MATHEMATICS

Quadratic equation

Solution to $ax^2+bx+c=0$: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Euler identities

$$Ae^{i\phi}=A(\cos\phi+i\sin\phi)$$
$$e^z=e^{x+iy}=e^x(\cos y+i\sin y)$$
$$\sin\phi=\frac{e^{i\phi}-e^{-i\phi}}{2i}$$
$$\cos\phi=\frac{e^{i\phi}+e^{-i\phi}}{2}$$

Trig

Identities:

$$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$$
$$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$$

$$\sin A\sin B=\frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right]$$
$$\cos A\cos B=\frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$$
$$\sin A\cos B=\frac{1}{2}\left[\sin(A+B)+\sin(A-B)\right]$$
$$\cos A\sin B=\frac{1}{2}\left[\sin(A+B)-\sin(A-B)\right]$$

$$\sin^2 A=\frac{1-\cos 2A}{2}$$
$$\cos^2 A=\frac{1+\cos 2A}{2}$$

$$\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\sin A-\sin B=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$
$$\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\cos A-\cos B=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Law of Sines: sides: A, B, & C; angles opposite: α, β, & γ

$$\frac{A}{\sin\alpha}=\frac{B}{\sin\beta}=\frac{C}{\sin\gamma}$$

Law of Cosines: sides: A, B, and C; angle opposite of $C=\gamma$

$$C^2=A^2+B^2-2AB\cos\gamma$$

Circular Arclength: $s=r\theta$; θ in rad

Calculus

Fundamental theorem of calculus

First part: Define $F(x)=\int_a^x f(t)dt$ in interval $[a,b]$ with f continuous and real-valued in $[a,b]$. Then, F is continuous on $[a,b]$, differentiable on (a,b), and $F'(x)=f(x)\forall x\in[a,b]$

Corollary: If f is a real-valued continuous function on $[a,b]$, and g is an antiderivative of f in $[a,b]$, then $\int_a^b f(x)dx=g(b)-g(a)$.

Second part (stronger than corollary): Let f be a real-valued function defined on $[a,b]$ with an antiderivative g on $[a,b]$ (i.e., $f(x)=g'(x)\;\forall x\in[a,b]$). If f is integrable on $[a,b]$ then $\int_a^b f(x)dx=g(b)-g(a)$. Note that here f needn't be continuous.

Basic theorems

Chain rule $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\cdot\frac{\mathrm{d}u}{\mathrm{d}x}$

Product rule $\dfrac{d}{dx}(u\cdot v)=u\cdot\dfrac{dv}{dx}+v\cdot\dfrac{du}{dx}$

Integration by parts $\int u\,\frac{dv}{dx}\,dx=uv-\int v\,\frac{du}{dx}\,dx$

Calculus of variations

$J=\int_{x_1}^{x_2} f\{y(x),y\prime(x);x\}$

$\frac{\partial f}{\partial y}-\frac{d}{dx}\frac{\partial f}{\partial y\prime}=0\Rightarrow$ stationary points of J

$\frac{\partial f}{\partial x}-\frac{d}{dx}\left(f-y\prime\frac{\partial f}{\partial y\prime}\right)=0$

$f-y\prime\frac{\partial f}{\partial y\prime}=const$ for $\frac{\partial f}{\partial x}=0$

Combinatorics

Permutations — Number of k-permutations of a set of n elements:

$\mathrm{Perm}(n,k)=(n)_k=n\times(n-1)\times\cdots\times(n-k+1)=\frac{n^k}{n!}$ if $k\le n$

Combinations — n choose k: $\mathrm{C}(n,k)=\binom{n}{k}=\frac{(n)_k}{n!}=\frac{n!}{k!(n-k)!}$ if $k\le n$

Probability — Divide multiplicity by total number of possiblities

examples

N coins, macrostate with n heads:
$\Omega(N,n)=\frac{N!}{n!(N-n)!}=C(N,n)$
$\mathbb{P}(N,n)=\Omega/2^n$

N kids, k oranges (distinguishable):
k^N ways to distribute oranges

N kids, k oranges (indistinguishable):
$C(N+k-1,k)$ ways to distribute oranges

52 cards, there are:
$C(52,5)$ possible hands
$10(4^5-4)$ straights
$10(4^5)$ straights + straight flushes
40 straight flushes
$13\cdot 12\cdot C(4,2)\cdot C(4,3)$ full houses

Stirling's Approximation (for $N\gg 1$):
$N!\sim N^N e^{-N}\sqrt{2\pi N}$ (strong)
$N!\sim N^N e^{-N}$ (weak)
$\ln(N!)\approx N\ln N-N$ (log of weak)

Vectors & vector theorems

Dot product: $\boldsymbol{A}\cdot\boldsymbol{B}=AB\cos\theta=A_xB_x+A_yB_y+A_zB_z$
Cross product: $\boldsymbol{A}\times\boldsymbol{B}=AB\sin\theta\hat{\boldsymbol{n}}$
Cross, squared $|\boldsymbol{A}\times\boldsymbol{B}|^2=A^2B^2\sin^2\theta=A^2B^2-(\boldsymbol{A}\cdot\boldsymbol{B})^2$
Triple product: $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=\boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C})-\boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B})$
Vector sum, squared $|\boldsymbol{A}+\boldsymbol{B}|^2=A^2+B^2+2\boldsymbol{A}\cdot\boldsymbol{B}=A^2+B^2+2AB\cos\theta$
Gradient theorem: $\int_{\boldsymbol{a}}^{\boldsymbol{b}}(\nabla f)\cdot d\boldsymbol{l}=f(\boldsymbol{b})-f(\boldsymbol{a})$
Divergence theorem (Gauss's thm): $\int_V(\nabla\cdot\boldsymbol{F})\,dV=\oint_S\boldsymbol{F}\cdot d\boldsymbol{S}$
Curl theorem (Stokes' thm): $\int_S(\nabla\times\boldsymbol{F})\cdot d\boldsymbol{S}=\oint_C\boldsymbol{F}\cdot d\boldsymbol{r}$

Coordinate systems & conversions

Coordinate Conversion

cartesian to...	*cylindrical to...*	*spherical to...*
cylindrical	**cartesian**	**cartesian**
$\rho=\sqrt{x^2+y^2}$ $\phi=\arctan(y/x)$ $z=z$	$x=\rho\cos\phi$ $y=\rho\sin\phi$ $z=z$	$x=r\sin\theta\cos\phi$ $y=r\sin\theta\sin\phi$ $z=r\cos\theta$
spherical	**spherical**	**cylindrical**
$r=\sqrt{x^2+y^2+z^2}$ $\theta=\arccos(z/r)$ $\phi=\arctan(y/x)$	$r=\sqrt{\rho^2+z^2}$ $\theta=\arctan(\rho/z)$ $\phi=\phi$	$\rho=r\sin\theta$ $\phi=r\sin\theta$ $z=r\cos\theta$

Unit Vector Conversion

cartesian to...	*cylindrical to...*	*spherical to...*
cylindrical	**cartesian**	**cartesian**
$\hat{\boldsymbol{\rho}}=\frac{x}{\rho}\hat{\boldsymbol{x}}+\frac{y}{\rho}\hat{\boldsymbol{y}}$	$\hat{\boldsymbol{x}}=cos\phi\hat{\boldsymbol{\rho}}-sin\phi\hat{\boldsymbol{\phi}}$	$\hat{\boldsymbol{x}}=\sin\theta\cos\phi\hat{\boldsymbol{r}}+$ $\cos\theta\cos\phi\hat{\boldsymbol{\theta}}-\sin\phi\hat{\boldsymbol{\phi}}$
$\hat{\boldsymbol{\phi}}=-\frac{y}{\rho}\hat{\boldsymbol{x}}+\frac{x}{\rho}\hat{\boldsymbol{y}}$	$\hat{\boldsymbol{x}}=cos\phi\hat{\boldsymbol{\rho}}-sin\phi\hat{\boldsymbol{\phi}}$	$\hat{\boldsymbol{y}}=\sin\theta\sin\phi\hat{\boldsymbol{r}}+$ $\cos\theta\sin\phi\hat{\boldsymbol{\theta}}+\cos\phi\hat{\boldsymbol{\phi}}$
$\hat{\boldsymbol{z}}=\hat{\boldsymbol{z}}$	$\hat{\boldsymbol{z}}=\hat{\boldsymbol{z}}$	$\hat{\boldsymbol{z}}=\cos\theta\hat{\boldsymbol{r}}-\sin\theta\hat{\boldsymbol{\theta}}$
spherical	**spherical**	**cylindrical**
$\hat{\boldsymbol{r}}=\frac{x\hat{\boldsymbol{x}}+y\hat{\boldsymbol{y}}+z\hat{\boldsymbol{z}}}{r}$	$\hat{\boldsymbol{r}}=\frac{\rho}{r}\hat{\boldsymbol{\rho}}+\frac{z}{r}\hat{\boldsymbol{z}}$	$\hat{\boldsymbol{\rho}}=\sin\theta\hat{\boldsymbol{r}}+\cos\theta\hat{\boldsymbol{\theta}}$
$\hat{\boldsymbol{\theta}}=\frac{xz\hat{\boldsymbol{x}}+yz\hat{\boldsymbol{y}}-\rho^2\hat{\boldsymbol{z}}}{r\rho}$	$\hat{\boldsymbol{\theta}}=\frac{z}{r}\hat{\boldsymbol{\rho}}-\frac{\rho}{r}\hat{\boldsymbol{z}}$	$\hat{\boldsymbol{\phi}}=\hat{\boldsymbol{\phi}}$
$\hat{\boldsymbol{\phi}}=\frac{-y\hat{\boldsymbol{x}}+x\hat{\boldsymbol{y}}}{\rho}$	$\hat{\boldsymbol{\phi}}=\hat{\boldsymbol{\phi}}$	$\hat{\boldsymbol{z}}=\cos\theta\hat{\boldsymbol{r}}-\sin\theta\hat{\boldsymbol{\theta}}$

