THREE-DIMENSIONAL VIV OF FLEXIBLE STRUCTURES

By George S. Triantafyllou

Department of Naval Architecture and Marine Engineering
National Technical University of Athens
Zografou 15773, Athens, Greece

E-mail: gtrian@deslab.ntua.gr

Abstract

We present a theoretical model of three-dimensional riser dynamics in a current.

1 Introduction

Existing models of VIV of flexible structures are one-dimensional, based on the assumption that the direction of the current along the length is constant, and that the structure vibrates perpendicularly to the current. In many case however the direction of the current varies along the length, and this necessitates the use of two-dimensional models of VIV.

2 Formulation

We consider a riser of diameter *d* and length *l* in a current of velocity *U*. The fluid has density and kinematic viscosity . The frame of reference has the *x* axis parallel to the axis of the riser, and the *y* and *z* axes in a horizontal plane.

The equations of motion along the *y*, *z* axes are:

$$m\frac{\partial^2 y}{\partial t^2} + b\frac{\partial y}{\partial t} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right) = F_y \tag{1}$$

$$m\frac{\partial^2 z}{\partial t^2} + b\frac{\partial z}{\partial t} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 z}{\partial x^2}\right) - \frac{\partial}{\partial x}\left(T\frac{\partial z}{\partial x}\right) = F_z \tag{2}$$

where *b* is the structural damping per unit length, *m* is the mass of the structure per unit length, and F_y, F_z are the fluid forces per unit length along the *y*, *z* directions respectively.

We assume that the angle between the direction of the current and the *y* axis is a known function of *x*. We denote by the direction parallel to the current and by the direction perpendicular to the current. Then we have the following coordinate transformation relations:

$$\xi = y\cos\theta + z\sin\theta \tag{3}$$

$$\eta = -y\sin\theta + z\cos\theta \tag{4}$$

$$y = \xi\cos\theta - \eta\sin\theta \tag{5}$$

$$z = \xi\sin\theta + \eta\cos\theta \tag{6}$$

It is convenient to work using complex variables. For the displacements of the structure *y*, *z* we set:

$$y = \mathrm{Re}[Y\exp(i\omega t)] \tag{7}$$

$$z = \mathrm{Re}[Z\exp(i\omega t)] \tag{8}$$

Likewise for the displacements of the structure parallel and normal to the current we set:

$$\xi = \mathrm{Re}[Q\exp(i\omega t)] \tag{9}$$

$$\eta = \mathrm{Re}[H\exp(i\omega t)] \tag{10}$$

We model the force per unit length in the direction as follows:

$$F_\eta = \mathrm{Re}[(a_\eta\omega^2 H + i\frac{1}{2}\rho U^2 dC_{Lv}\frac{H}{|H|})\exp(i\omega t)] \tag{11}$$

where *a* is the added mass per unit length for acceleration along the direction, and the ratio *H*/|*H*| which has magnitude one ensures that the force is in phase with the component of the velocity of the structure normal to the current.

In the direction we use a linearized formula:

$$F_\xi = \mathrm{Re}[(a_\xi\omega^2 Q + iC\omega Q)\exp(i\omega t)] \tag{12}$$

where a is the added mass per unit length for acceleration along the direction, and C is the linearized fluctuating component of the quadratic damping term for this oscillation. The linear damping coefficient C is given by:

$$C = C_D \rho d \omega U \tag{13}$$

Implicit in equations (12) and (11) is the assumption that the two axes and are the principal axes for the added mass tensor. This can be justified from the observation that, in a frame fixed on the cylinder, the flow is symmetric with respect to both axes , if averaged over one period of oscillation. The forces per unit length along the y and z directions can be found from F and F by using the transformation equations (3) and (4).

We substitute equations (7) through (12) into equations (1) and (2) and obtain:

(a) <u>y-axis</u>

$$\begin{aligned}
&\frac{d^2}{dx^2}(EI\frac{d^2Y}{dx^2}) - \frac{d}{dx}(T\frac{dY}{dx}) + ibY\omega - \omega^2 Y(m + a_{yy}) - \omega^2 Z a_{yz} \\
&= iY(\frac{1}{2|H|}C_{Lv}\rho U^2 d \sin^2\theta - C\omega\cos^2\theta) \\
&- iZ(\frac{1}{2|H|}C_{Lv}\rho U^2 d + C\omega)\sin\theta\cos\theta
\end{aligned} \tag{14}$$

