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A Branch-and-Cut Method for Dynamic Decision Making Under Joint Chance Constraints

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In this paper, we consider a finite-horizon stochastic mixed-integer program involving dynamic decisions under a constraint on the overall performance or reliability of the system. We formulate this problem as a multistage (dynamic) chance-constrained program, whose deterministic equivalent is a large-scale mixed-integer program. We study the structure of the formulation and develop a branch-and-cut method for its solution. We illustrate the efficacy of the proposed model and method on a dynamic inventory control problem with stochastic demand in which a specific service level must be met over the entire planning horizon. We compare our dynamic model with a static chance-constrained model, a dynamic risk-averse optimization model, a robust optimization model, and a pseudo-dynamic approach and show that significant cost savings can be achieved at high service levels using our model.

Data, as supplemental material, are available at http://dx.doi.org/10.1287/mnsc.2013.1822.

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1. Introduction

In this paper, we consider finite-horizon dynamic (multistage) decision-making problems under uncertainty such that a risk, reliability, or service level requirement must be met over the entire planning horizon. Such a restriction could be due to stipulations in a customer contract or certain regulations. Examples include restricting the loss-of-load probabilities in power systems, satisfying contractual service levels in inventory management, and limiting risk in certain financial portfolios. A common approach to limiting violations of such a requirement is to add a penalty term to the objective function. However, such penalties are intangible and inherently hard to estimate. The resulting solutions may be overly conservative under high penalties or may result in low reliability and contract violations under low penalties. A more direct approach is to introduce probabilistic (chance) constraints in the associated optimization model to ensure that the required reliability or service level is met.

Charnes et al. (1958) and Charnes and Cooper (1959, 1963) were the first to study an optimization problem with *individual* chance constraints. In a dynamic problem, satisfying a service level in each time period may result in a poor service level over the entire horizon. In addition, typically there are correlations between the random variables in multiple time periods. As a result, a *joint* chance constraint may be more appropriate to ensure an overall high service level. Miller and Wagner (1965) study probabilistic programming with joint chance constraints and independent random variables. Joint chance constraints with dependent random variables were introduced in Prékopa (1973). Miller and Wagner (1965) and Prékopa (1973) show that under certain assumptions on the distribution of the right-hand side vector in the chance constraint, the deterministic equivalent can be formulated as a convex program. Some recent applications of optimization problems with chance constraints include probabilistic set covering (Beraldi and Ruszczyński 2002b, Saxena et al. 2010); probabilistic production and distribution planning (Lejeune and Ruszczyński 2007); call center staffing (Gurvich et al. 2010); insuring critical paths (Shen et al. 2010); optimal vaccine allocation (Tanner and Ntaimo 2010); and reliable emergency medical service design (Beraldi et al. 2004). For a more thorough treatment of optimization under chance constraints, we refer the reader to Dentcheva (2009), Prékopa (2003), Kall and Wallace (1994), Birge and Louveaux (1997), and Prékopa (1995).

One challenge with linear programs with joint chance constraints is that the feasible region is nonconvex. Prékopa (1990) introduces the concept of p-efficient points, which define the extreme points of the nonconvex feasible region. There are several methods in the literature that rely on the enumeration of the exponentially many p-efficient points. Sen (1992) uses the p-efficient points to give a disjunctive programming reformulation of joint chance constraints with finite discrete distributions. Valid inequalities are proposed based on the extreme points of the reverse polar of the disjunctive program. Dentcheva et al. (2000) use p-efficient points to obtain various reformulations of probabilistic programs with discrete random variables and to derive valid bounds on the optimal objective function value. Ruszczyński (2002) uses the concept of p-efficient points to derive consistent orders on different scenarios representing the discrete distribution. The consistent ordering is represented with precedence constraints, and valid inequalities for the resulting precedence-constrained knapsack set are proposed. Beraldi and Ruszczyński (2002a) propose a branch-and-bound method for probabilistic integer programs using a partial enumeration of the p-efficient points. Cheon et al. (2006) give a global optimization algorithm that successively partitions the nonconvex feasible region until a global optimal solution is obtained. Tayur et al. (1995) give an algebraic geometry algorithm for a scheduling problem with joint chance constraints that solves a series of chance-constrained integer programs with varying reliability levels.

Alternatively, a deterministic equivalent model is obtained by adding additional binary variables. The linear programming relaxation of this formulation is weak in general but can be strengthened by adding valid inequalities obtained from the so-called mixing set substructure (Luedtke et al. 2010, Küçükyavuz 2012). Combining decomposition and cutting plane techniques, Luedtke (2011) proposes a branch-and-cut decomposition algorithm for solving a two-stage chance-constrained program, where the recourse decisions incur no additional costs.

Another challenge with the optimization problems with joint chance constraints is that in cases of continuous distributions, calculating the joint probability of several events requires the evaluation of a multidimensional integral, which is hard to compute accurately (Ahmed and Shapiro 2008). Ben-Tal and Nemirovski (1998), Calafiore and Campi (2005, 2006), and Nemirovski and Shapiro (2005, 2006) approximate the nonconvex chance constraint with convex constraints such that the solution to this approximation is feasible with a high probability. However, such methods could yield highly conservative solutions (Ahmed and Shapiro 2008). In this paper, we assume a finite discrete distribution (or a finite sample from the continuous distribution), circumventing the difficulty of evaluating high dimensional integrals.

Most of the literature on joint chance constraints considers *static* decisions (Birge and Louveaux 1997). In other words, the decisions are made once at the beginning of the horizon and are not updated as the uncertainty is revealed. An exception is Lulli and Sen (2004), who consider a probabilistic batch-sizing problem under a finite discrete demand distribution. In their model, nonanticipativity of decisions is enforced only for the scenarios that meet the desired service constraint. Andrieu et al. (2010) study chance constraints with dynamic (multistage) decisions that appear in hydro power reservoir management. The authors assume a continuous probability distribution and give a finite dimensional approximation of the infinite-dimensional chance constraint by discretizing the continuous decision variables. Andrieu et al. (2010) state that it is not clear if the recent advances in integer programming methods for chance constraints would apply to the multistage setting. In this paper, we show that because of the nonanticipativity of the decisions, branch-and-cut methods can indeed be developed for the multistage setting. We give valid inequalities for the dynamic model based on its mixing and continuous mixing substructures. We illustrate that significant cost savings can be achieved through dynamic decision making using an application in inventory management with random demands and costs and a prescribed service level requirement for meeting the demand on time over the planning horizon. Our computational experiments illustrate the effectiveness of the branch-and-cut algorithm using the continuous mixing inequalities for this inventory management problem. Finally, we compare our dynamic chance-constrained optimization model for the inventory management problem with a dynamic risk-averse model, a robust optimization model, and a pseudo-dynamic approach.

2. Static Model

Even though our focus is on dynamic decisions, we first review the more commonly studied static models for completeness.

Consider a dynamic (multistage) problem with n stages. Let ξ and μ denote N- and B-variate random variables representing the right-hand sides and costs, respectively, with a known finite discrete distribution function. Let

$$A = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & A_{n3} & \cdots & A_{nn} \end{pmatrix}, \quad x_t = \begin{pmatrix} x_{t1} \\ x_{t2} \\ \vdots \\ x_{tb_t} \end{pmatrix},$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \xi_t = \begin{pmatrix} \xi_{t1} \\ \xi_{t2} \\ \vdots \\ \xi_{t\eta_t} \end{pmatrix}, \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix},$$

where A_{ti} is an $\eta_t \times b_i$ matrix representing the constraint coefficients of decision vector $x_i \in \mathbb{R}_+^{b_i}$ in stage t,

$i \le t$ with $N = \sum_{t=1}^{n} \eta_t$ and $B = \sum_{t=1}^{n} b_t$. Note that the entries of the coefficient matrix A_{ti} with $t < i$ are all zero because the constraints in stage t do not depend on decisions after stage t.

Let $X(\xi) \subseteq \mathbb{R}_+^{\beta_1} \times \mathbb{Z}_+^{\beta_2}$ be a mixed-integer set dependent of ξ and defined by additional constraints on x, where $\beta_1 + \beta_2 = B$. Let τ be the required reliability level, $0 \le \tau \le 1$. The static chance-constrained program with joint chance constraints is

$$\min\{\mathbb{E}_{(\xi,\mu)}\mu^T x\colon \mathbb{P}(Ax \ge \xi) \ge \tau,\ x \in X(\xi)\}. \tag{1}$$

This is referred to as a static model because the decisions in stage t do not depend on the realizations in stages $i = 1, \ldots, t-1$. Letting $y = Ax$, the joint chance constraint can be rewritten as $\mathbb{P}(y \ge \xi) \ge \tau$. The formulation with individual chance constraints $\mathbb{P}(y_{ti} \ge \xi_{ti}) \ge \tau$ for all $t = 1, \ldots, n$, $i = 1, \ldots, \eta_t$, results in an overall reliability of the system to be much lower than that with joint chance constraints. In addition, in the case of individual chance constraints, we can linearize the constraints using quantiles (see Kall and Wallace 1994). Therefore, throughout this paper, we will be interested in optimization problems with *joint* chance constraints.

Assume that the random vector $\Gamma = (\xi, \mu)$ has finitely many realizations (scenarios) given by $(\mathbf{D}^1, \mathbf{c}^1)$, $(\mathbf{D}^2, \mathbf{c}^2), \ldots, (\mathbf{D}^m, \mathbf{c}^m)$ with probabilities $\pi^1, \pi^2, \ldots, \pi^m$, where $\mathbf{D}^i = (D_1^i, D_2^i, \ldots, D_n^i)$, $\mathbf{c}^i = (c_1^i, c_2^i, \ldots, c_n^i)$, and $c_t^i = \{c_{t1}^i, \ldots, c_{tb_t}^i\}$, for $i = 1, \ldots, m$, $t = 1, \ldots, n$. Let X^i represent the feasible region corresponding to D^i. Throughout, we let $[i, j] := \{t \in \mathbb{Z}\colon i \le t \le j\}$. We assume $y \ge 0$ without loss of generality (see Küçükyavuz 2012).

A deterministic equivalent of the static chance-constrained program is

$$\min \sum_{i=1}^{m}\sum_{t=1}^{n} \pi^i c_t^i x_t$$

$$\text{s.t.}\quad y = Ax, \tag{2}$$

$$y_t \ge D_t^i(1 - z^i) \quad t \in [1, n],\ i \in [1, m], \tag{3}$$

$$\sum_{i=1}^{m} \pi^i z^i \le 1 - \tau, \tag{4}$$

$$x \in X^i, \quad y \in \mathbb{R}_+^N, \tag{5}$$

$$z \in \{0, 1\}^m, \tag{6}$$

where $z^i = 0$ implies that under scenario i, we have no violated chance constraint (i.e., $y = Ax \ge \mathbf{D}^i$) at the solution (y, x). This formulation is from Luedtke et al. (2010). The set defined by the inequalities (3) for a fixed t is known as the *mixing set*. Atamtürk et al. (2000) and Günlük and Pochet (2001) give valid inequalities called star or mixing mixed-integer rounding inequalities for this set and show that these inequalities give the convex hull of solutions. In addition, Luedtke et al. (2010) and Küçükyavuz (2012) give strengthened mixing inequalities for the mixing set with an additional knapsack constraint, $\mathcal{Q}_t = \{(y_t, z) \in \mathbb{R}_+ \times \{0, 1\}^m\colon \sum_{i=1}^{m} \pi^i z^i \le 1 - \tau,\ y_t \ge D_t^i(1 - z^i),\ i \in [1, m]\}$ as it arises in static chance-constrained linear programs.

