Statistical Sciences

DoSS Summer Bootcamp Probability Module 2

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Recap

Learnt in last module:

- Measurable spaces
  - Sample Space
  - σ-algebra
- Probability measures
  - Measures on σ-field
  - Basic results
- Conditional probability
  - Bayes' rule
  - Law of total probability

Outline

- Independence of events
  - Pairwise independence, mutual independence
  - Conditional independence
- Random variables
- Distribution functions
- Density functions and mass functions
- Independence of random variables

Independence of events

Recall the Bayes rule:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

- What if B does not change our belief about A?

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- What if B does not change our belief about A?
- This means $P(A \mid B) = P(A)$.
- Equivalently, $P(A \cap B) = P(A)P(B)$.

$$\mathbb{P}(A \mid B) = \mathbb{P}(A) \iff \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \mathbb{P}(A)$$

$$\iff \mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$$

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- Equivalently, $P(A \cap B) = P(A)P(B)$.

Independence of two events

Two events A and B are independent if $P(A \cap B) = P(A)P(B)$.

Remark:

Independence of events

Consider more than 2 events:

Pairwise independence

We say that events $A_1, A_2, \cdots, A_n$ are pairwise independent if

$$P(A_i \cap A_j) = P(A_i) \cdot P(A_j), \quad \forall i \neq j$$

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Mutual independence

We say that events $A_1, A_2, \cdots, A_n$ are mutually independent or independent if for all subsets $I \in \{1, 2, \cdots, n\}$

$$P(\cap_{i \in I} A_i) = \prod_{i \in I} P(A_i)$$

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Independence of events

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Remark: mutual independence $\Longrightarrow$ pairwise independence is true

but $\Longleftarrow$ does not hold.

Independence of events

Example:

- Toss two fair coins.;
- $A = \{$ First toss is head$\}$, $B = \{$ Second toss is head $\}$, $C = \{$ Outcomes are the same $\}$;
- $A = \{HH, HT\}, B = \{HH, TH\}, C = \{HH, TT\}$;

Independence of events

Example:

- Toss two fair coins.;
- $A = \{$ First toss is head$\}$, $B = \{$ Second toss is head $\}$, $C = \{$ Outcomes are the same $\}$;
- $A = \{HH, HT\}, B = \{HH, TH\}, C = \{HH, TT\}$;
- $P(A \cap B) = P(A)P(B)$, $P(A \cap C) = P(A)P(C)$, $P(B \cap C) = P(B)P(C)$;

$A \cap B = \{HH\}$, $\mathbb{P}(A \cap B) = \frac{1}{4}$

$A \cap C = \{HH\}$, $\mathbb{P}(A \cap C) = \frac{1}{4}$

$B \cap C = \{HH\}$, $\mathbb{P}(B \cap C) = \frac{1}{4}$

A, B, C are pairwise independent.

Independence of events

Example:

- Toss two fair coins.;
- $A = \{$ First toss is head$\}$, $B = \{$ Second toss is head $\}$, $C = \{$ Outcomes are the same $\}$;
- $A = \{HH, HT\}, B = \{HH, TH\}, C = \{HH, TT\}$;
- $P(A \cap B) = P(A)P(B)$, $P(A \cap C) = P(A)P(C)$, $P(B \cap C) = P(B)P(C)$;
- $P(A \cap B \cap C) \neq P(A)P(B)P(C)$.

$= \frac{1}{4}$ $\quad = \frac{1}{8}$

$A \cap B \cap C = \{HH\}$.

Independence of events

Conditional independence

Two events A and B are conditionally independent given an event C if

$$P(A \cap B \mid C) = P(A \mid C)P(B \mid C).$$

definition is replaced by conditional probability

$\mathbb{P}(\cdot \mid C)$

Independence of events

Conditional independence

Two events A and B are conditionally independent given an event C if

$$P(A \cap B \mid C) = P(A \mid C)P(B \mid C).$$

Example:

Previous example continued:

- $A = \{HH, HT\}, B = \{HH, TH\}, C = \{HH, TT\}$;
- $P(A \cap B \mid C) = ?$, $P(A \mid C)P(B \mid C) = ?$

$$\mathbb{P}(A \cap B \mid C) = \frac{\mathbb{P}(A \cap B \cap C)}{\mathbb{P}(C)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

$$\mathbb{P}(A \mid C) = \frac{\mathbb{P}(A \cap C)}{\mathbb{P}(C)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}, \quad \mathbb{P}(B \mid C) = \frac{1}{2}.$$

$$\mathbb{P}(A \cap B \mid C) \neq \mathbb{P}(A \mid C)\,\mathbb{P}(B \mid C)$$

A and B are not conditionally independent

Independence of events

Conditional independence

Two events A and B are conditionally independent given an event C if

$$P(A \cap B \mid C) = P(A \mid C)P(B \mid C).$$

Example:

Previous example continued:

- $A = \{HH, HT\}, B = \{HH, TH\}, C = \{HH, TT\}$;
- $P(A \cap B \mid C) =?$, $P(A \mid C)P(B \mid C) =?$

Remark:

Equivalent definition:

$$P(A \mid B, C) = P(A \mid C).$$

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Idea:

Instead of focusing on each events themselves, sometimes we care more about functions of the outcomes.

