Adaptive Proximal Average based Variance Reducing Stochastic Methods for Optimization with Composite Regularization

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1. Introduction for basic methods

Traditional Formulation:

$$\min_{x\in\mathbb{R}^d} F(x) := \frac{1}{n}\sum_{i=1}^{n} f_i(x) + r(x), \tag{1}$$

- $f_i : \mathbb{R}^d \to \mathbb{R}$: the empirical loss of the i-th sample with regard to the parameter x.
- r: the regularization term, which is convex but possibly non-smooth.
- Examples: LASSO, sparse SVM, ℓ_1, ℓ_2-Logistic Regression.

Forward-Backward Splitting:

$$x^{k+1} = \mathrm{prox}_r^{\gamma}(x^k - \gamma\cdot\Box), \tag{2}$$

where $\Box$ can be $\nabla f(x^k)$ in GD, $\nabla f_i(x^k)$ in SGD, or $\nabla f_i(x^k) - \nabla f_i(\tilde{x}) + \nabla f(\tilde{x})$ in variance reducing stochastic gradient descent.

Proximal Operator:

$$\mathrm{prox}_r^{\gamma}(x) = \arg\min_{y\in\mathbb{R}^d}\Big(r(y) + \frac{1}{2\gamma}\|y-x\|^2\Big). \tag{3}$$

One requirement for using proximal operators is that $\mathrm{prox}_r^{\gamma}(x)$ can be calculated effectively.

2. More complex problem

Composite Regularization:

$$\begin{aligned}\min_{x\in\mathbb{R}^d} F(x) &= f(x) + r(x)\\ &= \frac{1}{n}\sum_{i=1}^{n} f_i(x) + \sum_{k=1}^{K} w_k r_k(x),\end{aligned}$$

where $w_k \ge 0$ and $\sum_{k=1}^K w_k = 1$.

Examples:

- Overlapping group lasso: $r(x) = \lambda\sum_{k=1}^{K}\|x_{g_k}\|$.
- Graph-guided fused lasso:

$$r(x) = \sum_{\{i,j\}\in\mathcal{E}} w_{ij}|x_i - x_j|.$$

Difficulty:

$\mathrm{prox}_r^{\gamma}(x)$ is hard to be calculated.

Drawbacks of Existing Methods

- ADMM: requires more space and involves complex implementation and convergence analysis.
- Three operator splitting: involves strong assumption.

3. Related works

Variance Reducing Stochastic Methods (Prox-SVRG, Prox-SAGA)

- Use $v^k = \nabla f_j(x^k) - \nabla f_j(\tilde{x}) + \frac{1}{n}\sum_{i=1}^{n}\nabla f_i(\tilde{x})$ to replace $\nabla f_j(x^k)$.

- Prox-SVRG:

Define $\theta = \frac{1}{\gamma\mu(1-4L\gamma)m} + \frac{4L\gamma(m+1)}{(1-4L\gamma)m}$, then

$$\mathbb{E}F(\tilde{x}_s) - F^* \le \theta[F(\tilde{x}_{s-1}) - F^*]. \tag{4}$$

If $0<\gamma<1/(4L)$ and m is large enough such that $\theta<1$, then Prox-SVRG can achieve the linear convergence rate.

- Prox-SAGA:

By the Lyapunov function $T^k = \frac{1}{n}\sum_{i=1}^{n} f_i(x_i^k) - f(x^*) - \frac{1}{n}\sum_{i=1}^{n}\langle\nabla f_i(x^*), x_i^k - x^*\rangle + c\|x^k - x^*\|^2$,

$$\mathbb{E}\|x^k - x^*\|^2 \le \Big(1 - \min\Big\{\frac{1}{4n}, \frac{\mu}{3L}\Big\}\Big)^k T^0. \tag{5}$$

Proximal Average (PA)

- Definition

The proximal average of r is the unique semi-continuous convex function $\hat{r}(x)$ such that

$$\mathrm{prox}_{\hat{r}}^{\gamma}(x) = \sum_{k=1}^{K} w_k\cdot\mathrm{prox}_{r_k}^{\gamma}(x). \tag{6}$$

- Lemma

Assume that each r_k is L_k-Lipschitz continuous, then $0 \le r(x) - \hat{r}(x) \le \frac{\gamma\bar{L}^2}{2}$, where $\bar{L}^2 = \sum_{k=1}^K w_k L_k^2$.

- Conclusion. As the stepsize γ gets smaller, $\hat{r}(x)$ would be closer to $r(x)$.

