it is a lower bound and for any other lower bound  $y \in X$ ,  $y \leq x$ .

Note that the supremum and infimum of a bounded set do not necessarily need to exist. However, if they do exist they are unique, which justifies the article *the* (exercise). Nevertheless, the reals have a remarkable property, which we will take as an axiom.

## Completeness Axiom

Let  $S \subseteq \mathbb{R}$  be bounded above. Then there exists  $r \in \mathbb{R}$  such that  $r = \sup S$ , i.e.  $S$  has a least upper bound.

By setting  $S' = -S := \{-s : s \in S\}$  and noting  $\inf S = -\sup S'$ , we obtain a similar statement for infima if  $S$  is bounded below. As mentioned above, this property is fairly special, for example it fails for the rationals.

## Example

Let  $S = \{q \in \mathbb{Q} : q^2 < 7\}$ . Then  $S$  is bounded above in  $\mathbb{Q}$ , but there exists no least upper bound in  $\mathbb{Q}$ .

There is a nice alternative characterization for suprema in the real numbers.

### Proposition

*Let  $S \subseteq \mathbb{R}$  be bounded above. Then  $r = \sup S$  if and only if  $r$  is an upper bound and for all  $\epsilon > 0$  there exists an  $s \in S$  such that  $r - \epsilon < s$ .*

*Proof. ( $\Rightarrow$ )*

*Proof.* ( $\Leftarrow$ )

Using the same trick, we may obtain a similar result for infima.

### Example

Consider  $S = \{1/n : n \in \mathbb{N}\}$ . Then  $\sup S = 1$  and  $\inf S = 0$ .

# Functions

## Definition

A function  $f$  from a set  $X$  to a set  $Y$  is a subset of  $X \times Y$  with the properties:

- ① For every  $x \in X$ , there exists a  $y \in Y$  such that  $(x, y) \in f$
- ② If  $(x, y) \in f$  and  $(x, z) \in f$ , then  $y = z$ .

$X$  is called the *domain* of  $f$ .

How does this connect to other descriptions of functions you may have seen?

## Example

For a set  $X$ , the identity function is:

$$1_X : X \rightarrow X, \quad x \mapsto x$$

## Definition (Image and pre-image)

Let  $f : X \rightarrow Y$  and  $A \subseteq X$  and  $B \subseteq Y$ .

- The *image* of  $f$  is the set  $f(A) := \{f(x) : x \in A\}$ .
- The *pre-image* of  $f$  is the set  $f^{-1}(B) := \{x : f(x) \in B\}$ .

Helpful way to think about it for proofs:

If  $y \in f(A)$ , then  $y \in Y$ , and there exists an  $x \in A$  such that  $y = f(x)$ .

If  $x \in f^{-1}(B)$ , then  $x \in X$  and  $f(x) \in B$ .



## Definition (Surjective, injective and bijective)

Let  $f : X \rightarrow Y$ , where  $X$  and  $Y$  are sets. Then

- $f$  is *injective* if  $x_1 \neq x_2$  implies  $f(x_1) \neq f(x_2)$
- $f$  is *surjective* if for every  $y \in Y$ , there exists an  $x \in X$  such that  $y = f(x)$
- $f$  is *bijective* if it is both injective and surjective

## Example

Let  $f : X \rightarrow Y$ ,  $x \mapsto x^2$ .

$f$  is surjective if

$f$  is injective if

$f$  is bijective if

$f$  is neither surjective nor injective if

# References

Marcoux, Laurent W. (2019). *PMATH 351 Notes*. url:  
<https://www.math.uwaterloo.ca/~lwmarcou/notes/pmath351.pdf>

Runde ,Volker (2005). *A Taste of Topology*. Universitext. url:  
<https://link.springer.com/book/10.1007/0-387-28387-0>

Zwiernik, Piotr (2022). *Lecture notes in Mathematics for Economics and Statistics*.  
url: <http://84.89.132.1/~piotr/docs/RealAnalysisNotes.pdf>