Exercises for Module 3: Set Theory II and Metric Spaces I

1. Show that for sets $A, B \subseteq X$ and $f : X \to Y$, $f(A \cap B) \subseteq f(A) \cap f(B)$.

2. Let $f : X \to Y$ and $B \subseteq Y$. Prove that $f(f^{-1}(B)) \subseteq B$, with equality iff f is surjective.

3. Prove that $f(\cup_{i \in I} A_i) = \cup_{i \in I} f(A_i)$, where $f : X \to Y$, $A_i \subseteq X \; \forall i \in I$.

4. Show that $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality.

5. Show that $|(0,1)| = |(1,\infty)|$.

6. Show that the infinity norm $||x||_\infty$, $x \in \mathbb{R}^n$, is a norm.

7. Let (X, d) be any metric space, and define $\tilde{d} : X \times X \to \mathbb{R}$ by

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \quad x, y \in X.$$

Show that $\tilde{d}$ is a metric on X.

8. Let X be a set and define $d \colon X \times X \to \mathbb{R}$ by $d(x, x) = 0$ and $d(x, y) = 1$ for $x \neq y \in X$. Prove that d is a metric on X. What do open balls look like for different radii $r > 0$? What does an arbitrary open set look like?