

# Quantum Computing

## A brief introduction

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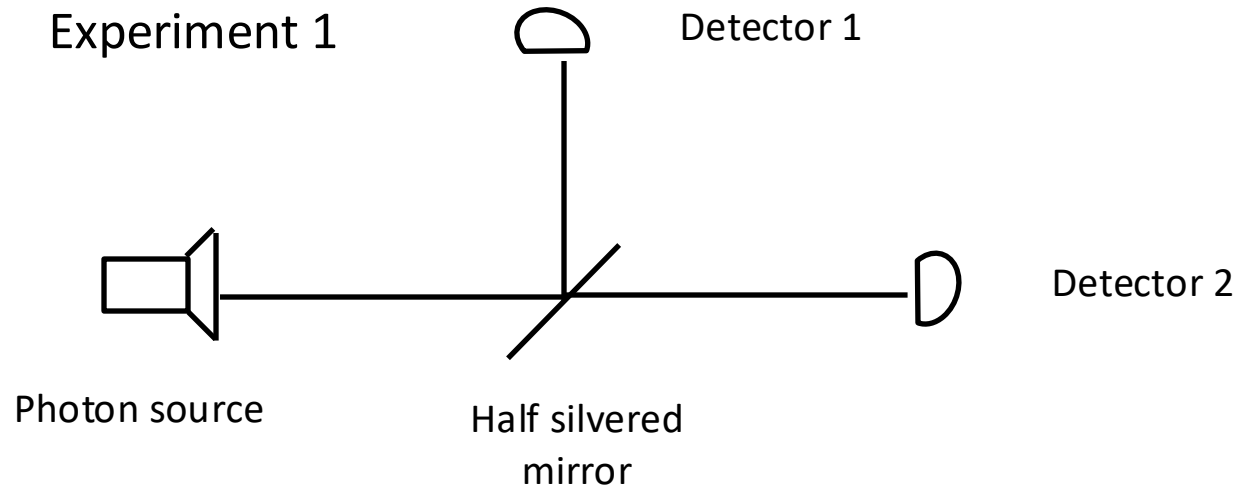
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# Beam splitters and QM

I can safely say that no one understands Quantum Mechanics - Feynman



Photon source emits stream of photons.

$P(\text{photon arrives at Detector 1}) = .5$

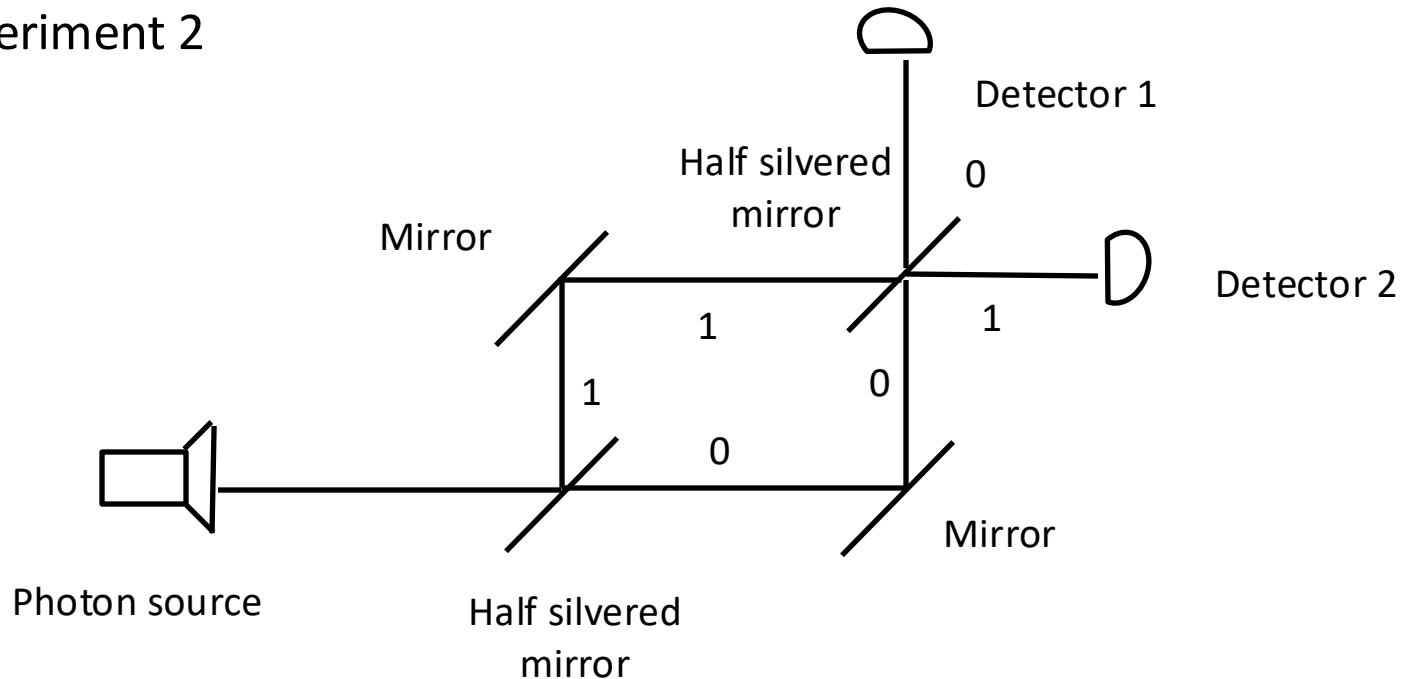
$P(\text{photon arrives at Detector 2}) = .5$

