Experiment 3
Bell Inequalities

1 Introduction

Consider two spin-$\frac{1}{2}$ particles combined to form a spin-0 "singlet" state,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle\,|-\rangle - |-\rangle\,|+\rangle\right). \tag{1}$$

Such a state can be created when a single spin-0 particle decays into two spin-$\frac{1}{2}$ particles. For example, a neutral pion (pi meson) can decay into an electron-positron pair,

$$\pi^0 \rightarrow e^+ + e^-.$$

We say that the spin states of the two individual particles are **entangled** because they are perfectly anticorrelated. While the spin of the electron is equally likely to be up or down, the spin of the positron is fully determined once the electron's spin is specified. The "spookiness" of this proposition follows from considering some idealized thought experiments. Suppose the electron and positron are sent to observers separated by vast distances, with (let's say) the electron's recipient slightly further from the source than the positron's recipient. After measuring $\hat{S}^z$ on the positron, suppose a measurement of $+\frac{\hbar}{2}$ is obtained. According to the postulates of quantum mechanics, the wave function *instantly* collapses to the state $|+\rangle\,|-\rangle$ and the other observer is *certain* to obtain $-\frac{\hbar}{2}$ when measuring $\hat{S}^z$ on the positron. Einstein and cronies took issue with this non-local collapse. If the observers are separated by light years, a *local* theory would require that the measurement of the electron's spin causes information to travel at super-luminal speeds to "tell" the positron which spin state has been decided by the first measurement. This is the essence of the **Einstein-Podolsky-Rosen (EPR) paradox**, which essentially states that either quantum mechanics is incomplete or it predicts faster-than-light propagation of information.

2 Bell's Inequality

Perhaps there are some **hidden variables** which we cannot directly observe but which fully fix the outcome of any measurement. **Bell's inequalities** concern some general statements about the results of measurements in the presence of such hidden variables. To illustrate just what Bell's inequalities say and how it relates to quantum mechanics, let us suppose both observers agree to measure spin only along one of three predetermined axes, $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, or $\hat{\mathbf{c}}$, shown below in Fig. 2.

To derive the punchline of Bell's theorem, we assume that many copies of this singlet state are created, with one particle going to each of the two observers. Upon receiving the particle, each observer *randomly* chooses one of these axes along which to measure spin. According to the hidden variable assumption, each measurement is deterministically fixed by some hidden degrees of freedom. We cannot know the details about the hidden variables, but we can enumerate the possible *types* of particles.

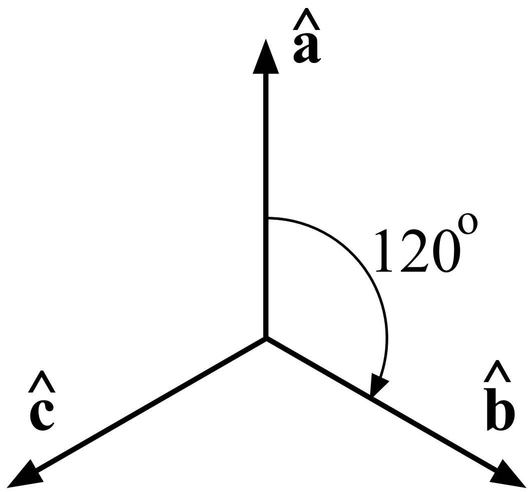

Fig. 1: Axes $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, $\hat{\mathbf{c}}$, mutually separated by 120°.

1. Show that the individual spins in the singlet state are perfectly anticorrelated with respect to *any* axis. Do this by constructing the operators $\hat{S}^{\hat{\mathbf{a}}} = \hat{\mathbf{a}} \cdot \hat{\mathbf{S}}$ and $\hat{S}^{\hat{\mathbf{b}}} = \hat{\mathbf{b}} \cdot \hat{\mathbf{S}}$ and considering the expectation value

$$\left\langle \hat{S}^{\hat{\mathbf{a}}} \cdot \hat{S}^{\hat{\mathbf{b}}} \right\rangle \;=\; \langle \psi | \, \hat{S}^{\hat{\mathbf{a}}} \otimes \hat{S}^{\hat{\mathbf{b}}} \, | \psi \rangle \;=\; -\frac{\hbar^2}{4} \cos \theta_{ab}, \tag{2}$$

