Mathematics

Calculus

Basic theorems

Chain rule $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Product rule $\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$

Integration by parts $\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx$

Calculus of variations

$J = \int_{x_1}^{x_2} f\{y(x), y\prime(x); x\}$

$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y\prime} = 0$

$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y\prime \frac{\partial f}{\partial y\prime}\right) = 0$

$f - y\prime \frac{\partial f}{\partial y\prime} = const$ for $\frac{\partial f}{\partial x} = 0$

Vectors

Divergence theorem (Gauss' thm): $\int_V (\nabla \cdot \boldsymbol{F})\, d_V = \oint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$

Stokes theorem: $\int_S \nabla \times \boldsymbol{F} \cdot dS = \oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$

Dot product: $\boldsymbol{A} \cdot \boldsymbol{B} = |A||B|\cos\theta\hat{\boldsymbol{n}} = A_x B_x + A_y B_y + A_z B_z$

Cross product: $\boldsymbol{A} \times \boldsymbol{B} = |A||B|\sin\theta\hat{\boldsymbol{n}}$

Coordinate systems & conversions

Coordinate Conversion

cartesian to...	*cylindrical to...*	*spherical to...*
cylindrical	**cartesian**	**cartesian**
$\rho = \sqrt{x^2+y^2}$	$x = \rho\cos\phi$	$x = r\sin\theta\cos\phi$
$\phi = \arctan(y/x)$	$y = \rho\sin\phi$	$y = r\sin\theta\sin\phi$
$z = z$	$z = z$	$z = r\cos\theta$
spherical	**spherical**	**cylindrical**
$r = \sqrt{x^2+y^2+z^2}$	$r = \sqrt{\rho^2+z^2}$	$\rho = r\sin\theta$
$\theta = \arccos(z/r)$	$\theta = \arctan(\rho/z)$	$\phi = r\sin\theta$
$\phi = \arctan(y/x)$	$\phi = \phi$	$z = r\cos\theta$

Unit Vector Conversion

cartesian to...	*cylindrical to...*	*spherical to...*
cylindrical	**cartesian**	**cartesian**
$\hat{\boldsymbol{\rho}} = \frac{x}{\rho}\hat{\boldsymbol{x}} + \frac{y}{\rho}\hat{\boldsymbol{y}}$	$\hat{\boldsymbol{x}} = cos\phi\hat{\boldsymbol{\rho}} - sin\phi\hat{\boldsymbol{\phi}}$	$\hat{\boldsymbol{x}} = \sin\theta\cos\phi\hat{\boldsymbol{r}} + \cos\theta\cos\phi\hat{\boldsymbol{\theta}} - \sin\phi\hat{\boldsymbol{\phi}}$
$\hat{\boldsymbol{\phi}} = -\frac{y}{\rho}\hat{\boldsymbol{x}} + \frac{x}{\rho}\hat{\boldsymbol{y}}$	$\hat{\boldsymbol{x}} = cos\phi\hat{\boldsymbol{\rho}} - sin\phi\hat{\boldsymbol{\phi}}$	$\hat{\boldsymbol{y}} = \sin\theta\sin\phi\hat{\boldsymbol{r}} + \cos\theta\sin\phi\hat{\boldsymbol{\theta}} + \cos\phi\hat{\boldsymbol{\phi}}$
$\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}$	$\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}$	$\hat{\boldsymbol{z}} = \cos\theta\hat{\boldsymbol{r}} - \sin\theta\hat{\boldsymbol{\theta}}$
spherical	**spherical**	**cylindrical**
$\hat{\boldsymbol{r}} = \frac{x\hat{\boldsymbol{x}}+y\hat{\boldsymbol{y}}+z\hat{\boldsymbol{z}}}{r}$	$\hat{\boldsymbol{r}} = \frac{\rho}{r}\hat{\boldsymbol{\rho}} + \frac{z}{r}\hat{\boldsymbol{z}}$	$\hat{\boldsymbol{\rho}} = \sin\theta\hat{\boldsymbol{r}} + \cos\theta\hat{\boldsymbol{\theta}}$
$\hat{\boldsymbol{\theta}} = \frac{xz\hat{\boldsymbol{x}}+yz\hat{\boldsymbol{y}}-\rho^2\hat{\boldsymbol{z}}}{r\rho}$	$\hat{\boldsymbol{\theta}} = \frac{z}{r}\hat{\boldsymbol{\rho}} - \frac{\rho}{r}\hat{\boldsymbol{z}}$	$\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$
$\hat{\boldsymbol{\phi}} = \frac{-y\hat{\boldsymbol{x}}+x\hat{\boldsymbol{y}}}{\rho}$	$\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$\hat{\boldsymbol{z}} = \cos\theta\hat{\boldsymbol{r}} - \sin\theta\hat{\boldsymbol{\theta}}$

