

Module 6: End of Topology, Start of Linear Algebra

Operational math bootcamp



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Outline

- Finish topology
 - Dense subsets
 - Compactness
 - Continuity
- Start linear algebra
 - Vector spaces and subspaces

Last time

Definition (Topology)

Let $\mathcal{T} \subseteq \mathcal{P}(X)$. We call \mathcal{T} a *topology* on X if the following holds:

- (i) $\emptyset, X \in \mathcal{T}$
- (ii) Let A be an arbitrary index set. If $U_\alpha \in \mathcal{T}$ for $\alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$ (\mathcal{T} is closed under taking arbitrary unions)
- (iii) Let $n \in \mathbb{N}$. If $U_1, \dots, U_n \in \mathcal{T}$, then $\bigcap_{i=1}^n U_i \in \mathcal{T}$ (\mathcal{T} is closed under taking finite intersections)

If $U \in \mathcal{T}$, we call U *open*. We call $U \subseteq X$ *closed*, if $U^c \in \mathcal{T}$. We call (X, \mathcal{T}) a *topological space*.

Definition

Let (X, \mathcal{T}) be a topological space and let $A \subseteq X$ be any subset.

- The *interior* of A is $\overset{\circ}{A} := \{a \in A : \exists U \in \mathcal{T} \text{ s.t. } U \subseteq A \text{ and } a \in U\}$.
- The *closure* of A is $\overline{A} := \{x \in X : \forall U \in \mathcal{T} \text{ with } x \in U, U \cap A \neq \emptyset\}$.
- The *boundary* of A is $\partial A := \{x \in X : \forall U \in \mathcal{T} \text{ with } x \in U, U \cap A \neq \emptyset \text{ and } U \cap A^c \neq \emptyset\}$.

Density

Definition

Let (X, \mathcal{T}) be a topological space. A subset $A \subseteq X$ is called *dense* if $\overline{A} = X$.

Using the definition of closure, we see that $A \subseteq X$ is dense if and only if for all non-empty $U \in \mathcal{T}$, $U \cap A \neq \emptyset$.

Example

- The rationals \mathbb{Q} are dense in the reals \mathbb{R} .
- The only dense subset in $(X, \mathcal{P}(X))$ is X itself.
- Any non-empty subset is dense in $(X, \{\emptyset, X\})$.

Separability

Definition

A topological space (X, \mathcal{T}) is *separable* if it contains a countable dense subset.

Example

Hausdorff space

Definition

A topological space (X, \mathcal{T}) is called *Hausdorff* if for all $x \neq y \in X$ there exist open sets U_x, U_y with $x \in U_x$ and $y \in U_y$ such that $U_x \cap U_y = \emptyset$.

So in a Hausdorff space, we can separate any two elements using open sets.

Example

Let (X, d) be a metric space. Then (X, \mathcal{T}_d) is Hausdorff, where \mathcal{T}_d is the topology induced by the metric d .

Example

Let X be an infinite set and $\mathcal{T} = \{U \subseteq X : U^c \text{ is finite}\} \cup \emptyset$. Then (X, \mathcal{T}) is not Hausdorff.

Compactness

Definition

Let (X, \mathcal{T}) be a topological space and $K \subseteq X$.

A collection $\{U_i\}_{i \in I}$ of open sets is called *open cover* of K if $K \subseteq \bigcup_{i \in I} U_i$.

The set K is called *compact* if for all open covers $\{U_i\}_{i \in I}$ there exists a finite subcover, meaning there exists an $n \in \mathbb{N}$ and $\{U_1, \dots, U_n\} \subseteq \{U_i\}_{i \in I}$ such that $K \subseteq \bigcup_{i=1}^n U_i$.

Example

Let $S \subseteq X$ where (X, \mathcal{T}) is a topological space. If S is finite, then it is compact.

Example

$(0, 1)$ is not compact.

Proposition

Let (X, \mathcal{T}) be a topological space and take a non-empty subset $K \subseteq X$. The following holds:

- 1 If X is compact and K is closed, then K is compact (i.e. closed subsets of compact sets are compact).
- 2 If (X, \mathcal{T}) is Hausdorff, then K being compact implies that K is closed.

Proof.

(1) If X is compact and $K \subseteq X$ is closed, then K is compact



Proof.

(2) If (X, \mathcal{T}) is Hausdorff, then $K \subseteq X$ compact $\Leftrightarrow K$ is closed.



Proof continued

Compactness on \mathbb{R}^n

Theorem (Heine-Borel Theorem)

Let $K \subseteq \mathbb{R}^n$. Then K is compact with respect to the topology induced by the Euclidean distance if and only if it is closed and bounded.

