Exercises for Module 7: Linear Algebra I

1. Suppose $\mathbf{v}_1, ..., \mathbf{v}_m$ is linearly independent in V and $\mathbf{w} \in V$. Prove that if $\mathbf{v}_1 + w, ..., \mathbf{v}_m + w$ is linearly dependent, then $\mathbf{w} \in \text{span}(\mathbf{v}_1, ..., \mathbf{v}_m)$.

2. Suppose that $\mathbf{v}_1, ..., \mathbf{v}_m$ is linearly independent in V and $\mathbf{w} \in V$. Show that $\mathbf{v}_1, ..., \mathbf{v}_m, w$ is linearly independent if and only if

$$\mathbf{w} \notin \text{span}\{\mathbf{v}_1, ..., \mathbf{v}_m\}$$

3. Let $T \in \mathcal{L}(\mathbb{P}(\mathbb{R}), \mathbb{P}(\mathbb{R}))$ be the map $T(p(x)) = x^2 p(x)$ (multiplication by x^2).

   (i) Show that T is linear.

   (ii) Find the null space and range of T.

4. Let U and V be finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Show that

$$\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T$$

5. Let $D \in \mathcal{L}(\mathbb{P}_4(\mathbb{R}), \mathbb{P}_3(\mathbb{R}))$ be the differentiation map, $Dp = p'$. Find bases of $\mathbb{P}_4(\mathbb{R})$ and $\mathbb{P}_3(\mathbb{R})$ such that the matrix representation of $\mathcal{M}(D)$ with respect to these basis is given by

$$\mathcal{M}(D) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

6. Show that matrix multiplication of square matrices is not commutative, i.e find matrices $A, B \in M_2$ such that $AB \neq BA$.