Differential Elements

cartesian	**cylindrical**	**spherical**
$d\boldsymbol{l}=dx\hat{\boldsymbol{x}}+dy\hat{\boldsymbol{y}}+dz\hat{\boldsymbol{z}}$	$d\boldsymbol{l}=d\rho\hat{\boldsymbol{\rho}}+\rho d\phi\hat{\boldsymbol{\phi}}+dz\hat{\boldsymbol{z}}$	$d\boldsymbol{l}=dr\hat{\boldsymbol{r}}+rd\theta\hat{\boldsymbol{\theta}}+r\sin\theta d\phi\hat{\boldsymbol{\phi}}$
$d\boldsymbol{A}=dy\,dx\hat{\boldsymbol{x}}+dx\,dz\hat{\boldsymbol{y}}+dx\,dy\hat{\boldsymbol{z}}$	$d\boldsymbol{A}=\rho d\phi\,dz\hat{\boldsymbol{\rho}}+d\rho\,dz\hat{\boldsymbol{\phi}}+\rho\,d\rho\,d\phi\hat{\boldsymbol{z}}$	$d\boldsymbol{A}=r^2\sin\theta\,d\theta\,d\phi\,\hat{\boldsymbol{r}}+r\sin\theta\,dr\,d\phi\,\hat{\boldsymbol{\theta}}+r\,dr\,d\theta\,\hat{\boldsymbol{\phi}}$
$dV=dx\,dy\,dz$	$dV=\rho\,d\rho\,d\phi\,dz$	$dV=r^2\sin\theta\,dr\,d\theta\,d\phi$

POSITIONS, VELOCITIES, & ACCELERATIONS

polar
$$\boldsymbol{r}=\rho\boldsymbol{e_\rho}$$
$$\boldsymbol{v}=\dot\rho\boldsymbol{e_\rho}+\rho\dot\theta\boldsymbol{e_\theta}$$
$$\boldsymbol{a}=\left(\ddot\rho-\rho\dot\theta^2\right)\boldsymbol{e_\rho}+\left(\rho\ddot\theta+2\dot\rho\dot\theta\right)\boldsymbol{e_\theta}$$

cylindrical
$$\boldsymbol{r}=\rho\boldsymbol{e_\rho}+z\boldsymbol{e_z}$$
$$\boldsymbol{v}=\dot\rho\boldsymbol{e_\rho}+\rho\dot\theta\boldsymbol{e_\theta}+\dot z\boldsymbol{e_z}$$
$$\boldsymbol{a}=\left(\ddot\rho-\rho\dot\theta^2\right)\boldsymbol{e_\rho}+\left(\rho\ddot\theta+2\dot\rho\dot\theta\right)\boldsymbol{e_\theta}+\ddot z\boldsymbol{e_z}$$

spherical
$$\boldsymbol{r}=\rho\boldsymbol{e_\rho}$$
$$\boldsymbol{v}=\dot\rho\boldsymbol{e_\rho}+\rho\dot\theta\boldsymbol{e_\theta}+\rho\dot\phi\sin\theta\boldsymbol{e_\phi}$$
$$\boldsymbol{a}=\left(\ddot\rho-\rho\dot\theta^2-\rho\dot\phi^2\sin^2\theta\right)\boldsymbol{e_\rho}+\left(\rho\ddot\theta+2\dot\rho\dot\theta-\rho\dot\phi^2\sin\theta\cos\theta\right)\boldsymbol{e_\theta}+\left(\rho\ddot\phi\sin\theta+2\dot\rho\dot\phi\sin\theta+2\rho\dot\theta\dot\phi\cos\theta\right)\boldsymbol{e_\phi}$$

del, ∇, in CARTESIAN

del operator: $\boldsymbol{\nabla}=\boldsymbol{e_x}\frac{\partial}{\partial x}+\boldsymbol{e_y}\frac{\partial}{\partial y}+\boldsymbol{e_z}\frac{\partial}{\partial z}$

gradient: $\boldsymbol{\nabla}\phi=grad\,\phi=\boldsymbol{e_x}\frac{\partial\phi}{\partial x}+\boldsymbol{e_y}\frac{\partial\phi}{\partial y}+\boldsymbol{e_z}\frac{\partial\phi}{\partial z}$

directional derivative: $\frac{d\phi}{ds}=\boldsymbol{\nabla}\phi\cdot\frac{\boldsymbol{A}}{|\boldsymbol{A}|}$

divergence: $\boldsymbol{\nabla}\cdot\boldsymbol{V}=div\,\boldsymbol{V}=\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}$

curl: $\boldsymbol{\nabla}\times\boldsymbol{V}=\boldsymbol{e_x}\left(\frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z}\right)+\boldsymbol{e_y}\left(\frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial x}\right)+\boldsymbol{e_z}\left(\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right)$

Laplacian: $\Delta f=\boldsymbol{\nabla}^2\phi=\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\phi)=div\,grad\,\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$

del, ∇, in CYLINDRICAL

gradient: $\boldsymbol{\nabla}f=grad\,f=\frac{\partial f}{\partial\rho}\boldsymbol{e_\rho}+\frac{1}{\rho}\frac{\partial f}{\partial\phi}\boldsymbol{e_\phi}+\frac{\partial f}{\partial z}\boldsymbol{e_z}$

divergence: $\boldsymbol{\nabla}\cdot\boldsymbol{V}=div\,\boldsymbol{V}=\frac{1}{\rho}\frac{\partial(\rho\boldsymbol{V}_\rho)}{\partial\rho}+\frac{1}{\rho}\frac{\partial\boldsymbol{V}_\phi}{\partial\phi}+\frac{\partial\boldsymbol{V}_z}{\partial z}$

curl: $\boldsymbol{\nabla}\times\boldsymbol{V}=\left(\frac{1}{\rho}\frac{\partial V_z}{\partial\phi}-\frac{\partial V_\phi}{\partial z}\right)\boldsymbol{e_\rho}+\left(\frac{\partial V_\rho}{\partial z}-\frac{\partial V_z}{\partial\rho}\right)\boldsymbol{e_\phi}+\frac{1}{\rho}\left(\frac{\partial(\rho V_\phi)}{\partial\rho}-\frac{\partial V_\rho}{\partial\phi}\right)\boldsymbol{e_z}$

Laplacian: $\Delta f=\boldsymbol{\nabla}^2f=\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}f)=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2}+\frac{\partial^2 f}{\partial z^2}$

del, ∇, in SPHERICAL

gradient: $\boldsymbol{\nabla}f=grad\,f=\frac{\partial f}{\partial r}\boldsymbol{e_r}+\frac{1}{r}\frac{\partial f}{\partial\theta}\boldsymbol{e_\theta}+\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\boldsymbol{e_\phi}$

divergence:
$$\boldsymbol{\nabla}\cdot\boldsymbol{V}=div\,\boldsymbol{V}=\frac{1}{r^2}\frac{\partial(r^2V_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(V_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi}$$

curl: $\boldsymbol{\nabla}\times\boldsymbol{V}=\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(V_\phi\sin\theta)-\frac{\partial V_\theta}{\partial\phi}\right)\boldsymbol{e_r}+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial V_r}{\partial\phi}-\frac{\partial}{\partial r}(rV_\phi)\right)\boldsymbol{e_\theta}+\frac{1}{r}\left(\frac{\partial}{\partial r}(rV_\theta)-\frac{\partial V_r}{\partial\theta}\right)\boldsymbol{e_\phi}$

Laplacian: $\Delta f=\boldsymbol{\nabla}^2f=\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}f)=$
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

Fourier series

real-valued functions, period of $2l$:
$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)$$
$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)\,dx$$
$$a_n=\frac{1}{l}\int_{-l}^{l}f(x)cos\frac{n\pi x}{l}\,dx,\;n\ge 1$$
$$b_n=\frac{1}{l}\int_{-l}^{l}f(x)sin\frac{n\pi x}{l}\,dx,\;n\ge 1$$

complex-valued functions, period of $2l$:
$$f(x)=\sum_{n=-\infty}^{\infty}c_ne^{in\pi x/l}$$
$$c_n=\frac{1}{2l}\int_{-l}^{l}f(x)e^{-in\pi x/l}\,dx,\;n\in\mathbb{Z}$$

***ANY* real-valued function** on interval $[0,L]$ **(Fourier sine series)**:
$$f(x)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right)$$
$$b_n=\frac{2}{L}\int_0^L f(x)sin\frac{n\pi x}{L}\,dx,\;\;n\ge 1$$

convergence (Dirichlet): If $f(x)$ is periodic of period $2l$, and if between $-l$ and l it is single-valued, has a finite number of max. and min. values, and a finite number of discont., and if $\int_{-l}^{l}|f(x)|\,dx$ is finite, Fourier series converges to $f(x)$ at all points where $f(x)$ is continuous. At discontinuities, series converges to midpoint of the jump.

example Fourier series

Sawtooth $x/2L\longmapsto\frac{1}{2}-\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{L}\right)$

Triangle $T(x)\longmapsto\frac{X_0}{2}-\sum_{n=1}^{\infty}\frac{2X_0}{(\pi k)^2}$, n odd

Square $2[H(x/L)-H(x/L-1)]-1\longmapsto\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{L}\right)$, for n odd and H is Heaviside step

Taylor series

Taylor series of $f(x)$ about $x=a$: $f(x)=\sum_{n=0}^{\infty}\frac{1}{n!}(x-a)^nf^{(n)}(x=a)$

example Maclaurin series (Taylor series with a=0)

$$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}\text{ for }-1<x\le 1;\text{ this }\approx x\text{ for }|x|\ll 1$$
$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$
$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n\text{ for }|x|<1$$
$$\frac{x^m}{1-x}=\sum_{n=m}^{\infty}x^n\text{ for }|x|<1\text{ and }m\in\mathbb{N}_0$$
$$\frac{x}{(1-x)^2}=\sum_{n=1}^{\infty}nx^n\text{ for }|x|<1$$
$$\sqrt{1+x}=\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{(1-2n)(n!)^2(4^n)}x^n\text{ for }|x|<1$$
$$(1+x)^\alpha=\sum_{n=0}^{\infty}\binom{\alpha}{n}x^n\approx 1+\alpha x\text{ for }|x|<1\text{ and }\forall\,\alpha\in\mathbb{C}$$
$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}\text{ for all }x$$
$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}\text{ for all }x$$

Green's method

$$x(t)=\int_{-\infty}^{t}F(t')G(t,t')dt'$$

Ordinary differential equations

Separable 1st-order

Equation can be written as $f(y)dy = f(x)dx$, such as $\frac{dy}{dx} = N(1-y)$. Evaluate integrals directly.

