# Exercises for Module 9: Linear Algebra III

1. Let $A, B \in M_n(\mathbb{F})$ be similar matrices. Show that their characteristic polynomials coincide.

Proof Let $A$ & $B$ be similar. Then there exists an invertible matrix $S$ such that $A = SBS^{-1}$.

Note that similar matrices have the same determinant:

$$\det(A) = \det(SBS^{-1}) = \det(S)\det(B)\det(S^{-1}) = \det(B)$$

since $\det(S^{-1}) = \det(S)^{-1}$ as $SS^{-1} = I$.

We can write

$$\begin{aligned} A - \lambda I &= SBS^{-1} - \lambda SIS^{-1} \\ &= S(BS^{-1} - \lambda IS^{-1}) \\ &= S(B - \lambda I)S^{-1}. \end{aligned}$$

Therefore if $A$ & $B$ are similar, then $A - \lambda I$ & $B - \lambda I$ are similar, and therefore $\det(A - \lambda I) = \det(B - \lambda I)$. So $A, B$ have the same char. poly.

2. Show that $A \in M_n(\mathbb{C})$ is invertible if and only if $0 \notin \sigma(A)$.

Recall that $\lambda \in \sigma(A)$ means $\lambda$ is an eigenvalue for $A$, i.e. $\det(A - \lambda I) = 0$.

($\Rightarrow$) By contrapositive.

Suppose $0 \in \sigma(A)$. Then $\det(A - 0I) = 0$.

$\Rightarrow \det(A) = 0$

$\Rightarrow A$ is not invertible by theorem from class.

($\Leftarrow$) By contrapositive.

Suppose $A$ is not invertible.

Then $\det(A) = 0$.

$\Rightarrow \det(A - 0I) = 0$

$\Rightarrow 0 \in \sigma(A)$

3. Suppose $N$ is a nilpotent matrix. Show that $\sigma(N) = \{0\}$.

Suppose $N$ is nilpotent. This means $\exists\, k \geq 1$ s.t. $N^k = 0$.

First, we show $\{0\} \subseteq \sigma(N)$.

Since $N$ is nilpotent, $N^k = 0 \Rightarrow \det(N^k) = 0 \Rightarrow \det(N)^k = 0 \Rightarrow \det(N) = 0$.

Thus $0 \in \sigma(N)$ by previous exercise.

To show $\sigma(N) \subseteq \{0\}$, first note that if $v \neq 0$ is an eigenvector associated with $\lambda$, then $N^k v = \lambda^k v$.

(By induction: $Nv = \lambda v$ by def of eigenvector, if $N^m v = \lambda^m v$ then $N^{m+1} v = N N^m v$
$= N \lambda^m v$
$= \lambda^m N v$
$= \lambda^m \lambda v$
$= \lambda^{m+1} v$ )

Then $N^k v = \lambda^k v \Rightarrow 0 = \lambda^k v \Rightarrow \lambda = 0$ since $v \neq 0$.

Thus if $\lambda$ is an eigenvalue of $N$, $\lambda = 0$, so $\sigma(N) \subseteq \{0\}$.

$\therefore \sigma(N) = \{0\}$

4. Let $A \in M_n(\mathbb{C})$ be an invertible matrix. Show that $\lambda$ is an eigenvalue of $A$ if and only if $\lambda^{-1}$ is an eigenvalue of $A^{-1}$.

Let $\lambda$ be an eigenvalue of $A$. $\lambda \neq 0$ by exercise 2.

$\iff Av = \lambda v$ where $v \neq 0$ by definition

$\iff A^{-1}Av = A^{-1}\lambda v$

$\iff Iv = \lambda A^{-1} v$

$\iff v = \lambda A^{-1} v$

$\iff \lambda^{-1} v = A^{-1} v$

$\iff \lambda^{-1}$ is an eigenvalue of $A^{-1}$ by definition

5. Suppose $A \in M_n(\mathbb{C})$ is Hermitian. Show that all the eigenvalues of $A$ are real. Hint: Note that if $\mathbf{x}$ is a normalized eigenvector of $A$ with eigenvalue $\lambda$, then $\langle A\mathbf{x}, \mathbf{x}\rangle = \lambda$.

Suppose $A$ is Hermitian. This means $A = A^*$.

Let $\lambda$ be an eigenvalue of $A$. Then $\exists v \neq 0$ s.t $Av = \lambda v$.

We can normalize $v$ by dividing by $\|v\| = \sqrt{\langle v, v\rangle}$, so

$\exists x \neq 0$ s.t $Ax = \lambda x$ & $\|x\| = 1$.

Then $\lambda = \lambda \|x\|^2 = \lambda \langle x, x\rangle$

$= \langle \lambda x, x\rangle$ by linearity of 1st argument of inner prod

$= \langle Ax, x\rangle$

$= \langle x, A^* x\rangle$ since $A^*$ is the adjoint

$= \langle x, Ax\rangle$ since $A = A^*$

$= \langle x, \lambda x\rangle$

$= \bar{\lambda}\langle x, x\rangle$ by conjugate symmetry & linearity of inner product

$= \bar{\lambda}$

Since $\lambda = \bar{\lambda}$, $\lambda \in \mathbb{R}$.

6. Let $A \in M_n(\mathbb{R})$. Show that the eigenvalues of $A^T A$ are non-negative.

Let $A \in M_n(\mathbb{R})$. Note that this means the adjoint of $A$ is $A^T$.

Let $\lambda$ be an eigenvalue of $A^T A$ with normalized eigenvector $x$, i.e. $A^T A x = \lambda x$ & $\|x\| = 1$.

Then $\lambda = \lambda \|x\|^2 = \lambda \langle x, x\rangle = \langle \lambda x, x\rangle$

$= \langle A^T A x, x\rangle$

$= \langle Ax, Ax\rangle$ since $(A^T)^* = A$

$= \|Ax\|^2 \geq 0$ by properties of norm