UNIVERSITY OF TORONTO | Statistical Sciences

DoSS Summer Bootcamp Probability Module 6

Ichiro Hashimoto

University of Toronto

July 20, 2023

Recap

Learnt in last module:

- Moments
  - Expectation, Raw moments, central moments
  - Moment-generating functions
- Change-of-variables using MGF
  - Gamma distribution
  - Chi square distribution
- Conditional expectation
  - Law of total expectation
  - Law of total variance

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Outline

- Covariance
  - Covariance as an inner product
  - Correlation
  - Cauchy-Schwarz inequality
  - Uncorrelatedness and Independence
- Concentration
  - Markov's inequality
  - Chebyshev's inequality
  - Chernoff bounds

Covariance

Recall the property of expectation:

$$\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y).$$

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Covariance

Recall the property of expectation:

$$\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y).$$

What about the variance?

$$\begin{aligned} Var(X+Y) &= \mathbb{E}(X+Y-\mathbb{E}(X)-\mathbb{E}(Y))^2 \\ &= \mathbb{E}(X-\mathbb{E}(X))^2 + \mathbb{E}(Y-\mathbb{E}(Y))^2 + 2\mathbb{E}((X-\mathbb{E}(X))(Y-\mathbb{E}(Y))) \\ &= Var(X) + Var(Y) + 2\,\underbrace{\mathbb{E}((X-\mathbb{E}(X))(Y-\mathbb{E}(Y)))}_{?} \end{aligned}$$

Covariance

Intuition:

A measure of how much X, Y change together.

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A measure of how much X, Y change together.

Covariance

For two jointly distributed real-valued random variables X, Y with finite second moments, the covariance is defined as

$$Cov(X, Y) = \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))).$$

Simplification:

$$Cov(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y).$$

$$Cov(X,Y) = EXY - EXEY - EYEX + EXEY$$

$$= EXY - EX \cdot EY.$$

Covariance

Properties:

- $Cov(X, X) = Var(X) \geq 0$;
- $Cov(X, a) = 0, \quad a$ is a constant;
- $Cov(X, Y) = Cov(Y, X)$;
- $Cov(X + a, Y + b) = Cov(X, Y)$;
- $Cov(aX, bY) = abCov(X, Y)$.

$$Cov(X, a) = E\left(X - EX\right)\cdot\big(\underbrace{a - Ea}_{=0}\big) = 0$$

$$Cov(X+a, Y+b) = E\left(X+a - E(X+a)\right)\cdot\left(Y+b - E(Y+b)\right)$$

$$= E\left(X - EX\right)\cdot\left(Y - EY\right) = Cov(X, Y)$$

$$Cov(aX, bY) = E\left(aX - EaX\right)\left(bY - EbY\right)$$

$$= ab\,E(X - EX)(Y - EY) = ab\,Cov(X, Y)$$

Covariance

Properties:

- $Cov(X, X) = Var(X) \geq 0$; (I)
- $Cov(X, a) = 0$, a is a constant;
- $Cov(X, Y) = Cov(Y, X)$;
- $Cov(X + a, Y + b) = Cov(X, Y)$; (IV)
- $Cov(aX, bY) = abCov(X, Y)$.

Corollary about variance:

$$Var(aX + b) = a^2 Var(X).$$

$$Var(aX+b) \overset{(I)}{=} Cov(aX+b, aX+b) \overset{(IV)}{=} Cov(aX, aX) \overset{(V)}{=} a^2 Cov(X,X) \overset{(I)}{=} a^2 Var(X)$$

Covariance

Relate covariance to inner product:

Inner product (not rigorous)

Inner product is a operator from a vector space V to a field F (use $\mathbb{R}$ here as an example): $<\cdot,\cdot>$: $V \times V \to \mathbb{R}$ that satisfies:

- Symmetry: $<x, y> = <y, x>$;
- Linearity in the first argument: $<ax + by, z> = a <x, z> + b <y, z>$;
- Positive-definiteness: $<x, x> \geq 0$, and $<x, x> = 0 \Leftrightarrow x = 0$

$V = \underbrace{L^2 \text{ space.}}_{\text{a space of}}$ square-integrable random variables.

