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Multi-period stochastic optimization models for dynamic asset allocation

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Abstract

Institutional investors manage their strategic asset mix over time to achieve favorable returns subject to various uncertainties, policy and legal constraints, and other requirements. One may use a multi-period portfolio optimization model in order to determine an optimal asset mix. The concept of scenarios is typically employed for modeling random parameters in a multi-period stochastic programming model, and scenarios are constructed via a tree structure. Recently, an alternative stochastic programming model with simulated paths was proposed by Hibiki [Hibiki, N., 2001b. A hybrid simulation/tree multi-period stochastic programming model for optimal asset allocation. In: Takahashi, H. (Ed.), The Japanese Association of Financial Econometrics and Engineering. JAFEE Journal 89–119 (in Japanese); Hibiki, N., 2003. A hybrid simulation/tree stochastic optimization model for dynamic asset allocation. In: Scherer, B. (Ed.), Asset and Liability Management Tools: A Handbook for Best Practice, Risk Books, pp. 269–294], and it is called a hybrid model. The advantage of the simulated path structure compared to the tree structure is to give a better accuracy to describe uncertainties of asset returns. In this paper, we compare the two types of multi-period stochastic optimization models, and clarify that the hybrid model can evaluate and control risk better than the scenario tree model using some numerical tests. According to the numerical results, an efficient frontier of the hybrid model with the fixed-proportion strategy dominates that of the scenario tree model when we evaluate them on simulated paths. Moreover, optimal solutions of the hybrid model are more appropriate than those of the scenario tree model.

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1. Introduction

Rational investors maximize the expected utility of return from their investment portfolio, or minimize their risk exposure of return, subject to their required expected return. They must decide on their optimal portfolio in securities in order to meet their satisfaction. This paper discusses optimal dynamic investment policies for investors, who make an investment decision in each asset category over time. This problem is called "dynamic asset allocation".

Asset allocation decisions are critical for investors with diversified portfolios. Institutional investors must manage their strategic asset mix over time to achieve favorable returns, in presence of uncertainties and subject to various legal constraints, policies, and other requirements. A multi-period portfolio optimization model can be used in order to determine an optimal asset mix.

It is critical for stochastic modeling to handle uncertainties and investment decisions appropriately. The decisions have to be independent from knowledge of actual paths that will occur. Thus, we must define a set of decision variables and a set of constraints to prevent an optimization model from being solved by anticipating events in the future. In addition, we need a sufficient number of paths to get a better accuracy with respect to the future possible events.

The concept of scenarios is typically employed for modeling random parameters in multi-period stochastic programming models. Scenarios are constructed via a tree structure (see Mulvey and Ziemba (1995, 1998) for a detailed discussion). The model is based on the expansion of the decision space, taking into account a conditional nature of the scenario tree. Conditional decisions are made at each node, subject to the modeling constraints. To ensure that the constructed representative set of scenarios covers the set of possibilities to a sufficient degree, the numbers of decision variables and constraints in the scenario tree may grow exponentially. This model is called a scenario tree model.

Recently, an alternative stochastic programming model using simulated paths was proposed by Hibiki (2001b). Hibiki (2003) developed a general formulation for several investment strategies, and highlighted its characteristics and properties by using some numerical tests. Scenarios are constructed via a simulated path structure. We can generate sample paths associated with asset returns using a Monte Carlo simulation method. The advantage of the simulated path structure compared to the tree structure is to give a better accuracy to describe uncertainties of asset returns. The model not only describes the uncertainties on the simulated path structure but also makes conditional decisions on the tree structure. Therefore, it is called a "hybrid" model.[1] It can be easily implemented and efficiently solved using a standard mathematical programming software package.

The hybrid model is developed to overcome the shortcoming of the scenario tree model associated with uncertainties. Therefore, it is important to answer the question how quantitatively the hybrid model is better than the scenario tree model, which was not shown in the previous papers (Hibiki, 2001b, 2003). In this paper, we compare the two types of multi-period stochastic optimization models, and clarify that the hybrid model can evaluate and control risk better than the scenario tree model by using some numerical tests.

We need the following developments to achieve our goal. At first, we develop an algorithm to solve the hybrid model with a fixed-proportion strategy, which is formulated as a non-convex program. This is because the two kinds of models should be compared using the same strategy. Secondly, we propose the procedure of comparing them in the simulated path framework.

The paper is organized as follows. Section 2 presents the concept and formulations of the two kinds of models, and develops the iterative algorithm to solve the hybrid model with the fixed-proportion strategy. In Section 3, we demonstrate a scenario generation process and a procedure of generating an extended decision tree, and explain how to generate a scenario tree from simulated paths. Section 4 presents some numerical tests for various cases to compare the scenario tree model with the hybrid model. Section 5 provides some concluding remarks and outlines our future research.

2. Multi-period stochastic programming models

2.1. Modeling for uncertainties and conditional decisions

Scenarios of asset returns are typically constructed via a tree structure in the multi-period stochastic programming problem as in the left-hand-side of Fig. 1. Meanwhile, 12 simulated paths over three periods give another description of scenarios shown as in the middle of Fig. 1.

Hibiki (2001b) developed the hybrid model in a multi-period optimization framework. Discrete values of asset returns are generated by Monte Carlo simulation to describe uncertainties more accurately than would the scenario tree as in the left-hand-side of Fig. 1. However, if a decision is made on the associated path, the model is solved anticipating the event in the future. Therefore, the rule that the same investment decision is made in similar states is defined to satisfy the non-anticipativity

[1] Hibiki (2000, 2001a) developed a simulated path model. The model also requires simulated paths to have the accuracy of uncertainties, but it cannot make conditional decisions. The hybrid model is allowed to expand the decision space and to make conditional decisions as in the scenario tree model. The simulated path model is a special version of the hybrid model.

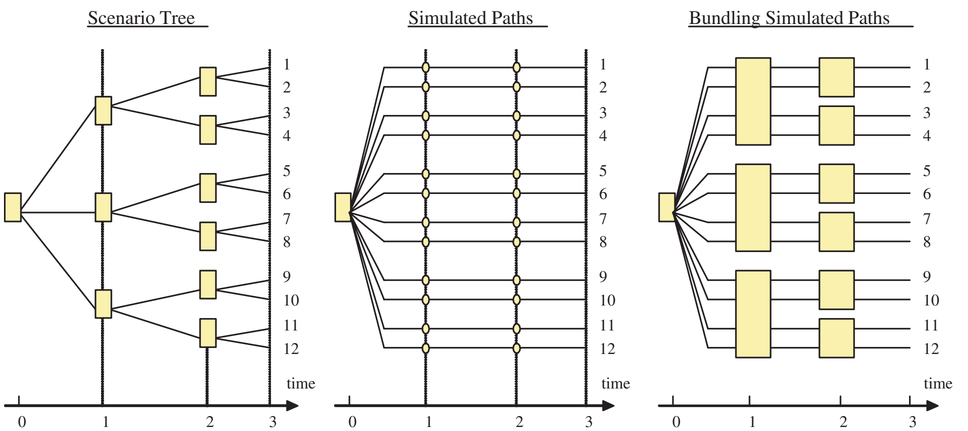

Fig. 1. A scenario tree and simulated paths.

condition[2] in the simulated path. An investment strategy that the same decision is made in similar states is called a "fixed strategy " in the simulated path approach. The word "fixed" does not mean "buy and hold (strategy)", or "constant rebalance (strategy)". Similar states at each time are bundled using one of some procedures shown in Section 3.2. Several bundles are made on simulated paths at each time to have the fixed strategy (decision rule) for risky assets.[3] An example of bundles, or "fixed-decision nodes", is shown as in the right-hand-side of Fig. 1. The important idea is the generation of decision nodes via a tree structure, while scenarios are generated via a simulated path structure. We call it an "extended decision tree". The tree in Fig. 1 is called a "3–2" branching tree, because it has three bundles at time 1, and paths through any bundle at time 1 pass two bundles at time 2.

2.2. Preparation

We invest in n risky assets and cash. The investment is made at time 0 (present), and time T is the planning horizon.

2.2.1. Notations

Some notations are used only in one model, and some are used in the both models. We attach '[Scenario]' to the explanation of notations of the scenario tree model,

[2] The condition which prevents an optimization model from being solved by anticipating the future is called "non-anticipativity condition".

[3] The path-dependent decisions can be made to cash variables, because cash return is risk-free at each time when we invest in assets.

'[Hybrid]' to the hybrid model, and '[Both]' to the both models. Notations are as follows.

