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# SHARP CONDITIONS FOR NONEXPLOSIONS AND EXPLOSIONS IN MARKOV JUMP PROCESSES

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We give sharp sufficient conditions for nonexplosions and explosions in Markov pure jump processes in terms of the holding time parameters and moments of the jump distributions.

**Introduction.** We consider a time homogeneous Markov jump process $\{Z_t\}$ in continuous time with the state space $S$, a subset of nonnegative reals. For general description and construction of such processes, see, for example, Breiman [(1968), pages 328–338] and Chung (1967). If the process is in state $z$, then it stays there for an exponential length of time with parameter $\lambda(z)$, after which it jumps from $z$. Let $T_n$ be the time of the $n$th jump of the process, $T_n = \inf\{t > T_{n-1}: Z_t \neq Z_{t-}\}$, $T_0 = 0$. It is said that the process does not explode if there are only finitely many jumps on finite time intervals, in other words, $T_n \to \infty$ as $n \to \infty$. The question of explosion is an important one, since when there are no explosions the forward and backward equations are satisfied and the solutions are unique. To shorten notations, denote by $Z_n$ the embedded jump chain, $Z_n = Z(T_n)$. The well-known necessary and sufficient condition for nonexplosion is

$$\sum_{n=1}^{\infty} \frac{1}{\lambda(Z_n)} = \infty \quad \text{a.s.}; \tag{1}$$

see, for example, Chung [(1967), pages 259–260] and Breiman [(1968), page 337]. However, this condition is hard to check, in general, because it involves the embedded chain. Sufficient conditions for nonexplosion and explosion given below are formulated in terms of the parameters of the process: $\lambda(z)$ and moments of jumps from $z$. Let $\mathscr{F}_t$ be the natural filtration of the process $Z_t$ and $\mathscr{F}_n = \mathscr{F}_{T_n}$.

**Results.**

THEOREM 1. *Let $\lambda(z)$, $z \geq 0$, be bounded on bounded intervals. Let $m(z) = E(Z_{n+1} - Z_n \mid Z_n = z)$ be the mean of jumps from $z$. Suppose there exists a positive, increasing function $f(z)$ such that for all $z \geq 0$,*

$$m(z)\lambda(z) \leq f(z) \quad \text{and} \quad \int_0^{\infty} \frac{dz}{f(z)} = \infty. \tag{2}$$

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*Then*

$$\sum_{n=0}^{\infty} \frac{1}{\lambda(Z_n)} = \infty \quad a.s.$$

The sufficient condition (2) is sharp in the sense that it is the best possible condition in terms of $\lambda(z)$ and $m(z)$ for a class of processes, as the following example shows.

EXAMPLE 1. Let $\lambda(z)$ and $m(z)$, $z \geq 0$, be positive. Define $0 = s_0 < s_1 < s_2 < \cdots$ by $s_{n+1} = s_n + m(s_n)$. Let $Z_t$ be the jump process that jumps from $s_n$ to $s_{n+1}$ after exponential waiting time with parameter $\lambda(s_n)$, $n = 0, 1, 2 \ldots$. Then

$$\sum_{n=0}^{\infty} \frac{1}{\lambda(Z_n)} = \sum_{n=0}^{\infty} \frac{1}{\lambda(s_n)} = \sum_{n=0}^{\infty} \frac{s_{n+1} - s_n}{\lambda(s_n)m(s_n)}. \tag{3}$$

If $m(z)$ is continuous, then $s_n \to \infty$. If, moreover, it satisfies the growth condition $s_{n+1} - s_n < C(s_n - s_{n-1})$ for some constant $C > 0$, then divergence of the series in (3) is equivalent to $\int_0^\infty dz/(m(z)\lambda(z)) = \infty$; for example, Knopp (1956).

As an application we now give an example of processes for which our theorems are directly applicable, whereas it is not easy, if at all possible, to verify explosions directly.

EXAMPLE 2. Consider population dependent Markov branching processes, where an individual in a population of size $z$ lives for an exponentially distributed length of time with parameter $a(z)$ and has a random number of offspring $X(z)$ with distribution on nonnegative integers $p_z(\cdot)$. Then the parameters of the process are $\lambda(z) = za(z)$ and $m(z) = \sum_i ip_z(i)$. Essentially divergence of the integral $\int_1^\infty dz/(zm(z)a(z))$ is necessary and sufficient for nonexplosions of such population processes.