4. Our methods

Alternative:

$$\min_{x\in\mathbb{R}^d}\hat{F}(x) = f(x) + \hat{r}(x), \tag{7}$$

in which r is replaced by its proximal average $\hat{r}$. Then the iteration becomes

$$\begin{aligned}x^{k+1} &= \mathrm{prox}_{\hat{r}}^{\gamma}(x^k - \gamma v^k)\\ &= \sum_{k=1}^{K} w_k\cdot\mathrm{prox}_{r_k}^{\gamma}(x^k - \gamma v^k).\end{aligned}$$

We need to decrease γ adaptively.

APA-SVRG

- ADA-SVRG Algorithm

```
Initialize: An initial number of inner loops m_0 > 0, decay rate 0 < ρ < 1, and an initial point x̃_0.
for s = 1, 2, ···, do
    x^0 = x̃_{s-1}, ṽ = Σ_{i=1}^n f_i(x̃_{s-1})/n;
    m_s = m_0 · ρ^{-s};
    γ_s = min{1/4L, ρ^s};
    for l = 1, 2, ···, m_s do
        Randomly pick j from {1, 2, ..., n};
        v^l = ∇f_j(x^{l-1}) − ∇f_j(x̃_{s-1}) + ṽ;
        x^l = Σ_{k=1}^K w_k · prox_{r_k}^{γ_s}(x^{l-1} − γ_s v^l);
    end for
    x̃_s = Σ_{l=1}^{m_s} x^l / n.
end for
```

- Theorem

Theorem 1 (APA-SVRG)**.** Suppose that L-smoothness, μ-strong convexity and L_k-Lipschitz continuous regularisers assumptions hold. Then for the update in APA-SVRG, it holds that

$$\begin{aligned}&\mathbb{E}F(\tilde{x}_s) - F^*\\ \le\ &\theta^s(\hat{F}_0(\tilde{x}_0) - F^*) + \frac{\gamma_0}{2}\bar{L}^2\frac{\theta}{\theta-\rho}(\theta^s - \rho^s).\end{aligned}$$

- Remarks

· When $\rho = 1$, i.e. the stepsize is fixed, $\mathbb{E}F(\tilde{x}_{s+1})$ will not converge to the minimum value.
· When $0<\rho<1$, $F(\tilde{x}_{s+1}) - F^*$ approaches 0 at the exponential rate.

- Complexity

Corollary 1. To achieve the ϵ-accurate solution, the overall iteration complexity of APA-SVRG is $\sum_{s=0}^{S}\mathcal{O}(n+2m_s) = \mathcal{O}(nS + \sum_{s=0}^{S} m_s) = \mathcal{O}(n\log\frac{1}{\epsilon} + m_0\frac{1}{\epsilon})$.

APA-SAGA

- APA-SAGA Algorithm

```
Initialize: An initial number of inner loops m_0 > 0, decay rate 0 < ρ < 1, an initial point x^0, and g_i^0 = ∇f(x^0), i = 1, 2, ..., n.
for s = 1, 2, ···, do
    m = m_0 · ρ^{-s};
    γ_s = (1/3L) · ρ^s;
    x^0 = x_s;
    for l = 1, ···, m do
        Randomly pick j from {1, 2, ..., n};
        v^l = ∇f_j(x^{l-1}) − g_j^l + Σ_{i=1}^n g_i^l / n;
        x^l = Σ_{k=1}^K w_k · prox_{r_k}^{γ_s}(x^{l-1} − γ_s v^l);
        Update g_i^l, i = 1, 2, ..., n:
            g_i^l = { ∇f_j(x^{l-1}),  if i = j,
                      g_i^{l-1},      otherwise.
    end for
    x_s = x^m.
end for
```

