



UNIVERSITY OF  
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# Statistical Sciences

## DoSS Summer Bootcamp Probability Module 4

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# Recap

Learnt in last module:

- Discrete probability
  - ▷ Classical probability
  - ▷ Combinatorics
  - ▷ Common discrete random variables
- Continuous probability
  - ▷ Geometric probability
  - ▷ Common continuous random variables
- Exponential family

# Outline

- Joint and marginal distributions
  - ▷ Joint cumulative distribution function
  - ▷ Independence of continuous random variables
- Conditional distribution
- Functions of random variables
  - ▷ Convolutions
  - ▷ Change of variables
  - ▷ Order statistics

# Joint and marginal distributions

$(\Omega, \mathcal{F}, (P))$  probability triple

**Random vector:** joint behaviour of multivariate random variables

Recap  $X$  is random variable  $\stackrel{\text{def}}{\iff} \boxed{\begin{array}{l} \forall B \in \mathcal{R} \\ \text{Borel sets} \end{array}}, X^{-1}(B) \in \mathcal{F}.$

Need to generalize Borel sets  $\mathcal{R}$  to  $\mathbb{R}^n$ .

Consider the collection of cubes

$$\{ (a_1, b_1] \times \cdots \times (a_n, b_n] \mid a_i < b_i, i=1, \dots, n \}$$

Define Borel sets  $\mathcal{R}^n$  as the smallest  $\sigma$ -algebra containing all those cubes.

Def  $X = (X_1, \dots, X_n) : \Omega \rightarrow \mathbb{R}^n$  is a random vector  
if  $X^{-1}(B) \in \mathcal{F}$  for any  $B \in \mathcal{B}^n$ .

Remark Similarly with  $n=1$ , we can check if  $X$  is a random vector  
by checking special form of  $B$ ,

$$B = (-\infty, x_1] \times \dots \times (-\infty, x_n].$$

This means

if  $X^{-1}((-\infty, x_1] \times \dots \times (-\infty, x_n]) \in \mathcal{F}$  for any  $x_1, \dots, x_n$ ,  
then  $X$  must be a random vector.

Cor If  $X_i$  are random variable on  $(\Omega, \mathcal{F})$ ,  
then  $X = (X_1, \dots, X_n)$  is a random vector.

$$\begin{aligned} \text{(pf)} \quad X^{-1}((-\infty, x_1] \times \dots \times (-\infty, x_n]) &= \{X_i \leq x_i, \forall i\} \\ &= \bigcap_{i=1}^n \{X_i \leq x_i\} \\ &= \bigcap_{i=1}^n \underbrace{X_i^{-1}((-\infty, x_i])}_{\substack{\in \mathcal{F} \\ \text{since each } X_i \text{ is} \\ \text{a random variable.}}} \in \mathcal{F}. \end{aligned}$$

# Joint and marginal distributions

**Random vector:** joint behaviour of multivariate random variables

## Joint cumulative distribution function

Consider a random vector  $(X_1, X_2, \dots, X_d)$ , the joint cumulative distribution function of  $(X_1, X_2, \dots, X_d)$  is defined by:

$$F_{X_1, X_2, \dots, X_d}(x_1, x_2, \dots, x_d) = \mathbb{P}[X_1 \leq x_1, X_2 \leq x_2, \dots, X_d \leq x_d].$$

$$\forall i, x_i \in (-\infty, \infty]$$

$$\Leftrightarrow (x_1, \dots, x_d) \in (-\infty, x_1] \times \dots \times (-\infty, x_d]$$

natural extension of  $(-\infty, x]$

# Joint and marginal distributions

**Random vector:** joint behaviour of multivariate random variables

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### Remark:

For discrete random vector, it suffices to find the joint probability mass function

$$p_{X_1, \dots, X_n}(x_1, \dots, x_n) = P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n), \quad x_i \in \mathbb{R},$$

and

$$\mathbb{P}((X_1, \dots, X_n) \in C) = \sum_{(x_1, \dots, x_n) \in C} p_{X_1, \dots, X_n}(x_1, \dots, x_n).$$

# Joint and marginal distributions

## Remark:

For continuous random vector, consider the joint probability density function.

