Exercises for Module 6: Topology and Linear Algebra

1. Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$ be dense. Show that if $A \subseteq B \subseteq X$, then B is dense as well.

2. Let $(X, \mathcal{T})$ be a Hausdorff topological space. Show that the singleton $\{x\}$ is closed for all $x \in X$. Hint: Show that the complement is open.

3. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$ be topological spaces and let $f\colon X \to Y$, $g\colon Y \to Z$ be continuous. Show that $g \circ f\colon X \to Z$ is continuous as well.

4. Let (X, d) be a metric space and $K \subset X$ compact. Show that for all $\epsilon > 0$ there exists $\{x_1, x_2, \ldots, x_n\} \subseteq K$ such that for all $y \in K$ we have $d(y, x_i) < \epsilon$ for some $i = 1, \ldots, n$.

5. Suppose that $\alpha \in \mathbb{F}, \mathbf{v} \in V$, and $\alpha \mathbf{v} = \mathbf{0}$. Prove that $a = 0$ or $v = 0$.

6. Prove the following: Let V be a vector space and let $U_1, U_2 \subseteq V$ be subspaces. Then $U_1 \cap U_2$ is also a subspace of V.

7. Let U_1 and U_2 be subspaces of a vector space V. Prove that $U_1 \cup U_2$ is a subspace of V if and only if $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$.