Linear 1st-order

Write the equation in the form $y' + P(x)y = Q(x)$ and then define

$$I = \int P(x)dx$$

and find y by solving

$$ye^I = \int Q(x)e^I dx + c$$

Linear 2nd-order homogeneous with constant coefficients

Equations of the form

$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0$$

Write the characteristic polynomial $a_2 D^2 y + a_1 D y + a_0 y = 0$ and factor into $(D-a)(D-b)y = 0$. In general, this can be solved by letting $u = (D-a)y$, solving the 1st-order diff eq $(D-b)u = 0$ for $u(x)$, substituting this solution into the equation $(D-a)y = u(x)$, and finally solving *this* linear 1st-order ODE. In fact, this method can be generalized to higher-order linear diff eq's. However, there are pre-determined solution forms based upon the relationships between a and b:

$$a, b \in \mathbb{R}, a \neq b \Rightarrow y = c_1 e^{ax} + c_2 e^{bx}$$
$$a, b \in \mathbb{R}, a = b \Rightarrow y = (Ax+B)e^{ax}$$

For

$$a, b \in \mathbb{C}, a = b^* = \alpha \pm i\beta,$$

any of the following forms are solutions:

$$y = Ae^{\alpha + i\beta x} + Be^{\alpha - i\beta x}$$
$$y = e^{\alpha x}\left(Ae^{i\beta x} + Be^{-i\beta x}\right)$$
$$y = e^{\alpha x}(c_1 \sin\beta x + c_2 \cos\beta x)$$
$$y = ce^{\alpha x}\sin(\beta x + \gamma)$$
$$y = ce^{\alpha x}\cos(\beta x + \delta)$$

Linear 2nd-order inhomogeneous with constant coefficients

Equations of one of the forms

$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = f(x)$$
$$\frac{d^2y}{dx^2} + \frac{a_1}{a_2}\frac{dy}{dx} + \frac{a_0}{a_2} y = F(x)$$

can be solved, generally, as described for the homogeneous case, but with $F(x)$ on the right-hand side when solving the first 1st-order ODE, $(D-b)u = F(x)$. (This gives both the particular *and* complementary solution.) Otherwise, find $y = y_c + y_p$ where y_c, the complementary solution, comes from solving the homogeneous equation and y_p is a particular solution from a pre-computed form for specific $F(x)$:

$(D-a)(D-b)y = F(x) = ke^{cx}$, particular solution y_p is given by:

$y_p = Ce^{cx}$ if c is not equal to either a or b;
$y_p = Cxe^{cx}$ if c equals a or b, $a \neq b$;
$y_p = Cx^2 e^{cx}$ if $c = a = b$

(For $F(x) = k\cos\alpha x$ or $F(x) = k\sin\alpha x$, solve the above with $F(x) = ke^{c = i\alpha x}$ and take the real or imag part, respectively. For $F(x) = const$, set $c = 0$.)

A more general form of this (called the *method of undetermined coefficients*) follows:

$(D-a)(D-b)y = F(x) = e^{cx}P_n(x)$; $P_n(x)$ is a polynomial of degree n:

$$y_p = \begin{cases} e^{cx}Q_n(x) & \text{if } c \neq a \text{ and } c \neq b \\ xe^{cx}Q_n(x) & \text{if } c = a \text{ or } c = b \text{ but } a \neq b \\ x^2 e^{cx}Q_n(x) & \text{if } c = a = b \end{cases}$$

CONSTANTS

$c =$ speed of light in vacuum $= 2.998 \times 10^8$ m/s
$\mu_0 =$ mag const / perm of vacuum $= 4\pi \times 10^{-7}$ N·A^{-2} or H m^{-1}
$\varepsilon_0 =$ elec const / permit of vacuum $= 8.854 \times 10^{-12}$ F m^{-1}
$Z_0 =$ char impedance of vacuum $= 376.73\ \Omega$
$h =$ Planck's const $= 6.626 \times 10^{-34}$ J·s $= 4.136 \times 10^{-15}$ eV·s

$e =$ charge of electron $= 1.602 \times 10^{-19}$ C
$m_e =$ mass of electron $= 9.109 \times 10^{-31}$ kg $= 0.511$ MeV/c^2
$m_n =$ mass of neutron or proton $= 1.67 \times 10^{-27}$ kg $= 938$ MeV/c^2
$\mu_B =$ Bohr magneton, $e\hbar/2m_e = 9.274 \times 10^{-24}$ J/T $= 5.7884 \times 10^{-4}$ eV/T
$\mu_N =$ Nuclear magneton, $e\hbar/2m_p = 5.051 \times 10^{-27}$ J/T

$G =$ gravit. constant $= 6.674 \times 10^{-11}$ N m^{2}kg^{-2}
$g =$ gravit. accel on Earth surface $= 9.8$ m/s^2

$R_S =$ mean radius of Sun $= 696 \times 10^6$ m
$R_E =$ mean radius of Earth $= 6.371 \times 10^6$ m
$R_M =$ mean radius of Moon $= 1.737 \times 10^6$ m

$R_{S,E} =$ mean distance, Earth to Sun $= 149.6 \times 10^9$ m
$R_{M,E} =$ mean distance, Earth to Moon $= 384.4 \times 10^6$ m

$M_S =$ mass of Sun $= 1.99 \times 10^{30}$ kg
$M_E =$ mass of Earth $= 5.98 \times 10^{24}$ kg
$M_M =$ mass of Moon $= 7.35 \times 10^{22}$ kg

$k_B =$ Boltzmann's constant $= 1.38 \times 10^{-23}$ J/K
$R =$ Ideal gas constant $= 8.315$ J/mol·K
$N_A =$ Avogadro's number $= 6.02214179 \times 10^{23}$ mol^{-1}

$c =$ Speed of sound in air @ STP $= 340.29$ m/s

jsection*Units & Conversions

Unit vec. — Have direction but NO units
Distance — Å$= 1 \times 10^{-10}$ m
Area
Volume — mL = cm^3; L $= 1 \times 10^{-3}$ m^3
Velocity
Mass — u $= 1.661 \times 10^{-27}$ kg
Pressure — Pa = N/m^2; atm $= 1.013 \times 10^5$ N/m^2; atm = 1.013bar; atm = 14.7 lb/in^2; atm = 760 mm Hg
Energy — J = kg·m^2/s^2 = N·m; eV $= 1.602 \times 10^{-19}$ J; Btu = 1054 J; cal = 4.186 J; Cal = 1 000 cal
Power — W = J/s
Force — N = kg·m/s^2
Temp — °R $= \frac{5}{9}T_{\text{kelvin}}$; °C $= T_{\text{kelvin}} - 273.15$; °F $= \frac{9}{5}T_{\text{celsius}} + 32$
STP — 300 K and 1 atm
Current — A $\approx 6.241 \times 10^{18}$ electrons/s
E-Field — N/C
B-Field — Particle w/ 1 C charge passing thru **B**-field of 1 T at 1 m/s expereinces 1 N force; T = N/(A·m); T = kg·A^{-1}·s^{-2}
V diff — / EMF (same units, different concepts); V = J/C; V = kg·m^2·s^{-3}·A^{-1}
Cap — F = s^4·A^2·m^{-2}·kg^{-1}
Ind — If $di/dt = 1$ A/s & the $emf = 1$ V, then $L = 1$ H; H = m^2·kg·s^{-2}·A^{-2}; H = V·s/A = V/(A/s)

COMMON PROPERTIES OF SUBSTANCES, LIGHT

$H_2 0$
$\mathbb{C}_V = 1$ cal/g°C ≈ 4.2 J/g°C
$H_{\text{liquid}\to\text{gas}} = 40.6$ kJ
$L_{\text{liquid}\to\text{gas}} = 2260$ J/g
$L_{\text{solid}\to\text{liquid}} = 333$ J/g

Air
$\approx 78\%\ {}^7N_2 + 21\%\ {}^8O_2 + 1\%\ {}^{18}Ar$
mass of Nitrogen gas$= (2 \times 14)$ u $\approx 4.76 \times 10^{-26}$ kg

EM Spectrum

CLASSICAL MECHANICS

Newton's laws

1st: Body remains at rest or in uniform motion unless acted upon by a force
2nd: $\boldsymbol{F}_{tot} = \frac{d\boldsymbol{p}}{dt} = m\boldsymbol{a}$
3rd: $\boldsymbol{F}_{A\to B} = -\boldsymbol{F}_{B\to A}$

Energy

potential energy: $\int_1^2 \boldsymbol{F}\cdot d\boldsymbol{r} \equiv U_1 - U_2$
(work, done by force $\boldsymbol{F}$, req'd to move particle from point 1 to point 2 with no change in kinetic energy); potential energy is the capacity to do work.
force due to the potential U**:** $\boldsymbol{F} = -\boldsymbol{\nabla}U$
kinetic energy: $T_{trans} \equiv \frac{1}{2}m|\boldsymbol{v}|^2$
$T_{rot} \equiv \frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{L}$
$T = \frac{\boldsymbol{p}^2}{2m}$
total energy: $E \equiv T + U$
1D solution given E **and** $U(x)$**,** for conservative force only:

$$t - t_0 = \int_{x_0}^{x}\frac{\pm dx}{\sqrt{\frac{2}{m}[E - U(x)]}}$$

Conservation theorems

linear momentum: $\frac{d}{dt}(p_1 + p_2) = 0$ (or $p_1 + p_2$ is const) if no external forces act upon system
angular momentum: $\dot{\boldsymbol{L}} = \boldsymbol{r}\times\dot{\boldsymbol{p}} = 0$ (or $\boldsymbol{L}$ is const) if no external torque acts upon system
energy: $\boldsymbol{F} + \nabla U = 0$; $\frac{dE}{dt} = 0$ if the force field represented by $\boldsymbol{F}$ is conservative

Lagrangian dynamics

Hamilton's principle — Nature minimizes (makes stationary) the action.
Constrained — If a 3D system of N particles has $n < 3N$ minimum generalized coordinates, the system is *constrained*.
Natural — The coordinates q_n are *natural* if the relationships of r_α (every particle's position) to q_n doesn't change with time.
Ignorable — a coordinate q_i is *ignorable* if the corresponding generalized momentum p_i is constant.

