

# **Statistical Sciences**

# DoSS Summer Bootcamp Probability Module 8

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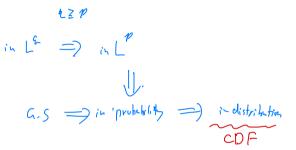
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## Recap

#### Learnt in last module:

- Stochastic convergence
  - ▷ Convergence in distribution
  - ▷ Convergence in probability
  - > Convergence almost surely
  - $\triangleright$  Convergence in  $L^p$
  - Relationship between convergences





### **Outline**

- Convergence of functions of random variables
  - ▷ Slutsky's theorem
  - ▷ Continuous mapping theorem
- Laws of large numbers
  - ▶ WLLN
  - ▷ SLLN
  - ▷ Glivenko-Cantelli theorem
- Central limit theorem



**Recall: Stochastic convergence** If  $X_n \to X$ ,  $Y_n \to Y$  in some sense, how is the limiting property of  $f(X_n, Y_n)$ ?



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### Convergence of functions of random variables (a.s.)

Suppose the probability space is complete, if  $X_n \xrightarrow{a.s.} X$ ,  $Y_n \xrightarrow{a.s.} Y$ , then for any real numbers a, b,

- $aX_n + bY_n \xrightarrow{a.s.} aX + bY$ :
- $X_n Y_n \xrightarrow{a.s.} XY$

#### Remark:

Still require all the random variables to be defined on the same probability space



(Pf) Snoe Xn x a-s, there exists = Nx C SL s.t.  $X_n \rightarrow X$  on  $N_k$  and  $P(N_k) = 1$ . Since In I Tais, there ensts Ny CD St. (n-) ( on Ny and P (Nx)=1. On Nx Ny, we know Xx -> X, Yx -> Y postusse Thus we have on NxNN7, a Kut bin -> a X+ by pointwise X. In -> X-7 pointwise. (P(Nx n Nx) = 1-1P((Nx nNx))) = 1- [P ( Nx C Nx C) apply union bond = 1- ( P(Nx) + P(Nx)) = (-0=1 =0 : [P ( Nx 1 Nz)= () Thus we have conford portwise convergence of axutby, -> X and Xu: Zy -> X. Y hold with probability.

### Convergence of functions of random variables (probability)

Suppose the probability space is complete, if  $X_n \xrightarrow{P} X$ ,  $Y_n \xrightarrow{P} Y$ , then for any real numbers a, b,

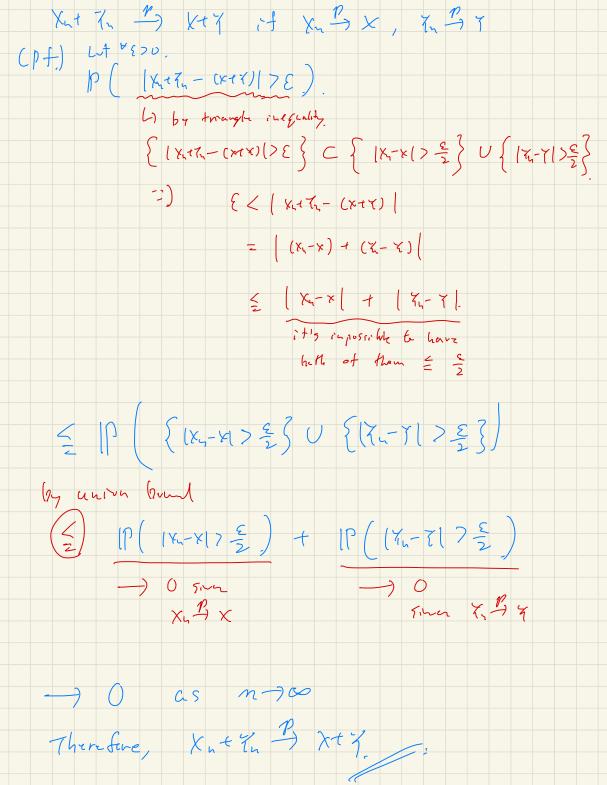
- $aX_n + bY_n \xrightarrow{P} aX + bY$ ;
- $X_n Y_n \xrightarrow{P} XY$ .