### Joint probability density function

$$f_{X_1, \dots, X_n}(x_1, \dots, x_n) = \frac{\partial^n F_{X_1, \dots, X_n}(x_1, \dots, x_n)}{\partial x_1 \dots \partial x_n}, \quad x_i \in \mathbb{R}.$$

Similarly,

$$\mathbb{P}((X_1, \dots, X_n) \in C) = \int_{(x_1, \dots, x_n) \in C} f_{X_1, \dots, X_n}(x_1, \dots, x_n) dx_1 dx_2 \dots dx_n.$$



# Joint and marginal distributions

Consider the special case of  $C$  where  $X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n$  are allowed to take any possible values:

*all  $x_j$ 's except for  $x_i$*

- Discrete case

$$\mathbb{P}(X_i = x_i) = \mathbb{P}(X_i = x_i, X_j \in \mathbb{R}, j \neq i) = \sum_{x_j, j \neq i} p_{X_1, \dots, X_n}(x_1, \dots, x_n).$$

- Continuous case

$$\begin{aligned} \mathbb{P}(X_i \leq x_i) &= \mathbb{P}(X_i \leq x_i, X_j \in \mathbb{R}, j \neq i) \\ &= \int_{-\infty}^{x_i} \left( \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1, \dots, X_n}(t_1, \dots, t_n) dt_1 \cdots dt_{i-1} dt_{i+1} \cdots dt_n \right) dt_i. \end{aligned}$$

*except for  $x_i$*

# Joint and marginal distributions

Taking the derivative regarding  $x_i$ , this gives us the marginal probability density function.

## Marginal probability density function

$$f_{X_i}(x_i) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1, \dots, X_n}(x_1, \dots, x_i, \dots, x_n) \underbrace{dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n}_{\text{except for } x_i}.$$

except for  $x_i$

# Joint and marginal distributions

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## Marginal probability density function

$$f_{X_i}(x_i) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1, \dots, X_n}(x_1, \dots, x_i, \dots, x_n) dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n.$$

### Remark:

Marginal probability mass function (density function) of  $X_i$  is obtained by summing (integrating) the joint probability over all the other dimensions.

# Joint and marginal distributions

## Example: Draws from an urn

Suppose each of two urns contains twice as many red balls as blue balls, and no others, and suppose one ball is randomly selected from each urn, with the two draws independent of each other. Let  $A$  and  $B$  be discrete random variables associated with the outcomes of the draw from the first urn and second urn respectively. 1 represents a draw of red ball, while 0 represents a draw of blue ball.

	1	0	$\mathbb{P}(B)$
1	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{2}{3}$
0	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{1}{3}$
$\mathbb{P}(A)$	$\frac{2}{3}$	$\frac{1}{3}$	1

Table: Joint and marginal pmf of draws from an urn

# Joint and marginal distributions

## Examples: continuous case

Consider the joint probability density function

$$f(x, y) = \begin{cases} kx & \text{for } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

### Remark:

- Find  $k$ .
- Compute the marginal probability density function of  $X$  and  $Y$ .

# Joint and marginal distributions

Integrate to find the value of  $k$

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = \int_0^1 \int_0^1 kx dx dy = \int_0^1 \frac{k}{2} dy = \frac{k}{2}$$

$$\therefore k = 2.$$

Marginal density

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_0^1 f(x, y) dy$$

$$= \begin{cases} \int_0^1 2x dy & \text{--- if } x \in (0, 1) \\ 0 & \text{--- otherwise} \end{cases} = \begin{cases} 2x & \text{--- if } x \in (0, 1) \\ 0 & \text{--- otherwise} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y) dx = \int_0^1 f(x,y) dx$$

$$= \begin{cases} \int_0^1 2x dx & \text{--- if } y \in (0,1) \\ 0 & \text{--- otherwise.} \end{cases}$$

$$= \begin{cases} 1 & \text{--- if } y \in (0,1) \\ 0 & \text{--- otherwise.} \end{cases}$$

# Joint and marginal distributions

## Recap: independence of random variables

### Corollary of independence

If  $X_1, \dots, X_n$  are random variables, then  $X_1, X_2, \dots, X_n$  are independent if

$$P(X_1 \leq x_1, \dots, X_n \leq x_n) = \prod_{i=1}^n P(X_i \leq x_i)$$



# Joint and marginal distributions

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### Remark:

Suppose  $X_1, \dots, X_n$  can only take values from  $\{a_1, \dots\}$ , then  $X_i$ 's are independent if

$$\underbrace{P(\cap \{X_i = a_i\})}_{\text{joint pmf}} = \prod_{i=1}^n \underbrace{P(X_i = a_i)}_{\text{marginal pmf}}.$$

# Joint and marginal distributions

## Remark:

This is equivalent to check whether the joint pmf is always the product of the corresponding marginal pmf.

