Random Walks in Number Theory: The Magic and Mystery of Number theoretic Fourier Series

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Weyl Sums

Weyl sums are sums of the form

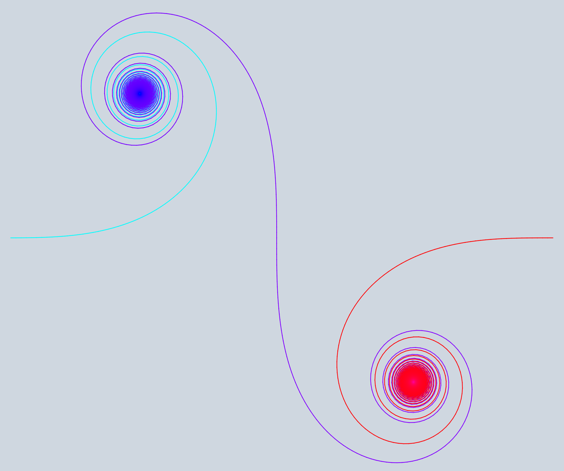
$$S(N)=\sum_{n=1}^{N} e^{2\pi i f(n)},$$

where $f(n)$ is some real-valued smooth function. Examples:

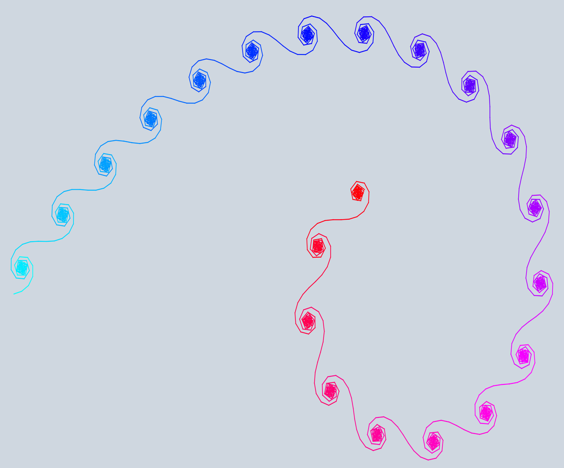
- **Gauss sums:** $f(n)=n^2/p$, p prime, $1\le N\le p$.
- **Generalized Gauss sums:** $f(n)=an^b$, a,b constants.

Properties: The graphs of these sums show fractal-like behavior, and swirls called "curlicues". The graphs arise in physics (e.g. Thermodynamics).