#### Remark:

• Still require all the random variables to be defined on the same probability space

Recall 
$$x_n \xrightarrow{p} x$$
 if,  $\forall \epsilon > 0$ ,  $p((x_n - x_1 > \epsilon) \rightarrow 0$  as  $n > 0$ .





### Convergence of functions of random variables $(L^p)$

Suppose the probability space is complete, if  $X_n \xrightarrow{L^p} X$ ,  $Y_n \xrightarrow{L^p} Y$ , then for any real numbers a, b,

•  $aX_n + bY_n \xrightarrow{L^p} aX + bY$ ;

#### Remark:

• Still require all the random variables to be defined on the same probability space



Xat In Les XeT of Xy X, Z, Z, Z (pf) Recall that 11 × 11, p = (E|x|) p is a ann if p21. Theretere we have triangle inequality, i.e. 11 x+ 7 11p = 11 x11p + 11711p 11 (xx+3,1- (xxx)) = 11 xx-x11p + 117x-711p -> 0. -) O -) O SILCE TO Y

Even Yntin d) Xti fails

### Slutsky's theorem

If  $X_n \xrightarrow{d} X$  and  $Y_n \xrightarrow{P} c$  (c is a constant), then

Remark: Convergence in distribution is different.

- $X_n + Y_n \xrightarrow{d} X + c$ :
- $X_n Y_n \xrightarrow{d} cX$ :
- $X_n/Y_n \xrightarrow{d} X/c$ , where  $c \neq 0$ .

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#### Remark:

• The theorem remains valid if we replace all the convergence in distribution with convergence in probability.



**Remark**: The requirement that  $Y_n \xrightarrow{P} c$  (c is a constant) is necessary.



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### **Examples:**

$$X_n \sim \mathcal{N}(0,1), Y_n = -X_n$$
, then  $Y_n \sim \mathcal{M}(0,1)$  as well.

- $X_n \xrightarrow{d} Z \sim \mathcal{N}(0,1), Y_n \xrightarrow{d} Z \sim \mathcal{N}(0,1);$
- $X_n + Y_n \xrightarrow{d} 0$ ;  $\approx 2 \approx$
- $X_n Y_n = -X_n^2 \xrightarrow{d} -\chi^2(1); \neq 2^2 \wedge \chi^*(1)$
- $X_n/Y_n = -1$ .



### Continuous mapping theorem

Let  $X_n$ , X be random variables, if  $g(\cdot): \mathbb{R} \to \mathbb{R}$  satisfies  $\mathbb{P}(X \in D_g) = 0$ , then

- $X_n \xrightarrow{a.s.} X \Rightarrow g(X_n) \xrightarrow{a.s.} g(X)$ :
- $X_n \xrightarrow{P} X \Rightarrow g(X_n) \xrightarrow{P} g(X)$ :
- $X_n \xrightarrow{d} X \Rightarrow g(X_n) \xrightarrow{d} g(X)$ :

where  $D_{\sigma}$  is the set of discontinuity points of  $g(\cdot)$ .

10 convergen fail in general. Lt Xh = X when X ∈ LD but € L2.

Lut g(x)= x2.



Then 9(X4) & P. So P convergence doesn't make

g is essentially Coutinuous writ X

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where  $D_g$  is the set of discontinuity points of  $g(\cdot)$ .

#### Remark:

- If  $g(\cdot)$  is continuous, then ...
- If X is a continuous random variable, and  $D_g$  only include countably many points, then ...



### Weak Law of Large Numbers (WLLN)

If  $X_1, X_2, \dots, X_n$  are i.i.d. random variables  $\mathbb{E}(|X_i|) < \infty$ , then

