Module 4: Metric Spaces and Sequences II

1. Find the closure, interior, and boundary of the following sets using Euclidean distance:

   (i) $A := \{(x, y) \in \mathbb{R}^2 : y < x^2\} \subseteq \mathbb{R}^2$

   (ii) $B := [0, 1) \times [0, 1) \subseteq \mathbb{R}^2$

   (iii) $C := \{0\} \cup \{1/n \colon n \in \mathbb{N}\} \subseteq \mathbb{R}$

(i) $\overline{A} = \{(x,y) \in \mathbb{R}^2 : y \leq x^2\}$

$\mathring{A} = A$

$\partial A = \{(x,y) \in \mathbb{R}^2 : y = x^2\}$

(ii) $\overline{B} = [0,1] \times [0,1]$

$\mathring{B} = (0,1) \times (0,1)$

$\partial B = \{0\} \times [0,1] \cup \{1\} \times [0,1] \cup [0,1] \times \{0\} \cup [0,1] \times \{1\}$

(iii) $\overline{C} = C$

$\mathring{C} = \emptyset$

$\partial C = C$

2. Prove the following: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space (X, d) that converges to a point $x \in X$. Then x is unique.

Proof. By contradiction.

Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in a metric space (X,d) that converges to both $x_1 \in X$ & $x_2 \in X$, where $x_1 \neq x_2$.

Note that since $x_1 \neq x_2$, by property (i) of metrics, $\exists\, \delta > 0$ s.t $d(x_1, x_2) = \delta$ (i.e. it is non-zero).

Let $\varepsilon > 0$ be arbitrary.

Since $x_n \to x_1$, $\exists\, n_1 \in \mathbb{N}$ s.t. $d(x_n, x_1) < \varepsilon/2$ $\forall n \geq n_1$.

Similarly, since $x_n \to x_2$, $\exists\, n_2 \in \mathbb{N}$ s.t. $d(x_n, x_2) < \varepsilon/2$ $\forall n \geq n_2$.

Let $n \geq \max\{n_1, n_2\}$. Then by the Δ inequality,

$$d(x_1, x_2) \leq d(x_1, x_n) + d(x_n, x_2) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this holds for $\varepsilon = \delta$, which is a contradiction. $\therefore x_1 = x_2$.

3. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}$ such that $x_n \to x$ and and $y_n \to y$, with $\alpha, x, y, \in \mathbb{R}$.

(i) Show that $\alpha\, x_n \to \alpha\, x$.

(i) Show that $x_n + y_n \to x + y$.

(i) Let $x_n \to x$.

Let $\varepsilon > 0$. Since $x_n \to x$, $\exists n_0 \in \mathbb{N}$ s.t $\forall n \geq n_0$, $|x_n - x| < \frac{\varepsilon}{|\alpha|}$.

This implies $\exists n_0 \in \mathbb{N}$ s.t $|\alpha||x_n - x| < \varepsilon$

$\Rightarrow \exists n_0 \in \mathbb{N}$ s.t $|\alpha x_n - \alpha x| < \varepsilon$.

$\therefore \alpha x_n \to \alpha x$

(ii) Let $\varepsilon > 0$ arbitrary.

Since $x_n \to x$, $\exists n_x \in \mathbb{N}$ s.t. $\forall n \geq n_x$, $|x_n - x| < \varepsilon/2$.

Since $y_n \to y$, $\exists n_y \in \mathbb{N}$ s.t. $\forall n \geq n_y$, $|y_n - y| < \varepsilon/2$.

Let $n^* = \max\{n_x, n_y\}$.

Then for $n \geq n^*$,
$$\begin{aligned} |x_n + y_n - (x+y)| &= |x_n - x + y_n - y| \\ &\leq |x_n - x| + |y_n - y| \quad \text{by } \Delta \text{ ineq} \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

$\therefore x_n + y_n \to x + y$.

4. Show that discrete metric spaces (i.e. those with the metric defined as define $d\colon X \times X \to \mathbb{R}$ by $d(x,x) = 0$ and $d(x,y) = 1$ for $x \neq y \in X$) are complete.

Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (X, d).

Then $\forall \varepsilon > 0$ $\exists n_\varepsilon \in \mathbb{N}$ s.t. $d(x_n, x_m) < \varepsilon$ $\forall n, m \geq n_\varepsilon$.

Since this holds for $\varepsilon = 1$, we must have that $\exists n_1 \in \mathbb{N}$ s.t. $x_n = x_m$ $\forall n, m \geq n_1$.

Therefore every Cauchy sequence in (X, d) is eventually constant, so every Cauchy sequence converges.

$\therefore (X, d)$ is complete

5. Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$. Prove that

$$f \text{ is Lipschitz continuous } \Rightarrow f \text{ is uniformly continuous } \Rightarrow f \text{ is continuous.}$$

Provide examples to show that the other directions do not hold.

(1) f is Lipschitz $\Rightarrow$ f is uniformly continuous

Suppose $f: X \to Y$ is Lipschitz with Lipschitz constant $K > 0$.
Let $\varepsilon > 0$ arbitrary. Choose $\delta = \varepsilon / K > 0$. Then if $x_1, x_2 \in X$ s.t. $d_X(x_1, x_2) < \delta = \varepsilon/K$, then $d_Y(f(x_1), f(x_2)) \le K d_X(x_1, x_2) < K \varepsilon / K = \varepsilon$. Thus f is uniformly cont. by def.

↳ by def of Lipschitz cont.

(2) Example of f that is unif cont but not Lipschitz.

Let $f(x) = \sqrt{x}$, $f: [0,1] \to [0,1]$

For $\varepsilon > 0$, choose $\delta = \varepsilon$. Then for any $x, y \in [0,1]$, if $|x-y| < \delta = \varepsilon^2$, then

$$|\sqrt{x} - \sqrt{y}|^2 \le |\sqrt{x} - \sqrt{y}|\,|\sqrt{x} + \sqrt{y}| = |x-y| < \varepsilon^2 \implies |\sqrt{x} - \sqrt{y}| < \varepsilon$$

$\therefore f(x) = \sqrt{x}$ is unif. cont. on $[0,1]$

However, f is not Lipschitz.

Proof Suppose in order to derive a contradiction that it is.
Then $\forall x, y \in [0,1]$, $|\sqrt{x} - \sqrt{y}| \le K|x-y|$. Take $y = 0 \Rightarrow \sqrt{x} \le \frac{K}{x} \Rightarrow \frac{1}{\sqrt{x}} \le K$.
But $\lim_{x \to 0} \frac{1}{\sqrt{x}} = \infty \not\le K$, which is a contradiction. $\therefore f$ is not Lipschitz

(3) f is unif cont. $\Rightarrow$ f is cont

This is clear from the definitions (using the ε-δ def of continuity).
Take δ to be the one from the def of uniformly continuous, and we are done.

(4) Example of a function which is continuous but not uniformly cont.

Choose $f: [0, \infty) \to [0, \infty)$, $f(x) = x^2$.

We know that f is continuous (prove it using ε-δ is you like).

Suppose in order to derive a contradiction that it is uniformly continuous.

Then for any $\varepsilon > 0$ $\exists \delta > 0$ s.t. $\forall x, y \in X$ with $|x-y| < \delta$, $|x^2 - y^2| < \varepsilon$.

$\Rightarrow |x-y|\,|x+y| < \varepsilon$. Choose $\varepsilon = 1$ and $y = x + \frac{\delta}{2}$ (okay since $|x-y| = \frac{\delta}{2} < \delta$)

Then $\frac{\delta}{2} \left| x + x + \frac{\delta}{2} \right| < 1$

$\Rightarrow \frac{\delta}{2}\left(2x + \frac{\delta}{2}\right) < 1$

$\Rightarrow x\delta + \frac{\delta^2}{4} < 1$

Choose $x = 1/\delta$

$\Rightarrow 1 + \frac{\delta^2}{4} < 1$

Contradiction.

