Exercises for Module 6: Metric Spaces IV

4. Let $(x_n)_{n\in N}$ be a sequence in $\mathbb{R}$. Show that $\lim_{n\to\infty} x_n = 0$ if and only if $\limsup_{n\to\infty} |x_n| = 0$.

1. Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$ be dense. Show that if $A \subseteq B \subseteq X$, then B is dense as well.

2. Let $(X, \mathcal{T})$ be a Hausdorff topological space. Show that the singleton $\{x\}$ is closed for all $x \in X$. Hint: Show that the complement is open.

3. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$ be topological spaces and let $f\colon X \to Y$, $g\colon Y \to Z$ be continuous. Show that $g \circ f\colon X \to Z$ is continuous as well.

4. Let (X, d) be a metric space and $K \subset X$ compact. Show that for all $\epsilon > 0$ there exists $\{x_1, x_2, \dots, x_n\} \subseteq K$ such that for all $y \in K$ we have $d(y, x_i) < \epsilon$ for some $i = 1, \dots, n$.