Number of points and lines in the projective plane over a finite field

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1 Projective space

Let k be a field and V a vector space over k.

Definition 1 (Projective space)**.** *On $(V \times k) \setminus \{0\}$ we define the following equivalence relation $(v, w \in V \times k)$:*

$$v \sim w : \Longleftrightarrow \exists \lambda \in k^*, w = \lambda \cdot v$$

The set of equivalence classes

$$\mathrm{P}(V) := ((V \times k) \setminus \{0\}) \sim$$

is called the projective space over V.

Two points $[v], [w] \in \mathrm{P}(V)$ are identical if and only if $v, w \in (V \times k) \setminus \{0\}$ lie on the same line throug the origin in $V \times k$. Hence the projective space is the set of lines through the origin in $V \times k$.

Definition 2 (Line in projective space)**.** *The line $g([v], [w])$ through two points $[v], [w] \in \mathrm{P}(V)$ is defined as the set of points*

$$g([v], [w]) := \{[v + t \cdot (w - v)] \mid t \in k\} \cup \{[w - v]\}$$

We have to show, that this definition is well-definied, i.e. that $g([v], [w]) = g([v'], [w'])$ holds if $[v] = [v']$ and $[w] = [w']$.

Proof. Let $\lambda, \mu \in k^*$ such that $v' = \lambda v$ and $w' = \mu w$.

If $\lambda = \mu$ we have directly $[v' + t(w' - v')] = [\lambda(v + t(w - v))] = [v + t(w - v)]$ and $[w' - v'] = [\lambda(w - v)] = [w - v]$.

If $\lambda \neq \mu$ the matter is a bit technical. We only show "$\subseteq$", the other direction is similar: let $[z] = [v + t(w - v)] \in g([v], [w])$. With $a := \frac{\lambda \mu}{\mu - \mu t + \lambda t}$ and $t' := \frac{\lambda t}{\mu - \mu t + \lambda t}$ we find for $\mu - \mu t + \lambda t \neq 0$ with a short calculation $az = v' + t'(w' - v')$ and hence $[z] \in g([v'], [w'])$. If $\mu - \mu t + \lambda = 0$ we have $[z] = [v + \frac{\mu}{\mu - \lambda}(w - v)] = [\frac{1}{\mu - \lambda}(\mu w - \lambda v)] = [w' - v'] \in g([v'], [w'])$ $\qquad \square$

TODO: homgenous coordinates

2 Projective plain over a finite field

Definition 3 (Projective plain). $P(k^2)$ *is called the projective plane over k.*

In the following let $k = \mathbb{F}_q$ be a finite field with q elements.

Propostition 1.

$$P(\mathbb{F}_q^2) = \{(x : y : 1) \mid x, y \in \mathbb{F}_q\} \cup \{(x : 1 : 0) \mid x \in \mathbb{F}_q\} \cup \{(1 : 0 : 0)\}$$

The set of lines in $P(\mathbb{F}_q^2)$ consists of three types of lines:

- *for every $y, m \in \mathbb{F}_q$:*

$$g_{y,m} := g((0 : y : 1), (1 : y+m : 1)) = \{(0 : y : 1)+t(1 : m : 1) \mid t \in \mathbb{F}_q\} \cup \{(1 : m : 0)\}$$

- *for every $x \in \mathbb{F}_q$:*

$$g_{x,\infty} := g((x : 0 : 1), (x : 1 : 1)) = \{(x : t : 1) \mid t \in \mathbb{F}_q\} \cup \{(0 : 1 : 0)\}$$

- $g_{\infty,\infty} := g((1 : 0 : 0), (0 : 1 : 0)) = \{(t : 1 : 0) \mid t \in \mathbb{F}_q\} \cup \{(1 : 0 : 0)\}$

Hence the number of points as well as the number of lines in $P(\mathbb{F}_q^2)$ is $q^2 + q + 1$.

Proof. TODO $\qquad\square$