Finding the electric field directly from a charge distribution: Coulomb's law and Gauss's law

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1 Background

There are two equations (or rather two forms of the same equation) that we can use to find $\boldsymbol{E}$ directly from a charge distribution: Coulomb's law and Gauss's law. Note first that $\boldsymbol{E}$ is a vector whose magnitude and direction change by coordinate; i.e., it's a vector function of three dimensions: $\boldsymbol{E}(\boldsymbol{r})$, or $\boldsymbol{E}(x, y, z)$, or $\boldsymbol{E}(r, \theta, z)$, or $\boldsymbol{E}(r, \theta, \phi)$, depending upon how you want to write the coordinate.

With certain symmetries in the problem, you can work out the integral that Coulomb's law leads to. Other situations lend themselves to use of Gauss's law. A third way, which I'm not addressing here, is to first find the electric potential and then use $\boldsymbol{E} = -\boldsymbol{\nabla} V$ to find the electric field. Something for another write-up, but note that the math I describe here generalizes to that method as well.

2 Coulomb's Law

First, starting with Coulomb's law for a point charge, we see that the point charge causes an electric field $\boldsymbol{E}_i$ at some point in space $\boldsymbol{r}$ according to the equation

$$\boldsymbol{E_i}(\boldsymbol{r}) = k\frac{Q}{r_i^2}\hat{\boldsymbol{r}}_i \tag{1}$$

where $\boldsymbol{r}_i$ is the vector pointing *from* the charge *to* the point in space at which we're finding the E-field (see Fig. 1).

But we have a whole distribution of charge, so instead of just a *single* charge Q, we need to add up (superposition's our friend!) the contributions to the electric field from every little bit of charge, $\mathrm{d}Q$, in the overall charge distribution. We could beat around the bush here, but there's a very nice mathematical tool for adding up lots of little things (the integral), so we're going to use it, without fear and without hate, where we can:

$$\boldsymbol{E}(\boldsymbol{r}) = \int_{\substack{\text{everywhere}\\ \text{there's charge}}} k\frac{\hat{\boldsymbol{r}}_i}{{r_i}^2}\,\mathrm{d}Q \tag{2}$$

(Refer to Fig. 2 for the geometry involved in this equation.) Unfortunately this can be a prohibitively tricky integral to work out since each different bit of charge $\mathrm{d}Q$ has a different $\boldsymbol{r}_i$ and so a different (up to) 3-component vector $\hat{\boldsymbol{r}}_i$ that must be summed over. Certain problems are still tractable with Coulomb's law, but this is typically because symmetry in the charge distribution (or simply low-dimensionality, such as in a line of charge) leads to the cancellation of all but one vector component of $\boldsymbol{E}$.

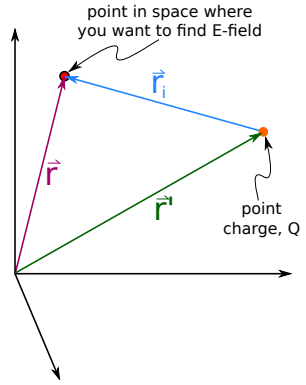

Figure 1: Geometry for analyzing the E-field due to a point charge. Note that only the vector $\boldsymbol{r}_i$ and its associated unit vector, $\hat{\boldsymbol{r}}_i$, enter into the RHS of Eqn. 1. That's where the *physics* is in this picture, which has nothing to do with where you chose your origin!

I'm about to delve deeper into the *process* and *math* involved in calculating the E-field from Coulomb's law. But don't tune out! It's similar to computing the electrostatic potential from a charge distribution, and the ideas behind working these integrals also generalize to the surface and volume integrals you'll find in Gauss's law.

It's your first job to think about what a small bit of charge looks like for the distribution you're given and in the coordinate system you've chosen. If you're given a charge density (linear, surface, or volume), then you can write $\mathrm{d}Q = (\text{charge dens}) \cdot \mathrm{d}V$ where $\mathrm{d}V$ is the differential "volume" element appropriate to the distribution's dimensionality and your coordinate system. Remember that $\mathrm{d}Q$ has units of charge (e.g., C in SI), so if you know the surface charge density η, which has units $\mathrm{C/m^2}$, you know that your differential "volume" element $\mathrm{d}V$ must consist of two lengths to make the overall units end up as C; for this example, and using Cartesian coordinates, you should get $\mathrm{d}Q = \eta\,\mathrm{d}x'\,\mathrm{d}y'$.

Note that η may change with location in space, and so be dependent upon some components (or all) of $\boldsymbol{r}'$. Note also that I use primes (') to indicate the coordinate that you're *summing over*, $\boldsymbol{r}'$, which has Cartesian coordinates (x', y') in 2D; refer again to Fig. 2.