Differential Elements

cartesian	cylindrical	spherical
$d\boldsymbol{l} = dx\hat{\boldsymbol{x}} + dy\hat{\boldsymbol{y}} + dz\hat{\boldsymbol{z}}$	$d\boldsymbol{l} = d\rho\hat{\boldsymbol{\rho}} + \rho d\phi\hat{\boldsymbol{\phi}} + dz\hat{\boldsymbol{z}}$	$d\boldsymbol{l} = dr\hat{\boldsymbol{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\phi}}$
$d\boldsymbol{A} = dy\,dx\,\hat{\boldsymbol{x}} + dx\,dz\,\hat{\boldsymbol{y}} + dx\,dy\,\hat{\boldsymbol{z}}$	$d\boldsymbol{A} = \rho d\phi\,dz\,\hat{\boldsymbol{\rho}} + d\rho\,dz\,\hat{\boldsymbol{\phi}} + \rho\,d\rho\,d\phi\,\hat{\boldsymbol{z}}$	$d\boldsymbol{A} = r^2\sin\theta\,d\theta\,d\phi\,\hat{\boldsymbol{r}} + r\sin\theta\,dr\,d\phi\,\hat{\boldsymbol{\theta}} + r\,dr\,d\theta\,\hat{\boldsymbol{\phi}}$
$dV = dx\,dy\,dz$	$dV = \rho\,d\rho\,d\phi\,dz$	$dV = r^2\sin\theta\,dr\,d\theta\,d\phi$

POSITIONS, VELOCITIES, & ACCELERATIONS

polar

$\boldsymbol{r} = \rho\boldsymbol{e_\rho}$

$\boldsymbol{v} = \dot{\rho}\boldsymbol{e_\rho} + \rho\dot{\theta}\boldsymbol{e_\theta}$

$\boldsymbol{a} = \left(\ddot{\rho} - \rho\dot{\theta}^2\right)\boldsymbol{e_\rho} + \left(\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta}\right)\boldsymbol{e_\theta}$

cylindrical

$\boldsymbol{r} = \rho\boldsymbol{e_\rho} + z\boldsymbol{e_z}$

$\boldsymbol{v} = \dot{\rho}\boldsymbol{e_\rho} + \rho\dot{\theta}\boldsymbol{e_\theta} + \dot{z}\boldsymbol{e_z}$

$\boldsymbol{a} = \left(\ddot{\rho} - \rho\dot{\theta}^2\right)\boldsymbol{e_\rho} + \left(\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta}\right)\boldsymbol{e_\theta} + \ddot{z}\boldsymbol{e_z}$

spherical

$\boldsymbol{r} = \rho\boldsymbol{e_\rho}$

$\boldsymbol{v} = \dot{\rho}\boldsymbol{e_\rho} + \rho\dot{\theta}\boldsymbol{e_\theta} + \rho\dot{\phi}\sin\theta\boldsymbol{e_\phi}$

$\boldsymbol{a} = \left(\ddot{\rho} - \rho\dot{\theta}^2 - \rho\dot{\phi}^2\sin^2\theta\right)\boldsymbol{e_\rho} + \left(\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta} - \rho\dot{\phi}^2\sin\theta\cos\theta\right)\boldsymbol{e_\theta} + \left(\rho\ddot{\phi}\sin\theta + 2\dot{\rho}\dot{\phi}\sin\theta + 2\rho\dot{\theta}\dot{\phi}\cos\theta\right)\boldsymbol{e_\phi}$

del, ∇, in CARTESIAN

del operator: $\nabla = \boldsymbol{e_x}\frac{\partial}{\partial x} + \boldsymbol{e_y}\frac{\partial}{\partial y} + \boldsymbol{e_z}\frac{\partial}{\partial z}$