Lagrangian: $\mathscr{L} = T - U$
Action: $S = \int_{t_1}^{t_2}\mathscr{L}(q_1, q_2, \ldots, q_N, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_N, t)dt$
Euler-Lagrange equations: $\frac{\partial\mathscr{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{q}_1}, \ldots$ etc.
Generalized forces: $F_i = \frac{\partial\mathscr{L}}{\partial q_i}$
Generalized momenta $p_i = \frac{\partial\mathscr{L}}{\partial\dot{q}_i}$

Stability

Lyupanov stablility — if all solutions of the dynamical system that start out near an equilibrium point x_e end up within ε of x_e forever, then x_e is Lyapunov stable
Asymptotic stability — If x_e is Lyapunov stable and all solutions that start out near x_e converge to x_e, then x_e is asymptotically stable

Conservative force

Conditions, given $\boldsymbol{F}$ *has continuous 1st partials in a simply connected region…*

No curl anywhere: $\boldsymbol{\nabla}\times\boldsymbol{F} = 0$
Equal work regardless of path :
$W_C = \int_C \boldsymbol{F}\cdot d\boldsymbol{s} = const.\ \forall$ paths C
$W_C = \oint_C \boldsymbol{F}\cdot d\boldsymbol{s} = 0\ \forall$ closed contours C
$\boldsymbol{F}\cdot d\boldsymbol{r}$ **is exact differential**
$\boldsymbol{F} = \nabla W$**,** W **single-valued**
Allows definition of potential: $\boldsymbol{F} = -\boldsymbol{\nabla}\boldsymbol{U}$

Particular forces explained

Fundamental forces

gravity
point mass or sph.-symm mass: $\boldsymbol{F} = -G\frac{Mm}{r^2}\boldsymbol{e_r} \approx -mg$ on earth
generally: $\boldsymbol{F} = -Gm\int_V \frac{\rho(\boldsymbol{r'})\boldsymbol{e_r}}{r^2}dv'$
grav field vector: $\boldsymbol{g} \equiv -\boldsymbol{\nabla}\Phi = \boldsymbol{F}/m$
grav potential, point mass: $\Phi = -G\frac{M}{r}$
grav potential, mass distr: $\Phi = -G\int_V \frac{\rho(\boldsymbol{r'})}{r}dv'$
potential energy: $U = m\Phi$
Gauss' law for grav, int: $\oint_S \boldsymbol{g}\cdot d\boldsymbol{A} = -4\pi GM$
Gauss' law for grav, dif: $\nabla\cdot\boldsymbol{g} = -4\pi G\rho$
Poisson's equation: $\nabla^2\phi = 4\pi G\rho$, for rad-sym system, this is $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = 4\pi G\rho(r)$ and $\boldsymbol{g}(r) = -\boldsymbol{e_r}\frac{\partial\phi}{\partial r}$

lorentz — charged particle in $\boldsymbol{E}$- and $\boldsymbol{B}$-fields
$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$
$\boldsymbol{F} = q\left(-\nabla\phi - \frac{\partial\boldsymbol{A}}{\partial t} + \boldsymbol{v}\times(\nabla\times\boldsymbol{A})\right)$
q =particle's charge
$\boldsymbol{v}$ =particle's velocity
$\boldsymbol{E}$ =electric field strength
$\boldsymbol{B}$ =magnetic field strength
ϕ =electric potential
$\boldsymbol{A}$ =magnetic potential

Purely inertial forces

linear
$\boldsymbol{F}_{\text{inert}} = -m\boldsymbol{A}$ ($\boldsymbol{A}$: frame's accel w.r.t. inertial frame)
centrifugal
$\boldsymbol{F}_{\text{centr}} = m(\boldsymbol{\Omega}\times\boldsymbol{r})\times\boldsymbol{\Omega}$ (generally)
$\boldsymbol{F}_{\text{centr}} = \frac{mv^2}{r}\boldsymbol{e_r} = mr\Omega^2\boldsymbol{e_r}$ (for circular motion)
$U_{centr}(r) = \frac{\ell^2}{2mr^2}$ (ℓ: angular momentum)
Free-fall accel (e.g., on Earth): $\boldsymbol{g} = \boldsymbol{g_0} + (\boldsymbol{\Omega}\times\boldsymbol{R})\times\boldsymbol{\Omega}$
coriolis
$\boldsymbol{F}_{\text{cor}} = 2m\dot{\boldsymbol{r}}\times\boldsymbol{\Omega}$

Derived forces

spring (simple, linear)
$F = -kx$ (x: displ from eq lib, k: spring const)
$U = \frac{1}{2}kx^2$
friction
$\boldsymbol{F}_f = \mu F_N$ (μ: static (μ_s) or kinetic (μ_k), F_N: normal force)
Angle of friction (obj starts to move): $tan\theta = \mu_s$
Energy converted to heat: $E_{th} = \mu_k\int F_n(x)dx$
general retarding
$\boldsymbol{F} = -bm\dot{x}^n$ (b: damping const, m: mass, n: power of velocity dep., just 1 in simple cases)
air resistance / drag
$W = \frac{1}{2}c_W\rho Av^2$, c_W: dimensionless drag coeff, ρ: air density, A: cross-sectional area perp. to velocity (v)
buoyant
$F = \rho_{fluid}Vg$, dir. opposite to grav.-induced pressure grad. in fluid; ρ_{fluid}: density, V: submerged volume, g: grav.
tidal
$\boldsymbol{e_R}$ points from Moon's center to test mass on Earth
$\boldsymbol{e_D}$ is from center of Moon to center of Earth
(x, y) ECEF coord of test mass
$\boldsymbol{F}_T = -GmM_m\left(\frac{\boldsymbol{e_R}}{R^2} - \frac{\boldsymbol{e_D}}{D^2}\right)$
$F_{T_x} \approx 2GmM_m x/D^3$
$F_{T_y} \approx -GmM_m y/D^3$

Harmonic oscillation

Simple harmonic oscillator

$$m\ddot{x} = -kx$$
$$\omega_0^2 \equiv k/m$$
$$\ddot{x} + \omega_0^2 x = 0$$
$$x(t) = A\sin(\omega_0 t - \delta)$$
$$E = T + U = \tfrac{1}{2}kA^2$$

Damped oscillator

equation of motion: $m\ddot{x} + b\dot{x} + kx = 0$; b is resisting force coeff, k is restoring force coeff

convenient substitutions: $\beta \equiv \frac{b}{2m}$ (damping), and $\omega_0^2 \equiv k/m$ (natural ang. freq, undamped sys)

new eqn of motion: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$

general sol'n:

$$x(t) = e^{-\beta t}\left[A_1 exp\left(\sqrt{\beta^2 - \omega_0^2}t\right) + A_2 exp\left(-\sqrt{\beta^2 - \omega_0^2}t\right)\right]$$

underdamping: $\omega_0^2 > \beta^2$

critical damping: $\omega_0^2 = \beta^2$

overdamping: $\omega_0^2 < \beta^2$

Sinusoidally-driven damped oscillator

eqn of motion: $m\ddot{x} + b\dot{x} + kx = F_0\cos\omega t$; b is resisting force coeff, k is restoring force coeff

convenient substitutions: $A = F_0/m$ (driving ampl), $\beta \equiv \frac{b}{2m}$ (damping), and $\omega_0^2 \equiv k/m$ (natural ang. freq, undamped sys)

new eqn of motion: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A\cos\omega t$

complementary solution:

$$x_c(t) = e^{-\beta t}\left[A_1 exp\left(\sqrt{\beta^2 - \omega_0^2}t\right) + A_2 exp\left(-\sqrt{\beta^2 - \omega_0^2}t\right)\right]$$

particular solution:

$$x_p(t) = \frac{A}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}}cos(\omega t - \delta)$$

$$\delta = \arctan\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right)$$

amplitude resonance frequency: $\omega_R = \sqrt{\omega_0^2 - 2\beta^2}$ $(\omega_r < \omega_1 < \omega_0)$

kinetic energy resonance frequency: $\omega_E = \omega_0$

quality factor: $Q \equiv \frac{\omega_R}{2\beta} \approx \frac{\omega_0}{\Delta\omega}$ (the latter is for lightly damped systems; $\Delta\omega$ is the distance between half-energy points — $D_res/\sqrt{2}$ — on the amplitude resonance curve)

Underdamped oscillator

pseudo-frequency of oscillation: $\omega_1^2 \equiv \omega_0^2 - \beta^2$

solution (form 1): $x(t) = e^{-\beta t}\left[A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t}\right]$

solution (form 2): $x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)$

phase plot: Use the var. subst. $u = \omega_1 x$, $w = \beta x + \dot{x}$ and plot w on the y-axis vs. u on the x-axis

response to δ force: $x(t) = \frac{b}{\omega_1}e^{-\beta(t-t_0)}\sin\omega_1(t - t_0)$

green's fn: $G(t,t') \equiv \frac{1}{m\omega_1}e^{-\beta(t-t')}\sin\omega_1(t - t')$, $t \geq t'$; 0 otherwise

Critically damped oscillator

qualitative behavior: System approaches equilibrium (natural solution dies out) faster than the others.

solution: $x(t) = (A + Bt)e^{-\beta t}$

Overdamped oscillator

pseudo-frequency of (non-)oscillation: $\omega_2^2 \equiv \beta^2 - \omega_0^2$

solution: $x(t) = e^{-\beta t}\left[A_1 e^{\omega_2 t} + A_2 e^{-\omega_2 t}\right]$

phase plot: Asymptotic behavior tends towards $\dot{x} = -(\beta - \omega_2)x$ unless $A_1 = 0$, then it goes to $\dot{x} = -(\beta + \omega_2)x$

Series RLC circuit

voltage across inductor: $V_L = L\frac{dI}{dt} = L\ddot{q}$

voltage across resistor: $V_R = IR = R\frac{dq}{dt} = R\dot{q}$

voltage across capacitor: $V_C = \frac{q}{C}$

diffeq of RLC circuit with driving power source:
$L\ddot{q} + R\dot{q} + q/C = V(t)$

Electrical–mechanical equivalents

	Mechanical		**Electrical**
x	Displacement	q	Charge
$\dot{x}$	Velocity	$\dot{q} = I$	Current
m	Mass	L	Inductance
b	Damping resistance	R	Resistance
l/k	Mech compliance	C	Capacitance
F	Ampl of impr. force	ε	Ampl of impr emf

Rigid body dynamics

center of mass:
$\boldsymbol{R} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \boldsymbol{r_\alpha}$ (discrete point masses)
$\boldsymbol{R} = \frac{1}{M}\int \boldsymbol{r}\,dm$ (continuous mass distr.)