(b) <u>z-axis</u>

$$\begin{aligned}
&\frac{d^2}{dx^2}(EI\frac{d^2Z}{dx^2}) - \frac{d}{dx}(T\frac{dZ}{dx}) + ibZ\omega - \omega^2 Z(m + a_{zz}) - \omega^2 Y a_{yz} \\
&= iZ(\frac{1}{2|H|}C_{Lv}\rho U^2 d \cos^2\theta - C\omega\sin^2\theta) \\
&- iY(\frac{1}{2|H|}C_{Lv}\rho U^2 d + C\omega)\sin\theta\cos\theta
\end{aligned} \tag{15}$$

where

$$a_{yy} = a_\eta \sin^2\theta + a_\xi \cos^2\theta \tag{16}$$

$$a_{zz} = a_\eta \cos^2\theta + a_\xi \sin^2\theta \tag{17}$$

$$a_{yz} = (a_\xi - a_\eta)\cos\theta\sin\theta \tag{18}$$

Regions of the structure with no current need some special consideration. Let *y*, *z* be the displacement of the center of the cross section. Then:

$$y = |Y|\cos(\omega t + \alpha_y) \tag{19}$$

$$z = |Z|\cos(\omega t + \alpha_z) \tag{20}$$

Then the components $_y$, $_z$ of the velocity of the center of the cross section are:

$$v_y = -\omega |Y| \sin(\omega t + \alpha_y) \tag{21}$$

$$v_z = -\omega |Z| \sin(\omega t + \alpha_z) \tag{22}$$

The magnitude of the drag force is $1/2 C_D\ d(\ _y^2 + \upsilon_z^2)$, and has direction opposite to the resultant velocity. Consequently, the two components of the force are:

$$F_y = -\frac{1}{2} C_D \rho d (v_y{}^2 + v_z{}^2) \frac{v_y}{|v_y|} \tag{23}$$

$$F_z = -\frac{1}{2} C_D \rho d (v_y{}^2 + v_z{}^2) \frac{v_z}{|v_z|} \tag{24}$$

Now we replace (23), (24) with the following linear forces:

$$F_{ly} = -C_y v_y \tag{25}$$

$$F_{lz} = -C_z v_z \tag{26}$$

The constants C_y, C_z are determined from the condition that (25), (26) dissipate the same amount of work as (23), (24):

$$\int_0^{2\pi/\omega} F_y v_y dt = \int_0^{2\pi/\omega} F_{ly} v_y dt \tag{27}$$

$$\int_0^{2\pi/\omega} F_z v_z dt = \int_0^{2\pi/\omega} F_{lz} v_z dt \tag{28}$$

The right-hand side of (27) becomes:

$$\int_0^{2\pi/\omega} F_{ly} v_y dt = \frac{\pi}{\omega} C |Y|^2 \omega^2 \tag{29}$$

The left-hand side of (27) becomes:

$$\int_0^{2\pi/\omega} F_y v_y dt = K\omega^2 (|Y|^3 \frac{8}{3} + |Z|^2 |Y| \frac{4}{3}(1+\cos^2(\alpha_z - \alpha_y))) \tag{30}$$

where

$$K = \frac{1}{2} C_D \rho d \tag{31}$$

Equating (30) and (29) we obtain the value of C_y:

$$C_y = (\frac{8}{3\pi}|Y| + \frac{4}{3\pi}(1+\cos^2(\alpha_z - \alpha_y))\frac{|Z|^2}{|Y|})\omega K \tag{32}$$

Likewise we obtain the following expression for C_z:

$$C_z = (\frac{8}{3\pi}|Z| + \frac{4}{3\pi}(1+\cos^2(\alpha_z - \alpha_y))\frac{|Y|^2}{|Z|})\omega K \tag{33}$$

The equations (32) and (33) are not very convenient to use because they contain the phase difference between *y* and *z* which is unknown beforehand. We can simplify things by making the reasonable assumption that *y* and *z* are 90 degrees out of phase, in which case cos($_z$ - $_y$) = 0. We still have however potential difficulties from the terms $|Y|^2/|Z|$, $|Z|^2/|Y|$. Another possibility which simplifies things considerably is to use the following simplifications:

$$C_y = \frac{8}{3\pi}|Y|\omega K = \frac{4}{3\pi} C_D \rho d |Y| \omega \tag{34}$$

$$C_z = \frac{8}{3\pi}|Z|\omega K = \frac{4}{3\pi} C_D \rho d |Z| \omega \tag{35}$$

We can now re-write the two equations in the following form valid for regions with or without current.

(a) <u>*y*-axis</u>

$$\begin{aligned} &\frac{d^2}{dx^2}(EI\frac{d^2Y}{dx^2}) - \frac{d}{dx}(T\frac{dY}{dx}) + ibY\omega - \omega^2 Y(m + a_{yy}) - \omega^2 Z a_{yz} \\ &= iY(\frac{1}{2|H|}C_{Lv}\rho U^2 d \sin^2\theta - C_{Ry}\omega) - iZ(\frac{1}{2|H|}C_{Lv}\rho U^2 d \sin\theta\cos\theta + C_{Rc}\omega) \end{aligned} \tag{36}$$

(b) z-axis

$$\frac{d^2}{dx^2}(EI\frac{d^2Z}{dx^2})-\frac{d}{dx}(T\frac{dZ}{dx})+ibZ\omega-\omega^2Z(m+a_{zz})-\omega^2Ya_{yz}$$
$$=iZ(\frac{1}{2|H|}C_{Lv}\rho U^2 d\cos^2\theta-C_{Rz}\omega)-iY(\frac{1}{2|H|}C_{Lv}\rho U^2 d\sin\theta\cos\theta+C_{Rc}\omega) \quad (37)$$

where:

(i) C_{Lv} is a known function of the local reduced frequency and amplitude for non-zero velocity of current, and $C_{Lv} = 0$ for zero velocity of current.