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Example:

- Toss a fair coin twice: $\{HH, HT, TH, TT\}$
- Care about the number of heads: $\{2, 1, 0\}$

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Random variables

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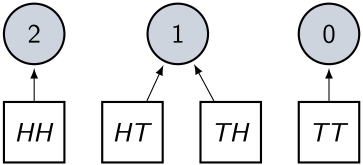

Figure: Mapping from the sample space to the numbers of heads

Random Variables

Example:

- Select twice from red and black ball with replacement: $\{RR, RB, BR, BB\}$
- Care about the number of red balls: $\{2, 1, 0\}$

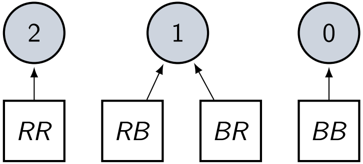

Figure: Mapping from the sample space to the numbers of red balls

Random Variables

Merits:

- Mapping the complicated events on σ-field to some numbers on real line.
- Simplify different events into the same structure

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Random Variables

Consider sample space Ω and the corresponding σ-field $\mathcal{F}$, for $X : \Omega \to \mathbb{R}$, if

$$A \in \mathcal{R} \quad \text{(Borel sets on } \mathbb{R}) \Rightarrow X^{-1}(A) \in \mathcal{F},$$

then we call X as a random variable.
Here $X^{-1}(A) = \{\omega : X(\omega) \in A\}$.
We can also say X is $\mathcal{F}$-measurable.

Distribution functions

$(\Omega, \mathcal{F}, \mathbb{P}) \xrightarrow{X} (\mathbb{R}, \mathcal{R}, \mu_X)$

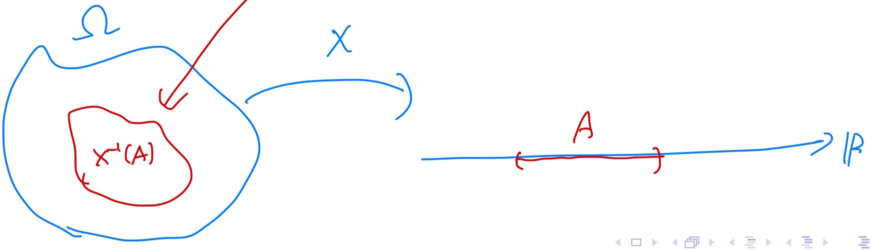
Probability measure $P(\cdot)$ on $\mathcal{F}$ can induce a measure $\mu(\cdot)$ on $\mathcal{R}$:

Probability measure on $\mathcal{R}$

We can define a probability μ on $(R, \mathcal{R})$ as follows:

$$\forall A \in \mathcal{R}, \quad \mu(A) := P(X^{-1}(A)) = P(X \in A).$$

Then μ is a probability measure and it is called the distribution of X.



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Remark:

Verify that μ is a probability measure.

- $\mu(\mathbb{R}) = 1$.
- If $A_1, A_2, \cdots \in \mathcal{R}$ are disjoint, then $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

Distribution functions

Consider the special set that belongs to $\mathcal{R}$, $(-\infty, x]$:

Cumulative Distribution Function

The cumulative distribution function of random variable X is defined as follows:

$$\underline{F(x) := P(X \le x) = P(X^{-1}((-\infty, x]))}, \quad \forall x \in \mathbb{R}.$$

$$= \mu((-\infty, x])$$

Distribution functions

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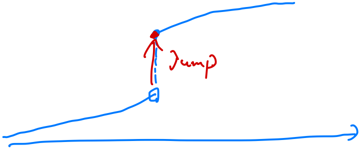
Properties of CDF:

- $\lim_{x \to \infty} F(x) = 1$, $\lim_{x \to -\infty} F(x) = 0$ ← can show by continuity of probability measure.
- $F(\cdot)$ is non-decreasing
- $F(\cdot)$ is right-continuous
- Let $F(x^-) = \lim_{y \nearrow x} F(y)$, then $F(x^-) = P(X < x)$
- $P(X = x) = F(x) - F(x^-)$ ←



Distribution functions

Proofs of properties of CDF (first 2 properties):

$F(x) = \mathbb{P}(X \leq x)$

When $x \nearrow \infty$ $\{X \leq x\} \nearrow \Omega$.