So far, so good

# Beam splitters and QM

## Mach-Zender Interferometer

### Experiment 2



Photon source emits stream of photons.

$P(\text{photon arrives at Detector 1}) = 0$

$P(\text{photon arrives at Detector 2}) = 1$

Huh?

# According to QM

## Analysis

Beam splitter causes the photon to go into superposition:

$$\alpha_1|0\rangle + \alpha_2|1\rangle, |\alpha_1|^2 = \frac{1}{2}, |\alpha_2|^2 = \frac{1}{2}. |0\rangle \text{ state is right, } |1\rangle \text{ is up.}$$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Beam splitter acts on incoming state via the matrix  $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ .

In experiment 1, if all photons leave source in state  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ , after the splitter they are in state  $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$ . So, they arrive at detector 1 with probability  $\frac{1}{2}$  and detector 2 with probability  $\frac{1}{2}$ .

However, going through another beam splitter, in experiment 2, yields the output state:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix}.$$

So, they always arrive at detector 2.

# Postulates

1. State of a system is a unit vector over  $\mathbb{C}$  in Hilbert space ( $\mathcal{H}$ ) of dimension  $2^n$ 
  - A qubit is a quantum system, with  $n = 1$ . A one qubit system is in general state  $|\psi\rangle = a|0\rangle + b|1\rangle$ ,  $a\bar{a} + b\bar{b} = 1$
2. A system, with state,  $|\psi(t)\rangle$ , evolves according to a unitary operator, namely,  $U(|\psi(0)\rangle)$ 
  - $U$  is unitary if  $(x, y) = (Ux, Uy)$ . Note  $UU^\dagger = I$
  - Example is a Hamiltonian:  $H(t)|\psi(t)\rangle = i\hbar \frac{d|\psi(t)\rangle}{dt}$
  - $|\varphi(t_2)\rangle = e^{-i\hbar H(t_2-t_1)}|\varphi(t_1)\rangle$
3. Each observable is represented by a Hermitian operator,  $\hat{Q}$ , the expectation value of  $\hat{Q}$  is  $\langle\psi|\hat{Q}|\psi\rangle$ . The outcome of a measurement of the operator is an eigenvalue of the operator. The probability of getting a particular eigenvalue,  $\lambda$ , is the square of the  $\lambda$ -component of  $|\psi\rangle$

# Postulates

4. Two physical systems  $\mathcal{H}_1$  and  $\mathcal{H}_2$  can be treated as a single system,  $\mathcal{H}_1 \otimes \mathcal{H}_2$ . If  $\mathcal{H}_1$  is in state,  $|\psi_1\rangle$  and  $\mathcal{H}_2$  is in state,  $|\psi_2\rangle$ , the joint state is  $|\psi_1\rangle \otimes |\psi_2\rangle$
5. Given an orthonormal basis  $\mathcal{B} = \{\varphi_i\}$ , one can perform a von-Neuman measurement  $\mathcal{H}_A$  on  $|\psi\rangle = \sum_i \alpha_i |\varphi_i\rangle$  that outputs  $i$  with probability  $|\alpha_i|^2$ . It is projective.

