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Template Library: Theory of Computation

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0.1 Sets and Set Operations

Membership and common sets:

$$x \in A, \quad y \notin B, \quad \mathbb{N}, \quad \mathbb{Z}, \quad \mathbb{R}, \quad \emptyset$$

Operations:

$$A \cup B, \quad A \cap B, \quad A \setminus B, \quad \overline{A}, \quad A \times B$$

Subset and proper subset:

$$A \subseteq B, \quad A \subset B, \quad A \supseteq B$$

Power set:

$$\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

Cardinality:

$$|A| = n, \quad |\mathcal{P}(A)| = 2^{|A|}$$

Set-builder notation:

$$A = \{x \in \mathbb{N} \mid x > 5\}$$

0.2 Logic and Quantifiers

Quantifiers:

$$\forall x \in A, \quad \exists y \in B, \quad \nexists z \in C$$

Logical connectives:

$$P \wedge Q, \quad P \vee Q, \quad \neg P, \quad P \implies Q, \quad P \iff Q$$

Contrapositive pattern (useful for proofs):

$$(P \implies Q) \iff (\neg Q \implies \neg P)$$

0.3 Functions

Function signature:

$$f : A \rightarrow B$$

Injective (one-to-one):

$$\forall a, b \in A, a \neq b \implies f(a) \neq f(b)$$

Surjective (onto):

$$\forall b \in B, \exists a \in A : f(a) = b$$

Bijjective:

f is injective and surjective

Composition:

$$(g \circ f)(x) = g(f(x))$$

0.4 Strings and Languages

Alphabet, string, empty string:

$$\Sigma = \{0, 1\}, \quad w \in \Sigma^*, \quad \varepsilon$$

String length and concatenation:

$$|w|, \quad w_1 \cdot w_2, \quad w^R \text{ (reversal)}$$

Language operations:

$$L_1 \cup L_2, \quad L_1 \cap L_2, \quad L_1 \cdot L_2, \quad L^*, \quad L^+, \quad \overline{L} = \Sigma^* \setminus L$$

Language defined by set-builder:

$$L = \{w \in \{0, 1\}^* \mid w \text{ contains an even number of 0s}\}$$

0.5 Direct Proof

Claim 1. *For all $n \in \mathbb{N}$, if n is even, then n^2 is even.*

Proof. Let n be an even natural number. Then $n = 2k$ for some $k \in \mathbb{N}$. Therefore $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$, which is even. ■

0.6 Proof by Contraposition

Claim 2. $\forall a, b \in \mathbb{N}, a \neq b \implies f(a) \neq f(b)$.

Proof. We prove the contrapositive: $f(a) = f(b) \implies a = b$.

Suppose $f(a) = f(b)$. Then

$$f(a) = f(b) \quad (\text{assumption})$$

$$a - 2 = b - 2 \quad (\text{by definition of } f)$$

$$a = b \quad (\text{adding 2 to both sides})$$

Since $(P \implies Q) \iff (\neg Q \implies \neg P)$, the original statement holds. ■

0.7 Proof by Contradiction

Claim 3. $\sqrt{2}$ is irrational.

Proof. Assume for contradiction that $\sqrt{2}$ is rational. Then $\sqrt{2} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$, and $\gcd(p, q) = 1$.

Squaring both sides: $2 = \frac{p^2}{q^2}$, so $p^2 = 2q^2$. Thus p^2 is even, so p is even. Write $p = 2k$. Then $4k^2 = 2q^2$, so $q^2 = 2k^2$, meaning q is also even.

But this contradicts $\gcd(p, q) = 1$. $\Rightarrow \times$ ■

(Note: define `\contradiction` in preamble as `\newcommand{\contradiction}{\rightarrow!}` or use `\lightning` from `stmaryrd`.)

0.8 Proof by Induction

Claim 4. $\forall n \geq 1, \sum_{i=1}^n i = \frac{n(n+1)}{2}$.

Proof. **Base case** ($n = 1$): $\sum_{i=1}^1 i = 1 = \frac{1 \cdot 2}{2}$. ✓

Inductive hypothesis: Assume the claim holds for some $k \geq 1$:

$$\sum_{i=1}^k i = \frac{k(k+1)}{2} \quad (\text{I.H.})$$

Inductive step: We show it holds for $k + 1$.

$$\begin{aligned}
\sum_{i=1}^{k+1} i &= \left(\sum_{i=1}^k i \right) + (k + 1) \\
&= \frac{k(k + 1)}{2} + (k + 1) \quad \text{by (I.H.)} \\
&= \frac{k(k + 1) + 2(k + 1)}{2} \\
&= \frac{(k + 1)(k + 2)}{2}
\end{aligned}$$

By the principle of mathematical induction, the claim holds for all $n \geq 1$. ■

0.9 Pumping Lemma Proof (Regular Languages)

Claim 5. $L = \{0^n 1^n \mid n \geq 0\}$ is not regular.