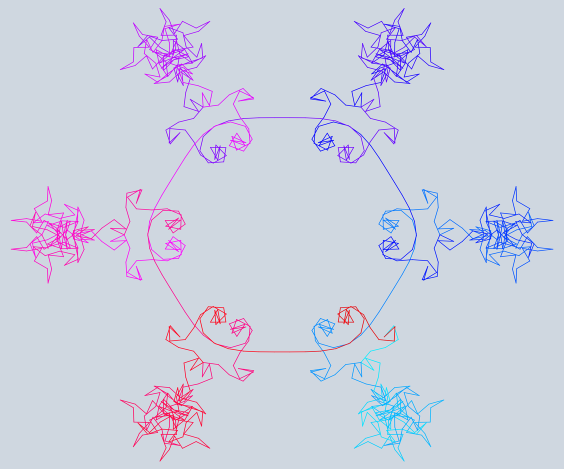
Weyl Sums Gallery



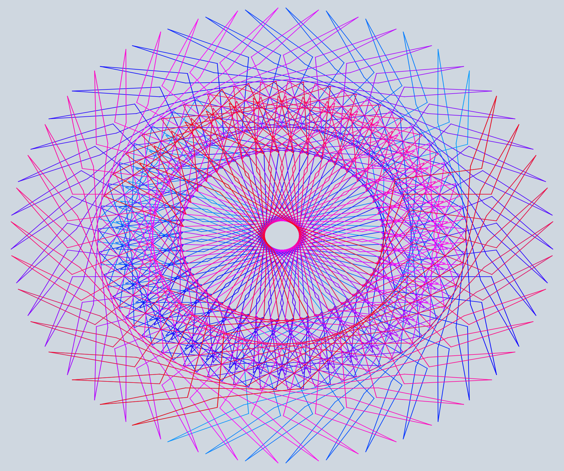
$f(n)=\frac{4n^2}{11213}$, $N\le 11213$



$f(n)=\frac{100n^2}{10001}$, $N\le 1040$



$f(n)=\frac{n^3}{1002}$, $N\le 1024$



$f(n)=\frac{n^7}{1050}$, $N\le 1200$

References

- Berry, M. V. ; Goldberg, J. *Renormalisation of curlicues*, Nonlinearity 1 (1988): 1-26
- Moore, Ross. R. ; van der Poorten, A. J. *On the thermodynamics of curves and other curlicues.* Proceedings of conference on geometry and physics (1989): 89-0031
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Numbertheoretic Fourier Series

Definition

We investigate the Fourier series of the form

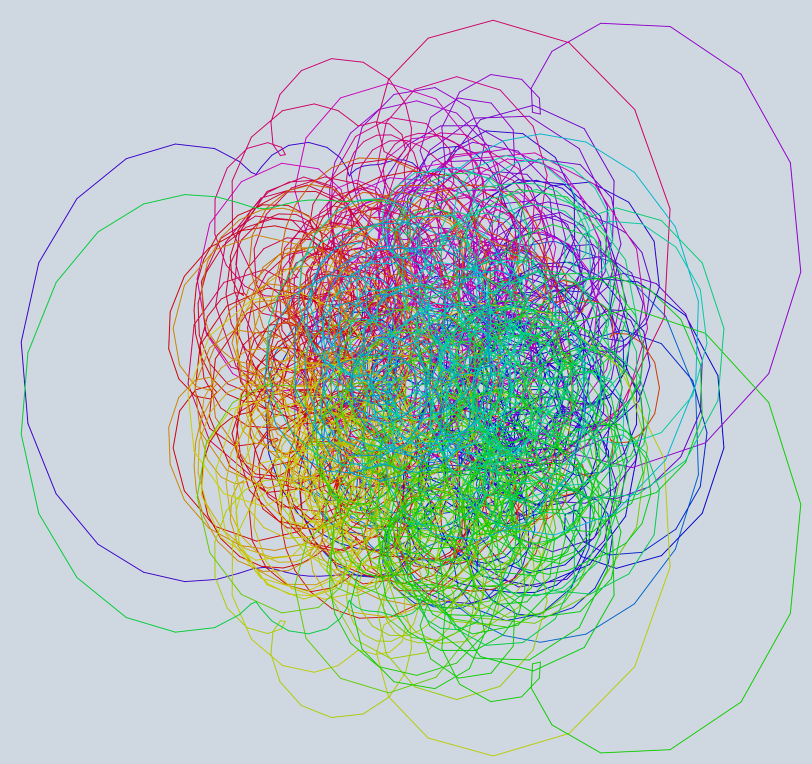
$$g(t)=\sum_{n=1}^{\infty}\frac{f(n)}{n}e^{2\pi int},$$

as $0\le t\le 1$, where $f(n)$ is some natural number theoretic function. In particular we iconsider the following functions : $\mu(n)n^b$, $\sin an^b$, $e^{2\pi i a n^b}$, as a and b varies we find some similar behaviour compared with random coefficients.