Further, if  $|\psi\rangle = \sum_i \alpha_i |\varphi_i\rangle |\gamma_i\rangle$   $\mathcal{H}_A \otimes \mathcal{H}_B$  measurement yields  $i$  with probability  $|\alpha_i|^2$  and leaves state in  $|\varphi_i\rangle |\gamma_i\rangle$ .  $M = \sum m_i P_i = \sum m_i |i\rangle\langle i|$

# Linear Algebra

- Dirac Notation: Element in Hilbert space of dimension  $2^n$  is represented by n-entry symbol.  $|000 \dots 00 \rangle \leftrightarrow (1, 0, \dots, 0)^T$ ,  $|000 \dots 01 \rangle \leftrightarrow (0, 1, \dots, 0)^T$ , ...,  $|111 \dots 1 \rangle \leftrightarrow (0, 0, \dots, 1)^T$  where column vectors have  $2^n$  coordinates.
- Notation:  $|0 \rangle \otimes |0 \rangle \otimes \dots \otimes |0 \rangle = |000 \dots 0 \rangle$
- $A$  is normal if  $AA^T = A^T A$
- Spectral Theorem: If  $T$  is a normal operator in the Hilbert space  $\mathcal{H}$ , there is an orthonormal basis  $v_i$ ; each is an eigenvector of  $T$ . For every such, there is a unitary matrix,  $P$ ,  $T = P\Lambda P^*$ , and  $\Lambda$  is diagonal.
- Dual basis
- Inner product:  $(v_1, v_2, \dots, v_n) \cdot (w_1, w_2, \dots, w_n) = \sum_{i=1}^n \bar{v}_i w_i$
- Outer product:  $(|\psi \rangle \langle \phi|)|\gamma \rangle = |\psi \rangle (\langle \phi|\gamma \rangle)$
- Theorem: Every linear operator can be written as  $T = T_{m,n} |b_m \rangle \langle b_n|$ ,
- $T_{m,n} = \langle b_m | T | b_n \rangle$

# Linear Algebra (continued)

Tensor product: If  $|\varphi_i\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$  is a basis for  $\mathcal{H}_1$  and  $|\phi_i\rangle = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$  is a basis for  $\mathcal{H}_2$ ,

$|\varphi_i\rangle \otimes |\phi_i\rangle$  is a basis for  $\mathcal{H}_1 \otimes \mathcal{H}_2$ .  $\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \otimes \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} =$   
 $(\alpha_0\beta_0, \alpha_0\beta_1, \alpha_1\beta_0, \alpha_1\beta_1)^T$ .  $A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \dots & \dots & \dots \\ a_{n1}B & \dots & a_{nn}B \end{pmatrix}$

Schmidt decomposition: If  $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ , there is an orthonormal basis  $|\varphi_i\rangle$  for  $\mathcal{H}_1$  and an orthonormal basis  $|\phi_i\rangle$  for  $\mathcal{H}_2$  and  $p_i \geq 0$  such that  $|\psi\rangle = \sum_i \sqrt{p_i} |\varphi_i\rangle |\phi_i\rangle$

$$\text{Tr}(A) = \langle b_n | A | b_n \rangle$$

Eigenvector:  $T|\psi\rangle = c|\psi\rangle$



# More notation

- $A \otimes B = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}$ ,  $x \otimes y = (x_1y_1, x_1y_2, \dots, x_ny_n)^T$
- $|v\rangle = (v_1, v_2, \dots, v_n)^T$ ,  $\langle w| = (w_1, w_2, \dots, w_n)$  then

$$|v\rangle\langle w| = \begin{pmatrix} v_1\overline{w_1} & v_1\overline{w_2} & \dots & v_1\overline{w_n} \\ \dots & \dots & \dots & \dots \\ v_n\overline{w_1} & v_n\overline{w_2} & \dots & v_n\overline{w_n} \end{pmatrix}, \text{ so } I = \sum |i\rangle\langle i| \text{ and } M = \sum M_{ij}|i\rangle\langle j|$$

- Pauli matrices

$$\begin{aligned} - \sigma_0 &= I, \sigma_X = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_Y = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_Z = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ - [X, Y] &= iZ, [Y, Z] = iX, [Z, X] = iY \end{aligned}$$

# Mixed states and density

- For pure states,  $|\psi\rangle$ , density is  $\rho = |\psi\rangle\langle\psi|$
- Mixed states:  $\{(p_1, |\psi_1\rangle), (p_2, |\psi_2\rangle), \dots, (p_n, |\psi_n\rangle)\}$ , where the probability that the system is in pure state  $|\psi_i\rangle$  is  $p_i$  and  $\sum p_i = 1$
- Density operator for mixed state is  $\sum p_i |\psi_i\rangle\langle\psi_i|$
- Bloch Sphere
  - Pure state in general position is  $|\psi\rangle = \cos\left(\frac{\theta}{2}\right) |0\rangle + e^{i\varphi} \sin\left(\frac{\theta}{2}\right) |1\rangle$ .
  - For mixed state  $|\psi_i\rangle = p_i(\alpha_{X,i}, \alpha_{Y,i}, \alpha_{Z,i})$  on interior of Bloch sphere
  - $\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$  evolves as  $\rho = \sum p_i U|\psi_i\rangle\langle\psi_i|U^\dagger$
  - $\rho = \frac{1}{2}I + \alpha_X X + \alpha_Y Y + \alpha_Z Z$
- $P(|0\rangle) = \langle 0|\psi\rangle\langle\psi|0\rangle = \text{Tr} \langle 0|\psi\rangle\langle\psi|0\rangle = \text{Tr}(|0\rangle\langle 0| |\psi\rangle\langle\psi|)$

# Mixed states and density

- Partial trace: Consider composite system  $AB$ .
  - $\rho^A = \text{Tr}_B(\rho^{AB})$
  - $\text{Tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) = |a_1\rangle\langle a_2| \text{Tr}(|b_1\rangle\langle b_2|) = |a_1\rangle\langle a_2| \langle b_2|b_1\rangle$
  - Example
    - $\rho = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$   
 $= \frac{1}{2} \text{Tr}(|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1|)$   
 $= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$

# Circuits and gates

- Universal gate set: A gate set is universal if  $\forall n > 0$ , any  $n$ -bit unitary operator can be approximated to arbitrary accuracy by a quantum circuit from this set
- An entangling gate is one that for an input product state  $|\alpha\rangle |\beta\rangle$ , the output state is not a product state (e.g.-CNOT).
  - Example:  $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
- Theorem: A set of states with an entangling 2-qubit gate together with all 1-qubit gates is universal.
- Theorem: If  $U$  is a 1-qubit gate,  $U = e^{ix}R_z(\beta)R_y(\gamma)R_z(\delta)$

# Gates and states

- General position on Bloch sphere:  $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$
- Measurement:  $I = \sum |i\rangle\langle i|$ ,  $M = \sum m_i P_i$ ,  $M$  is Hermitian,  $P_i = |i\rangle\langle i|$ .
- Controlled gates:
  - $c - U|0\rangle|\psi\rangle = |0\rangle|\psi\rangle$
  - $c - U|1\rangle|\psi\rangle = |1\rangle U|\psi\rangle$

# Common gates

- Pauli gates

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note:  $X^2 = Y^2 = Z^2 = I$

- Hadamard

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

$$H^{\otimes n}(|0000 \dots 0\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \dots \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

- Rotation

$$R_X(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{iX\theta} \end{pmatrix} = \begin{pmatrix} e^{-iX\theta/2} & 0 \\ 0 & e^{iX\theta/2} \end{pmatrix}$$