Proof. Assume for contradiction that L is regular. Let p be the pumping length given by the Pumping Lemma.

Choose $s = 0^p 1^p \in L$. Since $|s| = 2p \geq p$, the Pumping Lemma guarantees $s = xyz$ where:

1. $|y| > 0$,
2. $|xy| \leq p$,
3. $\forall i \geq 0, xy^i z \in L$.

Since $|xy| \leq p$, y consists entirely of 0s. Write $y = 0^k$ for some $k \geq 1$.

Consider $xy^2 z = 0^{p+k} 1^p$. Since $k \geq 1$, we have $p + k > p$, so $xy^2 z \notin L$. This contradicts condition 3.

Therefore L is not regular. ■

0.10 Pumping Lemma Proof (Context-Free Languages)

Claim 6. $L = \{a^n b^n c^n \mid n \geq 0\}$ is not context-free.

Proof. Assume for contradiction that L is context-free. Let p be the pumping length given by the Pumping Lemma for CFLs.

Choose $s = a^p b^p c^p \in L$. Since $|s| = 3p \geq p$, we can write $s = uvxyz$ where:

1. $|vy| > 0$,
2. $|vxy| \leq p$,
3. $\forall i \geq 0, uv^i xy^i z \in L$.

Since $|vxy| \leq p$, the substring vxy cannot span all three symbols a, b, c . Therefore pumping ($i = 2$) will increase the count of at most two of the three symbols, making $uv^2 xy^2 z \notin L$. Contradiction. ■

0.11 Equation Environments

Unnumbered display math (no label/ref possible):

$$a + b = c$$

Numbered equation (can label and reference):

$$E = mc^2 \tag{1}$$

Reference: Equation (1).

Multi-line aligned at equals, selectively numbered:

$$\begin{aligned} f(x) &= (x + 1)^2 \\ &= x^2 + 2x + 1 \end{aligned} \tag{2}$$

Fully unnumbered multi-line:

$$\begin{aligned} a &= b + c \\ d &= e + f \end{aligned}$$

Annotated derivation (double alignment):

$$\begin{aligned} f(c) &= f(d) && \text{(given)} \\ c - 2 &= d - 2 && \text{(by definition)} \\ \therefore c &= d && \text{(by algebra)} \end{aligned}$$

Custom-tagged equation:

$$a = k - 2 \quad \text{(Inductive Hypothesis)}$$

Cases (piecewise functions):

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ f(n-1) + f(n-2) & \text{if } n \geq 2 \end{cases}$$

Inline condition after display math:

$$\begin{array}{l} a = c \\ b = d \end{array} \quad \text{where } c, d \in \mathbb{N}.$$

0.12 Matrices

Parenthesized matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Bracketed matrix:

$$B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Determinant (vertical bars):

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Augmented matrix (useful for Gaussian elimination):

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array} \right]$$

0.13 Transition Table (DFA)

δ	0	1
$\rightarrow q_0$	q_1	q_2
q_1	q_1	q_3
q_2	q_3	q_2
$*q_3$	q_3	q_3

Convention: \rightarrow marks the start state, $*$ marks accepting states.

0.14 Transition Table (NFA)

δ	0	1	ε
$\rightarrow q_0$	$\{q_0, q_1\}$	$\{q_0\}$	\emptyset
q_1	\emptyset	$\{q_2\}$	$\{q_2\}$
$*q_2$	\emptyset	\emptyset	\emptyset

0.15 Truth Table

P	Q	$P \wedge Q$	$P \vee Q$	$P \implies Q$
T	T	T	T	T
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

0.16 Diagonalization Table

Used in Cantor's proof that the set of infinite binary sequences is uncountable. Suppose for contradiction that $f : \mathbb{N} \rightarrow \{0, 1\}^\omega$ is a bijection. List the supposed enumeration:

	b_1	b_2	b_3	b_4	b_5	\dots
$f(1)$	\bar{d}	0	1	0	1	\dots
$f(2)$	1	\bar{d}	0	1	1	\dots
$f(3)$	0	1	\bar{d}	0	0	\dots
$f(4)$	1	1	0	\bar{d}	1	\dots
$f(5)$	0	0	1	1	\bar{d}	\dots
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots

Construct $d = d_1d_2d_3\dots$ where $d_i \neq f(i)_i$ (i.e., flip the i -th bit of the i -th sequence). The diagonal entries \bar{d} are the bits we flip. Then d differs from every $f(n)$ in position n , so $d \notin \{f(1), f(2), \dots\}$, contradicting surjectivity.

