Fractals, Patterns and Randomness in Number Theory and Mathematical Physics: Unraveling the Mysteries of Number-Theoretic Fourier Series

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Number Theory

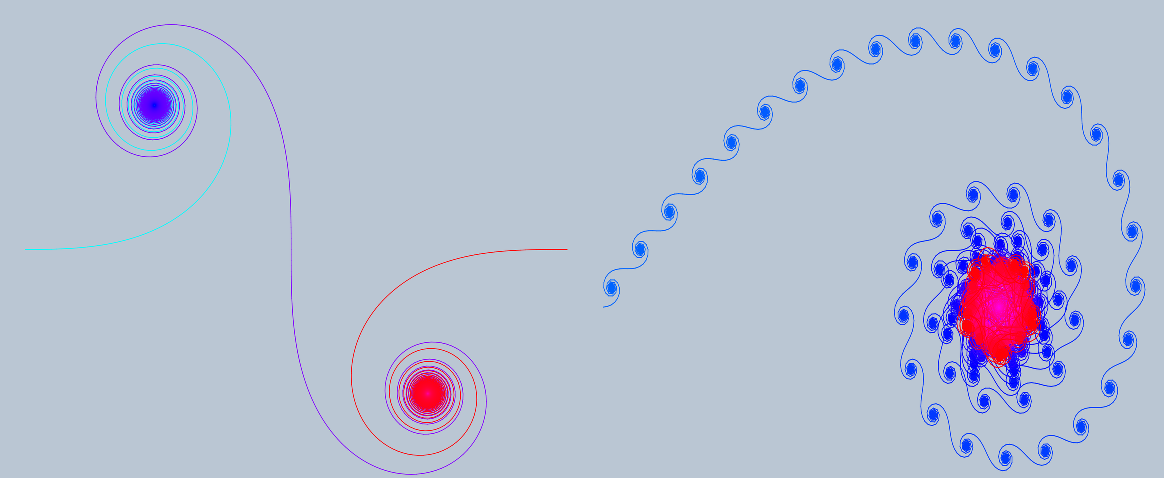
Gauss Sums

- Gauss Sums are sums of the form

$$S(N) = \sum_{n=1}^{p} e^{2\pi i \frac{n^2}{p}}$$

where p is prime.

- Plots of Gauss sums show a spiral structure called a Cornu Spiral
- Generalized Gauss sums include exponents other than 2 and scaling constants



Weyl Sums

- Weyl sums are of the form

$$S(N) = \sum_{n=1}^{N} e^{2\pi i f(n)}$$

where $f(n)$ is a "smooth" function

- Weyl sums exhibit more complex behavior, including fractal-like patterns known as "curlicues"

Renormalization

- Renormalization formulas developed by Berry and Goldberg can be used to "smooth out" curlicues and make the self similarity more apparent.

References

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Number-Theoretic Fourier Series

Definition

Number-theoretic Fourier series are sums of the form

$$g(t) = \sum_{n=1}^{\infty} \frac{f(n)}{n} e^{2\pi i n t},$$

as $0 \le t \le 1$, where $f(n)$ is some natural number-theoretic function. In particular we consider the following functions:

$$\mu(n)n^b, \sin(\alpha n^\beta), exp(2\pi i \alpha n^\beta), \{\sqrt{n}\}.$$

We compare their behavior with that of random coefficients $f(n)$.

Observations

- Fourier series with Moebius coefficients $\mu(n)$, or more generally $\mu(n)n^b$, behave like series with random coefficients. Their plots show chaotic behavior.
- Fourier series with coefficients $f(n) = exp(2\pi i \alpha n^\beta)$ show interesting spiral-type features.
- Close-up images of these graphs reveal intriguing fractal-type features.
- Fourier series with random complex coefficients $f(n)$ with $|f(n)| = 1$ show chaotic behavior.

Renormalization Formulas

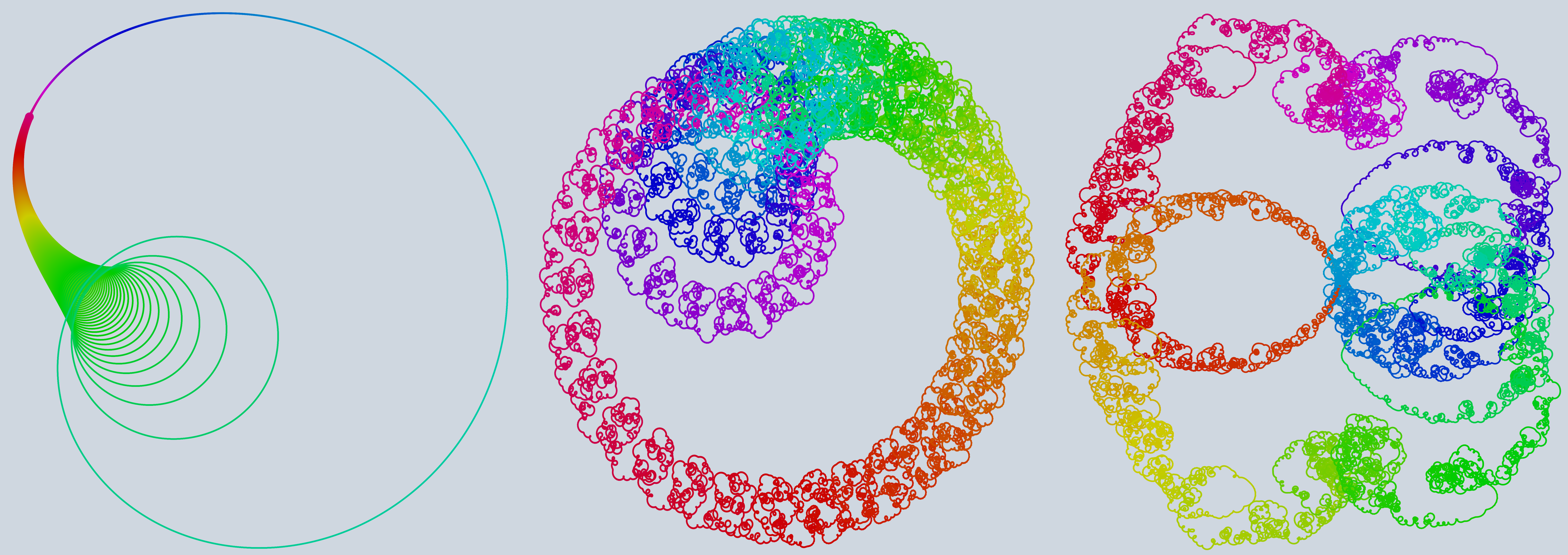
- Renormalization formula for Weyl Sums (Berry-Goldberg) :

$$\sum_{n=1}^{N} exp(i\pi\alpha n^2) \sim \frac{exp(\frac{i\pi}{4})}{|\alpha|^{\frac{1}{2}}} \sum_{n=1}^{\lfloor N\alpha \rfloor} exp(-i\pi\frac{1}{\alpha}n^2)$$

- Renormalization formula for Fourier Series :

$$\sum_{n=1}^{\infty} \frac{exp(i\pi\alpha n^2)}{n} e^{i\pi t n} \sim exp(\frac{i\pi}{4})|\alpha|^{\frac{1}{2}} exp(\frac{-i\pi t^2}{4\alpha}) \sum_{n=1}^{\infty} \frac{exp(i\pi\frac{1}{\alpha}n^2)}{n} e^{i\pi\frac{t}{\alpha}n}$$

Number-Theoretic Fourier Series Gallery



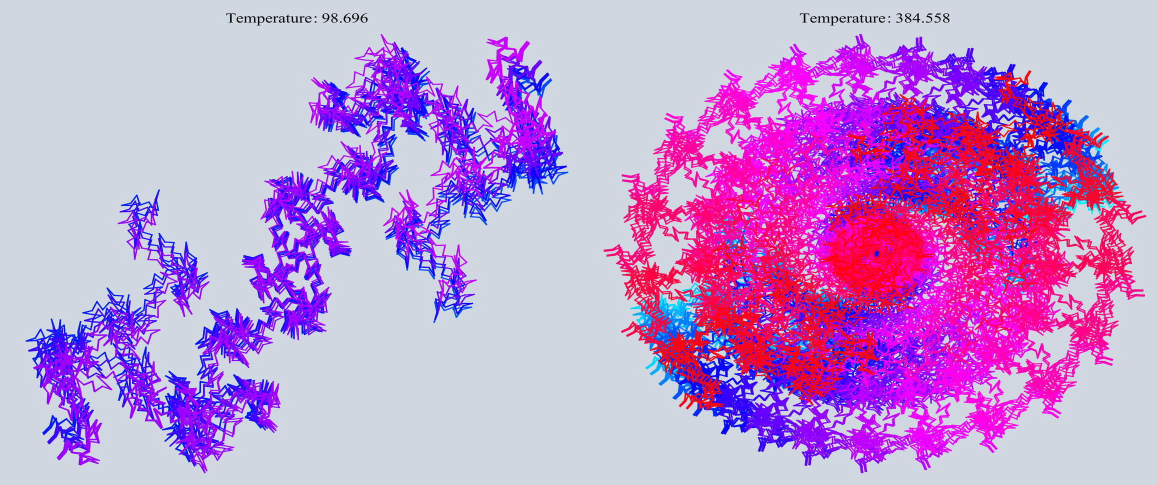
$f(n) = \exp(\frac{1}{5} i n^{1.3})$ $f(n) = \exp(2 i n^{1.3})$ f(n)= $\mu(n)$

Plots of Fourier series $\sum_{n=1}^{\infty} \frac{f(n)}{n} e^{2\pi i n t}$ for various number-theoretic functions f(n).

Physics

Thermodynamics

- We can describe curves using thermodynamics
- Entropy and temperature relate length of a curve to the area it covers
- Volume is defined to be the length of a curve
- Pressure measures the area covered



A "cool" curve with moderate entropy   A "hot" curve with a large amount of entropy

Optics

- Diffraction around opaque objects shows intensity patterns that can be modeled by the magnitude of a vector connecting parts of a Cornu spiral
- The Talbot effect is a diffraction effect caused by periodic gratings. An image of the grating reappears periodically as distance from the grating increases.

Engineering

- Talbot lasers are a theoretical laser design which use the effect to generate specific frequencies of light.
- Quantum lithography is photolithography using quantum particles. Using the Talbot effect, we can improve resolution by a factor of two.

References

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