(1) Sets

S_t [Scenario] set of states at time t ($s \in S_t$)
[Hybrid] set of fixed-decision nodes at time t ($s \in S_t$)

V_t^s [Hybrid] set of paths passing any fixed-decision node s at time t ($i \in V_t^s$)

(2) Parameters

p^s [Scenario] probability of scenario s at the planning horizon[4]

I [Hybrid] number of simulated paths

ρ_{j0} [Both] price of risky asset j at time 0 ($j = 1, \ldots, n$)

ρ_{jt}^s [Scenario] price of risky asset j of state s at time t ($j = 1, \ldots, n$; $t = 1, \ldots, T$; $s \in S_t$)

$\rho_{jt}^{(i)}$ [Hybrid] price of risky asset j of path i at time t ($j = 1, \ldots, n$; $t = 1, \ldots, T$; $i = 1, \ldots, I$)

r_0 [Both] interest rate in period 1 (the rate at time 0 is used)

$r_{t-1}^{s'}$ [Scenario] interest rate in period t, which is the rate of the predecessor state s' at time $t-1$ (of state s at time t) ($t = 2, \ldots, T$; $s' \in S_{t-1}$)

$r_{t-1}^{(i)}$ [Hybrid] interest rate in period t (the rate of path i at time $t-1$ is used) ($t = 2, \ldots, T$; $i = 1, \ldots, I$)

W_0 [Both] initial wealth

W_G [Both] target terminal wealth

γ [Both] risk aversion coefficient

(3) Decision variables

z_{j0} [Both] investment unit for asset j and time 0 ($j = 1, \ldots, n$)

z_{jt}^s [Scenario] investment unit for asset j, time t, and state s
[Hybrid] base investment unit[5] for asset j, time t, and node s ($j = 1, \ldots, n$; $t = 1, \ldots, T-1$; $s \in S_t$)

v_0 [Both] cash at time 0

v_t^s [Scenario] cash of state s at time t ($t = 1, \ldots, T-1$; $s \in S_t$)

$v_t^{(i)}$ [Hybrid] cash of path i at time t ($t = 1, \ldots, T-1$; $i = 1, \ldots, I$)

q^s [Scenario] shortfall below target terminal wealth of scenario s ($s \in S_T$)

$q^{(i)}$ [Hybrid] shortfall below target terminal wealth of path i ($i = 1, \ldots, I$)

The decision variables for all assets (including cash) are state-dependent for the scenario tree model. On the other hand, the decision variables for risky assets are node-dependent while cash variables are path-dependent for the hybrid model.

[4] The scenario s corresponds to the state s at the planning horizon.

[5] The base investment unit is defined as the control variable of the investment unit. Details are shown in Section 2.4.1.

2.2.2. Objective function

The objective is the maximization of the function which is defined using two kinds of measures; the expected terminal wealth $\mathrm{E}[W_T]$ as return measure, and the first-order lower partial moment LPM_1 of terminal wealth as risk measure (Harlow, 1991)

$$\text{Objective function} = \mathrm{E}[W_T] - \gamma \cdot \mathrm{LPM}_1 \tag{1}$$

The lower partial moment is one of downside risk measures, and expresses tail risk of the relevant distribution of wealth below target.

$\mathrm{E}[W_T]$ and LPM_1 for the both models are calculated as follows:

$$\text{[Scenario]: } \mathrm{E}[W_T] = \sum_{s \in S_T} p^s W_T^s, \quad \mathrm{LPM}_1 = \sum_{s \in S_T} p^s |W_T^s - W_G|_-, \tag{2}$$

$$\text{[Hybrid]: } \mathrm{E}[W_T] = \frac{1}{I}\sum_{i=1}^{I} W_T^{(i)}, \quad \mathrm{LPM}_1 = \frac{1}{I}\sum_{i=1}^{I} |W_T^{(i)} - W_G|_-, \tag{3}$$

where $|a|_- = \max(-a, 0)$. W_T^s is terminal wealth of scenario s in the scenario tree model, and $W_T^{(i)}$ is terminal wealth of path i in the hybrid model.

2.3. The scenario tree model

Investment units are mostly used as decision variables in the general literatures to formulate the scenario tree model, and the typical formulation is follows:[6]

$$\text{Maximize} \quad \sum_{s \in S_T} p^s W_T^s - \gamma \left(\sum_{s \in S_T} p^s q^s \right) \tag{4}$$

$$\text{subject to} \quad \sum_{j=1}^{n} \rho_{j0} z_{j0} + v_0 = W_0, \tag{5}$$

$$\sum_{j=1}^{n} \rho_{j1}^s z_{j1}^s + v_1^s = \sum_{j=1}^{n} \rho_{j1}^s z_{j0} + (1 + r_0) v_0 \quad (s \in S_1), \tag{6}$$

$$\sum_{j=1}^{n} \rho_{jt}^s z_{jt}^s + v_t^s = \sum_{j=1}^{n} \rho_{jt}^s z_{j,t-1}^{s'} + (1 + r_{t-1}^{s'}) v_{t-1}^{s'} \quad (t = 2, \ldots, T-1;\ s \in S_t), \tag{7}$$

$$W_T^s = \sum_{j=1}^{n} \rho_{jT}^s z_{j,T-1}^{s'} + (1 + r_{T-1}^{s'}) v_{T-1}^{s'} \quad (s \in S_T), \tag{8}$$

$$W_T^s + q^s \geqslant W_G \quad (s \in S_T), \tag{9}$$

$$z_{j0} \geqslant 0 \quad (j = 1, \ldots, n), \tag{10}$$

$$z_{jt}^s \geqslant 0 \quad (j = 1, \ldots, n;\ t = 1, \ldots, T-1;\ s \in S_t), \tag{11}$$

$$v_0 \geqslant 0; \quad v_t^s \geqslant 0 \quad (t = 1, \ldots, T-1;\ s \in S_t), \tag{12}$$

$$q^s \geqslant 0 \quad (s \in S_T). \tag{13}$$

[6] Other constraints such as boundary conditions, policy and legal constraints, and other requirements can be easily added.

Constraint (5) is a budget constraint at time 0. Constraints (6) and (7) are cash flow constraints at time t, and the values of both sides show wealth of state s at time t. Constraint (8) shows terminal wealth. It is assured that the second term of the objective function (except γ) represents the LPM_1 together with Constraint (9). Constraints (11)–(13) are non-negativity constraints.

We have an alternative formulation where investment proportions are used as decision variables. However, the formulation contains non-linear constraints. It can be transformed into the equivalent formulation where investment units are used as decision variables. It leads to equivalent optimal solutions, while the typical formulation with investment units does not contain non-linear constraints, and hence contains only linear constraints. This is the reason investment units are used.

2.4. The hybrid model

2.4.1. Investment strategies with investment unit functions

To help the reader to understand the fixed strategy used in the simulated path approach, we explain how investments are determined on each simulated path by using the 3–2 branching tree in Fig. 1.

Table 1 shows a typical example of four kinds of strategies; fixed-unit strategy, fixed-proportion strategy, buy-and-hold strategy, and constant rebalance strategy. The values in the table show an example of investments for one of risky assets.

The word 'fixed' means that investments must have the same value for all simulated paths passing any node (bundle). For example, we explain the fixed-unit strategy. At time 1, 40 units are invested for path 1–4 through the first node, respectively. Similarly, 25 units are invested for path 5–8 through the second node, and 35 units are invested for path 9–12 through the third node. Investment unit for buy-and-hold strategy is fixed over the period. Specifically, 30 units are invested for all paths and all periods in this example. But we do not call it the fixed strategy. The words 'fixed strategy' is newly defined in the simulated path approach.

Only one value such as an investment proportion or unit is the same for all paths through any node under the fixed strategy. For example, the fixed-proportion strategy requires that investment proportions have the same value, but they do not have the same unit for all paths passing any node. We formulate the model using the associated decision variables with the fixed strategy. Moreover, we introduce "investment unit function", $h^{(i)}(z_{jt}^s)$, which shows investment unit on the path i, for the purpose of the general formulation to the fixed strategy. To show that investment units are path-dependent while decision variables used to describe the investment units are node-dependent, the function is defined as follows:[7]

$$h^{(i)}(z_{jt}^s) = a_{jt}^{(i)} z_{jt}^s, \tag{14}$$

[7] The superscript (i) attaches to the function to describe explicitly that investment units have different values for all paths.

