Problem Set 4

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Problem 1

The error of the function obtained from the equation $|\frac{1}{3}(I_2 - I_1)|$ is approximately 0.026633333333333137, whereas the actual error of the function is approximately 0.026660000000000572 (results obtained from **newman_5.6.py**). The differences between the actual error and the error obtained using this equation can be partially explained by the fact that the trapezoidal rule has an error of $\mathcal{O}(h^2)$, and when relating I_2 to I_1, we neglect the higher-order terms.

Problem 2

(a) We are given

$$E = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + V(x)$$

Initially, at $t = 0$, $E = V(a)$. It follows

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + V(x) = V(a)$$

Isolating for $\frac{dx}{dt}$ and integrating to find T yields

$$\begin{aligned}
\frac{dx}{dt} &= \sqrt{\frac{2}{m}(V(a) - V(x))} \\
\frac{T}{4} &= \int_a^0 \frac{dx}{\sqrt{\frac{2}{m}(V(a) - V(x))}} \\
\frac{T}{4} &= \sqrt{\frac{m}{2}} \int_a^0 \frac{dx}{\sqrt{V(a) - V(x)}} \\
T &= \sqrt{8m} \int_a^0 \frac{dx}{\sqrt{V(a) - V(x)}}
\end{aligned}$$

Since potential is symmetrical around the origin, we have

$$T = \sqrt{8m} \int_0^a \frac{dx}{\sqrt{V(a) - V(x)}}$$

(b) From the previous part, we have

$$T = \sqrt{8m} \int_0^a \frac{dx}{\sqrt{V(a) - V(x)}}$$

Plugging in $V(x) = x^4$, $m = 1$ and $a = 2$, it follows that

$$V(x) = \sqrt{8} \int_0^2 \frac{dx}{\sqrt{a^4 - x^4}}$$

We can use change of variables to

$$\int_a^b f(x)\, dx = \int_{-1}^1 f\left(\frac{b-a}{2}\xi + \frac{a+b}{2}\right) \frac{dx}{d\xi}\, d\xi$$

When $a = 0$ and $b = 2$, we have

$$\int_0^2 f(x)\, dx = \int_{-1}^1 f\left(\xi + 1\right) \frac{dx}{d\xi}\, d\xi$$

The graph below was obtained from **newman_5.10.py**.

(c) At large amplitudes a, the initial energy $V(a)$ is quite large, so as the particle moves closer to the origin, $\sqrt{V(a) - V(x)}$ becomes larger, and there speed of the particle will grow significantly at the origin. Therefore, even for larger a, the longer distance is made up for the increase in speed at the origin. As a approaches 0, $V(a)$ also approaches 0. Therefore, the integrand $\frac{dx}{\sqrt{V(a)-V(x)}}$ becomes very large, and when x approaches 0, T becomes very large. Therefore, near the origin, the particle's motion is very slow, causing the period to diverge as amplitude approaches 0.

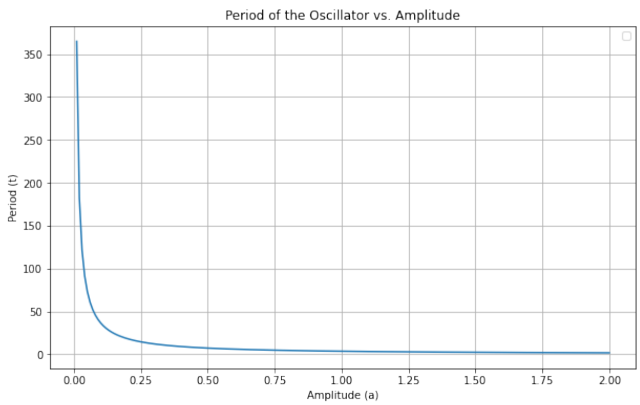

Figure 1: Period vs Amplitude

Problem 3

(a) An iterative approach (where the values of H_n are stored into memory) instead of a recursive approach was used to save time. Here is the diagram for $n = 4$.

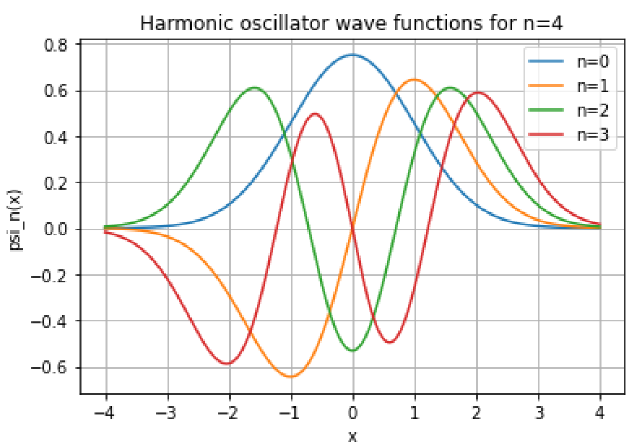

Figure 2: Harmonic Oscillation for $n = 4$

(b) Here is the diagram for $n = 30$

(c) The obtained answer from **newman_5.13.py** is 2.345207879911715.

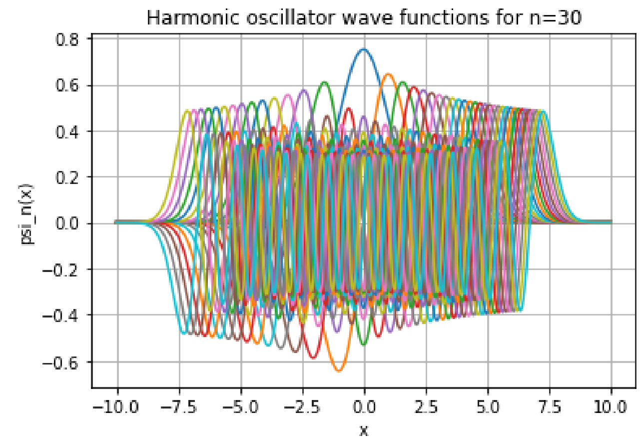

Figure 3: Harmonic Oscillation for $n = 30$

(d) Given the wave function

$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-\frac{x^2}{2}} H_n(x)$$

Squaring both sides yields

$$|\psi_n(x)|^2 = \left(\frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-\frac{x^2}{2}} H_n(x) \right)^2 = \frac{1}{2^n n! \sqrt{\pi}} e^{-x^2} H_n(x)^2$$

Simplifying ,we have

$$\begin{aligned} \langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 |\psi_n(x)|^2 \, dx \\ &= \int_{-\infty}^{\infty} x^2 \left(\frac{1}{2^n n! \sqrt{\pi}} e^{-x^2} H_n(x)^2 \right) dx \\ &= \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} x^2 H_n(x)^2 \, dx \end{aligned}$$

Since Gaussian Hermite quadrature approximates integrals of this kind,

$$\int_{-\infty}^{\infty} e^{-x^2} f(x) dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

the effective $f(x)$ is $x^2 H_n(x)^2$. The uncertainty for $n = 5$ using Gauss-Hermite quadrature and H_n is approximately 2.3452078799117144. (Results from **newman_5.13.py**)
Since Gauss Hermite calculates integrals of degrees $2n - 1$ exactly, for n nodes, it is an exact approximation.