gradient: $\nabla\phi = grad\,\phi = \boldsymbol{e_x}\frac{\partial\phi}{\partial x} + \boldsymbol{e_y}\frac{\partial\phi}{\partial y} + \boldsymbol{e_z}\frac{\partial\phi}{\partial z}$

directional derivative: $\frac{d\phi}{ds} = \nabla\phi \cdot \frac{\boldsymbol{A}}{|\boldsymbol{A}|}$

divergence: $\nabla \cdot \boldsymbol{V} = div\,\boldsymbol{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

curl: $\nabla \times \boldsymbol{V} = \boldsymbol{e_x}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \boldsymbol{e_y}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \boldsymbol{e_z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$

Laplacian: $\Delta f = \nabla^2\phi = \nabla \cdot (\nabla\phi) = div\,grad\,\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

del, ∇, in CYLINDRICAL

gradient: $\nabla f = grad\,f = \frac{\partial f}{\partial\rho}\boldsymbol{e_\rho} + \frac{1}{\rho}\frac{\partial f}{\partial\phi}\boldsymbol{e_\phi} + \frac{\partial f}{\partial z}\boldsymbol{e_z}$

divergence: $\nabla \cdot \boldsymbol{V} = div\,\boldsymbol{V} = \frac{1}{\rho}\frac{\partial(\rho\boldsymbol{V}_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial\boldsymbol{V}_\phi}{\partial\phi} + \frac{\partial\boldsymbol{V}_z}{\partial z}$

curl: $\nabla \times \boldsymbol{V} = \left(\frac{1}{\rho}\frac{\partial V_z}{\partial\phi} - \frac{\partial V_\phi}{\partial z}\right)\boldsymbol{e_\rho} + \left(\frac{\partial V_\rho}{\partial z} - \frac{\partial V_z}{\partial\rho}\right)\boldsymbol{e_\phi} + \frac{1}{\rho}\left(\frac{\partial(\rho V_\phi)}{\partial\rho} - \frac{\partial V_\rho}{\partial\phi}\right)\boldsymbol{e_z}$

Laplacian: $\Delta f = \nabla^2 f = \nabla \cdot (\nabla f) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}$

del, ∇, in SPHERICAL

gradient: $\nabla f = grad\,f = \frac{\partial f}{\partial r}\boldsymbol{e_r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\boldsymbol{e_\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\boldsymbol{e_\phi}$

divergence: $\nabla \cdot \boldsymbol{V} = div\,\boldsymbol{V} = \frac{1}{r^2}\frac{\partial(r^2 V_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(V_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi}$

curl: $\nabla \times \boldsymbol{V} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(V_\phi\sin\theta) - \frac{\partial V_\theta}{\partial\phi}\right)\boldsymbol{e_r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial V_r}{\partial\phi} - \frac{\partial}{\partial r}(rV_\phi)\right)\boldsymbol{e_\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial V_r}{\partial\theta}\right)\boldsymbol{e_\phi}$

Laplacian:

$$\Delta f = \nabla^2 f = \nabla \cdot (\nabla f) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

Taylor series

Taylor series of $f(x)$ about $x = a$:

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \cdots + \frac{1}{n!}(x-a)^n f^{(n)}(a) + \cdots$$

Ordinary differential equations

Separable 1st-order

Equation can be written as $f(y)dy = f(x)dx$, such as $\frac{dy}{dx} = N(1-y)$. Evaluate integrals directly.

Linear 1st-order

Write the equation in the form $y' + P(x)y = Q(x)$ and then define

$$I = \int P(x)dx$$

and find y by solving

$$ye^I = \int Q(x)e^I dx + c$$

Linear 2nd-order homogeneous with constant coefficients

Equations of the form

$$a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0 y = 0$$

Write the characteristic polynomial $a_2D^2y + a_1Dy + a_0y = 0$ and factor into $(D-a)(D-b)y = 0$. In general, this can be solved by letting $u = (D-a)y$, solving the 1st-order diff eq $(D-b)u = 0$ for $u(x)$, substituting this solution into the equation $(D-a)y = u(x)$, and finally solving *this* linear 1st-order ODE. In fact, this method can be generalized to higher-order linear diff eq's. However, there are pre-determined solution forms based upon the relationships between a and b:

$$a, b \in \mathbb{R}, a \neq b \Rightarrow y = c_1e^{ax} + c_2e^{bx}$$
$$a, b \in \mathbb{R}, a = b \Rightarrow y = (Ax+B)e^{ax}$$

