PHY 3310 - Neutrino oscillations

Fall 2019

We wish to investigate the oscillation of neutrino flavors by modeling the dynamics of *all* three flavors, $|\nu_e\rangle$, $|\nu_\mu\rangle$ and $|\nu_\tau\rangle$. Neutrinos are generally detected by indirect evidence of their interaction with other observable particles through the weak interaction. States of definite *flavor* take part in the weak interaction. Interestingly, the mass eigenstates are superpositions of definite flavor states. The transformation is between flavor eigenstates and mass eigenstates is described by the **Pontecorvo-Maki-Nakagawa-Sakata (PMNS)** matrix $\hat{U}$,

$$\begin{pmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \\ |\nu_\tau\rangle \end{pmatrix} = \begin{pmatrix} U_{e1}^* & U_{e2}^* & U_{e3}^* \\ U_{\mu 1}^* & U_{\mu 2}^* & U_{\mu 3}^* \\ U_{\tau 1}^* & U_{\tau 2}^* & U_{\tau 3}^* \end{pmatrix} \begin{pmatrix} |\nu_1\rangle \\ |\nu_2\rangle \\ |\nu_3\rangle \end{pmatrix},$$

where

$$\begin{aligned} \hat{U} &= \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta_{12}\cos\theta_{13} & \sin\theta_{12}\cos\theta_{13} & \sin\theta_{13}e^{-i\delta} \\ -\sin\theta_{12}\cos\theta_{23} - \cos\theta_{12}\sin\theta_{23}\sin\theta_{13}e^{i\delta} & \cos\theta_{12}\cos\theta_{23} - \sin\theta_{12}\sin\theta_{23}\sin\theta_{13}e^{i\delta} & \sin\theta_{23}\cos\theta_{13} \\ \sin\theta_{12}\sin\theta_{23} - \cos\theta_{12}\cos\theta_{23}\sin\theta_{13}e^{i\delta} & -\cos\theta_{12}\sin\theta_{23} - \sin\theta_{12}\cos\theta_{23}\sin\theta_{13}e^{i\delta} & \cos\theta_{23}\cos\theta_{13} \end{pmatrix} \\ &\times \begin{pmatrix} e^{i\alpha_1/2} & 0 & 0 \\ 0 & e^{i\alpha_2/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{1}$$

Aside from the complex phases, this is actually nothing more than a series of simple rotations (characterized by angle θ_{12}, θ_{13}, θ_{23}) in "flavor space." These angles are given by

- $\sin^2 2\theta_{12} \simeq 0.846 \pm 0.021$
- $\sin^2 2\theta_{13} \simeq 0.093 \pm 0.008$
- $\sin^2 2\theta_{23} > 0.92$
- Phases δ, $\alpha_{1,2}$ are currently unknown. Nonzero values would indicate beyond-standard-model physics, so we can imagine that they're zero for now.

1. Define the matrix $\hat{U}$ in a Jupyter notebook. The only preamble you'll need is something like `from pylab import *`. You can use the function `arcsin()` to extract the angles from the information given. Verify numerically that $\hat{U}^\dagger \hat{U} = \hat{U}\hat{U}^\dagger = \hat{I}$. I would define a matrix of zeros and then fill in the elements one by one:

```
U = matrix(zeros((3,3)),dtype='complex')

U[0,0] = cos(theta12)*cos(theta13)
...
```

Once you have $\hat{U}$, go ahead and define `Uc = conj(U.transpose())` to get the Hermitian conjugate defined as well.

2. It is known from experiments that $\Delta m^2_{12} \equiv m^2_2 - m^2_1 = 7.9 \times 10^{-5}$ eV^2 (we're going to work with $\hbar = c = 1$ for simplicity). One also has $\Delta m^2_{32} = 2.3 \times 10^3$ eV^2. Use this information (and a blind guess for m_1) to define the masses m_2 and m_3.

3. Define a Hamiltonian matrix in the mass basis. Using p and E interchangeably, we can write

```
H = matrix(zeros((3,3)))
E = 1.0
H[0,0] = E + (m1**2.0)/(2.0*E)
...
```

You can treat `E` as a free parameter that can be adjusted later.

4. The next step is to calculate time evolution of a state $|\psi(t)\rangle$ with $|\psi(0)\rangle = |\nu_e\rangle$. That is, an electron neutrino is created through a weak process. As it travels through space, the mass eigenstates evolve with different phases. One way to do this is to initialize an electron neutrino state (in the flavor basis)

$$|\psi(0)\rangle = |\nu_e\rangle \doteq \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

via `psi0 = array([0.0,0.0,1.0])`. You'll also want to convert this to a column vector, `psi0.shape = (3,1)`. The basic strategy is to employ the matrix representation,

$$|\psi(t)\rangle \;=\; \hat{U}^\dagger e^{-i\hat{H}t/\hbar}\hat{U}\,|\psi(0)\rangle\,. \tag{2}$$

Define an array of time values `t = linspace(0,tmax,Nt)` (you'll likely want `tmax` pretty large, around 4000 or so at least). Also define an array `p1` of the same length to store the probability of measuring $|\nu_e\rangle$ as a function of time. Loop over time points, and at each step define a new matrix whose diagonal entries are the phases $e^{-iE_it/\hbar}$,

```
eiHt = matrix(zeros((3,3)),dtype='complex')
eiHt[0,0] = exp(-1j*H[0,0]*t[n])
...
```

The punchline is that you can implement time evolution (for a fixed time t) in a single line,

```
psit = Uc*eiHt*U*psi0
```

The first (right to left) `U` converts from flavor to mass basis. In this basis, time evolution is provided by the diagonal matrix $e^{-i\hat{H}t/\hbar}$. Then we convert back to the flavor basis. The reason for the final transformation is so that components give us the probabilities directly, `p1[n] = abs(psit[0])**2`. You can also add `p2` and `p3` arrays to keep track of the probabilities of the various transitions.

5. Plot the probabilities as a function of time. Try adjusting various parameters (or adding nonzero *phases*) to see what affect these parameters have. Do the results depend on m_1? E?