(ii) For non-zero current, C_{Ry} is given by:

$$C_{Ry} = C\cos^2\theta \quad (38)$$

for zero current:

$$C_{Ry} = C_y \quad (39)$$

(iii) For non-zero current, C_{Rz} is given by:

$$C_{Rz} = C\sin^2\theta \quad (40)$$

for zero current:

$$C_{Ry} = C_z \quad (41)$$

(iv) Finally, for non-zero current, C_{Rc} is given by:

$$C_{Rc} = C\sin\theta\cos\theta \quad (42)$$

while for zero current:

$$C_{Rc} = 0 \quad (43)$$

3 Energy Relations

We multiply equation (36) by the complex conjugate of Y, Y^*, and we integrate along the length of the structure. Then after some integrations by parts we separate the real and imaginary parts and we obtain the following equations:

$$\int_0^l (EI\,|\frac{d^2Y}{dx^2}|^2 - T\,|\frac{dY}{dx}|^2)\,dx = \omega^2 \int_0^l (m + a_{yy})\,|Y|^2 dx + \omega^2 \int_0^l a_{yz}\,\mathrm{Re}[Y^* Z]\,dx$$
$$+\int_0^l \mathrm{Im}[Y^* Z](\frac{1}{2\,|H|} C_{Lv} \rho U^2 d \sin\theta\cos\theta + C_{Rc}\omega)\,dx \tag{44}$$

and

$$\int_0^l b\omega\,|Y|^2\;dx - \omega^2 \int_0^l a_{yz}\,\mathrm{Im}[Y^* Z]dx$$
$$= \int_0^l (\frac{|Y|^2}{2\,|Y|} C_{Lv} \rho U^2 d \sin^2\theta - C_{Ry}\,|Y|^2\;\cos^2\theta)\,dx \tag{45}$$
$$-\int_0^l \mathrm{Re}[Y^* Z](\frac{1}{2\,|H|} C_{Lv} \rho U^2 d \sin\theta\cos\theta + C_{Rc}\omega)\,dx$$

If we repeat the process with equation (37) we obtain the following equations:

$$\int_0^l (EI\,|\frac{d^2Z}{dx^2}|^2 - T\,|\frac{dZ}{dx}|^2)\,dx = \omega^2 \int_0^l (m + a_{zz})\,|Z|^2 dx + \omega^2 \int_0^l a_{yz}\,\mathrm{Re}[Y^* Z]\,dx$$
$$-\int_0^l \mathrm{Im}[Y^* Z](\frac{1}{2\,|H|} C_{Lv} \rho U^2 d \sin\theta\cos\theta + C_{Rc}\omega)\,dx \tag{46}$$

and

$$\int_0^l b\omega\,|Z|^2\;dx - \omega^2 \int_0^l a_{yz}\,\mathrm{Im}\left[Y^* Z\right]dx$$
$$= \int_0^l (\frac{|Z|^2}{2\,|H|} C_{Lv} \rho U^2 d \cos^2\theta - C_{Rz}\,|Z|^2\;\sin^2\theta)\,dx \tag{47}$$
$$-\int_0^l \mathrm{Re}[Y^* Z](\frac{1}{2\,|H|} C_{Lv} \rho U^2 d \sin\theta\cos\theta + C_{Rz}\omega)\,dx$$

From equations (44) and (46) it follows that for each mode there are two possible synchronization frequencies. This can be seen more readily in the special case where the two last terms at the right-hand side of equations (44) and (46) are small and the two equations decouple.

4 Appendix A

The following relations have been used for the derivation of the equivalent linear damping relations:

$$\int_0^{2\pi} \sin^2 x \,|\sin x|\, dx = \int_0^{2\pi} \cos^2 x \,|\cos x|\, dx = \frac{8}{3} \tag{48}$$

$$\int_0^{2\pi} \sin^2 x \,|\cos x|\, dx = \int_0^{2\pi} \cos^2 x \,|\sin x|\, dx = \frac{4}{3} \tag{49}$$

$$\int_0^{2\pi} \sin x \cos x \,|\sin x|\, dx = \int_0^{2\pi} \sin x \cos x \,|\cos x|\, dx = 0 \tag{50}$$

Contents: