MATH 317 Portfolio

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1 Two proofs I am proud of

1.1 Proof 1

Given Text (Ch 3.3, Problem 18): A magic square is a square array of natural numbers whose rows, columns, and diagonals all sum to the same number. Prove that the following 4 by 4 square cannot be completed to form a magic square.

-	**1**	**2**	**3**	**4**
A	-	1	-	2
B	3	4	5	-
C	6	7	-	8
D	9	-	10	-

Proof. We will use a proof by contradiction to prove that the above 4 by 4 square cannot be completed to form a magic square. So we assume that all rows, columns, and diagonals sum to the same number. The sum of the diagonal involving $A4$, $C3$, $B2$ and $D1$ can be written as

$$A4 + B3 + C2 + D1 = 9 + 7 + 5 + 2$$

$$A4 + B3 + C2 + D1 = 23$$

We have shown that the sum of all rows, columns, and diagonals sum to 23. Therefore, there exists some natural number p such that column B can be expressed as

$$23 = B1 + B2 + B3 + B4$$

$$23 + 1 + 4 + 7 + p$$

Using algebra, we obtain

$$p = 11.$$

Now, there exists some natural number q such that row 4 can be expressed as

$$23 = A4 + B4 + C4 + D4$$

$$23 = 9 + p + 10 + q$$

Using substitution and algebra we obtain

$$23 = 9 + 11 + 10 + q$$

$$q = -7$$

We have found q is not a natural number, which contradicts our assumption that q was a natural number. Additionally, we know that the sum of each row, diagonal, and column should sum to 34, so we know that this magic square is incorrect. Therefore, we have proven that the rows, columns, and diagonals of this square do not all sum to the same value, and this magic square cannot be completed.

□

Reflection I chose this proof because it was tricky for me to figure out the first time. However, after working with classmates in breakout rooms and thinking about it more, I understood it better. Also, it was a fun proof to do.

1.2 Proof 2

Given Text: (Ch 3.2 Problem 9) Is the following proposition true or false? Explain: For each positive real number x, if x is irrational, then $\sqrt{x}$ is irrational.

Proof. We will prove that for each positive real number x, if x is irrational, then $\sqrt{x}$ is irrational. However, we do not have the equation of an irrational number, so we will do a proof by contrapositive. If for some positive real number x, if $\sqrt{x}$ is rational, then we will prove x is rational. If $\sqrt{x}$ is rational for a positive real number x, we can write

$$\sqrt{x} = \frac{m}{n}$$

Using algebra, we obtain

$$(\sqrt{x})^2 = (\frac{m}{n})^2$$

$$x = \frac{m^2}{n^2}$$

Since m and n are integers conserved over multiplication, we conclude m^2 and n^2 are integers. Therefore, x has been written in the form $x = \frac{q}{d}$ for some integers q and d, and hence, x is a rational number. In conclusion, we have proved the statement that for each positive real number x, if x is irrational, then $\sqrt{x}$ is irrational via the contrapositive. □

Reflection I chose this proof because I wanted to revisit the contrapositive. I am proud of this proof because using the contrapositive is not intuitive. It was challenging but rewarding to understand and apply.

2 Two corrected proofs

2.1 Proof 1

2.1.1 Original proof

Given Text: (Ch 3.5 Problem 14) If an integer has a remainder of 6 when it is divided by 7, is it possible to determine the remainder of the square of that integer when it is divided by 7? If so, determine the remainder and prove that your answer is correct.

Proof. We assume n is an integer with a remainder of 6 when divided by 7, and will prove n^2 has a remainder of 6^2 when divided by 7. We can write that we assume $n = 6(\mod 7)$. According to Theorem 3.28, we know that

$$n * n = 6 * 6(mod 7)$$

$$n^2 = 6^2(mod 7)$$

Therefore, we have shown if n=6 (mod7) then n$^2 = 6^2(mod 7)$.

□

2.1.2 Corrected proof

Proof. We assume n is an integer with a remainder of 6 when divided by 7, and will prove n^2 has a remainder of 1 when divided by 7. We can write that we assume $n = 7k + 6$ where k is an integer. Therefore, we can square $n = 7k + 6$ to obtain

$$n^2 = (7k + 6)^2$$

$$n^2 = 49k^2 + 84k + 36$$
$$n^2 = 7(7k^2 + 12k + 5) + 1$$

Since k is an integer and is conserved under multiplication and addition, we can use the integer q to represent $(7k^2 + 12k + 5)$.

$$n^2 = 7q + 1$$

Therefore, we have found that if n is an integer with a remainder of 6 when divided by 7, then n^2 has a remainder of 1 when divided by 7.