Theorem 1 can be generalized in such a way that symmetric sufficient conditions can be also given for explosions. To do so, note that the embedded chain $\{Z_n\}$ is time homogeneous and denote for $\alpha > 0$,

$$n_\alpha(z) = E\left(\left(\frac{Z_1}{z}\right)^\alpha - 1 \mid Z_0 = z\right),$$

$$n_{-\alpha}(z) = E\left(1 - \left(\frac{z}{Z_1}\right)^\alpha \mid Z_0 = z\right).$$

THEOREM 2. *Let $\lambda(z)$, $z \geq 0$, be bounded on bounded intervals. Suppose there exists a positive function $f(z)$ such that for some $\alpha > 0$ $f(z)z^{\alpha-1}$ is* *increasing and*

$$zn_\alpha(z)\lambda(z) \le f(z) \quad \text{and} \quad \int_0^\infty \frac{dz}{f(z)} = \infty. \tag{4}$$

*Then*

$$\sum_{n=0}^{\infty} \frac{1}{\lambda(Z_n)} = \infty \quad a.s.$$

THEOREM 3. *Suppose there exists a positive function $f(z)$ such that for some $\alpha > 0$, $f(z)z^{-\alpha-1}$ is decreasing and for all $z$ sufficiently large,*

$$zn_{-\alpha}(z)\lambda(z) \ge f(z) \quad \text{and} \quad \int_0^\infty \frac{dz}{f(z)} < \infty. \tag{5}$$

*Then*

$$\sum_{n=0}^{\infty} \frac{1}{\lambda(Z_n)} < \infty \quad a.s. \text{ on } \{Z_n \to \infty\}.$$

REMARKS.

1. Theorem 1 is a particular case of Theorem 2 with $\alpha = 1$.
2. Conditions (4) and (5) get weaker with decrease in $\alpha$. To see this use an elementary inequality $x^\gamma - 1 \ge \gamma(x-1)$ for $\gamma \ge 1$ and all $x$, obtained from convexity. Letting $\gamma = \alpha/\beta$, $x = (Z_1/z)^\beta$, we obtain
$$\beta n_\alpha(z) \ge \alpha n_\beta(z) \quad \text{for } 0 < \beta < \alpha,$$
and with $x = (z/Z_1)^\beta$, we obtain
$$\beta n_{-\alpha}(z) \ge \alpha n_{-\beta}(z) \quad \text{for } 0 < \beta < \alpha.$$
3. Condition (5) implies that $\lambda(z)$ is unbounded. This follows from the bound $n_{-\alpha}(z) \le 1$, which implies $1/f(z) \ge 1/(z\lambda(z))$. Thus $\int_0^\infty dz/f(z) < \infty$ implies that $\lambda(z)$ is unbounded.
4. A relation between conditions (4) and (5) is provided by the inequality
$$\alpha n_\beta(z) \ge \beta n_{-\alpha}(z) \quad \text{for any } \alpha, \beta > 0,$$
which follows from an elementary inequality $1 - x^{-\gamma} \le \gamma(x-1)$, $\gamma > 0$, by taking $x = (Z_1/z)^\beta$ and $\gamma = \alpha/\beta$.
5. To see how well conditions of Theorems 2 and 3 fit together, notice that if $Z_1/z \Rightarrow 1$ in distribution as $Z_0 = z \to \infty$ and $Z_1$ has no particularly long tails, then the following approximations (from a second order Taylor expansion) are valid for $z \to \infty$:
$$zn_\alpha(z)/\alpha \approx m(z) - 1/2(1-\alpha)v(z)/z,$$
$$zn_{-\alpha}(z)/\alpha \approx m(z) - 1/2(1+\alpha)v(z)/z,$$
where $v(z) = E((Z_1 - z)^2 \mid Z_0 = z)$.

