Module 7: Linear Algebra I

Operational math bootcamp

Statistical Sciences
UNIVERSITY OF TORONTO

Emma Kroell

University of Toronto

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Outline

Today:

- Vector space
- Subspace
- Linear independence and bases
- Linear maps, null space, range, inverses
- Matrices as linear maps

Definition

We call V a **vector space** if the following hold:

(A) *Commutativity in addition:* $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ for all $\mathbf{u},\mathbf{v}\in V$

(B) *Associativity in addition:* $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$ for all $\mathbf{u},\mathbf{v},\mathbf{w}\in V$

(C) *Existence of a neutral element, addition:* There exists a vector $\mathbf{0}$ such that for any $\mathbf{v}\in V$, $\mathbf{0}+\mathbf{v}=\mathbf{v}$

(D) *Additive inverse:* For every $\mathbf{v}\in V$, there exists another vector, which we denote $-\mathbf{v}$, such that $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$.

(E) *Existence of a neutral element, multiplication:* For any $\mathbf{v}\in V$, $1\times\mathbf{v}=\mathbf{v}$

(F) *Associativity in multiplication:* Let $\alpha,\beta\in\mathbb{F}$. For any $\mathbf{v}\in V$, $(\alpha\beta)\mathbf{v}=\alpha(\beta\mathbf{v})$

(G) Let $\alpha\in\mathbb{F},\mathbf{u},\mathbf{v}\in V$. $\alpha(\mathbf{u}+\mathbf{v})=\alpha\mathbf{u}+\alpha\mathbf{v}$.

(H) Let $\alpha,\beta\in\mathbb{F},\mathbf{v}\in V$. $(\alpha+\beta)\mathbf{v}=\alpha\mathbf{v}+\beta\mathbf{v}$.

Elements of the vector space are called vectors.
Most often we will assume $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$.

Example

The following are vector spaces:

- $\mathbb{R}^n$
- $\mathbb{C}^n$
- $C(\mathbb{R}; \mathbb{R})$, continuous functions from $\mathbb{R}$ to $\mathbb{R}$
- $M_{n \times m}$, matrices of size $n \times m$
- $\mathbb{P}_n$ (polynomials of degree n, $p(x) = a_0 + a_1 x + \ldots + a_n x^n$).

Lemma

For every $\mathbf{v} \in V$, $0\mathbf{v} = \mathbf{0}$.

Proof.

$\square$

Lemma

For every $\mathbf{v} \in V$, we have $-\mathbf{v} = (-1) \times \mathbf{v}$.

Proof.

$\square$

Definition

A subset U of V is called a **subspace** of of V if U is also a vector space (using the same addition and scalar multiplication as on V).

Proposition

A subset U of V is a subspace of V if and only if U satisfies the following three conditions:

1. $\mathbf{0} \in U$
2. Closed under addition: $\mathbf{u}, \mathbf{v} \in U$ implies $\mathbf{u} + \mathbf{v} \in U$
3. Closed under scalar multiplication: $\alpha \in \mathbb{F}$ and $\mathbf{u} \in U$ implies $\alpha\mathbf{u} \in U$

Proof.

$(\Rightarrow)$

$(\Leftarrow)$

$\square$

Linear combinations

Definition

A linear combination of vectors $\mathbf{v}_1, ..., \mathbf{v}_n$ of vectors in V is a vector of the form

$$\alpha_1 \mathbf{v}_1 + ... + \alpha_n \mathbf{v}_n = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k$$

where $\alpha_1, ..., \alpha_m \in \mathbb{F}$.