momentum: $\boldsymbol{p} \equiv m\boldsymbol{v}$

kinetic energy:
$T_{tot} = T(\text{motion of CM}) + T(\text{rotation about CM})$
$T_{tot} = T_{rot}(\text{about an instantaneously fixed point in body})$
$T_{trans} = \frac{1}{2}m|\boldsymbol{v}|^2$
$T_{rot} = \frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{L}$
$T_{rot} = \frac{1}{2}\left(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2 + \right)$ (if coord. sys = principal axes) $T_{rot} = \frac{1}{2}I\omega^2$ (freshman physics model)

moment of inertia, point mass: $I = \int r^2 dm$ or, for a point mass, $I = r^2 m$, where r is the perp. distance to axis of rotation

moment of inertia, rigid body:

$$\boldsymbol{I} = \int_{\text{mass}}\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -yz & -xz & x^2+y^2 \end{pmatrix} dM$$

or

$$I_{ij} = \int_{\text{mass}}\left(r^2\delta_{ij} - r_i r_j\right) dM \quad \text{where } r_i \text{ is distance to } i^{th} \text{ axis}$$

EXAMPLES, with $\rho = const.$

Cylindrical tube, height=h, radii=r_1 & r_2; rot. about center:
$I_z = \frac{1}{2}m\left(r_1^2 + r_2^2\right)$; $I_x = I_y = \frac{1}{12}m\left[3\left(r_1^2 + r_2^2\right) + h^2\right]$
generalize to solid cylinder, cylindrical shell, disk, or ring

Ellipsoid with semiaxes $a\hat{\boldsymbol{x}}$, $b\hat{\boldsymbol{y}}$, and $c\hat{\boldsymbol{z}}$ rotating about center:
$I_x = \frac{1}{5}m\left(b^2 + c^2\right)$, $I_y = \frac{1}{5}m\left(a^2 + c^2\right)$, $I_z = \frac{1}{5}m\left(a^2 + b^2\right)$
generalize to solid sphere

Spherical shell with radius r:
$I_{x,y,z} = \frac{2}{3}mr^2$

Right circular cone of radius r & height h rotating about point:
$I_z = \frac{3}{10}mr^2$; $I_x = I_y = \frac{3}{5}m\left(\frac{r^2}{4} + h^2\right)$

Cuboid with side lengths $a\hat{\boldsymbol{x}}$, $b\hat{\boldsymbol{y}}$, and $c\hat{\boldsymbol{z}}$ rotating about center:
$I_x = \frac{1}{12}m\left(b^2 + c^2\right)$, $I_y = \frac{1}{12}m\left(a^2 + c^2\right)$, $I_z = \frac{1}{12}m\left(a^2 + b^2\right)$

Rod length L rotating about its end:
$I_{x,y} = \frac{1}{3}mL^2$

Torus tube radius a, cross-sectional radius b about its center:
$I_{\text{diameter}} = \frac{1}{8}m\left(4a^2 + 5b^2\right)$; $I_{\text{vertical}} = m\left(a^2 + \frac{3}{4}b^2\right)$

principal axes: Any axis through O with $\boldsymbol{\omega} \parallel \boldsymbol{L}$ when $\boldsymbol{\omega}$ points along that axis; i.e., $\boldsymbol{L} = \lambda\boldsymbol{\omega}$. Principal axes are eigenvectors of $\boldsymbol{I}$.

parallel axis theorem: $I_z = I_{cm} + md^2$; I_{cm}: inertia about center of mass, m: mass, d: distance between axes

angular momentum:
$\boldsymbol{L} \equiv \boldsymbol{r} \times \boldsymbol{p}$ ($\boldsymbol{r}$: position vec, $\boldsymbol{p}$: linear momentum)
$\boldsymbol{L} = \boldsymbol{I}\vec{\boldsymbol{\omega}}$
$\boldsymbol{L} = \boldsymbol{L}(\text{motion of CM}) + \boldsymbol{L}(\text{motion relative to CM})$

torque: $\boldsymbol{\tau} \equiv \boldsymbol{r} \times \boldsymbol{F} = \dot{\boldsymbol{L}} = \boldsymbol{r} \times \dot{\boldsymbol{p}}$

work: $W = \tau\theta$, θ in rad

angle: $\boldsymbol{\theta}$

angular velocity:
$\omega = \dot{\theta}$ (in one dimension)
$\boldsymbol{\omega} = \frac{\boldsymbol{r}\times\boldsymbol{v}}{|\boldsymbol{r}|^2}$

linear velocity: $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$

angular acceleration: $\boldsymbol{\alpha} = \boldsymbol{a}_T/r$; $\boldsymbol{a}_T$ is accel tangential to $\boldsymbol{r}$

newton's 2nd-law: $\boldsymbol{\tau} = I\boldsymbol{\alpha}$

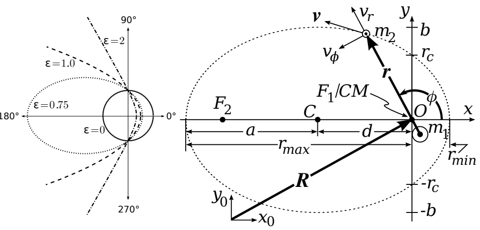
time derivatives for frame S rotating at $\boldsymbol{\omega}$ w.r.t. inertial frame S_0
Unit vector $\hat{\boldsymbol{e}}$ fixed in S: $\frac{d\hat{\boldsymbol{e}}}{dt} = \boldsymbol{\omega} \times \hat{\boldsymbol{e}}$
Vector $\boldsymbol{r}$ fixed in S: $\left(\frac{d\boldsymbol{r}}{dt}\right)_{S_0} = \left(\frac{d\boldsymbol{r}}{dt}\right)_S + \boldsymbol{\omega} \times \boldsymbol{r}$

Newton's 2^{nd} in rotating frame: $m\ddot{\boldsymbol{r}} = \boldsymbol{F} + 2m\dot{\boldsymbol{r}} \times \boldsymbol{\omega} + m(\boldsymbol{\omega} \times \boldsymbol{r}) \times \boldsymbol{\omega}$

Euler's equations of motion (*body frame*): $\dot{\boldsymbol{L}} + \boldsymbol{\omega} \times \boldsymbol{L} = \boldsymbol{\tau}\ldots$
$\lambda_1\dot{\omega}_1 - (\lambda_2 - \lambda_3)\,\omega_2\omega_3 = \tau_1$
$\lambda_2\dot{\omega}_2 - (\lambda_3 - \lambda_1)\,\omega_3\omega_1 = \tau_2$
$\lambda_3\dot{\omega}_3 - (\lambda_1 - \lambda_2)\,\omega_1\omega_2 = \tau_3$

stability: If $\lambda_1 < \lambda_2 < \lambda_3$, rotations about $\hat{\boldsymbol{e}}_1$ and $\hat{\boldsymbol{e}}_3$ are stable, while rotations about $\hat{\boldsymbol{e}}_2$ are not. If $\lambda_1 = \lambda_2$, rotations about all principal axes are stable.

Orbits



Definitions:
$M = m_1 + m_2$
$\mu = \frac{m_1 m_2}{m_1 + m_2}$
e.g., $U(\rho) = \frac{-Gm_1 m_2}{\rho}$
$\boldsymbol{r}$: vector from body 1 to body 2
$\boldsymbol{R}$: vector from origin in inertial frame to system's CM

Kinetic energy: $T = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu\dot{r}^2$

Lagrangian: $\mathscr{L} = \frac{1}{2}\mu\dot{\rho}^2 + \frac{1}{2}\mu\rho^2\dot{\phi}^2 - U(\rho)$

Solution in ϕ: $\dot{\phi} = \frac{\ell}{\mu\rho^2}$ (ℓ const — angular momentum)

Solution in ρ: $\mu\ddot{\rho} = -\frac{d}{d\rho}U(\rho) + \frac{\ell^2}{\mu\rho^3} = -\frac{d}{d\rho}\left[U(\rho) + \frac{\ell^2}{2\mu\rho^2}\right]$

Effective potential: $U_{eff} = U(\rho) + \frac{\ell^2}{2\mu\rho^2}$

Note cons. of energy: $\frac{d}{dt}\left(\frac{1}{2}\mu\dot{\rho}^2\right) = -\frac{d}{dt}U_{eff}(\rho)$;
$E = \frac{1}{2}\mu\dot{\rho}^2 + U_{eff}(\rho)$

Use: $u = 1/r$ and $\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \dot{\phi}\frac{d}{d\phi} = \frac{\ell}{\mu\rho^2}\frac{d}{d\phi} = \frac{\ell u^2}{\mu}\frac{d}{d\phi}$

u-equation: $u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2 F(u)}$

Use: $\gamma = Gm_1 m_2$ and $F(u) = -\gamma u^2$; then $U''(\phi) = -u(\phi) + \gamma\mu/\ell^2$; use $w(\phi) = u(\phi) - \gamma\mu/\ell^2$, so $W(\phi) = A\cos(\phi - \delta)$ ergo
$u(\phi) = \frac{\gamma\mu}{\ell^2} + A\cos\phi$

Radial eqn: $r(\phi) = \frac{r_c}{1 + \varepsilon\cos\phi}$

Cartesian: $\left(\frac{x + \frac{r_c\varepsilon}{1-\varepsilon^2}}{\frac{r_c}{1-\varepsilon^2}}\right)^2 + \left(\frac{y}{\frac{r_c}{\sqrt{1-\varepsilon^2}}}\right)^2 = 1$

Eccentricity: $\varepsilon = A \cdot r_c$ (A some constant)

Circular orbit: $r_c = \ell^2/\gamma\mu$

Min radius: $r_{min} = \frac{r_c}{1+\varepsilon} = \frac{\ell^2}{\gamma\mu(1+\varepsilon)}$ (at $\phi = 0$; **periapsis**); $\ell = \mu r v_{tan}$

Max radius: $r_{max} = \frac{r_c}{1-\varepsilon}$ (at $\phi = \pi$; **apoapsis**)

Radial ($\hat{r}$) velocity: $v_r = \sqrt{\frac{\mu}{r_c}} \cdot \varepsilon \cdot \sin\phi$

Tangential ($\hat{\phi}$) velocity: $v_\phi = \sqrt{\frac{\mu}{r_c}} \cdot (1 + \varepsilon \cdot \cos\phi)$