Table 1
A typical example of four kinds of strategies

Strategy	Fixed-unit		Fixed-proportion		Buy-and-hold		Constant rebalance	
$t=0$	30 units		30%		30 units		30%	
State (path)	$t=1$	$t=2$	$t=1$	$t=2$	$t=1$	$t=2$	$t=1$	$t=2$
1	40 units	50 units	40%	50%	30 units	30 units	30%	30%
2	40 units	50 units	40%	50%	30 units	30 units	30%	30%
3	40 units	55 units	40%	55%	30 units	30 units	30%	30%
4	40 units	55 units	40%	55%	30 units	30 units	30%	30%
5	25 units	20 units	25%	20%	30 units	30 units	30%	30%
6	25 units	20 units	25%	20%	30 units	30 units	30%	30%
7	25 units	30 units	25%	30%	30 units	30 units	30%	30%
8	25 units	30 units	25%	30%	30 units	30 units	30%	30%
9	35 units	40 units	35%	40%	30 units	30 units	30%	30%
10	35 units	40 units	35%	40%	30 units	30 units	30%	30%
11	35 units	20 units	35%	20%	30 units	30 units	30%	30%
12	35 units	20 units	35%	20%	30 units	30 units	30%	30%

where $a_{jt}^{(i)}$ is an investment unit parameter that must be independent on the rate of returns of path i after time t to keep non-anticipativity condition. The investment unit function can show various investment strategies. We consider two kinds of investment strategies.

(1) Fixed-unit strategy: $h^{(i)}(z_{jt}^{s}) = z_{jt}^{s}$

Investment units have the same value for all paths passing any node for risky assets. Cash has the different value for any path.

(2) Fixed-proportion strategy: $h^{(i)}(z_{jt}^{s}) = \left(\frac{W_t^{(i)}}{\rho_{jt}^{(i)}}\right) z_{jt}^{s}$

Investment proportions have the same value for all paths passing any node for any asset.

The function of the fixed-unit strategy is linear, while the function of the fixed-proportion strategy is non-convex because $W_t^{(i)}$ is a function of decision variables.

2.4.2. Formulation

We show a typical formulation, which structure is the same as that of the scenario tree model[8]

[8] Constraint (16) is a budget constraint at time 0. Constraints (17) and (18) are cash flow constraints at time t, and the values of both sides show wealth of path i at time t. Constraint (19) shows terminal wealth. It is assured that the second term of the objective function (except γ) represents the LPM_1 together with Constraint (20). Constraints (22)–(24) are non-negativity constraints.

$$\text{Maximize} \quad \frac{1}{I}\sum_{i=1}^{I} W_T^{(i)} - \gamma\left(\frac{1}{I}\sum_{i=1}^{I} q^{(i)}\right) \tag{15}$$

$$\text{subject to} \quad \sum_{j=1}^{n} \rho_{j0} z_{j0} + v_0 = W_0, \tag{16}$$

$$\sum_{j=1}^{n} \rho_{j1}^{(i)} z_{j0} + (1+r_0)v_0 = \sum_{j=1}^{n} \rho_{j1}^{(i)} h^{(i)}(z_{j1}^s) + v_1^{(i)} \quad (s \in S_1;\ i \in V_1^s), \tag{17}$$

$$\sum_{j=1}^{n} \rho_{jt}^{(i)} h^{(i)}(z_{j,t-1}^{s'}) + (1+r_{t-1}^{(i)})v_{t-1}^{(i)} = \sum_{j=1}^{n} \rho_{jt}^{(i)} h^{(i)}(z_{jt}^s) + v_t^{(i)}$$
$$(t = 2,\ldots,T-1;\ s \in S_t;\ i \in V_t^s), \tag{18}$$

$$W_T^{(i)} = \sum_{j=1}^{n} \rho_{jT}^{(i)} h^{(i)}(z_{j,T-1}^{s'}) + (1+r_{T-1}^{(i)})v_{T-1}^{(i)} \quad (s' \in S_{T-1};\ i \in V_{T-1}^{s'}), \tag{19}$$

$$W_T^{(i)} + q^{(i)} \geqslant W_G \quad (i = 1,\ldots,I), \tag{20}$$

$$z_{j0} \geqslant 0 \quad (j = 1,\ldots,n), \tag{21}$$

$$z_{jt}^s \geqslant 0 \quad (j = 1,\ldots,n;\ t = 1,\ldots,T-1;\ s \in S_t), \tag{22}$$

$$v_0 \geqslant 0; \quad v_t^{(i)} \geqslant 0 \quad (t = 1,\ldots,T-1;\ i = 1,\ldots,I), \tag{23}$$

$$q^{(i)} \geqslant 0 \quad (i = 1,\ldots,I). \tag{24}$$

If we select the strategy which has a linear investment unit function such as the fixed-unit strategy, we can formulate as a linear programming problem, and solve a large-scale problem easily in practical use.

2.5. Iterative algorithm to solve the hybrid model with the fixed-proportion strategy

In Section 4, we compare the hybrid model with the scenario tree model.[9] Because the hybrid model with the fixed-proportion strategy is a large-scale problem with numerous, non-linear, and non-convex constraints, it is difficult to solve the problem in practical use. More specifically, the solver does not deliver any solution. We attempt to develop an algorithm to solve it approximately[10] to compare the two models. The basic idea is as follows.

Suppose that we derive the optimal solutions of the hybrid model with the fixed-proportion strategy, and calculate the wealth of path i at time t, $W_t^{(i)*}$. If we set up $h^{(i)}(z_{jt}^s) = \left(\frac{W_t^{(i)*}}{\rho_{jt}^{(i)}}\right) z_{jt}^s$ as the investment unit function, and solve the problem, the same solutions are supposed to be obtained. This would only be true if the global solution

[9] Our numerical tests for the hybrid model show that the efficient frontier of the fixed-proportion strategy dominates the efficient frontier of the fixed-unit strategy [see Hibiki (2003) for details].

[10] In this paper, we could not develop an algorithm to derive a global optimal solution. This is our future research.

could be computed. However, using this characteristics we develop the iterative algorithm to solve it approximately. The algorithm has three steps.

Step 1. We solve a problem with the fixed-unit strategy and calculate wealth of path i at time t, $W^{(i)*}_{t(0)}$. Let Obj_0 denote the objective function value, and set $k = 1$.

Step 2. We set up $h^{(i)}(z^s_{jt}) = \left(\frac{W^{(i)*}_{t(k-1)}}{\rho^{(i)}_{jt}}\right) z^s_{jt}$ as the investment unit function at the kth iteration, and solve the problem. We calculate wealth of path i at time t, $W^{(i)*}_{t(k)}$, and the objective function value Obj_k.

Step 3. Stop if a value $Obj_k - Obj_{k-1}$ is lower than a tolerance. Otherwise, set $k \leftarrow k+1$, and return to Step 2.

The algorithm does not guarantee to derive the global optimal solutions for the fixed-proportion strategy. This algorithm is a heuristic one, and any solution derived may be locally optimal. However, it is expected that the solution derived is not so bad, but rather close to the global optimal solution because the efficient frontier derived by the algorithm dominates the efficient frontier by the fixed-unit strategy according to our numerical examples in Section 4. It is more useful in practical use that we have the ability to derive a solution by solving linear programming problems successively rather than the solver does not deliver any solution to a large-scale problem.

We evaluate the heuristic algorithm using two kinds of values; the objective function value and the optimal investment proportions. At first, we define the following improvement rate IR(k) to evaluate the algorithm

$$\mathrm{IR}(k) = \frac{Obj_k - Obj_{k-1}}{Obj_5 - Obj_0} \quad (k = 1, \ldots, 5),$$

where Obj_k is the objective function value of the kth iteration, and Obj_0 is the objective function value of the fixed-unit strategy. The reason to use Obj_5 is that the procedure almost converges less than five iterations according to our experience. Table 2 shows the improvement rate of the objective function for 15 kinds of risk aversion coefficients. The problem which is solved to obtain these results is an instance of the problems considered in Section 4. For example, when $\gamma = 0.6\%$, 94.3% of $Obj_5 - Obj_0$ goes up at the first iteration, and 5.5% at the second iteration. Cumulative improvement rate by the second iteration, CIR(2) ($= \mathrm{IR}(1) + \mathrm{IR}(2)$), is more than 99% for all γ as shown in the bottom line in Table 2.

Secondly, we examine investment proportions passing any node. Table 3 shows the average of standard deviation of investment ratios on the paths through each node, $\bar{\sigma}_{x,jt(k)}$ for asset j, time t, and the kth iteration. #k shows the kth iteration.[11] The results are shown for two kinds of γ due to lack of space. The values are calculated as follows:

[11] The value of #0 shows the average of standard deviation in the case that the problem of the fixed-unit strategy is solved.