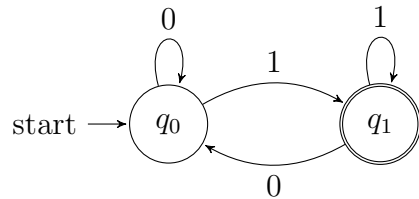
The same structure applies to proving A_{TM} is undecidable. Suppose H decides A_{TM} . Build a table of TMs vs. their own descriptions:

	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	\dots
M_1	accept	reject	accept	reject	\dots
M_2	accept	accept	accept	accept	\dots
M_3	reject	reject	reject	accept	\dots
M_4	accept	accept	reject	reject	\dots
\vdots	\vdots	\vdots	\vdots	\vdots	\ddots
D	reject	reject	accept	accept	\dots

Entry $(M_i, \langle M_j \rangle)$ shows whether M_i accepts $\langle M_j \rangle$. The diagonal is $H(M_i, \langle M_i \rangle)$ —does M_i accept its own description? The machine D flips the diagonal: $D(\langle M_i \rangle)$ does the opposite of M_i on $\langle M_i \rangle$. Then $D(\langle D \rangle)$ must both accept and reject—contradiction.

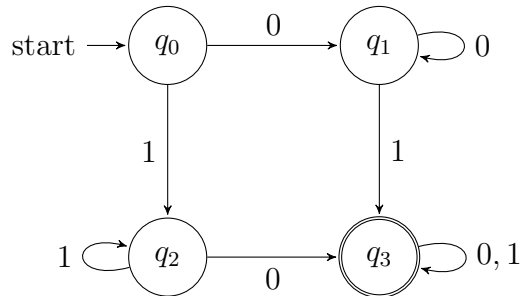
0.17 DFA: Simple Linear

A DFA that accepts strings ending in 1 over $\Sigma = \{0, 1\}$.



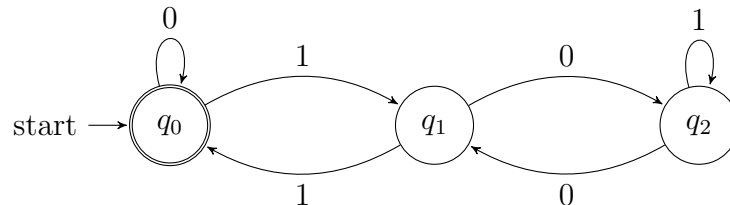
0.18 DFA: Grid Layout

A DFA with states in a 2x2 grid (e.g., tracking two properties).



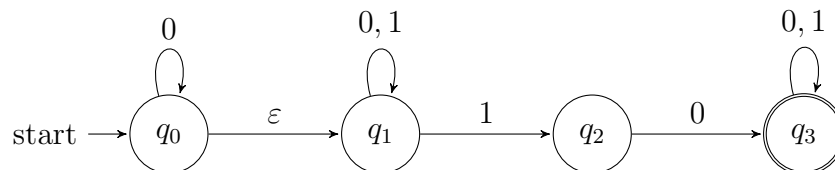
0.19 DFA: Modular Arithmetic (mod 3)

Accepts binary strings whose numeric value is divisible by 3.



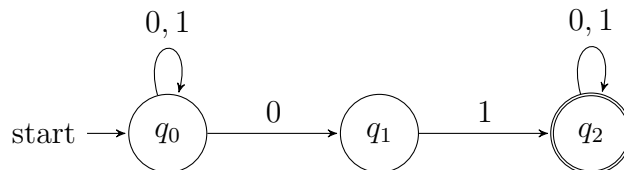
0.20 NFA: With Epsilon Transitions

An NFA with ε -transitions.



