MAT231 Project 2 Cover Page

Tutorial Section: TUT103		Group Number: 33
Name	**Student Number**	**Email**
Yuyang Zhang	1007925621	yuyangzhang.zhang@mail.utoronto.ca
Danny Wang	1008063435	dannyf.wang@mail.utoronto.ca
Houman Ebrahimi	1008071783	houman.ebrahimi@mail.utoronto.ca
Joaquin Lim	1008463440	joaquin.lim@mail.utoronto.ca

Introduction and summary

Sleeping quality is vital for our health and wellness, thus is it significant to gain an understanding of how our body's circadian clock functions. This project investigates and models a person's normal sleep timing (sleep/wake time) and sleep duration based on the circadian clock, as well as these parameters under different special conditions. This document defines our problems and assumptions, illustrates the process in constructing this model, how this model can be applied to account for different conditions, and a thorough analysis of the accuracy and viability of the model.

Defining the problem (list your applications)

Let t be the amount of time elapsed, in hours. Let H(t) be a periodic function representing the propensity of sleep, known as the sleep drive. Let $H^+(t)$ and $H^-(t)$ be sinusoidal functions, representing sleep thresholds. Additionally, let S(t) be a boolean function that has the value 1 or 0, representing whether the person is asleep or awake, known as the sleep indicator.

By constructing a phenomenological model using H(t), as well as sleep thresholds $H^+(t)$ and $H^-(t)$ and applying real world data on sleeping/waking times, we can address the problems below.

A. What does the model predict after sleep deprivation? Is the model consistent with the common experience of difficulty falling asleep after sunrise following being awake all night? If you need to be awake for 4 hours, is it better to stay awake for 4 hours or get up 4 hours early?
B. What does the model predict for the onset and duration of sleep on a day when a person takes an afternoon nap?
C. Many animals, dogs or infant humans for example, have multiple sleep bouts per day. This is known as polyphasic sleep. Explain how changing H^+_0 and H^-_0 can result in multiple sleeps per day. Does the total amount of sleep change with more sleep per day?
D. When someone changes time-zones (or daylight saving time starts or stops), α changes slowly. For a time change of τ hours, assume that α changes (linearly) from 0 to τ over τ days (i.e. 24 $\times\tau$ hours). What happens to the sleep times during this time change?
E. How would you account for caffeine or other stimulants in the model? Plot a solution H(t) that illustrates the effects of caffeine before bedtime. Account for the pharmacokinetics of caffeine - it takes about 30 minutes after ingestion to have an effect and caffeine has a half-life of about 5 hours in the body.

Assumptions

- The sleeping and waking time used is an average of all the sleeping and waking times from a Kaggle dataset, linked in our sources.
- The start time, t_0, is assumed to be at 7:00 am.
- Without external interference, the base model has the same waking and sleeping times every day (e.g., $s(0 + t) = s(T + t)$ for all $t > 0$).
- The base model does not account for any irregularities in sleep (i.e. conscious decision to nap, waking up in the middle of the night), any necessary changes to the model when addressing the application questions will be stated.

Constructing the model (list your model modifications)

To create the base version of the model, we started by finding a numerical solution for the piecewise ODE. The difficulty of this model is implementing a modified Euler's method, with a conditional differential equation. We made all of the formulas relative to parameter cells so that we could freely modify any parameters without having to recreate any formulas. A sample of our full table can be viewed in Appendix B.

We approached this modification using Google Sheets. In addition to the standard columns needed to Euler's method, we made additional columns for $H^+(t)$ and $H^-(t)$, which simply used the formula provided with our parameters, taking in the value of **t** from the current row.

To determine which differential equation to use in each row, we needed to consider whether the user was currently awake or asleep. To do this, we created a new column in our sheet to represent the sleep indicator variable **s(t)**, which had the value of 1 if the person was awake, and 0 if the person was asleep. Thus, we could create a condition for **dH/dt** that utilized a different equation depending on the value of s(t).

$$\frac{dH}{dt} = \begin{cases} (1-H)/\chi_{\mathrm{w}} & \text{if awake} \\ -H/\chi_{\mathrm{s}} & \text{if asleep} \end{cases}.$$

```
=IF(I17=TRUE,-C17/$B$3, (1-C17)/$B$2)
```

s(t) H χ_w H χ_s

The difficulty now was to find a formula that could autonomously change s(t) to the correct value based on the information available. We started this approach by considering what conditions would make someone awake or asleep (seen in the table below). We determined that being awake or asleep was mutually exclusive, so we only needed to implement one set of conditions, ultimately implementing the asleep conditions. If neither asleep condition was satisfied, we could conclude the person should be awake.

awake	1	awake prior	AND	within upper threshold
	2	asleep prior	AND	crossed lower threshold
asleep	3	asleep prior	AND	within lower threshold
	4	awake prior	AND	crossed upper threshold

$$\texttt{=IF(OR(AND(}\overbrace{\texttt{I17=FALSE,}\underbrace{\texttt{C18}}_{\text{H}}\texttt{>}\underbrace{\texttt{G18}}_{\text{H-}}}^{4}\texttt{),AND(}\overbrace{\texttt{I17=TRUE,}\underbrace{\texttt{C18}}_{\text{H}}\texttt{>}\underbrace{\texttt{H18}}_{\text{H+}}}^{3}\texttt{)),}\underbrace{\texttt{TRUE}}_{\text{asleep}}\texttt{,}\underbrace{\texttt{FALSE}}_{\text{awake}}\texttt{)}$$

