Exercises for Module 7: Linear Algebra I

1. Suppose $\mathbf{v}_1, ..., \mathbf{v}_m$ is linearly independent in V and $\mathbf{w} \in V$. Prove that if $\mathbf{v}_1 + \mathbf{w}, ..., \mathbf{v}_m + \mathbf{w}$ is linearly dependent, then $\mathbf{w} \in \mathrm{span}(\mathbf{v}_1, ..., \mathbf{v}_m)$.

Proof Suppose $v_1+w, \ldots, v_m+w$ are linearly dependent.
Then for $\alpha_i \in \mathbb{F}$, $i=1,\ldots,m$,

$$0 = \sum_{i=1}^{m} \alpha_i (v_i + w) \quad \text{has at least one } \alpha_i \neq 0$$

$$\Rightarrow \quad 0 = \sum_{i=1}^{m} \alpha_i v_i + w \sum_{i=1}^{m} \alpha_i$$

$$\Rightarrow \quad w = \frac{\sum_{i=1}^{m} \alpha_i v_i}{\sum_{i=1}^{m} \alpha_i}$$

$$\Rightarrow \quad w = \sum_{i=1}^{m} \frac{\alpha_i}{\sum_{j=1}^{m} \alpha_j} v_i$$

$$\Rightarrow w \in \mathrm{span}\{v_1, \ldots, v_m\}$$

2. Suppose that $\mathbf{v}_1, ..., \mathbf{v}_m$ is linearly independent in V and $\mathbf{w} \in V$. Show that $\mathbf{v}_1, ..., \mathbf{v}_m, \mathbf{w}$ is linearly independent if and only if

$$\mathbf{w} \notin \mathrm{span}\{\mathbf{v}_1, ..., \mathbf{v}_m\}$$

$(\Rightarrow)$ By contrapositive.
Suppose $w \in \mathrm{span}\{v_1, \ldots, v_m\}$. Then $\exists \alpha_i$, $i=1,\ldots,m$ s.t.
$w = \sum_{i=1}^{m} \alpha_i v_i$.

$$\Rightarrow \quad 0 = \sum_{i=1}^{m} \alpha_i v_i - w$$

$$\Rightarrow \quad 0 = \sum_{i=1}^{m} \beta_i v_i + \beta_{m+1} w \quad \text{has a non-trivial sol'n for } \beta_i \in \mathbb{F}$$

$\Rightarrow v_1, \ldots, v_m, w$ are lin. dependent

$(\Leftarrow)$ Also by contrapositive. Suppose $v_1, \ldots, v_m, w$ are linearly dependent. Then $\exists \alpha_i$, $i=1,\ldots,m+1$, s.t. $0 = \sum_{i=1}^{m} \alpha_i v_i + \alpha_{m+1} w$ has a non-trivial sol'n. Note that we must have $\alpha_{m+1} \neq 0$ because otherwise $v_1, \ldots, v_m$ would be linearly dependent. $\Rightarrow w = \sum_{i=1}^{m} \frac{-\alpha_i}{\alpha_{m+1}} v_i \Rightarrow w \in \mathrm{span}\{v_1, \ldots, v_m\}$

3. Let $T \in \mathcal{L}(\mathbb{P}(\mathbb{R}), \mathbb{P}(\mathbb{R}))$ be the map $T(p(x)) = x^2 p(x)$ (multiplication by x^2).

(i) Show that T is linear.

(ii) Find the null space and range of T.

(i) Let $\alpha, \beta \in \mathbb{F}$, $p, q \in \mathbb{P}(\mathbb{R})$.

$$\begin{aligned} T(\alpha p(x) + \beta q(x)) &= x^2(\alpha p(x) + \beta q(x)) \\ &= \alpha x^2 p(x) + \beta x^2 q(x) \\ &= \alpha T p(x) + \beta T q(x) \end{aligned}$$

(ii) Null space

We need polynomials $p(x)$ such that $x^2 p(x) = 0$ $(\forall x \in \mathbb{R})$.
This implies $p(x) = 0$ $\forall x \in \mathbb{R}$, so $\text{null } T = \{0\}$.

Range

We need to find all polynomials p s.t. $\exists$ polynomial q with
$p(x) = Tq(x) \Rightarrow p(x) = x^2 q(x)$

This holds as long as p has minimum degree ≥ 2, so
$\text{range } T = \{0\} \cup \{p(x) : \text{minimum degree of } p \text{ is at least } 2\}$.

4. Let U and V be finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Show that

$$\dim \text{null } ST \leq \dim \text{null } S + \dim \text{null } T$$

Proof By the rank-nullity thm, for $T : U \to V$, $\dim U = \dim \text{range } T + \dim \text{null } T$.
Note that $T : U \to V$, $S : V \to W$, $ST : U \to W$.
Also, null ST is a subspace of U. Let T' be T restricted to the subspace null ST.

$$\begin{aligned} \dim \text{null } ST &= \dim \text{null } T' + \dim \text{range } T' && \text{by rank nullity} \\ &= \dim \text{null } T + \dim \text{range } T' && \text{since null } T \subseteq \text{null } ST \\ &\leq \dim \text{null } T + \dim \text{null } S + \dim \text{range } S(T') && \text{by rank nullity applied to range } T \\ &= \dim \text{null } T + \dim \text{null } S && \text{by construction} \end{aligned}$$

5. Let $D \in \mathcal{L}(\mathbb{P}_4(\mathbb{R}), \mathbb{P}_3(\mathbb{R}))$ be the differentiation map, $Dp = p'$. Find bases of $\mathbb{P}_4(\mathbb{R})$ and $\mathbb{P}_3(\mathbb{R})$ such that the matrix representation of $\mathcal{M}(D)$ with respect to these basis is given by

$$\mathcal{M}(D) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Basis for $\mathbb{P}_4(\mathbb{R})$: $u_1 = \frac{1}{4}x^4$, $u_2 = \frac{1}{3}x^3$, $u_3 = \frac{1}{2}x^2$, $u_4 = x$, $u_5 = 1$

Basis for $\mathbb{P}_3(\mathbb{R})$: $v_1 = x^3$, $v_2 = x^2$, $v_3 = x$, $v_4 = 1$

Then

$$\left.\begin{aligned} T(u_1) &= \left(\tfrac{1}{4}x^4\right)' = x^3 = v_1 \\ T(u_2) &= \left(\tfrac{1}{3}x^3\right)' = x^2 = v_2 \\ T(u_3) &= \left(\tfrac{1}{2}x^2\right)' = x = v_3 \\ T(u_4) &= (x)' = 1 = v_4 \\ T(u_5) &= (1)' = 0 \end{aligned}\right\} \Rightarrow M(D) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

6. Show that matrix multiplication of square matrices is not commutative, i.e find matrices $A, B \in M_2$ such that $AB \neq BA$.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

$$AB = \begin{pmatrix} 19 & 24 \\ 43 & 50 \end{pmatrix}; \quad BA = \begin{pmatrix} 23 & 34 \\ 31 & 46 \end{pmatrix}$$