Random Walks in Number Theory: The Magic and Mystery of Numbertheoretic Fourier Series

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Illinois Geometry Lab

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Weyl Sums

Weyl sums are sums of the form

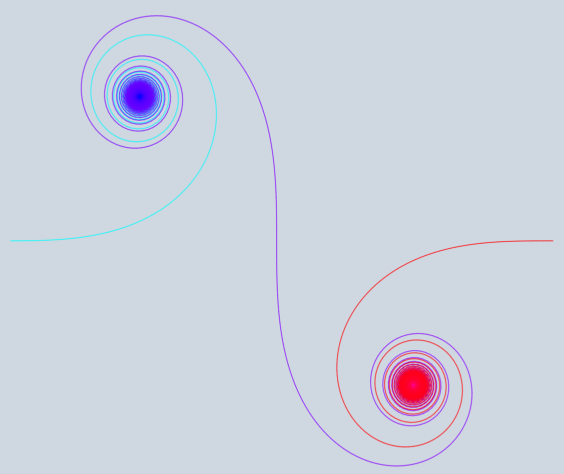
$$S(N)=\sum_{n=1}^{N} e^{2\pi i f(n)},$$

where $f(n)$ is some real-valued smooth function. **Examples:**

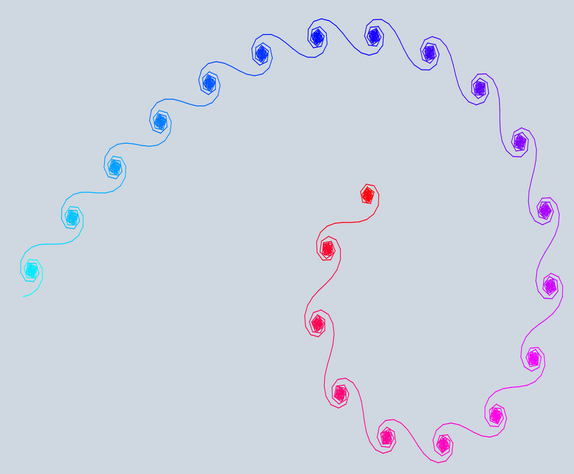
- **Gauss sums:** $f(n)=n^2/p$, p prime.
- **Generalized Gauss sums:** $f(n)=an^b$, a,b constants.

Properties: Such curves of these sums show fractal-like behavior, and swirls called "curlicues". The graphs arise in physics (e.g. thermodynamics).