where θ_{ab} is the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, and $|\psi\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle \, |-\rangle - |-\rangle \, |+\rangle \right)$. Write $\mathbf{a} = a_x \hat{\mathbf{x}} + a_y \hat{\mathbf{y}} + a_z \hat{\mathbf{z}}$ and $\mathbf{b} = b_x \hat{\mathbf{x}} + b_y \hat{\mathbf{y}} + b_z \hat{\mathbf{z}}$. Note that the expectation values for two-particle system are worked out as

$$\langle \phi_1 | \, \langle \phi_2 | \, \hat{S}^{\hat{\mathbf{a}}} \otimes \hat{S}^{\hat{\mathbf{b}}} \, |\psi_1\rangle \, |\psi_2\rangle \;=\; \langle \phi_1 | \, \hat{S}^{\hat{\mathbf{a}}} \, |\psi_1\rangle \, \langle \phi_2 | \, \hat{S}^{\hat{\mathbf{b}}} \, |\psi_2\rangle \, . \tag{3}$$

Since the spins are perfectly anti-correlated about any axis, we know that if particle 1 returns $+\frac{\hbar}{2}$ when measured along $\hat{\mathbf{a}}$ then particle 2 will return $-\frac{\hbar}{2}$ when measured along the same axis. However, each observer is choosing an axis randomly, so the measurement results are not guaranteed to be perfectly anticorrelated. For example, it is possible that observer 1 obtains $+\frac{\hbar}{2}$ for a measurement along $\hat{\mathbf{a}}$ while observer 2 measures $+\frac{\hbar}{2}$ for a measurement along $\hat{\mathbf{b}}$. We *will* assume by way of hidden variables that each particle carries a set of instructions which fix its state when measured along any of the three axes. Since each particle can be measured along each of three possible axes, there are 2^3 "types" of each particle, corresponding to the possible (well-defined) spin projections along each of the three axes. These rules are summarized in the table below.

Population	Particle 1	Particle 2
N_1	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+)$	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}-)$
N_2	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-)$	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+)$
N_3	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+)$	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-)$
N_4	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}-, \hat{\mathbf{c}}-)$	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+)$
N_5	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+)$	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}-, \hat{\mathbf{c}}-)$
N_6	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-)$	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+)$
N_7	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+)$	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-)$
N_8	$(\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}-)$	$(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}+)$

Take a moment to convince yourself that these possibilities reflect perfect anticorrelation about any particular axis and also capture all possible configurations. Now we wish to calculate the probability of the two observers obtaining the same result, $\mathcal{P}_{\text{same}}$. For "populations" 1 and 8, it is *impossible* for both observers to obtain the same result by assumption of anticorrelation,

$$\text{type 1, 8} \qquad \left\{ \begin{array}{rcl} \mathcal{P}^{(i)}_{\text{same}} & = & 0, \\ \mathcal{P}^{(i)}_{\text{opp}} & = & 1. \end{array} \right.$$

For the other states, note that particle 1 has two axes with one value ($\pm$) and one axis with the other. We *could* count up all the possibilities, but lets just take type 7 without loss of generality where the particle instructions are $(\hat{\mathbf{a}}-, \hat{\mathbf{b}}-, \hat{\mathbf{c}}+)$ and $(\hat{\mathbf{a}}+, \hat{\mathbf{b}}+, \hat{\mathbf{c}}-)$.

2. Consider a particle with hidden variables ("instructions") given by type 7. Write a short program to select axes for each of the particles randomly. Count whether the measurements obtained are the same or different, and loop until the fractions for each converge.

For a particle of type 7, we list the possible measurement outcomes for each of the observers below.

Axis 1	Axis 2	Measurements
$\hat{\mathbf{a}}$	$\hat{\mathbf{a}}$	$(+)(-)$
$\hat{\mathbf{a}}$	$\hat{\mathbf{b}}$	$(+)(-)$
$\hat{\mathbf{a}}$	$\hat{\mathbf{c}}$	$(+)(+)$
$\hat{\mathbf{b}}$	$\hat{\mathbf{a}}$	$(+)(-)$
$\hat{\mathbf{b}}$	$\hat{\mathbf{b}}$	$(+)(-)$
$\hat{\mathbf{b}}$	$\hat{\mathbf{c}}$	$(+)(+)$
$\hat{\mathbf{c}}$	$\hat{\mathbf{a}}$	$(-)(-)$
$\hat{\mathbf{c}}$	$\hat{\mathbf{b}}$	$(-)(-)$
$\hat{\mathbf{c}}$	$\hat{\mathbf{c}}$	$(-)(+)$