Table 2
Improvement rate of the objective function

	γ															
	10	5	4	3	2	1.5	1	0.8	0.6	0.5	0.4	0.3	0.2	0.1	0	Ave.
$k=1$	99.2%	96.1%	95.1%	95.3%	95.1%	95.6%	95.1%	95.0%	94.3%	94.4%	94.4%	94.3%	100.0%	91.2%	85.9%	94.7%
$k=2$	0.6%	3.8%	4.6%	4.6%	4.7%	4.2%	4.8%	4.9%	5.5%	5.4%	5.5%	5.6%	0.0%	8.3%	13.3%	5.1%
$k=3$	0.2%	0.1%	0.2%	0.1%	0.2%	0.2%	0.1%	0.2%	0.1%	0.1%	0.1%	0.1%	0.0%	0.4%	0.8%	0.2%
$k=4$	0.1%	−0.1%	0.0%	0.1%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
$k=5$	0.0%	0.0%	0.0%	−0.1%	0.0%	0.0%	0.0%	0.0%	0.0%	0.1%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
CIR(2)	99.8%	100.0%	99.7%	99.9%	99.8%	99.8%	99.8%	99.8%	99.8%	99.9%	99.9%	99.9%	100.0%	99.5%	99.2%	99.8%

Table 3
Average of standard deviation of investment ratios

$\gamma=1.5$		# 0	# 1	# 2	# 3	$\gamma=1$		# 0	# 1	# 2	# 3
Cash	$t=1$	1.63%	0.12%	0.01%	0.02%	Cash	$t=1$	1.70%	0.13%	0.01%	0.00%
	$t=2$	1.44%	0.04%	0.01%	0.00%		$t=2$	1.81%	0.06%	0.01%	0.00%
	$t=3$	2.23%	0.14%	0.01%	0.00%		$t=3$	2.55%	0.17%	0.01%	0.00%
Stock	$t=1$	2.03%	0.01%	0.00%	0.00%	Stock	$t=1$	0.57%	0.00%	0.02%	–
	$t=2$	1.55%	0.04%	0.02%	0.00%		$t=2$	1.95%	0.05%	0.02%	0.01%
	$t=3$	1.94%	0.12%	0.02%	0.02%		$t=3$	2.40%	0.18%	0.02%	0.01%
Bond	$t=1$	0.70%	0.04%	0.04%	0.02%	Bond	$t=1$	0.88%	0.02%	0.05%	0.01%
	$t=2$	0.92%	0.04%	0.01%	0.03%		$t=2$	1.08%	0.03%	0.02%	0.03%
	$t=3$	1.51%	0.08%	0.04%	0.03%		$t=3$	1.84%	0.09%	0.03%	0.03%
CB	$t=1$	0.97%	0.05%	0.00%	0.00%	CB	$t=1$	0.98%	0.02%	0.00%	0.05%
	$t=2$	1.30%	0.03%	0.03%	0.01%		$t=2$	1.41%	0.05%	0.02%	0.02%
	$t=3$	1.79%	0.09%	0.03%	0.04%		$t=3$	1.83%	0.10%	0.02%	0.02%

$$\bar{\sigma}_{x,jt(k)} = \frac{1}{|S_t|} \sum_{s \in S_t} \sigma^s_{x,jt(k)},$$

$$\sigma^s_{x,jt(k)} = \frac{1}{|V^s_t|} \sum_{i \in V^s_t} (x^{(i)*}_{jt(k)} - \bar{x}^{s*}_{jt(k)})^2, \quad \text{where } \bar{x}^{s*}_{jt(k)} = \frac{1}{|V^s_t|} \sum_{i \in V^s_t} x^{(i)*}_{jt(k)},$$

$$x^{(i)*}_{jt(k)} = \frac{\rho^{(i)}_{jt} \cdot h^{(i)}(z^{s*}_{jt})}{W^{(i)*}_{t(k)}} = \frac{\rho^{(i)}_{jt}\left(W^{(i)*}_{t(k-1)}/\rho^{(i)}_{jt}\right) z^{s*}_{jt}}{W^{(i)*}_{t(k)}} = \left(\frac{W^{(i)*}_{t(k-1)}}{W^{(i)*}_{t(k)}}\right) z^{s*}_{jt} \quad (i \in V^s_t),$$

where $x^{(i)*}_{jt(k)}$ is an optimal investment proportion at the kth iteration, and $\sigma^s_{x,jt(k)}$ is standard deviation of $x^{(i)*}_{jt(k)}$ in the set V^s_t. $|S_t|$ is the number of nodes at time t, and $|V^s_t|$ is the number of paths in the node.

If a value, $\sigma^s_{x,jt(k)}$, is equal to 0, investment proportions for asset j, time t, and node s have the same value. The smaller the value, the more similar investment proportions for all paths passing the node. Values of #1 become much smaller than those of #0, because investment proportions are different for all paths passing any node in the fixed-unit strategy. All of values are lower than 0.1% at the second iteration. Based on these results and because of saving computation time, we derive the solution of the fixed-proportion strategy by solving linear programming problems three times ($k = 0, 1, 2$) in this paper.

3. Scenario generation

3.1. Generating simulated paths for the hybrid model

In general, scenarios associated with asset returns are generated according to stochastic differential equations or time series models. Mulvey and Thorlacius (1998) use Towers Perrin's scenario generation system, "CAP: Link", to solve a multi-period stochastic programming problem for pension funds. A scenario system is based on a cascading set of stochastic differential equations. The Russell–Yasuda model (see Cariño et al., 1998a,b; Cariño and Ziemba, 1998) used for the ALM of casualty insurance company, generates scenarios whose returns are calculated from a factor model that incorporates dependence between periods.

The two kinds of models need different types of scenario structures. It is difficult to compare the results derived from the different scenario structures. Therefore, it is necessary to consider how to generate scenarios from the same possibility set and how to compare the results. In this paper, we choose the 'path to tree' procedure. First, we generate simulated paths, and secondly, we construct the scenario tree from the simulated paths.

Next, we need to select a return generation model appropriately because the characteristics of the model affect the optimal solutions. However, the main aim of this paper is to compare the two kinds of multi-period optimization models and to clarify the difference between them. Therefore, we use the following simple procedure with the statistics associated with asset returns (expected rate of return, standard deviation and correlation matrix of rate of return) to generate scenarios of rates of returns of n risky assets and call rate.[12]

A rate of return $\mu_{jt}^{(i)}$ is generated as follows, where asset 0 ($j = 0$) is assigned to call loan.

(1) The rate of return of asset j in period t is normally distributed with mean $\bar{\mu}_{jt}$ and standard deviation σ_{jt}, and it is generated by

$$\mu_{jt}^{(i)} = \bar{\mu}_{jt} + \sigma_{jt}\varepsilon_{jt}^{(i)},$$

where $\varepsilon_{jt}^{(i)}$ is a random sample from a multi-variate standardized normal distribution.

(2) The random variable ε_{jt} ($j = 0, \ldots, n$; $t = 1, \ldots, T$) follows that

$$\varepsilon_{jt} \sim N(\mathbf{o}, \Sigma),$$

where Σ is $(n+1)T \times (n+1)T$ correlation matrix. $\mu_{0t}^{(i)}$ is a rate of change of a call rate. The call rate $r_t^{(i)}$ is calculated by

$$r_1^{(i)} = r_0 \times \left(1 + \mu_{01}^{(i)}\right),$$

$$r_t^{(i)} = r_{t-1}^{(i)} \times \left(1 + \mu_{0t}^{(i)}\right) \quad (t = 2, \ldots, T-1).$$

We illustrate the summary statistics for numerical tests in Section 4 to help the reader to understand. Random samples are generated from two kinds of summary statistic. One is the summary statistics calculated by the available market data; Nikko stock performance index (TSE 1), Nikko bond performance index, Nikko CB performance index, and call rate.

The other is the virtual statistics considering serial correlations between two different periods. It is one of the important concerns in a multi-period model to take a serial correlation of the asset price into consideration. Let c denote a parameter associated with a serial correlation. Eleven cases of different parameters are tested to examine the effect of serial correlation as in Table 4. The parameter c is also the autocorrelation of each asset itself between period t and period $t + 1$ ($t = 1, 2, 3$).

3.2. Procedure of generating the extended decision tree

We need to classify and bundle simulated paths to make conditional decisions in the hybrid model. Hibiki (2003) showed two kinds of classifying methods as follows.[13]

[12] Using the normal distribution simply for generating samples for returns may be unrealistic. We can generate samples using another models as shown in the previous literatures.

[13] See Pflug (2001) for another clustering methods for generating scenario tree.

Table 4
Eleven kinds of correlation parameters

Case	cm5	cm4	cm3	cm2	cm1	
Parameter	$c = -0.5$	$c = -0.4$	$c = -0.3$	$c = -0.2$	$c = -0.1$	
Case	cp0	cp1	cp2	cp3	cp4	cp5
Parameter	$c = 0.0$	$c = 0.1$	$c = 0.2$	$c = 0.3$	$c = 0.4$	$c = 0.5$

(1) Sequential clustering method (SQC method).
The method is applied to the data set of simulated paths over the planning period by using the well-known hierarchical clustering method in each period sequentially. Generated clusters represent the fixed-decision nodes. The method implemented is based on similarities calculated by distances between sampled return vectors.

