MATHEMATICS

Trig

Identities:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$
$$\cos A \cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$
$$\sin A \cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$
$$\cos A \sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 A = \frac{1 - \cos 2A}{2}$$
$$\cos^2 A = \frac{1 + \cos 2A}{2}$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$
$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Law of Sines: sides: A, B, and C; angles opposite: α, β, and γ

$$\frac{A}{\sin\alpha} = \frac{B}{\sin\beta} = \frac{C}{\sin\gamma}$$

Law of Cosines: sides: A, B, and C; angle opposite of $C = \gamma$

$$C^2 = A^2 + B^2 - 2AB\cos\gamma$$

Circular Arclength: $s = r\theta$; θ in rad

Vectors

Divergence theorem (Gauss' thm): $\int_V (\nabla \cdot \mathbf{F})\, d_V = \oiint_S \mathbf{F} \cdot \mathbf{n}\, dS$
Stokes theorem: $\int_S \nabla \times \mathbf{F} \cdot dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$
Dot product: $\mathbf{A} \cdot \mathbf{B} = |A||B|\cos\theta\hat{\mathbf{n}} = A_x B_x + A_y B_y + A_z B_z$
Cross product: $\mathbf{A} \times \mathbf{B} = |A||B|\sin\theta\hat{\mathbf{n}}$

Coordinate systems & conversions

COORDINATE CONVERSION

cartesian to...	cylindrical to...	spherical to...
cylindrical	cartesian	cartesian
$\rho = \sqrt{x^2 + y^2}$ $\phi = \arctan(y/x)$ $z = z$	$x = \rho\cos\phi$ $y = \rho\sin\phi$ $z = z$	$x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$
spherical	spherical	cylindrical
$r = \sqrt{x^2 + y^2 + z^2}$ $\theta = \arccos(z/r)$ $\phi = \arctan(y/x)$	$r = \sqrt{\rho^2 + z^2}$ $\theta = \arctan(\rho/z)$ $\phi = \phi$	$\rho = r\sin\theta$ $\phi = r\sin\theta$ $z = r\cos\theta$

UNIT VECTOR CONVERSION

cartesian to...	cylindrical to...	spherical to...
cylindrical	cartesian	cartesian
$\hat{\boldsymbol{\rho}} = \frac{x}{\rho}\hat{\mathbf{x}} + \frac{y}{\rho}\hat{\mathbf{y}}$	$\hat{\mathbf{x}} = cos\phi\hat{\boldsymbol{\rho}} - sin\phi\hat{\boldsymbol{\phi}}$	$\hat{\mathbf{x}} = \sin\theta\cos\phi\hat{\mathbf{r}} +$ $\cos\theta\cos\phi\hat{\boldsymbol{\theta}} - \sin\phi\hat{\boldsymbol{\phi}}$
$\hat{\boldsymbol{\phi}} = -\frac{y}{\rho}\hat{\mathbf{x}} + \frac{x}{\rho}\hat{\mathbf{y}}$	$\hat{\mathbf{x}} = cos\phi\hat{\boldsymbol{\rho}} - sin\phi\hat{\boldsymbol{\phi}}$	$\hat{\mathbf{y}} = \sin\theta\sin\phi\hat{\mathbf{r}} +$ $\cos\theta\sin\phi\hat{\boldsymbol{\theta}} + \cos\phi\hat{\boldsymbol{\phi}}$
$\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$
spherical	spherical	cylindrical
$\hat{\mathbf{r}} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{r}$	$\hat{\mathbf{r}} = \frac{\rho}{r}\hat{\boldsymbol{\rho}} + \frac{z}{r}\hat{\mathbf{z}}$	$\hat{\boldsymbol{\rho}} = \sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\boldsymbol{\theta}}$
$\hat{\boldsymbol{\theta}} = \frac{xz\hat{\mathbf{x}} + yz\hat{\mathbf{y}} - \rho^2\hat{\mathbf{z}}}{r\rho}$	$\hat{\boldsymbol{\theta}} = \frac{z}{r}\hat{\boldsymbol{\rho}} - \frac{\rho}{r}\hat{\mathbf{z}}$	$\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$
$\hat{\boldsymbol{\phi}} = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{\rho}$	$\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}$

