Mathematics Bootcamp

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Preface

These notes were prepared for the inaugaral Department of Statistical Sciences Graduate Student Bootcamp at the University of Toronto, which is to be held in July 2022.

References are provided for each section. All references are freely available online, though some may require a University of Toronto library log-in to access.

1 Review of proof techniques with examples from algebra and analysis

1.1 Axioms of a field

(A1) *Commutativity in addition:* $x + y = y + x$

(A2) *Commutativity in multiplication:* $x \times y = y \times x$

(B1) *Associativity in addition:* $x + (y + z) = (x + y) + z$

(B2) *Associativity in multiplication:* $x \times (y \times z) = (x \times y) \times z$

(C) *Distributivity:* $x \times (y + z) = x \times y + x \times z$

(D1) *Existence of a neutral element, addition:* There exists a number 0 such that $x + 0 = x$ for every x.

(D2) *Existence of a neutral element, multiplication:* There exists a number 1 such that $x \times 1 = x$ for every x.

(E1) *Existence of an inverse, addition:* For each number x, there exists a number $-x$ such that $x + (-x) = 0$.

(E2) *Existence of an inverse, multiplication:* For each number $x \neq 0$, there exists a number $1/x$ such that $x \times 1/x = 1$.

[**EK:** *This section to be worked on later*]

1.1.1 Exercises

1. For any $a, b \neq 0$, $1/(ab) = 1/a \times 1/b$

2. For $a > 0$, $1/(-a) = -1/a$.

3. For $a, b \neq 0$, $1/(a/b) = b/a$

2 Linear Algebra

2.1 Vector spaces

2.1.1 Axioms of a vector space

Let V be a set and let $\mathbb{F}$ be a field.

Definition 2.1 *We call V a **vector space** if the following hold:*
Addition:

(A) *Commutativity in addition:* $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ *for all* $\mathbf{u}, \mathbf{v} \in V$

(B) *Associativity in addition:* $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ *for all* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$

(C) *Existence of a neutral element, addition: There exists a vector $\mathbf{0}$ such that for any $\mathbf{v} \in V$, $\mathbf{0} + \mathbf{v} = \mathbf{v}$*

(D) *Additive inverse: For every $\mathbf{v} \in V$, there exists another vector, which we denote $-\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.*

Multiplication by a scalar:

(E) *Existence of a neutral element, multiplication: For any $\mathbf{v} \in V$, $1 \times \mathbf{v} = \mathbf{v}$*

(F) *Associativity in multiplication: Let $\alpha, \beta \in \mathbb{F}$. For any $\mathbf{v} \in V$, $(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$*

Associativity:

(G) *Let $\alpha \in \mathbb{F}, \mathbf{u}, \mathbf{v} \in V$. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \beta\mathbf{v}$.*

(H) Let $\alpha, \beta \in \mathbb{F}, \mathbf{v} \in V$. $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$.

Elements of the vector space are called vectors.

Most often we will assume $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$.

Examples of vector spaces: $\mathbb{R}^n$. $\mathbb{C}^n$, $M_{m \times n}$ (matrices of size $m \times n$), $\mathbb{P}_n$ (polynomials of degree n, $p(x) = a_0 + a_1 x + \ldots + a_n x^n$).

Lemma 2.2 *For every $\mathbf{v} \in V$, we have $-\mathbf{v} = (-1) \times \mathbf{v}$.*

Proof. Our goal is to show that $(-1) \times \mathbf{v}$ is the additive inverse of $\mathbf{v}$. We show this as follows:

$$\mathbf{v} + (-1) \times \mathbf{v} = \mathbf{v} \times (1 + (-1)) = \mathbf{v} \times 0 = 0$$

The last step uses exercise 2.7. [**EK:** *Done by hand in class*] $\qquad\qquad\square$

2.1.2 Subspaces

Definition 2.3 *A subset U of V is called a **subspace** of of V if U is also a vector space (using the same addition and scalar multiplication as on V).*

Proposition 2.4 *A subset U of V is a subspace of V if and only if U satisfies the following three conditions:*

1. $\mathbf{0} \in U$

2. *Closed under addition: $u, w \in U$ implies $\mathbf{u} + \mathbf{v} \in U$*

3. *Closed under scalar multiplication: $\alpha \in \mathbb{F}$ and $u \in U$ implies $\alpha\mathbf{u} \in U$*

Proof. $\Rightarrow$ If U is a subspace of V, then U satisfies these 3 properties by Definition 2.1.
$\Leftarrow$ Suppose U satisfies the given 3 conditions. Then for any $\mathbf{v} \in U$, there must exist $-\mathbf{v} \in U$ by property 3, since $-\mathbf{v} = (-1) \times \mathbf{v}$ by Lemma 2.2 (property D). Property 1 assures property C. Properties 2 and 3, and the fact that $U \subset V$, assure the remaining properties hold.