For

$$a, b \in \mathbb{C}, a = b^* = \alpha \pm i\beta,$$

any of the following forms are solutions:

$$y = Ae^{\alpha+i\beta x} + Be^{\alpha-i\beta x}$$
$$y = e^{\alpha x}\left(Ae^{i\beta x} + Be^{-i\beta x}\right)$$
$$y = e^{\alpha x}(c_1\sin\beta x + c_2\cos\beta x)$$
$$y = ce^{\alpha x}\sin(\beta x + \gamma)$$
$$y = ce^{\alpha x}\cos(\beta x + \delta)$$

Linear 2nd-order inhomogeneous with constant coefficients

Equations of one of the forms

$$a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0y = f(x)$$

$$\frac{d^2y}{dx^2} + \frac{a_1}{a_2}\frac{dy}{dx} + \frac{a_0}{a_2}y = F(x)$$

can be solved, generally, as described for the homogeneous case, but with $F(x)$ on the right-hand side when solving the first 1st-order ODE, $(D-b)u = F(x)$. (This gives both the particular *and* complementary solution.) Otherwise, find $y = y_c + y_p$ where y_c, the complementary solution, comes from solving the homogeneous equation and y_p is a particular solution from a pre-computed form for specific $F(x)$:

$(D-a)(D-b)y = F(x) = ke^{cx}$, particular solution y_p is given by:

$y_p = Ce^{cx}$ if c is not equal to either a or b;

$y_p = Cxe^{cx}$ if c equals a or b, $a \neq b$;

$y_p = Cx^2e^{cx}$ if $c = a = b$

(For $F(x) = k\cos\alpha x$ or $F(x) = k\sin\alpha x$, solve the above with $F(x) = ke^{c=i\alpha x}$ and take the real or imag part, respectively. For $F(x) = const$, set $c = 0$.)

A more general form of this (called the *method of undetermined coefficients*) follows:

$(D-a)(D-b)y = F(x) = e^{cx}P_n(x)$; $P_n(x)$ is a polynomial of degree n:

$$y_p = \begin{cases} e^{cx}Q_n(x) & \text{if } c \neq a \text{ and } c \neq b \\ xe^{cx}Q_n(x) & \text{if } c = a \text{ or } c = b, a \neq b \\ x^2e^{cx}Q_n(x) & \text{if } c = a = b \end{cases}$$

Classical Mechanics

Newton's laws

1st: Body remains at rest or in uniform motion unless acted upon by a force

2nd: $\boldsymbol{F}_{tot} = \frac{d\boldsymbol{p}}{dt} = m\boldsymbol{a}$

3rd: $\boldsymbol{F}_{A\to B} = -\boldsymbol{F}_{B\to A}$

Lagrangian dynamics

Hamilton's principle — Nature minimizes (makes stationary) the action.

Constrained — If a 3D system of N particles has $n < 3N$ minimum generalized coordinates, the system is *constrained*.

Natural — The coordinates q_n are *natural* if the relationships of r_α (every particle's position) to q_n doesn't change with time.

Ignorable — a coordinate q_i is *ignorable* if the corresponding generalized momentum p_i is constant.

Lyupanov Stability — If x_e is Lyapunov stable and all solutions that start out near x_e converge to x_e, then x_e is asymptotically stable

Lagrangian: $\mathscr{L} = T - U$

Action: $S = \int_{t_1}^{t_2}\mathscr{L}(q_1, q_2, \ldots, q_N, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_N, t)dt$

Euler-Lagrange equations: $\frac{\partial\mathscr{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{q}_1}, \ldots$ etc.

