

Module 2: Set Theory

Operational math bootcamp



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Outline

- Review of basic set theory
- Ordered Sets
- Functions
- Cardinality
- The Axiom of Choice

Recall that for sets S, T :

- the *union* of S and T , denoted $S \cup T$, is all the elements that are in *either* S and T
- and the *intersection* of S and T , denoted $S \cap T$, is all the elements that are in *both* S and T .

We extend the definition of union and intersection to an arbitrary family of sets as follows:

Definition

Let S_α , $\alpha \in A$, be a family of sets. A is called the *index set*. We define

$$\bigcup_{\alpha \in A} S_\alpha := \{x : \exists \alpha \text{ such that } x \in S_\alpha\}$$

$$\bigcap_{\alpha \in A} S_\alpha := \{x : x \in S_\alpha \forall \alpha \in A\}$$

Example

$$\bigcup_{n=1}^{\infty} [-n, n] = \mathbb{R}$$

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

Theorem (De Morgan's Laws)

Let $\{S_\alpha\}_{\alpha \in A}$ be an arbitrary collection of sets. Then

$$\left(\bigcup_{\alpha \in A} S_\alpha \right)^c = \bigcap_{\alpha \in A} S_\alpha^c \quad \text{and} \quad \left(\bigcap_{\alpha \in A} S_\alpha \right)^c = \bigcup_{\alpha \in A} S_\alpha^c$$

Proof.



Since a set is itself a mathematical object, a set can itself contain sets.

Definition

The power set $\mathcal{P}(S)$ of a set S is the set of all subsets of S .

Example

Let $S = \{a, b, c\}$.

Then $\mathcal{P}(S) =$

Another way of building a new set from two old ones is the Cartesian product of two sets.

Definition

Let S, T be sets. The *Cartesian product* $S \times T$ is defined as the set of tuples with elements from S, T , i.e

$$S \times T = \{(s, t) : s \in S \text{ and } t \in T\}.$$

Ordered set

Definition

A *relation* R on a set X is a subset of $X \times X$. A relation \leq is called a *partial order* on X if it satisfies

- ① reflexivity: $x \leq x$ for all $x \in X$
- ② transitivity: for $x, y, z \in X$, $x \leq y$ and $y \leq z$ implies $x \leq z$
- ③ anti-symmetry: for $x, y \in X$, $x \leq y$ and $y \leq x$ implies $x = y$

The pair (X, \leq) is called a *partially ordered set*.

A *chain* or *totally ordered set* $C \subseteq X$ is a subset with the property $x \leq y$ or $y \leq x$ for any $x, y \in C$.

Example

The real numbers with the usual ordering, (\mathbb{R}, \leq) are totally ordered.

Example

The power set of a set X with the ordering given by subsets, $(\mathcal{P}(X), \subseteq)$ is partially ordered set.

Example

Let $X = \{a, b, c, d\}$. What is $\mathcal{P}(X)$? Find a chain in $\mathcal{P}(X)$.

Example

Consider the set $C([0, 1], \mathbb{R}) := \{f : [0, 1] \rightarrow \mathbb{R} : f \text{ is continuous}\}$.

For two functions $f, g \in C([0, 1], \mathbb{R})$, we define the ordering as $f \leq g$ if $f(x) \leq g(x)$ for $x \in [0, 1]$. Then $(C([0, 1], \mathbb{R}), \leq)$ is a partially ordered set. Can you think of a chain that is a subset of $(C([0, 1], \mathbb{R}))$?

Definition

A function f from a set X to a set Y is a subset of $X \times Y$ with the properties:

- ① For every $x \in X$, there exists a $y \in Y$ such that $(x, y) \in f$
- ② If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

X is called the *domain* of f .

How does this connect to other descriptions of functions you may have seen?

Example

For a set X , the identity function is:

$$1_X : X \rightarrow X, \quad x \mapsto x$$

Definition (Image and pre-image)

Let $f : X \rightarrow Y$ and $A \subseteq X$ and $B \subseteq Y$. The image of f is the set $f(A) := \{f(x) : x \in A\}$ and the pre-image of f is the set $f^{-1}(B) := \{x : f(x) \in B\}$

Helpful way to think about it for proofs:

If $y \in f(A)$, then $y \in Y$, and there exists an $x \in A$ such that $y = f(x)$.

If $x \in f^{-1}(B)$, then $x \in X$ and $f(x) \in B$.

Definition (Surjective, injective and bijective)

Let $f : X \rightarrow Y$, where X and Y are sets. Then

- f is *injective* if $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$
- f is *surjective* if for every $y \in Y$, there exists an $x \in X$ such that $y = f(x)$
- f is *bijective* if it is both injective and surjective

Example

Let $f : X \rightarrow Y$, $x \mapsto x^2$.

f is surjective if

f is injective if

f is bijective if

f is neither surjective nor injective if

Proposition

Let $f : X \rightarrow Y$ and $A \subseteq X$. Prove that $A \subseteq f^{-1}(f(A))$, with equality iff f is injective.

Proof.



Cardinality

Definition

The *cardinality* of a set A , denoted $|A|$, is the number of elements in the set.

We say that the empty set has cardinality 0 and is finite.