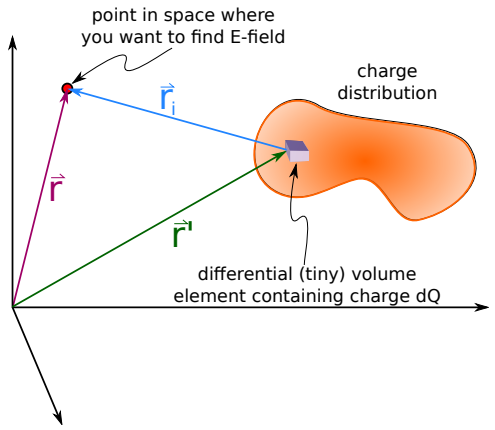

Figure 2: Geometry for analyzing the E-field due to a charge distribution. The *physics* for each little bit of the sum is due to the differential charge element $\mathrm{d}Q$ and the vector $\boldsymbol{r}_i$ pointing from $\mathrm{d}Q$ to the point $\boldsymbol{r}$ where you're trying to find the E-field; hence, these quantities are what show up explicitly in the integrand of Eqn. 2. The integral is taken over all of the charge distribution, so the coordinates that make up $\boldsymbol{r}'$ show up when we break $\mathrm{d}Q$ down and we get a purely geometrical part ($\mathrm{d}x'\,\mathrm{d}y'$ in plane-Cartesian coordinates). So $\boldsymbol{r}'$ is the variable of integration (and so also shows up in the limits of integration), as can be seen in Eqn. 3.

Summarizing this discussion of Coulomb's law for a surface charge density in Cartesian coordinates is the equation

$$\boldsymbol{E}(\boldsymbol{r}) = \int_{y'=?}^{?}\int_{x'=?}^{?} k\frac{\eta(x',y')\,\hat{\boldsymbol{r}}_i}{{r_i}^2}\,\mathrm{d}x'\,\mathrm{d}y'. \tag{3}$$

If you're using polar coordinates, e.g. plane polar coordinates (r',θ'), remember that *an angle has no units* (sadly for radians, they just don't count!), so $\mathrm{d}\theta'$ has no units and hence the differential "volume" **cannot just involve** $\mathrm{d}r'$ **and** $\mathrm{d}\theta'$. Of course all confusion is avoided when you draw a picture of your differential "volume" element and you identify what the lengths of each side are, based upon that picture; see Fig. 3.

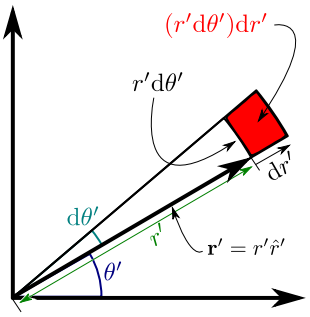

Figure 3: Geometry for finding a differential "volume" element in 2D using plane-polar coordinates. Tweak the vector's coordinate θ' (angle) a little (by an amount $\mathrm{d}\theta'$), and the vector rotates a bit, tracing out an arc of length $r'\,\mathrm{d}\theta'$. Tweak the vector's coordinate r' (magnitude) a little (by an amount $\mathrm{d}r'$), and the vector gets longer, tracing out the length $\mathrm{d}r'$. Multiply these, and the differential "volume" element has an area $(r'\,\mathrm{d}\theta')\,\mathrm{d}r'$.

3 Gauss's law

If a charge Q is present somewhere in space, it "produces" an E-field. Surrounding this charge with a closed surface, and using Coulomb's Law along with the divergence theorem from vector calculus, it can be shown that the *total electric flux*, $\Phi_E = \oint \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{A}$, flowing out through that surface is just Q/ϵ_0:

$$\Phi_{\mathrm{E,\,through\,closed\,surf}} = \frac{1}{\epsilon_0}Q, \tag{4}$$

which is the same as saying

$$\oint_{\substack{\text{closed}\\\text{surf}}} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{A} = \frac{1}{\epsilon_0}Q. \tag{5}$$

This is true no matter *where* the charge is inside the surface or what *shape* or *size* the surface has, so long as it's a **closed surface that contains the charge**.

