Module 4: Metric Spaces II
Operational math bootcamp

Statistical Sciences
UNIVERSITY OF TORONTO

Emma Kroell

University of Toronto

July 17, 2023

Outline

- Open and closed sets

- Sequences
  - Cauchy sequences
  - subsequences

- Continuous functions

Statistical Sciences
UNIVERSITY OF TORONTO

Definition (Open and closed sets)

Let (X, d) be a metric space.

- A set $U \subseteq X$ is *open* if for every $x \in U$ there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$.
- A set $F \subseteq X$ is *closed* if $F^c := X \setminus F$ is open.

Note:

Proposition

Let (X, d) be a metric space.

1. *Let $A_1, A_2 \subseteq X$. If A_1 and A_2 are open, then $A_1 \cap A_2$ is open.*
2. *If $A_i \subseteq X$, $i \in I$ are open, then $\cup_{i \in I} A_i$ is open.*

Proof. (1) Let $A_1, A_2 \subseteq X$. If A_1 and A_2 are open, then $A_1 \cap A_2$ is open.

(2) If $A_i \subseteq X$, $i \in I$ are open, then $\cup_{i \in I} A_i$ is open.

Using DeMorgan, we immediately have the following corollary:

Corollary

Let (X, d) be a metric space.

1. Let $A_1, A_2 \subseteq X$. If A_1 and A_2 are closed, then $A_1 \cup A_2$ is closed.
2. If $A_i \subseteq X$, $i \in I$ are closed, then $\cap_{i \in I} A_i$ is closed.

Definition (Interior and closure)

Let $A \subseteq X$ where (X, d) is a metric space.

- The *closure* of A is $\overline{A} :=$

- The *interior* of A is $\mathring{A} :=$

- The *boundary* of A is $\partial A :=$

Example

Let $X = (a, b] \subseteq \mathbb{R}$ with the ordinary (Euclidean) metric. Then

Statistical Sciences
UNIVERSITY OF TORONTO

Proposition

Let $A \subseteq X$ where (X, d) is a metric space. Then $\mathring{A} = A \setminus \partial A$.

Proof.

Sequences

Definition (Sequence)

Let (X, d) be a metric space. A *sequence* is an ordered list of points x_n, $n \in \mathbb{N}$, in X, denoted $(x_n)_{n \in \mathbb{N}}$. We say that a sequence $(x_n)_{n \in \mathbb{N}}$ *converges* to a point $x \in X$ if

Recall: $\overline{A} =$

Proposition

Let (X, d) be a metric space, and let $A \subseteq X$. Then $\overline{A}$ is equal to the set of points in X which are limits of a sequence in A.

Proof.

Corollary

A set $F \subseteq X$, where (X, d) is a metric space, is closed if and only if every sequence in F which converges in X converges to a point in F.

Remark:

Cluster points of a set

Definition

Let (X, d) be a metric space and $A \subseteq X$. A point $x \in X$ is a *cluster point* of A (also called accumulation point) if for every $\epsilon > 0$, $B_\epsilon(x)$ contains infinitely many points in A.

Statistical Sciences
UNIVERSITY OF TORONTO

Proposition

$x \in X$ is a cluster point of $A \subseteq X$ where (X, d) is a metric space if and only if there exists a sequence of points $x_n \in A$, $n \in \mathbb{N}$, such that $x_n \to x$.

Proof.

Combining the previous result with the limit characterization of closure gives the following:

Corollary

For $A \subseteq X$, (X, d) a metric space, we have

$$\overline{A} = A \cup \{x \in X : x \text{ is a cluster point of } A\}.$$

Cauchy sequences

Definition (Cauchy sequence)

Let (X, d) be a metric space. A sequence denoted $(x_n)_{n \in \mathbb{N}} \in X$ is called a *Cauchy sequence* if

Proposition

Let (X, d) be a metric space, and let $(x_n)_{n \in \mathbb{N}}$ be a convergent sequence in X. Then $(x_n)_{n \in \mathbb{N}}$ is Cauchy.

Proof.

Definition

A metric space where every Cauchy sequence converges (to a point in the space) is called *complete*.

Example:

Proposition

Let (X, d) be a metric space, and let $Y \subseteq X$.

(i) If X is complete and if Y is closed in X, then Y is complete.

(ii) If Y is complete, then it is closed in X.

Proof.

Subsequences

Definition

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in a metric space (X, d). Let $(n_k)_{k \in \mathbb{N}}$ be a sequence of natural numbers with $n_1 < n_2 < \cdots$. The sequence $(x_{n_k})_{k \in \mathbb{N}}$ is called a *subsequence* of $(x_n)_{n \in \mathbb{N}}$. If $(x_{n_k})_{k \in \mathbb{N}}$ converges to $x \in X$, we call x a *subsequential limit*.

Example

$((-1)^n)_{n \in \mathbb{N}}$

> **Proposition**
>
> A sequence $(x_n)_{n \in \mathbb{N}}$ in a metric space (X, d) converges to $x \in X$ if and only if every subsequence of $(x_n)_{n \in \mathbb{N}}$ also converges to x.

Proof.

Statistical Sciences
UNIVERSITY OF TORONTO

Proof continued

Continuity

> **Definition**
>
> Let (X, d_X) and (Y, d_Y) be metric spaces, let $x_0 \in X$, and let $f : X \to Y$. f is *continuous* at x_0 if for every sequence $(x_n)_{n \in \mathbb{N}}$ in X that converges to x_0, we have $\lim_{n \to \infty} f(x_n) = f(x_0)$.
>
> We say that f is continuous if it is continuous at every point in X.

Theorem

Let (X, d_X) and (Y, d_Y) be metric spaces, let $x_0 \in X$, and let $f : X \to Y$. The following are equivalent:

(i) *f is continuous at x_0*

(ii) *for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0))) < \epsilon$ for all $x \in X$ with $d_X(x, x_0) < \delta$*

(iii) *for each $\epsilon > 0$, there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$*

Statistical Sciences
UNIVERSITY OF TORONTO

(i) f is continuous at x_0

(ii) for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0))) < \epsilon$ for all $x \in X$ with $d_X(x, x_0) < \delta$

(iii) for each $\epsilon > 0$, there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$

Proof. (i) $\Rightarrow$ (ii)

(ii) $\Rightarrow$ (iii)

(iii) $\Rightarrow$ (i)

References

Charles C. Pugh (2015). *Real Mathematical Analysis*. Undergraduate Texts in Mathematics. https://link-springer-com.myaccess.library.utoronto.ca/book/10.1007/978-3-319-17771-7

Runde ,Volker (2005). *A Taste of Topology*. Universitext. url: https://link.springer.com/book/10.1007/0-387-28387-0

Zwiernik, Piotr (2022). *Lecture notes in Mathematics for Economics and Statistics*. url: http://84.89.132.1/ piotr/docs/RealAnalysisNotes.pdf

Statistical Sciences
UNIVERSITY OF TORONTO