Distributed Systems Assignment 2
Part 2

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1. 1) The Petri net given in this question has only four reachable markings, M_1, M_2, M_3, and M_4, as demonstrated in Figures 1 and 2. Asterisks ('*') denote enabled transitions. In the initial marking, M_1, only the transition t_2 may fire, which leads the Petri net into marking M_2. In marking M_2, only t_1 may fire, so the Petri net transitions into marking M_3. In marking M_3, the only enabled transition is t_3, which transforms the Petri net into marking M_4. After marking M_4, the Petri net transitions back into marking M_1, as t_1 is the only enabled transition.

   Since in every one of these four markings, there is only one enabled transition, the Petri net is deterministic and the sequence of markings repeats infinitely

   $$M_1 \to M_2 \to M_3 \to M_4 \to M_1 \to ... \tag{1}$$

   According to the English Wikipedia article on Petri nets, a Petri net is L_k-live if and only if all of its transitions are L_k live. In this case, the definition of L_3-liveness applies to the Petri net under investigation, as there is the infinite firing sequence (1) in which every transition fires infinitely often.

   2) We produced a reachability graph of the second Petri net in this exercise by using WoPed. The result can be found in the bitmap image `Question2.png`. As the reachability graph shows, all of the 17 markings reachable from the initial marking belong to the same strongly connected graph (component). This Petri net is at least L_2-live, because every transition can be fired arbitrarily often, as the reachability graph shows.

   Consider the vertex "`( p1 p3 p6...`". All cycles in the reachability graph pass through it. The only incoming edge to this vertex is labeled t_3, which means that all cycles have to contain t_3, making t_3 satisfy L_3-liveness. This vertex has two outgoing edges labeled t_1 and t_4. The destination of the latter has only one outgoing edge, labeled as t_1. Thus, the transition t_1 also satisfies L_3-liveness. A similar argument starting from the vertex "`( p1 p3 p6...`" on the bottom-right side of the reachability graph shows that t_5 is also L_3 live. It can be reasoned further, that t_2 and t_4 are also L_3-live, by considering one step farther in the reachability graph.

   3) We also produced a reachability graph for the third Petri net using WoPeD, and saved it to the bitmap image `Question3.png`. It is more complex than in the previous case, but it can be seen that for instance, the edges 1 and

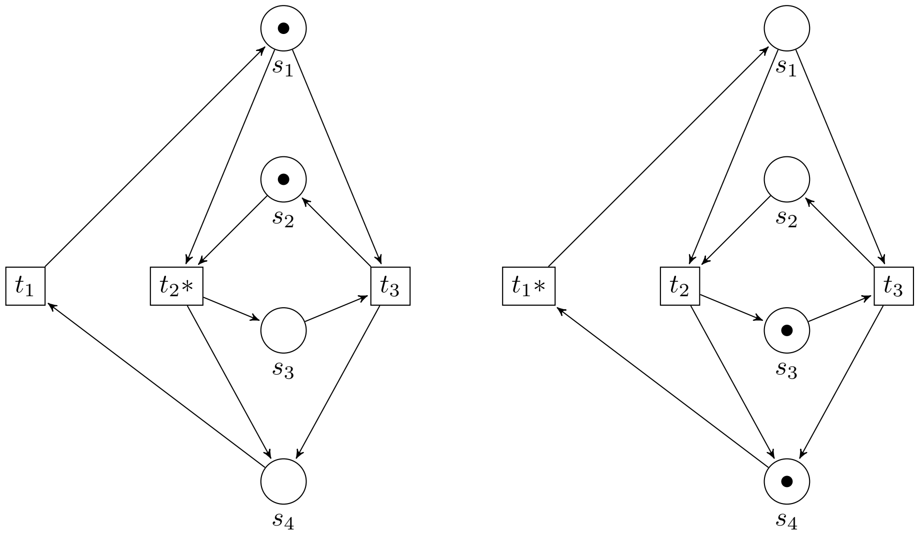

Figure 1: The fist two markings (M_1 on the left and M_2 on the right).