It is not hard to imagine, just like with Coulomb's law, that this can generalize from a single point charge to a continuous distribution of charge. For a given closed surface, the electric flux through it will just be the total of all the bits of charge contained within the surface scaled by the factor $1/\epsilon_0$. And a closed surface that lives in 3D encloses a *volume* there, so the basic generalization of Eqn. 5,

$$\oint_{\substack{\text{closed}\\\text{surf}}} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{A} = \frac{1}{\epsilon_0}Q_{\mathrm{encl}}, \tag{6}$$

can also be written with Q_{encl} in terms of a volume integral over the differential bits of charge enclosed by the surface,

$$\oint_{\substack{\text{closed}\\\text{surf}}} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{A} = \frac{1}{\epsilon_0}\iiint_{\mathrm{encl.vol}} \mathrm{d}Q. \tag{7}$$

With Coulomb's law, you probably just integrated over all the charge (or the unique part of a symmetrical distribution). With Gauss's law, you get to choose a surface over which to integrate—but the key is to choose a surface that leaves just one component $\boldsymbol{E}$ (or some symmetric components of E) as *constant* and the *only unknown* on the surface you've chosen. This is necessary because you only have one equation (Eqn. 6/7) to work with. If a charge distribution doesn't exhibit symmetry, or if you choose a Gaussian surface that doesn't take advantage of a distribution's symmetry, you won't end up with just one part of $\boldsymbol{E}$ as unknown, and the integral on the LHS of (6) can't be done.

For example: With a spherically-symmetric charge distribution, choosing a spherical surface centered at the charge distribution's center leaves only the magnitude of the E-field's radial component unknown. The E-field vectors *only* point radially (due to symmetry), so there's nothing else about $\boldsymbol{E}$ to figure out. See Fig. 4 for more details on choosing a Gaussian surface for spherically symmetric charge

distribution. Recall that $\mathrm{d}Q = \rho\,\mathrm{d}V$, and in spherical coordinates, $\rho = \rho(r,\theta,\phi) \equiv \rho_{r,\theta\,\phi}$ and $\mathrm{d}V = \mathrm{d}r\,(r\,\mathrm{d}\theta)\,(r\sin\theta\,\mathrm{d}\phi)$ in general, so Eqn. 7 becomes

$$\begin{aligned}
\oint_{\substack{\text{spherical}\\\text{shell}}} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{A} &= \frac{1}{\epsilon_0}\iiint_{\substack{\text{inside}\\\text{sphere}}} \mathrm{d}Q \\
\oint_{\theta,\phi} (\|\boldsymbol{E}\|\hat{\boldsymbol{r}})\cdot(\mathrm{d}A\,\hat{\boldsymbol{r}}) &= \frac{1}{\epsilon_0}\iiint_{r,\theta,\phi} \rho_{r,\theta,\phi}\,\mathrm{d}r\,(r\,\mathrm{d}\theta)\,(r\sin\theta\,\mathrm{d}\phi) \\
\oint_{\theta,\phi} \|\boldsymbol{E}\|\,\mathrm{d}A &= \frac{1}{\epsilon_0}\iiint_{r,\theta,\phi} \rho_{r,\theta,\phi}\,r^2\,\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
\|\boldsymbol{E}\| \oint_{\theta,\phi} dA &= \frac{1}{\epsilon_0}\iiint_{r,\theta,\phi} \rho_{r,\theta,\phi}\,r^2\,\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
\|\boldsymbol{E}\|\,(4\pi r^2) &= \frac{1}{\epsilon_0}\iiint_{r,\theta,\phi} \rho_{r,\theta,\phi}\,r^2\,\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
\|\boldsymbol{E}\| &= \frac{1}{4\pi\epsilon_0 r^2}\iiint_{r,\theta,\phi} \rho_{r,\theta,\phi}\,r^2\,\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi.
\end{aligned}$$

The last integral on the right might look intimidating, but it will almost always simplify (in this class at least). If $\rho = \text{const}$, then $\rho_{r,\theta\,\phi} = \rho = \text{const}$ and

$$\begin{aligned}
\iiint_{r,\theta,\phi} \rho_{r,\theta,\phi}\,r^2\,\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi &= \iiint_{r,\theta,\phi} \rho\,\mathrm{d}r\,(r\,\mathrm{d}\theta)\,(r\sin\theta\,\mathrm{d}\phi) \\
&= \rho\underbrace{\iiint_{r,\theta,\phi} \mathrm{d}r\,(r\,\mathrm{d}\theta)\,(r\sin\theta\,\mathrm{d}\phi)}_{\text{volume of a sphere}} \\
&= \rho\times\left(\frac{4}{3}\pi r^3\right).
\end{aligned}$$

Even easier than starting with the expression in spherical coordinates, though, you could start way back at $\mathrm{d}Q = \rho\,\mathrm{d}V$, and

$$\begin{aligned}
\int_{\text{vol}} \rho\,\mathrm{d}V &= \rho\int_{\text{vol}} \mathrm{d}V \\
&= \rho\times(\text{vol}) \\
&= \rho\times\left(\frac{4}{3}\pi r^3\right).
\end{aligned}$$