Ellipse params: $a = \frac{r_c}{1-\varepsilon^2}$; $b = \frac{r_c}{\sqrt{1-\varepsilon^2}}$; $d = a\varepsilon$; $\varepsilon = \sqrt{1 - (b/a)^2}$

Orbital period: $\tau = 2\pi\sqrt{\frac{a^3}{\mu}}$

Energy: $E = \frac{\gamma^2\mu}{2\ell^2}\left(\varepsilon^2 - 1\right)$

Kepler's 1^{st} law: Orbits: ellipses w/ sun at a focus (approx. true)

Kepler's 2^{nd} law: Line from Sun to planet, const. area/time
$dA = \frac{1}{2}r^2 d\phi$; $\frac{dA}{dt} = \frac{1}{2}\frac{\ell}{\mu}$, inep. of time

Kepler's 3^{rd} law:

$$\tau = \frac{A}{dA/dt} = \frac{2\pi ab\mu}{\ell} \Rightarrow \tau^2 = 4\pi^2\frac{a^3 r_c \mu^2}{\ell^2} = 4\pi^2\frac{a^3\mu}{\gamma} \approx \frac{4\pi^2}{GM_s}a^3$$

Coupled oscillators

$\boldsymbol{M}\ddot{\boldsymbol{x}} = -\boldsymbol{K}\boldsymbol{x}$, with $(\boldsymbol{M},\boldsymbol{K}) \in \mathbb{R}^{N\times N}$ and $\boldsymbol{x} \in \mathbb{R}^{N\times 1}$

assume solution: $\boldsymbol{x} = \text{Re}\{z(t)\}$, $z(t) = \boldsymbol{a}_n e^{i(\omega_n t - \delta_n)}$
$n \in \mathbb{Z} \cap [1, N]$
$\boldsymbol{a}_n \in \mathbb{R}^{N\times 1}$ = eigenvectors
$\omega_n \in \mathbb{R}$ = eigenvalues
$\delta_n \in \mathbb{R}$ = phase term (can be excluded, whereupon $\boldsymbol{a}_n \in \mathbb{C}$)

actual solution: $\boldsymbol{x} = \text{Re}\left\{\sum_n A_n \boldsymbol{a}_n e^{i(\omega_n t - \delta_n)}\right\}$, $A_n \in \mathbb{R}$

normal frequencies: ω_n; are the generalized eigenvalues of system

normal modes: solutions to equations of motion only containing one of the $\{\omega_n\}$; *all* motion can be described as a weighted sum of the normal modes; equations of motion written in terms of ξ_n diagonalize both $\boldsymbol{M}$ and $\boldsymbol{K}$

normal coordinates: ξ_n; vary independently of one another
e.g.: 2 m's, 3 k's, $k_1 = k_3$: $\xi_1 = \frac{1}{2}(x_1 + x_2)$ & $\xi_2 = \frac{1}{2}(x_1 - x_2)$

General case:

Deformable solids (linear, isotropic)

Continuum hypothesis: Matter can be treated as continuous on a large enough scale ($\geq \mu$m)

Moduli

Bulk modulus BM: measures substance's resistance to uniform compression. Defined as pressure increase needed to cause a given relative decrease in volume. SI unit is pascal.
$\text{BM} = -V\frac{\partial P}{\partial V}$

Young's modulus YM: measure of stiffness of an isotropic elastic material. Defined as ratio of uniaxial stress over the uniaxial strain. SI unit is pascal.
$\text{YM} = \frac{F/A}{\Delta L/L}$ where F=force, A=area force is applied to, & $\Delta L/L$=fractional change of length

Shear modulus SM: deformation of a solid experiencing a force $\|$ to one of its surfaces while its opposite face experiences an opposing force (such as friction). SI unit is pascal.
$\text{SM} = \frac{F/A}{\Delta x/l}$ where F=force, A=area force is applied to, Δx is transverse displacement, & l=initial length

Poisson's ratio ν: ratio, when a sample object is stretched, of the contraction or transverse strain (perpendicular to the applied load), to the extension or axial strain (in the direction of the applied load)
$\nu = -\frac{\varepsilon_{\text{trans}}}{\varepsilon_{\text{axial}}}$
$\varepsilon_{\text{trans}}$=trans. strain ($-$ for axial tension, $+$ for axial compr)
$\varepsilon_{\text{axial}}$=axial strain ($+$ for axial tension, $-$ for axial compr)

Interrelationships
$\text{YM} = 2\text{SM}(1 + \nu)$
$\text{YM} = 3\text{BM}(1 - 2\nu)$
$\text{YM} = \frac{9\text{BM}\cdot\text{SM}}{3\text{BM}+\text{SM}}$

Wave equation in taut string

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

Definitions
$c = \sqrt{T/\mu}$ = speed of propagation
T = tension
μ = linear density
$k = \omega/c = 2\pi/\lambda = n\pi/L$ (for finite string) = wave number, $n \geq 1$
$\omega = 2\pi c/\lambda = n\pi c/L$ (for finite string) = circ. freq
$\nu = c/\lambda = nc/2L$ (for finite string) = ang. freq

Infinitely-long string

General sol'n: Wave f moving $(\rightarrow)$ & wave g moving $(\leftarrow)$

$u(x,t) = f(x-ct) + g(x+ct)$

$\frac{\partial u}{\partial t} = -cf'(x-ct) + cg'(x+ct)$

Standing waves:

$u(x,t) = A\sin(kx-\omega t) + A\sin(kx+\omega t) = 2A\sin(kx)\cos(\omega t)$

Finite string

$u(x,t) = \sum_{n=1}^{\infty} \sin k_n x (B_n \cos\omega_n t + C_n \sin\omega_n t)$

Solve for initial conditions:

$u(x,0) = \sum_{n=1}^{\infty} B_n \sin k_n x$

$$B_n = \frac{2}{L}\int_0^L u(x,0)\, sin\left(\frac{n\pi x}{L}\right) dx, \ n \geq 1$$

$\dot{u}(x,0) = \sum_{n=1}^{\infty} \omega_n C_n \sin k_n x$

$$C_n = \frac{2}{\omega_n L}\int_0^L \dot{u}(x,0)\, sin\left(\frac{n\pi x}{L}\right) dx, \ n \geq 1$$

ELECTROMAGNETICS

Maxwell's equations

Name	In general:	In matter:
Coulomb	$\nabla\cdot\boldsymbol{E} = \frac{1}{\varepsilon_0}\rho$	$\nabla\cdot\boldsymbol{D} = \rho_f$
Faraday	$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$	$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$
No name	$\nabla\cdot\boldsymbol{B} = 0$	$\nabla\cdot\boldsymbol{B} = 0$
Ampere+Maxwell	$\nabla\times\boldsymbol{B} = \mu_0\boldsymbol{J} + \mu_0\varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t}$	$\nabla\times\boldsymbol{H} = \boldsymbol{J}_f + \frac{\partial\boldsymbol{D}}{\partial t}$

Boundary conditions

Electric	Magnetic
$\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_f$	$B_1^{\perp} - B_2^{\perp} = 0$
$\boldsymbol{E}_1^{\parallel} - \boldsymbol{E}_2^{\parallel} = 0$	$\frac{1}{\mu_1}\boldsymbol{B}_1^{\parallel} - \frac{1}{\mu_2}\boldsymbol{B}_2^{\parallel} = \boldsymbol{K}_f \times \hat{\mathbf{n}}$
$V_1 = V_2$	$\boldsymbol{A}_1 = \boldsymbol{A}_2$
$\frac{\partial V_1}{\partial n} - \frac{\partial V_2}{\partial n} = -\frac{1}{\varepsilon_0}\sigma$	for $\boldsymbol{\nabla}\cdot A = 0$, $\frac{\partial \boldsymbol{A}_1}{\partial n} - \frac{\partial \boldsymbol{A}_2}{\partial n} = -\mu_0 K$

Maxwell stress tensor

Auxiliary fields

Definitions:	Linear media:
$\boldsymbol{D} = \varepsilon_0\boldsymbol{E} + \boldsymbol{P}$	$\boldsymbol{P} = \varepsilon_0\chi_e\boldsymbol{E}$, $\boldsymbol{D} = \varepsilon\boldsymbol{E}$
$\boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}$	$\boldsymbol{M} = \chi_m\boldsymbol{H}$, $\boldsymbol{H} = \frac{1}{\mu}\boldsymbol{B}$

Potentials

$\boldsymbol{E} = -\nabla V - \frac{\partial A}{\partial t}$, $\boldsymbol{B} = \nabla\times\boldsymbol{A}$

Energy, momentum, and power

energy/work $U = \frac{1}{2}\int\left(\varepsilon E^2 + \frac{1}{\mu}B^2\right)d\tau$ or $U = \frac{1}{2}\int \rho V d\tau$

superposition: $U_{sup} = U_1 + U_2 + \varepsilon\int \boldsymbol{E_1}\cdot\boldsymbol{E_2}$

momentum $\boldsymbol{P} = \varepsilon\int(\boldsymbol{E}\times\boldsymbol{B})d\tau$

Poynting vector $\boldsymbol{S} = \frac{1}{\mu}(\boldsymbol{E}\times\boldsymbol{B})$

Larmor formula $P = \frac{\mu}{6\pi c}q^2a^2$

Devices

capacitor Holding voltage constant, $U_{\text{tot}} = U_{\text{batt}} + U_{\text{push}}$

$dU = VdQ + F_{\text{push}}dx$

$\boldsymbol{F_E} = +$¡++¿

electromagnet Holding current constant, $U_{\text{tot}} = U_{\text{batt}} + U_{\text{push}}$;

$dU = VdQ + F_{\text{push}}dx$

OPTICS

Geometric

Diffraction

Scattering

Nonlinear

SPECIAL RELATIVITY

Axioms

Principle of Relativity: The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems in uniform translatory motion relative to each other.

Principle of Invariant Light Speed: Light travels at c regardless of motion of the source of the light or the inertial frame in which it is observed

Isotropy & Homogeneity of Space

Independence of Measuring Devices on their History

Consequences

- Time dilation
- Lorentz (length) contraction
- Relativity of simultaneity
- Composition of velocities
- Inertia and momentum
- Equivalence of mass and energy

Lorentz transformations

$t' = \gamma\left(t - vx/c^2\right)$

$x' = \gamma(x - vt)$

$y' = y$

$z' = z$

$\gamma = \frac{1}{\sqrt{1-(v/c)^2}}$

Generally...

