UNIVERSITY OF TORONTO | Statistical Sciences

# DoSS Summer Bootcamp Probability
# Module 10


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# Recap

Learnt in last module:

- Markov Chain
  - ▷ Markov Property
- Discrete-time Markov Chain
  - ▷ Transition probability
  - ▷ Chapman-Kolmogorov equation
- Continuous-time Markov Chain
  - ▷ Transition probability
  - ▷ Chapman-Kolmogorov equation
  - ▷ Generator matrix

# Outline

- Poisson process
  - ▷ Poisson-Gamma relationship
  - ▷ Properties of Poisson Process
- Brownian motion
  - ▷ Properties of Brownian motion
  - ▷ Brownian motion with drift
  - ▷ Geometric Brownian motion

# Poisson process

## Poisson process: an example of CTMC

### Poisson process

A Poisson process $\{N(t)\}_{t\geq 0}$ with intensity $\lambda > 0$ is a collection of non-decreasing integer-valued random variables satisfying the properties that

- $N(0) = 0$;
- Independent increments: $N(t)$ is independent of $N(t+s) - N(t)$;
- $N(t+s) - N(s) \sim Poisson(\lambda t), \quad t \geq 0, s \geq 0.$

**Remark:**

- Easy to verify the Markov property of Poisson process;
- $N(t) \sim Poisson(\lambda t)$.

# Poisson process

**Examples:**

- The number of customers arriving at a grocery store with intensity $\lambda = 5$ customers per hour;
- The number of students coming to the TA session with intensity $\lambda = 3$ students per hour;
- The number of births in Canada with intensity $\lambda = 40$ per hour.

**The probability that more than 60 babies are born between 9 to 11 AM in Canada:**

$$\mathbb{P}(N(t+2) - N(t) > 60) = \mathbb{P}(N(2) > 60) = 1 - \sum_{k=0}^{60} \frac{e^{-40\cdot 2}(40 \cdot 2)^k}{k!}$$

# Poisson process

**Think about the waiting time for the event:**

**Inter-arrival time for Poisson process**

Consider a Poisson process $\{N(t)\}_{t\geq 0}$ with intensity $\lambda$, and let $T_1$ be the time for the first event. Sequentially, let $T_n$ denote the time between the $(n-1)$-th and the $n$-th event. Then $\{T_n\}_{n\geq 1}$ are i.i.d. exponential random variables with parameter $\lambda$, e.g.

$$\mathbb{P}(T_n \leq t) = 1 - e^{-\lambda t}.$$

**Proof:**

# Poisson process

**Arrival time for Poisson process:**

**Poisson-Gamma relationship**

Consider a Poisson process $\{N(t)\}_{t\geq 0}$ with intensity $\lambda$, then the total time until $n$ events is $\sum_{i=1}^{n} T_i \sim \Gamma(n, \lambda)$.

**Proof:**

# Poisson process

A plot about reverse the time and number of events

# Poisson process

## Useful Properties:

### $T_1 \mid N(s) = 1 \sim U[0, s]$

Consider a Poisson process $\{N(t)\}_{t \geq 0}$ with intensity $\lambda$, then

$$\mathbb{P}(T_1 < t \mid N(s) = 1) = \frac{t}{s}, \quad t < s.$$

## Proof:

# Poisson process

**$N(s) \mid N(t) = n \sim B(n, p = \frac{s}{t})$ for $s < t$**

Consider a Poisson process $\{N(t)\}_{t \geq 0}$ with intensity $\lambda$, then for $s < t$,

$$N(s) \mid N(t) = n \sim B(n, p = \frac{s}{t}).$$

**Proof:**

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# Poisson process

## Superposition

If $\{N_1(t)\}_{t\geq 0}$ and $\{N_2(t)\}_{t\geq 0}$ are independent Poisson processes with intensities $\lambda_1$ and $\lambda_2$, respectively, then $\{N(t) := N_1(t) + N_2(t)\}_{t\geq 0}$ is also a Poisson process with intensity $\lambda_1 + \lambda_2$.

**Proof:**

# Poisson process

**Thinning**

Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with intensity $\lambda$. Suppose each event is independently of type $i$ with probability $p_i$ for $i = 1, \cdots, k$ with $\sum_{i=1}^{k} p_i = 1$. If $N_i(t)$ is the number of events of type $i$ happen up to time $t$, then $\{N_i(t)\}$ is a Poisson process with rate $\lambda p_i$.

