Exercises for Module 9: Linear Algebra III

1. Let V be an inner product space and $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be an orthonormal basis and $\mathbf{y} \in V$. Then, $\mathbf{x}$ has a unique representation $\mathbf{y} = \sum_{i=1}^{n} \alpha_i \mathbf{x}_i$. Show that $\alpha_i = \langle \mathbf{y}, \mathbf{x}_i \rangle$ for all $i = 1, \ldots, n$.

2. Let V be an inner product space and $U \subseteq V$ a subset. Show that $U^\perp$ is a subspace of V.

3. Let $A, B \in M_n(\mathbb{F})$ be similar matrices. Show that their characteristic polynomials coincide.

4. Show that $A \in M_n(\mathbb{C})$ is invertible if and only if $0 \notin \sigma(A)$.

5. Suppose N is a nilpotent matrix. Show that $\sigma(N) = \{0\}$.

6. Let $A \in M_n(\mathbb{C})$ be an invertible matrix. Show that λ is an eigenvalue of A if and only if λ^{-1} is an eigenvalue of A^{-1}.

7. Suppose $A \in M_n(\mathbb{C})$ is Hermitian. Show that all the eigenvalues of A are real. Hint: Note that if $\mathbf{x}$ is a normalized eigenvector of A with eigenvalue λ, then $\langle A\mathbf{x}, \mathbf{x} \rangle = \lambda$.

8. Let $A \in M_n(\mathbb{R})$. Show that the eigenvalues of $A^T A$ are non-negative.