(2) Portfolio based clustering method (PBC method).
The method is applied to a set of wealth of paths at time t which is calculated by any portfolio over the planning period. While we can use a portfolio, such as an equally weighted portfolio, an optimal portfolio derived by solving the simulated path model, we need to find an appropriate portfolio.

Because Hibiki (2003) showed the PBC method with an optimal portfolio for the simulated path model (S-PBC method) is the best method among some methods, the S-PBC method is used in this paper. This method is applied not only to bundle simulated paths but also to make a scenario tree.

3.3. Generating a scenario tree from simulated paths to compare the scenario tree model with the hybrid model

We should not compare the efficient frontiers derived from the two kinds of models, because they use different scenario structures. The following steps are proposed to generate a scenario tree based on the 'path to tree' procedure as mentioned before, and to compare the two kinds of models.

Step 1: Simulated paths are generated and bundled using the S-PBC method. Using the iterative algorithm in Section 2.5, we solve the hybrid model with the fixed-proportion strategy for several risk aversion coefficients. We calculate several expected terminal wealth and risk to illustrate an efficient frontier. This step is the standard procedure for the hybrid model.

Step 2: We generate a scenario tree from the simulated paths in Step 1, and calculate prices on the scenario tree.

Step 3: We solve the scenario tree model for several risk aversion coefficients, and derive optimal investment units. We calculate optimal investment ratios from optimal investment units.

Step 4: We apply the optimal investment ratios derived from the scenario tree model to the hybrid model, and calculate several expected terminal wealth and risk to illustrate the relationship between them.

Step 5: We compare the efficient frontier derived in Step 1 with the curve calculated in Step 4.

We explain the procedures from Step 2 to Step 4 in detail.

Step 2: Procedure of generating a scenario tree from simulated paths.

(1) Calculation of the average value.

We compute $\bar{\mu}^s_{jt}$, the average rate of return at node s from period 1 to period $T-1$

$$\bar{\mu}^s_{jt} = \frac{1}{|V^s_t|}\sum_{i\in V^s_t}\mu^{(i)}_{jt} \quad (j=1,\ldots,n;\ t=1,\ldots,T-1;\ s\in S_t),$$

where $\mu^{(i)}_{jt}$ is a rate of return for asset j, period t, and path i. A set of paths V^s_t generated in Step 1 is used.

(2) Matching of the first two moments.

$\bar{\mu}^s_{jt}$ is adjusted so that the expected value and the standard deviation of μ^s_{jt} calculated are equivalent to those of $\mu^{(i)}_{jt}$

$$\mu^s_{jt} = \left(\frac{\bar{\mu}^s_{jt} - \bar{\mu}_{jt}}{\breve{\sigma}_{jt}}\right)\times\sigma_{jt} + \bar{\mu}_{jt} \quad (j=1,\ldots,n;\ t=1,\ldots,T-1;\ s\in S_t),$$

where $\bar{\mu}_{jt}$ is the expected rate of return of $\mu^{(i)}_{jt}$, σ_{jt} is the standard deviation of rate of return of $\mu^{(i)}_{jt}$, and $\breve{\sigma}_{jt}$ is the standard deviation of $\bar{\mu}^s_{jt}$. The mean and standard deviation of the asset return in the scenario tree model match those in the hybrid model for each asset and period.[14] However, correlations between assets and serial correlations used in the scenario tree model do not match those in the hybrid model. This is because of the technical difficulties.

(3) Calculation of the asset price.

We compute ρ^s_{jt}, the asset price from the rate of return μ^s_{jt}

$$\begin{aligned}
\rho^s_{j1} &= \rho_{j0}\left(1+\mu^s_{j1}\right) \quad (j=1,\ldots,n;\ s\in S_1),\\
\rho^s_{jt} &= \rho^{s'}_{j,t-1}\left(1+\mu^s_{jt}\right) \quad (j=1,\ldots,n;\ t=2,\ldots,T-1;\ s\in S_t),\\
\rho^{(i)}_{jT} &= \rho^{s'}_{j,T-1}\left(1+\mu^{(i)}_{jT}\right) \quad (j=1,\ldots,n;\ i\in V^{s'}_{T-1};\ s'\in S_{T-1}).
\end{aligned}$$

The number of scenarios is also I in the scenario tree model. The number of paths through the node s' at time $T-1$ is $|V^{s'}_{T-1}|$ $(s'\in S_{T-1})$, which depends on the branching tree.

Step 3: Solving the scenario tree model, and calculating optimal investment proportions.

We can derive optimal investment units by solving the scenario tree model; z^*_{j0}, z^{s*}_{jt} (optimal solutions of risky asset j) and v^*_0, v^{s*}_t (optimal solutions of cash). Optimal investment proportions are computed as follows.

[14] This is a method for matching the first two moments. See Høyland et al. (2003) for a heuristic scenario generation method for matching the first four moments.

$w_{j0}^* \equiv \frac{\rho_{j0} z_{j0}^*}{W_0}$: Investment proportion of risky asset j at time 0
$c_0^* \equiv \frac{v_0^*}{W_0}$: Investment proportion of cash at time 0
$w_{jt}^{s*} \equiv \frac{\rho_{jt}^s z_{jt}^{s*}}{W_t^{s*}}$: Investment proportion of risky asset j for time t and state s
$c_t^{s*} \equiv \frac{v_t^{s*}}{W_t^{s*}}$: Investment proportion of cash for time t and state s

Step 4: Evaluation of optimal solutions of the scenario tree model on the simulated paths.

We substitute the optimal investment proportions derived from the scenario tree model for the expressions of portfolio return (Eqs. (27) and (28)) on the simulated paths. The expected terminal wealth and risk (LPM_1) are calculated as in Equations (25) and (26) using terminal wealth on simulated paths to illustrate the relationship between them

$$\text{Expected terminal wealth:} \quad \overline{W}_T \equiv \frac{1}{I} \sum_{i=1}^{I} W_T^{(i)*}, \tag{25}$$

$$\text{Risk:} \quad \text{LPM}_1 \equiv \frac{1}{I} \sum_{i=1}^{I} \max(W_G - W_T^{(i)*}, 0), \tag{26}$$

where

$$R_1^{(i)*} = \sum_{j=1}^{n} \left(1 + \mu_{j1}^{(i)}\right) w_{j0}^* + (1 + r_0) c_0^* \quad (i \in V_1^s;\ s \in S_1), \tag{27}$$

$$R_t^{(i)*} = \sum_{j=1}^{n} \left(1 + \mu_{jt}^{(i)}\right) w_{j,t-1}^{s'*} + (1 + r_{t-1}^{(i)}) c_{t-1}^{s'*} \quad (i \in V_t^s;\ t = 2, \ldots, T;\ s' \in S_{t-1}), \tag{28}$$

$$W_T^{(i)*} = \left(\prod_{t=1}^{T} R_t^{(i)*}\right) W_0 \quad (i = 1, \ldots, I). \tag{29}$$

4. Numerical tests: Comparison of the models

We report some results of numerical tests.[15] We compare the two models numerically; the hybrid model with the fixed-proportion strategy and the scenario tree model. In addition, the hybrid model with the fixed-unit strategy is also tested for the purpose of reference. Four assets (stock, bond, convertible bond(CB), and cash) are considered over four periods. The number of scenarios (simulated paths) is

[15] All of the problems are solved using NUOPT (Ver. 5.1.0a) – mathematical programming software package developed by Mathematical System, Inc. – on Windows 2000 personal computer which has 1.8 GHz CPU and 768 MB memory.

10,000. The number of constraints except non-negativity constraints is about 50,000, and the number of decision variables is also about 50,000. The size of a branching tree depends on the case.

Initial prices of stock, bond, and CB are assumed to be 1 without loss of generality. The initial call rate is 0.44%. The initial wealth is 100 million Japanese yen, and the target terminal wealth is also 100 million Japanese yen.

We have performed four kinds of numerical tests.

Case A1. Basic results for the 5–4–3 branching tree using statistics of historical data.

Case B1. Basic results for the 5–4–3 branching tree using virtual statistics considering various serial correlations.

Case A2. Comparison of the results for various numbers of *N*–*N*–*N* branching trees ($N = 2, 3, \ldots, 13$) using statistics of historical data.

Case A3. Comparison of the results for various structures of branching trees under the same number of nodes at time 3 using statistics of historical data.