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \xrightarrow{P} \mu.$$

~ M= EX

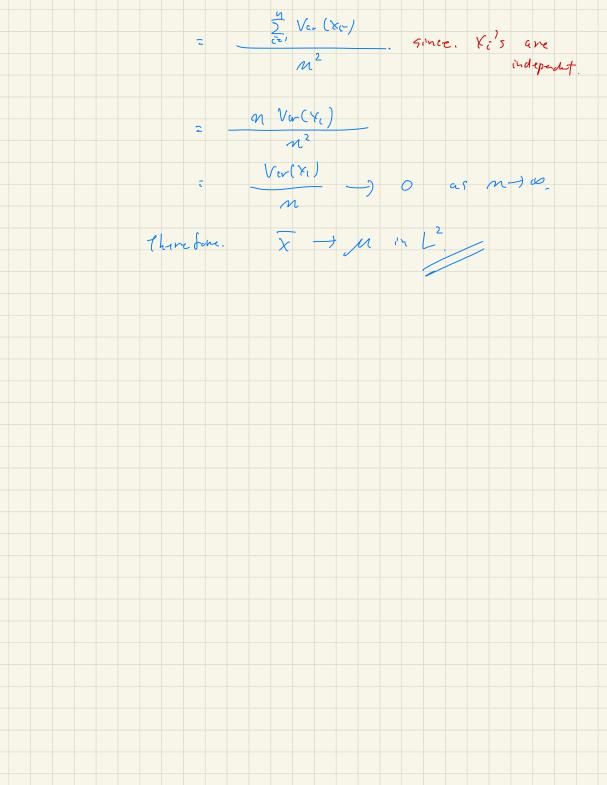
#### Remark:

A more easy-to-prove version is the  $L^2$  weak law, where an additional assumption  $Var(X_i) < \infty$  is required.

Sketch of the proof:

$$\mathbb{E}\left[\overline{X} - \mu\right]^{2} = \operatorname{Vor}\left(\overline{X}\right)$$

$$= \operatorname{Vor}\left(\frac{\overline{X}}{n}\right)$$



A generalization of the theorem: triangular array

#### Triangular array

A triangular array of random variables is a collection  $\{x_{n,k}\}_{1 \le k \le n}$ .

$$M_{2}(\longrightarrow X_{1,1} \xrightarrow{Sun} S_{1})$$

$$m_{2} \longrightarrow X_{2,1}, X_{2,2} \xrightarrow{Sun} S_{2}$$

$$m_{3} \longrightarrow X_{3,1}, X_{3,2}, X_{3,3} \xrightarrow{Sun} S_{3}$$

$$\vdots$$

$$M \longrightarrow X_{n,1}, X_{n,2}, \cdots, X_{n,n} \xrightarrow{Sun} S_{n} = \prod_{k \geq 1} X_{n,k}$$

**Remark:** We can consider the limiting property of the row sum  $S_n$ .



## Law of Large Numbers

### $L^2$ weak law for triangular array

Suppose  $\{X_{n,k}\}$  is a triangular array,  $n=1,2,\cdots,k=1,2,\cdots,n$ . Let  $S_n=\sum_{k=1}^n X_{n,k},\ \mu_n=\mathbb{E}(S_n),\ \text{if}\ \sigma_n^2/b_n^2\to 0,\ \text{where}\ \sigma_n^2=Var(S_n)\ \text{and}\ b_n\ \text{is a sequence}$  of positive real numbers, then

$$\frac{S_n-\mu_n}{b_n} \quad \xrightarrow{P} \quad 0.$$

#### Remark:

The  $L^2$  weak law for i.i.d. random variables is a special case of that for triangular array.





**Proof:** 

### Remark:

A more generalized version incorporates truncation, then the second-moment constraint is relieved.



### Strong Law of Large Numbers (SLLN)

Let  $X_1, X_2, \cdots$  be an i.i.d. sequence satisfying  $\mathbb{E}(X_i) = \mu$  and  $\mathbb{E}(|X_i|) < \infty$ , then  $\bar{X} = \frac{\sum_{i=1}^n X_i}{n} \xrightarrow{a.s.} \mu$ .

**Remark:** The proof needs Borel-Cantelli lemma.

### Strong Law of Large Numbers (SLLN)

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**Remark:** The proof needs Borel-Cantelli lemma.

#### Glivenko-Cantelli theorem

Let  $X_i$ ,  $i = 1, \dots, n$  i.i.d. with distribution function  $F(\cdot)$ , and let

$$F_n(x) = \frac{1}{n} \sum_{i=1}^n I(X_i \le x)$$
, then as  $n \to \infty$ ,

$$\sup_{x\in\mathbb{R}}|F(x)-F_n(x)|\to 0,\quad a.s.$$



Note that 
$$0 \leq I(X_0 \leq x) \leq 1$$

Cinite

$$\int_{\mathcal{O}} \int_{\mathcal{O}} \left\{ \left\{ \mathbb{E} \left[ \left( X_{i} \leq X \right) = \left[ P \left( X_{i} \leq X \right) = F \left( \mathcal{O} \right) \leq l \right] \right\} \right\}$$

