Module 4: Metric Spaces and Sequences II

Operational math bootcamp

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Outline

- Open and closed sets
- Sequences
  - Cauchy sequences
  - subsequences
- Continuous functions
  - Contractions
- Equivalence of metrics

Definition (Open and closed sets)

Let (X, d) be a metric space.

- A set $U \subseteq X$ is *open* if for every $x \in U$ there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$.
- A set $F \subseteq X$ is *closed* if $F^c := X \setminus F$ is open.

Proposition

Let (X, d) be a metric space.

1. *Let $A_1, A_2 \subseteq X$. If A_1 and A_2 are open, then $A_1 \cap A_2$ is open.*
2. *If $A_i \subseteq X$, $i \in I$ are open, then $\cup_{i \in I} A_i$ is open.*

Proof.

(1) Let $A_1, A_2 \subseteq X$. If A_1 and A_2 are open, then $A_1 \cap A_2$ is open.

(2) If $A_i \subseteq X$, $i \in I$ are open, then $\cup_{i \in I} A_i$ is open.

$\square$

Using DeMorgan, we immediately have the following corollary:

Corollary

Let (X, d) be a metric space.

1. *Let $A_1, A_2 \subseteq X$. If A_1 and A_2 are closed, then $A_1 \cup A_2$ is closed.*
2. *If $A_i \subseteq X$, $i \in I$ are closed, then $\cap_{i \in I} A_i$ is closed.*

Definition (Interior and closure)

Let $A \subseteq X$ where (X, d) is a metric space.

- The *closure* of A is $\overline{A} :=$
- The *interior* of A is $\mathring{A} :=$
- The *boundary* of A is $\partial A :=$

Example

Let $X = (a, b] \subseteq \mathbb{R}$ with the ordinary (Euclidean) metric. Then

Proposition

Let $A \subseteq X$ *where* (X, d) *is a metric space. Then* $\mathring{A} = A \setminus \partial A$*.*

Proof.

$\square$

Sequences

Definition (Sequence)

Let (X, d) be a metric space. A *sequence* is an ordered list of points x_n, $n \in \mathbb{N}$, in X, denoted $(x_n)_{n\in\mathbb{N}}$. We say that a sequence $(x_n)_{n\in\mathbb{N}}$ *converges* to a point $x \in X$ if

Proposition

Let (X, d) be a metric space, and let $A \subseteq X$. Then $\overline{A}$ is equal to the set of points in X which are limits of a sequence in A.

Proof.

$\square$

Proof continued

Corollary

A set $F \subseteq X$, where (X, d) is a metric space, is closed if and only if every sequence in F which converges in X converges to a point in F.

Cluster points of a set

Definition

Let (X, d) be a metric space and $A \subseteq X$. A point $x \in X$ is a *cluster point* of A (also called accumulation point) if for every $\epsilon > 0$, $B_\epsilon(x)$ contains infinitely many points in A.

Proposition

$x \in X$ is a cluster point of $A \subseteq X$ where (X, d) is a metric space if and only if there exists a sequence of points $x_n \in A$, $n \in \mathbb{N}$, such that $x_n \to x$.

Proof.

$\square$

Combining the previous result with the limit characterization of closure gives the following:

Corollary

For $A \subseteq X$, (X, d) a metric space, we have

$$\overline{A} = A \cup \{x \in X : x \text{ is a cluster point of } A\}.$$

Cauchy sequences

Definition (Cauchy sequence)

Let (X, d) be a metric space. A sequence denoted $(x_n)_{n\in\mathbb{N}} \in X$ is called a *Cauchy sequence* if

Proposition

Let (X, d) be a metric space, and let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in X. Then $(x_n)_{n\in\mathbb{N}}$ is Cauchy.

Proof.

$\square$

Definition

A metric space where every Cauchy sequence converges (to a point in the space) is called *complete*.

Proposition

Let (X, d) be a metric space, and let $Y \subseteq X$.

(i) If X is complete and if Y is closed in X, then Y is complete.

(ii) If Y is complete, then it is closed in X.

Proof.