**Properties of Poisson process:**

Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with intensity $\lambda$, then

- $T_1 \mid N(s) = 1 \sim U[0, s]$;
- $N(s) \mid N(t) = n \sim B(n, p = \frac{s}{t})$ for $s < t$;
- Superposition:
- Thinning.

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# Brownian motion

## Brownian motion: an example of process with continuous time and continuous state

### Brownian motion

Standard Brownian motion is a continuous-time process $\{B(t)\}_{t\geq 0}$ satisfying that

- $B(0) = 0$;
- Independent increments: for $0 \leq q < r \leq s < t$, $B(t) - B(s)$ and $B(r) - B(q)$ are independent random variables;
- $B(t+s) - B(s) \sim \mathcal{N}(0, t)$, $s \geq 0, t > 0$;
- $B(t)$ is almost surely continuous.

**Remark:**

Easy to verify the Markov property.

# Brownian motion

## Useful properties of Brownian motion:

**Joint distribution regarding Brownian motion**

For $0 < t_1 < \cdots < t_n$, $(B(t_1), B(t_2), \cdots, B(t_n))^\top$ follows a multivariate normal distribution.

**Proof:**

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# Brownian motion

**$Cov(B(s), B(t)) = \min(t, s)$**

For a standard Brownian motion $\{B(t)_{t \geq 0}\}$, the covariance satisfies

$$Cov(B(s), B(t)) = \min(t, s).$$

**Proof:**

**Remark:**

Useful technique: rearrange into independent parts

# Brownian motion

Note when

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{MVN}\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}\right),$$

the conditional distribution satisfies

$$X \mid Y = y \sim \mathcal{N}\left(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(y - \mu_2), \left(1 - \rho^2\right)\sigma_1^2\right).$$

**Conditional distribution regarding Brownian motion**

For $0 < s < t$, we have

- $B(s) \mid B(t) = a \sim \mathcal{N}(\frac{s}{t}a, (1 - \frac{s}{t})s)$;
- $B(t) \mid B(s) = a \sim \mathcal{N}(a, t - s)$.

# Brownian motion

**Proof:**

# Brownian motion

## Brownian motion with drift

For $\mu \in \mathbb{R}$ and $\sigma > 0$, the process defined by $\{D(t) = \mu t + \sigma B(t)\}$ is called the Brownian motion with drift. $\mu$ is the drift parameter and $\sigma^2$ is the variance parameter.

**Remark:**

- $D(0) = 0$;
- $D(t) \sim \mathcal{N}(\mu t, \sigma^2 t^2)$.

**Example:**

Find the probability that Brownian motion with drift takes value between 1 and 2 at time $t = 4$, when $\mu = 0.6, \sigma^2 = 0.25$.

# Brownian motion

**Geometric Brownian Motion**

Let $\{D(t) = \mu t + \sigma B(t)\}$ be a Brownian motion with drift, the process $\{G(t) = G(0)e^{D(t)}\}_{t\geq 0}$ is called Geometric Brownian motion, provided that $G(0) > 0$.

**Remark:**

$\mathbb{E}(G(t)) = G(0)e^{t(\mu + \frac{\sigma^2}{2})}$.

# Problem Set

**Problem 1:** The Poisson process with intensity $\lambda$ is an example of CTMC.

- Find $P^{(t)}$;
- Compute the generator matrix $G$.

**Problem 2:** If $\{N(t)\}_{t\geq 0}$ is a Poisson process with $\lambda = 3$, compute the probability $\mathbb{P}(N(2) = 4, N(4) = 8)$.

**Problem 3:** Suppose that undergraduate students and graduate students arrive for office hours according to a Poisson process with rate $\lambda_1 = 5$ and $\lambda_2 = 3$ respectively. What is the expected time until the first student arrives?

# Problem Set

**Problem 4:** Let $\{B(t)\}_{t\geq 0}$ be a standard Brownian motion. Show that the followings are Brownian motions.

- $\{Y(t) = B(t+\alpha) - B(\alpha)\}_{t\geq 0}$ for all $\alpha \geq 0$;
- $\{Y(t) = \alpha B(t/\alpha^2)\}_{t\geq 0}$ for all $\alpha \geq 0$.