Written Homework 1

Exercise 1

Verify that the function $y(x) = x + Cx^{-2}$ is a solution of the differential equation $xy' + 2y = 3x$. In addition, find the value of $C \in \mathbb{R}$ so that we have $y(1) = 5$.

Answer We first find $y'(x)$ by differentiating $y(x) = x + Cx^{-2}$:

$$y'(x) = 1 + C(-2)x^{-3}$$

Next we substitute y and y' in the Left Hand Side the differential equation $xy' + 2y = 3x$:

$$\begin{aligned} LHS = xy' + 2y &= x\left(1 + C(-2)x^{-3}\right) + 2\left(x + Cx^{-2}\right) \\ &= x + xC(-2)x^{-3} + 2x + 2Cx^{-2} = x + 2x = 3x = RHS \end{aligned}$$

Therefore, $y(x) = x + Cx^{-2}$ is a solution to the ODE $xy' + 2y = 3x$. To find the particular solution that satisfies $y(1) = 5$, we plug in $x = 1$ and $y = 5$ into $y(x) = x + Cx^{-2}$:

$$5 = 1 + C(1)^{-2} \Rightarrow C = 4$$

Exercise 2

Solve the following IVP:

1. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x+2}}, \quad y(2) = -1$
2. $\dfrac{dy}{dx} = x\sqrt{x^2+9}, \quad y(-4) = 0$
3. $\dfrac{dy}{dx} = xe^{-x}, \ \ y(0) = 1$

Answer You can also solve this by directly integrating $\int y'dx$, but I accidently used separation of variables.

1.

$$\begin{aligned} dy &= \frac{1}{\sqrt{x+2}}dx \\ \Longrightarrow \int dy &= \int \frac{1}{\sqrt{x+2}}dx \\ \Longrightarrow \ y &= 2(x+2)^{1/2} + C \end{aligned}$$

Plug in $y(2) = -1$, we get

$$y(2) = 2(4)^{1/2} + C = -1 \implies C = -5$$

so a particular solution is

$$y = 2(x+2)^{1/2} - 5$$

2.

$$\begin{aligned} dy &= x\sqrt{x^2+9}dx \\ \implies \int dy &= \frac{1}{2}\int \sqrt{x^2+9}d(x^2+9)dx \\ \implies y &= \frac{1}{3}(x^2+9)^{3/2}+C \end{aligned}$$

Plug in the initial value condition we get

$$y = \frac{1}{3}(x^2+9)^{3/2} - \frac{125}{3}$$

3.

$$\begin{aligned} dy &= xe^{-x}dx \\ \implies \int dy &= \int xe^{-x}dx \\ \implies y &= -xe^{-x} - \int -e^{-x}dx \\ \implies y &= -xe^{-x} - e^{-x} + C \end{aligned}$$

Plug in the initial value condition we get

$$y = -xe^{-x} - e^{-x} + 1$$

$\boxed{\textbf{Exercise 3}}$

Draw the solution curves of

$$y' = \frac{-x}{\sqrt{1-x^2}}$$

for the initial conditions $y(0) = 1, y(0) = 2, y(0) = 3$. Draw the curves in the same $xy-$plane.

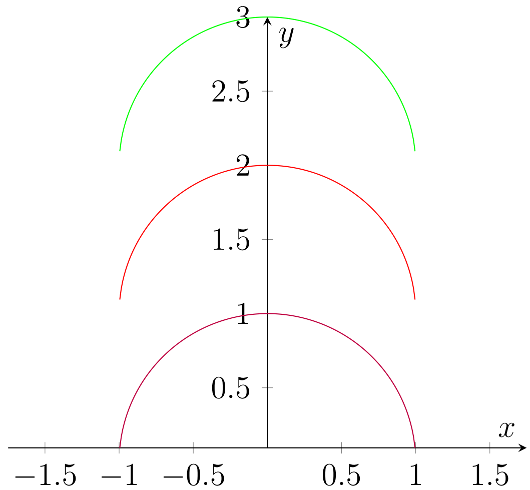
Answer First we solve for a general solution:

$$\begin{aligned} y &= \int y'dx = \frac{1}{2}\int (1-x^2)^{-1/2}d(1-x^2) \\ \implies y &= (1-x^2)^{1/2} + C \end{aligned}$$

