Written Homework 2

Exercise 1

[Temperature model] When Dr. Watson picks up the loot, he realizes that the loot is an ice cube of temperature $-20°$, in which some curious object is encapsulated. In order to obtain what is inside the ice cube, Holmes and Watson put the ice cube in a room of constant temperature $T_R = 30°$. It is known in this situation, the rate of change of the temperature of the ice cube T_C satisfies

$$\frac{dT_C}{dt} = -k(T_C - T_R),$$

where k is a constant.

After one minute, the temperature of the ice cube rises to $-15°$. How long will it take for the temperature of the ice cube to become $0°$ so that it melts?

Hint: we need to figure out the constant k in order to solve this problem.

Exercise 2

Find the general solution of

1. $xy' + y = x^3$
2. $y' + ay = b$
3. $\dfrac{dr}{d\theta} = (r + e^\theta)\tan\theta$
4. $\dfrac{dy}{dx} - \dfrac{2xy}{x^2+1} = 1$

Exercise 3

Find a particular solution of

1. $xy' + 3y = \dfrac{\sin x}{x^2}, \;\; x \neq 0, \;\; y(\dfrac{\pi}{2}) = 1$
2. $y' - y = e^x, y(0) = 1$

The differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{1}$$

is called the **Bernoulli Equation**. If $n = 0$, then this is the linear first order differential equation; otherwise, we can not solve this ODE by the method of integrating factor. However, if we multiply

$$(1-n)y^{-n}$$

on both sides of (1), we get

$$(1-n)y^{-n}\frac{dy}{dx} + (1-n)P(x)y^{1-n} = (1-n)Q(x).$$

Notice that

$$\frac{d(y^{1-n})}{dx} = (1-n)y^{-n}\frac{dy}{dx},$$

Equation (1) becomes

$$\frac{d(y^{1-n})}{dx} + (1-n)P(x)y^{1-n} = (1-n)Q(x). \tag{2}$$

Now let $u = y^{n-1}$, then (2) becomes

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x) \tag{3}$$

which can be solved by the integrating factor method.

Use the above method to solve Exercise 4 and 5:

Exercise 4

Solve

1. $xy' + y = y^2 \log x$
2. $y' + xy = \dfrac{x}{y^3}, \quad y \neq 0$

Exercise 5

Find a particular solution of $y' + \dfrac{1}{x}y = \dfrac{y^2}{x}, \quad y(0) = 1.$

Exercise 6

Solve

$$\frac{dx}{dy} + 2yx = e^{-y^2}.$$

Exercise 7

Solve for a particular solution

$$(x - \sin y)dy + (\tan y)dx = 0, \quad y(1) = \frac{\pi}{6}$$

Hint: Look at the last exercise.