- 2 qubit gate

$$CNOT(|xy\rangle) = |x, x \oplus y\rangle$$

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

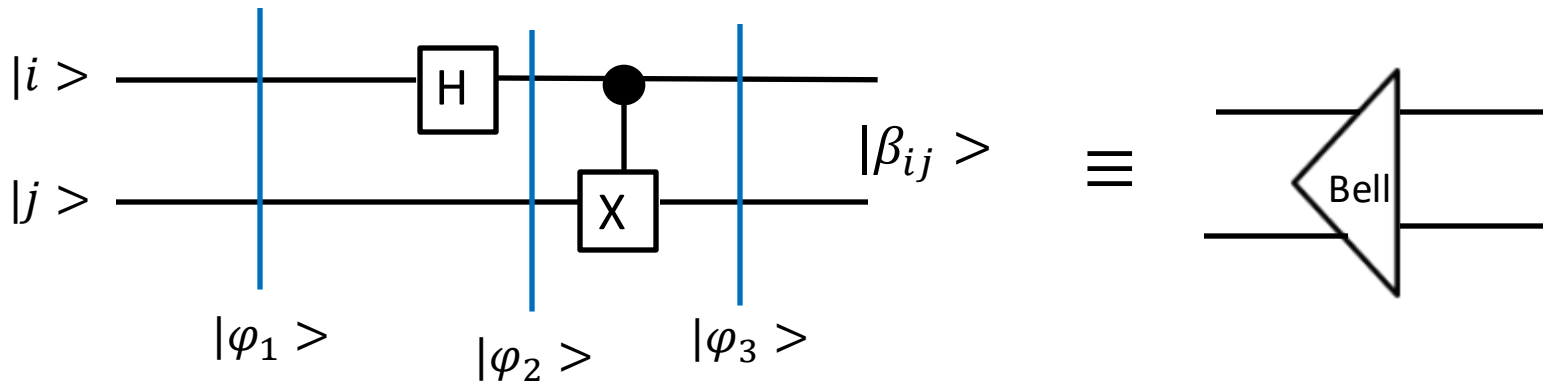
- If  $A^2 = 1$ ,  $e^{i\theta X} = I \cos(\theta) + iX \sin(\theta)$

# Measurement in alternate basis

- Computational basis is  $|i\rangle$ .  $U|\varphi_j\rangle = |j\rangle$
- Suppose we want to measure  $|\psi\rangle$  with respect to basis  $B = \{|\varphi_j\rangle\}$
- $|\psi\rangle = \sum \alpha_j |\varphi_j\rangle$
- To measure wrt  $B = \{|\varphi_j\rangle\}$ , Project  $|\psi\rangle$  onto  $|\varphi_j\rangle\langle\varphi_j|$
- $(\text{Tr}(|\psi\rangle\langle\psi||\varphi_j\rangle\langle\varphi_j|)) = \text{Tr}(\langle\varphi_j|\psi\rangle\langle\psi|\varphi_j\rangle) = \alpha_j^2$
- $\rho = |\psi\rangle\langle\psi|$  is density operator for the pure state  $|\psi\rangle$ .
- $\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$  is the density operator for mixed states  $\{(p_i, |\psi_i\rangle)\}$

# Converting to Bell Basis

- Computational basis is  $|i\rangle$ ,  $U|\varphi_j\rangle = |j\rangle$
- $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ ,  $|\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$
- $|\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ ,  $|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

