

Module 6: Linear Algebra I

Operational math bootcamp



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Outline

- Vector spaces and subspaces
- Linear combinations and bases
- Linear transformations

Definition

We call V a **vector space** if the following hold:

- (A) *Commutativity in addition:* $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$
- (B) *Associativity in addition:* $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$
- (C) *Existence of a neutral element, addition:* There exists a vector $\mathbf{0}$ such that for any $\mathbf{v} \in V$, $\mathbf{0} + \mathbf{v} = \mathbf{v}$
- (D) *Additive inverse:* For every $\mathbf{v} \in V$, there exists another vector, which we denote $-\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
- (E) *Existence of a neutral element, multiplication:* For any $\mathbf{v} \in V$, $1 \times \mathbf{v} = \mathbf{v}$
- (F) *Associativity in multiplication:* Let $\alpha, \beta \in \mathbb{F}$. For any $\mathbf{v} \in V$, $(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$
- (G) Let $\alpha \in \mathbb{F}, \mathbf{u}, \mathbf{v} \in V$. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.
- (H) Let $\alpha, \beta \in \mathbb{F}, \mathbf{v} \in V$. $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$.

Definition

A subset U of V is called a **subspace** of V if U is also a vector space (using the same addition and scalar multiplication as on V).

Proposition

A subset U of V is a subspace of V if and only if U satisfies the following three conditions:

- ① $\mathbf{0} \in U$
- ② Closed under addition: $u, w \in U$ implies $\mathbf{u} + \mathbf{v} \in U$
- ③ Closed under scalar multiplication: $\alpha \in \mathbb{F}$ and $u \in U$ implies $\alpha \mathbf{u} \in U$

Proof.

\Rightarrow If U is a subspace of V , then U satisfies these 3 properties by the definition of a vector space.

\Leftarrow Suppose U satisfies the given 3 conditions.

Then for any $\mathbf{v} \in U$, there must exist $-\mathbf{v} \in U$ by property 3, since $-\mathbf{v} = (-1) \times \mathbf{v}$ (exercise). Property 1 assures property C. Properties 2 and 3, and the fact that $U \subset V$, assure the remaining properties hold. □

Linear combinations

Definition

A linear combination of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ of vectors in V is a vector of the form

$$\alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n = \sum_{k=1}^n \alpha_k \mathbf{v}_k$$

where $\alpha_1, \dots, \alpha_m \in \mathbb{F}$.

Span

Definition

The set of all linear combinations of a list of vectors v_1, \dots, v_m in V is called the **span** of v_1, \dots, v_m , denoted $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$. In other words,

$$\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\} = \{\alpha_1 \mathbf{v}_1 + \dots + \alpha_m \mathbf{v}_n : \alpha_1, \dots, \alpha_n \in \mathbb{F}\}$$

The span of the empty list is defined to be $\{\mathbf{0}\}$.

We say a vector space is *finite dimensional* if it can be spanned by a finite list of vectors; otherwise it is *infinite dimensional*.

Linear independence

Definition

A list of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{F}^n$ is said to be *linearly independent* if

$$0 = \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n,$$

where the α_i , $i = 1, \dots, n$ are scalars, admits only the solution $\alpha_1 = \dots = \alpha_n = 0$.

Otherwise we say the vectors are *linearly dependent*.

Definition

A system of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ is called a basis (for the vector space V) if any vector $\mathbf{v} \in V$ admits a unique representation as a linear combination

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n = \sum_{k=1}^n \alpha_k \mathbf{v}_k$$

In undergrad, you likely thought about this as: the equation $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n$, where the α_i are unknown, has a unique solution.

Claim

All bases of a vector space V have the same length.

Proof.

Definition

The *dimension* of a vector space V , denoted $\dim V$, is the length of any basis of V .



Bases

Example of bases:

For \mathbb{R}^n : $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, \dots , $e_n = (0, \dots, 0, 1)$

For \mathbb{P}^n : $1, x, x^2, \dots, x^n$

Definition

The linear combination $\alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n$ is called trivial if $\alpha_k = 0$ for every k .

proposition

A system of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$ is a basis if and only if it is linearly independent and complete (generating).

Linear transformations

Definition

A **transformation** T from domain X to codomain Y is a rule that assigns an output $y = T(x) \in Y$ to each input $x \in X$

Definition

A transformation from a vector space U to a vector space V is **linear** if

$$T(\alpha \mathbf{u} + \beta \mathbf{v}) = \alpha T(\mathbf{u}) + \beta T(\mathbf{v}) \quad \text{for any } \mathbf{u}, \mathbf{v} \in V, \alpha, \beta \in \mathbb{F}$$

Examples

- Differentiation
- Integration
- Rotation of vectors
- Reflection of vectors



Linear transformations

Definition

Let $T : U \rightarrow V$ be a linear transformation. We define the following important subspaces:

- *Kernel or Null space:*

$$\text{Ker } T = \{\mathbf{u} \in U : T\mathbf{u} = \mathbf{0}\}$$

- *Range*

$$\text{Range } T = \{\mathbf{v} \in V : \exists \mathbf{u} \in U \text{ such that } \mathbf{v} = T\mathbf{u}\}$$

The dimensions of these spaces are often called the following:

- *Nullity*

$$\text{Nullity}(T) = \dim(\text{Ker}(T))$$

- *Rank*

$$\text{Rank}(T) = \dim(\text{Range}(T))$$

Linear transformations

Rank Theorem

For a matrix A or equivalently a linear transformation $A : \mathbb{F}^n \rightarrow \mathbb{F}^m$:

$$\text{Rank } A = \text{Rank } A^T$$

Rank Nullity Theorem

Let $T : U \rightarrow V$ be a linear transformation, where U and V are finite-dimensional vector spaces. Then

$$\text{Rank } T + \text{Nullity } T = \dim U.$$

References

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