DIFFERENTIAL ELEMENTS

cartesian	cylindrical	spherical
$d\mathbf{l} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$	$d\mathbf{l} = d\rho\hat{\boldsymbol{\rho}} + \rho d\phi\hat{\boldsymbol{\phi}} + dz\hat{\mathbf{z}}$	$d\mathbf{l} = dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}} +$ $r\sin\theta d\phi\hat{\boldsymbol{\phi}}$
$d\mathbf{A} = dy\,dx\,\hat{\mathbf{x}} +$ $dx\,dz\,\hat{\mathbf{y}} +$ $dx\,dy\,\hat{\mathbf{z}}$	$d\mathbf{A} = \rho d\phi\,dz\,\hat{\boldsymbol{\rho}} +$ $d\rho\,dz\,\hat{\boldsymbol{\phi}} +$ $\rho\,d\rho\,d\phi\,\hat{\mathbf{z}}$	$d\mathbf{A} = r^2\sin\theta\,d\theta\,d\phi\,\hat{\mathbf{r}} +$ $r\sin\theta\,dr\,d\phi\,\hat{\boldsymbol{\theta}} +$ $r\,dr\,d\theta\,\hat{\boldsymbol{\phi}}$
$dV = dx\,dy\,dz$	$dV = \rho\,d\rho\,d\phi\,dz$	$dV = r^2\sin\theta\,dr\,d\theta\,d\phi$

POS, VEL, ACCEL

polar:
$\mathbf{r} = \rho\mathbf{e}_\rho$
$\mathbf{v} = \dot{\rho}\mathbf{e}_\rho + \rho\dot{\theta}\mathbf{e}_\theta$
$\mathbf{a} = \left(\ddot{\rho} - \rho\dot{\theta}^2\right)\mathbf{e}_\rho + \left(\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta}\right)\mathbf{e}_\theta$
cylindrical:
$\mathbf{r} = \rho\mathbf{e}_\rho + z\mathbf{e}_z$
$\mathbf{v} = \dot{\rho}\mathbf{e}_\rho + \rho\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{e}_z$
$\mathbf{a} = \left(\ddot{\rho} - \rho\dot{\theta}^2\right)\mathbf{e}_\rho + \left(\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta}\right)\mathbf{e}_\theta + \ddot{z}\mathbf{e}_z$
spherical:
$\mathbf{r} = \rho\mathbf{e}_\rho$
$\mathbf{v} = \dot{\rho}\mathbf{e}_\rho + \rho\dot{\theta}\mathbf{e}_\theta + \rho\dot{\phi}\sin\theta\mathbf{e}_\phi$
$\mathbf{a} = \left(\ddot{\rho} - \rho\dot{\theta}^2 - \rho\dot{\phi}^2\sin^2\theta\right)\mathbf{e}_\rho + \left(\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta} - \rho\dot{\phi}^2\sin\theta\cos\theta\right)\mathbf{e}_\theta +$
$\left(\rho\ddot{\phi}\sin\theta + 2\dot{\rho}\dot{\phi}\sin\theta + 2\rho\dot{\theta}\dot{\phi}\cos\theta\right)\mathbf{e}_\phi$

del, ∇, in CARTESIAN

del operator: $\nabla = \mathbf{e}_x\frac{\partial}{\partial x} + \mathbf{e}_y\frac{\partial}{\partial y} + \mathbf{e}_z\frac{\partial}{\partial z}$

gradient: $\nabla\phi = grad\,\phi = \mathbf{e}_x\frac{\partial\phi}{\partial x} + \mathbf{e}_y\frac{\partial\phi}{\partial y} + \mathbf{e}_z\frac{\partial\phi}{\partial z}$

directional derivative: $\frac{d\phi}{ds} = \nabla\phi \cdot \frac{\mathbf{A}}{|\mathbf{A}|}$

divergence: $\nabla \cdot \mathbf{V} = div\,\mathbf{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$

curl: $\nabla \times \mathbf{V} = \mathbf{e}_x\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \mathbf{e}_y\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \mathbf{e}_z\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$