3. Dynamic Model

Most models in the literature, given by the chance-constrained program in §2, are static in nature. The decisions are made here and now and do not change as the uncertain parameters are revealed over time. However, a more flexible and efficient planning model would be to allow the decisions in period t to take the observed data in periods $1, \ldots, t-1$ into account. This gives rise to a multistage model under joint chance constraints in which the decisions at every stage are adaptive to the realizations of the uncertain data. In this model, the order of events is as follows: at the beginning of the planning horizon, first-stage decisions, given by the vector x_1, are made. Next, the random events occurring in the first stage, Γ_1, are observed. Based on the observed outcomes, the decisions at the second stage, $x_2(\Gamma_1)$, are made, and so on. More generally, let $\Gamma_{t-1} := (\xi_1, \mu_1, \xi_2, \mu_2, \ldots, \xi_{t-1}, \mu_{t-1})$. Let $x_t(\Gamma_{t-1})$ be the decision vector at stage $t \in [2, n]$, whose value is determined after the random variables Γ_{t-1} are observed. Then instead of the static chance constraints in (1), we have a dynamic chance constraint:

$$\mathbb{P}\left[A\begin{pmatrix} x_1 \\ x_2(\Gamma_1) \\ \vdots \\ x_n(\Gamma_{n-1}) \end{pmatrix} \ge \xi\right] \ge \tau. \tag{7}$$

Let $x^i = (x_1^i, \ldots, x_n^i)$, where $x_t^i = (x_{t1}^i, \ldots, x_{tb_t}^i)$ is the vector of decisions made in stage t under scenario i, $t \in [1, n]$, $i \in [1, m]$. Also let $y^i = Ax^i$ for $i \in [1, m]$, $y^i = (y_1^i, \ldots, y_n^i)$, $y_t^i = (y_{t1}^i, \ldots, y_{t\eta_t}^i)$, $t \in [1, n]$. Note that in the dynamic model, the nonanticipativity of the decisions must be enforced; i.e., the scenarios that have the same set of past outcomes until time t should have the same action in period t. Let $S_t^\ell = \{k \in [1, m]\colon D_j^\ell = D_j^k,\ c_j^\ell = c_j^k,\ j \in [1, t-1]\}$ be the set of scenarios that share the same history with scenario ℓ until time t. The deterministic equivalent of the dynamic model with chance constraint (7) is given by

$$\min \sum_{i=1}^{m}\sum_{t=1}^{n} \pi^i c_t^i x_t^i \tag{8}$$

$$\text{s.t.}\quad y^i = Ax^i \quad i \in [1, m], \tag{9}$$

$$y_t^\ell \ge D_t^\ell(1 - z^\ell) \quad t \in [1, n],\ \ell \in [1, m], \tag{10}$$

$$y_t^\ell = y_t^k \quad t \in [1, n],\ \ell \in [1, m],\ k \in S_t^\ell \setminus \{\ell\}, \tag{11}$$

$$x_t^\ell = x_t^k \quad t\in[1,n],\ \ell\in[1,m],\ k\in S_t^\ell\backslash\{\ell\}, \tag{12}$$

$$\sum_{i=1}^m \pi^i z^i \le 1-\tau, \tag{13}$$

$$y^i\in\mathbb{R}_+^N, \quad z^i\in\{0,1\} \quad i\in[1,m], \tag{14}$$

$$x^i\in X^i \quad i\in[1,m]. \tag{15}$$

Constraints (11) and (12) are the nonanticipativity constraints. Note that constraints (9) and (12) imply inequality (11). However, in the next section, we will use inequality (11) to obtain strong valid inequalities. Observe that a more compact representation with variables corresponding to the nodes of a scenario tree could be used in practice. All of our results apply to the more compact model, and we test our results on the compact formulation in §6. In what follows, we use the noncompact model for notational convenience.

4. A Branch-and-Cut Method for the Dynamic Model

In this section, we give valid inequalities for the dynamic joint chance-constrained model based on two substructures: mixing set and continuous mixing set. Throughout this section, for ease of exposition, we assume that there is a single constraint in each stage; i.e., $\eta_t=1$, $t\in[1,n]$ and $N=n$. Therefore, $y^i\in\mathbb{R}_+^n$, $i\in[1,m]$. Our results apply to the general case with $\eta_t>1$ by considering one constraint at a time in each stage.

4.1. Mixing Inequalities

For any fixed stage $t\in[1,n]$ and scenario $\ell\in[1,m]$ with $S_t^\ell\neq\varnothing$, suppose, without loss of generality, that $S_t^\ell=\{1,\dots,m_t^\ell\}$ and D^i for all $i\in S_t^\ell$ are sorted in nonincreasing order as $D_t^1\ge D_t^2\ge\cdots\ge D_t^{m_t^\ell}$. As observed by Luedtke et al. (2010), if there exists $\nu_t^\ell\le m_t^\ell$ such that $\sum_{i=1}^{\nu_t^\ell}\pi^i\le 1-\tau$ and $\sum_{i=1}^{\nu_t^\ell+1}\pi^i>1-\tau$, then we must have $y_t^\ell\ge D_t^{\nu_t^\ell+1}$. In addition, let $\langle 1\rangle,\langle 2\rangle,\dots,\langle m_t^\ell\rangle$ be a nondecreasing order of scenario probabilities, i.e., $\pi^{\langle 1\rangle}\le\pi^{\langle 2\rangle}\le\cdots\le\pi^{\langle m_t^\ell\rangle}$. Also let p_t^ℓ be such that $\sum_{i=1}^{p_t^\ell}\pi^{\langle i\rangle}\le 1-\tau$ and $\sum_{i=1}^{p_t^\ell+1}\pi^{\langle i\rangle}>1-\tau$. Then the extended (knapsack) cover inequality

$$\sum_{i=1}^{m_t^\ell} z^i\le p_t^\ell$$

is valid (Küçükyavuz 2012).

Proposition 1. *For any fixed t and scenario ℓ with $S_t^\ell\neq\varnothing$, and $D_t^1\ge D_t^2\ge\cdots\ge D_t^{m_t^\ell}$ for $i\in S_t^\ell$, and for $q_t^\ell\in\mathbb{Z}_+$ such that $q_t^\ell\le\nu_t^\ell$, let $\bar T=\{i_1,i_2,\dots,i_a\}\subseteq\{1,\dots,q_t^\ell\}$ with $i_1<i_2<\cdots<i_a$, $L\subseteq S_t^\ell\backslash\{1,\dots,q_t^\ell+1\}$ and a permutation of the elements in L, $\Pi_L=\{l_1,l_2,\dots,l_{p_t^\ell-q_t^\ell}\}$ such that $l_j\ge q_t^\ell+1+j$. For $\nu_t^\ell<p_t^\ell$, the $(\bar T,\Pi_L)$ inequalities*

$$y_t^\ell+\sum_{j=1}^a(D_t^{i_j}-D_t^{i_{j+1}})z^{i_j}+\sum_{j=1}^{p_t^\ell-q_t^\ell}\alpha'_j(1-z^{l_j})\ge D_t^{i_1}, \tag{16}$$

are valid for the set given by (10)–(14), *where $D_t^{i_{a+1}}=D_t^{q_t^\ell+1}$ if $q_t^\ell+1\le\nu_t^\ell$, $D_t^{i_{a+1}}=0$ if $q_t^\ell+1>\nu_t^\ell$, and $\alpha'_1=D_t^{q_t^\ell+1}-D_t^{\min\{\nu_t^\ell+1,q_t^\ell+2\}}$, and for $j=2,\dots,p_t^\ell-q_t^\ell$,*

$$\alpha'_j=\max\left\{\alpha'_{j-1},D_t^{q_t^\ell+1}-D_t^{\min\{\nu_t^\ell+1,\,q_t^\ell+1+j\}}-\sum_{i:\,i<j\ \text{and}\ l_i\ge q_t^\ell+1+j}\alpha'_i\right\}.$$

Proof of Proposition 1. Note that for some $t\in[1,n]$ and $\ell\in[1,m]$ such that $S_t^\ell\neq\varnothing$, from (10), (11), and a relaxation of (13), we obtain a set $\mathcal{Q}_t^\ell=\{(y_t^\ell,z)\in\mathbb{R}_+\times\{0,1\}^{|S_t^\ell|}\colon \sum_{i=1}^{m_t^\ell}\pi^iz^i\le 1-\tau,\ y_t^\ell\ge D_t^i(1-z^i),\ i\in S_t^\ell\}$ for which $(\bar T,\Pi_L)$ inequalities of Küçükyavuz (2012) given by (16) are valid. □

In other words, nonanticipativity of decisions allows us to use the valid inequalities proposed for the static problem in a dynamic setting.

Corollary 1. *Given that $y_t^\ell\ge D_t^{\nu_t^\ell+1}$, a special case of inequalities* (16) *with $q_t^\ell=\nu_t^\ell$ is the so-called mixing inequalities,*

$$y_t^\ell+\sum_{j=1}^a(D_t^{i_j}-D_t^{i_{j+1}})z^{i_j}\ge D_t^{i_1}, \tag{17}$$

where $1\le i_1<i_2<\cdots<i_a\le\nu_t^\ell$ and $D_t^{i_{a+1}}=D_t^{\nu_t^\ell+1}$, and $D_t^{i_{a+1}}=0$ if $\nu_t^\ell=m_t^\ell$.

Note that $i_1=1$ is needed for inequalities (17) to be strong. We use inequalities (17) in our computational experiments in §6 because of their polynomial time separation (see Günlük and Pochet 2001).