$E^2 = (pc)^2 + \left(mc^2\right)^2$; E is total energy of particle

$p = \frac{1}{c}\sqrt{E^2 - \left(mc^2\right)^2}$

$\boldsymbol{p}c^2 = E\boldsymbol{v}$

$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}$

Moving mass

$E = \gamma mc^2$

$\boldsymbol{p} = \gamma m\boldsymbol{v}$

$\gamma = 1/\sqrt{1-(v/c)^2}$

Photons

$E = pc$, or $E = \hbar\omega$, or $E = hc/\lambda$

$\boldsymbol{p} = \hbar\boldsymbol{k} = h\nu/c = h/\lambda$

$k = 2\pi/\lambda$

QUANTUM MECHANICS

de Broglie

for ALL things, light & matter:

$\lambda = h/p$

$\nu = E/h$

Schrödinger Equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$$

this is non-relativistic

Ψ can be complex-valued

boundary conditions lead to energy quantization

not derived (initially) but fit to reality

Time Independent Schrödinger Equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = E\psi$$

Quantum Things

- Mass, charge, etc.
- Energy (cf. blackbody radiation, photoelectric effect, photons & other fields)
- Interference – 2 slit experiment
- Tunneling (radioactive decay, electronic devices)
- Zero-point motion, i.e., electron never at 0 energy
- Diffraction in matter

Operators

position, $\langle x\rangle$: $\hat{x} = x$

a function of position, $\langle f(\boldsymbol{r})\rangle$: $\hat{f} = f(\boldsymbol{r})$

velocity, $\langle v\rangle = \frac{d\langle x\rangle}{dt}$: $\hat{\boldsymbol{v}} = \frac{\hbar}{im}\boldsymbol{\nabla}$

Note that this is velocity of expectation, but gives velocity in QM

momentum, $m\frac{d\langle x\rangle}{dt}$: $\hat{\boldsymbol{p}} = \frac{\hbar}{i}\boldsymbol{\nabla}$

energy: $\hat{H} = -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + V$

WAVES

Forms

$$\boldsymbol{\nabla}^2\boldsymbol{A} = \frac{1}{v^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2}$$

For any $f(x)$, $f(x\pm vt)$ is a traveling wave and solves the wave eqn.

$\sin\frac{2\pi}{\lambda}(x - vt) = \sin\left(\frac{2\pi}{\lambda}x \pm \frac{2\pi}{T}t\right) = \sin\left(\frac{2\pi}{\lambda}x \pm 2\pi f t\right) = \sin(kx\pm\omega t)$

Simple Diffraction

$\sin\theta = n\lambda/2\pi$

THERMODYNAMICS & STATISTICAL MECHANICS

Fundamentals

0th Law—If sys A in therm eqlib w/ sys B & B in therm eqlib w/ sys C, A in eqlib w/ C

1st Law—Cons. of energy; $\Delta U_{tot} = Q + W_{tot}$

2nd Law—S_{tot} for an isolated sytem always stays the same or increases

3rd Law—As $T\to 0$, $S\to 0$; OR as $T\to 0$, $C_V\to 0$

Fundamental Assumption of Stat Mech—Given an isolated system in equilibrium, it is found with equal probability in each of its accessible microstates

Definitions

Random walk $\langle\vec{L}\rangle = 0$ $\sqrt{\langle\vec{L}^2\rangle} = \ell\sqrt{N}$, independent of number of dimensions

mean free path: $\ell \approx \frac{1}{4\pi r^2}\frac{V}{N}$

time between collisions: $\langle\Delta t\rangle \approx \frac{\ell}{v_{\text{rms}}}$

Temperature—$T \equiv \left(\frac{\partial S}{\partial U}\right)^{-1}$

1. Measure of the tendency of an object to spontaneously give up energy to its surroundings.
2. That which is the same for two systems in thermal equilibrium.

NOTE A negative temperature is *hotter* than an infinite temperature.

Heat—Any spontaneous flow of energy from one object to another caused by a difference in temp between the objects.
Mechanisms: conduction, convection, radiation

Work—Any other transfer of energy into or out of a system.

quasistatic compression: $W = -P\Delta V = -\int_{V_i}^{V_f} P(V)\,dV$

Heat Capacity—Heat required to increase the temp of a substance

$\mathbb{C} \equiv \frac{Q}{\Delta T}$

specific heat capacity: $c = \frac{Q}{m\Delta T} = \frac{\mathbb{C}}{m}$

$\mathbb{C}_V = \left(\frac{\partial U}{\partial T}\right)_{V,N}$ (since $Q = \Delta U - W$ and $W = -P\Delta V$); "energy capacity"

$\mathbb{C}_P = \left(\frac{\partial U + P\partial V}{\partial T}\right)_{P,N} = \left(\frac{\partial H}{\partial T}\right)_{P,N}$; "enthalpy capacity"

Latent Heat—For a phase transition (1st-order), energy goes into / comes out of molecular rearrangement, *not* a change in KE, so $Q \not\Rightarrow \Delta U$, and $\mathbb{C}\to\infty$.

$L \equiv \frac{Q}{m}$

Entropy—$S \equiv k_B \ln\Omega$; SI units of J/K

$\Delta S = \Delta U/T = Q/T$ (const. temp & vol, no work)

$dS = Q/T = \frac{C_V\,dT}{T} \Rightarrow \Delta S = S_f - S_i = \int_{T_i}^{T_f}\frac{Q}{T}dT = \int_{T_i}^{T_f}\frac{C_V}{T}dT$ (V const, W=0, T varying)

or $\Delta S = \int_{T_i}^{T_f}\frac{Q}{T}dT$ if $W \neq 0$

Mixing *different* gasses: $\Delta S_A = N_A\ln(V_{f,A}/V_{i,A})$; $\Delta S_{tot} = \Delta S_A + \Delta S_B$

Einstein solid, high-T: $S \approx Nk_B\ln U - Nk_B\ln(\varepsilon N) + Nk_B$

Monatomic ideal gas: $S \approx Nk_B\ln V + NK\ln U^{3/2} + f(N)$

Enthalpy—H; ΔH is equal to the change in the internal energy of the system, plus the work that the system has done on its surroundings.

$H \equiv U + P\mathrm{d}V$

$\Delta H = \Delta U + P\Delta V + W_{non-compr/exp}$

If P is const., $\Delta H = Q + W_{non-compr/exp}$

Gibbs Free Energy—$G \equiv U + PV - TS = H - TS$

Energy TS into rabbit from surroundings; extra energy PV to displace atmosphere

Helmholtz Free Energy—$F \equiv U - TS$

Energy TS into rabbit from surroundings; NO accounting for PV to displace atmosphere

$F = -k_B T\ln Z$

$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N}$

$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N}$

$\mu = +\left(\frac{\partial F}{\partial N}\right)_{T,V}$

Chemical potential—$\mu \equiv \left(\frac{\partial E}{\partial N}\right)_{S,V}$

Change in energy with addition of a particle, other vars fixed

$\mu = \left(\frac{\partial E}{\partial N}\right)_{S,V}$

$\mu = -T\left(\frac{\partial S}{\partial N}\right)_{U,V}$

$\mu = \left(\frac{\partial F}{\partial N}\right)_{T,V}$

$\mu = -kT\ln(Z_1/N)$ for single-particle-state Z

Ideal Gas—No interaction among particles, random motion

$PV = Nk_BT$

mean translational velocity= $\sqrt{\frac{6}{5}\frac{U}{mN}}$

Note that *non*-ideal gas yields reduced pressure due to interactions

Equipartition Theorem

$U_{\text{therm}} = Nf\frac{1}{2}k_BT$

Thermal energy distributes evenly to each quadratic degree of freedom (true even for relativistic systems, but *not* true for quantum-dominated systems).

Diatomic gas at low temp has 3 DOF (3 transl)

Diatomic gas at room temp has 5 DOF (3 transl + 2 rot)

Diatomic gas at high temp has 5 DOF (3 transl + 2 rot + 2 vibr)

Einstein solid has 6 DOF (3 springs = 3×(1 PE + 1 KE))

Liquid has 3 quad DOF and other DOF non-quadratic

Sackur-Tetrode equation

$S = NK\left[ln\left(\frac{V}{N}\left(\frac{4\pi mU}{3Nh^2}\right)^{3/2}\right) + \frac{5}{2}\right]$ (Valid for ideal, monatomic gas)

Isothermal Compression/Expansion

(Quasistatic) $\Delta T = 0$, or $T = const.$; slow heat exchange w/ outside equalizes temp

$\Rightarrow \Delta U = 0$

$\Rightarrow Q = -W$

For ideal gas:

$P = \frac{const.}{V}$, where $const. = Nk_BT$

$W = -\int_{V_i}^{V_f} PdV = -\int_{V_i}^{V_f} Nk_BT/V\,dV = -Nk_BT\ln(Vf/Vi)$

$\Rightarrow W > 0$ if $V_i > V_f$

$\Rightarrow W < 0$ if $V_i < V_f$

Adiabatic Compression/Expansion

(Quasistatic) $Q = 0$ (Isolated system can't exchange heat)

$\Rightarrow \Delta U = W$

For ideal gas:

$\Delta U = -P\Delta V$

$Nf\frac{1}{2}k_BdT = -P\Delta V$; since $P = Nk_BV/T$, $Nf\frac{1}{2}k_BdT = -\frac{Nk_BT}{V}dT$

$T^{f/2}V = const.$

$PV^{(f+2)/f} = const.$
Define $\gamma = (f+2)/f$, and $PV^{\gamma} = const.$

Heat Conduction Law (Fourier)

Generally: $Q = -k\nabla T$
One dimension: $Q_x = -k\frac{dT}{dx}$
One dimensional heat equation: $\frac{\partial T}{\partial t} = K\frac{\partial^2 T}{\partial x^2}$ with $K = \frac{k_t}{c\rho}$;
k_t =therm conductivity, ρ =density, and c =specific heat capacity
Solution to heat eqn: $T(x,t) = T_0 + \frac{A}{\sqrt{t}}e^{-x^2/4Kt}$, $A = const.$