Laplacian: $\Delta f = \nabla^2\phi = \nabla \cdot (\nabla\phi) = div\,grad\,\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

del, ∇, in CYLINDRICAL

gradient: $\nabla f = grad\,f = \frac{\partial f}{\partial\rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial\phi}\mathbf{e}_\phi + \frac{\partial f}{\partial z}\mathbf{e}_z$

divergence: $\nabla \cdot \mathbf{V} = div\,\mathbf{V} = \frac{1}{\rho}\frac{\partial(\rho\mathbf{V}_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial\mathbf{V}_\phi}{\partial\phi} + \frac{\partial\mathbf{V}_z}{\partial z}$

curl: $\nabla \times \mathbf{V} = \left(\frac{1}{\rho}\frac{\partial V_z}{\partial\phi} - \frac{\partial V_\phi}{\partial z}\right)\mathbf{e}_\rho + \left(\frac{\partial V_\rho}{\partial z} - \frac{\partial V_z}{\partial\rho}\right)\mathbf{e}_\phi + \frac{1}{\rho}\left(\frac{\partial(\rho V_\phi)}{\partial\rho} - \frac{\partial V_\rho}{\partial\phi}\right)\mathbf{e}_z$

Laplacian: $\Delta f = \nabla^2 f = \nabla \cdot (\nabla f) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}$

del, ∇, in SPHERICAL

gradient: $\nabla f = grad\,f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\mathbf{e}_\phi$

divergence: $\nabla \cdot \mathbf{V} = div\,\mathbf{V} = \frac{1}{r^2}\frac{\partial(r^2 V_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(V_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi}$

curl: $\nabla \times \mathbf{V} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(V_\phi\sin\theta) - \frac{\partial V_\theta}{\partial\phi}\right)\mathbf{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial V_r}{\partial\phi} - \frac{\partial}{\partial r}(rV_\phi)\right)\mathbf{e}_\theta +$
$\frac{1}{r}\left(\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial V_r}{\partial\theta}\right)\mathbf{e}_\phi$

Laplacian:
$$\Delta f = \nabla^2 f = \nabla \cdot (\nabla f) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

Taylor series

Taylor series of $f(x)$ about $x = a$:

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \cdots + \frac{1}{n!}(x-a)^n f^{(n)}(a) + \cdots$$

Ordinary differential equations

Separable 1st-order

Equation can be written as $f(y)dy = f(x)dx$, such as $\frac{dy}{dx} = N(1-y)$. Evaluate integrals directly.

Linear 1st-order

Write the equation in the form $y' + P(x)y = Q(x)$ and then define $I = \int P(x)dx$ and find y by solving $ye^I = \int Q(x)e^I dx + c$

Linear 2nd-order homogeneous with constant coefficients

Equations of the form $a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0 y = 0$
a and b:

$$a, b \in \Re, a \neq b \Rightarrow y = c_1 e^{ax} + c_2 e^{bx}$$
$$a, b \in \Re, a = b \Rightarrow y = (Ax + B)e^{ax}$$

For $a, b \in \Im, a = b^* = \alpha \pm i\beta$, any of the following forms are solutions

$$y = Ae^{\alpha + i\beta x} + Be^{\alpha - i\beta x}$$
$$y = e^{\alpha x}\left(Ae^{i\beta x} + Be^{-i\beta x}\right)$$
$$y = e^{\alpha x}\left(c_1\sin\beta x + c_2\cos\beta x\right)$$
$$y = ce^{\alpha x}\sin(\beta x + \gamma)$$
$$y = ce^{\alpha x}\cos(\beta x + \delta)$$

Linear 2nd-order inhomogeneous with constant coefficients

Equations of one of the forms

$$a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0 y = f(x)$$
$$\frac{d^2y}{dx^2} + \frac{a_1}{a_2}\frac{dy}{dx} + \frac{a_0}{a_2}y = F(x)$$

$(D-a)(D-b)y = F(x) = ke^{cx}$, particular solution y_p is given by:

$y_p = Ce^{cx}$	if c is not equal to either a or b;
$y_p = Cxe^{cx}$	if c equals a or b, $a \neq b$;
$y_p = Cx^2e^{cx}$	if $c = a = b$

(For $F(x) = k\cos\alpha x$ or $F(x) = k\sin\alpha x$, solve the above with $F(x) = ke^{c = i\alpha x}$ and take the real or imag part, respectively. For $F(x) = const$, set $c = 0$.)