4.2. Continuous Mixing Inequalities

For a stage $t\in[1,n]$ and scenario $\ell\in[1,m]$, if $S_t^\ell\neq\varnothing$, then we call the pair (t,ℓ) a *nonanticipative node* of the scenario tree representing the random data. At a nonanticipative node (t,ℓ), for a fixed stage $T\in[t,n]$, inequality (10) can be rewritten as

$$y_T^\ell=\sum_{j=1}^t A_{Tj}x_j^\ell+\sum_{j=t+1}^T A_{Tj}x_j^\ell\ge D_T^\ell(1-z^\ell). \tag{18}$$

In this section, we assume that $\sum_{j=t+1}^T A_{Tj}x_j^\ell\ge\mathbf{0}$. This holds, for example, when $A_{Tj}\ge\mathbf{0}$. Let $\bar s=\sum_{j=1}^t A_{Tj}x_j^\ell$. Note that $x_j^\ell=x_j^k$ for $k\in S_t^\ell$, $j\in[1,t]$. Therefore, under this assumption, a relaxation of the feasible set defined by constraints (9), (10), (11), and (14) with respect to nonanticipative node (t,ℓ), stage $T\in[t,n]$, and $R_t^\ell \subseteq S_t^\ell$, is

$$Q_{tT}^\ell = \{(\bar{s},\{y_T^i\}_{i\in R_t^\ell},\{z^i\}_{i\in R_t^\ell})\in\mathbb{R}\times\mathbb{R}_+^{|R_t^\ell|}\times\{0,1\}^{|R_t^\ell|}:\ y_T^i + D_T^i z^i \ge D_T^i,\ y_T^i \ge \bar{s},\ i\in R_t^\ell\}. \tag{19}$$

Let $D_{tT}^\ell := \max\{D_T^i:\ i\in R_t^\ell\}$.

Proposition 2. *At a nonanticipative node (t,ℓ) with $R_t^\ell\subseteq S_t^\ell\neq\varnothing$, for a fixed stage $t\in[1,n]$ and $T\in[t,n]$, let $\sigma=\bar{s}/D_{tT}^\ell$; $r^i=(y_T^i-\bar{s})/D_{tT}^\ell$; $\bar{z}^i=z^i$ for $i\in R_t^\ell$ with $D_T^i<D_{tT}^\ell$, and $\bar{z}^i=z^i-1$ for $i\in R_t^\ell$ with $D_T^i=D_{tT}^\ell$; and $f^i=D_T^i/D_{tT}^\ell$ for $i\in R_t^\ell$ with $D_T^i<D_{tT}^\ell$, and $f^i=0$ for $i\in R_t^\ell$ with $D_T^i=D_{tT}^\ell$. Also let the digraph $G=(V,E)$, where $V=R_t^\ell$ and $E=\{(i,j):\ i,j\in R_t^\ell,\ i\neq j,\ f^i\neq f^j\}\cup\{(i,i):\ i\in R_t^\ell\}$. The arc length $\phi_{jk}(\sigma,r,\bar{z})$ associated with $(j,k)\in E$ with respect to the point $(\bar{s},\{y_T^i\}_{i\in R_t^\ell},\{z^i\}_{i\in R_t^\ell})\in Q_{tT}^\ell$ is*

$$\phi_{jk}(\sigma,r,\bar{z})=\begin{cases}\sigma+r^j+(f^j-f^k+1)\bar{z}^j-f^k & \text{for } (j,k)\in E \text{ if } f^j<f^k,\\ r^j+(f^j-f^k)\bar{z}^j & \text{for } (j,k)\in E \text{ if } f^j>f^k,\\ \sigma+r^j+\bar{z}^j-f^j & \text{for } (j,k)\in E \text{ if } j=k.\end{cases}$$

Then the inequality

$$\sum_{(j,k)\in C}\phi_{jk}(\sigma,r,\bar{z})\ge 0, \tag{20}$$

where $C\subseteq E$ is an elementary cycle in G, is valid for Q_{tT}^ℓ.

Proof of Proposition 2. Weakening the coefficients of z^i, $i\in R_t^\ell$ in (19), we obtain an equivalent set of feasible solutions:

$$Q_{tT}^\ell = \big\{(\bar{s},\{y_T^i\}_{i\in R_t^\ell},\{z^i\}_{i\in R_t^\ell})\in\mathbb{R}\times\mathbb{R}_+^{|R_t^\ell|}\times\{0,1\}^{|R_t^\ell|}:\ y_T^i + D_{tT}^\ell z^i \ge D_T^i,\ y_T^i\ge\bar{s},\ i\in R_t^\ell\big\},$$

or equivalently,

$$Q_{tT}^\ell = \left\{(\bar{s},\{y_T^i\}_{i\in R_t^\ell},\{z^i\}_{i\in R_t^\ell})\in\mathbb{R}\times\mathbb{R}_+^{|R_t^\ell|}\times\{0,1\}^{|R_t^\ell|}:\ \frac{\bar{s}}{D_{tT}^\ell}+\frac{y_T^i-\bar{s}}{D_{tT}^\ell}+z^i\ge\frac{D_T^i}{D_{tT}^\ell},\ y_T^i\ge\bar{s},\ i\in R_t^\ell\right\}. \tag{21}$$

Note that Q_{tT}^ℓ given by (21) is the first type of continuous mixing set (Van Vyve 2005), which is defined as $P^{\text{CMIX}}=\{(\sigma,r,\bar{z})\in\mathbb{R}\times\mathbb{R}_+^{|R_t^\ell|}\times\mathbb{Z}^{|R_t^\ell|}:\ \sigma+r^i+\bar{z}^i\ge f^i,\ i\in R_t^\ell\}$, where $0\le f^j<1$, $j\in R_t^\ell$. Van Vyve (2005) proposes a linear description of $\text{conv}(P^{\text{CMIX}})$. The linear description includes bound constraints $r\ge 0$ and cycle inequalities given by (20). □

At a nonanticipative node (t,ℓ) for a fixed stage $t\in[1,n]$, $T\in[t,n]$, and $\ell\in[1,m]$, if a lower bound for $\bar{s}=\sum_{j=1}^t A_{Tj}x_j^\ell$ is known, then continuous mixing inequalities can be strengthened. In other words, suppose that as in §4.1, we have $\bar{s}=\sum_{j=1}^t A_{Tj}x_j^\ell\ge\bar{D}_t^\ell$ in all feasible solutions for some $\bar{D}_t^\ell\ge 0$. Then for $R_t^\ell\subseteq\{k\in S_t^\ell:\ D_T^k\ge\bar{D}_t^\ell\}$, let

$$\bar{Q}_{tT}^\ell = \{(\bar{s},\{y_T^i\}_{i\in R_t^\ell},\{z^i\}_{i\in R_t^\ell})\in\mathbb{R}\times\mathbb{R}_+^{|R_t^\ell|}\times\{0,1\}^{|R_t^\ell|}:\ y_T^i+D_T^i z^i\ge D_T^i,\ y_T^i\ge\bar{s}\ge\bar{D}_t^\ell,\ i\in R_t^\ell\}. \tag{22}$$

Let $\bar{D}_{tT}^\ell := \max\{D_T^i:\ i\in R_t^\ell\}-\bar{D}_t^\ell$. Then (22) is equivalent to

$$\bar{Q}_{tT}^\ell = \left\{(\bar{s},\{y_T^i\}_{i\in R_t^\ell},\{z^i\}_{i\in R_t^\ell})\in\mathbb{R}\times\mathbb{R}_+^{|R_t^\ell|}\times\{0,1\}^{|R_t^\ell|}:\ \frac{\bar{s}-\bar{D}_t^\ell}{\bar{D}_{tT}^\ell}+\frac{y_T^i-\bar{s}}{\bar{D}_{tT}^\ell}+z^i\ge\frac{D_T^i-\bar{D}_t^\ell}{\bar{D}_{tT}^\ell},\ y_T^i\ge\bar{s}\ge\bar{D}_t^\ell,\ i\in R_t^\ell\right\}.$$

Proposition 3. *At a nonanticipative node (t,ℓ) with $R_t^\ell\subseteq\{k\in S_t^\ell:\ D_T^k\ge\bar{D}_t^\ell\}\neq\varnothing$, for a fixed stage $t\in[1,n]$ and $T\in[t,n]$, let $\sigma=(\bar{s}-\bar{D}_t^\ell)/\bar{D}_{tT}^\ell$; $r^i=(y_T^i-\bar{s})/\bar{D}_{tT}^\ell$; $\bar{z}^i=z^i$ for $i\in R_t^\ell$ with $D_T^i-\bar{D}_t^\ell<\bar{D}_{tT}^\ell$, and $\bar{z}^i=z^i-1$ for $i\in R_t^\ell$ with $D_T^i-\bar{D}_t^\ell=\bar{D}_{tT}^\ell$; and $f^i=(D_T^i-\bar{D}_t^\ell)/\bar{D}_{tT}^\ell$ for $i\in R_t^\ell$ with $D_T^i-\bar{D}_t^\ell<\bar{D}_{tT}^\ell$, and $f^i=0$ for $i\in R_t^\ell$ with $D_T^i-\bar{D}_t^\ell=\bar{D}_{tT}^\ell$; and $f^0=0$. Also let the digraph $G^+=(V^+,E^+)$, where $V^+=\{0\}\cup R_t^\ell$ and $E^+=\{(i,j):\ i,j\in V^+,\ i\neq j,\ f^i\neq f^j\}\cup\{(i,i):\ i\in R_t^\ell\}$. The arc length $\phi_{jk}^+(\sigma,r,\bar{z})$ associated with each arc with respect to the point $(\bar{s},\{y_T^i\}_{i\in R_t^\ell},\{z^i\}_{i\in R_t^\ell})\in\bar{Q}_{tT}^\ell$ is*

$$\phi_{jk}^+(\sigma,r,\bar{z})=\begin{cases}\sigma+r^j+(f^j-f^k+1)\bar{z}^j-f^k & \text{for } (j,k)\in E^+ \text{ if } f^j<f^k,\ j\neq 0,\\ r^j+(f^j-f^k)\bar{z}^j & \text{for } (j,k)\in E^+ \text{ if } f^j>f^k,\\ \sigma-f^k & \text{for } (0,k)\in E^+\\ \sigma+r^j+\bar{z}^j-f^j & \text{for } (j,k)\in E^+ \text{ if } j=k\end{cases}$$

where $\bar{z}^0=r^0=f^0=0$. Then the cycle inequality

$$\sum_{(j,k)\in C}\phi_{jk}^+(\sigma,r,\bar{z})\ge 0, \tag{23}$$

where $C\subseteq E^+$ is an elementary cycle in G^+, is valid for $\bar{Q}_{tT}^\ell$.

Proof of Proposition 3. Note that in this case, because $\sigma\ge 0$ in all feasible solutions to $\bar{Q}_{tT}^\ell$, we have the second type of continuous mixing set (Van Vyve 2005), which is defined as $P_+^{\text{CMIX}}=\{(\sigma,r,\bar{z})\in\mathbb{R}_+\times\mathbb{R}_+^{|R_t^\ell|}\times\mathbb{Z}^{|R_t^\ell|}:\ \sigma+r^i+\bar{z}^i\ge f^i,\ i\in R_t^\ell\}$, where $0\le f^j<1$, $j\in R_t^\ell$. Van Vyve (2005) proposes a linear description of $\text{conv}(P_+^{\text{CMIX}})$ that includes bound constraints

σ, $r \geq 0$ and cycle inequalities defined on the digraph G^+ given by inequalities (23). □

We refer to cycle inequalities (20) and (23) as continuous mixing inequalities (cuts). Note that continuous mixing inequalities (20) and (23) subsume mixing inequalities (17) (Van Vyve 2005). For $t = T$, we have $y^i_T = \bar{s}$, and the continuous mixing inequalities reduce to the mixing inequalities.