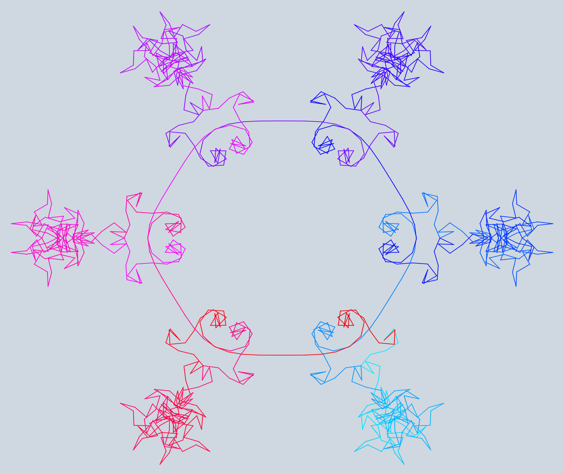
Weyl Sums Gallery



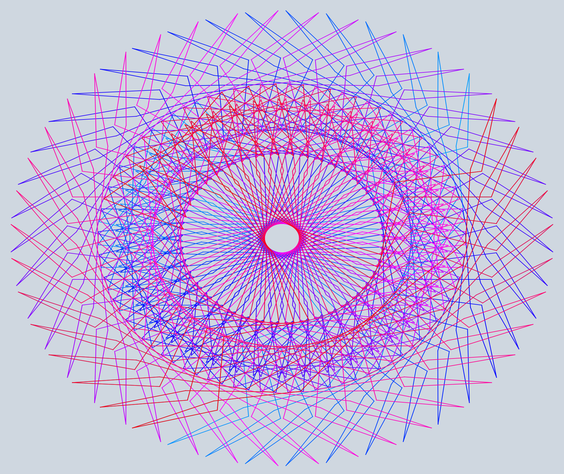
$f(n)=\frac{4n^2}{11213}$, $N\le 11213$



$f(n)=\frac{100n^2}{10001}$, $N\le 1040$



$f(n)=\frac{n^3}{1002}$, $N\le 1024$



$f(n)=\frac{n^7}{1050}$, $N\le 1200$

References

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- Moore, Ross. R. ; van der Poorten, A. J. *On the thermodynamics of curves and other curlicues.* Proceedings of conference on geometry and physics (1989): 89-0031
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Numbertheoretic Fourier Series

Definition

Numbertheoretic Fourier series are sums of the form

$$g(t)=\sum_{n=1}^{\infty}\frac{f(n)}{n}e^{2\pi int},$$

as $0\le t\le 1$, where $f(n)$ is some natural numbertheoretic function. In particular we consider the following functions:

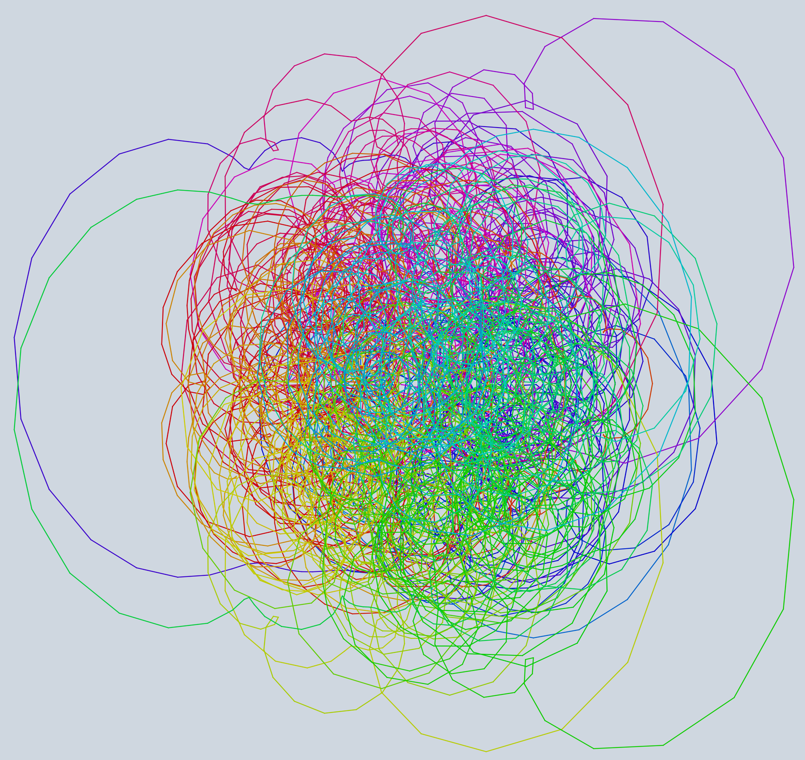
$$\mu(n)n^b, \sin(an^b), e^{2\pi ian^b}, \{\sqrt{n}\}.$$

We compare their behavior with that of random coefficients $f(n)$.