Span

Definition

The set of all linear combinations of a list of vectors $\mathbf{v}_1, ..., \mathbf{v}_n$ in V is called the **span** of $\mathbf{v}_1, ..., \mathbf{v}_n$, denoted $\text{span}\{\mathbf{v}_1, ..., \mathbf{v}_n\}$. In other words,

$$\text{span}\{\mathbf{v}_1, ..., \mathbf{v}_n\} = \{\alpha_1\mathbf{v}_1 + ... + \alpha_m\mathbf{v}_n : \alpha_1, ..., \alpha_n \in \mathbb{F}\}$$

The span of the empty list is defined to be $\{\mathbf{0}\}$.

Basis

Definition

A system of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ is called a basis (for the vector space V) if any vector $\mathbf{v} \in V$ admits a unique representation as a linear combination

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k.$$

Example

- For $\mathbb{F}^n$, $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, $\dots$, $e_n = (0, \dots, 0, 1)$ is a basis
- The monomials $1, x, x^2, \dots, x^n$ form a basis for $\mathbb{P}_n$.

Linear independence

Definition

A system of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in V is called *linearly independent* if

$$\sum_{i=1}^{n} \alpha_i \mathbf{v}_i = \mathbf{0}$$

implies $\alpha_i = 0$ for all $i = 1, \ldots, n$.

Otherwise, we call the system *linearly dependent*.

Linear combinations $\alpha_1 \mathbf{v}_1 + \ldots + \alpha_n \mathbf{v}_n$ such that $\alpha_k = 0$ for every k are called trivial.

Spanning set

Definition

A system of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in V is called *spanning* if any vector in V can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$. In other words,

$$V = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}.$$

Such a system is also often called generating or complete. The next proposition relates spanning and linearly independent to a basis.

Proposition

A system of vectors $\mathbf{v}_1, \ldots \mathbf{v}_n \in V$ is a basis if and only if it is linearly independent and spanning.

Proof.

$(\Rightarrow)$

Proof continued

$(\Leftarrow)$

Proposition

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ be spanning. Then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ contains a basis.

Sketch of proof.

Definition

An $\mathbb{F}$-vector space V is called *finite dimensional* if there exists a finite list of vectors that span it, i.e. there exist $n \in \mathbb{N}$ and $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ such that $V = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Otherwise, we call V *infinite dimensional*.

Example

- $\mathbb{F}^n$, $M_{m \times n}$, $\mathbb{P}_n$ are examples of finite dimensional vector spaces
- The $\mathbb{F}$-vector space $\mathbb{P} = \{\sum_{i=1}^{n} \alpha_i x^i \;:\; n \in \mathbb{N}, \alpha_i \in \mathbb{F}, i = 1, \ldots, n\}$ is infinite dimensional.

Corollary

Every finite dimensional vector space has a basis.

This follows from the fact that every spanning set for a vector space contains a basis.

This can also be extended to infinite dimensional vector spaces, i.e. when we do not assume that there exists a finite spanning set. However, this relies on the Axiom of Choice and is beyond the scope of this course.

Proposition

Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

Proof.

Dimension

Proposition

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and $\mathbf{u}_1, \ldots, \mathbf{u}_m$ be a basis for V. Then $m = n$.

The proof is omitted,. It relies on the fact that the number of elements in linearly independent systems are always less than or equal to the number of elements in spanning systems.

Definition

Let V be a finite dimensional $\mathbb{F}$-vector space. The number of elements in a basis of V is called the *dimension* of V and is denoted $\dim(V)$.

By the previous definition, the notion of dimension is well-defined.

Dimension

Example

- $\dim(\mathbb{F}^n) =$
- $\dim(\mathbb{P}_n) =$
- $\dim\{\mathbf{0}\} =$

Linear Maps

Definition

A map from a vector space U to a vector space V is **linear** if

$$T(\alpha \mathbf{u} + \beta \mathbf{v}) = \alpha T(\mathbf{u}) + \beta T(\mathbf{v}) \quad \text{for any } \mathbf{u}, \mathbf{v} \in V,\ \alpha, \beta \in \mathbb{F}$$

Notation: $\mathcal{L}(U, V)$ is the set of all linear maps from $\mathbb{F}$-vector space U to $\mathbb{F}$-vector space V