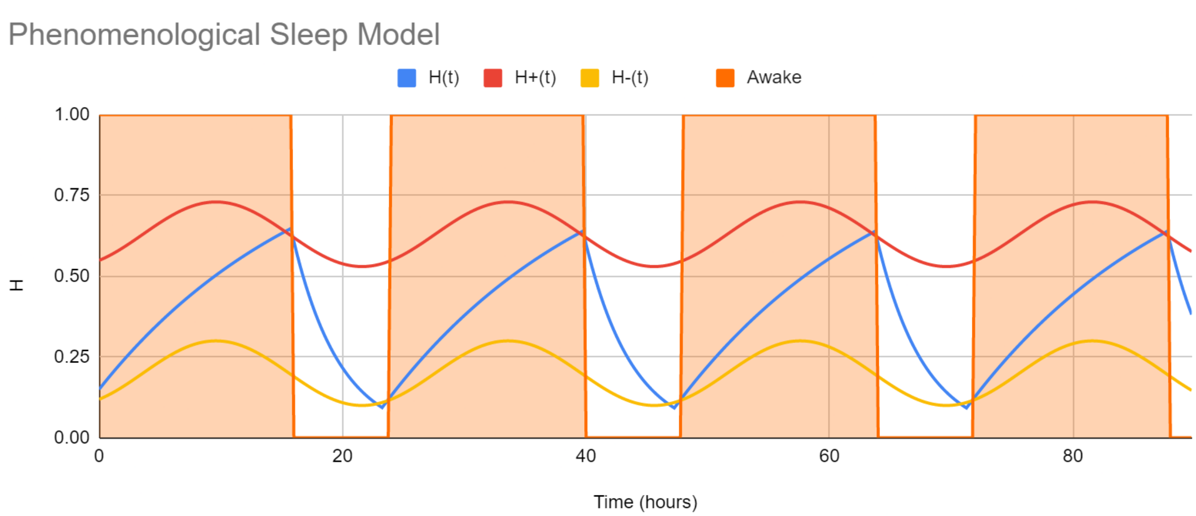
Following this, we found a Kaggle dataset [1] that had information about ~800 days of sleeping and waking times. We determined that using an average of all the sleeping and waking times would be a reasonable representation to use in our model. To do so, we extracted the time as a decimal and processed the data to get an average value. Since the decimal system treats any time after 12:00 am to be reset from 0:00, we accounted for this by adding 1 (equivalent to 24 hours) to all times later than 12:00am. A sample of our data is provided below. Further samples of the data table and processed data can be viewed in Appendix C.

Start	**Start Time**	**Start Time (Processed)**	**End**	**End Time**	**Time in bed**	
12/29/2014 22:57:49	10:57:49 PM	10:57:49 PM	2014-12-30 7:30:13	7:30:13 AM	8:32	
12/30/2014 21:17:50	9:17:50 PM	9:17:50 PM	2014-12-30 21:33:54	9:33:54 PM	0:16	Average Waking Time
12/30/2014 22:42:49	10:42:49 PM	10:42:49 PM	2014-12-31 7:13:31	7:13:31 AM	8:30	6:27:44 AM
12/31/2014 22:31:01	10:31:01 PM	10:31:01 PM	2015-01-01 6:03:01	6:03:01 AM	7:32	Average Sleeping Time
1/1/2015 22:12:10	10:12:10 PM	10:12:10 PM	2015-01-02 4:56:35	4:56:35 AM	6:44	10:47:04 PM

1/3/2015 0:34:57	12:34:57 AM	12:34:57 AM	2015-01-03 7:47:23	7:47:23 AM	7:12	`
1/4/2015 0:23:06	12:23:06 AM	12:23:06 AM	2015-01-04 7:37:09	7:37:09 AM	7:14	

To find a T-periodic solution numerically, we opted to fit the parameters by eye. Using the initial parameters provided in the instructions as a starting guide, we modified the parameters to end up with a solution that aligned with our researched sleep and wake times, and additionally ensured that the sleep indicator was consistent every day. Our final parameters, along with the produced graph are shown below.

Parameter	Value
χ_w	18
χ_s	4
a	0.1
α	3.6
T	24
H_0+	0.63
H_0-	0.2
Δt	0.25
t_0	0
$H_0(t)$	0.15



The plot of our base model

To modify the base model, we made changes to various formulas in our Google Sheet:

I. To make H(t) piecewise linear, we modified the formula for dH/dt from the original piecewise function to a constant value by removing H. We kept the denominators as the respective χ values, and changed the numerators to 1 and -1 for increasing and decreasing respectively.

```
=IF(I17=TRUE,-1/$B$3, 1/$B$2)
```

χ_w χ_s

II. Our sheet enabled us to freely change parameters, so we simply changed the alpha parameter to be equal to 0

III. To include higher harmonics, we modified the formula for $H^+(t)$ and $H^-(t)$ to include more sinusoidal terms:

```
(0.97*SIN((2*PI()/$B$6)*(B17-$B$5))+0.22*SIN((4*PI()/$B$6)*(B17-$B$5))
```

Analyzing the model and key results

Once a model was constructed, the data obtained from the equations were used and modified to address the application questions.

A: Sleep Deprivation and Sleeping Early

A sleep-deprived individual experiences a lack of sleep, hence in order to adjust our data for self-deprived individuals, the graph should continue to increase even after crossing the upper bound, until the person goes to sleep. First, we determined the quantity of sleep lost (chosen to be 4 hours), and manually altered the graph by changing the sleep indicator to awake, or FALSE at the correct time. The same was done to model an individual sleeping early, manually altering the sleep indicator to asleep 4 hours before they would have naturally fallen asleep.

These modifications were successful, shown in Figure 1.2 against the original model. As a result, we can see that this model is consistent with common experiences. The body's circadian rhythm is shifted, which is demonstrated through the graph when the person's sleep is naturally corrected overtime. This creates irregular conditions if the person attempts to fall asleep after sunrise following staying up for a longer period of time.

Furthermore, to conclude whether sleeping 4 hours late is better than 4 hours early, we overlaid the graphs in which an individual wakes up 4 hours early and goes to sleep 4 hours late onto the base model, as seen in Figure 1.2 in Appendix A. Visually, if an individual sleeps 4 hours later, their body reverts back to its normal routine in a shorter period of time. Thus, in the case of which option is less detrimental to a body's natural circadian rhythm, sleeping later is actually better.

