Module 6: End of Topology, Start of Linear Algebra

Operational math bootcamp

Emma Kroell

University of Toronto

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Outline

- Finish topology
  - Dense subsets
  - Compactness
  - Continuity
- Start linear algebra
  - Vector spaces and subspaces

Last time

Definition (Topology)

Let $\mathcal{T} \subseteq \mathcal{P}(X)$. We call $\mathcal{T}$ a *topology* on X if the following holds:

(i) $\emptyset, X \in \mathcal{T}$

(ii) Let A be an arbitrary index set. If $U_\alpha \in \mathcal{T}$ for $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$ ($\mathcal{T}$ is closed under taking arbitrary unions)

(iii) Let $n \in \mathbb{N}$. If $U_1, \ldots, U_n \in \mathcal{T}$, then $\bigcap_{i=1}^{n} U_i \in \mathcal{T}$ ($\mathcal{T}$ is closed under taking finite intersections)

If $U \in \mathcal{T}$, we call U *open*. We call $U \subseteq X$ *closed*, if $U^c \in \mathcal{T}$. We call $(X, \mathcal{T})$ a *topological space*.

Definition

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$ be any subset.

- The *interior* of A is $\mathring{A} := \{a \in A \colon \exists U \in \mathcal{T} \text{ s.t. } U \subseteq A \text{ and } a \in U\}$.
- The *closure* of A is $\overline{A} := \{x \in X \colon \forall U \in \mathcal{T} \text{ with } x \in U, U \cap A \neq \emptyset\}$.
- The *boundary* of A is $\partial A := \{x \in X \colon \forall U \in \mathcal{T} \text{ with } x \in U,\ U \cap A \neq \emptyset \text{ and } U \cap A^c \neq \emptyset\}$.

Density

Definition

Let $(X, \mathcal{T})$ be a topological space. A subset $A \subseteq X$ is called *dense* if $\overline{A} = X$.

Using the definition of closure, we see that $A \subseteq X$ is dense if and only if for all non-empty $U \in \mathcal{T}$, $U \cap A \neq \emptyset$.

Example

- The rationals $\mathbb{Q}$ are dense in the reals $\mathbb{R}$.
- The only dense subset in $(X, \mathcal{P}(X))$ is X itself.
- Any non-empty subset is dense in $(X, \{\emptyset, X\})$.

Separability

Definition

A topological space $(X, \mathcal{T})$ is *separable* if it contains a countable dense subset.

Example

Hausdorff space

Definition

A topological space $(X, \mathcal{T})$ is called *Hausdorff* if for all $x \neq y \in X$ there exist open sets U_x, U_y with $x \in U_x$ and $y \in U_y$ such that $U_x \cap U_y = \emptyset$.

So in a Hausdorff space, we can separate any two elements using open sets.

Example

Let (X, d) be a metric space. Then $(X, \mathcal{T}_d)$ is Hausdorff, where $\mathcal{T}_d$ is the topology induced by the metric d.

Example

Let X be an infinite set and $\mathcal{T} = \{U \subseteq X \colon U^c \text{ is finite}\} \cup \emptyset$. Then $(X, \mathcal{T})$ is not Hausdorff.

Compactness

Definition

Let $(X, \mathcal{T})$ be a topological space and $K \subseteq X$.

A collection $\{U_i\}_{i \in I}$ of open sets is called *open cover* of K if $K \subseteq \cup_{i \in I} U_i$.

The set K is called *compact* if for all open covers $\{U_i\}_{i \in I}$ there exists a finite subcover, meaning there exists an $n \in \mathbb{N}$ and $\{U_1, \ldots, U_n\} \subseteq \{U_i\}_{i \in I}$ such that $K \subseteq \cup_{i=1}^{n} U_i$.

Example

Let $S \subseteq X$ where $(X, \mathcal{T})$ is a topological space. If S is finite, then it is compact.

Example

$(0,1)$ is not compact.

Proposition

Let $(X, \mathcal{T})$ be a topological space and take a non-empty subset $K \subseteq X$. The following holds:

1. If X is compact and K is closed, then K is compact (i.e. closed subsets of compact sets are compact).
2. If $(X, \mathcal{T})$ is Hausdorff, then K being compact implies that K is closed.

Proof.

(1) If X is compact and $K \subseteq X$ is closed, then K is compact

$\square$

Proof.

(2) If $(X, \mathcal{T})$ is Hausdorff, then $K \subseteq X$ compact $\Leftrightarrow$ K is closed.