A more general form of this (called the *method of undetermined coefficients*) follows:

$(D-a)(D-b)y = F(x) = e^{cx}P_n(x)$; $P_n(x)$ is a polynomial of degree n:

$$y_p = \begin{cases} e^{cx}Q_n(x) & \text{if } c \neq a \text{ and } c \neq b \\ xe^{cx}Q_n(x) & \text{if } c = a \text{ or } c = b, a \neq b \\ x^2e^{cx}Q_n(x) & \text{if } c = a = b \end{cases}$$

Calculus of variations

$J = \int_{x_1}^{x_2} f\{y(x), y\prime(x); x\}$

$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y\prime} = 0$

$\frac{\partial f}{\partial x} - \frac{d}{dx}\left(f - y\prime\frac{\partial f}{\partial y\prime}\right) = 0$

$f - y\prime\frac{\partial f}{\partial y\prime} = const$ for $\frac{\partial f}{\partial x} = 0$

Fundamental (and not so fundamental) Constants

$G =$ gravit. constant $= 6.674 \times 10^{-11}$ N m^{2}kg^{-2}
$g =$ gravit. accel on Earth surface $= 9.8$ m/s^2
Note: $GM_e = gR_e^2$

$R_S =$ mean radius of Sun $= 696 \times 10^6$ m
$R_E =$ mean radius of Earth $= 6.371 \times 10^6$ m
$R_M =$ mean radius of Moon $= 1.737 \times 10^6$ m

$R_{S_E} =$ mean distance, Earth to Sun $= 149.6 \times 10^9$ m
$D =$ mean distance, Earth to Moon $= 384.4 \times 10^6$ m

$M_S =$ mass of Sun $= 1.99 \times 10^{30}$ kg
$M_E =$ mass of Earth $= 5.98 \times 10^{24}$ kg
$M_m =$ mass of Moon $= 7.35 \times 10^{22}$ kg

Physics

Newton's laws

1st: Body remains at rest or in uniform motion unless acted upon by a force
2nd: $\mathbf{F}_{tot} = \frac{d\mathbf{p}}{dt} = m\mathbf{a}$
3rd: $\mathbf{F}_{A\to B} = -\mathbf{F}_{B\to A}$

Lagrangian dynamics

Hamilton's principle — Nature minimizes (makes stationary) the action.
Constrained — If a 3D system of N particles has $n < 3N$ minimum generalized coordinates, the system is *constrained*.
Natural — The coordinates q_n are *natural* if the relationships of r_α (every particle's position) to q_n doesn't change with time.
Ignorable — a coordinate q_i is *ignorable* if the corresponding generalized momentum p_i is constant.
Lyupanov Stability — If x_e is Lyapunov stable and all solutions that start out near x_e converge to x_e, then x_e is asymptotically stable

Lagrangian: $\mathscr{L} = T - U$
Action: $S = \int_{t_1}^{t_2} \mathscr{L}(q_1, q_2, \ldots, q_N, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_N, t)dt$
Euler-Lagrange equations: $\frac{\partial \mathscr{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}_1}, \ldots$ etc.
Generalized forces: $F_i = \frac{\partial \mathscr{L}}{\partial q_i}$
Generalized momenta $p_i = \frac{\partial \mathscr{L}}{\partial \dot{q}_i}$