Generalize this to the continuous version:

# Joint and marginal distributions

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Generalize this to the continuous version:

## Independence of continuous random variables

Suppose  $X_1, \dots, X_n$  are continuous random variables, then  $X_i$ 's are independent if

$$\underbrace{f_{(X_1, \dots, X_n)}(x_1, x_2, \dots, x_n)}_{\text{joint density}} = \prod_{i=1}^n \underbrace{f_{X_i}(x_i)}_{\text{marginal density}}.$$

joint density

marginal density

# Conditional distribution

## Remark:

Given joint and marginal distributions, consider the conditional behaviour:

# Conditional distribution

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Given joint and marginal distributions, consider the conditional behaviour:

## Conditional distribution

For random variables  $X$  and  $Y$ , the conditional distribution of  $Y$  given  $X = x$  is defined by

- Discrete case

$$p_{Y|X=x}(y) = \underbrace{\mathbb{P}(Y = y \mid X = x)}_{\substack{\frac{\mathbb{P}(Y=y, X=x)}{\mathbb{P}(X=x)}}} = \frac{p_{X,Y}(x, y)}{p_X(x)}.$$

- Continuous case

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x, y)}{f_X(x)}.$$

# Conditional distribution

## Remark:

Another look at independence:

- Discrete case:

$X$  and  $Y$  are independent

$$\Leftrightarrow p_{Y|X=x}(y) = p_Y(y), \quad \forall x, y$$

$$\Leftrightarrow p_{X,Y}(x, y) = p_X(x)p_Y(y), \quad \forall x, y.$$

- Continuous case:

$X$  and  $Y$  are independent

$$\Leftrightarrow f_{Y|X}(y | x) = f_Y(y), \quad \forall x, y$$

$$\Leftrightarrow f_{X,Y}(x, y) = f_X(x)f_Y(y), \quad \forall x, y.$$

# Functions of random variables

Suppose we know the joint distribution of  $(X, Y)$ , what is the distribution of  $Z = X + Y$ ?

- Discrete case

$$\mathbb{P}(Z \in z) = \sum_{x+y=z} \mathbb{P}(X = x, Y = y)$$

- Continuous case

$$\mathbb{P}(Z \leq z) = \int_{x+y \leq z} f_{X,Y}(x, y) \, dx dy$$

## Remark:

This can be simplified in the independent case.

# Functions of random variables

## Convolutions of independent random variables

Suppose  $X$  and  $Y$  are independent, then for  $Z = X + Y$ ,

- Discrete case

$$\mathbb{P}(Z = z) = \sum_{k=-\infty}^{\infty} \mathbb{P}(X = k) \mathbb{P}(Y = z - k).$$

- Continuous case

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x) dx$$

$\mathbb{P}(X + Y = z, X = k)$

**Sketch of proof:**



(Discrete case.)

$$P(Z=z) = \sum_{x+y=z} \underbrace{P(X=x, Y=y)}$$

↓  $X, Y$  are independent

$$= \sum_{x+y=z} P(X=x) P(Y=y)$$

$$= \sum_{k_2} P(X=k_2) P(Y=z-k_2)$$

# Functions of random variables

Consider a function of random variable, and try to obtain the corresponding distribution function.

## Multivariate change-of-variables formula

Suppose  $\mathbf{X}$  is an  $n$ -dimensional random variable with joint density  $f_{\mathbf{X}}(\mathbf{x})$ . If  $\mathbf{Y} = H(\mathbf{X})$ , where  $H$  is a bijective, differentiable function, then  $\mathbf{Y}$  has density  $g_{\mathbf{Y}}(\mathbf{y})$ :

$$g(\mathbf{y}) = f\left(H^{-1}(\mathbf{y})\right) \left| \det \left[ \frac{dH^{-1}(\mathbf{z})}{d\mathbf{z}} \Big|_{\mathbf{z}=\mathbf{y}} \right] \right|$$

with the differential regarded as the Jacobian of  $H(\cdot)$ , evaluated at  $\mathbf{y}$ .

# Functions of random variables

Consider a function of random variable, and try to obtain the corresponding distribution function.

