

Problem 1

Prove that $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$ when X and Y are independent.

(Hint: simply consider the continuous case, use the independent property of the joint pdf)

Solution:

Proof. Assume both X, Y are continuous random variables and denote the joint PDF as $f_{(X,Y)}(x, y)$, then by independence, $f_{(X,Y)}(x, y) = f_X(x) \cdot f_Y(y)$

$$\begin{aligned}\mathbb{E}(XY) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{(X,Y)}(x, y) \, dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(y) \, dx dy \\ &= \left(\int_{-\infty}^{\infty} f_X(x) \, dx \right) \left(\int_{-\infty}^{\infty} f_Y(y) \, dy \right) \\ &= \mathbb{E}(X)\mathbb{E}(Y).\end{aligned}$$

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Problem 2

For $X \sim \text{Uniform}(a, b)$, compute $\mathbb{E}(X)$ and $\text{Var}(X)$.

Solution:

Proof. We have $f_X(x) = \frac{1}{b-a}$, $a < x < b$.

$$\mathbb{E}(X) = \int_a^b x \cdot \frac{1}{b-a} \, dx = \frac{b-a}{2},$$

and

$$\mathbb{E}(X^2) = \int_a^b x^2 \cdot \frac{1}{b-a} \, dx = \frac{(b-a)^2}{3},$$

therefore,

$$\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{(b-a)^2}{12}.$$

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Problem 3

Determine the MGF of $X \sim \mathcal{N}(\mu, \sigma^2)$.

(Hint: Start by considering the MGF of $Z \sim \mathcal{N}(0, 1)$, and then use the transformation $X = \mu + \sigma Z$)

Solution:

Proof. For $Z \sim \mathcal{N}(0, 1)$,

$$\begin{aligned}
 M_Z(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(tx - \frac{x^2}{2}\right) dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-t)^2}{2} + \frac{t^2}{2}\right) dx \\
 &= e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \mathcal{N}(t, 1) dx \\
 &= e^{\frac{t^2}{2}}.
 \end{aligned}$$

Then for $X = \mu + \sigma Z \sim \mathcal{N}(\mu, \sigma^2)$,

$$M_X(t) = \mathbb{E}(e^{t(\mu + \sigma Z)}) = e^{t\mu} \cdot M_Z(\sigma t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$

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Problem 4

The citizens of Remuera withdraw money from a cash machine according to $X = 50, 100, 200$ with probability 0.3, 0.5, 0.2, respectively. The number of customers per day has the distribution $N \sim \text{Poisson}(\lambda = 10)$. Let $T_N = X_1 + X_2 + \dots + X_N$ be the total amount of money withdrawn in a day, where each X_i has the probability above, and X_i 's are independent of each other and of N .

- Find $\mathbb{E}(T_N)$,
- Find $\text{Var}(T_N)$.

Solution:

Proof. Conditional on $N = n$, we have $\mathbb{E}(T_N \mid N = n) = \sum_{i=1}^n \mathbb{E}(X_i) = 105n$, and $\text{Var}(T_N \mid N = n) = \text{Var}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \text{Var}(X_i) = 2725n$. Therefore, by law of total expectation and variance,

$$\mathbb{E}(T_N) = \mathbb{E}(\mathbb{E}(T_N \mid N)) = \mathbb{E}(105N) = 105\mathbb{E}(N) = 1050,$$

and

$$\begin{aligned}
 \text{Var}(T_N) &= \text{Var}(\mathbb{E}(T_N \mid N)) + \mathbb{E}(\text{Var}(T_N \mid N)) \\
 &= \text{Var}(105N) + \mathbb{E}(2725N) \\
 &= 105^2 \text{Var}(N) + 2725\mathbb{E}(N) \\
 &= 137500.
 \end{aligned}$$

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