Orbits

Definitions: $M = m_1 + m_2$; $\mu = \frac{m_1 m_2}{m_1 + m_2}$; e.g., $U(\rho) = \frac{-Gm_1 m_2}{\rho}$
Kinetic energy: $T = \frac{1}{2}M\dot{r}^2 + \frac{1}{2}\mu\dot{r}^2$
Lagrangian: $\mathscr{L} = \frac{1}{2}\mu\dot{\rho}^2 + \frac{1}{2}\mu\rho^2\dot{\phi}^2 - U(\rho)$
Solution in ϕ: $\dot{\phi} = \frac{\ell}{\mu\rho^2}$ (ℓ const — angular momentum)
Solution in ρ: $\mu\ddot{\rho} = -\frac{d}{d\rho}U(\rho) + \frac{\ell^2}{\mu\rho^3} = -\frac{d}{d\rho}\left[U(\rho) + \frac{\ell^2}{2\mu\rho^2}\right]$
Effective potential: $U_{eff} = U(\rho) + \frac{\ell^2}{2\mu\rho^2}$
Note cons. of energy: $\frac{d}{dt}\left(\frac{1}{2}\mu\dot{\rho}^2\right) = -\frac{d}{dt}U_{eff}(\rho)$; $E = \frac{1}{2}\mu\dot{\rho}^2 + U_{eff}(\rho)$
Use: $u = 1/r$ and $\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \dot{\phi}\frac{d}{d\phi} = \frac{\ell}{\mu\rho^2}\frac{d}{d\phi} = \frac{\ell u^2}{\mu}\frac{d}{d\phi}$
u-equation: $u''(\phi) = -u(\phi) - \frac{\mu}{\ell^2 u(\phi)^2 F(u)}$
Use: $\gamma = Gm_1m_2$ and $F(u) = -\gamma u^2$; then $U''(\phi) = -u(\phi) + \gamma\mu/\ell^2$; use $w(\phi) = u(\phi) - \gamma\mu/\ell^2$, so $W(\phi) = A\cos(\phi - \delta)$ ergo $u(\phi) = \frac{\gamma\mu}{\ell^2} + A\cos\phi$
Radial eqn: $r(\phi) = \frac{r_c}{1 + \varepsilon\cos\phi}$
Cartesian:

$$\left(\frac{x + \frac{r_c\varepsilon}{1-\varepsilon^2}}{\frac{r_c}{1-\varepsilon^2}}\right)^2 + \left(\frac{y}{\frac{r_c}{\sqrt{1-\varepsilon^2}}}\right)^2$$

Eccentricity: $\varepsilon = A \cdot r_c$ (A some constant)
Circular orbit: $r_c = \ell^2/\gamma\mu$
Min radius: $r_{min} = \frac{r_c}{1+\varepsilon} = \frac{\ell^2}{\gamma\mu(1+\varepsilon)}$ (at $\phi = 0$; **perihelion**); $\ell = \mu r v_{tan}$
Max radius: $r_{max} = \frac{r_c}{1-\varepsilon}$ (at $\phi = \pi$; **aphelion**)
Radial velocity: $v_r = \sqrt{\frac{\mu}{r_c}} \cdot \varepsilon \cdot \sin\phi$
Tangential velocity: $v_t = \sqrt{\frac{\mu}{r_c}} \cdot \left(1 + \varepsilon \cdot \cos\phi\right)$
Ellipse params: $a = \frac{r_c}{1-\varepsilon^2}$; $b = \frac{r_c}{\sqrt{1-\varepsilon^2}}$; $d = a\varepsilon$; $\varepsilon = \sqrt{1 - (b/a)^2}$
Orbital period: $\tau = 2\pi \cdot a \cdot \sqrt{\frac{a}{\mu}}$
Energy: $E = \frac{\gamma^2\mu}{2\ell^2}(\varepsilon^2 - 1)$
Kepler's 1st law: Orbits are ellipses w/ sun at a focus (approx. true)
Kepler's 2nd law: Line from Sun to planet, equal areas in equal times
$dA = \frac{1}{2}r^2 d\phi$; $\frac{dA}{dt} = \frac{1}{2}\frac{\ell}{\mu}$, inep. of time
Kepler's 3rd law: $\tau = \frac{A}{dA/dt} = \frac{2\pi ab\mu}{\ell} \Rightarrow \tau^2 = 4\pi^2\frac{a^3 r_c \mu^2}{\ell^2} = 4\pi^2\frac{a^3\mu}{\gamma} \approx \frac{4\pi^2}{GM_s}a^3$