Generalized forces: $F_i = \frac{\partial\mathscr{L}}{\partial q_i}$

Generalized momenta $p_i = \frac{\partial\mathscr{L}}{\partial\dot{q}_i}$

Orbits

Definitions:

$M = m_1 + m_2$

$\mu = \frac{m_1 m_2}{m_1+m_2}$; e.g., $U(\rho) = \frac{-Gm_1m_2}{\rho}$

$\boldsymbol{r}$: vector from body 1 to body 2

$\boldsymbol{R}$: vector from origin in inertial frame to system's CM

Kinetic energy: $T = \frac{1}{2}M\dot{r}^2 + \frac{1}{2}\mu\dot{r}^2$

Lagrangian: $\mathscr{L} = \frac{1}{2}\mu\dot{\rho}^2 + \frac{1}{2}\mu\rho^2\dot{\phi}^2 - U(\rho)$

Solution in ϕ: $\dot{\phi} = \frac{\ell}{\mu\rho^2}$ (ℓ const — angular momentum)

Solution in ρ: $\mu\ddot{\rho} = -\frac{d}{d\rho}U(\rho) + \frac{\ell^2}{\mu\rho^3} = -\frac{d}{d\rho}\left[U(\rho) + \frac{\ell^2}{2\mu\rho^2}\right]$

Effective potential: $U_{eff} = U(\rho) + \frac{\ell^2}{2\mu\rho^2}$

Note cons. of energy: $\frac{d}{dt}\left(\frac{1}{2}\mu\dot{\rho}^2\right) = -\frac{d}{dt}U_{eff}(\rho)$; $E = \frac{1}{2}\mu\dot{\rho}^2 + U_{eff}(\rho)$

Use: $u = 1/r$ and $\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \dot{\phi}\frac{d}{d\phi} = \frac{\ell}{\mu\rho^2}\frac{d}{d\phi} = \frac{\ell u^2}{\mu}\frac{d}{d\phi}$

***u*-equation:** $u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2 F(u)}$

Use: $\gamma = Gm_1m_2$ and $F(u) = -\gamma u^2$; then $U''(\phi) = -u(\phi) + \gamma\mu/\ell^2$; use $w(\phi) = u(\phi) - \gamma\mu/\ell^2$, so $W(\phi) = A\cos(\phi-\delta)$ ergo $u(\phi) = \frac{\gamma\mu}{\ell^2} + A\cos\phi$

Radial eqn: $r(\phi) = \frac{r_c}{1+\varepsilon\cos\phi}$

Cartesian: $\left(\frac{x+\frac{r_c\varepsilon}{1-\varepsilon^2}}{\frac{r_c}{1-\varepsilon^2}}\right)^2 + \left(\frac{y}{\frac{r_c}{\sqrt{1-\varepsilon^2}}}\right)^2$

Eccentricity: $\varepsilon = A\cdot r_c$ (A some constant)

Circular orbit: $r_c = \ell^2/\gamma\mu$

Min radius: $r_{min} = \frac{r_c}{1+\varepsilon} = \frac{\ell^2}{\gamma\mu(1+\varepsilon)}$ (at $\phi = 0$; **perihelion**); $\ell = \mu r v_{tan}$

Max radius: $r_{max} = \frac{r_c}{1-\varepsilon}$ (at $\phi = \pi$; **aphelion**)

Radial velocity: $v_r = \sqrt{\frac{\mu}{r_c}}\cdot\varepsilon\cdot\sin\phi$

Tangential velocity: $v_t = \sqrt{\frac{\mu}{r_c}}\cdot(1+\varepsilon\cdot\cos\phi)$

Ellipse params: $a = \frac{r_c}{1-\varepsilon^2}$; $b = \frac{r_c}{\sqrt{1-\varepsilon^2}}$; $d = a\varepsilon$; $\varepsilon = \sqrt{1-(b/a)^2}$

Orbital period: $\tau = 2\pi\cdot a\cdot\sqrt{\frac{a}{\mu}}$

Energy: $E = \frac{\gamma^2\mu}{2\ell^2}(\varepsilon^2 - 1)$

Kepler's 1st law: Orbits: ellipses w/ sun at a focus (approx. true)

Kepler's 2nd law: Line from Sun to planet, const. area/time

$dA = \frac{1}{2}r^2 d\phi$; $\frac{dA}{dt} = \frac{1}{2}\frac{\ell}{\mu}$, inep. of time

Kepler's 3rd law: $\tau = \frac{A}{dA/dt} = \frac{2\pi ab\mu}{\ell} \Rightarrow \tau^2 = 4\pi^2\frac{a^3 r_c\mu^2}{\ell^2} = 4\pi^2\frac{a^3\mu}{\gamma} \approx \frac{4\pi^2}{GM_s}a^3$