We examine some basic characteristics about the difference between the scenario tree model and the hybrid model in the Case A1. The number of states in the scenario tree model or the number of paths in the hybrid model which comes out of each state or node at time $T-1$ is 166 or 167 $\left(=\frac{10{,}000}{5\times4\times3}=166.7\right)$. We test how the serial correlations affect the difference between the two kinds of models in the Case B1. The larger the size of the branching tree, the larger the number of states from time 1 to time 3 in the scenario tree model. On the other hand, the number of states or paths remains the same in the hybrid model even if the size of the branching tree becomes larger. How does the size of the branching tree affect the optimal solution for the two kinds of models? We examine this question in the Case A2. It is also important to compare the two models under various structures of branching trees. In the Case A3, we examine the difference of these models under the condition that the number of decision nodes(states) remains the same at time 3.[16]

The meaning of legend symbols in the graphs such as in Fig. 2 is shown in Table 5. The same short titles are used in the sentences to avoid redundancy and to keep clarity.

4.1. Case A1: Basic results for the 5–4–3 branching tree using statistics of historical data

Fig. 2 shows the four kinds of curves for the 5–4–3 branching tree using statistics of historical data. When we evaluate optimal solutions of the scenario tree model on the simulated paths, the efficient frontier of 'Scenario' moves downwards to

[16] The number of states or paths which comes out of each state or node at time 3 is N^3 in the Case A2, but they have the same number in the Case A3.

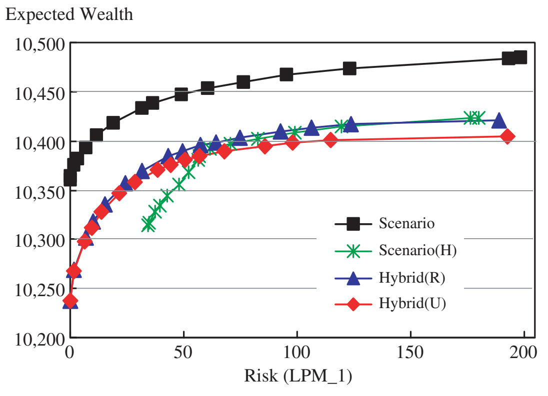

Fig. 2. Case A1: Efficient frontier.

Table 5
Legend symbols in the figures

Scenario	Efficient frontier when the scenario tree model is solved
Scenario(H)	Relationship between the expected terminal wealth and risk when the optimal solutions of scenario tree model are evaluated on simulated paths
Hybrid(R)	Efficient frontier when the hybrid model with the fixed-proportion strategy is solved
Hybrid(U)	Efficient frontier when the hybrid model with the fixed-unit strategy is solved

'Scenario(H)'. The problem solved by the scenario tree model is over-evaluated because of the insufficient description of uncertainties associated with asset returns. The efficient frontier of 'Hybrid(R)' is better than the curve of 'Scenario(H)', which cannot have low risk. The hybrid model can evaluate and control risk better than the scenario tree model.

The efficient frontier of 'Hybrid(R)' dominates the efficient frontier of 'Hybrid(U)'. This is because we need to hold cash after time 1 to execute transactions for the fixed-unit strategy in the simulated path approach, while we do not always have to hold cash for the fixed-proportion strategy. When γ is large, the two strategies have almost the same expected wealth and LPM_1, because cash is held to reduce risk. When γ is small, the efficient frontier of 'Hybrid(U)' is worse than the curve of 'Scenario(H)' due to the same reason.

We can verify our explanation by illustrating the average investment ratios at each time in Fig. 3. The horizontal axis is a risk aversion coefficient (γ), and the vertical axis is an average investment ratio. The smaller the risk aversion coefficient (γ), the more cash the fixed-unit strategy holds than the fixed-proportion strategy. Dynamic portfolios of the two strategies of the hybrid model are similar each other except cash.

Optimal solutions of the scenario tree model have very extreme solutions at time 0, and they are different from those of the hybrid model. This is because the scenario tree model has much less number of states in period 1 than those of the hybrid model.

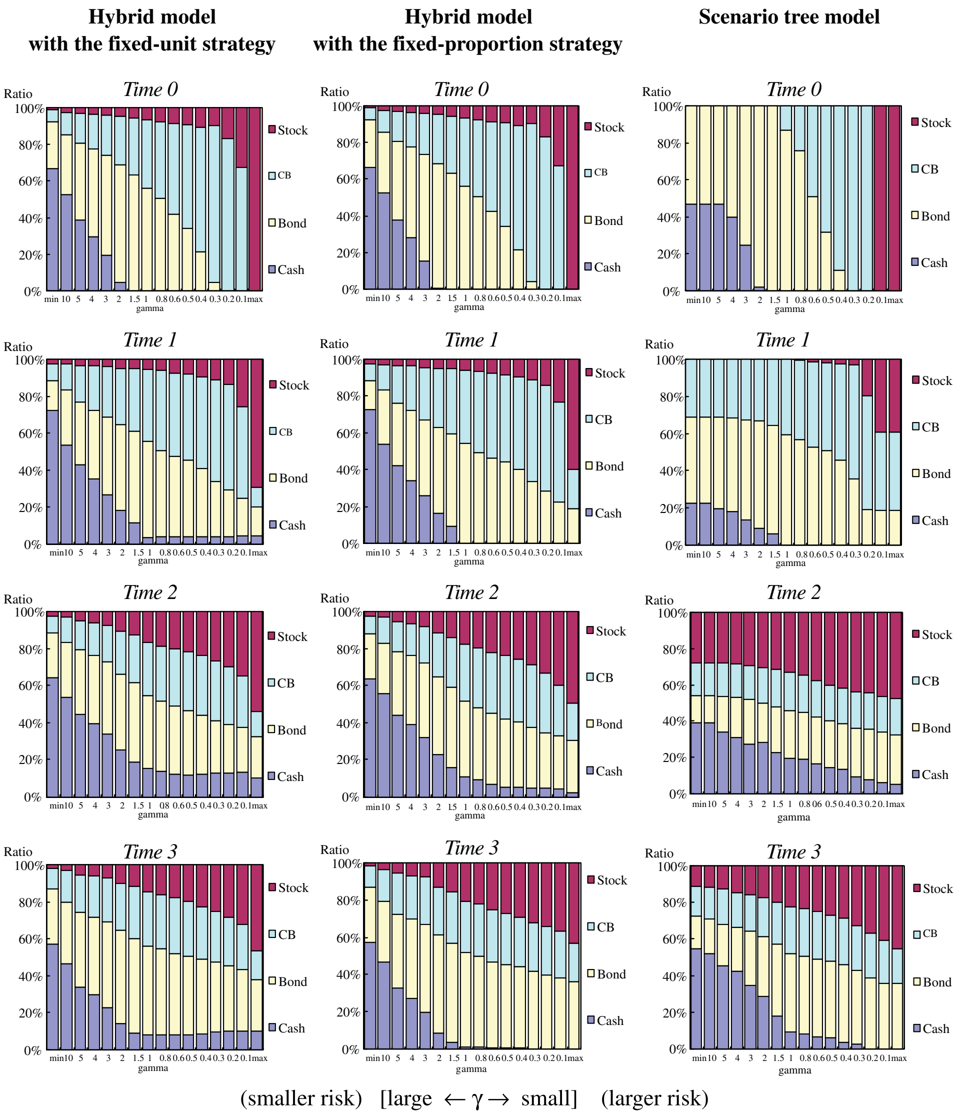

Fig. 3. Case A1: Average investment ratios.

4.2. Case B1: Basic results for the 5–4–3 branching tree using virtual statistics considering the serial correlation

We show eleven graphs for the 5–4–3 branching tree using virtual statistics considering the various serial correlations in Fig. 4. The larger the parameter c, the closer the efficient frontier of ‘Hybrid(R)’ to the curve of ‘Scenario(H)’, while the solutions of ‘Scenario(H)’ cannot also have low risk in this case. The smaller the

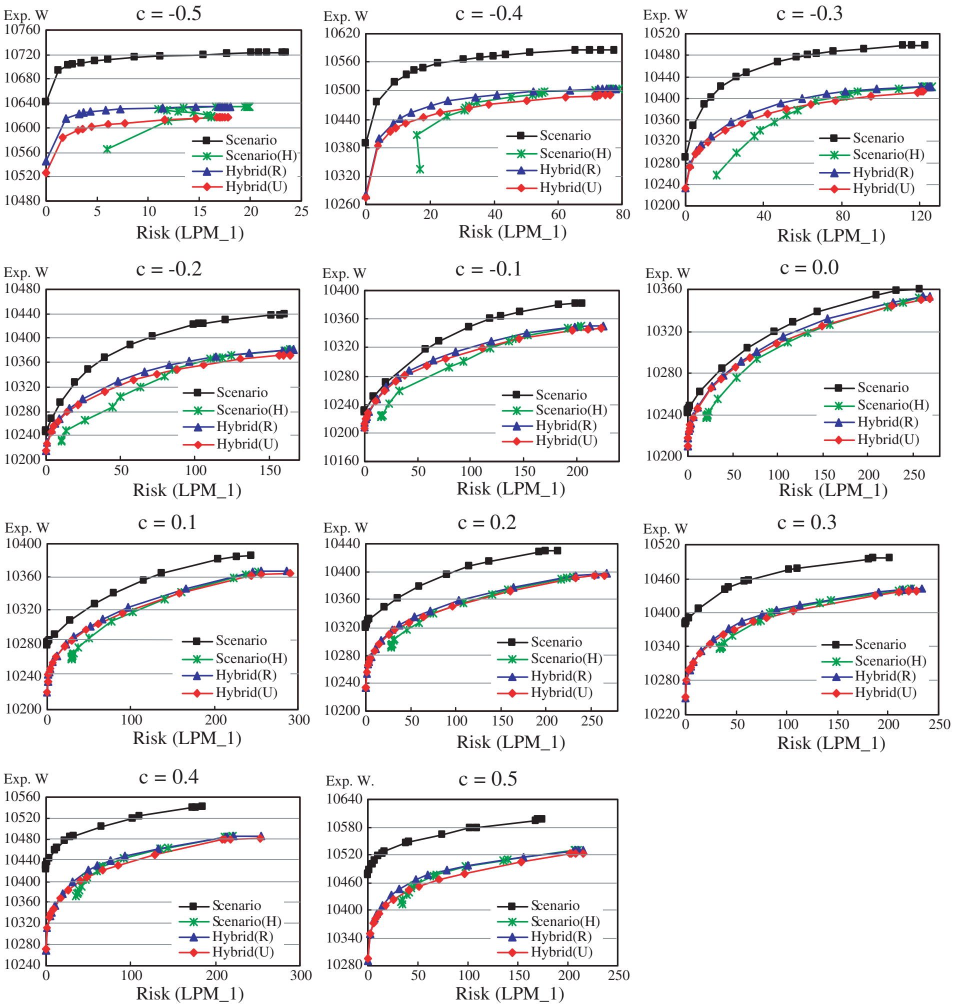

Fig. 4. Case B1: Efficient frontiers.

absolute value of the parameter c is, the closer the efficient frontier of 'Scenario' is to both the efficient frontier of 'Hybrid(R)' and the curve of 'Scenario(H)'. On the other hand, the larger the absolute value of the parameter c is, the more over-evaluated the efficient frontier of 'Scenario' is.

Fig. 5 shows investment proportions at time 0 for 16 kinds of risk aversion coefficients (γ). The horizontal axis is a risk aversion coefficient(γ). The optimal solutions of the hybrid model are different from those of the scenario tree model for all serial correlation parameters as well as the result of the Case A1. The optimal solutions of the two strategies of the hybrid model are similar as well.

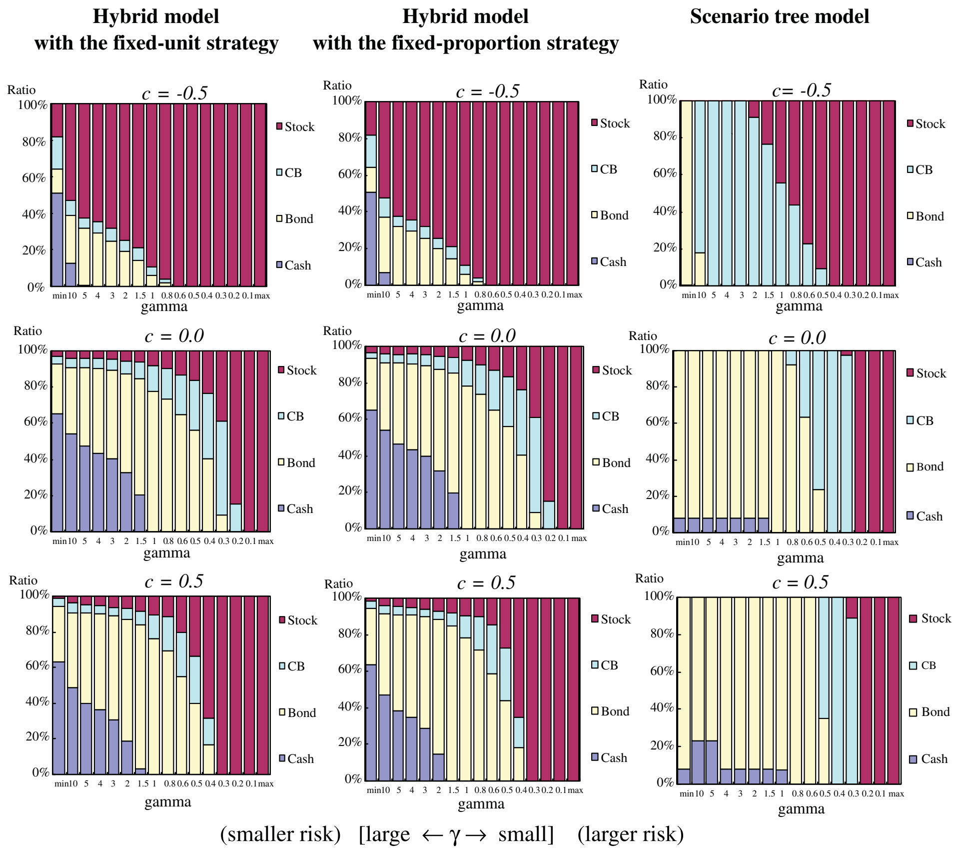

Fig. 5. Case B1: Investment proportions at time 0.

4.3. Case A2: Comparison of the N–N–N branching trees using statistics of historical data

Fig. 6 has 12 graphs for various *N–N–N* branching trees using statistics of historical data, and each graph shows the four kinds of the curves. The efficient frontier of 'Scenario' is also over-evaluated as well as the result of Case A1. Even if the size of branching trees becomes larger, the degree of difference between the two models is similar to that of the 5–4–3 branching tree. But the larger the size of the branching tree, not only the better the curve of 'Scenario(H)' but also the smaller the minimum risk of 'Scenario(H)'. This reason is that the scenario tree model that is solved with a larger branching tree can describe more accuracy of uncertainties, and control risk. However, the curve of 'Scenario(H)' is still dominated by the efficient frontier of 'Hybrid(R)'.

Fig. 7 shows investment ratios at time 0 of the two models. The horizontal axis is a risk aversion coefficient (γ), and the vertical axis is an average investment ratio.

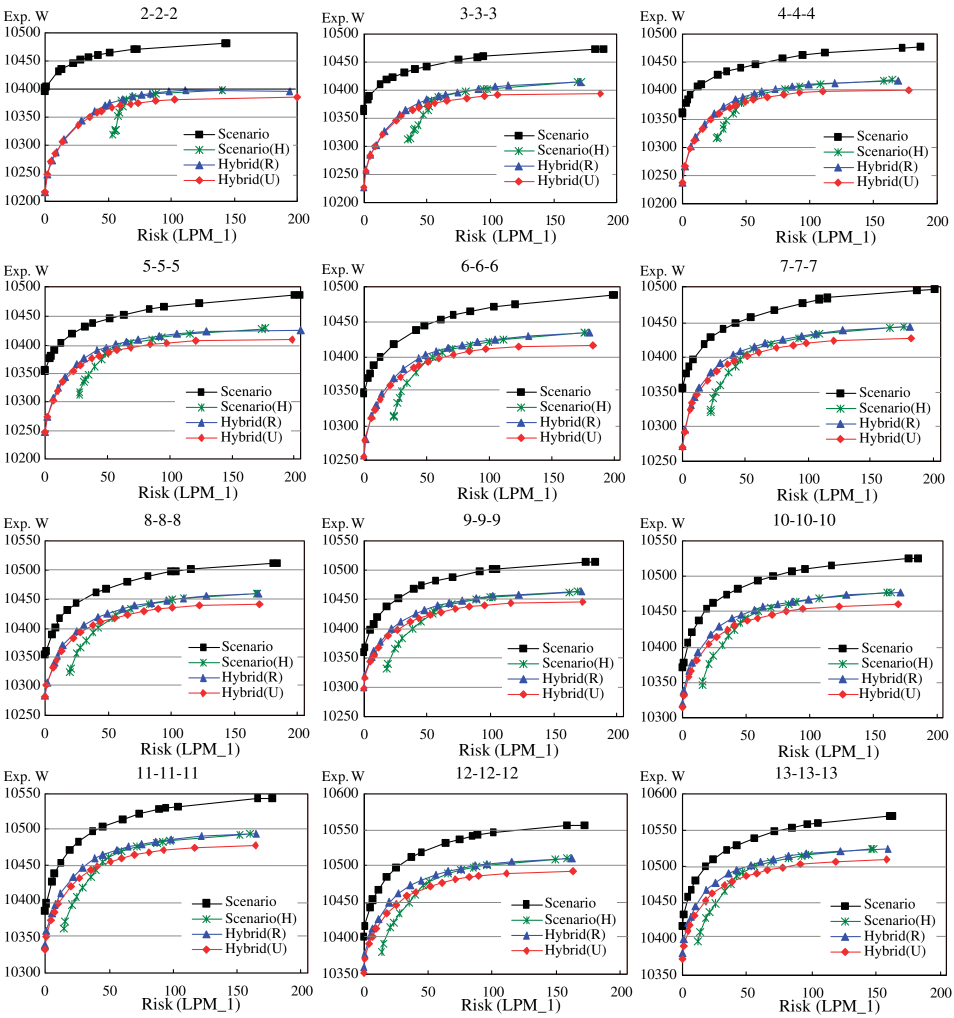

Fig. 6. Case A2: Efficient frontiers.

