



UNIVERSITY OF
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Statistical Sciences

DoSS Summer Bootcamp Probability Module 1

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Roadmap

This can be completed afterwards.

Basically we want to give a blueprint about what we are going to cover in 10 modules.

For reference: CS6015 Lecture 28

Outline

- Introduction: example
- Measurable spaces
 - ▷ Sample Space
 - ▷ Sigma-algebra
- Probability measures
 - ▷ Measures on σ -field
 - ▷ Basic results
- Conditional probability
 - ▷ Bayes' rule
 - ▷ Law of total probability

Introduction: example

Toss a coin

Measurable spaces

Sample Space

The sample space Ω is the set of all possible outcomes of an experiment.

Examples:

- Toss a coin: $\{H, T\}$
- Roll a die: $\{1, 2, 3, 4, 5, 6\}$

Event

An event is a collection of possible outcomes (subset of the sample space).

Examples:

- Get head when tossing a coin: $\{H\}$
- Get an even number when rolling a die: $\{2, 4, 6\}$

Measurable spaces

σ -algebra

A σ -algebra (σ -field) \mathcal{F} is a non-empty collection of subsets of such that

- If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$,
- If $A_1, A_2, \dots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Remark: $\emptyset, \Omega \in \mathcal{F}$

Probability measures

Measures on σ -field

A function $\mu : \mathcal{F} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ is called a measure if

- $\mu(\emptyset) = 0$,
- If $A_1, A_2, \dots \in \mathcal{F}$ and $A_i \cap A_j = \emptyset$, then $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

If $\mu(\Omega) = 1$, then μ is called a probability measure.

Properties:

- Monotonicity: $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$
- Subadditivity: $A \subseteq \cup_{i=1}^{\infty} A_i \Rightarrow \mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$
- Continuity from below: $A_i \nearrow A \Rightarrow \mu(A_i) \nearrow \mu(A)$
- Continuity from above: $A_i \searrow A$ and $\mu(A_i) < \infty \Rightarrow \mu(A_i) \searrow \mu(A)$

Probability measures

Proof of continuity from below:

Probability measures

Proof of continuity from above:

Remark: $\mu(A_i) < \infty$ is vital.

Probability measures

Examples:

$$\Omega = \{\omega_1, \omega_2, \dots\}, A = \{\omega_{a_1}, \dots, \omega_{a_i}, \dots\} \Rightarrow \mu(A) = \sum_{j=1}^{\infty} \mu(\omega_{a_j}).$$

Therefore, we only need to define $\mu(\omega_j) = p_j \geq 0$.

If further $\sum_{i=1}^{\infty} p_j = 1$, then μ is a probability measure.

- Toss a coin:

- Roll a die:

Conditional probability

Original problem:

- What is the probability of some event A ?
- $P(A)$ is determined by our probability measure.

New problem:

- Given that B happens, what is the probability of some event A ?
- $P(A \mid B)$ is the conditional probability of the event A given B .

Example:

- Roll a die: $P(\{2\} \mid \text{even number})$

Conditional probability

Bayes' rule

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

Remark: Does conditional probability $P(\cdot \mid B)$ satisfy the axioms of a probability measure?

Conditional probability

Multiplication rule

$$P(A \cap B) = P(A | B)P(B) = P(B | A)P(A)$$

Generalization:

Law of total probability

Let A_1, A_2, \dots, A_n be a partition of ω , such that $P(A_i) > 0$, then

$$P(B) = \sum_{i=1}^n P(A_i)P(B | A_i)$$

Problem Set

Problem 1: Prove that for a σ -field \mathcal{F} , if $A_1, A_2, \dots \in \mathcal{F}$, then $\cap_{i=1}^{\infty} A_i \in \mathcal{F}$.

Problem 2: Prove monotonicity and subadditivity of measure μ on σ -field.

Problem 3: (Monty Hall problem) Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?