Example

- Zero map
- Identity map
- Differentiation

Theorem

Suppose $\mathbf{u}_1, \ldots, \mathbf{u}_n$ *is a basis for* U *and* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *is a basis for* V*. Then there exists a unique linear map* $T : U \to V$ *such that* $T\mathbf{u}_j = \mathbf{v}_j$ *for* $j = 1, \ldots, n$*.*

Proof in book

Theorem

Let $S, T \in \mathcal{L}(U, V)$ *and* $\alpha \in \mathbb{F}$*.* $\mathcal{L}(U, V)$ *is a vector space with addition defined as the sum* $S + T$ *and multiplication as the product* αT*.*

The proof follows from properties of linear maps and vector spaces. Note that the additive identity is the zero map.

Lemma

Let $T \in \mathcal{L}(U, V)$. Then $T(\mathbf{0}) = \mathbf{0}$.

Proof.

$\square$

Null space and range

Definition

Let $T : U \to V$ be a linear transformation. We define the following important subspaces:

- *Kernel or null space*: $\text{null } T = \{\mathbf{u} \in U : T\mathbf{u} = 0\}$
- *Range*: $\text{range } T = \{\mathbf{v} \in V : \exists \mathbf{u} \in U \text{ such that } \mathbf{v} = T\mathbf{u}\}$

The dimensions of these spaces are often called the following:

- *Nullity*: $\text{nullity}(T) = \dim(\text{null}(T))$
- *Rank*: $\text{rank}(T) = \dim(\text{range}(T))$

Proposition

Let $T : U \to V$. The null space of T is a subspace of U and the range of T is a subspace of V.

Proof.

$\square$

Example

Zero map from a vector space U to a vector space V:

- The null space is
- The range is

Differentiation map from $\mathbb{P}(\mathbb{R})$ to $\mathbb{P}(\mathbb{R})$:

- The null space is
- The range is

Definition (Injective and surjective)

Let $T : U \to V$. T is *injective* if $T\mathbf{u} = T\mathbf{v}$ implies $\mathbf{u} = \mathbf{v}$ and T is *surjective* if $\forall \mathbf{u} \in U$, $\exists \mathbf{v} \in V$ such that $\mathbf{v} = T\mathbf{u}$, i.e. if range $T = V$.

Theorem

$T \in \mathcal{L}(U, v)$ is injective if and only if null $T = \{\mathbf{0}\}$.

Proof.

$(\Rightarrow)$

$(\Leftarrow)$

$\square$

Theorem (Rank Nullity Theorem)

Let $T : U \to V$ be a linear transformation, where U and V are finite-dimensional vector spaces. Then

$$\operatorname{rank} T + \operatorname{nullity} T = \dim U.$$

Proof.

Proof continued

Definition (Product of linear maps)

Let $S \in \mathcal{L}(U, V)$ and $T \in \mathcal{L}(V, W)$. We define the product $ST \in \mathcal{L}(U, W)$ for $\mathbf{u} \in U$ as $ST(\mathbf{u}) = S(T(\mathbf{u}))$.

Definition

A linear map $T : U \to V$ is *invertible* if there exists a linear map $S : V \to U$ such that ST is the identity map on U and TS is the identity map on V. Such a map S is called the *inverse* of T.

If T is invertible, we denote the inverse by T^{-1}. This is justified by the fact that the inverse is unique:

Proposition

Any invertible linear map has a unique inverse.

Proof.

$\square$

Theorem

A linear map is invertible if and only if it is injective and surjective.

See proof in the book.

Definition

An invertible linear map is called an *isomorphism*. If there exists an isomorphism from one vector space to another, we say that the vector spaces are *isomorphic*.

Theorem

Two finite-dimensional vector spaces over $\mathbb{F}$ are isomorphic if and only if they have the same dimension.

Proof.