B: Napping

To modify the model to account for naps, we manually adjusted the graph such that during the period of the nap occurring, H(t) would be decreasing. We selected the napping period to be one hour, and manually forced dH/dt to be in the "asleep" state. After 1 hour, we changed dH/dt to be in its awake state, then reverted dH/dt to the base conditional formula. The graph of the model overlaid on top of the base model can be viewed in Figure 1.4 of Appendix A. Visually, we can see that short naps will not alter an individual's rhythm of sleep. However, if the napping period increases in length, it will take a longer time for the individual to return to their normal sleep routine, resulting in fluctuations on the graph.

Additionally, we utilized modification i: making H(t) piecewise linear, described in the construction of the model, and illustrated in Figure 1.5 of Appendix A. In general, utilizing the same values for χ_w and χ_s resulted in more frequent sleeps and wakes. In the specific case of napping, the constant derivative had an impact. The base derivative of H(t) while an individual was awake decreased overtime, becoming less steep. This means that as an individual is awake for longer, their sleep drive flattens out as an individual is closer to falling asleep. However, since H(t) is now linear, there is no impact on dH/dt from how long an individual stays awake, meaning it does not matter what time an individual starts napping, only how long they nap for.

C: Multiple Sleeps

Since H^+ and H^- are the thresholds of when a being wakes up and falls asleep, moving these thresholds closer together would result in more frequent sleeping/waking cycles. If we define one day to be 24 hours, we can reduce parameter H^+_0 and increase parameter H^-_0 to achieve this desired effect (see Figure 1.6 in Appendix A). Using our multiple sleeps model, we can determine that the amount of sleep is identical to our base model, which comes out to 8 hours. Note that visually, the initial starting condition of H(t) skews the first 24 hours of sleeping/wake, but every following 24 hour interval results in the same amount of sleep of 8 hours.

Additionally, we also decided to apply modification ii: changing alpha to 0 such that the thresholds would be constant, rather than sinusoidal (see Figure 1.7 in Appendix A). This created the effect of consistently spaced sleeping and waking hours, without any variation. This resulted in less sleep than our base model. Realistically, this type of constant sleep is not realistic to humans, as the time our body feels naturally sleep or awake fluctuates by a certain amount.

D: Time Zones

The test application topic involved changing time zones, where alpha would change linearly depending on how many hours the person traveled forward/backwards. In our specific case, we decided to select a time change of 3 hours to examine what happens to a person's sleep. Comparing the resulting graph to our base graph, we see that there is simply a gradual change in a person's natural sleep and wake time (See Figure 1.7 in Appendix A). This implies that our body naturally adjusts our circadian rhythm based on a time zone, in a gradual way such that there is not a sudden drastic change.

E: Caffeine

The test application examined the effects of caffeine on the circadian rhythm, where the decay of caffeine follows a half-life exponential model. Once the person consumes coffee, as time passes, the amount of coffee inside the body decreases according to the half-life formula:

$$N(t) = N_0 \left(\frac{1}{2}\right)^{\frac{t}{t_{1/2}}}$$

Where t(½) is the half-life of caffeine (in this case, assumed to be 5 hours). However, for the purpose of clearer visualization, we decided to increase the half-life to better illustrate the situation. Caffeine is known for having disruptive effects on sleep, where an individual might experience trouble falling asleep. Based on Figure 1.8 in Appendix A, caffeine will disrupt the circadian rhythm and will cause disruptions in the sleep schedule. This is illustrated in the graph, as H(t) will surpass the upper threshold, demonstrating that the individual will go to sleep past their normal sleep schedule. This modification has been done by selecting a period when the person consumes caffeine, and applying the half-life formula to the function H(t) to examine alterations.

Strengths and limitations of the model(s)

Strengths:

- The model represents the general experience of sleep relatively accurate, with a reasonable periodic solution
- Our model does not only demonstrate the sleep timing pattern based on a normal circadian clock, but it also addresses the sleep timing under different special circumstances and these include: sleep deprivation, taking naps, sleeping multiple times during a day, time zone difference, and consuming caffeine. The success of implementing these applications makes the model much more realistic and widely usable.

- Additionally, our model is very flexible, with a modular system to change parameters and adjust dH/dt, enabling the exploration of different situations.

Limitations:

- The model can be further improved in terms of its predictability. For example, the model is limited in accounting for fluctuations in sleep and wake times. This model assumes the individual sleeps and wakes at the same time everyday.
- This model does not account for multiple sleep periods, if people suddenly wake up in the middle of the night.
- Furthermore, this model is not robust in modeling sleeping or waking against the natural circadian rhythm. For example, the data needs to be manually changed in order to account for napping. If the person were to take multiple naps a day, there will be difficulty in representing this situation with our model.

Further research

- The sleep and wake times for the model were based on the averages of data from >800 participants who participated in a sleep study. Data was extracted from the iOS Sleep Cycle App from 2014-2018. [1] This sample is limited to users of this app, which might have bias towards a certain demographic. More data from different sources may be needed to account for the differences in sleep amongst different populations.
- For application E, we model the amount of caffeine as a half-life exponential decay function implemented in the general model. However, caffeine decay varies across sex, age, and etc. Therefore, the realistic model accounts for this application can be much more complicated., and further researched. [2]
- The model is constructed based on given parameters and additional applications are mainly implemented by adjusting those parameters. This limits the functionality and flexibility of the model to some extent. Realistically, sleep routine is ensured by multiple parameters, an ideal model is better constructed in a way that fully combines more parameters with their optimal values. [3]
- As stated in our limitations, a “randomizer” could be implemented to purposely fluctuate the sleeping and wake times to match the realism of the varying starting and wake times.