- Complexity

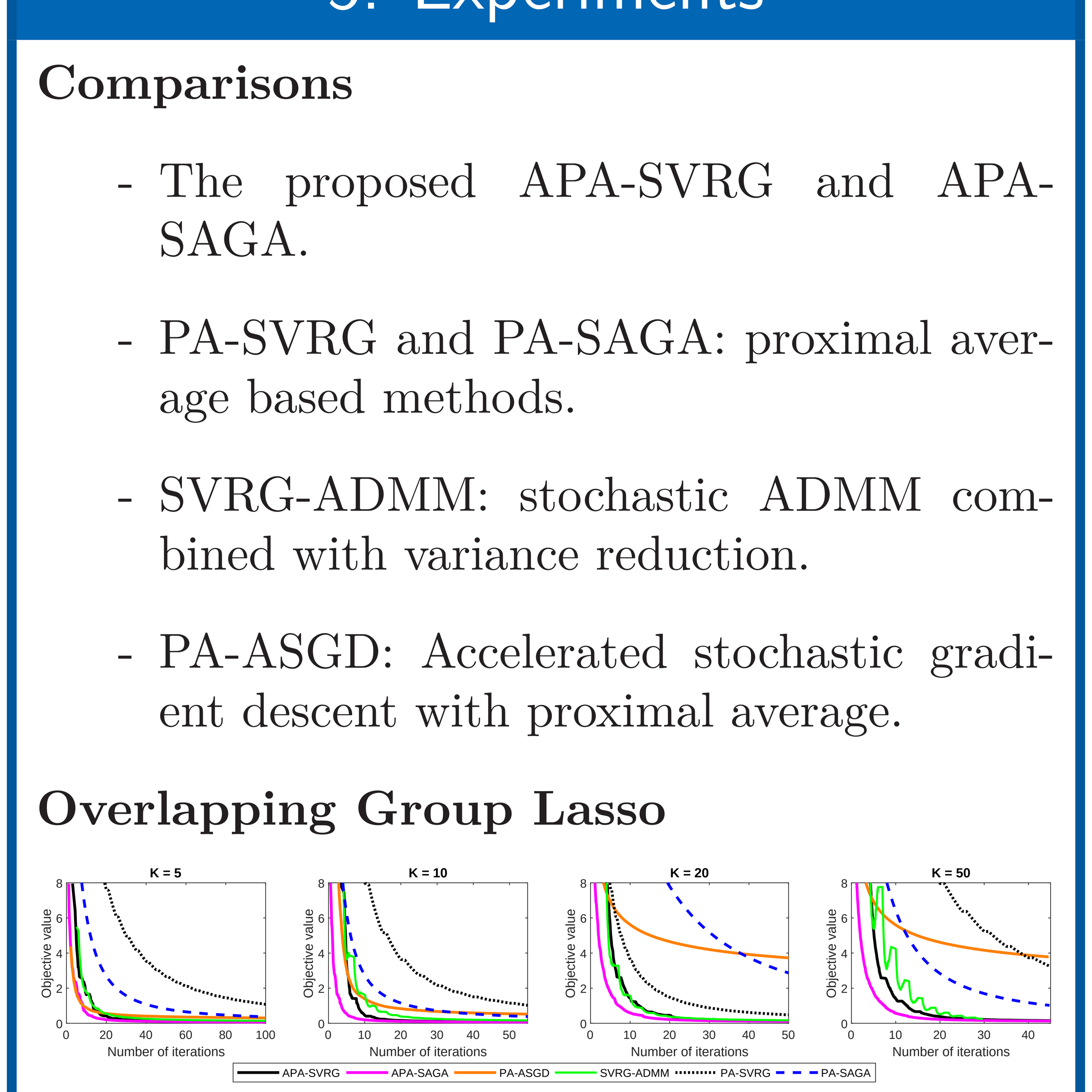
Corollary 2. To achieve the ϵ-accurate solution, the overall iteration complexity of APA-SAGA is $\mathcal{O}(n\log\frac{1}{\epsilon} + m_0\frac{1}{\epsilon})$.

5. Experiments

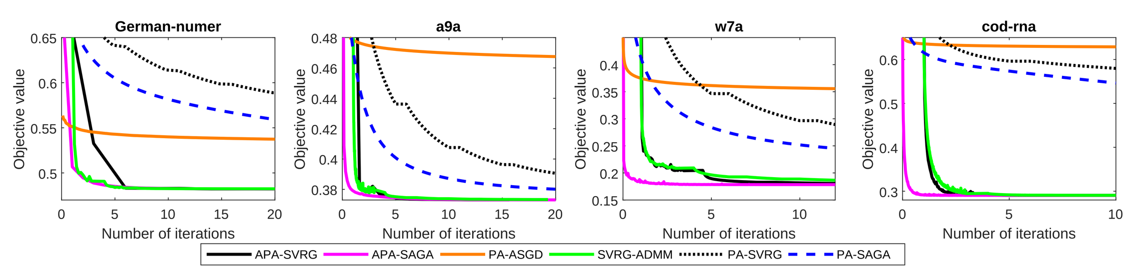
Comparisons

- The proposed APA-SVRG and APA-SAGA.
- PA-SVRG and PA-SAGA: proximal average based methods.
- SVRG-ADMM: stochastic ADMM combined with variance reduction.
- PA-ASGD: Accelerated stochastic gradient descent with proximal average.

Overlapping Group Lasso



Graph-Guided Logistic Regression



thankyou