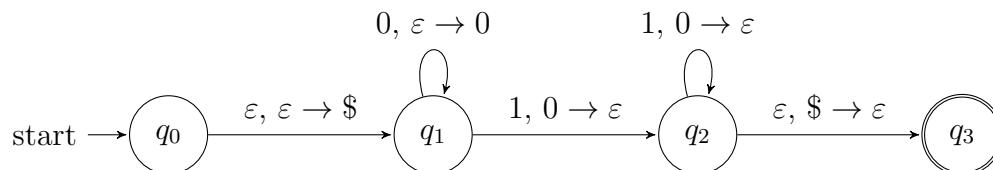
0.21 NFA: Nondeterministic Branching

An NFA that accepts strings containing the substring “01”.



0.22 PDA: Pushdown Automaton

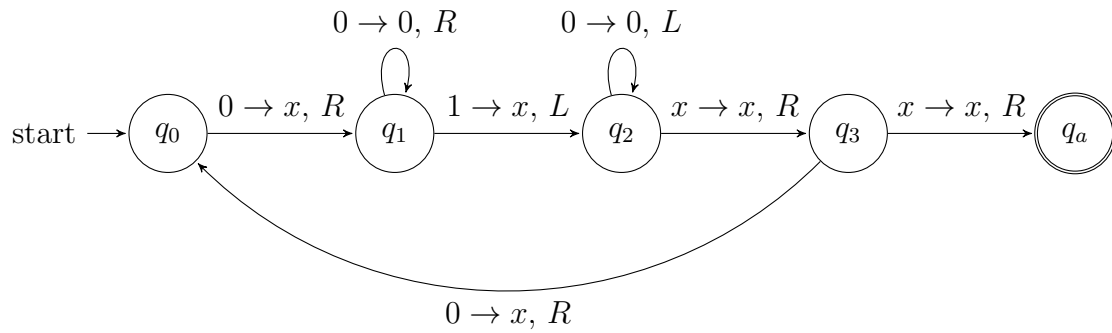
A PDA for $\{0^n 1^n \mid n \geq 0\}$.



PDA transitions use the notation: input, pop \rightarrow push.

0.23 Turing Machine

A TM that accepts $\{0^n 1^n \mid n \geq 1\}$.



TM transitions use the notation: read \rightarrow write, direction.

0.24 Context-Free Grammar

A CFG for $\{0^n 1^n \mid n \geq 0\}$:

$$S \rightarrow 0S1 \mid \varepsilon$$

A CFG for balanced parentheses:

$$S \rightarrow SS \mid (S) \mid \varepsilon$$

A more elaborate grammar with multiple productions:

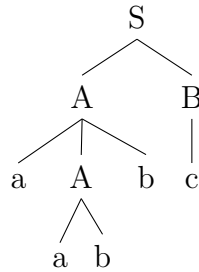
$$S \rightarrow AB \mid \varepsilon$$

$$A \rightarrow aAb \mid ab$$

$$B \rightarrow cB \mid c$$

0.25 Parse Tree

Using `tikz-qtree` for parse trees:



0.26 Derivation Steps

A leftmost derivation:

$$\begin{aligned}
 S &\Rightarrow AB && \text{(rule } S \rightarrow AB\text{)} \\
 &\Rightarrow aAbB && \text{(rule } A \rightarrow aAb\text{)} \\
 &\Rightarrow aabbB && \text{(rule } A \rightarrow ab\text{)} \\
 &\Rightarrow aabbc && \text{(rule } B \rightarrow c\text{)}
 \end{aligned}$$

Using \Rightarrow^* for multi-step derivation:

$$S \Rightarrow^* aabbc$$

0.27 Regular Expressions

Basic notation:

$$\begin{aligned}
 \Sigma &= \{0, 1\} \\
 L_1 &= 0^*10^* && \text{(strings with exactly one 1)} \\
 L_2 &= (0 \cup 1)^* && \text{(all strings over } \Sigma\text{)} \\
 L_3 &= \Sigma^*01\Sigma^* && \text{(contains substring 01)}
 \end{aligned}$$

Common patterns:

$$\begin{aligned}
 \text{At least one } a: & \quad a\Sigma^* \cup \Sigma^*a \\
 \text{Even length:} & \quad (\Sigma\Sigma)^* \\
 \text{Starts and ends same:} & \quad 0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1 \cup \varepsilon
 \end{aligned}$$

0.28 Complexity Classes

Big-O and related notation:

$$f(n) = O(g(n)), \quad f(n) = \Omega(g(n)), \quad f(n) = \Theta(g(n))$$

Class definitions:

$$\mathbf{P} = \bigcup_{k \geq 0} \text{TIME}(n^k)$$
$$\mathbf{NP} = \bigcup_{k \geq 0} \text{NTIME}(n^k)$$

0.29 Reduction Notation

Mapping reduction:

$$A \leq_m B$$

If $A \leq_m B$ and B is decidable, then A is decidable.