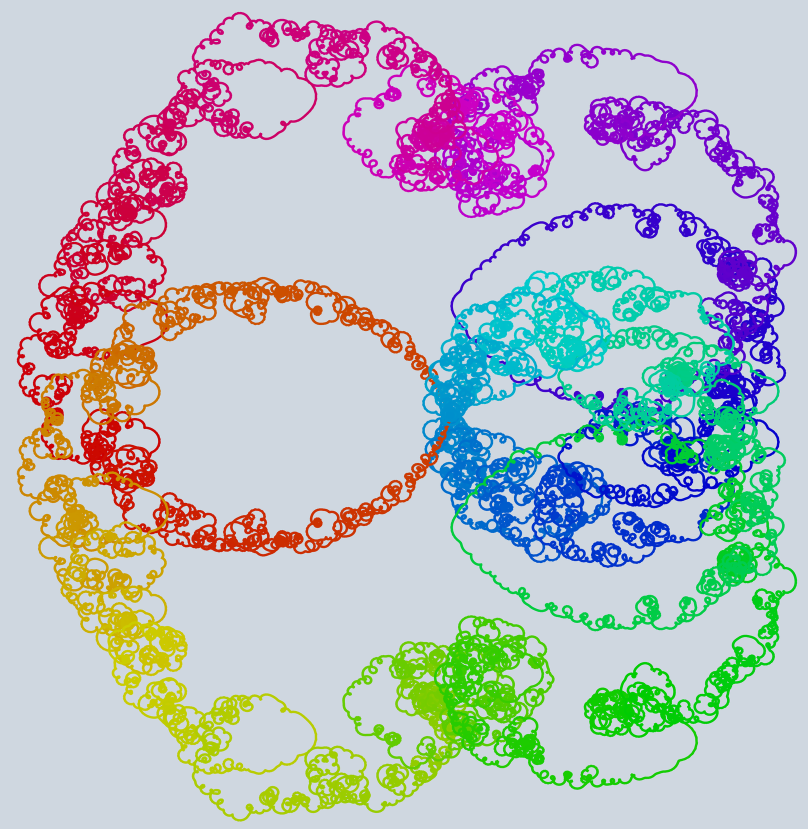
Observations

- Fourier series with Moebius coefficients $\mu(n)$, or more generally $\mu(n)n^b$, behave like series with random coefficients. Their plots show chaotic behavior.
- Fourier series with coefficients $f(n)=e^{2\pi ian^b}$ show interesting spiral-type features.
- Close-up images of these graphs reveal intriguing fractal-type features.
- Fourier series with random complex coefficients $f(n)$ with $|f(n)|=1$ show chaotic behavior.