Sources

[1] Diotte, D. (2022, April 25). *Sleep data*. Kaggle. Retrieved December 5, 2022, from https://www.kaggle.com/datasets/danagerous/sleep-data

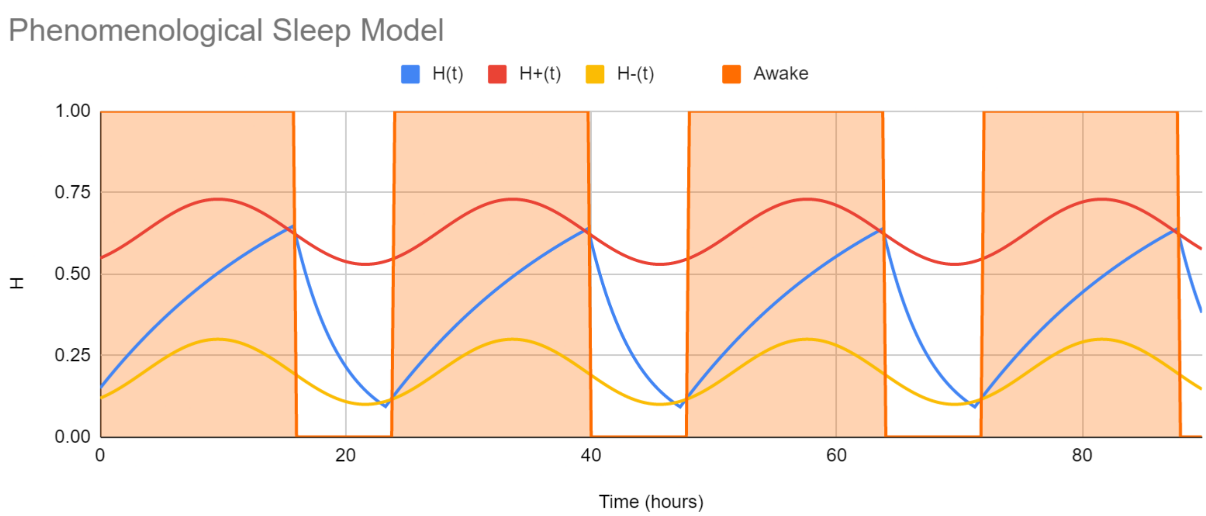
[2] *Gender differences in coffee consumption and its effects in young people*. (n.d.). Retrieved December 6, 2022, from

https://www.researchgate.net/publication/273986248_Gender_Differences_in_Coffee_Consumption_and_Its_Effects_in_Young_People

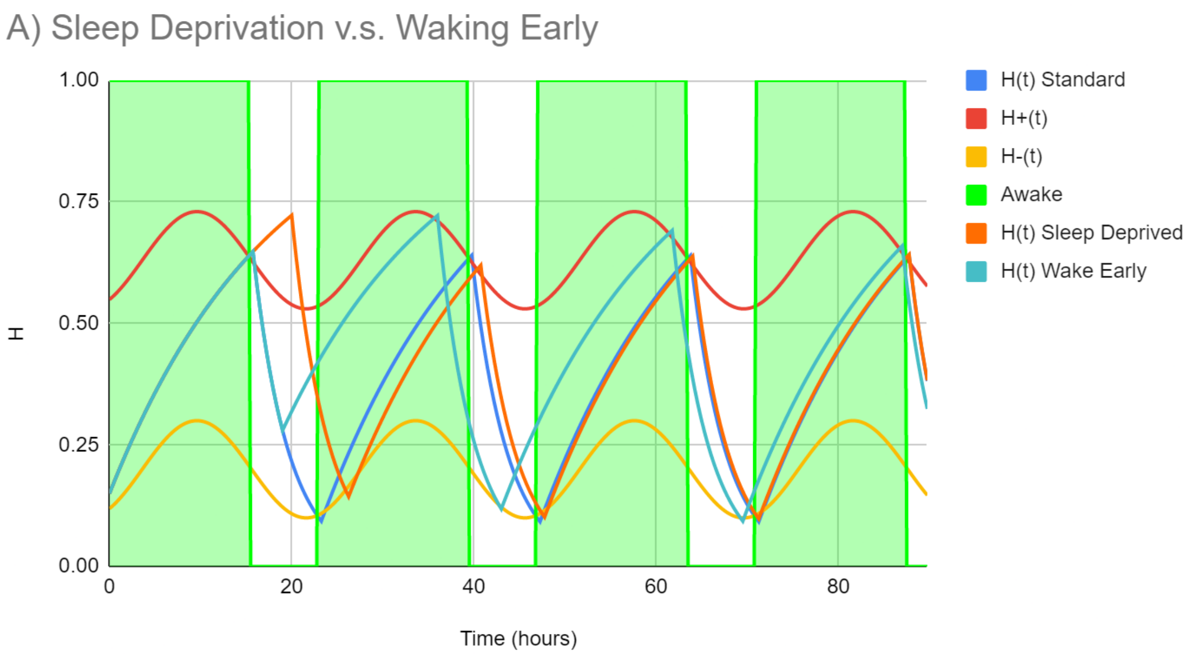
[3] Zwarensteyn, J. (2022, October 26). *12 factors affecting sleep: External and internal sleep quality elements*. Sleep Advisor. Retrieved December 5, 2022, from https://www.sleepadvisor.org/sleep-factors/