Cartesian system

inertia: $I = \int_V r^2 dM$, $dM = \rho(x, y, z)dxdydz$
momentum: $\mathbf{p} \equiv m\mathbf{v}$
kinetic energy: $T = \frac{1}{2}mv^2$

Rotating system

moment of inertia: $I = \int r^2 dm$ or, for a point mass, $I = r^2 m$, where r is the perp. distance to axis of rotation
parallel axis theorem: $I_z = I_{cm} + md^2$; I_{cm}: inertia about center of mass, m: mass, d: distance between axes
angular momentum: $\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} = I\mathbf{\Omega}$; $\mathbf{r}$: position vec, $\mathbf{p}$: linear momentum
torque: $\mathbf{N} \equiv \mathbf{r} \times \mathbf{F} = \dot{\mathbf{L}} = \mathbf{r} \times \dot{\mathbf{p}}$
work: $W = N\theta$, θ in rad
angle: θ
angular velocity: $\mathbf{\Omega} = \dot{\theta} = \frac{\mathbf{r}\times\mathbf{v}}{|\mathbf{r}|^2}$
linear velocity: $\mathbf{v} = \mathbf{\Omega} \times \mathbf{r}$
angular acceleration: $\alpha = \dot{\mathbf{\Omega}} = \ddot{\theta} = \mathbf{a}_T/r$; $\mathbf{a}_T$ is tangential acceleration
newton's 2$^{\text{nd}}$-law: $\mathbf{N} = I\alpha$
time deriv, unit vec in rotating frame: $\frac{d\mathbf{e}}{dt} = \mathbf{\Omega} \times \mathbf{e}$ ($\mathbf{e}$ fixed in body)
time deriv, vec in rotating frame: $\left(\frac{d\mathbf{r}}{dt}\right)_{S_0} = \left(\frac{d\mathbf{r}}{dt}\right)_S + \mathbf{\Omega} \times \mathbf{r}$ (S_0: inert, S: rot)
Newton's 2nd in rotating frame: $m\ddot{\mathbf{r}} = \mathbf{F} + 2m\dot{\mathbf{r}} \times \mathbf{\Omega} + m(\mathbf{\Omega} \times \mathbf{r}) \times \mathbf{\Omega}$

Conservative Force

Conditions, given that **F** has continuous first partial derivatives in a simply connected region...

No curl anywhere: $\nabla \times \mathbf{F} = 0$
Equal work regardless of path :
$W_C = \int_C \mathbf{F} \cdot d\mathbf{s} = constant\ \forall\, C$
$W_C = \oint_C \mathbf{F} \cdot d\mathbf{s} = 0\ \forall\, C$
$\mathbf{F} \cdot d\mathbf{r}$ is exact differential
$\mathbf{F} = \nabla W$, W single-valued
Allows definition of potential: $\mathbf{F} = -\nabla\mathbf{U}$

Specific Forces

gravity :
point mass or sph.-symm mass: $\mathbf{F} = -G\frac{Mm}{r^2}\mathbf{e}_r \approx -mg$ on earth
generally: $\mathbf{F} = -Gm\int_V \frac{\rho(\mathbf{r}')\mathbf{e}_r}{r^2}dv'$
grav field vector: $\mathbf{g} \equiv -\nabla\Phi = \mathbf{F}/m$
grav potential, point mass: $\Phi = -G\frac{M}{r}$
grav potential, mass distr: $\Phi = -G\int_V \frac{\rho(\mathbf{r}')}{r}dv'$
potential energy: $U = m\Phi$
Gauss' law for grav, int: $\oint_S \mathbf{g} \cdot d\mathbf{A} = -4\pi GM$
Gauss' law for grav, dif: $\nabla \cdot \mathbf{g} = -4\pi G\rho$
Poisson's equation: $\nabla^2\phi = 4\pi G\rho$, for rad-sym system, this is $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = 4\pi G\rho(r)$ and $\mathbf{g}(r) = -\mathbf{e}_r\frac{\partial\phi}{\partial r}$