Moving C to the LHS and then squaring the both sides, we get

$$(y-C)^2 = 1 - x^2 \implies (y-C)^2 + x^2 = 1$$

This is a radius 1 circle centered at $(C, 0)$, so the graph of y is part of this circle. Notice that since $(1-x^2)^{1/2} \geq 0$, we have $y \geq C$, and therefore y is the upper half of the circle. Plugging in the initial conditions to the general solution of y, we get $C = 0, 1, 2$. So the graph looks like



Exercise 4

Find the function $y = f(x)$ that satisfies the IVP

$$y' = (2 + 5x)e^{\frac{1}{3}x}, \;\; y(0) = 5.$$

Answer Let us find the general solution first by taking the anti-derivative of y':

$$y = \int y' \, dx = \int (2 + 5x)e^{\frac{1}{3}x} \, dx = \int 2e^{\frac{1}{3}x} \, dx + \int 5xe^{\frac{1}{3}x} \, dx + C$$

The first anti-derivative can be computed as:

$$\int 2e^{\frac{1}{3}x} \, dx = \frac{2}{1/3} e^{\frac{1}{3}x} = 6e^{\frac{1}{3}x}$$

The second anti-derivative can be computed using integration by parts with $u = 5x$ and $dv = e^{\frac{1}{3}x} \, dx$, which means $du = 5 \, dx$ and $v = \frac{e^{\frac{1}{3}x}}{\frac{1}{3}}$, as follows:

$$\int 5xe^{\frac{1}{3}x} \, dx = 5x\frac{e^{\frac{1}{3}x}}{\frac{1}{3}} - \int 5\frac{e^{\frac{1}{3}x}}{\frac{1}{3}} \, dx = 5x\frac{e^{\frac{1}{3}x}}{\frac{1}{3}} - 5\frac{e^{\frac{1}{3}x}}{\frac{1}{3^2}} = 15xe^{\frac{1}{3}x} - 45e^{\frac{1}{3}x}$$

Therefore, $y = 6e^{\frac{1}{3}x} + 15xe^{\frac{1}{3}x} - 45e^{\frac{1}{3}x} = 15xe^{\frac{1}{3}x} - 39e^{\frac{1}{3}x} + C$. We want a particular function satisfying $y(0) = 5$ from this family of solutions. Substituting $x = 0$ and $y = 5$ we get

$$5 = 15(0)e^{\frac{1}{3}0} - 39e^{\frac{1}{3}0} + C \Rightarrow 5 = 0 - 39 + C \Rightarrow C = 44$$

This gives us that the solution to the IVP $y' = (2 + 5x)e^{\frac{1}{3}x}, \; y(0) = 5$ is $y = 15xe^{\frac{1}{3}x} - 39e^{\frac{1}{3}x} + 44$.

Exercise 5

Right before Sherlock Holmes and Watson catch the robber, the robber throws away the loot with a launch speed $v_0 = 30m/s$ and a launch angle $\theta = \pi/3$. The position of the loot at time t is given by the Newtonian dynamics

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = -g$$

where g is the gravitational acceleration and x, y are the horizontal and vertical displacement of the loot, respectively. Also recall that the relation between the initial velocities in the horizontal and vertical directions are given by

$$v_x = v_0 \cos(\theta), \qquad v_y = v_0 \sin(\theta).$$

1. At what time does the loot reach its maximum height? What is the maximum height?

2. Watson, who is at the same position as the robber, is trying to catch the loot on a bike. What is the acceleration of Watson's bike if he wants to catch the loot when it falls to the ground, assuming the relation between the distance s and the acceleration a is given by
$$\frac{d^2 s}{dt^2} = a.$$

Answer In order solve these questions, we first need to find the expression of x and y. We have

$$x' = C_1$$

and

$$y' = -gt + C_2.$$

We also have $x'(0) = v_x = 30 \cdot \cos(\pi/3) = 15, y'(0) = v_y = 30 \cdot \sin(\pi/3) = 15\sqrt{3}$, so we have paticular solutions

$$x' = 15$$

and

$$y' = -gt + 15\sqrt{3}.$$

Integrate again we get

$$x = 15t + C_1$$

and

$$y = -\frac{1}{2}gt^2 + 15\sqrt{3}t + C_2.$$

We have $x(0) = 0$ and $y(0) = 0$ (for simplicity I assume the height of the robber is 0, but you can assume otherwise and change your answer accordingly) at the beginning and therefore

$$x = 15t$$

and

$$y = -\frac{1}{2}gt^2 + 15\sqrt{3}t.$$

1. When the loot reaches the maximum height, $y' = -gt + 15\sqrt{3} = 0$ and therefore $t = \dfrac{15\sqrt{3}}{g}$. So the maximum height is $y(\dfrac{15\sqrt{3}}{g}) = \dfrac{675}{2g}$.