If $A \leq_m B$ and A is undecidable, then B is undecidable.

0.30 Labeled Equation References

Define and reference equations:

$$\forall a, b \in A, f(a) = f(b) \implies a = b \tag{3}$$

As shown in (3), the function is injective.

0.31 Enumerate with Custom Labels

Roman numerals:

- (i) First condition
- (ii) Second condition
- (iii) Third condition

Alphabetical:

- (a) Case a
- (b) Case b

0.32 Verbatim / Pseudocode

```
DECIDE(w):  
  Simulate M on w for  $|w|^2$  steps  
  If M accepts, ACCEPT  
  If M rejects or loops, REJECT
```

0.33 Including External Files

Include a full PDF:

```
\includepdf[pages=-,pagecommand={},width=\textwidth]{solution.pdf}
```

Include an image:

```
\begin{center}  
  \includegraphics[width=0.8\linewidth]{screenshot.png}  
\end{center}
```

0.34 Spacing Reference

<code>\quad</code>	medium space (1em)
<code>\qquad</code>	large space (2em)
<code>\,</code>	thin space
<code>\;</code>	medium-thin space
<code>\medskip</code>	vertical medium skip
<code>\bigskip</code>	vertical big skip
<code>\pagebreak</code>	force new page

0.35 Citations and Bibliography

To cite sources, create a `.bib` file (e.g., `references.bib`) with entries like:

```
@book{sipser2013,  
  author    = {Michael Sipser},  
  title     = {Introduction to the Theory of Computation},  
  edition   = {3},  
  publisher = {Cengage Learning},
```

```

    year      = {2013},
    isbn      = {978-1-133-18779-0}
}

@book{hopcroft2007,
  author      = {John E. Hopcroft and Rajeev Motwani
                 and Jeffrey D. Ullman},
  title       = {Introduction to Automata Theory, Languages,
                 and Computation},
  edition     = {3},
  publisher   = {Pearson},
  year        = {2007},
  isbn       = {978-0-321-45536-9}
}

```

Then add to your preamble:

```

\usepackage[
  backend=biber,
  style=verbose,
  sorting=nyt           % sort by name, year, title
]{biblatex}
\addbibresource{references.bib}

```

Cite in text:

```

By the Pumping Lemma \cite[Theorem 1.70]{sipser2013}, ...
Sipser shows that  $A_{TM}$  is undecidable
\cite[pp.~207--209]{sipser2013}.
See also \cite{hopcroft2007}.

```

Print the bibliography at the end of your document:

```

\printbibliography

```

Build with: `pdflatex main.tex && biber main && pdflatex main.tex`

Alternative (simpler, no biber): Use `\thebibliography` directly—
no `.bib` file needed:


```

\begin{thebibliography}{9}
  \bibitem{sipser2013}
    Michael Sipser,
    \textit{Introduction to the Theory of Computation},
    3rd ed., Cengage Learning, 2013.

  \bibitem{hopcroft2007}
    John E. Hopcroft, Rajeev Motwani, Jeffrey D. Ullman,
    \textit{Introduction to Automata Theory, Languages,
    and Computation},
    3rd ed., Pearson, 2007.
\end{thebibliography}

```

Then cite with `\cite{sipser2013}` or `\cite[Theorem~1.70]{sipser2013}`.
This approach works with a single `pdflatex` pass.

0.36 Useful Symbols Quick Reference

Symbol	Command	Usage
ε	<code>\varepsilon</code>	empty string
\emptyset	<code>\emptyset</code>	empty set
Σ	<code>\Sigma</code>	alphabet
δ	<code>\delta</code>	transition function
\vdash	<code>\vdash</code>	yields / proves
\therefore	<code>\therefore</code>	therefore
\implies	<code>\implies</code>	implies (long arrow)
\iff	<code>\iff</code>	if and only if
\leq_m	<code>\le_m</code>	mapping reducible
\Rightarrow	<code>\Rightarrow</code>	derives (grammar)
$\langle M \rangle$	<code>\langle M \rangle</code>	encoding of M
\overline{L}	<code>\overline{L}</code>	complement
\mathcal{P}	<code>\mathcal{P}</code>	power set
\mathbb{N}	<code>\mathbb{N}</code>	natural numbers
∞	<code>\infty</code>	infinity