$\qquad\qquad\square$

[**EK:** *Add intersections and unions of subspaces*]

Definition 2.5 *Suppose $U_1, ..., U_m$ are subsets of V. The sum of $U_1, ..., U_m$, denoted $U_1 + ... + U_m$, is the set of all possible sums of elements of $U_1, ..., U_m$. More precisely,*

$$U_1 + ... + U_m = \{\mathbf{u}_1 + ... + \mathbf{u}_m : \mathbf{u}_1 \in U_1, ..., \mathbf{u}_m \in U_m\}$$

Proposition 2.6 *Suppose $U_1, ..., U_m$ are subspaces of V. Then $U_1 + ... + U_m$ is the smallest subspace of V containing $U_1, ..., U_m$.*

2.1.3 Exercises

Exercise 2.7 (1.7 in [1]) Show that $0\mathbf{v} = \mathbf{0}$ for $\mathbf{v} \in V$.

Exercise 2.8 (1.B.1 in [2]) Show that $-(-v) = v$ for $\mathbf{v} \in V$.

Exercise 2.9 (1.B.2 in [2]) Suppose that $\alpha \in \mathbb{F}, \mathbf{v} \in V$, and $\alpha\mathbf{v} = 0$. Prove that $a = 0$ or $v = 0$.

Exercise 2.10 (1.B.4 in [2]) Why is the empty space not a vector space?

Exercise: Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbb{R}$.

Exercise: A function $f : \mathbb{R} \to \mathbb{R}$ is called periodic if there exists a positive number such that $f(x) = f(x+p)$ for all $x \in \mathbb{R}$. Is the a set of periodic functions from $\mathbb{R}$ to $\mathbb{R}$ a subspace of $\mathbb{R}^{\mathbb{R}}$?

Exercise: A function $f : \mathbb{R} \to \mathbb{R}$ is called odd if

$$f(-x) = -f(x)$$

for all $x \in \mathbb{R}$. Let U_e denote the set of real-valued even functions on $\mathbb{R}$ and let U_o denote the set of real-valued odd functions on $\mathbb{R}$. Show that $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$.

2.2 Linear (in)dependence and bases

Definition 2.11 *A linear combination of a list* $\mathbf{v}_1, ..., \mathbf{v}_n$ *of vectors in* V *is a vector of the form*

$$\alpha_1 \mathbf{v}_1 + ... + \alpha_n \mathbf{v}_n = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k$$

where $\alpha_1, ..., \alpha_m \in \mathbb{F}$.

Definition 2.12 *The set of all linear combinations of a list of vectors* $v_1, ..., v_m$ *in* V *is called the* **span** *of* $v_1, ..., v_m$, *denoted* $span\{\mathbf{v}_1, ..., \mathbf{v}_n\}$. *In other words,*

$$span\{\mathbf{v}_1, ..., \mathbf{v}_n\} = \{\alpha_1 \mathbf{v}_1 + ... + \alpha_m \mathbf{v}_n : \alpha_1, ..., \alpha_n \in \mathbb{F}\}$$

The span of the empty list is defined to be $\{\mathbf{0}\}$.

Definition 2.13 *A system of vectors* $\mathbf{v}_1, ..., \mathbf{v}_n$ *is called a basis (for the vector space* V *) if any vector* $\mathbf{v} \in V$ *admits a unique representation as a linear combination*

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + ... + \alpha_n \mathbf{v}_n = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k$$

Definition 2.14 *The linear combination* $\alpha_1 \mathbf{v}_1 + ... + \alpha_n \mathbf{v}_n$ *is called trivial if* $\alpha_k = 0$ *for every* k.

Proposition 2.15 *A system of vectors* $\mathbf{v}_1, ... \mathbf{v}_n \in V$ *is a basis if and only if it is linearly independent and complete (generating).*

[**EK:** *Proof done by hand*]

2.3 Exercises

From Harvard: Exercise: Suppose v_1, v_2, v_3, v_4 (a) spans V and (b) is linearly independent. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also (a) spans V and (b) is linearly independent.

Exercise: Suppose $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, ..., v_m + w$ is linearly dependent, then $w \in \text{span}(v_1, ..., v_m)$.

Exercise: Suppose that $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Show that $v_1, ..., v_m, w$ is linearly independent if and only if

$$w \notin \text{span}(v_1, ..., v_m)$$

onExercises Exercise: Suppose v_1, v_2, v_3, v_4 (a) spans V and (b) is linearly independent. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also (a) spans V and (b) is linearly independent.
Exercise: Suppose $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, ..., v_m + w$ is linearly dependent, then $w \in \text{span}(v_1, ..., v_m)$.
Exercise: Suppose that $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Show that $v_1, ..., v_m, w$ is linearly independent if and only if

$$w \notin \text{span}(v_1, ..., v_m)$$

[**EK:** *Add a few from books*]

The following texts:
Linear Algebra Done Right [2]
Linear Algebra Done Wrong [1]

3   Set theory

4   Metric spaces

5   Topology

6   Functions on $\mathbb{R}$

7   Multivariable calculus

8   Functional analysis

9   Complex analysis and Fourier analysis

References

1. Treil S. Linear Algebra Done Wrong. 2017. Available from: `https://www.math.brown.edu/streil/papers/LADW/LADW.html`

2. Axler S. Linear Algebra Done Right. 3rd ed. Undergraduate Texts in Mathematics. Springer, 2015. Available from: `https://link-springer-com.myaccess.library.utoronto.ca/book/10.1007/978-3-319-11080-6`