The separation problem of inequalities (20) and (23) for a given set R^ℓ_t is to find a negative cost cycle in graphs G and G^+, respectively (Van Vyve 2005). Therefore, it can be solved in polynomial time using the Bellman–Ford algorithm (Ahuja et al. 1993). In graph G^+, for a given set S^ℓ_t, to select the best subset $R^\ell_t \subseteq \{k \in S^\ell_t\colon D^k_T \geq \bar{D}^\ell_t\} = \{k_1, \ldots, k_v\}$, where $D^{k_i}_T \leq D^{k_{i+1}}_T$ for $i \in [1, v-1]$, also takes polynomial time because we only need to consider v possibilities of R^ℓ_t, which are $\{k_1\}, \{k_1, k_2\}, \ldots, \{k_1, \ldots, k_v\}$. In other words, if there exists a violated continuous mixing inequality corresponding to a set $R^\ell_t \subseteq \{k \in S^\ell_t\colon D^k_T \geq \bar{D}^\ell_t\}$, then it is also a violated continuous mixing inequality corresponding to set $\{k_1, \ldots, k_{h(R^\ell_t)}\}$, where $h(R^\ell_t) = \max\{i\colon k_i \in R^\ell_t\}$, because the graph corresponding to the set R^ℓ_t is a subgraph of the graph corresponding to set $\{k_1, \ldots, k_{h(R^\ell_t)}\}$. The algorithm to find the best subset R^ℓ_t of a given S^ℓ_t is similar for the graph G.

Next, we study a dynamic probabilistic lot-sizing (DPLS) problem to illustrate additional modeling considerations for dynamic problems under joint chance constraints. In §6, we present our computational experience with a branch-and-cut method using mixing inequalities (17) and continuous mixing inequalities (23) for DPLS. Next, we compare the dynamic joint chance-constrained program with its static counterpart, a risk-averse optimization model, a robust optimization model, and a pseudo-dynamic approach for an application in inventory control.

5. An Application: Probabilistic Lot Sizing with Service Levels

Starting with the seminal work of Wagner and Whitin (1958), dynamic inventory control models typically rely on the assumption that demand is known for all successive time periods a priori with certainty. This is a restrictive assumption because the future demand can be influenced by many factors, most of which cannot be quantified ahead of time, such as recession, energy prices, etc. In addition, most deterministic inventory control models assume that in case backlogging is allowed, it is penalized with a shortage cost (Zangwill 1966, Pochet and Wolsey 1988, Küçükyavuz and Pochet 2009) with the exception of Gade and Küçükyavuz (2013) who limit the number of periods in which shortages occur.

Stochastic lot-sizing models address randomness in demands and costs. Guan and Miller (2008) study the stochastic uncapacitated lot-sizing problem with zero lead times and give a backward dynamic programming recursive algorithm that is polynomial in time with respect to the size of the scenario tree for cases when backlogging is not allowed or is prohibitively expensive. Huang and Küçükyavuz (2008) study stochastic lot-sizing problem with random lead times and give an algorithm that is polynomial with respect to the size of the scenario tree for cases with no backlogging. Guan (2011) studies the stochastic capacitated lot-sizing problems with zero lead times and gives a dynamic programming algorithm for cases in which backlogging is allowed and is penalized with backlogging costs.

The calculation of backordering cost involves certain intangible factors that are difficult to quantify, such as the cost of lost customer goodwill. In classical stochastic inventory control literature, an alternative model is to set a required service level corresponding to a maximum stockout probability (Nahmias 2005). The service level, τ, indicates the overall joint probability of meeting demand on time and can be a very crucial element of the customer contract as it limits the demand that is backlogged. Bitran and Yanasse (1984) give deterministic approximations for capacitated and static probabilistic production problems using these service levels. Lasserre et al. (1985) use a stochastic optimal control approach for probabilistic lot-sizing problems based on required service levels. In their deterministic equivalent model, violations of chance constraints are penalized in the cost function. In our research, we propose exact methods for dynamic probabilistic lot-sizing models that meet a required service level under a finite discrete distribution on the demands and costs. Unlike classical inventory control models, we assume that the discrete demand distribution over the planning horizon may be nonstationary and correlated.

In more recent literature, Beraldi and Ruszczyński (2002a) discuss a static, probabilistic version of the lot-sizing problem with a required service level and give a branch-and-bound algorithm for solving it. Demand is assumed to follow a finite and discrete distribution, and the production schedule is determined at the beginning of the planning horizon. (See also Lejeune and Ruszczyński 2007 for a more general probabilistic production and distribution planning problem.) This is a static model because it assumes that the production schedule cannot be updated during the planning horizon based on how the demands and costs unfold over time. In reality, many production schedules have the flexibility that the production level can be updated depending on the demands and costs observed in the past periods.

In this section, we consider a single-product multistage probabilistic lot-sizing problem in a dynamic setting. Based on the demands and costs we observed in the previous time periods, we determine an order schedule that minimizes the total expected cost while satisfying both the flow balance and service level constraints. This "wait-and-see" approach allows for the order quantities to be determined as demand and cost evolve. Bookbinder and Tan (1988) present an intermediate "static-dynamic" approach that uses a heuristic based algorithm to yield approximate results. In batch-sizing problems, production occurs in integral multiples of a given batch size. Lulli and Sen (2004) study a related dynamic probabilistic batch-sizing problem and propose a branch-and-price algorithm for its solution. We discuss the assumptions of their model in more detail in §5.2.

We evaluate the overall probability of stocking out over the horizon, instead of maintaining the service level for each period individually. In other words, we consider joint chance constraints instead of the much easier case of individual chance constraints, which call for quantile-based linear reformulations. Although our focus is on the DPLS in this section, for the sake of completeness we first review the static version of the probabilistic lot-sizing (SPLS) problem.

5.1. Static Probabilistic Lot Sizing

Beraldi and Ruszczyński (2002a) study stochastic integer problems under probabilistic constraints and present SPLS as an example. In their model, they approximate the total expected cost by eliminating the holding cost and inventory variables from the objective function. The model with the inventory costs is solved using a branch-and-cut algorithm in Küçükyavuz (2012). In SPLS, the order quantities, x_t, $t = 1, \ldots, n$, are determined at the beginning of the planning horizon. It is assumed that these order quantities cannot be changed during the planning horizon as some of the demands and costs are revealed. The objective is to minimize the expected total cost while complying with the inventory balance and joint probabilistic service level constraints.

Let δ_t, ξ_t, μ_t, γ_t, and ν_t be the random variables for demand, cumulative demand, variable and fixed order costs, and the holding cost in period t, $t \in [1, n]$, respectively. Thus, $\xi_t = \sum_{j=1}^{t} \delta_j$ for $t \in [1, n]$. Suppose that the joint distribution of these random variables is discrete and has finite support. Then the probability distribution can be represented by a finite number, m, of scenarios, with probabilities $\pi^1, \ldots, \pi^m$. The stochastic program for SPLS is

$$\min \ \mathbb{E}_\Gamma\left(\sum_{t=1}^{n} (\mu_t x_t + \nu_t s_t(\xi_t) + \gamma_t w_t)\right)$$

$$\text{s.t. } \mathbb{P}\begin{pmatrix} x_1 & \geq \xi_1 \\ x_1 + x_2 & \geq \xi_2 \\ x_1 + x_2 + x_3 & \geq \xi_3 \\ \vdots & \vdots \ \vdots \\ x_1 + x_2 + x_3 + \cdots + x_n & \geq \xi_n \end{pmatrix} \geq \tau,$$

$$0 \leq x_t \leq M_t w_t \quad t \in [1, n],$$

$$s_t(\xi_t) \geq \sum_{j=1}^{t} x_j - \xi_t \quad t \in [1, n-1], \tag{24}$$

$$s_n(\xi_n) = \sum_{j=1}^{n} x_j - \xi_n, \tag{25}$$

$$s_t(\xi_t) \geq 0 \quad t \in [1, n], \tag{26}$$

$$w_t \in \{0, 1\} \quad t \in [1, n],$$

where $s_t(\xi_t)$ is the inventory at the end of period t and M_t is the order capacity in period t. Inequalities (24) and (26) ensure that $s_t(\xi_t) = (\sum_{j=1}^{t} x_j - \xi_t)^+$, for $t \in [1, n-1]$. Inequality (25) is the end of horizon inventory constraint, which together with (26) ensures that all demand is delivered by the end of the planning horizon. Observe that the random right-hand sides in the chance constraint, $\xi_1, \ldots, \xi_n$, are highly correlated for this problem.

Let D_t^i be the cumulative demand until period t and c_t^i and g_t^i be the variable and fixed costs of ordering, and h_t^i be the variable holding cost in period t under scenario i. Finally, let x_t be the decision variable representing order quantity in period t, s_t^i be the inventory level at the end of period t under scenario i, and w_t be 1 if an order setup is made and 0 otherwise. The deterministic equivalent of the SPLS model is

$$\min \ \sum_{i=1}^{m} \sum_{t=1}^{n} \pi^i (c_t^i x_t + h_t^i s_t^i + g_t^i w_t) \tag{27}$$

$$\text{s.t. } y_t = \sum_{i=1}^{t} x_i \geq D_t^i (1 - z^i) \quad t \in [1, n], \ i \in [1, m], \tag{28}$$

$$\sum_{j=1}^{n} x_j \geq D_n^i \quad i \in [1, m], \tag{29}$$

$$s_t^i \geq \sum_{j=1}^{t} x_j - D_t^i \quad t \in [1, n], \ i \in [1, m], \tag{30}$$

$$0 \leq x_t \leq M_t w_t \quad t \in [1, n], \tag{31}$$

$$\sum_{i=1}^{m} \pi^i z^i \leq 1 - \tau, \tag{32}$$

$$s_t^i \geq 0 \quad t \in [1, n], \ i \in [1, m], \tag{33}$$

$$z^i \in \{0, 1\} \quad i \in [1, m], \tag{34}$$

$$w_t \in \{0, 1\} \quad t \in [1, n], \tag{35}$$

where z^i is 1 if there is unmet demand in scenario i and is 0 otherwise, following the convention of

Luedtke et al. (2010). This formulation has the mixing set (28) and a knapsack constraint (32) as its substructure for which strong valid inequalities are described in §4.1.