□

Reflection: I chose this proof because I did not understand what the question was asking the first time I proved it. In my original proof, I was trying to prove that n^2 is congruent to 6^2, which we already knew from a previous theorem. I also did not find that the remainder =1 the first time I proved this problem. I think my new work shows a better understanding of the question.

2.2 Proof 2

2.2.1 Original proof

Given Text: (Ch 13.2 Problem 13) Prove the following proposition:If $p, q \in \mathbb{Q}$ with $p < q$, then there exists an $x \in \mathbb{Q}$ with $p < x < q$.

Proof. We assume that p and q are rational numbers with $p < q$, and will show there exists a rational number x with $p < x < q$. If p and q are rational numbers, there exists integers m, n, t, and s with $n \neq 0$ and $s \neq 0$ such that

$$p = \frac{m}{n}$$

$$q = \frac{t}{s}$$

Therefore, average of $p + q$ will lie between p and q. The average can be expressed as a number x, where

$$x = \frac{1}{2}(p + q).$$

Since rational numbers are conserved under multiplication and addition, then we have proved that there exists a rational number x with $p < x < q$. □

2.2.2 Corrected proof

Proof. We assume that p and q are rational numbers with $p < q$, and will show there exists a rational number x with $p < x < q$. If p and q are rational numbers, there exists integers m, n, t, and s with $n \neq 0$ and $s \neq 0$ such that

$$p = \frac{m}{n}$$

$$q = \frac{t}{s}$$

Therefore, the average of $p + q$ will lie between p and q. The average can be expressed as a number x, where

$$x = \frac{1}{2}(p + q).$$

Since rational numbers are conserved under multiplication and addition, we know that x is a rational number. Additionally, we can rewrite the above equation to

$$x = \frac{p + q}{2}.$$

We also need to show that x lies between p and q, such that $p < x$ and $x < q$.

$$x = \frac{p + q}{2} = \frac{2p + (-p + q)}{2} = p + \frac{-p + q}{2}.$$

Additionally, since $p < q$, we know
$$\frac{-p+q}{2} > 0.$$
Therefore,
$$p + \frac{-p+q}{2} > p$$
and x> p.
We will show that $x < q$.
$$x = \frac{p+q}{2} = \frac{2q+(p-q)}{2} = q - \frac{-p+q}{2}$$
Therefore, $x < q$.
We have proved that if p and q are rational numbers such that $p < q$, then there exists a rational number x with $p < x < q$. □

Reflection: I chose this proof because it was missing an explanation of how we can know that the average (represented by x) is between p and q. Additionally, there were some typos in the original. I added information to make the proof more clear and easy to follow.

3 Three additional proofs

3.1 Proof 1

Given Text: (Problem 1 Portfolio Worksheet) Suppose x is an integer. Prove that x is even iff x^2 is even.

Proof. We assume that x is an integer, and will show that if and only if x^2 is even, then x is even. We will use a proof by contrapositive. The contrapositive of the original conjecture is if and only if x is odd, then x^2 is odd. If the contrapositive is proved true, then this proves the original conjecture true as well. If x is an odd integer, x can be expressed as
$$x = 2n + 1$$
where n is some integer. Using algebra, we obtain
$$x^2 = (2n+1)^2$$
$$x^2 = 4n^2 + 4n + 1$$
$$x^2 = 2(2n^2 + 2n) + 1$$

Since n is an integer and is therefore conserved under multiplication and addition, the term $(2n^2 + 2n)$ can be expressed by the integer q. We rewrite this as:
$$x^2 = 2q + 1$$

According to the definition of odd integers, this shows that if x is odd, then x^2 is odd. Since this statement used the term if and only if, we also need to consider the case that x is even, and show that in turn x^2 will not be odd, and will therefore be even. If x is an even integer, x can be expressed as

$$x^2 = 2m$$

where m is some integer. Using algebra we obtain

$$x^2 = (2m)^2$$

$$x^2 = 4m^2$$

$$x^2 = 2(2m^2)$$

According to the definition of even integers, this shows that if x is even, then x^2 is even. Therefore, we have proved that if and only if x is odd, then x^2 is odd. Additionally, by proof by contrapositive we have also shown that if and only if x^2 is even, then x is even.

□

3.2 Proof 2

Given Text: (Problem 11 Section 3.2) Prove that for each integer a, if $a^2 - 1$ is even, then 4 divides $a^2 - 1$.