tidal (due to Moon's gravity)
$\mathbf{e}_R$ points from Moon's center to test mass on Earth
$\mathbf{e}_D$ is from center of Moon to center of Earth
(x, y) ECEF coord of test mass
$\mathbf{F}_T = -GmM_m\left(\frac{\mathbf{e}_R}{R^2} - \frac{\mathbf{e}_D}{D^2}\right)$
$F_{T_x} \approx 2GmM_m x/D^3$
$F_{T_y} \approx -GmM_m y/D^3$

spring (simple, linear)
$F = -kx$ (x: displ from eq lib, k: spring const)
$U = \frac{1}{2}kx^2$

inertial force, linear accel: $\mathbf{F}_{\text{inert}} = -m\mathbf{A}$ ($\mathbf{A}$: frame's accel w.r.t. inertial frame)

centrifugal (inertial force)
$\mathbf{F}_{\text{centr}} = m(\mathbf{\Omega} \times \mathbf{r}) \times \mathbf{\Omega}$ (generally)
$\mathbf{F}_{\text{centr}} = \frac{mv^2}{r}\mathbf{e}_r = mr\Omega^2\mathbf{e}_r$ (for circular motion)
$U_{centr}(r) = \frac{\ell^2}{2mr^2}$ (ℓ: angular momentum)
Free-fall accel (e.g., on Earth): $\mathbf{g} = \mathbf{g}_0 + (\mathbf{\Omega} \times \mathbf{R}) \times \mathbf{\Omega}$

coriolis (inertial force)
$\mathbf{F}_{\text{cor}} = 2m\dot{\mathbf{r}} \times \mathbf{\Omega}$

buoyant —
$F = \rho_{fluid}Vg$, dir. opposite to grav.-induced pressure grad. in fluid; ρ_{fluid}: density, V: submerged volume, g: grav.

lorentz —
$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, q is charge of particle, $\mathbf{v}$ its velocity, $\mathbf{B}$ is mag. field strength

electrostatic —
$\mathbf{F} = q\mathbf{E}$, q is charge, $\mathbf{E}$ electric field

Energy

potential energy: $\int_1^2 \mathbf{F} \cdot d\mathbf{r} \equiv U_1 - U_2$
(work, done by force $\mathbf{F}$, req'd to move particle from point 1 to point 2 with no change in kinetic energy); potential energy is the capacity to do work.
force due to the potential U: $\mathbf{F} = -\nabla U$
kinetic energy: $T \equiv \frac{1}{2}mv^2$
total energy: $E \equiv T + U$
1D solution given E and $U(x)$, for conservative force only:

$$t - t_0 = \int_{x_0}^{x} \frac{\pm dx}{\sqrt{\frac{2}{m}[E - U(x)]}}$$

Conservation theorems

linear momentum: $\frac{d}{dt}(p_1 + p_2) = 0$ (or $p_1 + p_2$ is const) if no external forces act upon system
angular momentum: $\dot{\mathbf{L}} = \mathbf{r} \times \dot{\mathbf{p}} = 0$ (or $\mathbf{L}$ is const) if no external torque acts upon system
energy: $\mathbf{F} + \nabla U = 0$; $\frac{dE}{dt} = 0$ if the force field represented by $\mathbf{F}$ is conservative

Harmonic oscillation

Simple harmonic oscillator

$$m\ddot{x} = -kx$$
$$\omega_0^2 \equiv k/m$$
$$\ddot{x} + \omega_0^2 x = 0$$
$$x(t) = A\sin(\omega_0 t - \delta)$$
$$E = T + U = \tfrac{1}{2}kA^2$$