5.2. Dynamic Probabilistic Lot Sizing

In DPLS, the order decision for a time period $t = 1, \ldots, n$ is dependent on the demands and costs revealed in periods 1 to $t-1$. This gives us the flexibility to update the order schedule over time, based on what we have already observed. Let $\Gamma_{t-1} := (\xi_1, \mu_1, \gamma_1, \nu_1, \ldots, \xi_{t-1}, \mu_{t-1}, \gamma_{t-1}, \nu_{t-1})$ be the random vector representing the demands and costs until time $t \in [2, n]$. Let $x_t(\Gamma_{t-1})$ be the decision variable at stage $t \in [2, n]$ whose value is determined after the random variables, Γ_{t-1}, are observed and x_1 be the initial order quantity. Then the chance constraint is updated as

$$\mathbb{P}\begin{pmatrix} x_1 & \geq \xi_1 \\ x_1 + x_2(\Gamma_1) & \geq \xi_2 \\ x_1 + x_2(\Gamma_1) + x_3(\Gamma_2) & \geq \xi_3 \\ \vdots & \vdots\ \vdots \\ x_1 + x_2(\Gamma_1) + x_3(\Gamma_2) + \cdots + x_n(\Gamma_{n-1}) & \geq \xi_n \end{pmatrix} \geq \tau. \quad (36)$$

Let x_t^i represent the quantity ordered in period t in scenario i to allow order decisions to be dependent on the demand and cost realizations until time t. Also, let $w_t^i = 1$ if an order setup is made and $w_t^i = 0$ otherwise. Let $S_t^\ell = \{k \in [1, m]: D_j^\ell = D_j^k, h_j^\ell = h_j^k, g_j^\ell = g_j^k, c_j^\ell = c_j^k, j \in [1, t-1]\}$ be the set of scenarios that share the same demand and cost history with scenario ℓ until period t. The deterministic equivalent of the DPLS model is

$$\min \sum_{i=1}^{m} \sum_{t=1}^{n} \pi^i (c_t^i x_t^i + g_t^i w_t^i + h_t^i s_t^i)$$

s.t. (32)–(34),

$$y_t^i = \sum_{j=1}^{t} x_j^i \geq D_t^i (1 - z^i) \quad t \in [1, n],\ i \in [1, m], \quad (37)$$

$$\sum_{j=1}^{n} x_j^i \geq D_n^i \quad i \in [1, m], \quad (38)$$

$$s_t^i \geq \sum_{j=1}^{t} x_j^i - D_t^i \quad t \in [1, n],\ i \in [1, m], \quad (39)$$

$$0 \leq x_t^i \leq M_t w_t^i \quad t \in [1, n],\ i \in [1, m], \quad (40)$$

$$x_t^\ell = x_t^k \quad t \in [1, n],\ \ell \in [1, m],\ k \in S_t^\ell \setminus \{\ell\}, \quad (41)$$

$$w_t^i \in \{0, 1\} \quad t \in [1, n],\ i \in [1, m]. \quad (42)$$

Constraints (37)–(40) are the dynamic versions of constraints (28)–(31). Constraints (41) enforce nonanticipativity. Constraints (37) for a fixed $T \in [1, n]$ and (41) for some $t \leq T$ with $S_t^\ell \neq \varnothing$ have the continuous mixing set as their substructure for which strong valid inequalities are described in §4.2.

Note that for this model, for $1 \leq t \leq T \leq n$, we have $\bar{s} = \sum_{j=1}^{t} A_{Tj} x_j^\ell = y_t^\ell \geq \bar{D}_t^\ell = D_t^{\nu_t^\ell + 1}$, where ν_t^ℓ is as defined in §2. Also, $\sum_{j=t+1}^{T} A_{Tj} x_j^\ell = \sum_{j=t+1}^{T} x_j^\ell \geq \mathbf{0}$. Therefore, the stronger continuous mixing inequalities (23) are valid.

Example 1. To highlight the difference between static and dynamic models, let us consider a small test case with five time periods and five scenarios as given in Figure 1. The two numbers in the nodes of the scenario tree indicate the demand and unit order cost for each scenario in that time period. For example, the demand and unit order cost at period 2 in scenario 1 are 73 and 48, respectively. Each outcome of the future is represented by a scenario path. Note

Figure 1 Scenario Tree Representation

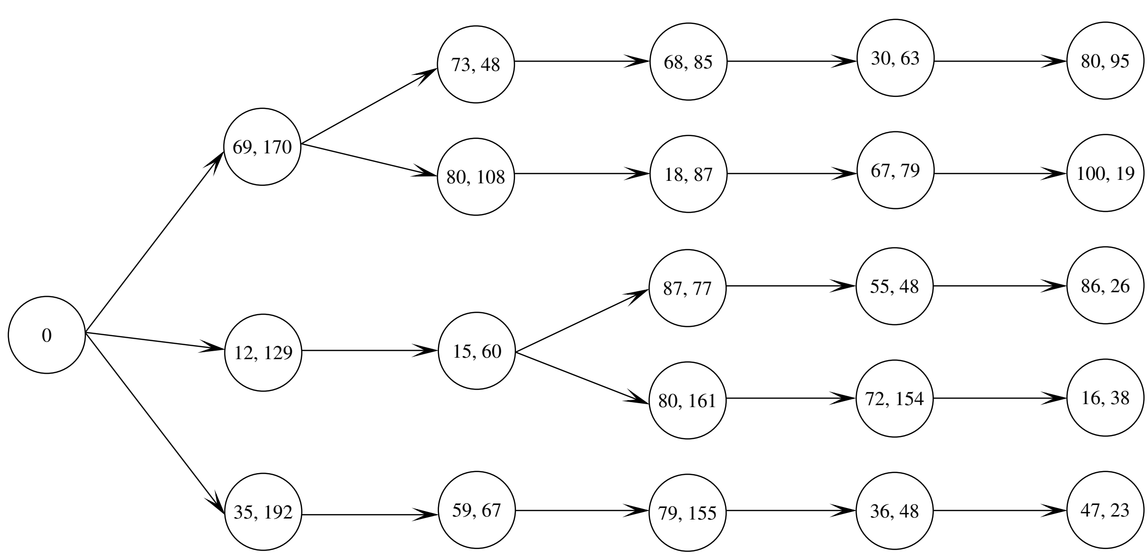


Table 1 Quantities Produced in DPLS and SPLS Models

	DPLS					
	Scenario					
Time	1	2	3	4	5	SPLS
1	69	69	69	69	69	69
2	73	73	110	110	104	141
3	68	0	0	0	0	0
4	110	0	0	0	36	30
5	0	192	76	16	47	94

that $S_1^j = \{1, 2, \ldots, 5\}$, for $j = 1, \ldots, 5$ (because at the first period we did not see any demand); $S_2^1 = S_2^2 = \{1, 2\}$; $S_2^3 = S_2^4 = \{3, 4\}$; and $S_3^3 = S_3^4 = \{3, 4\}$. Let the probability of occurrence of each scenario be $\pi = (0.25, 0.15, 0.10, 0.10, 0.40)$. Let the holding cost be 10% of the unit order cost and the required service level τ be 85%. Finally, suppose that there are no fixed order costs. We report the optimal order quantities for the SPLS and DPLS models in Table 1. The optimal cost given by the SPLS model is 29212.6. In addition, because the order quantities are decided ahead of time and are independent of the scenario path realized, the observed service level is 100%, even though backorders are acceptable. On the other hand, the optimal cost given by the DPLS model is 24143.0 and the observed service level is 85%. As a result, significant cost savings are achieved when order quantities are determined based on the demand and cost history.

Lulli and Sen (2004) consider the dynamic probabilistic batch-sizing problem (DPBS) and propose a branch-and-price algorithm for solving it. In the Dantzig–Wolfe reformulation of DPBS, the authors introduce one variable for each feasible order schedule for every scenario and another variable that is equal to 1 if a scenario is not violated and 0 otherwise. Then the master problem enforces nonanticipativity and calculates cost only for the scenarios that are not violated. In contrast, in our model, we assume that the nonanticipativity holds even for scenarios which have a stockout, which results in solutions that are implementable. We observe from Figure 1 that scenarios 3 and 4 have same demand and cost history until period 3. Therefore same decisions must be made for these scenarios until time period 3. If we do not enforce the nonanticipativity constraints (41) for the scenarios that are violated, we get the order quantities shown in Table 2. Note that this order schedule is not implementable. The order quantity at the first time period has to be the same across all scenarios because so far we have seen no demand and order costs. Therefore, at this point (the first time period) our decision should be independent of which scenario path we will follow in future.

Table 2 Quantities Produced in the DPLS Model with Relaxed (41)

	Scenario				
Time	1	2	3	4	5
1	69	0	12	12	35
2	251	0	102	167	138
3	0	0	0	0	0
4	0	0	55	0	36
5	0	334	86	16	47

In addition, we assume that all demand has to be met at the end of the time horizon in every scenario by introducing the end of horizon constraints (38). In Table 3 we show the order quantities in the DPLS model with no end of horizon constraints. As shown in the table, the scenarios that have stockouts also have unfulfilled demand at the end of planning horizon. For example, total demand in scenario 2 is 334 units, whereas total order quantity in scenario 2 is only 142 units, leading to a shortage at the end. Once a stockout occurs at period 3, no further orders take place and all future demand is lost. Therefore, we need to add these constraints to ensure that all demand is satisfied at the end of contract horizon for all scenarios.

Next we give an example for a valid mixing inequality. Let $t = 2$, $\ell = 1$, $S_2^1 = S_2^2 = \{1, 2\}$, $\bar{T} = \{2\}$, and $D_t^{\nu_t^\ell + 1} = 69 + 73$; then a valid mixing inequality (17) is

$$x_1^1 + x_2^1 + [(69 + 80) - (69 + 73)]z^2 \geq 69 + 80.$$

To see the validity of this inequality, note that if $z^2 = 0$, then the demand is not backlogged in scenario 2. Therefore, we must have $x_1^1 + x_2^1 \geq 149$. On the other hand, if $z^2 = 1$, then the demand can be backlogged in scenario 2; however, because $\pi^1 = 0.25$ and $\tau = 0.85$, demand cannot be backlogged in scenario 1. Hence, we must have $x_1^1 + x_2^1 \geq 142$.

Now we give an example for a valid continuous mixing inequality. Let $t = 1$, $\ell = 1$, $T = 3$, and $S_1^1 = \{1, 2, 3, 4, 5\}$. Note that $\bar{D}_1^1 = 69$. Consider the constraints associated with set $R_t^\ell = \{2, 3\} \subseteq S_1^1$:

$$x_1^2 + x_2^2 + x_3^2 + 167z^2 \geq 167,$$

$$x_1^3 + x_2^3 + x_3^3 + 114z^3 \geq 114.$$

Table 3 Quantities Produced in the DPLS Model with Relaxed (38)

	Scenario				
Time	1	2	3	4	5
1	69	69	69	69	69
2	73	73	110	110	104
3	68	0	0	0	0
4	110	0	0	0	36
5	0	0	76	16	47

Note that $\bar{D}^1_{1,2} = \max\{167, 114\} - \bar{D}^1_1 = 98$. Then

$$x_1^2 - 69 + x_2^2 + x_3^2 + 98z^2 \geq 167 - 69 = 98,$$
$$x_1^3 - 69 + x_2^3 + x_3^3 + 98z^3 \geq 114 - 69 = 45,$$

or

$$\frac{x_1^2 - 69}{98} + \frac{x_2^2 + x_3^2}{98} + z^2 - 1 \geq 0,$$
$$\frac{x_1^3 - 69}{98} + \frac{x_2^3 + x_3^3}{98} + z^3 \geq \frac{45}{98},$$

where $x_1^2 = x_1^3$ because of nonanticipativity.

