A Quadratic Model for Piano Tuning

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9 July 2016

Assumption #1: The octave around middle C (or C4) is roughly equal-tempered.
Assumption #2: Due to the inharmonicity of the strings, each octave above or below middle C should be "stretched" incrementally wider than the preceding octave.

Define a "stretch factor" *s* in semitones per octave. Then each note *n*, in semitones above middle C, should be tuned $\frac{s}{2}\left(\frac{n}{12}\right)^2$ semitones sharper than equal temperament (and each note below middle C flatter by the same amount).

For my piano, a stretch factor of *s* = 0.05 semitones (or 5 cents) seems to work well. In other words, the octave around C5 will be tuned 5 cents wider than equal temperament, the octave around C6 will be tuned 10 cents wider, and so on. Using the above formula, we find that each C above and below middle C should be tuned as follows:

C5	$^5/_2$ = 2.5 cents sharp	C3	2.5 cents flat
C6	5 + $^{10}/_2$ = 10 cents sharp	C2	10 cents flat
C7	5 + 10 + $^{15}/_2$ = 22.5 cents sharp	C1	22.5 cents flat

Now how do we get actual frequencies from this?

In an equal-tempered tuning, the frequency of a note *n* is $x = C4 \cdot 2^{\frac{n}{12}}$ Hz. Adding in our adjustment term, we get $x = C4 \cdot 2^{\frac{n + \left(\frac{s}{2}\right)\left(\frac{n}{12}\right)^2}{12}}$ Hz. (Notes below middle C should be flat rather than sharp, so subtract the adjustment term rather than adding it.)

For concert pitch (A = 440 Hz), the correct frequency for middle C depends on the choice of stretch factor *s*. Substitute *x* = 440 Hz and *n* = 9 semitones in the above formula, then solve for C4. For *s* = 0.05, the correct frequency is 261.41 Hz.

Plugging that value for C4 back into the formula, we can then compute:

C5	523.58 Hz	C3	130.52 Hz
C6	1051.71 Hz	C2	64.98 Hz
C7	2118.66 Hz	C1	32.25 Hz

The inverse formula $n = \frac{24\sqrt{6 s \log_2 \frac{x}{C4} + 36} - 144}{s}$ can be used to find the note *n* which has a frequency of *x* Hz. (For notes below middle C, replace $\frac{x}{C4}$ with $\frac{C4}{x}$ and negate *n*.)