Appendix

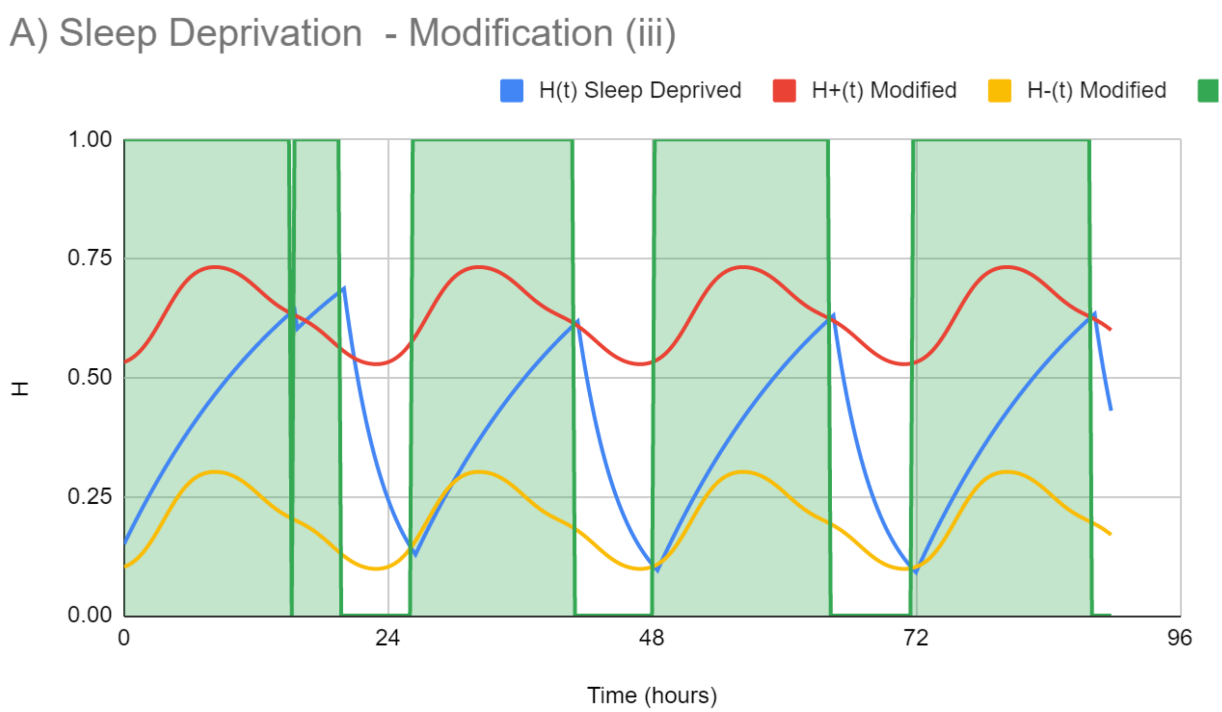
Appendix A: Plots of base and modified models



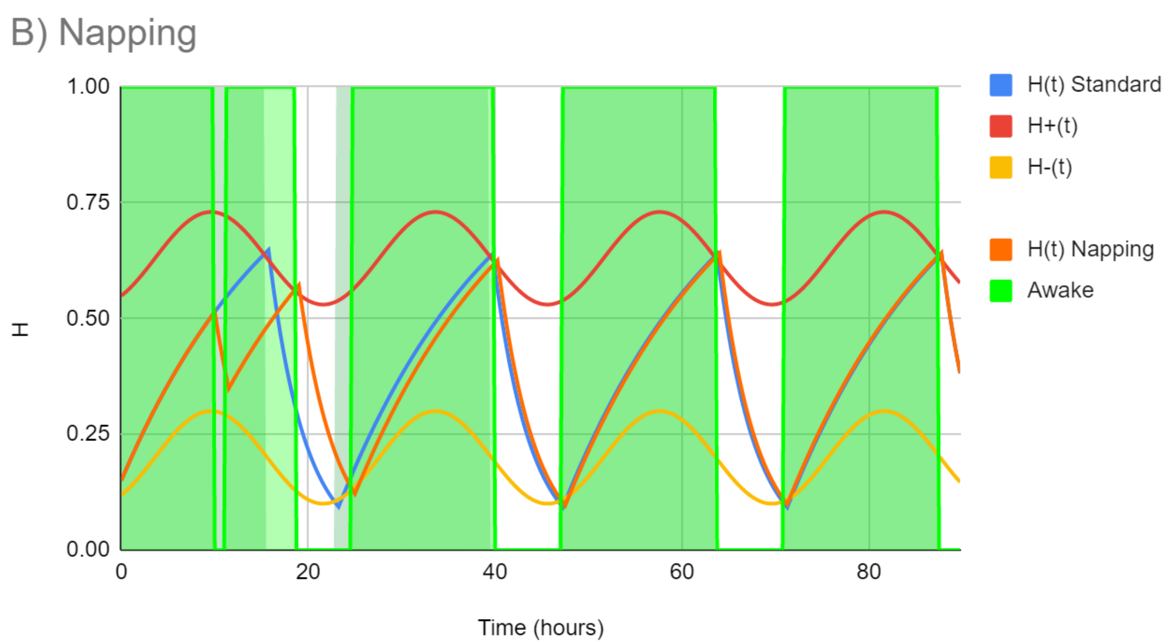
The base model (Figure 1.1)



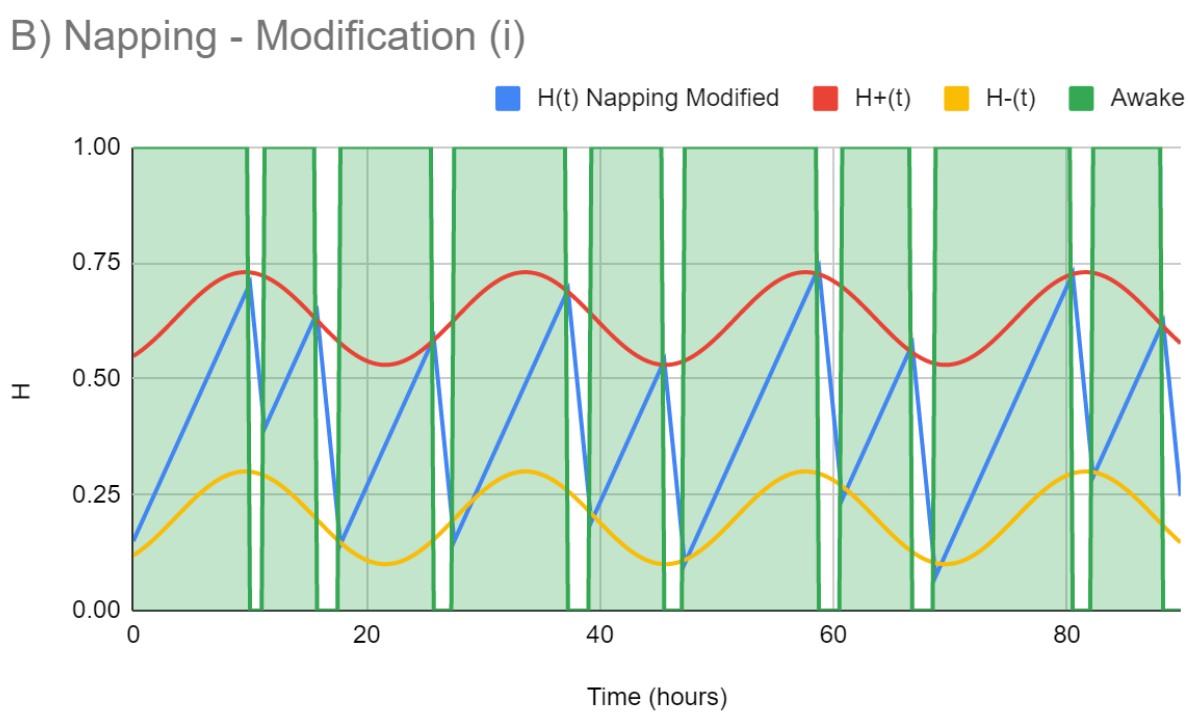
Addressing sleeping later or waking up earlier (Figure 1.2)



