Mathematics Bootcamp

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1 Review of proof techniques with examples from algebra and analysis

1.1 Axioms of a field

(A1) *Commutativity in addition:* $x + y = y + x$

(A2) *Commutativity in multiplication:* $x \times y = y \times x$

(B1) *Associativity in addition:* $x + (y + z) = (x + y) + z$

(B2) *Associativity in multiplication:* $x \times (y \times z) = (x \times y) \times z$

(C) *Distributivity:* $x \times (y + z) = x \times y + x \times z$

(D1) *Existence of a neutral element, addition:* There exists a number 0 such that $x + 0 = x$ for every x.

(D2) *Existence of a neutral element, multiplication:* There exists a number 1 such that $x \times 1 = x$ for every x.

(E1) *Existence of an inverse, addition:* For each number x, there exists a number $-x$ such that $x+(-x) = 0$.

(E2) *Existence of an inverse, multiplication:* For each number $x \neq 0$, there exists a number $1/x$ such that $x \times 1/x = 1$.

1.1.1 Exercises

1. For any $a, b \neq 0$, $1/(ab) = 1/a \times 1/b$
2. For $a > 0$, $1/(-a) = -1/a$.
3. For $a, b \neq 0$, $1/(a/b) = b/a$

2 Linear Algebra Part I

2.1 Vector spaces

2.1.1 Axioms of a vector space

Let V be a set and let $\mathbb{F}$ be a field. We call V a *vector space* if the following hold:
Addition:

(A) *Commutativity in addition:* $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$

(B) *Associativity in addition:* $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$

(C) *Existence of a neutral element, addition:* There exists a vector $\mathbf{0}$ such that for any $\mathbf{v} \in V$, $\mathbf{0} + \mathbf{v} = \mathbf{v}$

(D) *Additive inverse:* For every $\mathbf{v} \in V$, there exists another vector, which we denote $-\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

Multiplication by a scalar:

(E) *Existence of a neutral element, multiplication:* For any $\mathbf{v} \in V$, $1 \times \mathbf{v} = \mathbf{v}$

(F) *Associativity in multiplication:* Let $\alpha, \beta \in \mathbb{F}$. For any $\mathbf{v} \in V$, $(\alpha\beta)\mathbf{v} = \alpha(\beta\mathbf{v})$

Associativity:

(G) Let $\alpha \in \mathbb{F}, \mathbf{u}, \mathbf{v} \in V$. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \beta\mathbf{v}$.

(H) Let $\alpha, \beta \in \mathbb{F}, \mathbf{v} \in V$. $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$.

Elements of the vector space are called vectors.
Most often we will assume $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$.
Examples of vector spaces: $\mathbb{R}^n$. $\mathbb{C}^n$, $M_{m\times n}$ (matrices of size $m \times n$), $\mathbb{P}_n$ (polynomials of degree n, $p(x) = a_0 + a_1 x + \ldots + a_n x^n$).

2.1.2 Subspaces

Definition 2.1 *A subset U of V is called a* ***subspace*** *of of V if U is also a vector space (using the same addition and scalar multiplication as on V).*

Proposition 2.2 *A subset U of V is a subspace of V if and only if U satisfies the following three conditions:*

- *Additive identity:* $\mathbf{0} \in U$
- *Closed under addition:* $u, w \in U$ *implies* $\mathbf{u} + \mathbf{v} \in U$
- *Closed under scalar multiplication:* $\alpha \in \mathbb{F}$ *and* $u \in U$ *implies* $\alpha\mathbf{u} \in U$

[**EK:** *Add intersections and unions of subspaces*]

Definition 2.3 *Suppose $U_1, ..., U_m$ are subsets of V. The sum of $U_1, ..., U_m$, denoted $U_1 + ... + U_m$, is the set of all possible sums of elements of $U_1, ..., U_m$. More precisely,*

$$U_1 + ... + U_m = \{\mathbf{u}_1 + ... + \mathbf{u}_m : \mathbf{u}_1 \in U_1, ..., \mathbf{u}_m \in U_m\}$$