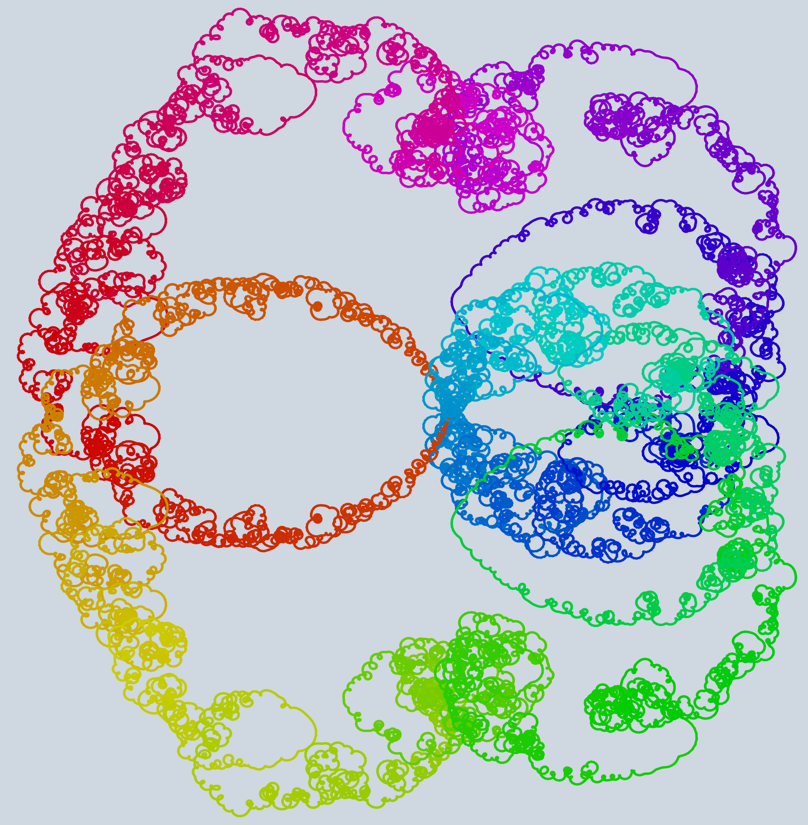
Observations

- Fourier series with Moebius coefficients $\mu(n)$, or more generally $\mu(n)n^b$, behave like series with random coefficients. Their plots show chaotic behavior.
- Fourier series with coefficients $f(n)=an^b$ show interesting spiral-type features.
- Close-up images of these graphs reveal intriguing fractal-type features.

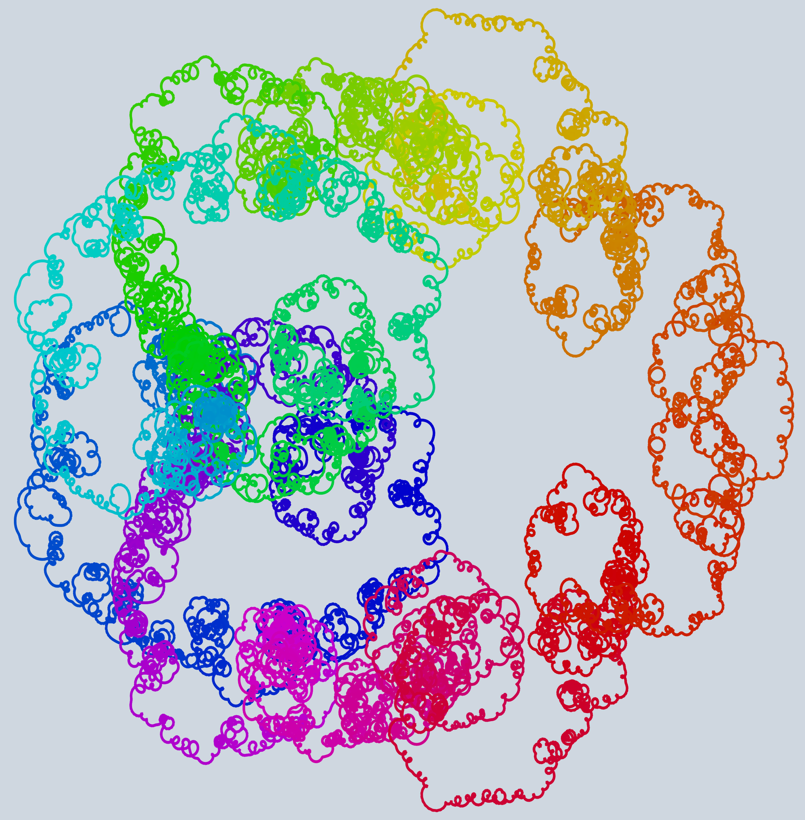
Number Theoretic Fourier Series Gallery



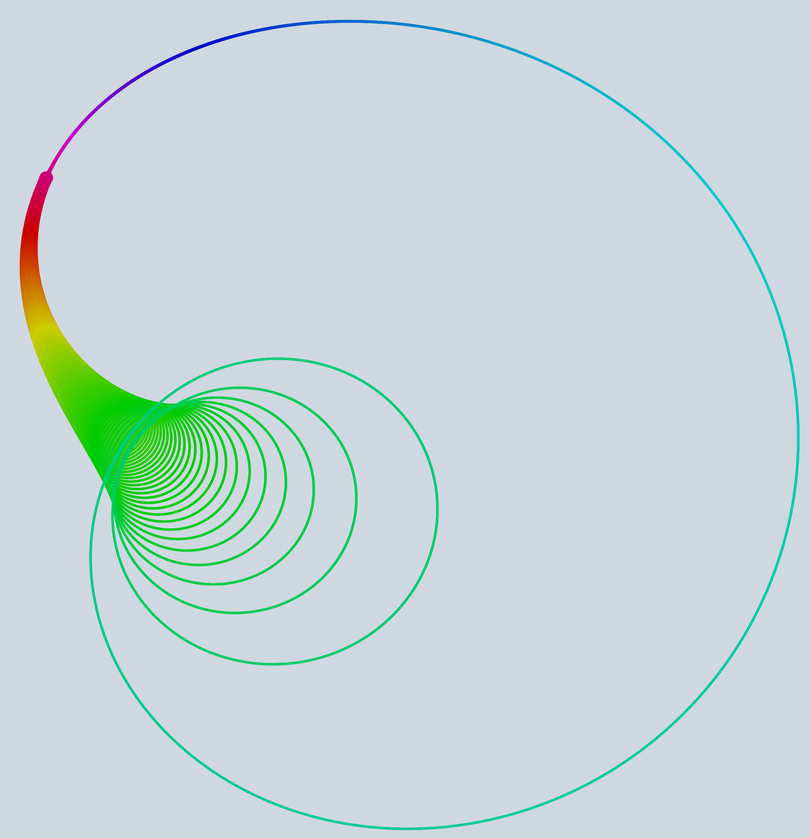
$f(n)=\mu(n)\sqrt{n}$



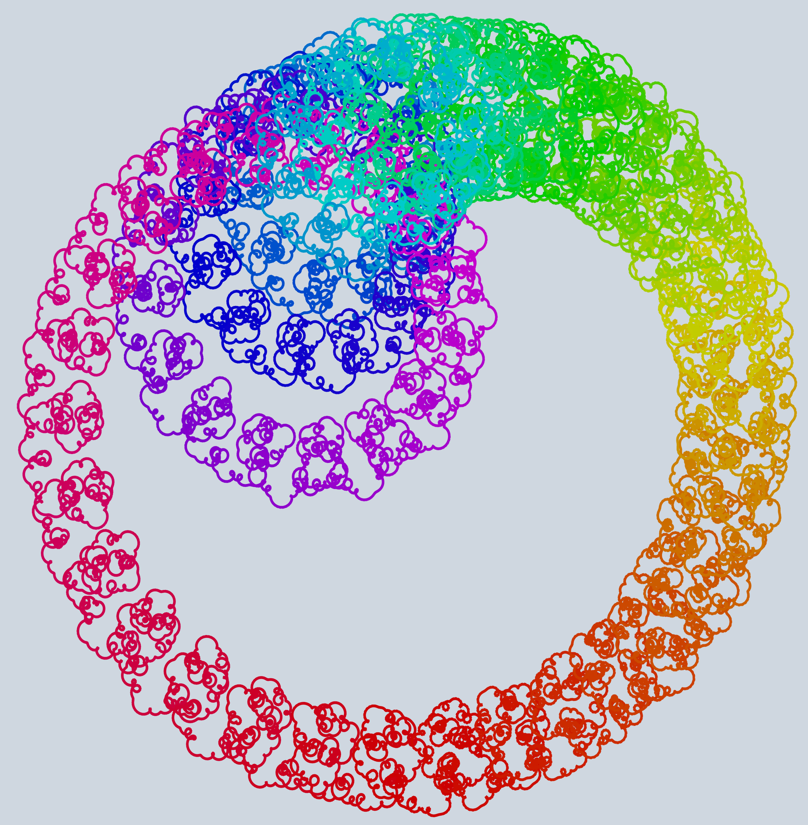
$f(n)=\mu(n)$



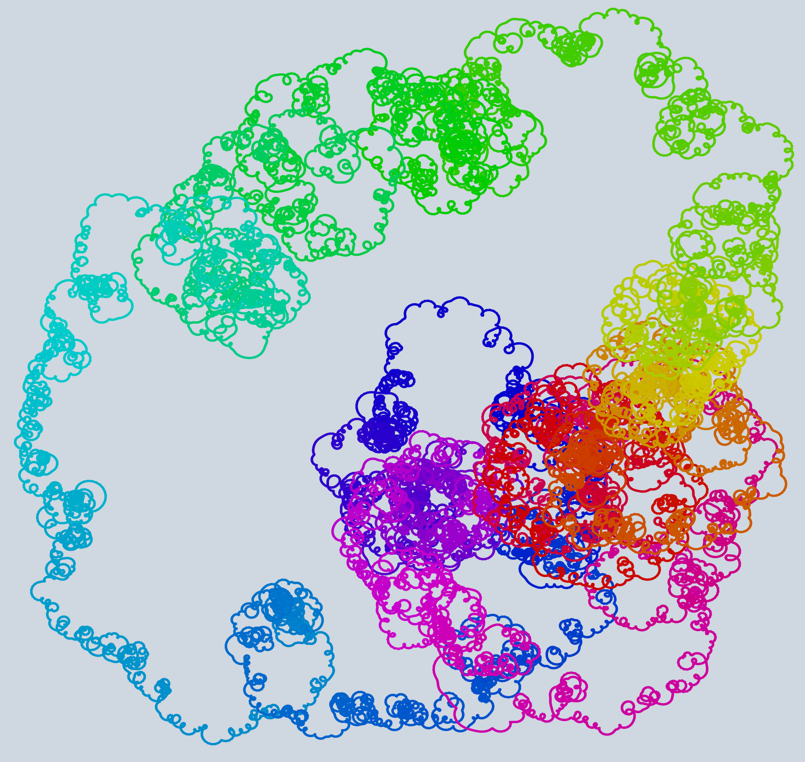
$f(n)$ random ± 1 coefficients



$f(n)=\exp(\frac{1}{5}in^{1.3})$



$f(n)=\exp(2in^{1.3})$



$f(n)=\exp(2\pi i a(n))$, a(n) random real number in [-1,1]

Motivation and Background

- If $f(n)=\mu(n)$, then

$$g(t)=\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{2\pi int},$$

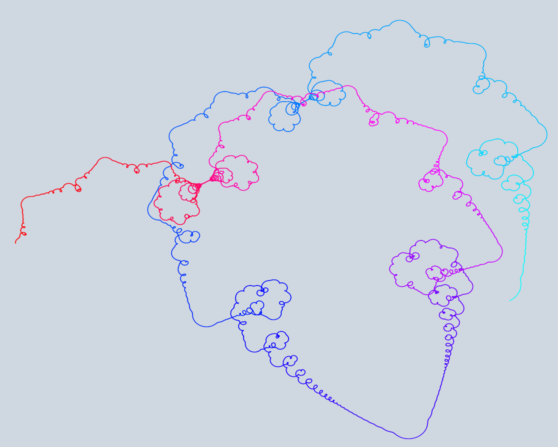
converges for all t (Bateman-Chowla (1963)).