By the continuity from below,

$$\lim_{x \to \infty} \mathbb{P}(X \leq x) = \mathbb{P}(\Omega) = 1$$

When $x \searrow -\infty$, $\{X \leq x\} \searrow \emptyset$.

By the continuity from above,

$$\lim_{x \to -\infty} \mathbb{P}(X \leq x) = \mathbb{P}(\emptyset) = 0.$$

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Let $x_1 \leq x_2$ . .

Then $\{X \leq x_1\} \subset \{X \leq x_2\}$

So, $\{X \leq x_2\} = \{X \leq x_1\} \cup \{x_1 < X \leq x_2\}$

disjoint.

By countable additivity

$$F(x_2) = \mathbb{P}(X \leq x_2) = \mathbb{P}(X \leq x_1) + \underbrace{\mathbb{P}(x_1 < X \leq x_2)}_{\geq 0}$$

$$\geq \mathbb{P}(X \leq x_1) = F(x_1).$$

Density functions and mass functions

Classification of the random variables:

- Discrete random variable: X takes either a finite or countable number of possible numbers.
- Continuous random variable: The CDF is continuous everywhere.

Density functions and mass functions

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- Discrete random variable: X takes either a finite or countable number of possible numbers.
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Another perspective (function):

- Discrete random variable: focus on the probability assigned on each possible values
- Continuous random variable: consider the derivative of the CDF (The continuous monotone CDF is differentiable almost everywhere)

Density functions and mass functions

Probability mass function

The probability mass function of X at some possible value x is defined by

$$p_X(x) = P(X = x).$$

Relationship between PMF and CDF:

$$F(x) = P(X \leq x) = \sum_{y \leq x} p_X(y)$$

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Example:

Toss a coin

Density functions and mass functions

Probability density function

The probability density function of X at some possible value x is defined by

$$f_X(x) = \frac{d}{dx} F(x).$$

Relationship between PDF and CDF:

$$F(x) = P(X \leq x) = \int_{y \leq x} f_X(y)\, dy = \int_{-\infty}^{x} f_X(y)\, dy$$

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Example:

Independence of random variables

Define independence of random variables based on independence of events:

Independence of random variables

Suppose $X_1, X_2, \cdots, X_n$ are random variables on $(\Omega, \mathcal{F}, P)$, then

$$
\begin{aligned}
& X_1, X_2, \cdots, X_n \text{ are independent} \\
\Leftrightarrow \quad & \{X_1 \in A_1\}, \{X_2 \in A_2\}, \cdots, \{X_n \in A_n\} \text{ are independent}, \quad \forall A_i \in \mathcal{R} \\
\Leftrightarrow \quad & P(\cap_{i=1}^n \{X_i \in A_i\}) = \prod_{i=1}^n P(\{X_i \in A_i\})
\end{aligned}
$$

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Independence of random variables

Example:

Toss a fair coin twice, denote the number of heads of the i-th toss as X_i, then X_1 and X_2 are independent.

- A_i can be $\{0\}$ or $\{1\}$
- $\{(0,0),(0,1),(1,0),(1,1)\}$
- $P(\{X_1 \in A_1\} \cap \{X_2 \in A_2\}) = \frac{1}{4}$
- $P(\{X_1 \in A_1\}) = P(\{X_2 \in A_2\}) = \frac{1}{2}$

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- $P(\{X_1 \in A_1\}) = P(\{X_2 \in A_2\}) = \frac{1}{2}$

Remark:
How to check independence in practice?

Independence of random variables

Corollary of independence

If $X_1, \cdots, X_n$ are random variables, then $X_1, X_2, \cdots, X_n$ are independent if

$$P(X_1 \leq x_1, \cdots, X_n \leq x_n) = \prod_{i=1}^{n} P(X_i \leq x_i)$$

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Remark:

Independence of discrete random variables

Suppose $X_1, \cdots, X_n$ can only take values from $\{a_1, \cdots\}$, then X_i's are independent if

$$P(\cap\{X_i = a_i\}) = \prod_{i=1}^{n} P(X_i = a_i)$$

Problem Set

Problem 1: Give an example where the events are pairwise independent but not mutually independent.

Problem 2: Verify that the measure $\mu(\cdot)$ induced by $P(\cdot)$ is a probability measure on $\mathcal{R}$.

Problem 3: Prove properties 3 - 5 of CDF $F(\cdot)$.

Problem 4: Bob and Alice are playing a game. They alternatively keep tossing a fair coin and the first one to get a H wins. Does the person who plays first have a better chance at winning?