Damped oscillator

equation of motion: $m\ddot{x} + b\dot{x} + kx = 0$; b is resisting force coeff, k is restoring force coeff

convenient substitutions: $\beta \equiv \frac{b}{2m}$ (damping), and $\omega_0^2 \equiv k/m$ (natural ang. freq, undamped sys)

new eqn of motion: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$

general sol'n:

$$x(t) = e^{-\beta t}\left[A_1 exp\left(\sqrt{\beta^2 - \omega_0^2}t\right) + A_2 exp\left(-\sqrt{\beta^2 - \omega_0^2}t\right)\right]$$

underdamping: $\omega_0^2 > \beta^2$

critical damping: $\omega_0^2 = \beta^2$

overdamping: $\omega_0^2 < \beta^2$

Sinusoidally-driven damped oscillator

eqn of motion: $m\ddot{x} + b\dot{x} + kx = F_0\cos\omega t$; b is resisting force coeff, k is restoring force coeff

convenient substitutions: $A = F_0/m$ (driving ampl), $\beta \equiv \frac{b}{2m}$ (damping), and $\omega_0^2 \equiv k/m$ (natural ang. freq, undamped sys)

new eqn of motion: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A\cos\omega t$

complementary solution:

$$x_c(t) = e^{-\beta t}\left[A_1 exp\left(\sqrt{\beta^2 - \omega_0^2}t\right) + A_2 exp\left(-\sqrt{\beta^2 - \omega_0^2}t\right)\right]$$

particular solution:

$$x_p(t) = \frac{A}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}} cos(\omega t - \delta)$$

$$\delta = \arctan\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right)$$

amplitude resonance frequency: $\omega_R = \sqrt{\omega_0^2 - 2\beta^2}$ ($\omega_r < \omega_1 < \omega_0$)

kinetic energy resonance frequency: $\omega_E = \omega_0$

quality factor: $Q \equiv \frac{\omega_R}{2\beta} \approx \frac{\omega_0}{\Delta\omega}$ (the latter is for lightly damped systems; $\Delta\omega$ is the distance between half-energy points — $D_r es/\sqrt{2}$ — on the amplitude resonance curve)

Underdamped oscillator

pseudo-frequency of oscillation: $\omega_1^2 \equiv \omega_0^2 - \beta^2$

solution (form 1): $x(t) = e^{-\beta t}\left[A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t}\right]$

solution (form 2): $x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)$

phase plot: Use the var. subst. $u = \omega_1 x$, $w = \beta x + \dot{x}$ and plot w on the y-axis vs. u on the x-axis

response to δ force: $x(t) = \frac{b}{\omega_1}e^{-\beta(t-t_0)}\sin\omega_1(t - t_0)$

green's fn: $G(t, t') \equiv \frac{1}{m\omega_1}e^{-\beta(t-t')}\sin\omega_1(t - t')$, $t \geq t'$; 0 otherwise

Critically damped oscillator

qualitative behavior: System approaches equilibrium (natural solution dies out) faster than the others.

solution: $x(t) = (A + Bt)e^{-\beta t}$

Overdamped oscillator

pseudo-frequency of (non-)oscillation: $\omega_2^2 \equiv \beta^2 - \omega_0^2$

solution: $x(t) = e^{-\beta t}\left[A_1 e^{\omega_2 t} + A_2 e^{-\omega_2 t}\right]$

phase plot: Asymptotic behavior tends towards $\dot{x} = -(\beta - \omega_2)x$ unless $A_1 = 0$, then it goes to $\dot{x} = -(\beta + \omega_2)x$

Series RLC circuit

voltage across inductor: $V_L = L\frac{dI}{dt} = L\ddot{q}$

voltage across resistor: $V_R = IR = R\frac{dq}{dt} = R\dot{q}$

voltage across capacitor: $V_C = \frac{q}{C}$

diffeq of RLC circuit with driving power source: $L\ddot{q} + R\dot{q} + q/C = V(t)$

Electrical–mechanical equivalents

	Mechanical		**Electrical**
x	Displacement	q	Charge
$\dot{x}$	Velocity	$\dot{q} = I$	Current
m	Mass	L	Inductance
b	Damping resistance	R	Resistance
l/k	Mechanical compliance	C	Capacitance
F	Ampl of impressed force	ε	Ampl of impressed emf