**Proofs.**

PROOF OF THEOREM 2. Let $F(z) = \int_0^z 1/(f(x))\,dx$. By monotonicity of $x^{\alpha-1}f(x)$, for any $y > x$,

$$\frac{1}{f(y)} \le \frac{1}{f(x)}\frac{y^{\alpha-1}}{x^{\alpha-1}}.$$

Hence we obtain for *any* $x, z > 0$,

$$F(z) - F(x) = \int_x^z \frac{1}{f(y)}\,dy \le \frac{1}{f(x)}x^{1-\alpha}\frac{(z^\alpha - x^\alpha)}{\alpha} = \frac{x}{\alpha f(x)}\left(\left(\frac{z}{x}\right)^\alpha - 1\right).$$

Let now $U_n = F(Z_n)$ and use the above inequality to obtain

$$E(U_{n+1} \mid \mathscr{F}_n) \le U_n + \frac{Z_n n_\alpha(Z_n)}{\alpha f(Z_n)}.$$

Assumption (4) now gives, with $C = \alpha^{-1}$,

$$E(U_{n+1} \mid \mathscr{F}_n) \le U_n + C/\lambda(Z_n).$$

Thus

$$V_n = U_n - \sum_{k=0}^{n-1}\frac{C}{\lambda(Z_k)}$$

is a supermartingale. Now let $D > 0$ and define a stopping time

$$\tau_D = \inf\{n: \sum_{k=0}^{n}\frac{C}{\lambda(Z_k)} \ge D\}.$$

Then $V_{n\wedge\tau_D}$ is also a supermartingale. Moreover, it is bounded from below by $-D$; thus it converges almost surely to a finite limit. On the event $\{\sum_{k=0}^\infty C/(\lambda(Z_k)) < D\}$: $\tau_D = \infty$ and $\lambda(Z_n) \to \infty$, implying $Z_n \to \infty$ and $F(Z_n) \to F(\infty) = \infty$. Thus $V_n \ge F(Z_n) - D \to \infty$ and $V_{n\wedge\tau_D} \to \infty$. Hence $P(\sum_{k=0}^\infty 1/(\lambda(Z_k)) < D) = 0$. Because $D$ was arbitrary, the result follows. □

PROOF OF THEOREM 3. Because $f(x)x^{-1-\alpha}$ is decreasing, we have the inequality for $y > x$,

$$\frac{1}{f(y)} \ge \frac{1}{f(x)}\left(\frac{y}{x}\right)^{-1-\alpha},$$

from which we obtain for *any* $x, z \ge 0$,

$$F(z) - F(x) \ge \frac{1}{f(x)}x^{1+\alpha}\int_x^z y^{-1-\alpha}\,dy = \frac{x}{\alpha f(x)}\left(1 - \left(\frac{x}{z}\right)^\alpha\right).$$

We obtain from this inequality with $U_n = F(Z_n)$,

$$E(U_{n+1} \mid \mathscr{F}_n) \ge U_n + \frac{Z_n n_{-\alpha}(Z_n)}{\alpha f(Z_n)}.$$

Assumption (5) gives

$$E(U_{n+1} \mid \mathscr{F}_n) \geq U_n + C/\lambda(Z_n) \tag{6}$$

for sufficiently large $Z_n$, $Z_n > C_1$, say. Consider the martingale

$$W_n = -U_n + \sum_{k=0}^{n-1}(E(U_{k+1} \mid \mathscr{F}_k) - U_k). \tag{7}$$

Let $\tau_N$ be the $N$th time that sequence $Z_n$ falls below $C_1$,

$$\tau_N = \inf\{n > \tau_{N-1}: Z_n \leq C_1\},$$

and as usual $\tau_N = \infty$ if $\{Z_n\}$ falls below $C_1$ less than $N$ times. Then as $-U_n \geq -F(\infty)$, it follows from (6) and (7) and condition (5) that

$$W_{n\wedge\tau_N} \geq -(N+1)F(\infty) > -\infty.$$

Thus $W_{n\wedge\tau_N}$ converges a.s. Thus $W_n$ converges a.s. on the event $\{\tau_N = \infty\}$. Hence $\sum_{k=0}^{\infty}(E(U_{k+1} \mid \mathscr{F}_k) - U_k)$ converges a.s. on this event. Letting $N \to \infty$, we obtain that $\sum_{k=0}^{\infty}(E(U_{k+1} \mid \mathscr{F}_k) - U_k)$ converges a.s. on $\{Z_n \to \infty\}$. Using the bound in (6), the result follows.

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