Let's do the same thing, but with the charge density a function of only the radial distance, r: $\rho(r,\theta\,\phi) = \rho(r) \equiv \rho_r$. First, recognize that ρ is constant everywhere on a spherical shell (since r is constant on the shell), so your differential volume element can be a spherical shell with just a tiny (differential) bit of thickness: $\mathrm{d}r$. So $\mathrm{d}Q = \rho(r)\,\mathrm{d}V$ still, but now $\mathrm{d}V = (\text{surface area of shell})\cdot\mathrm{d}r = 4\pi r^2\,\mathrm{d}r$.

$$\begin{aligned}
\iiint_{r,\theta,\phi} \cdots &= \iiint_{r,\theta,\phi} \rho\,\mathrm{d}r\,(r\,\mathrm{d}\theta)\,(r\sin\theta\,\mathrm{d}\phi) \\
&= \rho\underbrace{\iiint_{r,\theta,\phi} \mathrm{d}r\,(r\,\mathrm{d}\theta)\,(r\sin\theta\,\mathrm{d}\phi)}_{\text{volume of a sphere}} \\
&= \rho\times(\text{volume of a sphere}) \\
&= \rho\times\left(\frac{4}{3}\pi r^3\right)
\end{aligned}$$

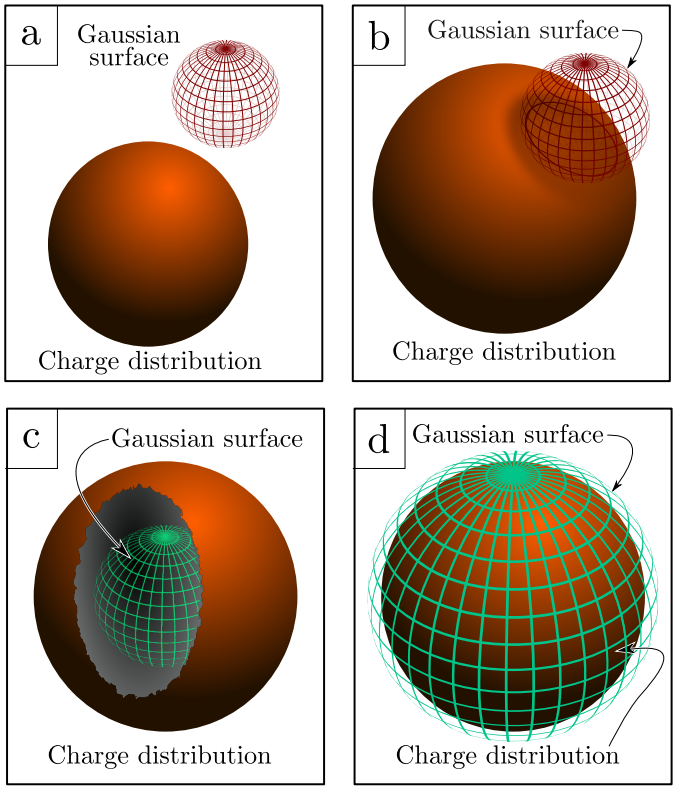

Figure 4: Choices of Gaussian surface for spherically-symmetric charge distribution.
(a) No enclosed charge and *not* capturing symmetry. There is no net flux in or out, but there is non-zero E-field on its surface; $\Phi_E = 0$ does *not necessarily imply* $\boldsymbol{E} = 0$ for any Gaussian surface you've chosen!
(b) Some enclosed charge and *not* capturing symmetry. Net flux is non-zero and E-field is non-zero on the surface, but $\|\boldsymbol{E}\|$ varies across the surface and the direction $\boldsymbol{E}$ points is not just radial with respect to the surface. Therefore, you cannot find the E-field using this surface and Gauss's law.
(c) May enclose charge and captures symmetry (shares center with charge distribution). If charge doesn't exist inside this surface, the net flux is zero *and* the E-field is zero—due to the symmetry. If there *is* charge inside, the symmetry indicates that $\boldsymbol{E}$ has the same magnitude everywhere on the surface and is pointing radially outward, parallel with the surface normals *everywhere* on the surface. Only one value, $\|\boldsymbol{E}\| = \text{const}$, is unknown, and Gauss's law gives you that.
(d) Encloses charge and captures symmetry. Analysis is the same as in (b), but surface area and so $\|\boldsymbol{E}\|$ will have different values as in that case.