$\therefore f(x) = x^2$ not unif. cont.

6. Show that the function $f(x) = \frac{1}{2}\left(x + \frac{5}{x}\right)$ has a unique fixed point on $(0, \infty)$. What is it? (Hint: you will have to restrict the interval.)

We need $|f(x) - f(y)| \leq K|x-y|$ for $K \in [0,1)$ & $x, y \in X$. We can pick $X \subset (0, \infty)$.

$$
\begin{aligned}
|f(x) - f(y)| &= \left|\tfrac{1}{2}\left(x + \tfrac{5}{x}\right) - \tfrac{1}{2}\left(y + \tfrac{5}{y}\right)\right| \\
&= \tfrac{1}{2}\left| x - y + \tfrac{5}{x} - \tfrac{5}{y}\right| \\
&= \tfrac{1}{2}\left| x - y + \tfrac{5y - 5x}{xy}\right| \\
&= \tfrac{1}{2}\left| x - y - 5\,\tfrac{x-y}{xy}\right| \\
&= \tfrac{1}{2}|x-y|\left|1 - \tfrac{5}{xy}\right|
\end{aligned}
$$

So we need $\frac{1}{2}\left|1 - \frac{5}{xy}\right| \leq K$, $K \in [0,1)$. Take $K = 4/5$.

$\Rightarrow \left|1 - \frac{5}{xy}\right| \leq 8/5$

$\Rightarrow -\frac{8}{5} \leq 1 - \frac{5}{xy} \leq \frac{8}{5}$

$-\frac{8}{5} \leq 1 - \frac{5}{xy}$ | $1 - \frac{5}{xy} \leq \frac{8}{5}$
$\Rightarrow \frac{-13}{5} \leq \frac{-5}{xy}$ | $\Rightarrow \frac{-5}{xy} \leq \frac{4}{5}$
$\Rightarrow \frac{13}{5} \geq \frac{5}{xy}$ | $\Rightarrow \frac{-25}{4} \leq xy$
$\Rightarrow xy \geq 25/13$ | always true

If $x = y$, need $x^2 \geq \frac{25}{13}$

If $x = y$, need $x \geq \frac{5}{\sqrt{13}}$.

Proof: Let $X = \left[\frac{5}{\sqrt{13}}, \infty\right)$. X is complete since it is a closed subset of $\mathbb{R}$. Let $x, y \in X$. Then

$$
\begin{aligned}
|f(x) - f(y)| &= \left|\tfrac{1}{2}\left(x - \tfrac{5}{x}\right) - \tfrac{1}{2}\left(y - \tfrac{5}{y}\right)\right| = \tfrac{1}{2}\left|x - \tfrac{5}{x} - y + \tfrac{5}{y}\right| \\
&= \tfrac{1}{2}|x-y|\left|1 - \tfrac{5}{xy}\right| \\
&\leq \tfrac{1}{2}|x-y|\left|1 - \tfrac{5}{5/\sqrt{13} \cdot 5/\sqrt{13}}\right| \\
&= \tfrac{1}{2}|x-y|\left|1 - \tfrac{13}{5}\right| \\
&= \tfrac{1}{2}\,\tfrac{8}{5}\,|x-y| \\
&= 4/5\,|x-y|
\end{aligned}
$$

Thus f is a contraction w/ constant $K = 4/5$.

$\therefore$ By the contraction mapping Thm, f has a unique fixed point in $\left[\frac{5}{\sqrt{13}}, \infty\right)$. (It is $\sqrt{5}$)

To justify that there is no other fixed point on $(0, \frac{5}{\sqrt{13}})$, we note that

$$f\left(\frac{5}{\sqrt{13}}\right) = \frac{1}{2}\left(\frac{5}{\sqrt{13}} + \sqrt{13}\right) > \frac{5}{\sqrt{13}}$$

and since the function is decreasing on $(0, 5/\sqrt{13})$ $\left(f'(x) = \frac{1}{2}(1 - 5x^{-2}) < 0 \text{ if } x < 5/\sqrt{13}\right)$,