Covariance

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Remark:

Covariance defines an inner product over the quotient vector space obtained by taking the subspace of random variables with finite second moment and identifying any two that differ by a constant.

Covariance

Properties inherited from the inner product space

Recall in Euclidean vector space:

- $< x, y >= x^\top y = \sum_{i=1}^{n} x_i y_i$;
- $||x||_2 = \sqrt{< x, x >}$;
- $< x, y >= ||x||_2 \cdot ||y||_2 \cos(\theta)$.

$$\cos\theta = \frac{\langle x, y \rangle}{\|x\|_2 \ \|y\|_2}$$

Respectively:

- $< X, Y >= Cov(X, Y)$;
- $||X|| = \sqrt{Var(X)}$;

Covariance

A substitute for $\cos(\theta)$**:**

Correlation

For two jointly distributed real-valued random variables X, Y with finite second moments, the correlation is defined as

$$Corr(X, Y) = \rho_{XY} = \frac{Cov(X, Y)}{\sqrt{Var(X) \cdot Var(Y)}}.$$

Covariance

A substitute for $\cos(\theta)$:

Correlation

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Uncorrelatedness:

$$X, Y \text{ uncorrelated} \quad \Leftrightarrow \quad Corr(X, Y) = 0.$$

Covariance $\left|ax+by\right| = \left|\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle\right| \le \sqrt{a^2+b^2}\sqrt{x^2+y^2}$

Cauchy-Schwarz inequality

$$|Cov(X, Y)| \le \sqrt{Var(X)Var(Y)}.$$

Proof: Let $X - EX = \hat{X}$, $Y - E = \tilde{Y}$.

$$0 \le E(\hat{X} + t\tilde{Y})^2 = E\hat{X}^2 + 2t\,E\hat{X} \cdot E\tilde{Y} + t^2 E\tilde{Y}^2.$$

$$= Var(X) + 2\,Cov(X, Y) \cdot t + Var(Y)\,t^2.$$

This holds for any $t \in \mathbb{R}$,

$$D/4 = Cov(X, Y)^2 - Var(X)\,Var(Y) \le 0$$

$$\therefore \quad Cov(X, Y)^2 \le Var(X)\,Var(Y)$$

Covariance

Uncorrelatedness and Independence:

Observe the relationship:

$$Corr(X, Y) = 0 \quad \Leftrightarrow \quad Cov(X, Y) = 0 \quad \Leftrightarrow \quad \mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(X)$$

This can happen

when X and Y are independent

Covariance

Uncorrelatedness and Independence:

Observe the relationship:

$$Corr(X, Y) = 0 \quad \Leftrightarrow \quad Cov(X, Y) = 0 \quad \Leftrightarrow \quad \mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(X)$$

Conclusions:

- Independence $\Rightarrow$ Uncorrelatedness
- Uncorrelatedness $\not\Longrightarrow$ Independence

Remark:

Independence is a very strong assumption/property on the distribution.

Covariance

Special case: multivariate normal

Multivariate normal

A k-dimensional random vector $\mathbf{X} = (X_1, X_2, \cdots, X_k)^\top$ follows a multivariate normal distribution $\mathbf{X} \sim \mathcal{N}(\underline{\boldsymbol{\mu}}, \underline{\boldsymbol{\Sigma}})$, if

mean   covariance. $\Sigma \in \mathbb{R}^{k \times k}$

$$f_{\mathbf{X}}(x_1, \ldots, x_k) = \frac{\exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)}{\sqrt{(2\pi)^k|\boldsymbol{\Sigma}|}},$$

where $\boldsymbol{\mu} = \mathbb{E}[\mathbf{X}] = (\mathbb{E}[X_1], \mathbb{E}[X_2], \ldots, \mathbb{E}[X_k])^\top$, and $\underline{[\boldsymbol{\Sigma}]_{i,j} = \Sigma_{i,j} = Cov(X_i, X_j)}$.

