# Exercises for Module 8: Linear Algebra II

1. A square matrix is called *nilpotent* if $\exists k \in \mathbb{N}$ such that $A^k = 0$. Show that for a nilpotent matrix $A$, $|A| = 0$.

Suppose $A$ is nilpotent, i.e. $\exists k \in \mathbb{N}$ s.t $A^k = 0$.

Then $\det(A^k) = 0 \Rightarrow \det(\underbrace{A \cdots A}_{k \text{ times}}) = 0$

$\Rightarrow \underbrace{\det(A) \cdots \det(A)}_{k \text{ times}} = 0$ by properties of determinant

$\Rightarrow \det(A)^k = 0$

2. A real square matrix $Q$ is called *orthogonal* if $Q^T Q = I$. Prove that if $Q$ is orthogonal, then $|Q| = \pm 1$.

$$Q^T Q = I$$

$\Rightarrow \det(Q^T Q) = \det(I)$

$\Rightarrow \det(Q^T)\det(Q) = 1$

$\Rightarrow \det(Q)\det(Q) = 1$

$\Rightarrow \det(Q)^2 = 1$

$\Rightarrow \det(Q) = \pm 1$

3. An $n \times n$ matrix is called *antisymmetric* if $A^T = -A$. Prove that if $A$ is antisymmetric and $n$ is odd, then $|A| = 0$.

$$A^T = -A$$

$\Rightarrow \det(A^T) = \det(-A)$

$\Rightarrow \det(A) = (-1)^n \det(A)$

$\Rightarrow \det(A) = 0$ if $n$ is odd

4. Let $V$ be an inner product space, $U$ a vector space and $S\colon U \to V$, $S\colon U \to V$ be linear maps . Show that $\langle S\mathbf{u}, \mathbf{v}\rangle = \langle T\mathbf{u}, \mathbf{v}\rangle$ for all $\mathbf{u} \in U$ and $\mathbf{v} \in V$ implies $S = T$.

Proof

Suppose $\langle Su, v\rangle = \langle Tu, v\rangle \quad \forall u \in U, v \in V$

$\Rightarrow \langle Su, v\rangle - \langle Tu, v\rangle = 0$

$\Rightarrow \langle Su - Tu, v\rangle = 0$ by linearity in $1^{st}$ argument

$\Rightarrow Su - Tu = 0$ by proposition 5.57

$\Rightarrow Su = Tu \quad \forall u \in U$

$\Rightarrow S = T$

5. Let $V$ be an inner product space and $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be an orthonormal basis and $\mathbf{y} \in V$. Then, $\mathbf{x}$ has a unique representation $\mathbf{y} = \sum_{i=1}^{n} \alpha_i \mathbf{x}_i$. Show that $\alpha_i = \langle \mathbf{y}, \mathbf{x}_i\rangle$ for all $i = 1, \ldots, n$.

$$\langle y, x_i\rangle = \langle \sum_{j=1}^{n} \alpha_j x_j, x_i\rangle$$

$$= \sum_{j=1}^{n} \alpha_j \langle x_j, x_i\rangle \quad \text{by linearity in } 1^{st} \text{ argument}$$

$$= \alpha_i \quad \text{since } \langle x_j, x_i\rangle = 1 \text{ if } i=j \text{ \& } 0 \text{ otherwise}$$

6. Let $V$ be an inner product space and $U \subseteq V$ a subset. Show that $U^{\perp}$ is a subspace of $V$.

$U^{\perp} := \{x \in V : \langle x, u\rangle = 0 \quad \forall u \in U\}$

We must show $U^{\perp}$ is a subspace of $V$.

First of all, $0 \in U^{\perp}$ since $\langle 0, u\rangle = 0 \quad \forall u \in U$.

Let $x, y \in U^{\perp}$. Then $\langle x+y, u\rangle = \langle x, u\rangle + \langle y, u\rangle$ by linearity in 1st argument
$= 0$

so $x+y \in U^{\perp}$.

Also, if $\alpha \in \mathbb{F}$, $x \in U^{\perp}$, then $\langle \alpha x, u\rangle = \alpha\langle x, u\rangle = 0$, so $\alpha x \in U^{\perp}$.

$\therefore U^{\perp} \subseteq V$ is a subspace.

7. Let $U, V, W$ be inner product spaces and $S, T \in \mathcal{L}(U, V)$ and $R \in \mathcal{L}(V, W)$. Show that the following holds

1. $(S + \alpha T)^* = S^* + \overline{\alpha} T^*$ for all $\alpha \in \mathbb{F}$
2. $(S^*)^* = S$
3. $(RS)^* = S^* R^*$
4. $I^* = I$, where $I \colon U \to U$ is the identity operator on $U$

1\. Let $u \in U, v \in V$.

$\langle u, (S+\alpha T)^* v\rangle = \langle (S+\alpha T)u, v\rangle$ by def'n of adjoint

$= \langle Su + \alpha Tu, v\rangle$ by def of linear map

$= \langle Su, v\rangle + \alpha\langle Tu, v\rangle$ by lin. of 1st arg.

$= \langle u, S^* v\rangle + \alpha\langle u, T^* v\rangle$

$= \langle u, S^* v\rangle + \langle u, \bar{\alpha} T^* v\rangle$ by linearity + conjugate symmetry

$= \langle u, (S^* + \bar{\alpha} T^*) v\rangle$

$\therefore (S+\alpha T)^* = S^* + \bar{\alpha} T^*$, by exercise 4

2. Let $u \in U$, $v \in V$.

$$\langle u, (S^*)^* v\rangle = \langle S^* u, v\rangle$$
$$= \overline{\langle v, S^* u\rangle} \quad \text{by conjugate symmetry}$$
$$= \overline{\langle Sv, u\rangle} \quad \text{by def of adjoint}$$
$$= \langle u, Sv\rangle \quad \text{by conjugate symmetry}$$

$\therefore (S^*)^* = S$ by exercise 4

3. Let $u \in U$, $w \in W$.

$$\langle u, (RS)^* w\rangle = \langle RSu, w\rangle$$
$$= \langle Su, R^* w\rangle$$
$$= \langle u, S^* R^* w\rangle$$

$$\therefore (RS)^* = S^* R^*$$

4. Let $u_1, u_2 \in U$.

Then $\langle u_1, I^* u_2\rangle = \langle I u_1, u_2\rangle$
$$= \langle u_1, u_2\rangle$$
$$= \langle u_1, I u_2\rangle$$

$$\therefore I = I^*$$