Rigid Body Dynamics

center of mass:

$\boldsymbol{R} = \frac{1}{M}\sum_{\alpha=1}^N m_\alpha \boldsymbol{r_\alpha}$ (discrete point masses)

$\boldsymbol{R} = \frac{1}{M}\int \boldsymbol{r}\,dm$ (continuous mass distr.)

momentum: $\boldsymbol{p} \equiv m\boldsymbol{v}$

kinetic energy:

$T_{tot} = T(\text{motion of CM}) + T(\text{rotation about CM})$

$T_{tot} = T_{rot}(\text{about an instantaneously fixed point in body})$

$T_{trans} = \frac{1}{2}m|\boldsymbol{v}|^2$

$T_{rot} = \frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{L}$

$T_{rot} = \frac{1}{2}\left(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2 +\right)$ (if coord. sys = principal axes) $T_{rot} = \frac{1}{2}I\omega$ (freshman physics model)

moment of inertia, point mass: $I = \int r^2 dm$ or, for a point mass, $I = r^2 m$, where r is the perp. distance to axis of rotation

moment of inertia, rigid body: $\mathbb{I} = \int\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -yz & -xz & x^2+y^2\end{pmatrix}dM$

principal axes: Any axis through O with $\boldsymbol{\omega} \parallel \boldsymbol{L}$ when $\boldsymbol{\omega}$ points along that axis; i.e., $\boldsymbol{L} = \lambda\boldsymbol{\omega}$. Principal axes are eigenvectors of $\mathbb{I}$.

parallel axis theorem: $I_z = I_{cm} + md^2$; I_{cm}: inertia about center of mass, m: mass, d: distance between axes

angular momentum:

$\boldsymbol{L} \equiv \boldsymbol{r}\times\boldsymbol{p}$ ($\boldsymbol{r}$: position vec, $\boldsymbol{p}$: linear momentum)

$\boldsymbol{L} = \mathbb{I}\vec{\boldsymbol{\omega}}$

$\boldsymbol{L} = \boldsymbol{L}(\text{motion of CM}) + \boldsymbol{L}(\text{motion relative to CM})$

torque: $\boldsymbol{N} \equiv \boldsymbol{r}\times\boldsymbol{F} = \dot{\boldsymbol{L}} = \boldsymbol{r}\times\dot{\boldsymbol{p}}$

work: $W = N\theta$, θ in rad

angle: $\boldsymbol{\theta}$

angular velocity: $\boldsymbol{\omega} = \dot{\boldsymbol{\theta}} = \frac{\boldsymbol{r}\times\boldsymbol{v}}{|\boldsymbol{r}|^2}$

linear velocity: $\boldsymbol{v} = \boldsymbol{\omega}\times\boldsymbol{r}$

angular acceleration: $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \ddot{\boldsymbol{\theta}} = \boldsymbol{a}_T/r$; $\boldsymbol{a}_T$ is tangential acceleration

newton's 2$^{\text{nd}}$-law: $\boldsymbol{N} = I\boldsymbol{\alpha}$

time deriv, unit vec in rotating frame: $\frac{d\boldsymbol{e}}{dt} = \boldsymbol{\omega}\times\boldsymbol{e}$ ($\boldsymbol{e}$ fixed in body)

time deriv, vec in rotating frame: $\left(\frac{d\boldsymbol{r}}{dt}\right)_{S_0} = \left(\frac{d\boldsymbol{r}}{dt}\right)_S + \boldsymbol{\omega}\times\boldsymbol{r}$ (S_0: inertial frame, S: rotating frame)

Newton's 2nd in rotating frame: $m\ddot{\boldsymbol{r}} = \boldsymbol{F} + 2m\dot{\boldsymbol{r}}\times\boldsymbol{\omega} + m(\boldsymbol{\omega}\times\boldsymbol{r})\times\boldsymbol{\omega}$

Euler's equations of motion (*body frame*): $\dot{\boldsymbol{L}} + \boldsymbol{\omega}\times\boldsymbol{L} = \boldsymbol{\tau}\ldots$

$\lambda_1\dot{\omega}_1 - (\lambda_2-\lambda_3)\,\omega_2\omega_3 = \tau_1$

$\lambda_2\dot{\omega}_2 - (\lambda_3-\lambda_1)\,\omega_3\omega_1 = \tau_2$

$\lambda_3\dot{\omega}_3 - (\lambda_1-\lambda_2)\,\omega_1\omega_2 = \tau_3$

stability: If $\lambda_1 < \lambda_2 < \lambda_3$, rotations about $\hat{\boldsymbol{e}}_1$ and $\hat{\boldsymbol{e}}_3$ are stable, while rotations about $\hat{\boldsymbol{e}}_2$ are not. If $\lambda_1 = \lambda_2$, rotations about all principal axes are stable.