Consider the graph $G^+ = (V^+, E^+)$, where $V^+ = \{0, 2, 3\}$ and $E^+ = \{(0,3), (3,0), (0,2), (2,0), (2,2), (2,3), (3,3), (3,2)\}$ and the nonelementary cycle $\{(2,3), (3,2)\} \subseteq E^+$. The length of arc $(2,3)$ is $(x_1^2 - 69)/98 + (x_2^2 + x_3^2)/98 + (0 - 45/98 + 1)(z^2 - 1) - 45/98$ because $f^2 = 0 < 45/98 = f^3$. The length of arc $(3,2)$ is $(x_2^3 + x_3^3)/98 + (45/98 - 0)z^3$ because $f^3 = 45/98 = f^2 > 0$. The corresponding continuous mixing inequality (23) is

$$\frac{x_1^2 - 69}{98} + \frac{x_2^2 + x_3^2}{98} + \left(1 - \frac{45}{98}\right)(z^2 - 1) - \frac{45}{98} + \frac{x_2^3 + x_3^3}{98} + \frac{45}{98}z^3 \geq 0,$$

or

$$x_1^2 + x_2^2 + x_3^2 + 53z^2 + x_2^3 + x_3^3 + 45z^3 \geq 167. \quad (43)$$

To see the validity of inequality (43), note that we cannot have $z^2 = z^3 = 1$, because the total probability of scenarios 2 and 3 is $0.25 \geq 1 - \tau$. If $z^2 = 0$, then we must have $x_1^2 + x_2^2 + x_3^2 \geq 167$; thus, inequality (43) is satisfied. Finally, if $z^2 = 1$ and $z^3 = 0$, then $x_1^2 + x_2^3 + x_3^3 = x_1^3 + x_2^3 + x_3^3 \geq 114$, where the first equality follows from nonanticipativity. Observe that for the same choice of t, T, ℓ, and R_t^ℓ, inequality (20) is given by

$$x_1^2 + x_2^2 + x_3^2 + 53z^2 + x_2^3 + x_3^3 + 114z^3 \geq 167,$$

which is clearly weaker than (43).

5.3. Comparison with a Dynamic Risk-Averse Optimization Model

Risk-averse optimization has been receiving increasing attention in recent years (Ruszczyński and Shapiro 2006a, b; Shapiro 2009). In dynamic risk-averse optimization, decisions are made in multiple stages such that a nested dynamic risk measure often involving the costs is minimized (see Ruszczyński and Shapiro 2009). In contrast, the dynamic joint chance-constrained model (8)–(15) minimizes a risk-neutral objective (the expected total cost) while controlling the risk by a constraint that ensures that every feasible solution meets the required service level. In another line of research, Dentcheva and Ruszczyński (2008) propose a stochastic dynamic optimization problem in which the risk aversion is expressed by a stochastic ordering constraint.

We now compare a dynamic risk-averse optimization model for multistage inventory problems (Ahmed et al. 2007) with the joint chance-constrained DPLS model. As in previous sections, δ_t, ξ_t, μ_t, γ_t, and ν_t denote the random demand, cumulative demand, variable cost, fixed charge, and holding cost in period t. Let d_t denote the realization of δ_t in period t. We assume that δ_t has a discrete distribution and is independent of $(\delta_1, \ldots, \delta_{t-1})$. In addition, let ζ_t denote the random backlogging cost in period t, whose realization is denoted as b_t. Ahmed et al. (2007) propose a multistage risk-averse inventory model, where the decision makers decide the order quantity at the beginning of each period after observing the inventory level at the end of the previous period. Based on the model in Ahmed et al. (2007), but using the same decision variables x_t and w_t, $t \in [1, n]$ as in our SPLS model, consider the dynamic risk-averse optimization model

$$\begin{aligned}\min_{\substack{0 \leq x_t \leq M w_t,\, t \in [1,n] \\ \sum_{t=1}^n x_t \geq \xi_n \\ y_t = \sum_{i=1}^t x_i,\, t\in[1,n] \\ w_t \in \{0,1\},\, t \in [1,n]}} & \rho_1\big[\mu_1 x_1 + \gamma_1 w_1 + \nu_1(y_1 - \xi_1)^+ + \zeta_1(\xi_1 - y_1)^+ \\ & + \rho_{2|\Gamma_1}\big[\mu_2 x_2 + \gamma_2 w_2 + \nu_2(y_2 - \xi_2)^+ + \zeta_2(\xi_2 - y_2)^+ + \cdots \\ & + \rho_{t|\Gamma_{t-1}}\big[\mu_t x_t + \gamma_t w_t + \nu_t(y_t - \xi_t)^+ + \zeta_t(\xi_t - y_t)^+ + \cdots \\ & + \rho_{n|\Gamma_{n-1}}\big[\mu_n x_n + \gamma_n w_n + \nu_n(y_n - \xi_n)^+ \\ & + \zeta_n(\xi_n - y_n)^+\big] \cdots\big] \cdots\big]\big], \end{aligned} \quad (44)$$

where $\rho_{t|\Gamma_{t-1}}(Z)$ is a risk measure. In the following example, we use the mean-absolute deviation (MAD) as the risk measure, which is defined as $\mathrm{MAD}_\lambda(Z) := \mathbb{E}[Z] + \lambda\mathbb{E}|Z - \mathbb{E}[Z]|$, $\lambda \in [0, 1/2]$.

Example 1 (Continued). Let us solve the five-period test case with the risk-averse model (44). To illustrate the effect of the backlogging cost, we consider alternative backlogging costs given by 0, $0.5\mu_t$, $1.50\mu_t$, or $2\mu_t$. The comparison between the solutions, service levels, and the expected total order and holding costs of the DPLS, SPLS, and the risk-averse models is shown in Table 4. In our expected cost comparison, we recalculate the expected total cost of the risk-averse model as $\sum_{i=1}^m \sum_{t=1}^n \pi_t^i (c_t^i x_t + g_t^i w_t + h_t^i s_t)$ to exclude the artificial backlogging costs. When the backlogging costs are taken as zero, we see that optimizing the multistage risk measure on the order and holding costs cannot guarantee a feasible solution that meets the required service level. Increasing the backlogging cost restricts backlogging

Table 4 Order Quantities for DPLS, SPLS, and Risk-Averse Models in Example 1

	DPLS						Risk-averse			
	Scenario						ζ_t/μ_t			
Time	1	2	3	4	5	SPLS	0	0.5	1.5	2
1	69	69	69	69	69	69	0	0	35	69
2	73	73	110	110	104	141	0	210	205	171
3	68	0	0	0	0	0	0	0	0	0
4	110	0	0	0	36	30	0	0	0	0
5	0	192	76	16	47	94	334	124	94	94
Service level (%)			85			100	0	25	60	100
Cost increase from DPLS (%)			—			21.00	−32.07	−12.80	7.48	23.49

quantity to some extent. For example, the service level increases from 0 to 25% when the backlogging cost increases from 0 to $0.5\mu_t$. However, even when we let the backlogging costs equal 1.5 times the unit order costs ($1.5\mu_t$), we are not able to find a feasible solution to our original problem because the service level is only at 60%. We cannot find a feasible solution until we increase the backlogging costs to twice the unit order cost ($2\mu_t$), but the expected order and holding cost is 23.49% higher than that of DPLS. Clearly, the risk-averse model optimizes a different objective than the DPLS model. However, we provide the expected cost comparison to show that even when a high enough backlogging penalty is used in the risk-averse model that yields a feasible solution to DPLS, the solution quality with respect to the original objective can be far from optimal. Comparing the solutions to the DPLS, SPLS, and the risk-averse models, we can conclude that the decisions made by the DPLS model always satisfy the required service level at significantly lower expected order and holding costs. It is worth noting that in our model, the risk measure of meeting a service level over the entire horizon is a static risk measure, unlike those in dynamic risk-averse optimization. Shapiro (2009) discusses a desirable time-invariance property for certain dynamic risk measures.

Next we consider a pseudo-dynamic approach, which solves a series of static joint chance-constrained programs in an effort to obtain feasible solutions that are adaptive to the past observations with less computational effort than the fully dynamic model.

5.4. Comparison with a Pseudo-Dynamic Approach

The rolling horizon method (Baker 1977) is one of the widely used methodologies to solve a multistage problem. In solving a problem with n stages, at the beginning of the first period, we solve a subproblem involving periods 1 to $1+n_1$, $1+n_1 \le n$, and fix the solution to the first period. Then the input to the second period is updated. At the beginning of the second period, we solve a subproblem on periods 2 to $2+n_2$, where $2+n_2 \le n$ and n_2 is not necessarily equal to n_1, and fix the solution to the second period. We repeat this process until the solution to the last period is fixed. The rolling horizon method significantly reduces the computational burden for problems with a long or infinite horizon. The rolling horizon method has been applied to the lot-sizing problem with time-varying costs and demands (see, e.g., Morton 1977, Blackburn and Millen 1980).

In this subsection, we apply a pseudo-dynamic approach that is inspired by the rolling horizon method to the probabilistic stochastic lot-sizing problem. Initially, we solve the SPLS model and obtain the static decisions $x=(x_1, x_2, \ldots, x_n)$. We choose x_1 as the order quantity in the first period. In period $t \in [2, n]$, for each nonanticipative node in period t, we consider a similar static decision-making problem over the planning subhorizon over periods t to n, and we choose x_t as the order quantity in period t. Note that the resulting decisions are dynamic (adaptive to the past observations), but we solve a series of static decision-making models, which are easier to solve. During this process, we update the required service level for each subproblem. In period $t \in [1, n]$, given the solution x_j^i for $j \in [1, t-1]$ and $i \in [1, m]$, we define a set $V_t := \{i\colon\ i \in [1, m], \sum_{k=1}^{j} x_k^i \ge D_j^i \text{ for } j \in [1, t-1]\}$ as the set of scenarios that do not have stockout in periods 1 to $t-1$. Moreover, we define the updated required service level for the nonanticipative set S_t^ℓ as $\tilde{\tau}_t^\ell$, where $\tilde{\tau}_t^\ell = 0$ if $\ell \notin V_t$; otherwise $\tilde{\tau}_t^\ell = 1 - (\sum_{i \in V_t} \pi^i - \tau)^+$.