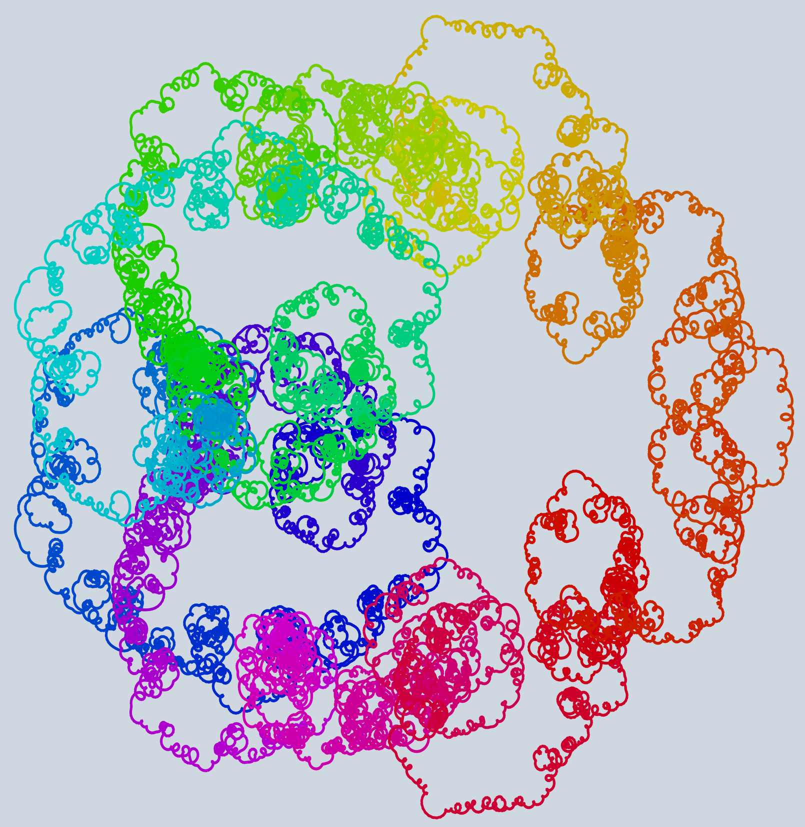
Numbertheoretic Fourier Series Gallery



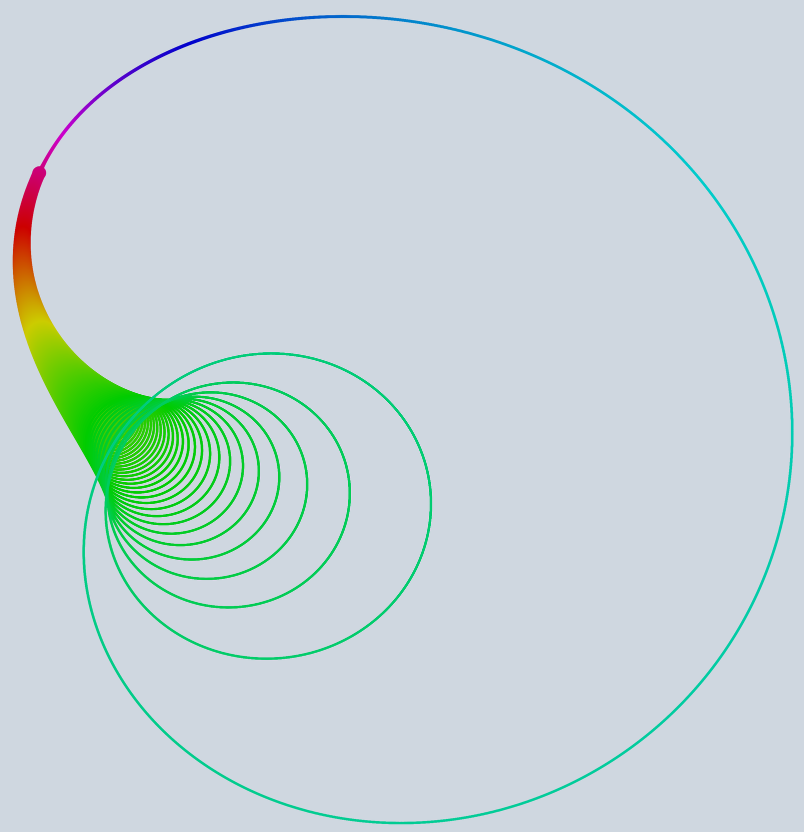
$f(n)=\mu(n)\sqrt{n}$



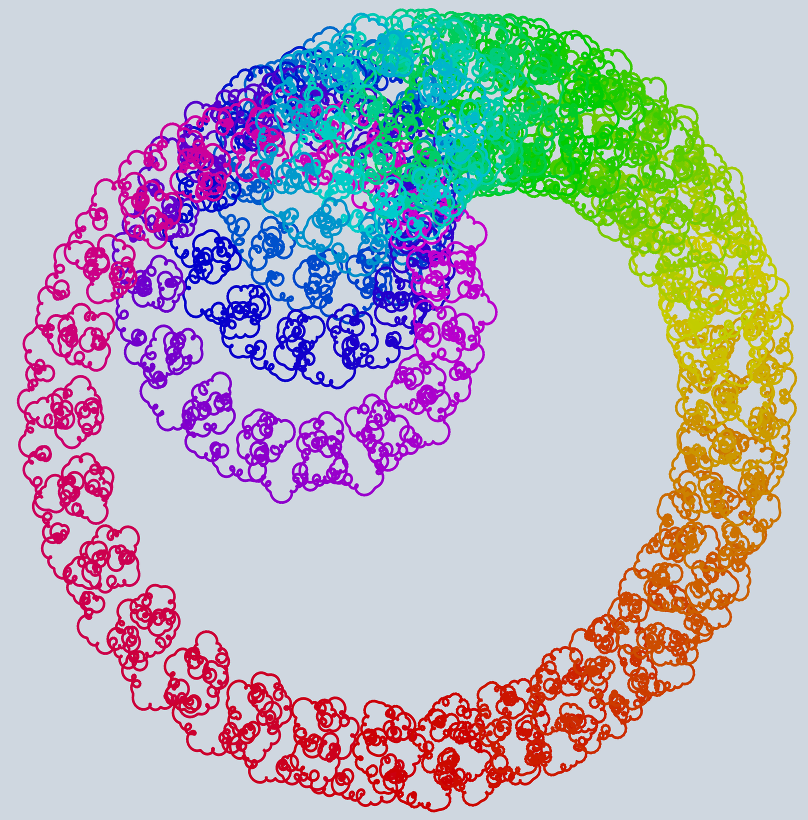
$f(n)=\mu(n)$