Coupled Oscillators

$\boldsymbol{M}\ddot{\boldsymbol{x}} = -\boldsymbol{K}\boldsymbol{x}$, with $(\boldsymbol{M},\boldsymbol{K}) \in \mathbb{R}^{N\times N}$ and $\boldsymbol{x}\in\mathbb{R}^{N\times 1}$

assume solution: $\boldsymbol{x} = \Re\left\{\boldsymbol{z}(t) = \boldsymbol{a}_n e^{i(\omega_n t - \delta_n)}\right\}$

$\boldsymbol{a}_n \in \mathbb{R}^{N\times 1}$ = eigenvectors

$\omega_n \in \mathbb{R}$ = eigenvalues

$\delta_n \in \mathbb{R}$ = phase term (can be excluded, whereupon $a_n \in \mathbb{C}$)

actual solution: $\boldsymbol{x} = \Re\left\{\sum_n A_n\boldsymbol{a}_n e^{i(\omega_n t - \delta_n)}\right\}$, $A_n \in \mathbb{R}$

normal frequencies: ω_n, $n\in[1,N]$ where N is dimension of K; generalized eigenvalues of system.

normal modes: solutions to equations of motion with only one of $\{\omega_n\}$; *all* motion can be described as a weighted sum of the normal modes; equations of motion written in terms of ξ_n diagonalize both $\boldsymbol{M}$ and $\boldsymbol{K}$

normal coordinates: vary independently of one another

e.g.: 2 m's, 3 k's, $k_1 = k_3$: use $\xi_1 = \frac{1}{2}(x_1+x_2)$ and $\xi_2 = \frac{1}{2}(x_1 - x_2)$

Conservative Force

Conditions, given F has continuous first partials in a simply connected region…

No curl anywhere: $\boldsymbol{\nabla}\times\boldsymbol{F} = 0$

Equal work regardless of path :

$W_C = \int_C \boldsymbol{F}\cdot d\boldsymbol{s} = const.\ \forall$ paths C

$W_C = \oint_C \boldsymbol{F}\cdot d\boldsymbol{s} = 0\ \forall$ closed contours C

$\boldsymbol{F}\cdot d\boldsymbol{r}$ is exact differential

$\boldsymbol{F} = \nabla W$, W single-valued

Allows definition of potential: $\boldsymbol{F} = -\nabla \boldsymbol{U}$

Specific Forces

gravity

point mass or sph.-symm mass: $\boldsymbol{F} = -G\frac{Mm}{r^2}\boldsymbol{e_r} \approx -mg$ on earth

generally: $\boldsymbol{F} = -Gm\int_V \frac{\rho(\boldsymbol{r}')\boldsymbol{e_r}}{r^2}dv'$

grav field vector: $\boldsymbol{g} \equiv -\boldsymbol{\nabla}\Phi = \boldsymbol{F}/m$

grav potential, point mass: $\Phi = -G\frac{M}{r}$

grav potential, mass distr: $\Phi = -G\int_V \frac{\rho(\boldsymbol{r}')}{r}dv'$

potential energy: $U = m\Phi$

Gauss' law for grav, int: $\oint_S \boldsymbol{g}\cdot d\boldsymbol{A} = -4\pi GM$

Gauss' law for grav, dif: $\nabla\cdot\boldsymbol{g} = -4\pi G\rho$

Poisson's equation: $\nabla^2\phi = 4\pi G\rho$, for rad-sym system, this is $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = 4\pi G\rho(r)$ and $\boldsymbol{g}(r) = -\boldsymbol{e_r}\frac{\partial\phi}{\partial r}$

tidal (due to Moon's gravity)