For a given $t \in [1, n]$ and $\ell \in [1, m]$, the initial inventory/stockout quantity is $\tilde{s}_{t-1}^\ell := \sum_{j=1}^{t-1} x_j - D_{t-1}$. Note that $\tilde{s}_{t-1}^\ell$ is negative when there is stockout. At each nonanticipative node associated with S_t^ℓ, given V_t and

$\tilde{s}_{t-1}$, the problem we need to solve, $SP_t^\ell(\tilde{s}_{t-1}, \tilde{\tau}_t^\ell)$, is defined as

$$\min \sum_{i\in S_t^\ell}\sum_{j=t}^{n} \tilde{\pi}^i (c_j^i x_j + h_j^i s_j^i + g_j^i w_j)$$

$$\text{s.t } \tilde{s}_{t-1}^\ell + \sum_{k=t}^{j} x_k \geq \sum_{k=t}^{j} d_k^i (1-z^i) \quad j\in[t,n],\ i\in S_t^\ell,$$

$$\tilde{s}_{t-1}^\ell + \sum_{k=t}^{n} x_k \geq \sum_{k=t}^{n} d_k^i \quad i\in S_t^\ell,$$

$$s_j^i \geq \sum_{k=t}^{j}(x_k - d_k^i) + \tilde{s}_{t-1}^\ell \quad j\in[t,n],\ i\in S_t^\ell,$$

$$0\leq x_j \leq M_j w_j \quad j\in[t,n],$$

$$\sum_{i\in S_t^\ell} \tilde{\pi}^i z^i \leq 1-\tilde{\tau}_t^\ell,$$

$$s_j^i \geq 0 \quad j\in[t,n],\ i\in S_t^\ell,$$

$$z^i\in\{0,1\} \quad i\in S_t^\ell,$$

$$w_j\in\{0,1\} \quad j\in[t,n],$$

where $\tilde{\pi}^i$ is the probability of scenario i. There are two options for $\tilde{\pi}^i$: (1) we can keep $\tilde{\pi}^i = \pi^i$ when we solve every subproblem; (2) we can update the probability of scenario i in S_t^ℓ for the problem $SP_t^\ell(\tilde{s}_{t-1}, \tilde{\tau}_t^\ell)$ by the conditional probability $\tilde{\pi}^i = \pi^i/(\sum_{j\in S_t^\ell}\pi^j)$. Now we look into these two cases.

Case 1. Table 5 gives the solution to the pseudo-dynamic model when we do not update the probabilities of each scenario at each stage. The service level reached is 75%, which is lower than the required service level. The expected total cost is 24,561.71. From this case, we can reach the conclusion that using the pseudo-dynamic approach without updating the probabilities of each scenario, we may obtain infeasible solutions violating the required service level constraint.

Case 2. Table 6 gives the solution to the pseudo-dynamic model when we update the probabilities of each scenario in each stage using conditional probabilities. The service level reached is 100%, and the expected total cost is 26,841.59, which is higher than that of the DPLS model but lower than that of the SPLS model. Thus, for this instance, the pseudo-dynamic approach finds a more conservative decision compared with the DPLS model and a less conservative decision compared with the SPLS model.

Interestingly, the solution given by the pseudo-dynamic approach using conditional probabilities can give more conservative solutions than that of SPLS. For example, if we consider the instance shown in Table 7 with two scenarios and three periods with the required service level at 80%, the solution given by the static model is $x=(2,0,10)$ with the expected total cost of 13. The solution given by the pseudo-dynamic approach is $x^1=(2,9,1)$, $x^2=(2,0,1)$, with the expected total cost of 22. This comparison indicates that although the pseudo-dynamic approach requires more computational effort than the static model does, it cannot guarantee a better solution than that of the static model.

Table 5 Quantities Produced by the Pseudo-Dynamic Approach Without Updating Probabilities

	DPLS				
	Scenario				
Time	1	2	3	4	5
1	69	69	69	69	69
2	141	141	100	100	104
3	0	0	0	0	0
4	110	0	0	0	36
5	0	124	86	26	47

Table 6 Quantities Produced by the Pseudo-Dynamic Approach With Updated Probabilities

	DPLS				
	Scenario				
Time	1	2	3	4	5
1	69	69	69	69	69
2	141	141	110	110	104
3	0	0	0	0	0
4	110	24	0	0	36
5	0	100	76	16	47

Table 7 Costs and Demands for a Two-Scenario Instance

	Scenario 1			Scenario 2		
	Time			Time		
Parameters	1	2	3	1	2	3
c	1	10	1	1	1	1
d	1	10	1	1	1	1
f	0	0	0	0	0	0
h	1	1	0	1	1	0
π		0.2			0.8	

5.5. Comparison with a Robust Optimization Model

In the dynamic joint chance-constrained model (8)–(15), we assume that the random variables have finitely many realizations; i.e., they have known discrete distributions. Another important method addressing the uncertainty of data is robust optimization, which assumes that the random variables belong to an uncertainty set and optimizes against the worst case. For example, Soyster (1973) proposes a formulation for a convex mathematical programming problem in which random variables can take any value in the given ranges. Ben-Tal and Nemirovski (1998, 1999, 2000), El Ghaoui and Lebret (1998), and El Ghaoui et al. (1997) assume that the uncertain variables are in ellipsoidal uncertainty sets. Erdoğan and Iyengar (2006) study the ambiguous chance-constrained program where the distributions of the random variables are uncertain. They propose a robust sampled problem to approximate the ambiguous chance-constrained program with a high probability. To avoid making overconservative decisions by considering the worst case, Bertsimas and Sim (2003) propose a robust integer programming model that allows controlling the degree of conservatism of the solutions by limiting the extent that the variables deviate from their mean values. Bertsimas and Thiele (2006) apply this technique to inventory problems under the assumption that the variable costs, fixed charges, holding costs, and backlogging costs are constant. Bertsimas and Brown (2009) show that for a linear optimization problem with uncertain data, given a coherent risk measure and the realizations of the uncertain data, a convex uncertainty set can be constructed to obtain the robust counterpart of an individual constraint.

For multistage robust optimization, Ben-Tal et al. (2009) propose an adjustable robust counterpart of multistage stochastic linear program in which the decision variables are affinely dependent on the past observations. Bertsimas et al. (2010) prove the optimality of the control policies that are affine in the realizations of random data in multistage robust optimization with convex state and linear control costs and a worst case objective. Bertsimas et al. (2009) demonstrate the application of this result in a single-echelon (one-dimensional) multiperiod robust inventory problem, with uncertain demand, convex inventory costs, and linear ordering costs. Bertsimas et al. (2011) propose a hierarchy of suboptimal polynomial control policies parameterized in the revealed uncertainty for the multidimensional, constrained, multistage robust optimization. In these models, there does not exist any constraint restricting the service performance over the entire planning horizon. In contrast, in our dynamic decision-making model, guaranteeing the service level over the entire planning horizon is one of the requirements. In addition, we assume linear holding costs and concave (fixed plus linear) order costs as in classical lot sizing (Wagner and Whitin 1958, Bitran and Yanasse 1982).

Finally, safe tractable approximations of joint chance-constrained linear programs are proposed in Chen et al. (2010) and Zymler et al. (2013). Chen et al. (2010) propose a second-order cone programming (SOCP) reformulation to approximate a joint chance constraint. Zymler et al. (2013) propose a tractable semidefinite programming (SDP) reformulation to approximate the distributionally robust joint chance constraints, where the set of all probability distributions for the random parameters is known. Although SOCP and SDP based approximations are computationally tractable for joint chance-constrained linear programs with continuous variables, these approximations are not tractable in the existence of binary variables as is the case in our original dynamic decision-making program. Note that the lot-sizing problem with time-varying capacities is NP-hard even when the demand and costs are deterministic (Bitran and Yanasse 1982).

In this section, we compare our joint chance-constrained program with a simplified robust counterpart. Let the uncertainty set for the cumulative demands be U, which is defined as $U=\{\{\xi_t\}_{t=1}^n\colon \underline{D}_t \le \xi_t \le \bar{D}_t,\ t\in[1,n]\}$. Here we use static decision variables as in the SPLS model. To compare with the DPLS model, we minimize the expected total cost, i.e., the objective function (27). Note that $s_t^i = (\sum_{i=1}^t x_t - \xi_t)^+$. One may consider a simplified robust counterpart of the joint chance constraint as $\sum_{j=1}^t x_j \ge \kappa \bar{D}_t$ for $t\in[1,n]$, where κ is an appropriately selected fraction of the maximum demand. The motivation is to meet the required service level while avoiding conservative solutions. Note that $\kappa = 100\%$ corresponds to the robust counterpart of the stochastic lot-sizing problem, guaranteeing a 100% service level, which may be overly conservative. A simplified robust optimization model of the lot-sizing problem (ROLS) is then

$$\min \quad \sum_{i=1}^m \sum_{t=1}^n \pi^i \left(c_t^i x_t + g_t^i w_t + h_t^i \left(\sum_{i=1}^t x_t - \xi_t \right)^+ \right)$$

$$\text{s.t.} \quad \sum_{j=1}^t x_j \ge \kappa \bar{D}_t \quad t\in[1,n-1], \tag{45}$$

$$\sum_{j=1}^n x_j \ge \bar{D}_n, \tag{46}$$

$$0 \le x_t \le M_t w_t \quad t\in[1,n], \tag{47}$$

$$w_t \in \{0,1\} \quad t\in[1,n]. \tag{48}$$

Example 1 (Continued). Consider the robust optimization model for the five-period test case. We first process the given data for the DPLS and SPLS models to obtain a valid uncertainty set. We let $\bar{D}_t = \max_{i\in[1,m]} D_t^i$. We solve the ROLS model with alternative $\kappa \in \{85\%, 90\%, 95\%, 100\%\}$. We do not consider the case that $\kappa < \tau = 85\%$ because the required service level is 85% in this example. The solutions are shown in Table 8.

We observe that until κ reaches 100%, we do not have a feasible solution meeting the required service level. When $\kappa = 100\%$, the expected total cost is 21.00% higher than that of DPLS. It is also 1.00%

Table 8 Solutions to ROLS with Alternative κ

κ (%)	x_1	x_2	x_3	x_4	x_5	Service level (%)	Cost increase from DPLS (%)
85	58.65	119.85	0	25.5	130	20	9.68
90	62.1	126.9	0	27	118	60	13.42
95	65.55	133.95	0	28.5	106	60	17.19
100	69	141	0	30	94	100	21.00

higher than that of SPLS because of the conservatism of the solution.