Probabilities and Multiplicities

Combinations:
N **coins, multiplicity of macrostate with** n **heads:**
$\Omega(N,n) = \frac{N!}{n!(N-n)!} = comb(N,n)$
Einstein solid w/ N **osc (**$N/3$ **atoms) and** q **units of energy (**hf**):**
$\Omega(N,q) = \frac{(q+N-1)!}{q!(N-1)!} = comb(q+N-1,q)$
2 Einstein solids w/ N_A**,** N_B**, and** q_{tot}**:**
Solid A can have $q_A \in [0, q_{tot}] \Longrightarrow q_{tot}+1$ different energy levels
Solid B has $q_B = q_{tot} - q_A$ energy
Multiplicity for a given macrostate (def'd by q_A): $\Omega_{tot} = \Omega_A\Omega_B$
Total microstates: $\Omega_{\text{grand}} = \text{comb}(N_A + N_B + q_{tot} - 1,\ q_{tot})$
Prob of microstate: $\mathbb{P} = \Omega_{tot}/\Omega_{grand}$
Most prob. microstate: $q_A = \frac{N_A}{N_A+N_B}q_{tot}$
Multipicities under various approximations:
Einstein solid, large & high temp ($N \gg 1$; $q \gg N$):
$\Omega(N,q) \approx \left(\frac{eq}{N}\right)^N$
Einstein solid, large & low(er) temp ($N \gg 1$; $q \gg 1$):
$\Omega(N,q) \approx \left(\frac{eN}{q}\right)^q$
Einstein solid any large N and q:
$$\Omega(N,q) \approx \frac{\left(\frac{q+N}{q}\right)^q\left(\frac{q+N}{N}\right)^N}{\sqrt{2\pi q(q+N)/N}} \approx \left(\frac{q+N}{q}\right)^q\left(\frac{q+N}{N}\right)^N$$
2 Einstein solids, large & high temp:
$\Omega_{tot} \approx \left(\frac{eq_A}{N_A}\right)^{N_A}\left(\frac{eq_B}{N_B}\right)^{N_B}$; if $N_A = N_B$, $\Omega_{tot} \approx (e/N)^{2N}(q_Aq_B)^N$
2-state paramagnet, large ($N_\uparrow \ll N$):
$\Omega(N,N_\uparrow) \approx \left(\frac{eN}{N_\uparrow}\right)^{N_\uparrow}$
Ideal Gas, d-Dimensional:
$\Omega_N = \frac{1}{N!}\frac{L^{dN}}{h^{dN}}\frac{2\pi^{dN/2}}{(dN/2-1)!}(2mU)^{(dN-1)/2}$
(d=# dimensions, L=length)
Ideal Gas, 3D:
$\Omega_N \approx \frac{1}{N!}\frac{V^N}{h^{dN}}\frac{\pi^{3N/2}}{(3N/2)!}(2mU)^{3N/2}$
$\Omega(U,V,N) \approx f(N)V^N U^{3N/2}$
Sharpness of multiplicity function:
Let $q_A = q/2 + x$, where x is the offset from the midpoint...
2 large Einstein solids, high temp; $N_A = N_B$
$\Omega = \left(\frac{e}{N}\right)^{2N} e^{N\ln(q/2)^2} - N(2x/q)^2$
$\Omega = \Omega_{\max}e^{-N(2x/q)^2}$
Ω drops to $\Omega_{\max}/e$ at $x = \pm\frac{q}{2\sqrt{N}}$
so width $= \frac{q}{\sqrt{N}}$
2 ideal gasses; $N_A = N_B = N \gg 1$
$\Omega_{\text{tot}} = [f(N)]^2(V_AV_B)^N(U_AU_B)^{3N/2}$
width of distr. in avg energy is $\frac{U_{\text{tot}}}{\sqrt{3N/2}}$
width of distr. in avg velocity is $\frac{V_{\text{tot}}}{\sqrt{N}}$
Boltzmann Statistics:
$\mathbb{P}(\text{state} = s) = \frac{1}{Z}e^{-E(s)/k_BT}$; $Z = \sum_s e^{-E(s)/k_BT}$
$\mathbb{P}(\text{energy} = s) = \frac{1}{Z}g(s)e^{-E(s)/k_BT}$; $Z = \sum_s g(s)e^{-E(s)/k_BT}$
(g=degeneracy)
For non-interacting, indistinguishable particles: $Z_tot = Z_1 \cdot Z_2 \ldots Z_N$
For non-interacting, distinguishable particles: $Z_{tot} = \frac{1}{N!}Z_1^N$
$<E>= \frac{1}{Z}\sum_s E(s)e^{-\beta E(s)} = -\frac{1}{Z}\frac{\partial Z}{\partial\beta}$ ($\beta = 1/k_BT$)
$\sigma_E = k_BT\sqrt{C/k_B}$
Maxwell speed distribution: $D(v) = \left(\frac{m}{2\pi k_BT}\right)^{3/2}4\pi v^2 e^{-mv^2/2k_BT}$
Note from equipartition thm, $v_{rms} = \sqrt{3k_BT/m}$ for ideal gas
From Maxwell, $<v>= \sqrt{8k_BT/\pi m}$ for ideal gas

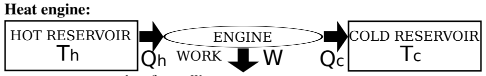
Thermodynamic interactions

Thermodynamic identity: $dU = T\,dS - P\,dV + \mu\,dN$ (any large system)
Simplifies to 1st law if ΔV is quasi-static, no other forms of work are done, and no other relevant variables are changed.

Interaction	Exchanges	Governed by	Formula
therm	energy, U	temp, T	$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_{V,N}$
mech	volume, V	pressure, P	$\frac{P}{T} = \left(\frac{\partial S}{\partial V}\right)_{U,N}$
diffusive	particles, N	chem pot, μ	$\frac{\mu}{T} = -\left(\frac{\partial S}{\partial N}\right)_{U,V}$

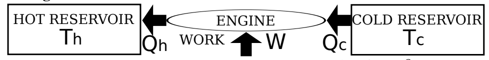
Heat engines, heat pumps, & refrigerators

Heat engine:



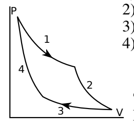
Efficiency: $e \equiv \frac{\text{benefit}}{\text{cost}} = \frac{W}{Q_h}$
since it's a *cycle*, $Q_h = Q_c + W \Rightarrow e = 1 - \frac{Q_c}{Q_h}$
in terms of temp, since 2nd law says $\frac{Q_c}{T_c} \geq \frac{Q_h}{T_h}$, $e \leq 1 - \frac{T_c}{T_h}$

Refrigerator:



Efficiency (Coefficient of Performance): COP $\equiv \frac{\text{benefit}}{\text{cost}} = \frac{Q_c}{W}$
since it's a *cycle*, $Q_h = Q_c + W \Rightarrow \text{COP} = \frac{1}{Q_h/Q_c - 1}$
in terms of temp, since 2nd law says $\frac{Q_h}{T_h} \geq \frac{Q_c}{T_c}$, COP $\leq \frac{T_c}{T_h - T_c}$

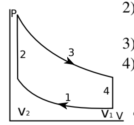
Carnot Cycle



1) isothermal expansion at T_h taking in Q_h
2) adiabatic expansion to T_c
3) isothermal compression at T_c expelling Q_c
4) adiabatic compression to T_h

$e = 1 - \frac{T_c}{T_h}$
Note: S const for quasi-static adiabatic and isothermal processes

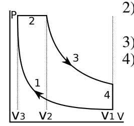
Otto Cycle



1) *compression*: adiabatic compr of gas+fuel in piston
2) *ignition*: fuel ignited while piston static (V const; T & P incr)
3) *power*: adiabatic exp of gas in cylinder does work
4) *exhaust*: hot gasses replaced by lower P, lower T gas (V const); fuel injected

$eff = 1 - \left(\frac{V_2}{V_1}\right)^{\gamma-1} = 1 - \frac{T_1}{T_2} = 1 - \frac{T_4}{T_3}$ where $\gamma = (f+2)/f$

Diesel Cycle



1) *compression*: (isentropic) compression of gas in piston
2) *injection/ignition*: fuel injected & ignites; P const while $V\uparrow$ & (piston moves)
3) *expansion*: isentropic expansion w/ $P\downarrow$
4) *exhaust*: hot gasses replaced by lower P, lower T gas (V const)

$e = 1 - \frac{1}{r^{\gamma-1}}\left(\frac{\alpha^\gamma - 1}{\gamma(\alpha-1)}\right)$
$r = V_1/V_2$=compr. ratio; $\alpha = V_3/V_2$=cut-off ratio

Paramagnets

Energy

$U = \mu B(N_\downarrow - N_\uparrow) = \mu B(N - 2N_\uparrow) = N\mu B\tanh(\mu B/k_BT)$

Magnetization

$M = mu(N_\uparrow - N_\downarrow) = -U/B = N\mu\tanh(\mu B/k_BT)$

Multiplicity & entropy

$\Omega(N_\uparrow) = \binom{N}{N_\uparrow} = \frac{N!}{N_\uparrow!N_\downarrow!}$
$S/k_B = \ln N! - \ln N_\uparrow! - \ln(N - N_\uparrow)! \approx$
$N\ln N - N_\uparrow\ln N_\uparrow - (N - N_\uparrow)\ln(N - N_\uparrow)$
$S = Nk_B[\ln(2\cosh x) - x\tanh x]$, where $x = \mu B/k_BT$
sharpness
$\Omega_{\max} = \Omega(N,N/2) \approx 2^N\sqrt{2/\pi N}$
$N_\uparrow = N/2 + x$ and $N_\downarrow = N/2 - x$
$\Omega \approx 2^N\sqrt{2/\pi N}\,e^{-2x^2/N}$
width$= \sigma = \sqrt{N/2}$

Heat capacity

$C_B = \left(\frac{\partial U}{\partial T}\right)_{N,B} = Nk_B\frac{(\mu B/k_BT)^2}{\cosh^2(muB/k_BT)}$

Temperature

$\frac{1}{T} = (\frac{\partial S}{\partial U})_{N,B} = \frac{\partial N_\uparrow}{\partial U}\frac{\partial S}{\partial N_\uparrow} = -\frac{1}{2\mu B}\frac{\partial S}{\partial N_\uparrow} = \frac{k_B}{2\mu B}\ln\left(\frac{N - U/\mu B}{N + U/\mu B}\right)$