Proposition

If X is finite set of cardinality n , then the cardinality of $\mathcal{P}(X)$ is 2^n .

Proof.



Definition

Two sets A and B have same cardinality, $|A| = |B|$, if there exists bijection $f : A \rightarrow B$.

Example

Which is bigger, \mathbb{N} or \mathbb{N}_0 ?

Cantor-Schröder-Bernstein

Definition

We say that the cardinality of a set A is less than the cardinality of a set B , denoted $|A| \leq |B|$ if there exists an injection $f : A \rightarrow B$.

Theorem (Cantor-Bernstein)

Let A, B , be sets. If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Example

$$|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$$

Proof.



Definition

Let A be a set.

- ① A is *finite* if there exists an $n \in \mathbb{N}$ and a bijection $f : \{1, \dots, n\} \rightarrow A$
- ② A is *countably infinite* if there exists a bijection $f : \mathbb{N} \rightarrow A$
- ③ A is *countable* if it is finite or countably infinite
- ④ A is *uncountable* otherwise

Example

The rational numbers are countable, and in fact $|\mathbb{Q}| = |\mathbb{N}|$.

Let's look at $\mathbb{Q}^+ := \{x \in \mathbb{Q} : x > 0\}$. The fact that the rationals are countable relies on this famous way of listing the rational numbers:

1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	\dots
2	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	\dots
3	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	\dots
4	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$	\dots
\vdots	\vdots	\vdots	\vdots	\vdots	\ddots

Example

This is a map from \mathbb{N} to \mathbb{Q}^+ . As long as we skip any fraction that is already in our list as we go along, it is injective. Since we can find an injection from \mathbb{Q}^+ to $\mathbb{N} \times \mathbb{N}$ (exercise), we have that $|\mathbb{Q}^+| = |\mathbb{N}|$. We can extend this to \mathbb{Q} . To do so, let $f: \mathbb{N} \rightarrow \mathbb{Q}^+$ be a bijection (which exists by the previous part). Then we can define another bijection $g: \mathbb{N} \rightarrow \mathbb{Q}$ by setting $g(1) = 0$ and

$$g(n) = \begin{cases} f(n) & \text{if } n \text{ is even,} \\ -f(n) & \text{if } n \text{ is odd,} \end{cases}$$

for $n > 1$.

Theorem

The cardinality of \mathbb{N} is smaller than that of $(0, 1)$.

Proof.

First, we show that there is an injective map from \mathbb{N} to $(0, 1)$.

Next, we show that there is no surjective map from \mathbb{N} to $(0, 1)$. We use the fact that every number $r \in (0, 1)$ has a binary expansion of the form $r = 0.\sigma_1\sigma_2\sigma_3\dots$ where $\sigma_i \in \{0, 1\}$, $i \in \mathbb{N}$. □

Proof.

Now we suppose in order to derive a contradiction that there does exist a surjective map f from \mathbb{N} to $(0, 1)$., i.e. for $n \in \mathbb{N}$ we have $f(n) = 0.\sigma_1(n)\sigma_2(n)\sigma_3(n) \dots$. This means we can list out the binary expansions, for example like

$$f(1) = 0.\textcolor{red}{0}0000000 \dots$$

$$f(2) = 0.1\textcolor{red}{1}11111111 \dots$$

$$f(3) = 0.01\textcolor{red}{0}1010101 \dots$$

$$f(4) = 0.101\textcolor{red}{0}101010 \dots$$

We will construct a number $\tilde{r} \in (0, 1)$ that is not in the image of f . □

Proof.

Define $\tilde{r} = 0.\tilde{\sigma}_1\tilde{\sigma}_2\dots$, where we define the n th entry of \tilde{r} to be the opposite of the n th entry of the n th item in our list:

$$\tilde{\sigma}_n = \begin{cases} 1 & \text{if } \sigma_n(n) = 0, \\ 0 & \text{if } \sigma_n(n) = 1. \end{cases}$$

Then \tilde{r} differs from $f(n)$ at least in the n th digit of its binary expansion for all $n \in \mathbb{N}$. Hence, $\tilde{r} \notin f(\mathbb{N})$, which is a contradiction to f being surjective. This technique is often referred to as Cantor's diagonal argument. □

Proposition

$(0,1)$ and \mathbb{R} have the same cardinality.

Proof.

The map $f : \mathbb{R} \rightarrow (0,1)$ defined by $x \mapsto \frac{1}{\pi} (\arctan(x) + \frac{\pi}{2})$ is a bijection. □

We have shown that there are different sizes of infinity, as the cardinality of \mathbb{N} is infinite but still smaller than that of \mathbb{R} or $(0,1)$. In fact, we have

$$|\mathbb{N}| = |\mathbb{N}_0| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}|.$$

Because of this, there are special symbols for these two cardinalities: The cardinality of \mathbb{N} is denoted \aleph_0 , while the cardinality of \mathbb{R} is denoted \mathfrak{c} . There is even a relationship between them: $\mathfrak{c} = 2^{\aleph_0}$, i.e. the cardinality of \mathbb{R} is the same as the cardinality of $\mathcal{P}(\mathbb{N})$.

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