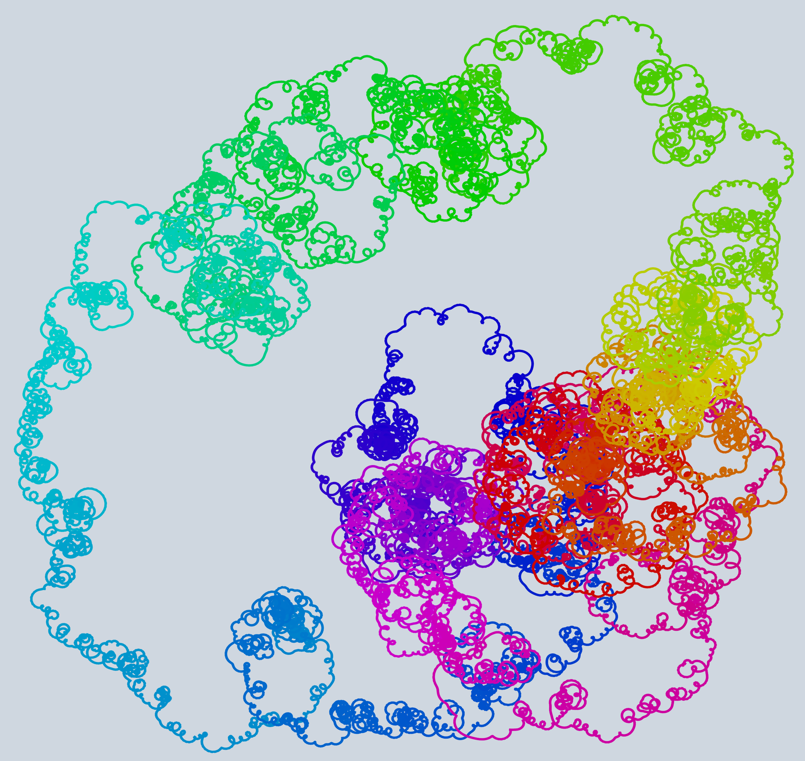
$f(n)$ random ± 1 coefficients



$f(n)=\exp(\frac{1}{5}in^{1.3})$



$f(n)=\exp(2in^{1.3})$

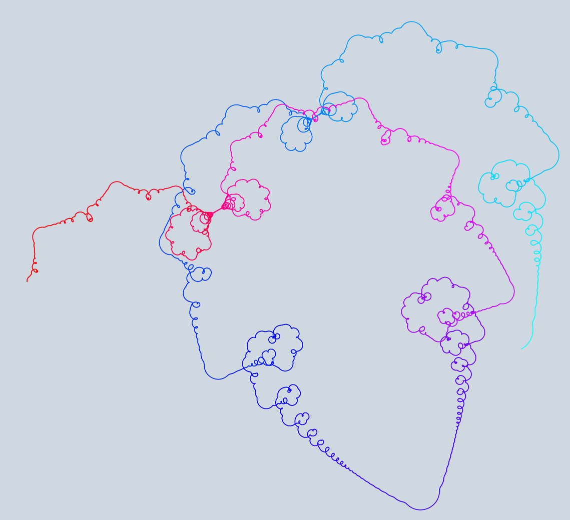


$f(n)=\exp(2\pi ia(n))$, a(n) random real number in [-1,1]