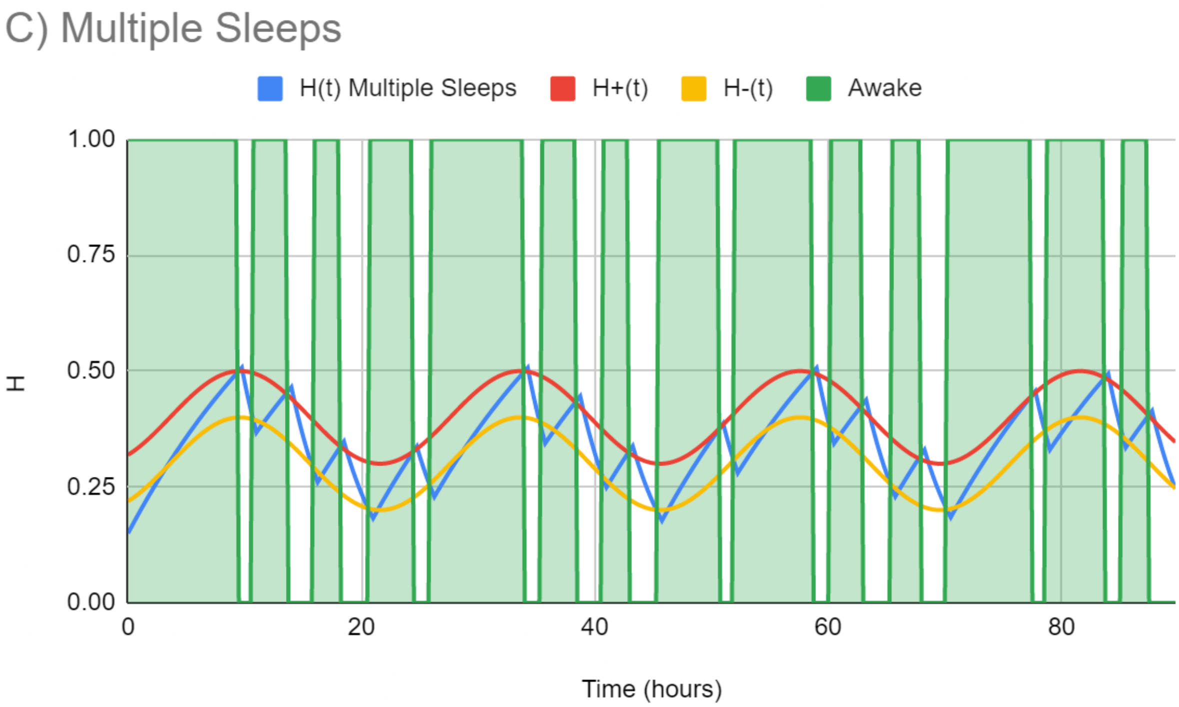
Looking at the effect of higher harmonics on thresholds (Figure 1.3)



The time it takes to adjust from a nap (Figure 1.4)



The effect of a linear H(t) on napping (Figure 1.5)



Graph for multiple sleep periods (Figure 1.6)