Just as we had a sequential characterization of the closure of a set in metric spaces, we similarly have a sequential characterization of compactness.

Theorem

Let (X, d) be a metric space. Then $K \subset X$ is compact with respect to the metric induced by d if and only if every sequence in K admits a subsequence converging to some point in K .

A corollary of this statement together with Heine-Borel is the Bolzano-Weierstrass theorem.

Corollary (Bolzano-Weierstrass)

Any bounded sequence in \mathbb{R}^n has a convergent subsequence.

Continuity on a topological space

Definition

Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. A map $f: X \rightarrow Y$ is called *continuous* if for all $U \in \mathcal{T}_Y$, $f^{-1}(U) \in \mathcal{T}_X$, i.e. the pre-image of open sets is open.

We can also specify continuity at a point $x_0 \in X$.

Definition

Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. A map $f: X \rightarrow Y$ is called *continuous at* $x_0 \in X$ if for all $U \in \mathcal{T}_Y$ with $f(x_0) \in U$, $f^{-1}(U) \in \mathcal{T}_X$, i.e. the preimage of open sets containing $f(x_0)$ is open (and contains x_0).

Proposition

Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. Suppose $K \subset X$ is compact and let $f: K \rightarrow Y$ be continuous. Then $f(K)$ is compact.

Recall from the set theory section:

If $f: X \rightarrow Y$:

- ① $A \subseteq B \subseteq Y \Rightarrow f^{-1}(A) \subseteq f^{-1}(B)$ and $A \subseteq B \subseteq X \Rightarrow f(A) \subseteq f(B)$
- ② $f^{-1}(\cup_{i \in I} A_i) = \cup_{i \in I} f^{-1}(A_i)$, where $A_i \subseteq Y \forall i \in I$
- ③ $f(\cup_{i \in I} A_i) = \cup_{i \in I} f(A_i)$, where $A_i \subseteq X \forall i \in I$
- ④ $A \subseteq X \Rightarrow A \subseteq f^{-1}(f(A))$
- ⑤ $B \subseteq Y \Rightarrow f(f^{-1}(B)) \subseteq B$

Proof.

Proof continued

Linear Algebra

Definition

We call V a **vector space** if the following hold:

- (A) *Commutativity in addition:* $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$
- (B) *Associativity in addition:* $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$
- (C) *Existence of a neutral element, addition:* There exists a vector $\mathbf{0}$ such that for any $\mathbf{v} \in V$, $\mathbf{0} + \mathbf{v} = \mathbf{v}$
- (D) *Additive inverse:* For every $\mathbf{v} \in V$, there exists another vector, which we denote $-\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
- (E) *Existence of a neutral element, multiplication:* For any $\mathbf{v} \in V$, $1 \times \mathbf{v} = \mathbf{v}$
- (F) *Associativity in multiplication:* Let $\alpha, \beta \in \mathbb{F}$. For any $\mathbf{v} \in V$, $(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$
- (G) Let $\alpha \in \mathbb{F}, \mathbf{u}, \mathbf{v} \in V$. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.
- (H) Let $\alpha, \beta \in \mathbb{F}, \mathbf{v} \in V$. $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$.

Elements of the vector space are called vectors.
Most often we will assume $\mathbb{F} = \mathbb{C}$ or \mathbb{R} .

Example

The following are vector spaces:

- \mathbb{R}^n
- \mathbb{C}^n
- $C(\mathbb{R}; \mathbb{R})$, continuous functions from \mathbb{R} to \mathbb{R}
- $M_{n \times m}$, matrices of size $n \times m$
- \mathbb{P}_n (polynomials of degree n , $p(x) = a_0 + a_1x + \dots + a_nx^n$).

Lemma

For every $\mathbf{v} \in V$, $0\mathbf{v} = \mathbf{0}$.

Proof.



Lemma

For every $\mathbf{v} \in V$, we have $-\mathbf{v} = (-1) \times \mathbf{v}$.

Proof.



Definition

A subset U of V is called a **subspace** of V if U is also a vector space (using the same addition and scalar multiplication as on V).

Proposition

A subset U of V is a subspace of V if and only if U satisfies the following three conditions:

- ① $\mathbf{0} \in U$
- ② Closed under addition: $u, w \in U$ implies $\mathbf{u} + \mathbf{v} \in U$
- ③ Closed under scalar multiplication: $\alpha \in \mathbb{F}$ and $u \in U$ implies $\alpha \mathbf{u} \in U$

Proof.

(\Rightarrow)

(\Leftarrow)



References

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