$\square$

Subsequences

Definition

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space (X, d). Let $(n_k)_{k\in\mathbb{N}}$ be a sequence of natural numbers with $n_1 < n_2 < \cdots$. The sequence $(x_{n_k})_{k\in\mathbb{N}}$ is called a *subsequence* of $(x_n)_{n\in\mathbb{N}}$. If $(x_{n_k})_{k\in\mathbb{N}}$ converges to $x \in X$, we call x a *subsequential limit*.

Example

$((-1)^n)_{n\in\mathbb{N}}$

Proposition

A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X, d) converges to $x \in X$ if and only if every subsequence of $(x_n)_{n\in\mathbb{N}}$ also converges to x.

Proof.

$\square$

Proof continued

Continuity

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces, let $x_0 \in X$, and let $f : X \to Y$. f is *continuous* at x_0 if for every sequence $(x_n)_{n \in \mathbb{N}}$ in X that converges to x_0, we have $\lim_{n \to \infty} f(x_n) = f(x_0)$.

We say that f is continuous if it is continuous at every point in X.

Theorem

Let (X, d_X) and (Y, d_Y) be metric spaces, let $x_0 \in X$, and let $f : X \to Y$. The following are equivalent:

(i) *f is continuous at x_0*

(ii) *for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0))) < \epsilon$ for all $x \in X$ with $d_X(x, x_0) < \delta$*

(iii) *for each $\epsilon > 0$, there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$*

(i) f is continuous at x_0
(ii) for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \epsilon$ for all $x \in X$ with $d_X(x, x_0) < \delta$

Proof.

(i) $\Rightarrow$ (ii)

$\square$

(i) f is continuous at x_0
(ii) for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0))) < \epsilon$ for all $x \in X$ with $d_X(x, x_0) < \delta$
(iii) for each $\epsilon > 0$, there is $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$

Proof continued

(ii) $\Rightarrow$ (iii)

(iii) $\Rightarrow$ (i)

Corollary

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$. The following are equivalent:

(i) f is continuous

(ii) if $U \subseteq Y$ is open, then $f^{-1}(U)$ is open

(iii) if $F \subseteq Y$ is closed, then $f^{-1}(F)$ is closed

We need the following results about sets and functions:

Let X and Y be sets and $f : X \to Y$. Let $A, B \subseteq Y$. Then

1. $A \subseteq B \implies f^{-1}(A) \subseteq f^{-1}(B)$
2. $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$

Proof.

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$.

(i) $\Rightarrow$ (ii):

$\square$

Proof continued

(ii) $\Rightarrow$ (i)

(ii) $\Rightarrow$ (iii)

(iii) $\Rightarrow$ (ii)

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$.

- f is *uniformly continuous* if for all $\epsilon > 0$, there exists $\delta > 0$ such that for every $x_1, x_2 \in X$ with $d_X(x_1, x_2) < \delta$, we have $d_Y(f(x_1), f(x_2))) < \epsilon$
- f is *Lipschitz continuous* if there exists a $K > 0$ such that for every $x_1, x_2 \in X$ we have $d_Y(f(x_1), f(x_2))) \leq K d_X(x_1, x_2)$

Proposition

Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$.

$$f \text{ is Lipschitz continuous } \Rightarrow \text{ f is uniformly continuous } \Rightarrow \text{ f is continuous}$$

Proof is one of your exercises.

Contraction Mapping Theorem

Definition

Let (X, d) be a metric space and let $f : X \to X$. We say that $x^* \in X$ is a *fixed point* of f if $f(x^*) = x^*$.

Definition

Let (X, d) be a metric space and let $f : X \to X$. f is a *contraction* if there exists a constant $k \in [0, 1)$ such that for all $x, y \in X$, $d(f(x), f(y))) \leq kd(x, y)$.

Observe that a function is a contraction if and only if it is Lipschitz continuous with constant $K < 1$.

Theorem (Contraction Mapping Theorem)

Suppose that $f : X \to X$ is a contraction and the metric space X is complete. Then f has a unique fixed point x^*.

Example

Let $f : \left[-\frac{1}{3}, \frac{1}{3}\right] \to \left[-\frac{1}{3}, \frac{1}{3}\right]$ be defined by the mapping $x \mapsto x^2$. Assume we use the standard Euclidean metric, $d(x, y) = |x - y|$. f has a unique fixed point because

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