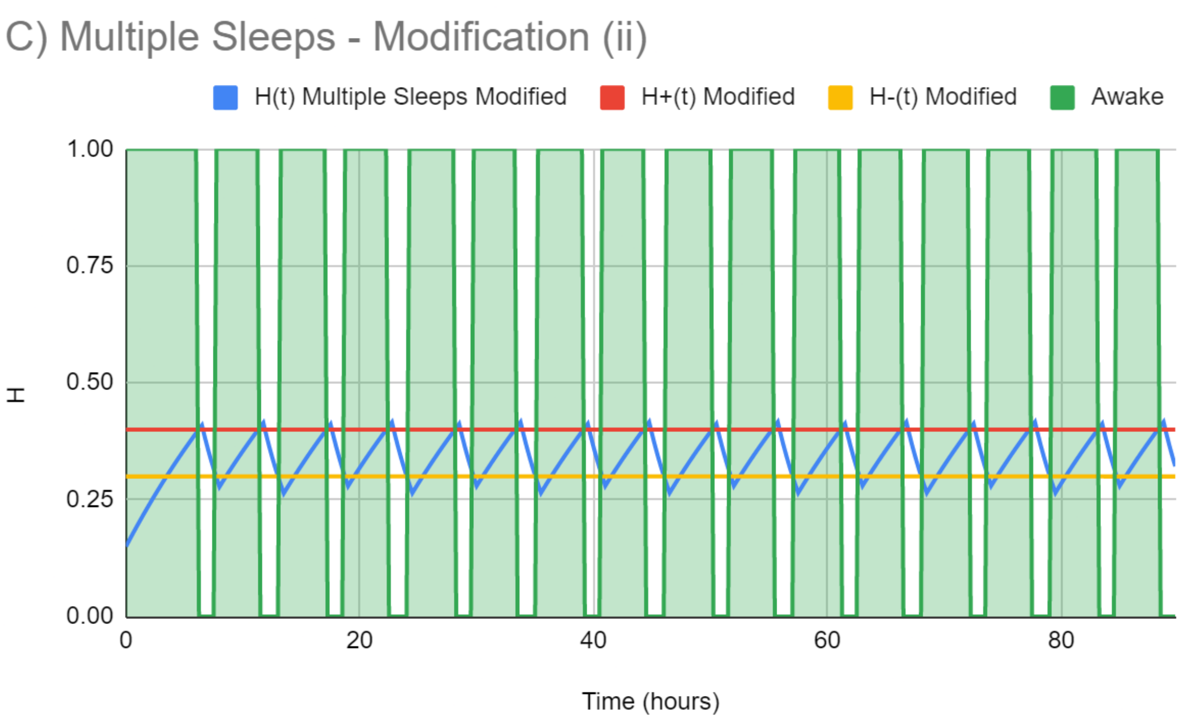
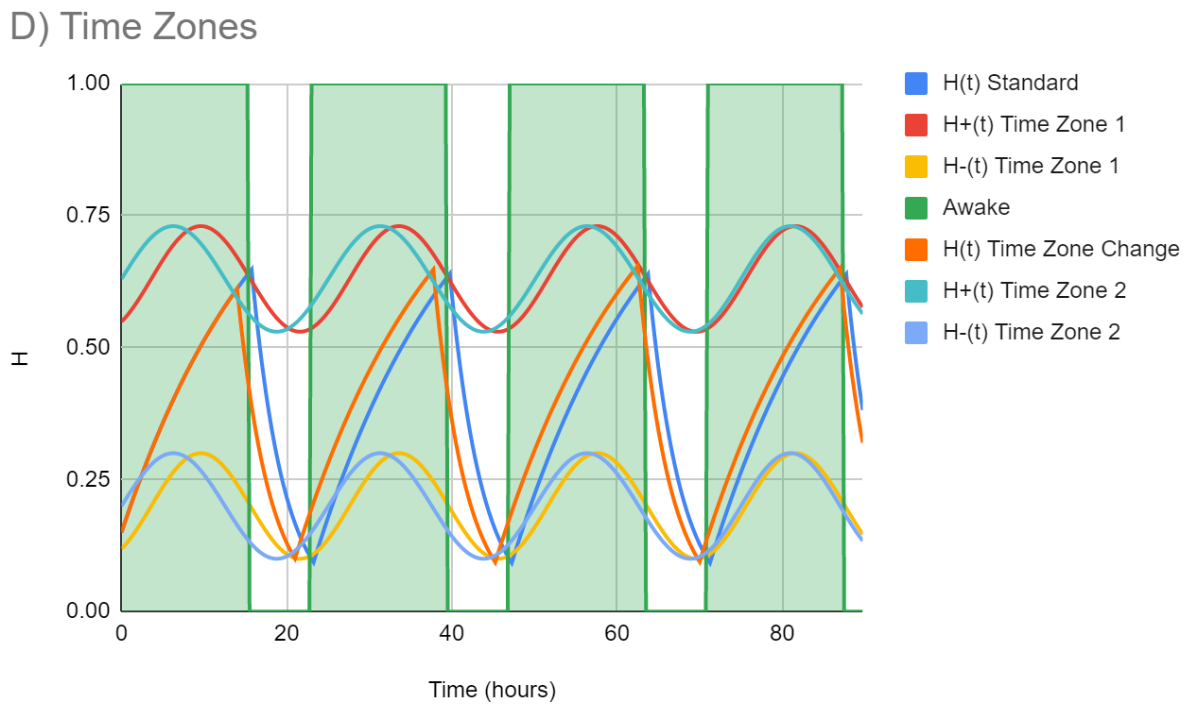
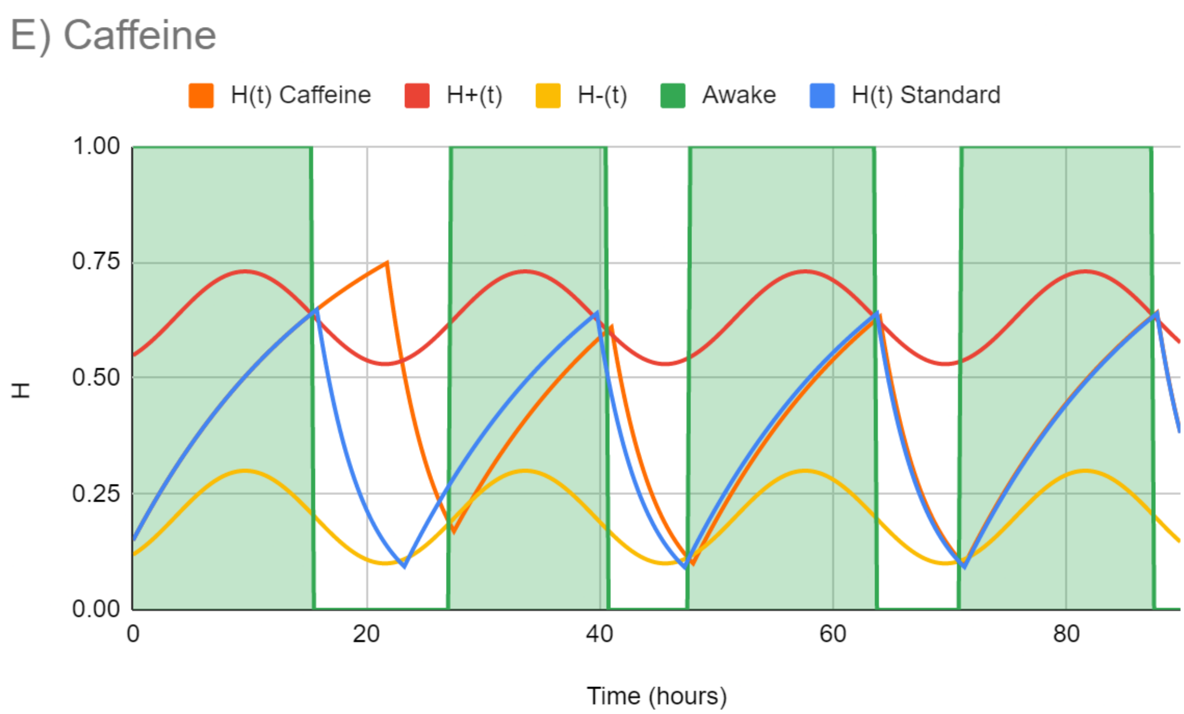