Proof. We will prove that for each integer a, if $a^2 - 1$ is even, then 4 divides $a^2 - 1$. We will use a proof by contrapositive. The contrapositive of the original conjecture is for each integer a, if 4 does not divide $a^2 - 1$, then $a^2 - 1$ is odd. Since a is an integer, there are 2 cases to consider, a could be even, or a could be odd.
Case 1: a is even
If a is even, according to the definition of even, a can be expressed as

$$a = 2q$$

where q is an integer. Using substitution and algebra we obtain

$$a^2 - 1 = (2q)^2 - 1$$

$$= 4q^2 - 1$$

$$= 2(2q^2) - 1$$

Since q is an integer and is therefore conserved under multiplication and addition, the term $(2q^2)$ can be expressed by an integer n.

$$a^2 - 1 = 2(n) - 1$$

Therefore, we can see if a is even, then 4 does not divide $a^2 - 1$. Additionally, this also shows that if a is even, (and therefore 4 does not divide $a^2 - 1$) then $a^2 - 1$ is odd.

Case 2: a is odd
If a is odd, according to the definition of odd, a can be expressed as

$$a = 2q + 1$$

where q is an integer. Using substitution and algebra we obtain

$$\begin{aligned} a^2 - 1 &= (2q+1)^2 - 1 \\ &= 4q^2 + 4q + 1 - 1 \\ &= 4q^2 + 4q \\ &= 4(q^2 + q) \end{aligned}$$

Since q is an integer and is therefore conserved under multiplication and addition, the term $(q^2 + q)$ can be expressed by an integer m.

$$a^2 - 1 = 4m$$

Therefore, we see if a is odd, then 4 divides $a^2 - 1$. Also, we can see that if a is odd, (and therefore 4 divides $a^2 - 1$) then $a^2 - 1$ is even.
Therefore, we have shown that if 4 does not divide $a^2 - 1$, then $a^2 - 1$ is odd. In only Case 1, when a is even, 4 does not divide $a^2 - 1$. In only Case 1, it was found that $a^2 - 1$ is odd. We have proven our contrapositive conjecture, and in turn also proven our original statement, that if $a^2 - 1$ is even, then 4 divides $a^2 - 1$. $\square$

3.3 Proof 3

Given Text: (Problem 15 Section 4.1 -Induction)
(a) What is $\frac{d^2}{dx^2}(e^{ax})$, the second derivative of e^{ax}?
Using the chain rule, we find that

$$\begin{aligned} \frac{d^2}{dx^2}(e^{ax}) &= \frac{d}{dx} * (\frac{de^{ax}}{dx}) \\ &= \frac{d^n}{dx^n}(ae^{ax}) \\ &= a^2 e^{ax} \end{aligned}$$

(b) What is, $\frac{d^3}{dx^3}(e^{ax})$ the third derivative of e^{ax}?

$$\frac{d^3}{dx^3}(e^{ax}) = a^3 e^{ax}$$

(c) Let n be a natural number. Make a conjecture about the nth derivative of the function $f(x) = e^{ax}$. That is, what is $\frac{d^n}{dx^n}(e^{ax})$? This con-jecture should be written as a self-contained proposition including an appropriate quantifier. Use mathematical induction to prove your conjecture.

Proof. We will use proof by induction to show that for all natural numbers n, $\frac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$.
We must first prove that the base case is true. The base case is when n=1.

$$\frac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$$

$$\frac{d^1}{dx^1}(e^a x) = a^1 e^{ax}$$

For the inductive step, we will prove that for all $k \in \mathbb{N}$, if P(k) is true, then P(k+1) is true. We will let k be a natural number, and assume that P(k) is true. Therefore, we assume

$$\frac{d^k}{dx^k}(e^{ax}) = a^k e^{ax}$$

In order to prove P(k+1) is true, we need to show that

$$\frac{d^{k+1}}{dx^{k+1}}(e^{ax}) = a^{k+1} e^{ax}$$

Using the chain rule of calculus, we obtain

$$\frac{d^{k+1}}{dx^{k+1}}(e^{ax}) = \frac{d^1}{dx^1} * \frac{d^k}{dx^k}(e^{ax})$$

$$= \frac{d}{dx} * (\frac{d^k e^{ax}}{dx^k})$$

Using our assumption stated above, we can substitute our assumption into the equation, and use the chain rule to simplify the equation.

$$\frac{d^{k+1}}{dx^{k+1}}(e^{ax}) = \frac{d}{dx} * (a^k e^{ax})$$

$$= \frac{da^k e^{ax}}{dx}$$

$$= a^k * ae^{ax}$$

$$= a^{k+1} e^{ax}$$

Therefore, we have shown that if P(k) is true, then P(k+1) is true and the inductive step has been established. Therefore, by induction, for all natural numbers n, $\frac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$. $\square$

4 Article summary

Article: Of Ants and Universes

1. **What was the goal of the article?** The goal of this article was to use a metaphor of an ant on a string to model the stretching of the universe. The article used mathematics to help quantitatively show that the universe is stretching, and also tell us some of the effects that this stretching has.