$\square$

Proof continued

Compactness on $\mathbb{R}^n$

Theorem (Heine-Borel Theorem)

Let $K \subseteq \mathbb{R}^n$. Then K is compact with respect to the topology induced by the Euclidean distance if and only if it is closed and bounded.

Just as we had a sequential characterization of the closure of a set in metric spaces, we similarly have a sequential characterization of compactness.

Theorem

Let (X, d) be a metric space. Then $K \subset X$ is compact with respect to the metric induced by d if and only if every sequence in K admits a subsequence converging to some point in K.

A corollary of this statement together with Heine-Borel is the Bolzano-Weierstrass theorem.

Corollary (Bolzano-Weierstrass)

Any bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Continuity on a topological space

Definition

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f: X \to Y$ is called *continuous* if for all $U \in \mathcal{T}_Y$, $f^{-1}(U) \in \mathcal{T}_X$, i.e. the pre-image of open sets is open.

We can also specify continuity at a point $x_0 \in X$.

Definition

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. A map $f: X \to Y$ is called *continuous at* $x_0 \in X$ if for all $U \in \mathcal{T}_Y$ with $f(x_0) \in U$, $f^{-1}(U) \in \mathcal{T}_X$, i.e. the preimage of open sets containing $f(x_0)$ is open (and contains x_0).

Proposition

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Suppose $K \subset X$ is compact and let $f : K \to Y$ be continuous. Then $f(K)$ is compact.

Recall from the set theory section:
If $f : X \to Y$:

1. $A \subseteq B \subseteq Y \Rightarrow f^{-1}(A) \subseteq f^{-1}(B)$ and $A \subseteq B \subseteq X \Rightarrow f(A) \subseteq f(B)$
2. $f^{-1}(\cup_{i \in I} A_i) = \cup_{i \in I} f^{-1}(A_i)$, where $A_i \subseteq Y \; \forall i \in I$
3. $f(\cup_{i \in I} A_i) = \cup_{i \in I} f(A_i)$, where $A_i \subseteq X \; \forall i \in I$
4. $A \subseteq X \Rightarrow A \subseteq f^{-1}(f(A))$
5. $B \subseteq Y \Rightarrow f(f^{-1}(B)) \subseteq B$

Proof.

Proof continued

Right and left continuous

Recall: $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|x_0 - y| < \delta$ implies $|f(x_0) - f(y)| < \epsilon$.

Definition

Let $f : \mathbb{R} \to \mathbb{R}$.

- f is *left continuous* at $x_0 \in \mathbb{R}$ if for all $\epsilon > 0$ there exists a $\delta > 0$, such $|f(x_0) - f(x)| < \epsilon$ whenever $x_0 - \delta < x < x_0$.
- f is *right continuous* at $x_0 \in \mathbb{R}$ if for all $\epsilon > 0$ there exists a $\delta > 0$, such $|f(x_0) - f(x)| < \epsilon$ whenever $x_0 < x < x_0 + \delta$.

We say that f is left continuous if it is left continuous at all points in the domain, and similar for right continuous.

Proposition

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if it is left and right continuous.

Proof.

$\square$

Proof continued

Bounded sequences and monotone convergence

Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. We call $(x_n)_{n\in\mathbb{N}}$ *bounded* if there exists an $M > 0$ such that $|x_n| < M$ for all $n \in \mathbb{N}$.

Theorem (Monotone convergence theorem)

(i) *Suppose* $(x_n)_{n\in\mathbb{N}}$ *is an increasing sequence, i.e.* $x_n \leq x_{n+1}$ *for all* $n \in \mathbb{N}$*, and that it is bounded (above). Then the sequence converges. Furthermore,* $\lim_{n\to\infty} x_n = \sup_{n\in\mathbb{N}} x_n$*, where* $\sup_{n\in\mathbb{N}} x_n := \sup\{x_n : n \in \mathbb{N}\}$.

(ii) *Suppose* $(x_n)_{n\in\mathbb{N}}$ *is a decreasing sequence, i.e.* $x_n \geq x_{n+1}$ *for all* $n \in \mathbb{N}$*, which is bounded (below). Then the sequence converges and* $\lim_{n\to\infty} x_n = \inf_{n\in\mathbb{N}} x_n := \inf\{x_n : n \in \mathbb{N}\}$.

Convention: $\sup A = \infty$ if $A \subseteq \mathbb{R}$ is not bounded above and $\inf A = -\infty$ if A is not bounded below.

Lemma

If $A \subseteq B \subseteq \mathbb{R}$ is non-empty, then $\inf A \leq \sup A$, $\sup A \leq \sup B$, *and* $\inf A \geq \inf B$.

The proof of this follows from the definition of greatest lower and least upper bound.

Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. We define the *limit superior* of $(x_n)_{n\in\mathbb{N}}$ as

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \sup_{k\geq n} x_k.$$

Similarly we define the *limit inferior* of $(x_n)_{n\in\mathbb{N}}$ as

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} \inf_{k\geq n} x_k.$$

If the sequence $(x_n)_{n\in\mathbb{N}}$ is not bounded above, then $\limsup_{n\to\infty} x_n = \infty$. Similarly, if the sequence $(x_n)_{n\in\mathbb{N}}$ is not bounded below, then $\liminf_{n\to\infty} x_n = -\infty$.

Proposition

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}$.

- The sequence of suprema, $s_n = \sup_{k \geq n} x_k$, is decreasing and the sequence of infima, $i_n = \inf_{k \geq n} x_k$, is increasing.
- The limit superior and the limit inferior of a bounded sequence always exist and are finite.

Proof.

The first part is true by the previous Lemma. The second follows by the Monotone Convergence Theorem. $\square$

Theorem

Let $(x_n)_{n \in \mathbb{N}}$ *be a sequence in* $\mathbb{R}$*. Then the sequence converges to* $x \in \mathbb{R}$ *if and only if* $\limsup_{n\to\infty} x_n = x = \liminf_{n\to\infty} x_n$.

Proof.

Proof continued

We can extend this easily to a sequence of functions $f_n \colon X \to \mathbb{R}$ as follows:

Define $f = \limsup_{n \to \infty} f_n$ to be the function defined pointwise by $f(x) = \limsup_{n \to \infty}(f_n(x))$ and similar for the limit inferior.

There also exists a set theoretic version in terms of unions and intersections which you will encounter in probability.

Linear Algebra

Definition

We call V a **vector space** if the following hold:

(A) *Commutativity in addition:* $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ for all $\mathbf{u},\mathbf{v}\in V$

(B) *Associativity in addition:* $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$ for all $\mathbf{u},\mathbf{v},\mathbf{w}\in V$

(C) *Existence of a neutral element, addition:* There exists a vector $\mathbf{0}$ such that for any $\mathbf{v}\in V$, $\mathbf{0}+\mathbf{v}=\mathbf{v}$

(D) *Additive inverse:* For every $\mathbf{v}\in V$, there exists another vector, which we denote $-\mathbf{v}$, such that $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$.

(E) *Existence of a neutral element, multiplication:* For any $\mathbf{v}\in V$, $1\times\mathbf{v}=\mathbf{v}$

(F) *Associativity in multiplication:* Let $\alpha,\beta\in\mathbb{F}$. For any $\mathbf{v}\in V$, $(\alpha\beta)\mathbf{v}=\alpha(\beta\mathbf{v})$

(G) Let $\alpha\in\mathbb{F},\mathbf{u},\mathbf{v}\in V$. $\alpha(\mathbf{u}+\mathbf{v})=\alpha\mathbf{u}+\alpha\mathbf{v}$.

(H) Let $\alpha,\beta\in\mathbb{F},\mathbf{v}\in V$. $(\alpha+\beta)\mathbf{v}=\alpha\mathbf{v}+\beta\mathbf{v}$.

Elements of the vector space are called vectors.
Most often we will assume $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$.

Example

The following are vector spaces:

- $\mathbb{R}^n$
- $\mathbb{C}^n$
- $C(\mathbb{R}; \mathbb{R})$, continuous functions from $\mathbb{R}$ to $\mathbb{R}$
- $M_{n \times m}$, matrices of size $n \times m$
- $\mathbb{P}_n$ (polynomials of degree n, $p(x) = a_0 + a_1 x + \ldots + a_n x^n$).

Lemma

For every $\mathbf{v} \in V$, $0\mathbf{v} = \mathbf{0}$.

Proof.

$\square$

Lemma

For every $\mathbf{v} \in V$, we have $-\mathbf{v} = (-1) \times \mathbf{v}$.

Proof.

$\square$

Definition

A subset U of V is called a **subspace** of of V if U is also a vector space (using the same addition and scalar multiplication as on V).

Proposition

A subset U of V is a subspace of V if and only if U satisfies the following three conditions:

1. $\mathbf{0} \in U$
2. Closed under addition: $\mathbf{u}, \mathbf{v} \in U$ implies $\mathbf{u} + \mathbf{v} \in U$
3. Closed under scalar multiplication: $\alpha \in \mathbb{F}$ and $\mathbf{u} \in U$ implies $\alpha\mathbf{u} \in U$

Proof.

$(\Rightarrow)$

$(\Leftarrow)$

$\square$

References

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