Figure 1.6 modified with alpha=0



Time it takes to adjust to different time zones (Figure 1.7)



Time it takes for the body to return back to the normal sleep routine after consuming caffeine (Figure 1.8)

Appendix B: Sample of Euler’s Method

t	H(t)	dH/dt	new H(t)		H+(t)	H-(t)	Asleep?	S(t)
0	0.150000	0.047222	0.161806		0.5490983006	0.1190983006	FALSE	1
0.25	0.161806	0.046566	0.173447		0.5531158168	0.1231158168	FALSE	1
0.5	0.173447	0.045920	0.184927		0.5574625629	0.1274625629	FALSE	1
0.75	0.184927	0.045282	0.196248		0.5621199254	0.1321199254	FALSE	1
1	0.196248	0.044653	0.207411		0.5670679609	0.1370679609	FALSE	1
1.25	0.207411	0.044033	0.218419		0.572285481	0.142285481	FALSE	1
1.5	0.218419	0.043421	0.229274		0.5777501435	0.1477501435	FALSE	1
1.75	0.229274	0.042818	0.239979		0.583438548	0.153438548	FALSE	1
2	0.239979	0.042223	0.250535		0.5893263357	0.1593263357	FALSE	1
2.25	0.250535	0.041637	0.260944		0.5953882943	0.1653882943	FALSE	1
2.5	0.260944	0.041059	0.271208		0.6015984655	0.1715984655	FALSE	1
2.75	0.271208	0.040488	0.281331		0.6079302565	0.1779302565	FALSE	1
3	0.281331	0.039926	0.291312		0.6143565535	0.1843565535	FALSE	1
3.25	0.291312	0.039372	0.301155		0.6208498381	0.1908498381	FALSE	1
3.5	0.301155	0.038825	0.310861		0.6273823052	0.1973823052	FALSE	1
3.75	0.310861	0.038285	0.320433		0.6339259816	0.2039259816	FALSE	1
4	0.320433	0.037754	0.329871		0.6404528463	0.2104528463	FALSE	1
4.25	0.329871	0.037229	0.339178		0.6469349504	0.2169349504	FALSE	1

Appendix C: Sample of Kaggle Sleep Data

Start	Start Time	Start Time (Processed)	End	End Time	Time in bed	
12/29/2014 22:57:49	10:57:49 PM	10:57:49 PM	2014-12-30 7:30:13	7:30:13 AM	8:32	
12/30/2014 21:17:50	9:17:50 PM	9:17:50 PM	2014-12-30 21:33:54	9:33:54 PM	0:16	Average Waking Time
12/30/2014 22:42:49	10:42:49 PM	10:42:49 PM	2014-12-31 7:13:31	7:13:31 AM	8:30	6:27:44 AM
12/31/2014 22:31:01	10:31:01 PM	10:31:01 PM	2015-01-01 6:03:01	6:03:01 AM	7:32	Average Sleeping Time
1/1/2015 22:12:10	10:12:10 PM	10:12:10 PM	2015-01-02 4:56:35	4:56:35 AM	6:44	10:47:04 PM
1/3/2015 0:34:57	12:34:57 AM	12:34:57 AM	2015-01-03 7:47:23	7:47:23 AM	7:12	`
1/4/2015 0:23:06	12:23:06 AM	12:23:06 AM	2015-01-04 7:37:09	7:37:09 AM	7:14	
1/4/2015 21:34:44	9:34:44 PM	9:34:44 PM	2015-01-05 4:53:34	4:53:34 AM	7:18	
1/5/2015 21:32:25	9:32:25 PM	9:32:25 PM	2015-01-06 5:00:03	5:00:03 AM	7:27	
1/6/2015 21:24:56	9:24:56 PM	9:24:56 PM	2015-01-07 5:00:02	5:00:02 AM	7:35	
1/7/2015 20:59:58	8:59:58 PM	8:59:58 PM	2015-01-08 6:19:20	6:19:20 AM	9:19	
1/8/2015 22:58:18	10:58:18 PM	10:58:18 PM	2015-01-09 6:14:58	6:14:58 AM	7:16	
1/9/2015 22:27:58	10:27:58 PM	10:27:58 PM	2015-01-10 7:29:00	7:29:00 AM	9:01	
1/10/2015 22:38:24	10:38:24 PM	10:38:24 PM	2015-01-11 7:28:59	7:28:59 AM	8:50	
1/11/2015 22:12:07	10:12:07 PM	10:12:07 PM	2015-01-12 6:20:29	6:20:29 AM	8:08	
1/12/2015 21:01:46	9:01:46 PM	9:01:46 PM	2015-01-13 6:13:23	6:13:23 AM	9:11	
1/13/2015 22:14:29	10:14:29 PM	10:14:29 PM	2015-01-14 6:20:33	6:20:33 AM	8:06	
1/14/2015 21:48:26	9:48:26 PM	9:48:26 PM	2015-01-15 5:02:05	5:02:05 AM	7:13	

Appendix D: Mind Map