2. As we have seen in Quiz 1, we can express Watson's displacement as

$$s = \frac{1}{2}at^2$$

When the loot falls to the ground,

$$y = -\frac{1}{2}gt^2 + 15\sqrt{3}t = 0 \implies t = 0, \frac{30\sqrt{3}}{g}.$$

So after $\dfrac{30\sqrt{3}}{g}$ seconds,

$$x(\frac{30\sqrt{3}}{g}) = \frac{450\sqrt{3}}{g},$$

and

$$s(\frac{30\sqrt{3}}{g}) = \frac{2700}{2g^2}a.$$

Therefore if Watson were to catch the loot when it falls to the ground,

$$\begin{aligned}\frac{450\sqrt{3}}{g} &= \frac{2700}{2g}a \\ \implies a &= \frac{\sqrt{3}}{3}g\end{aligned}$$

$\boxed{\textbf{Exercise 6}}$

Use separation of variables to find the general solutions (implicit if necessary, explicit if it can be done).

1. $y' = \dfrac{1+\sqrt{x}}{1+\sqrt{y}}$
2. $y' = \dfrac{(x-1)y^5}{x^2(2y^3-y)}$
3. $y' = 1 + x + y + xy$

Answer

1.

$$\begin{aligned}(1+\sqrt{y})dy &= (1+\sqrt{x})dx \\ \int(1+\sqrt{y})dy &= \int(1+\sqrt{x})dx \\ y + \frac{2}{3}y^{3/2} &= x + \frac{2}{3}x^{3/2} + C\end{aligned}$$

2.

$$\begin{aligned}\frac{2y^3-y}{y^5}dy &= \frac{x-1}{x^2}dx \\ \int\frac{2}{y^2}dy - \int\frac{1}{y^4}dy &= \int\frac{1}{x}dx - \int\frac{1}{x^2}dx \\ \implies \frac{-2}{y} + \frac{1}{3y^3} &= \ln|x| + \frac{1}{x} + C\end{aligned}$$

3.

$$\begin{aligned}
y' &= 1 + xy(1+x) \\
\implies \frac{dy}{dx} &= (1+x)(1+y) \\
\implies \frac{1}{1+y}dy &= (1+x)dx \\
\implies \ln(1+y) &= x + \frac{1}{2}x^2 + C \\
\implies 1+y &= e^{x+\frac{1}{2}x^2+C} \\
\implies y &= Ae^{x+\frac{1}{2}x^2} - 1
\end{aligned}$$

$\boxed{\textbf{Exercise 7}}$

Determine if the following IVP has a unique solution:

1. $y' = \sqrt[3]{y}, \ \ y(0) = 1$
2. $y' = \sqrt[3]{y}, \ \ y(0) = 0$
3. $y' = \ln(1+y^2), \ \ y(0) = 0$

Answer Recall that by the theorem of existence and uniqueness of solution, we need to identify $y' = f(x,y)$ and calculate the partial derivative $D_y(f(x,y))$:

1. $f(x,y) = \sqrt[3]{y}, \quad D_y(f) = \dfrac{1}{\sqrt[3]{y^2}}$. Since both f and $D_y(f)$ are continuous on $y = 1$, we can apply the theorem and conclude that there exists a unique solution.
2. On the other hand, $D_y(f)$ is not continuous at $y = 0$. Therefore we can not apply the theorem. If we try to solve this ODE by separation of variables, we get a particular solution
$$y = (\frac{3}{2}x)^{2/3}.$$
However, notice that $y = 0$ is also a solution to this ODE. So we see that even though solution exists, it is not unique.
3. $f(x,y) = \ln(1+y^2), D_y(f) = \dfrac{2y}{1+y^2}$, we see that both f and $D_y(f)$ are continuous at $y = 0$, we can apply the theorem and conclude that there exists a unique solution.