$\boldsymbol{e_R}$ points from Moon's center to test mass on Earth

$\boldsymbol{e_D}$ is from center of Moon to center of Earth

(x,y) ECEF coord of test mass

$\boldsymbol{F}_T = -GmM_m\left(\frac{\boldsymbol{e_R}}{R^2} - \frac{\boldsymbol{e_D}}{D^2}\right)$

$F_{T_x} \approx 2GmM_m x/D^3$

$F_{T_y} \approx -GmM_m y/D^3$

spring (simple, linear)

$F = -kx$ (x: displ from eq lib, k: spring const)

$U = \frac{1}{2}kx^2$

inertial force, linear accel: $\boldsymbol{F}_{inert} = -m\boldsymbol{A}$ ($\boldsymbol{A}$: frame's accel w.r.t. inertial frame)

centrifugal (inertial force)

$\boldsymbol{F}_{centr} = m(\boldsymbol{\Omega}\times\boldsymbol{r})\times\boldsymbol{\Omega}$ (generally)

$\boldsymbol{F}_{centr} = \frac{mv^2}{r}\boldsymbol{e_r} = mr\Omega^2\boldsymbol{e_r}$ (for circular motion)

$U_{centr}(r) = \frac{\ell^2}{2mr^2}$ (ℓ: angular momentum)

Free-fall accel (e.g., on Earth): $\boldsymbol{g} = \boldsymbol{g}_0 + (\boldsymbol{\Omega}\times\boldsymbol{R})\times\boldsymbol{\Omega}$

coriolis (inertial force)

$\boldsymbol{F}_{cor} = 2m\dot{\boldsymbol{r}}\times\boldsymbol{\Omega}$

friction (pseudo-force)

$\boldsymbol{F}_f = \mu F_N$ (μ: static (μ_s) or kinetic (μ_k), F_N: normal force)

Angle of friction (obj starts to move): $tan\theta = \mu_s$

Energy converted to heat: $E_{th} = \mu_k\int F_n(x)dx$

general retarding

$\boldsymbol{F} = -bm\dot{\boldsymbol{x}}^n$ (b: damping const, m: mass, n: power of velocity dep., just 1 in simple cases)

air resistance / drag

$W = \frac{1}{2}c_W\rho Av^2$, c_W: dimensionless drag coeff, ρ: air density, A: cross-sectional area perp. to velocity (v)

buoyant

$F = \rho_{fluid}Vg$, dir. opposite to grav.-induced pressure grad. in fluid; ρ_{fluid}: density, V: submerged volume, g: grav.

lorentz

$\boldsymbol{F} = q\boldsymbol{v}\times\boldsymbol{B}$, q is charge of particle, $\boldsymbol{v}$ its velocity, $\boldsymbol{B}$ is mag. field strength

electrostatic

$\boldsymbol{F} = q\boldsymbol{E}$, q is charge, $\boldsymbol{E}$ electric field

Energy

potential energy: $\int_1^2 \boldsymbol{F}\cdot d\boldsymbol{r} \equiv U_1 - U_2$

(work, done by force $\boldsymbol{F}$, req'd to move particle from point 1 to point 2 with no change in kinetic energy); potential energy is the capacity to do work.

force due to the potential U: $\boldsymbol{F} = -\boldsymbol{\nabla}U$

kinetic energy: $T_{trans} \equiv \frac{1}{2}m|\boldsymbol{v}|^2$

$T_{rot} \equiv \frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{L}$

$T = \frac{\boldsymbol{p}^2}{2m}$

total energy: $E \equiv T + U$

1D solution given E and $U(x)$, for conservative force only:

$$t - t_0 = \int_{x_0}^{x}\frac{\pm dx}{\sqrt{\frac{2}{m}\left[E - U(x)\right]}}$$

Conservation theorems

linear momentum: $\frac{d}{dt}(p_1+p_2) = 0$ (or p_1+p_2 is const) if no external forces act upon system

angular momentum: $\dot{\boldsymbol{L}} = \boldsymbol{r}\times\dot{\boldsymbol{p}} = 0$ (or $\boldsymbol{L}$ is const) if no external torque acts upon system

energy: $\boldsymbol{F} + \nabla U = 0$; $\frac{dE}{dt} = 0$ if the force field represented by $\boldsymbol{F}$ is conservative