Proposition 2.4 *Suppose $U_1, ..., U_m$ are subspaces of V. Then $U_1 + ... + U_m$ is the smallest subspace of V containing $U_1, ..., U_m$.*

2.1.3 Exercises

From Harvard Bootcamp:
Exercise: Prove that $-(-v) = v$ for every $v \in V$.
Exercise: Suppose that $a \in \mathbf{F}, v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.
Exercise: The empty set is not a vector space because it fails to satisfy only one of the requirements listed above. Which one?
Exercise: Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbb{R}$.
Exercise: A function $f : \mathbb{R} \to \mathbb{R}$ is called periodic if there exists a positive number such that $f(x) = f(x+p)$ for all $x \in \mathbb{R}$. Is the a set of periodic functions from $\mathbb{R}$ to $\mathbb{R}$ a subspace of $\mathbb{R}^{\mathbb{R}}$?
Exercise: A function $f : \mathbb{R} \to \mathbb{R}$ is called odd if

$$f(-x) = -f(x)$$

for all $x \in \mathbb{R}$. Let U_e denote the set of real-valued even functions on $\mathbb{R}$ and let U_o denote the set of real-valued odd functions on $\mathbb{R}$. Show that $\mathbb{R}^{\mathbb{R}} = U_e \oplus U_o$.

2.2 Linear (in)dependence and bases

Definition 2.5 *A linear combination of a list $\mathbf{v}_1, ..., \mathbf{v}_n$ of vectors in V is a vector of the form*

$$\alpha_1\mathbf{v}_1 + ... + \alpha_n\mathbf{v}_n = \sum_{k=1}^{n} \alpha_k\mathbf{v}_k$$

where $\alpha_1, ..., \alpha_m \in \mathbb{F}$.

Definition 2.6 *The set of all linear combinations of a list of vectors $v_1, ..., v_m$ in V is called the* ***span*** *of $v_1, ..., v_m$, denoted span$\{\mathbf{v}_1, ..., \mathbf{v}_n\}$. In other words,*

$$span\{\mathbf{v}_1, ..., \mathbf{v}_n\} = \{\alpha_1\mathbf{v}_1 + ... + \alpha_m\mathbf{v}_n : \alpha_1, ..., \alpha_n \in \mathbb{F}\}$$

The span of the empty list is defined to be $\{\mathbf{0}\}$.

Definition 2.7 *A system of vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_n$ *is called a basis (for the vector space* V *) if any vector* $\mathbf{v} \in V$ *admits a unique representation as a linear combination*

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \ldots + \alpha_n \mathbf{v}_n = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k$$

Definition 2.8 *The linear combination* $\alpha_1 \mathbf{v}_1 + ... + \alpha_n \mathbf{v}_n$ *is called trivial if* $\alpha_k = 0$ *for every* k*.*

Proposition 2.9 *A system of vectors* $\mathbf{v}_1, \ldots \mathbf{v}_n \in V$ *is a basis if and only if it is linearly independent and complete (generating).*

[**EK:** *Proof done by hand*]

2.3 Exercises

From Harvard: Exercise: Suppose v_1, v_2, v_3, v_4 (a) spans V and (b) is linearly independent. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also (a) spans V and (b) is linearly independent.

Exercise: Suppose $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, ..., v_m + w$ is linearly dependent, then $w \in \text{span}(v_1, ..., v_m)$.
Exercise: Suppose that $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Show that $v_1, ..., v_m, w$ is linearly independent if and only if

$$w \notin \text{span}(v_1, ..., v_m)$$

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Exercise: Suppose that $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Show that $v_1, ..., v_m, w$ is linearly independent if and only if

$$w \notin \text{span}(v_1, ..., v_m)$$

[**EK:** *Add a few from books*]