Assignment 5

Problem 1:

(a) Let X be time to first patient

$X \sim Exp(5)$

$P(X \leq 1) = \int_0^1 5e^{-5t} dt$

$= -e^{-5t} \Big|_0^1$

$= -(e^{-5} - e^{0})$

$= 0.993262$

$P(X > 1) = 1 - P(X \leq 1)$

$= \underline{6.74 \cdot 10^{-3}}$

(b) $X \sim Poisson(5/hr \cdot 8hr)$: day, 9am - 5pm

$E(X) = \lambda t$

$E(24) = 5(8) = \underline{40}$

5 min / session $\therefore$ X cannot be > 2 before David arrives at 9:10

(c) 3 scenarios: $\begin{cases} 0 \le X \le 3 \text{ at } 9{:}00,\ 0 \text{ at } 9{:}05 \\ 0 \le X \le 2 \text{ at } 9{:}00,\ 1 \text{ at } 9{:}05 \\ 0 \le X \le 1 \text{ at } 9{:}00,\ 2 \text{ at } 9{:}05 \end{cases}$

$X \sim \text{Poisson}(5\ \text{hr}^{-1} \cdot \frac{1}{12}\ \text{hr})$ divide into 5 min. segments

$$P(X \le 10) = P(X \le 3) \cdot P(X=0) + P(X \le 2) \cdot P(X=1) + P(X \le 1) \cdot P(X=2)$$

$$= \left(\frac{(\frac{5}{12})^3 e^{-5/12}}{3!} + \frac{(\frac{5}{12})^2 e^{-5/12}}{2!} + \frac{(\frac{5}{12}) e^{-5/12}}{1!} + \frac{(\frac{5}{12})^0 e^{-5/12}}{0!} \right) \frac{(\frac{5}{12})^0 e^{-5/12}}{0!}$$

$$+ \left(\frac{(\frac{5}{12})^2 e^{-5/12}}{2!} + \frac{(\frac{5}{12}) e^{-5/12}}{1!} + \frac{(\frac{5}{12})^0 e^{-5/12}}{0!} \right) \frac{(\frac{5}{12}) e^{-5/12}}{1!}$$

$$+ \left(\frac{(\frac{5}{12}) e^{-5/12}}{1!} + \frac{(\frac{5}{12})^0 e^{-5/12}}{0!} \right) \frac{(\frac{5}{12})^2 e^{-5/12}}{2!}$$

$$= \underline{\underline{0.984}}$$

Problem 2:

(a) $\mu = 30 cm^3/s \quad \sigma = 5 cm^3/s$

$X \sim N(30, 5^2)$

$P(X > 37) = P(Z > \frac{37-30}{5}) = P(Z > 1.4)$

$= 1 - P(Z \leq 1.4)$

$= 1 - 0.9192$

$= \underline{0.0808}$

(b) $P(X \leq 25) = P(Z \leq \frac{25-30}{5}) = P(Z \leq -1)$

$= \underline{0.1587}$

(c) $P(30-f \leq X \leq 30+f) = P(\frac{30-f}{5} \leq Z \leq \frac{30+f}{5})$

$= P\left(\frac{30+f}{5}\right) - \left(\frac{30-f}{5}\right)$ $\mu = 30$ median

$= P\left(\frac{f}{5}\right) - P\left(-\frac{f}{5}\right)$

$= 0.6$

Trial & Error for z, for which $P(z) - P(-z) = 0.6$

$\hookrightarrow P(0.84) - P(-0.84) = 0.7995 - 0.2005$

$= 0.599$

≈ 0.6

$\frac{f}{5} = 0.84$

$\underline{f = 4.2}$

Problem 3:

(a) $\lambda = \frac{1}{5}\ min^{-1}$ Let X represent time to first arrival

$X \sim Exp(1/5)$

$$P(X<2) = \int_0^2 \frac{1}{5} e^{-\frac{1}{5}t}\,dt$$

$$= -e^{-\frac{1}{5}t}\Big|_0^2$$

$$= -(e^{-\frac{2}{5}} - e^0)$$

$$= \underline{0.3297}$$

(b) $$P(X>10) = 1 - P(X \leq 10)$$

$$= 1 - \int_0^{10} \frac{1}{5}e^{-\frac{1}{5}t}\,dt$$

$$= 1 - (-e^{-\frac{1}{5}t})\Big|_0^{10}$$

$$= 1 + (e^{-2} - e^0)$$

$$= \underline{0.1353}$$

(c) $$\underbrace{P(X>5)}_{\text{late for a day}} = 1 - P(X \leq 5)$$

$$= 1 - \int_0^5 \frac{1}{5} e^{-\frac{1}{5}t}\,dt$$

$$= 1 - (-e^{-\frac{1}{5}t})_0^5$$

$$= 1 + (e^{-1} - e^0)$$

$$= 0.3679$$

Let L represent # days late

$$P(L \geq 1) = 1 - P(L<1) \qquad L \sim Bin(5; 1-0.3679$$

$$= 1 - P(L=0)$$

$$= 1 + \binom{5}{5}(1-0.3679)^5 (0.3679)^0$$

$$= \underline{0.8991}$$

(d) $X \sim Geo(0.3679)$ } until first late

$$E(X) = \frac{1}{0.3679} = \underline{2.7183}$$

Problem 4:

(a) $f(x,y) = \begin{cases} 24xy & 0 \le x, y \le 1,\ x+y \le 1 \\ 0 & \text{elsewhere} \end{cases}$

$Z_1 = X + Y$

Introduce 2nd mapping $Z_2 = Y$

$Z_1 = X + Y \to X = Z_1 - Y$
$= Z_1 - Z_2$

$Z_2 = Y \to Y = Z_2$

$24xy \to 24(z_1 - z_2)(z_2)$

$$J = \begin{bmatrix} \frac{\partial x}{\partial z_1} & \frac{\partial x}{\partial z_2} \\ \frac{\partial y}{\partial z_1} & \frac{\partial y}{z_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial (z_1 - z_2)}{\partial z_1} & \frac{\partial (z_1 - z_2)}{\partial z_2} \\ \frac{\partial z_2}{\partial z_1} & \frac{\partial z_2}{\partial z_1} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$y = 0 = z_2$

$\det J = (1)(1) - (-1)(0) = 1$

$x = 0 = z_1 - z_2$
$z_1 = z_2$

$x + y \le 1$
$z_1 - z_2 + z_2 \le 1 \quad z_1 \le 1$

$$g(z_1, z_2) = \begin{cases} 24(z_1 - z_2)(z_2)(1) & 0 < z_1 < 1,\ 0 < z_2 < z_1 \\ 0 & \text{elsewhere} \end{cases}$$

$$h(z_1) = \int_0^{z_1} 24(z_1 - z_2)(z_2)\, dz_2$$
$$= 24\int_0^{z_1} (z_1 z_2 - z_2^2)\, dz_2$$
$$= 24\left(\frac{z_1 z_2^2}{2} - \frac{z_2^3}{3}\right)\Big|_0^{z_1}$$
$$= 24\left(\frac{z_1^3}{2} - \frac{z_1^3}{3}\right)$$
$$= 12 z_1^3 - 8 z_1^3$$
$$= \underline{4 z_1^3}$$

(b) $\underbrace{\text{cyan} + \text{magenta}}_{z_1} + \underbrace{\text{yellow}}_{\le 0.1} = 1$

$\therefore z_1 \ge 0.9$

$$h(z_1 \ge 0.9) = \int_{0.9}^{1} 4 z_1^3\, dz_1$$
$$= z_1^4 \Big|_{0.9}^{1} = (1)^4 - (0.9)^4 = \underline{0.3439}$$