- If $f(n)$ is randomly chosen in $\{$-1,1$\}$ or uniformly distributed in [-1,1], then

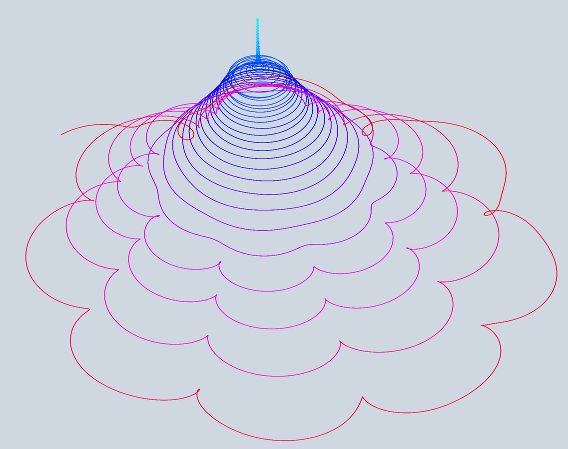
$$g(t)=\sum_{n=1}^{\infty}\frac{f(n)}{n}e^{2\pi int},$$

converges for all t with probability 1.

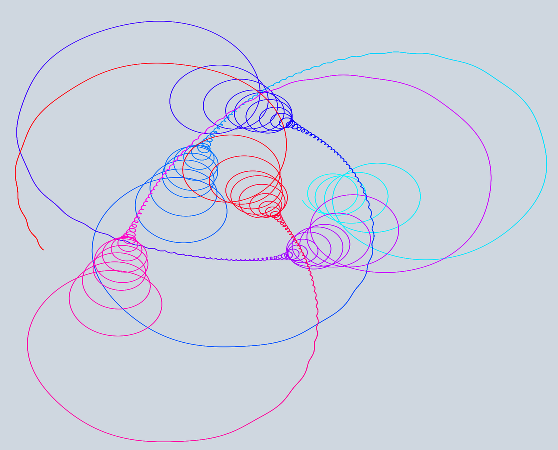
Fourier Series Close-ups Gallery



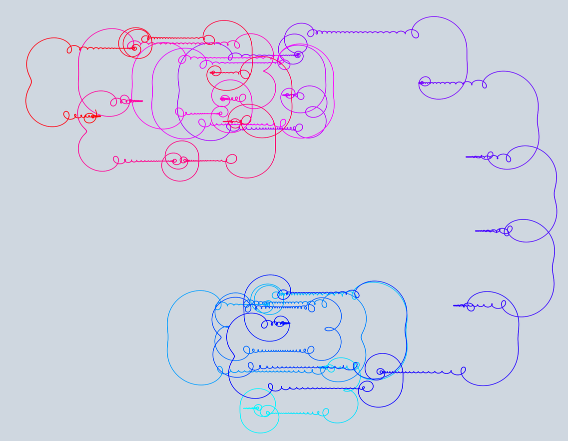
$f(n)=\{\sqrt{n}\}$, close to $\pi/6$



$f(n)=\{\sqrt{n}\}$, close to π



$f(n)=\exp(2in^2)$ close to $\pi/2$



$f(n)=\sin(n^2)$, close to 0.3

References

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- Bohman, J. ; Froberg, C. -E. *Heuristic investigation of chaotic mapping producing fractal objects.* BIT Numerical Mathematics, (1995): 609-615
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