$\Longleftrightarrow \quad \Sigma = \mathbb{E}\,(X-\mu)(X-\mu)^T$

Observation:

The distribution is decided by the covariance structure.

$$f_X(x_1,\cdots,x_d) = (2\pi)^{-\frac{d}{2}} \, |\det \Sigma|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right)$$

Note Σ is symmetric matrix.

$\Rightarrow$ There exists an $\underbrace{\text{orthogonal matrix } V}_{(VV^T = V^TV = I)}$ and diagonal $\Lambda = \begin{pmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_d^2 \end{pmatrix}$

when $\underbrace{\lambda_i \geq 0}$ (We can say this because Σ is positive semi definite)

Further assume that $\lambda_i > 0$ (it ensures Σ and Λ to be invertible)

s.t. $\Sigma V = V\Lambda \Longleftrightarrow \underbrace{\Sigma = V\Lambda V^T}$

spectral decomposition of symmetric matrix.

$\Longleftrightarrow \Sigma^{-1} = V\Lambda^{-1}V^T$

$$(x-\mu)^T \Sigma^{-1}(x-\mu) = (x-\mu)^T V \Lambda^{-1} \underbrace{V^T(x-\mu)}_{= z}$$

change variables by $z = V^T(x-\mu)$.

Note that Jacobian of $x \to z$ is 1

since $|\det V| = 1$.

therefore

$$p(z) = p(x) = (2\pi)^{-\frac{k}{2}} \underbrace{|\det \Lambda|}_{} ^{-\frac{1}{2}} \exp\left[-\frac{1}{2} z^T \Lambda^{-1} z\right]$$

$|\det \Lambda| = |\det \Sigma|$

due to $|\det V| = 1$

Recall that $\Lambda = \begin{pmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_k^2 \end{pmatrix}$,

So, $$p(z) = (2\pi)^{-\frac{k}{2}} \left| \prod_{i=1}^{k} \lambda_i^2 \right|^{-\frac{1}{2}} \exp\left[-\sum_{i=1}^{k} \frac{z_i^2}{2\lambda_i^2}\right]$$

$$= \prod_{i=1}^{k} \left[\underbrace{\frac{1}{\sqrt{2\pi}\,\lambda_i} \exp\left(-\frac{z_i^2}{2\lambda_i^2}\right)}_{\text{1-dimensional normal}} \right]$$

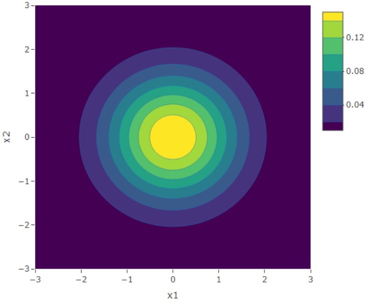
Covariance

$$X_i, i = 1, \cdots k \text{ independent} \Leftrightarrow f_{\mathbf{X}}(x_1, \ldots, x_k) = \prod_{i=1}^{k} f_{X_i}(x_i)$$

$$\Leftrightarrow \boldsymbol{\Sigma} = I_k \Leftrightarrow Cov(X_i, X_j) = 0, i \neq j.$$

Example:

- $Corr(X, Y) = 0$



Covariance

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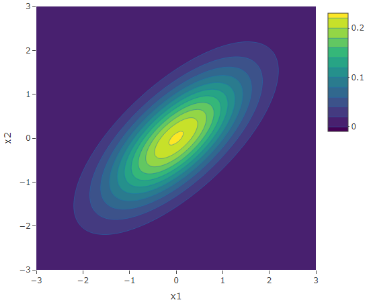
Example:

- $Corr(X, Y) = 0.7$



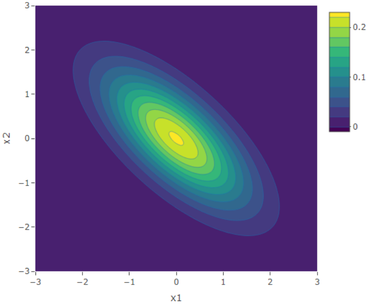
Covariance

$$X_i, i = 1, \cdots k \text{ independent} \Leftrightarrow f_{\mathbf{X}}(x_1, \ldots, x_k) = \prod_{i=1}^{k} f_{X_i}(x_i)$$

$$\Leftrightarrow \boldsymbol{\Sigma} = I_k \Leftrightarrow Cov(X_i, X_j) = 0, i \neq j.$$

Example:

- $Corr(X, Y) = -0.7$



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Concentration

Measures of a distribution:

- $\mathbb{E}(X^k)$, $\mathbb{E}(X)$, $Var(X)$;
- $Cov(X, Y)$ and $Corr(X, Y)$.

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- $\mathbb{E}(X^k)$, $\mathbb{E}(X)$, $Var(X)$;
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Tail probability: $\mathbf{P}(|X| > t)$

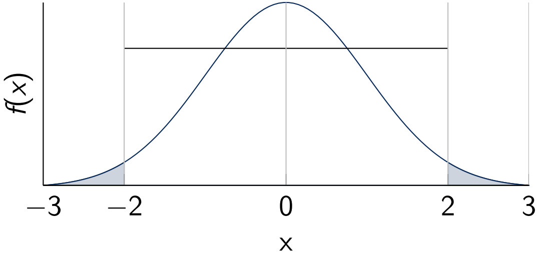

Figure: Probability density function of $\mathcal{N}(0, 1)$

Concentration

Concentration inequalities:

- Markov inequality
- Chebyshev inequality
- Chernoff bounds

Concentration

Concentration inequalities:

- Markov inequality
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Markov inequality

Let X be a random variable that is non-negative (almost surely). Then, for every constant $a > 0$,

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}.$$

Proof:

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Concentration

Markov inequality (continued)

Let X be a random variable, then for every constant $a > 0$,

$$\mathbb{P}(|X| \geq a) \leq \frac{\mathbb{E}(|X|)}{a}.$$

A more general conclusion:

Markov inequality (continued)

Let X be a random variable, if $\Phi(x)$ is monotonically increasing on $[0, \infty)$, then for every constant $a > 0$,

$$\mathbb{P}(|X| \geq a) = \mathbb{P}(\Phi(|X|) \geq \Phi(a)) \leq \frac{\mathbb{E}(\Phi(|X|))}{\Phi(a)}.$$

Concentration

Chebyshev inequality

Let X be a random variable with finite expectation $\mathbb{E}(X)$ and variance $Var(X)$, then for every constant $a > 0$,

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq a) \leq \frac{Var(X)}{a^2},$$

or equivalently,

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq a\sqrt{Var(X)}) \leq \frac{1}{a^2}.$$

Example:

Take $a = 2$,

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq 2\sqrt{Var(X)}) \leq \frac{1}{4}.$$

Concentration

Chernoff bound (general)

Let X be a random variable, then for $t \geq 0$,

$$\mathbb{P}(X \geq a) = \mathbb{P}(e^{t \cdot X} \geq e^{t \cdot a}) \leq \frac{\mathbb{E}\left[e^{t \cdot X}\right]}{e^{t \cdot a}},$$

and

$$\mathbb{P}(X \geq a) \leq \inf_{t \geq 0} \frac{\mathbb{E}\left[e^{t \cdot X}\right]}{e^{t \cdot a}}.$$

Remark:

This is especially useful when considering $X = \sum_{i=1}^{n} X_i$ with X_i's independent,

$$\mathbb{P}(X \geq a) \leq \inf_{t \geq 0} \frac{\mathbb{E}\left[\prod_i e^{t \cdot X_i}\right]}{e^{t \cdot a}} = \inf_{t \geq 0} e^{-t \cdot a} \prod_i \mathbb{E}\left[e^{t \cdot X_i}\right].$$

Problem Set

Problem 1: Let

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases},$$

compute $Cov(X, Y)$.

Problem 2: For $X \sim \mathcal{N}(0,1)$, compute the Chernoff bound.