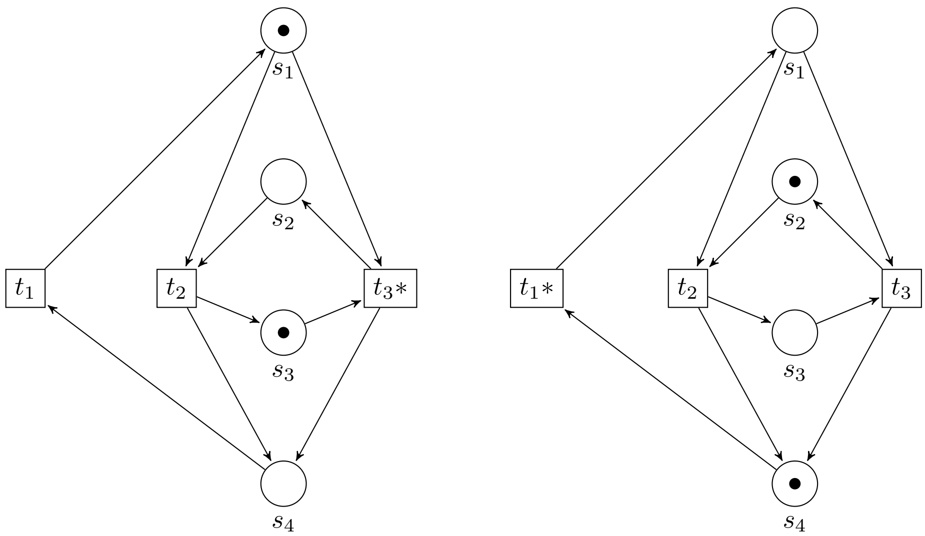

Figure 2: The next two markings (M_3 on the left and M_4 on the right).

106 both represent the marking

$$p_1, p_2, p_5, p_6, p_9, p_{10}$$

and the path between these edges is non-trivial. This means that the path represents a cycle in a firing sequence, so the second Petri net is live at least in the sense that there exists an infinite firing sequence, namely the one that repeats this cycle forever.

2. 1) From the Figures 1 and 2 it shows that all of the places in the first Petri net are 1-bounded.

   2) Because the reachability graph of the second Petri net is finite, it follows that every firing sequence keeps cycling through a finite number of markings. For every place, the maximum bound of tokens can be determined from the reachability graph, by checking which of the markings place the largest number of tokens in that place. In this case, all places happen to be 1-bounded.

   3) The transitions of the third Petri net can be represented as the following 12×7 matrix:

$$\mathbf{C} = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 \end{pmatrix}$$

   Since all columns sum to zero, it holds that

$$(1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1) \times \mathbf{C} = \mathbf{0},$$

   regardless of the initial marking. It follows by a known result[1] that the third Petri net is bounded.

3. 1) The marking, in which all odd-numbered places have tokens, is M_3. The shortest firing sequence that reaches M_3 from the initial marking M_1 is

$$M_1 \to M_2 \to M_3.$$

   2) For the second petri net, the marking in which all the odd-numbered places have a token is the one where the tokens are in places p_1, p_3, p_5, and p_7. A shortest firing sequence to this marking has to remove the token from s_6. The only transition that can do this is t_4. Similarly, the only

[1] http://wwwis.win.tue.nl/~wvdaalst/old/courses/BIScourse/BIS-12-structural-subclasses.pdf, p. 35

transition that can remove the token from s_2 is t_2. These two transitions are independent of each other, so it doesn't matter which one fires first in a shortest firing sequence. After t_2, t_1 needs to fire and the place s_8 has to have a token in order to enable t_3. After t_3 has fired, the part of the Petri net to the right of t_3 has the correct marking. At this point, it remains to remove tokens from s_2 and s_4 and move tokens to places s_1 and s_3. This can be accomplished by firing the transitions t_1, t_2, and t_1 again.

The three firing sequence fitting the description above, described in terms of transitions, are

$$\begin{aligned}&t_2, t_1, t_4, t_3, t_1, t_2, t_1;\\&t_2, t_4, t_1, t_3, t_1, t_2, t_1; \text{ and}\\&t_4, t_2, t_4, t_3, t_1, t_2, t_1.\end{aligned}$$

Therefore, the length of the shortest firing sequences leading to the marking where all odd-numubered places have a token is 7.

3) Following the approach described above, for the shortest firing sequence leading to the marking that puts tokens in odd-numbered places starts with any permutation of t_2, t_4, and t_6 followed by the firing sequence

$$t_1, t_3, t_5, t_1, t_2, t_1, t_3, t_4, t_1, t_2, t_1, t_3, t_1, t_2, t_1.$$

All of these six shortest paths have length 18.

4. To mark the odd-numbered places in the nth member, the following algorithm can be used:

- The two places with the highest indices have to be marked in the nth member by firing the transition t_{2n}.
- Also, the transitions t_{2n-2} and t_{2n-3} must be fired.
- After performing the previous two steps, firing the transition t_{2n-1} marks the odd-numbered places in the nth member.

There are two possible firing sequences that accomplish the desired marking:

$$\begin{aligned}&t_{2n}, t_{2n-2}, t_{2n-3}, t_{2n-1}\\&t_{2n-2}, t_{2n-3}, t_{2n}, t_{2n-1}\end{aligned}$$

Repeating this algorithm for all members in decreasing order produces a shortest firing sequence to a marking with tokens in all odd-numbered places.