Of the nine axis configurations, four lead to the same result and five involve the observers obtaining different values. The same breakdown is obtained for any of the "mixed" rules, so for truly random axis selection,

$$\text{type 2–7} \qquad \left\{ \begin{array}{rcl} \mathcal{P}^{(i)}_{\text{same}} & = & \frac{4}{9}, \\ \mathcal{P}^{(i)}_{\text{opp}} & = & \frac{5}{9}. \end{array} \right.$$

While we have the probability of same/opposite measurements for *each* type of system, a realistic experiment would not be restricted to just one type. How do we know the breakdown of how many particles are in population 1 versus population 6? If we knew this, we might be able to work out a Bell *equality*. Fortunately, we don't need much to work out a powerful Bell *inequality*. Assume we take a sample of N spin-0 states. Of these N states, N_i are of the type i. The overall probability of both observers obtaining the *same* spin measurement is then the probability of obtaining a system of type i, or $\frac{N_i}{N}$, times the probability of obtaining the same measurements for a system of type i,

$$\begin{aligned} \mathcal{P}_{\text{same}} &= \sum_{i=1}^{8} \frac{N_i}{N} \mathcal{P}^{(i)}_{\text{same}}, \\ &= \frac{N_1 \cdot 0 + \left(N_2 + N_3 + N_4 + N_5 + N_6 + N_7\right) \frac{4}{9} + N_8 \cdot 0}{N} \\ &\leq \frac{4}{9}. \end{aligned} \tag{4}$$

The inequality follows from adding $\frac{4}{9}\left(N_1 + N_8\right)/N$ to the right-hand side (which must be no less than zero) and using $N_1 + \cdots + N_8 = N$. The probability of obtaining *opposite* measurements follows similarly,

$$\begin{aligned} \mathcal{P}_{\text{opp}} &= \frac{N_1 \cdot 1 + \left(N_2 + N_3 + N_4 + N_5 + N_6 + N_7\right) \frac{5}{9} + N_8 \cdot 1}{N} \\ &\geq \frac{5}{9}. \end{aligned} \tag{5}$$

The relations in Eqs. (4)–(5) are known as **Bell inequalities**. These expressions are specific to our setup, but similar such relations can be derived for other experimental protocols. They must be respected by any hidden variable model and have been derived under rather general assumptions. The reason we *care* is because it's not obvious that the predictions of quantum mechanics will respect these conditions. If so, then no problem. But if not, then either quantum mechanics is wrong[1] or local hidden variable theories are not realized by nature.

3 Predictions of quantum mechanics

We would like to calculate the probabilities $\mathcal{P}_{\text{same}}$ and $\mathcal{P}_{\text{opp}}$ according to quantum mechanics. We assume the total state of the system is the spin singlet

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle\,|-\rangle - |-\rangle\,|+\rangle\right). \tag{6}$$

Observer 1 has three choices for the axis along which to measure spin. We are free to label whichever axis is chosen as the z direction, so we shall do that. Observer 2 also has a choice of three possibilities, and so we will treat the general case of $\hat{\mathbf{n}} = \hat{\mathbf{n}}(\theta, \phi)$.

3. The singlet state already defines the z axis. Why are we still free to choose any axis to label z? *Hint:* Consider the state

$$\frac{1}{\sqrt{2}} \left(|+\rangle_{\hat{\mathbf{n}}}\,|-\rangle_{\hat{\mathbf{n}}} - |-\rangle_{\hat{\mathbf{n}}}\,|+\rangle_{\hat{\mathbf{n}}}\right),$$

   for arbitrary unit vector $\hat{\mathbf{n}}$. How does it compare to the singlet?