$$\frac{a.s.}{e}$$
 E  $I(x \le x) = F(x)$ 



## **Limit Theorems and Counterexamples**

**Recall:** For the law of large numbers to hold, the assumption  $E|X| < \infty$  is crucial.

### Law of Large Numbers fail for infinite mean i.i.d. random variables

If  $X_1X_2,\ldots$  are i.i.d. to X with  $E|X_i|=\infty$ , then for  $S_n=X_1+\cdots+X_n$ ,  $P(\lim_{n\to\infty}S_n/n\in(-\infty,\infty))=0$ .

**Proof: Omitted** 



### **Central Limit Theorem**

$$Ver(\overline{X}) = \frac{\sigma^2}{N}$$

Sta( $\overline{X}$ ) =  $\frac{\sigma}{N}$   $\Leftrightarrow$  Sta( $\overline{X}$ ) =  $\sigma$ 

What is the limiting distribution of the sample mean?

#### Classic CLT

Suppose  $X_1, \dots, X_n$  is a sequence of i.i.d. random variables with  $\mathbb{E}(X_i) = \mu$ ,

$$Var(X_i) = \sigma^2 < \infty$$
, then

$$\frac{\sqrt{n}(\lambda_{n}-\mu)}{\sigma} \xrightarrow{d} \mathcal{N}(0,1).$$

$$\frac{\sqrt{n}}{\sigma} \cdot \frac{1}{n} \sum_{i=1}^{n} (\chi_{i}-\mu)_{i} = \frac{\sum_{i=1}^{n} (\chi_{i}-\mu)}{\sigma}$$

#### Remark:

- The proof involves characteristic function.
- A more generalized CLT is referred to as "Lindeberg CLT".



### **Central Limit Theorem**

#### **Example:**

Suppose  $X_i \sim Bernoulli(p)$ , i.i.d., consider  $Z_n = \frac{\sum_{i=1}^n X_i - np}{\sqrt{np(1-p)}}$ , then by CLT,  $Z_n \sim \mathcal{N}(0,1)$  asymptotically.

# **Monotone Convergence Theorem**

### Monotone Convergence Theorem

If  $X_n \ge c$  and  $X_n \nearrow X$ , then  $EX_n \nearrow EX$ 

**Usage:** 



# **Dominate Convergence Theorem**

### Dominated Convergence Theorem

If  $X_n \to X$  a.s. and  $|X_n| \le Y$  a.s. for all n and Y is integrable, then  $EX_n \to EX$ 

**Usage:** 



#### **Delta Method**

#### More about CLT: Delta method

Suppose  $X_n$  are i.i.d. random variables with  $EX_n=0$ ,  $VAR(X_n)=\sigma^2>0$ . Let g be a measurable function that is differentiable at 0 with  $g'(0)\neq 0$ . Then

$$\sqrt{n}\left(g\left(rac{\sum_{k=1}^{n}X_{k}}{n}-g(0)
ight)
ight)
ightarrow \mathcal{N}(0,\sigma^{2}g'(0)^{2})$$
 weakly.

**Proof under stronger assumption:** Here, we suppose g is continuously differentiable on  $\mathbb{R}$ . If you are interested in a general proof refer to Robert Keener's *Theoretical Statistics*.



### **Problem Set**

**Problem 1:** Prove that on a complete probability space, if  $X_n \xrightarrow{a.s.} X$ ,  $Y_n \xrightarrow{a.s.} Y$ , then  $X_n + Y_n \xrightarrow{a.s.} X + Y$ .

**Problem 2:** Prove that on a complete probability space, if  $X_n \xrightarrow{P} X$ ,  $Y_n \xrightarrow{P} Y$ , then  $X_n + Y_n \xrightarrow{P} X + Y$ .

**Problem 3:** A bank teller serves customers standing in the queue one by one. Suppose that the service time  $X_i$  for customer i has mean  $\mathbb{E}(X_i)=2$  (minutes) and  $Var(X_i)=1$ . We assume that service times for different bank customers are independent. Let Y be the total time the bank teller spends serving 50 customers. Find  $\mathbb{P}(90 < Y < 110)$ .