$(\Rightarrow)$

$\square$

Proof continued

$(\Leftarrow)$

Linear maps and matrices

Example

Let $A \in M_{m \times n}$ be a fixed matrix. Then, we can define a linear map $T_A \colon \mathbb{F}^n \to \mathbb{F}^m$ via $T_A(\mathbf{v}) = A\mathbf{v}$, where we recall matrix vector multiplication $(A\mathbf{v})_i = \sum_{k=1}^{n} A_{ik} v_k$ for $i = 1, \ldots, m$.

Next we will see that we can use matrices to represent linear maps between finite dimensional vector spaces.

Definition

Let $T \in \mathcal{L}(U, V)$ where U and V are vector spaces. Let $\mathbf{u}_1, \dots, \mathbf{u}_n$ and $\mathbf{v}_1, \dots, \mathbf{v}_m$ be bases for U and V respectively. The matrix of T with respect to these bases is the $m \times n$ matrix $\mathcal{M}(T)$ with entries A_{ij}, $i = 1, \dots, m$, $j = 1, \dots, n$ defined by

$$T\mathbf{u}_k = A_{1k}\mathbf{v}_1 + \cdots + A_{mk}\mathbf{v}_m$$

i.e. the kth column of A is the scalars needed to write $T\mathbf{u}_k$ as a linear combination of the basis of V:

$$T\mathbf{u}_k = \sum_{i=1}^{m} A_{ik}\mathbf{v}_i$$

Note that since a linear map $T \in \mathcal{L}(U, V)$ is uniquely determined by its image on a basis of U, we see that once we pick basis of U and V its matrix representation is uniquely determined.

Example

Let $D \in \mathcal{L}(\mathbb{P}_4(\mathbb{R}), \mathbb{P}_3(\mathbb{R}))$ be the differentiation map, $Dp = p'$. Find the matrix of D with respect to the standard bases of $\mathbb{P}_3(\mathbb{R})$ and $\mathbb{P}_4(\mathbb{R})$.

Standard basis: $1, x, x^2, x^3, (x^4)$

$T(u_1)$

$T(u_2)$

$T(u_3)$

$T(u_4)$

$T(u_5)$

The matrix is:

- Observe that if we choose bases $\mathbf{u}_1, \ldots, \mathbf{u}_n$ and $\mathbf{v}_1, \ldots, \mathbf{v}_m$ for U, V and represent $T \in \mathcal{L}(U, V)$ as a matrix $\mathcal{M}(T)$, then the corresponding map can be obtained by just working with the coordinates of vectors in U, V with respect to the chosen basis
- If $\mathbf{u} = \sum_{i=1}^{n} \alpha_i \mathbf{u}_i$, then the coordinates of $T(\mathbf{u})$ with respect to $\mathbf{v}_1, \ldots, \mathbf{v}_m$ can be obtained by the matrix vector multiplication $\mathcal{M}(T)\boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is the $n \times 1$ matrix with entries α_i

Example

If we want to find the derivative of $p = x^4 + 12x^3 - 5x^2 + 7$ with respect to the standard monomial basis of $\mathbb{P}_4(\mathbb{R})$, we use $\mathcal{M}(D)$ from the previous example to obtain

$$\mathcal{M}(D)\boldsymbol{\alpha} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 7 \\ 0 \\ -5 \\ 12 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -10 \\ 36 \\ 4 \end{pmatrix}.$$

Thus, translating back into the monomial basis of $\mathbb{P}_3(\mathbb{R})$ gives $D(p) = -10x + 36x^2 + 4x^3$.

Other points

- Looking at matrices as representations of linear maps gives us an intuitive explanation for why we do matrix multiplication the way we do! In fact, we want matrix multiplication to represent composition of linear maps
- We can use matrices to solve linear systems.

Next time

- Determinants
- Eigenvalues and eigenvectors
- Inner product spaces

References

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