[1] ...or at the very least, incomplete

There are now four possible cases to consider for the pair of measurement results, $++$, $+-$, $-+$ and $--$ with

$$
\begin{aligned}
\mathcal{P}_{++} &= \left|\langle +|_{\hat{\mathbf{n}}} \langle +| \, |\psi\rangle\right|^2, \\
\mathcal{P}_{+-} &= \left|\langle +|_{\hat{\mathbf{n}}} \langle -| \, |\psi\rangle\right|^2, \\
\mathcal{P}_{-+} &= \left|\langle -|_{\hat{\mathbf{n}}} \langle +| \, |\psi\rangle\right|^2, \\
\mathcal{P}_{--} &= \left|\langle -|_{\hat{\mathbf{n}}} \langle -| \, |\psi\rangle\right|^2.
\end{aligned} \tag{7}
$$

The notation employed is somewhat misleading because $|\psi\rangle$ itself is superposition of tensor product states, so it really should be represented by two kets. Recall

$$
\begin{aligned}
|+\rangle_{\hat{\mathbf{n}}} &= \cos\frac{\theta}{2}\,|+\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|-\rangle, \\
|-\rangle_{\hat{\mathbf{n}}} &= \sin\frac{\theta}{2}\,|+\rangle - e^{i\phi}\cos\frac{\theta}{2}\,|-\rangle.
\end{aligned} \tag{8}
$$

4. For specified θ and ϕ, define numerical vectors corresponding to the states $|\pm\rangle$, $|\pm\rangle_{\hat{\mathbf{n}}}$ and $|\psi\rangle$. The following is one way to accomplish this:

```
plus = array([1.0,0.0])
minus = array([0.0,1.0])

np = array([cos(theta/2.0),exp(1j*phi)*sin(theta/2.0)])
nm = array([sin(theta/2.0),-exp(1j*phi)*cos(theta/2.0)])

psi = (1.0/sqrt(2.0))*(kron(plus,minus)-kron(minus,plus))

psipp = kron(plus,np)
psipm = kron(plus,nm)
...
```

   For purposes of debugging, you can set `phi = 0`, `theta = 0`. Use these basis vectors to calculate the probabilities $\mathcal{P}_{++}$, $\mathcal{P}_{+-}$, $\mathcal{P}_{-+}$ and $\mathcal{P}_{--}$. The inner product $\langle\phi|\psi\rangle$ can be obtained through

```
asscalar(dot(conj(phi.transpose()),psi))
```

   Be *careful* when converting to a probability, because these amplitudes may be complex.

5. Calculate the quantities $\mathcal{P}_{\text{same}}$ and $\mathcal{P}_{\text{opp}}$. For a specific choice of θ and ϕ, $\mathcal{P}_{\text{same}}(\theta,\phi) = \mathcal{P}_{++}(\theta,\phi) + \mathcal{P}_{--}(\theta,\phi)$ and $\mathcal{P}_{\text{opp}}(\theta,\phi) = \mathcal{P}_{+-}(\theta,\phi) + \mathcal{P}_{-+}(\theta,\phi)$. To make connection with the three-axis experiments, you'll have to sum over all *three* orientations that the second observer uses, $\theta = 0, \pm 120^\circ$ with $\phi = 0$.

$$
\begin{aligned}
\mathcal{P}_{\text{opp}} &= \frac{1}{3}\sum_{\phi=0,\theta=0,\pm 120^\circ} \mathcal{P}_{\text{opp}}(\theta,\phi), \\
\mathcal{P}_{\text{same}} &= \frac{1}{3}\sum_{\phi=0,\theta=0,\pm 120^\circ} \mathcal{P}_{\text{same}}(\theta,\phi).
\end{aligned} \tag{9}
$$

   Are your results consistent with the Bell inequalities?

6. Do the results depend on the value(s) chosen for ϕ?

4 Simulation

So far, while we've employed numerical methods to do the calculations, everything has been theoretical. Sometimes it's useful to *simulate* a measurement process repeatedly to see the probability distribution emerge from a large number of measurements.

The JUPYTER notebook `BellTest.ipynb` allows the user to define a set of three axes through three choices of θ and ϕ and perform repeated double-spin measurements for random choices of both observers' axes. For a specified number of measurements, the number of same and opposite results are counted, and the resulting fractions, or experimental probabilities, are printed along with a check of whether the Bell inequality has been violated.

7. Run the program for $\theta = 0, \pm 120°$ and determine whether this is consistent with your previous results.

8. Find at least one other set of axes for which the Bell inequality is (a) violated and (b) satisfied.

9. Experiments have **consistently** shown that the rules of quantum mechanics lead to correct predictions. Given this, what does it mean for the Bell inequality to be violated (at least in some cases)?