Twelve broken lines in each graph of Fig. 7 show changes of investment proportions with risk aversion coefficients to a ‘*N*–*N*–*N*’ branching trees. Optimal solutions of the scenario tree model are different from those of the hybrid model like the Case A1, or the case of the 5–4–3 branching tree. Optimal solutions of the scenario tree model are more sensitive to the change of the size of branching tree than those of the hybrid model. The larger the size of the branching tree, the riskier assets we tend to invest in at time 0 for the both models. This is because we have the ability and flexibility to control risk under a large branching tree, even if riskier assets are invested in at time 0.

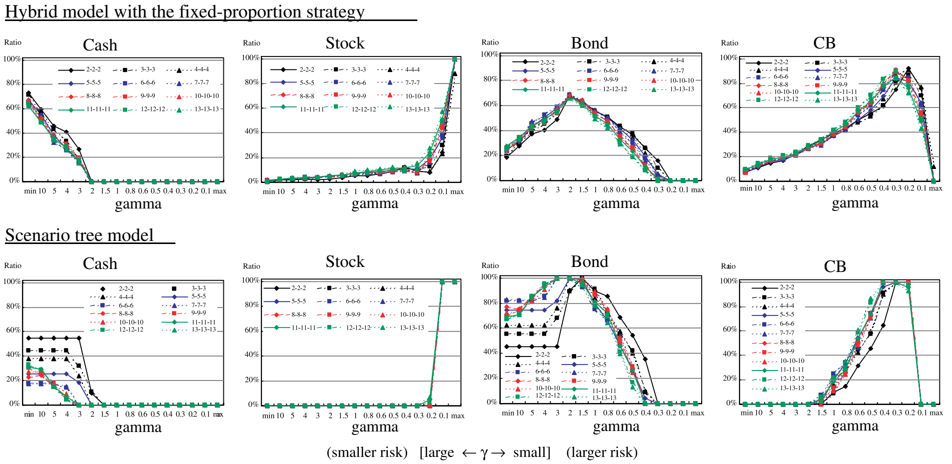

Fig. 7. Case A2: Investment proportions at time 0.

4.4. Case A3: Comparison of the branching trees under the same number of nodes at time 3 using statistics of historical data

We examine how the efficient frontiers of the two kinds of models differ by the branching tree structures. We solve the problems for four kinds of structures which number of nodes is 2000; 20–10–10, 20–20–5, 40–10–5, and 80–5–5 branching trees. We consider three groups that contain some pairs of branching tree. The group is defined by the characteristics of tree structures. We compare the following pairs of branching trees.

(1) The first group of pairs (Group 1), which pairs have the different number of nodes at both time 1 and time 2. Pairs of branching trees are 20–10–10 & 40–10–5, and 20–10–10 & 80–5–5.
(2) The second group of pairs (Group 2), which pairs have the same number of nodes at time 1, but the different number of nodes at time 2. A pair of branching trees is 20–10–10 & 20–20–5.
(3) The third group of pairs (Group 3), which pairs have the different number of nodes at time 1, but the same number of nodes at time 2. Pairs of branching trees are 20–20–5 & 40–10–5, 20–20–5 & 80–5–5, and 40–10–5 & 80–5–5.

Fig. 8 shows the curves and the investment ratios at time 0 for four kinds of branching trees. The horizontal axis of the top four graphs is the LPM_1 value, and the vertical axis is the expected terminal wealth. The horizontal axis of other eight graphs in the middle and bottom of Fig. 8 is the risk aversion coefficient (γ), and the vertical axis is the investment ratio at time 0. The larger the number of nodes is, the more upwards the efficient frontier moves. However, we almost have the similar relationship between the two models as in the previous cases.

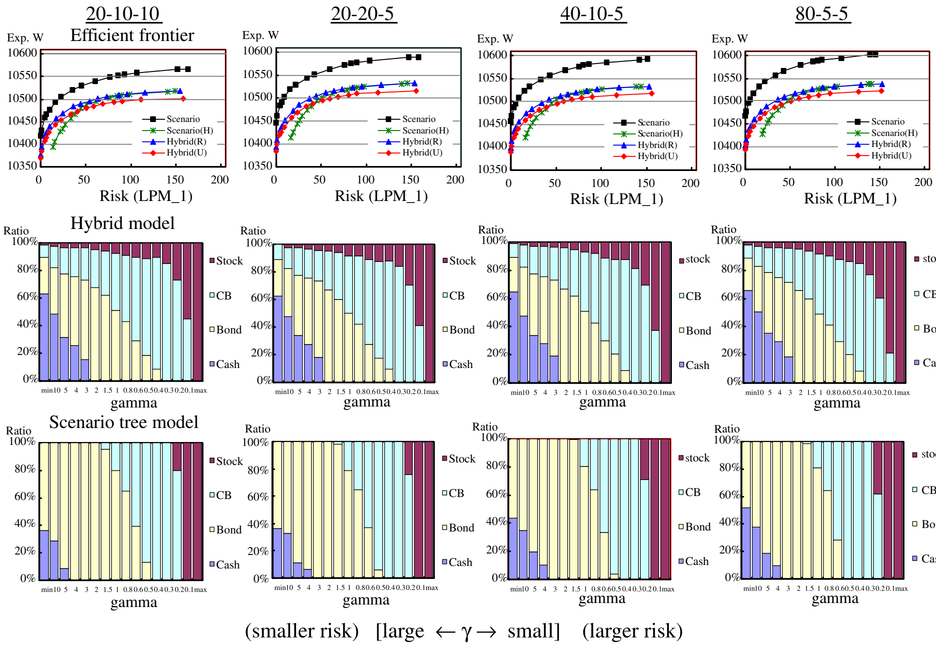

Fig. 8. Case A3: The graphs that the branching trees have 2000 nodes at time 3.

The pairs of efficient frontiers in Group 1 and Group 2 are different from each other. The pairs of efficient frontiers in Group 3 are more similar than those of the other groups, because there are a lot of paths through any node at the time 1.

5. Concluding remarks

The scenario tree model has been typically used for the dynamic portfolio optimization. However, We have some difficulties of how the set of scenarios covers the set of possibilities to a sufficient degree to describe the uncertainties. The hybrid optimization model using simulated paths and tree structures allows both describing of the uncertainties with high accuracy and making of conditional decisions. The previous papers (Hibiki, 2001b, 2003) show some characteristics of the hybrid model, but do not examine how the hybrid model is better than the scenario tree model. In this paper, we compare the two types of multi-period stochastic optimization models by using numerical tests, and illustrate the difference between them. Our contributions and related future research of this paper are as follows.

(1) We develop the specific algorithm to solve the hybrid model with the fixed-proportion strategy. We can derive an approximate solution to a non-convex dynamic asset allocation problem by solving linear programming problems

successively. This algorithm seems to work well, however it is a heuristic one. It is an important task to develop an alternative algorithm because it does not guarantee to derive a global optimal solution.

(2) We develop the method of comparing the two kinds of models with different scenario structures. In this method, the mean and standard deviation of asset returns of simulated paths match those of the scenario tree model in each period, but correlations between assets and serial correlations do not match each other in the scenario tree model because of the technical difficulties. We need to develop a correlation matching method to construct a scenario tree from simulated paths. Moreover, it is an interesting research to develop the 'tree to path' procedure, and to compare it with the 'path to tree' procedure we choose in this paper. If an appropriate scenario tree is constructed, it is easy to generate simulated paths from the scenario tree. It may be difficult to construct the appropriate scenario tree.

(3) We test some numerical examples to compare the two kinds of models. We show that the hybrid model can evaluate and control risk better than the scenario tree model. The scenario generation model with the two kinds of summary statistics is tested. We should compare the hybrid model with the scenario tree model under the various scenario generation models.

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