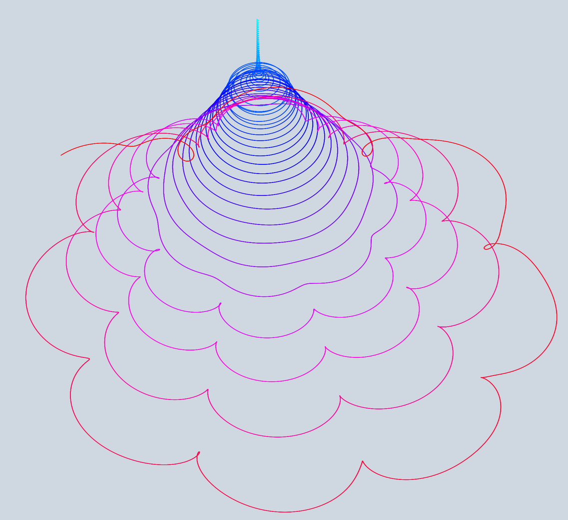
Motivation and Background

- If $f(n)$ is randomly chosen in {-1,1} or uniformly distributed in [-1,1], then
$$\sum_{n=1}^{\infty}\frac{f(n)}{n}e^{2\pi int}$$
converges for all t with probability 1.
- If $f(n)=\mu(n)$, then $\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{2\pi int}$ converges for all t (Bateman-Chowla (1963)).
- The geometric behavior of the latter series was investigated by Froberg (1966) and Bohman-Froberg (1995).

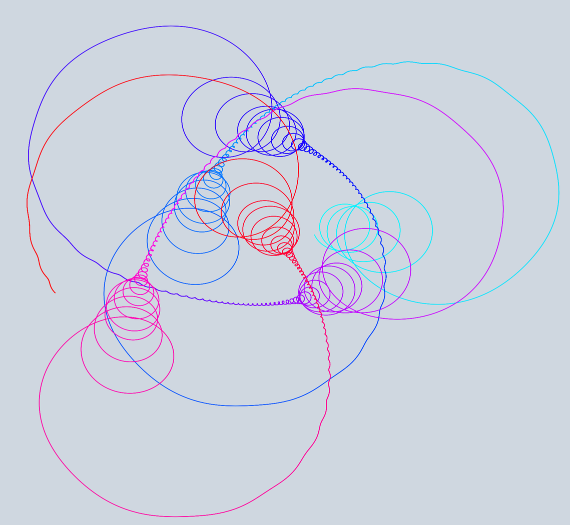
Fourier Series Close-ups Gallery



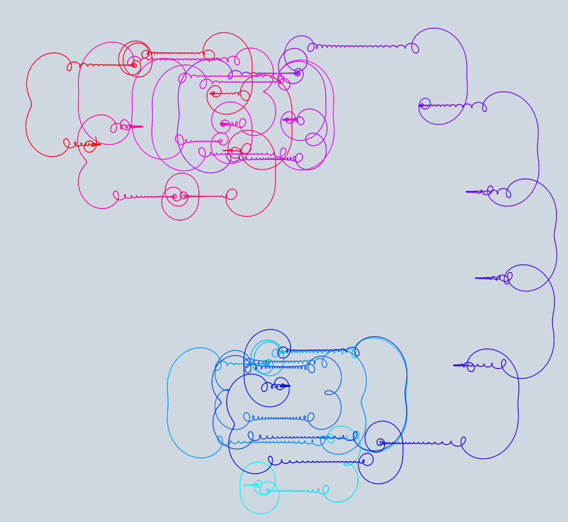
$f(n)=\{\sqrt{n}\}$, close to $\pi/6$



$f(n)=\{\sqrt{n}\}$, close to π



$f(n)=e^{2in^2}$ close to $\pi/2$



$f(n)=\sin(n^2)$, close to 0.3

References

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- Bohman, J. ; Froberg, C. -E. *Heuristic investigation of chaotic mapping producing fractal objects.* BIT Numerical Mathematics, (1995): 609-615
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