## Multivariate change-of-variables formula

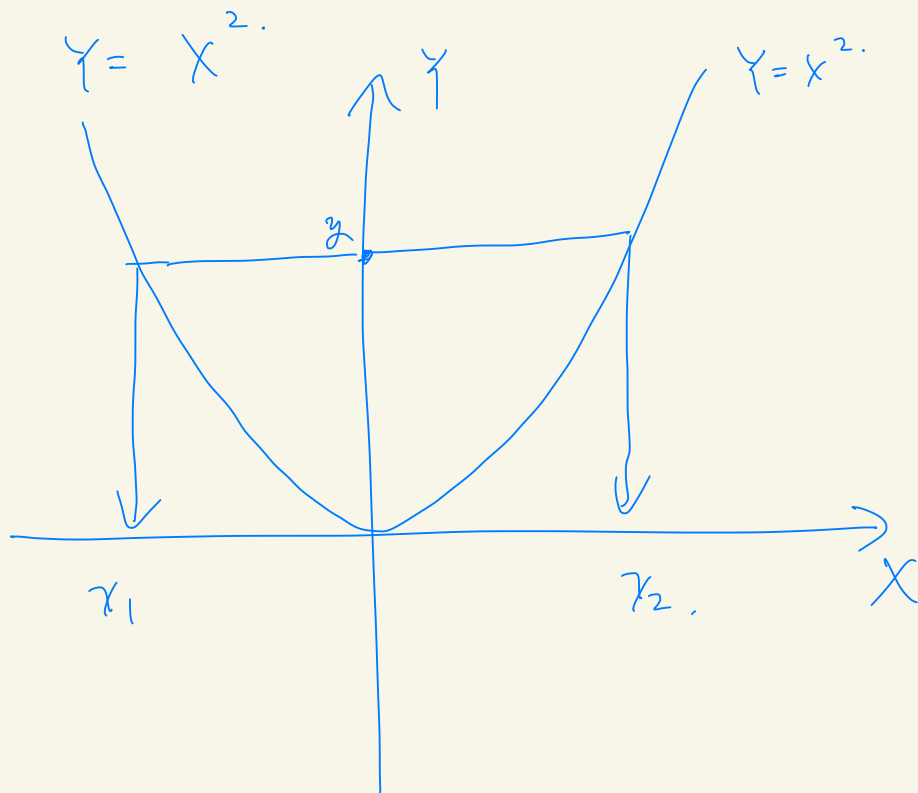
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with the differential regarded as the Jacobian of  $H(\cdot)$ , evaluated at  $\mathbf{y}$ .

### Remark:

Bijective property is important.



# Functions of random variables

## Special case of 2-dimensional vectors

### 2-dimensional change-of-variables formula

Suppose  $\mathbf{X} = (X_1, X_2)$  with joint density  $f_{X_1, X_2}(x_1, x_2)$ . If  $Y_1 = H_1(X_1, X_2)$ ,  $Y_2 = H_2(X_1, X_2)$ , where  $H$  is a bijective, differentiable function, then  $\mathbf{Y} = (Y_1, Y_2)$  has density  $g_{\mathbf{Y}}(y_1, y_2)$ :

$$g(y_1, y_2) = f_{X_1, X_2}(H_1^{-1}(y_1, y_2), H_2^{-1}(y_1, y_2)) \left| \frac{\partial H_1^{-1}}{\partial y_1} \frac{\partial H_2^{-1}}{\partial y_2} - \frac{\partial H_1^{-1}}{\partial y_2} \frac{\partial H_2^{-1}}{\partial y_1} \right|.$$

Remark:

Jacobian

# Functions of random variables

## Remark:

Every continuous bijective function from  $\mathbb{R}$  to  $\mathbb{R}$  is strictly monotonic.

# Functions of random variables

## Remark:

Every continuous bijective function from  $\mathbb{R}$  to  $\mathbb{R}$  is strictly monotonic.

**Special case of 1-dimensional random variable: generalize to monotonic functions**

## Univariate change-of-variables formula

Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a monotonic function on the support of  $f_X(x)$ , then for  $Y = g(X)$ , the density is:

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}(g^{-1}(y)) \right|.$$

*Jacobian*

# Functions of random variables

## Proof of univariate change-of-variable formula:

Assume  $g$  : monotone, increasing and differentiable.