6. Computations

We test the effectiveness of a branch-and-cut algorithm using continuous mixing cuts, which include mixing cuts as special cases, in solving the capacitated dynamic probabilistic lot-sizing problem described in §5.2. We also compare the optimal objective function values to the DPLS and SPLS models for these instances. The scenario tree generated is a full binary tree, where every node other than the leaf nodes has two child nodes; thus the number of scenarios is $m = 2^n$. We use the notation $Z \sim DU(u_1, u_2)$ to represent that the random variable Z follows a discrete uniform distribution in the interval $[u_1, u_2]$. For $t \in [1, n]$, the unit order cost $\mu_t \sim DU(10, 20)$ and the demand $\delta_t \sim DU(50, 100)$. To generate the scenario probabilities, we first randomly generate m numbers in the interval $[1, 201]$, then normalize them to have m numbers in $(0, 1)$, which sum to 1. We consider alternative time horizons $n \in \{7, 8\}$, required service levels $\tau \in \{0.75, 0.85, 0.9\}$, and ratios of fixed and variable order costs $\theta \in \{100, 150\}$. Let the capacity $M_t = 0.95 \max_{i\in[1,m]} D_n^i$, $t \in [1, n]$. For each combination, we report the average of five randomly generated instances. We conduct all the experiments on a 1-GHz Dual-Core AMD Opteron™ Processor 1,218 with 2 GB RAM. We use IBM ILOG CPLEX 12.2 as the mixed-integer programming solver and impose an hour time limit. We test three alternative branch-and-cut methods for the DPLS model:

1. Algorithm 1: Branch-and-cut with CPLEX cuts.
2. Algorithm 2: Branch-and-cut with CPLEX cuts and mixing cuts (17).
3. Algorithm 3: Branch-and-cut with CPLEX cuts and continuous mixing cuts (23) with $R_t^\ell = \{k \in S_t^\ell : D_T^k \ge \bar{D}_t^\ell = D_t^{\nu_t^\ell + 1}\}$. In this case, we separately report the number of continuous mixing cuts that are equivalent to mixing cuts (17), to highlight the usefulness of the more general class of continuous mixing inequalities.

We solve the SPLS model using the branch-and-cut method with mixing cuts (17).

We report our results in Tables 9 and 10, for $n = 7$ and $n = 8$, respectively. In columns "RGap" and "EGap", we report the average percentage integrality gap at the root node before branching and the end gap output by CPLEX, respectively. Column "Cuts" reports the average number of CPLEX cuts, mixing

Table 9 Computational Results for Seven-Stage Instances

		Cuts							
$n.m.\tau.\theta$	Alg.	CPLEX	MIX	CMIX	RGap (%)	Node	Time	EGap (%)	Cost (%)
7.128.90.100	1	3,151.6	0	0	11.84	7,129	73.5	0.00	
	2	2,386.4	260.8	0	11.63	3,643	36.9	0.00	
	3	2,160.6	260.2	234.4	10.67	2,410.6	29.5	0.00	
	SPLS	106.4	74.8	0	0.54	2.2	0.6	0.00	9.57
7.128.90.150	1	2,913.4	0	0	8.21	24,679.2	169.1	0.00	
	2	2,265.2	123.4	0	6.51	5,995.2	56.2	0.00	
	3	2,392.2	121.6	149.4	6.44	7,090.4	53.7	0.00	
	SPLS	1,006.4	79.6	0	2.34	12	1.8	0.00	11.36
7.128.85.100	1	3,804.2	0	0	10.59	31,351.6	206.9	0.00	
	2	3,555.6	194.6	0	10.47	12,223	107.5	0.00	
	3	3,483	187	206.2	9.50	25,103	116.7	0.00	
	SPLS	1,048	200.6	0	1.82	15.2	2.1	0.00	11.73
7.128.85.150	1	3,048	0	0	10.08	48,891.8	579.9	0.00	
	2	2,822.8	341.2	0	7.56	58,266.6	471.5	0.00	
	3	2,689.8	448.6	218.2	7.42	54,996.8	410.8	0.00	
	SPLS	1,633.6	78	0	2.46	69.4	2.3	0.00	9.01
7.128.75.100	1	4,599	0	0	15.69	81,906.6	826.9	0.00	
	2	4,204	395	0	13.74	47,180	466.0	0.00	
	3	4,072.4	399.6	311.8	13.34	29,835	329.5	0.00	
	SPLS	3,018.2	452.2	0	5.26	227.8	7.4	0.00	13.81
7.128.75.150	1	4,599	0	0	15.11	99,812.2	712.9	0.00	
	2	3,586.8	427.4	0	10.97	18,376.2	214.9	0.00	
	3	3,253.6	477.8	270	9.22	17,698.6	153.4	0.00	
	SPLS	2,888.2	522.2	0	4.11	151.8	5.5	0.00	11.95

Table 10 Computational Results for Eight-Stage Instances

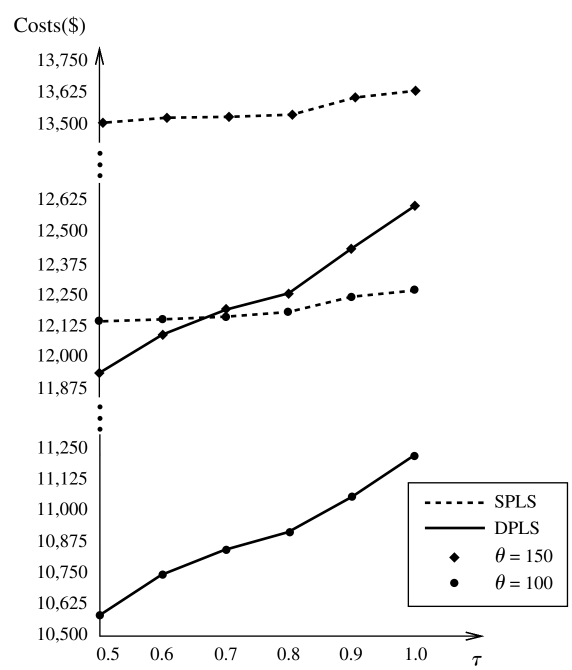
		Cuts							
$n.m.\tau.\theta$	Alg.	CPLEX	MIX	CMIX	RGap (%)	Node	Time	EGap (%)	Cost (%)
8.128.90.100	1	9,278	0	0	21.36	90,485.6	2,152.7(3)	1.12	
	2	8,147.2	1,661.6	0	21.16	73,827.2	897.1(3)	0.85	
	3	8,000.4	1,646.4	650.2	14.10	73,901.4	744.9(3)	0.71	
	SPLS	4,112.8	655	0	6.65	153	19.4	0.00	17.98
8.128.90.150	1	9,975	0	0	17.93	61,122.2	$\geq$ 3,600	1.60	
	2	9,023.6	951.4	0	15.96	85,794.6	$\geq$ 3,600	1.14	
	3	8,792.2	950.6	548	15.22	107,789	3,101.0(4)	1.12	
	SPLS	2,436.8	189	0	4.22	128.6	7.6	0.00	10.73
8.128.85.100	1	9,975	0	0	21.21	57,165.8	$\geq$ 3,600	2.59	
	2	8,512.8	1,462.2	0	20.02	83,876.6	$\geq$ 3,600	1.57	
	3	8,266.8	1,455.6	693.6	19.95	75,115.8	$\geq$ 3,600	1.53	
	SPLS	6,572	871	0	8.88	291.4	25.8	0.00	15.73
8.128.85.150	1	9,975	0	0	17.29	55,803.8	$\geq$ 3,600	3.42	
	2	8,878.6	1,096.4	0	14.25	41,197.4	1,843.5(4)	2.06	
	3	8,634.2	1,082.6	668.6	14.12	75,253.6	677.1(4)	1.82	
	SPLS	4,996.8	1,730.6	0	7.30	1,440.4	36.1	0.00	12.55
8.128.75.100	1	9,975	0	0	23.07	59,764.8	$\geq$ 3,600	4.22	
	2	8,330.6	1,418	0	20.53	48,209.2	$\geq$ 3,600	2.78	
	3	8,318.6	1,428.2	686.6	20.09	65,938.2	$\geq$ 3,600	2.55	
	SPLS	5,634.8	1,836.6	0	10.88	2,734.2	82.8	0.00	15.41
8.128.75.150	1	8,200.8	0	0	19.99	174,420.8	$\geq$ 3,600	3.73	
	2	6,653	1,986.4	0	18.11	108,442.2	1,460.5(4)	2.95	
	3	6,351	1,865.6	510.6	17.80	101,584.6	1,382.2(3)	2.30	
	SPLS	5,715.2	1,781.8	0	8.54	1,588	82.4	0.00	16.60

cuts, and continuous mixing cuts with $t < T$ added in columns "CPLEX", "MIX", and "CMIX", respectively. Columns "Time" and "Node" compare the average solution time (in seconds) and number of branch-and-cut tree nodes explored, respectively. In cases when some instances are not solved within the time limit, we report, in parentheses, the number of unsolved instances (unslvd). Column "Cost" reports the percentage cost increase of the SPLS model from that of the DPLS model. In the cases that none of the Algorithms 1, 2, and 3 can solve the DPLS model to optimality within the time limit, we compare the cost of the SPLS model with the best cost of the DPLS model obtained among these three algorithms.

Note that all of the seven-stage problems with 128 scenarios can be solved to optimality in less than half an hour with all four algorithms. However, our branch-and-cut algorithm with continuous mixing cuts, which adds a few hundred continuous mixing cuts, has smaller root gaps, and solves these instances faster than CPLEX default. For eight-stage problems with 256 scenarios, all but two of the instances reach the time limit with CPLEX default. However, using our branch-and-cut algorithm with continuous mixing cuts, more of these instances can be solved to optimality. Clearly, there is a trade-off between solution time and quality (with respect to expected costs) comparing the static and dynamic models. More cuts are added when the service level, τ, is smaller. This may be because the number of fractional z variables increases under low service levels. Note that the branch-and-price algorithm proposed in Lulli and Sen (2004) solves the related probabilistic batch-sizing instances with up to 16 scenarios.

In Figure 2, we depict the optimal expected cost of the SPLS and DPLS models as a function of the

Figure 2 Optimal Expected Cost vs. τ



required service level $\tau \in \{0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$ for the instances with $n = 7$, for $\theta = 100$ and $\theta = 150$. Clearly, the expected cost increases when the required service level increases, and the expected costs of the SPLS model are significantly higher than those of the DPLS model. We also observe that because of the limited flexibility of the static model, the expected costs remain relatively flat as the required service level, τ, changes. However, the solution to the dynamic model provides larger cost savings with small decreases in the service level.

7. Conclusions

In this paper, we present exact models and methods for a finite-horizon multistage (dynamic) stochastic mixed-integer programs under a joint chance constraint on the performance of the system over the entire horizon. This generalizes the more common single-stage (static) models with joint chance constraints prevalent in the literature. Our experiments illustrate that significant cost savings can be achieved when the decisions are adaptive to the past observations of random data. We show that the deterministic equivalent problem has the so-called continuous mixing substructure, and we develop a branch-and-cut algorithm based on this observation. We illustrate the proposed models and methods on a DPLS problem. Our computations demonstrate that the continuous mixing cuts are effective in solving DPLS. We also compare our DPLS model with the existing dynamic risk-averse optimization and robust optimization models and a pseudo-dynamic approach on an inventory control problem under uncertainty.

Supplemental Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/mnsc.2013.1822.

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