2. **What did the article show?** To begin with, this article illustrated a scenario of an ant on a string that is attached to a wall. The ant moves forward 1cm/s, while the entire string is stretched uniformly by an additional 1m. Through the harmonic series, this article showed that the ant will actually reach the end, despite what you might initially think. The article applied this concept to the universe, with the assumption that the universe stretches linearly. Then, the article explained that the universe doesn't stretch linearly, and then modeled what would happen if the universe stretched exponentially. If the universe stretched exponentially, we would get a different series, and could imagine that the ant would never reach the end, and every galaxy would end up isolated. However, the universe stretches at a rate faster than linear, but not exponentially.

3. **If there is a proof in the article, explain it in your own words (and equations). If there are multiple proofs given, explain the central one.** This article used a series that is $\frac{1}{100}$ th of the harmonic series. The harmonic series is:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ...$$

   We can see that the harmonic series diverges, which means when you add up all the values in the harmonic series it will go to infinity.
   The series that is $\frac{1}{100}$ th of the harmonic series is:

$$\frac{1}{100} + \frac{1}{200} + \frac{1}{300} + \frac{1}{400} + \frac{1}{500} + ...$$

   We can see that this series diverges as well. However, when the harmonic series reaches 100, the above series will reach 1. This series was used to model an ant walking along a 1m string at a rate of 1 cm/s, while the string is being uniformly stretched by 1 m every second. Therefore, the ant covers $\frac{1}{100}$ of the string in the first second. However, since the string is also being stretched, in the 2nd second the ant only covers $\frac{1}{200}$ of the string. This pattern gives us the above series.

4. **If a proof is implied but not given, or if the article asks you to complete a proof, please do so.** n/a

5. **What is one new and interesting thing that you learned from reading this article?** I knew that the universe was expanding, however I had never really thought about how. I think I probably thought that it is expanding because of leftover momentum from the big bang. However, I learned that this expansion is more like the stretching of a string. I learned that this stretching is very complicated. The stretching isn't linear, but is expanding faster than a linear rate. The expansion has to do with a "cosmological constant" based off of the presence of "dark energy". On a more concrete note, I also learned how harmonic series can be applied to linear expansion. While the universe isn't expanding linearly, it can still be helpful to understand the linear model as a starting point.

6. **How did you work through the mathematically difficult parts?** I talked through this article with a classmate. It helped me figure out what parts I was confused on. Also, I added a few terms from the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + ...$ to see for myself that the series will diverge to an infinitely large value. When the article started to apply the ant on a string model to the universe, I drew sketches to visualize the concepts.

5 Reflection on proofs

1. **Why do we prove in mathematics?** We prove in mathematics for a few reasons. First of all, a thorough proof is a great way to to show that a statement is true or false. Also, proofs are necessary to communicate with other mathematicians and readers. Additionally, "the efforts to prove a conjecture, may sometimes require a deeper understanding of the theory in question. A mathematician that tries to prove something may gain a great deal of understanding and knowledge, even if his/her efforts to prove that conjecture will end with failure."[1] This quote is spot-on; through the act of explaining your problem solving process, you most definitely gain a deeper sense of understanding. Another article I read discussed how proofs are just the answer to the question "Why?" when you want a convincing answer.[2] Therefore, we use proofs all the time in real life. Doing proofs in a math class is a way to slow down and ask "Why?" about topics that we've built our mathematical basis on, along with new topics.

2. **Are proofs essential to mathematics? Can we call something mathematics if it's not proof-based?** I think proofs are essential to mathematics. One article stated "don't consider "proof" as something different from other mathematical activities – obviously it is about reasoning, calculating, being ingenious/creative, using one's knowledge and experiences and then drawing conclusions."[3] I think proofs are intertwined with the mathematics I have been learning in other classes. When you show your work for a calculation, that is kind of proving how you got the answer. If something is not proof-based, I do not think it is mathematics.

3. **What makes writing proofs challenging?** One reason why proof writing is difficult is because you first have to figure out the answer to the problem, and then your second step is to write the proof. Therefore, your written proof is often more in depth, in a different order, and overall just different from your original thought process. Writing proofs involves a planning step. It can be difficult to plan out your proof if you are not used to general styles and typical methods of proofs, so proofs take practice.

References:

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