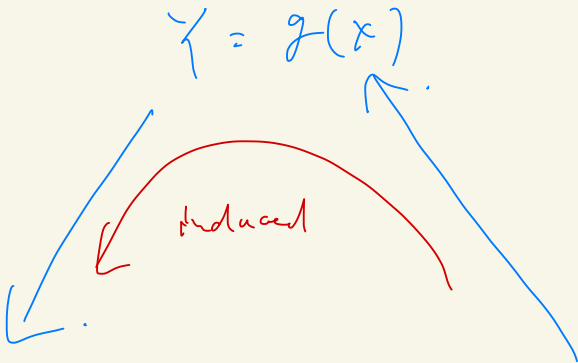
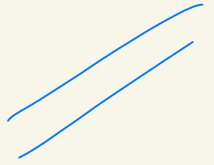
$$\begin{aligned} \int_{-\infty}^{y_0} \underbrace{f_X(z)}_{\substack{\text{density} \\ \text{of } X}} dz &= P(X \leq y_0) \\ &= P(g(X) \leq y_0) \\ &= P(X \leq g^{-1}(y_0)) \end{aligned}$$



$$= \int_{-\infty}^{g^{-1}(y_0)} f_x(x) dx$$

$$\left[ \begin{array}{l} \text{Change of variable from real analysis} \\ x = g^{-1}(y), \quad dx = \frac{d}{dy} g^{-1}(y) dy \end{array} \right]$$

$$= \int_{-\infty}^{y_0} \underbrace{f_x(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|}_{= f_Y(y)} dy$$



$\mu_X$  on  $\mathbb{R}$   $\longleftrightarrow$   $\mu_Y$  on  $\mathbb{R}$   
 Relationship between them

is given by change of variable.

# Functions of random variables

## Order statistics:

For random variables  $X_1, X_2, \dots, X_n$ , the order statistics are  $X_{(1)} \leq X_{(2)} \leq \dots X_{(n)}$ .

## Cumulative distribution functions of order statistics

Consider the case where  $X_i$ 's are independent identically distributed (i.i.d.) samples with cumulative distribution  $F_X(x)$ , then the CDF of  $X_{(r)}$  satisfies

$$F_{X_{(r)}}(x) = \sum_{j=r}^n \binom{n}{j} [F_X(x)]^j [1 - F_X(x)]^{n-j},$$

the corresponding probability density function is

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x) [F_X(x)]^{r-1} [1 - F_X(x)]^{n-r}.$$

defn in  
Keener's Theoretical  
Statistics

# Functions of random variables

Special cases of  $X_{(1)}$  and  $X_{(n)}$ :

$\min\{X_1, \dots, X_n\}$   
 $\max\{X_1, \dots, X_n\}$

$$F_{X_{(n)}}(x) = \mathbb{P}(\max\{X_1, \dots, X_n\} \leq x) = [F_X(x)]^n.$$

$$F_{X_{(1)}}(x) = \mathbb{P}(\min\{X_1, \dots, X_n\} \leq x) = 1 - [1 - F_X(x)]^n.$$

**Remark:**

For continuous random variable, taking derivatives to obtain the probability density function.

$$F_{X_{(r)}}(x) = \mathbb{P}(\underbrace{X_{(r)} \leq x}_{\text{rth smallest } X_i \leq x})$$

$\hookrightarrow$  rth smallest  $X_i \leq x$

$\Leftrightarrow$  there are "at least"  $r$   $X_i$ 's  $\leq x$ .

$$= \sum_{j=r}^n \mathbb{P}(\text{there are exactly } j, X_i \leq x)$$

$$= \sum_{j=r}^n \binom{n}{j} F_X(x)^j (1-F_X(x))^{n-j}$$

# Problem Set

**Problem 1:** Show that the probability density function of normal distribution  $N(\mu, \sigma^2)$  integrates to 1.

(Hint: consider two normal random variables  $X, Y$ )

**Problem 2:** Prove that for  $X$  with density function  $f_X(x)$ , the density function of  $y = X^2$  is

$$f_Y(y) = \frac{1}{2\sqrt{y}}(f_X(-\sqrt{y}) + f_X(\sqrt{y})), \quad y \geq 0.$$

(Hint: start by considering the CDF)

# Problem Set

**Problem 3:** Suppose  $X_1, \dots, X_n$  are i.i.d. sample following Uniform $[0, 1]$  distribution, find the joint probability density function of  $(X_{(1)}, X_{(n)})$ .  
(Hint: start by considering the CDF)