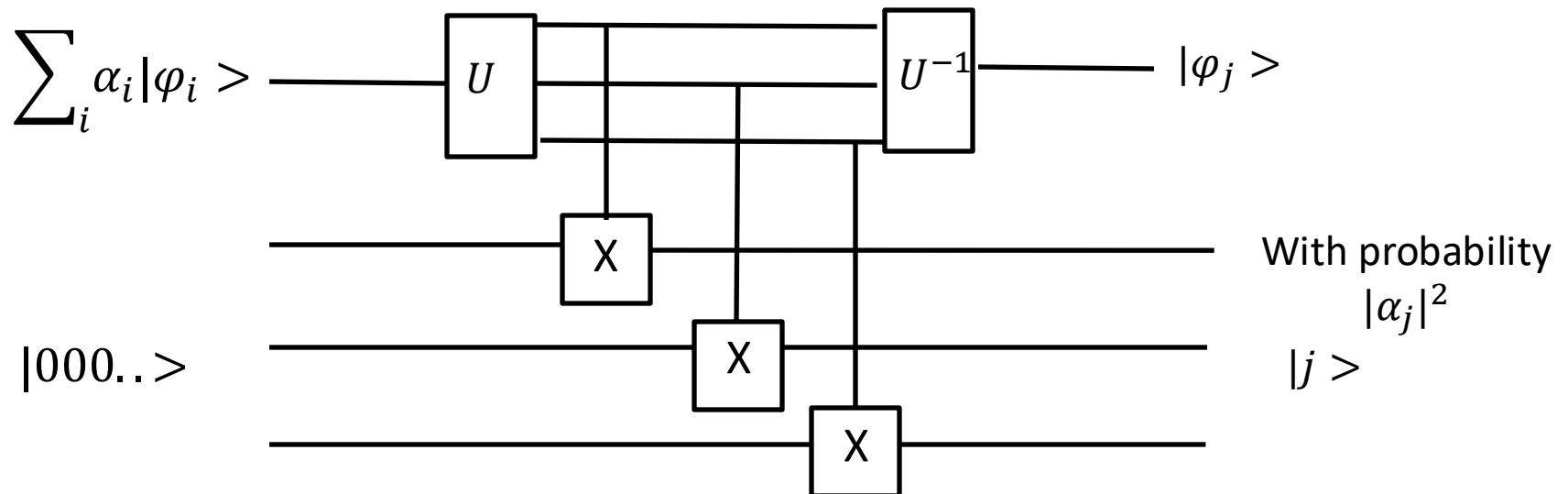
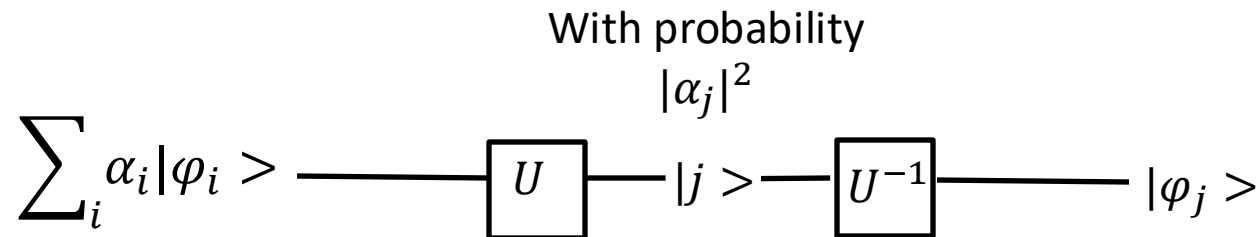


- $|\varphi_1\rangle = |00\rangle$
- $|\varphi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$
- $|\varphi_3\rangle = |\beta_{00}\rangle$

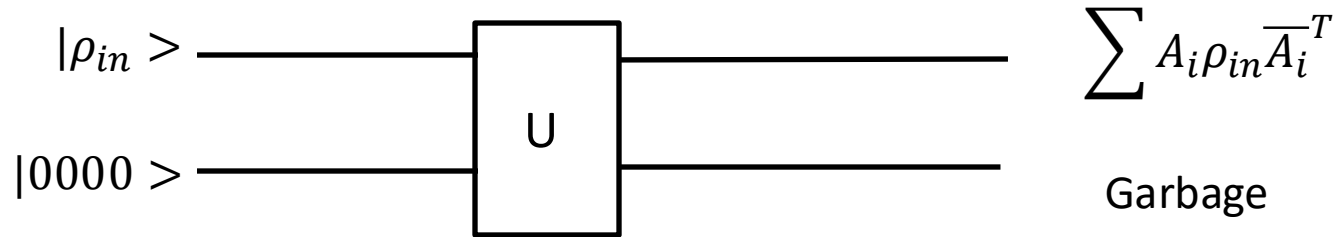


# Changing Measurement Basis

- Suppose  $|\varphi_i\rangle$  is a basis and our measurement basis is  $|i\rangle$ ,  $U|\varphi_i\rangle = |i\rangle$



# Super-operator and mixed states



- $\rho = |\psi\rangle\langle\psi|$ ,  $U|\psi\rangle$  has density  $\rho = U|\psi\rangle\langle\psi|\bar{U}^T = U\rho U^\dagger$
- $\langle 0|\psi\rangle\langle\psi|0\rangle = \langle 0|\rho|0\rangle = P(|0\rangle)$
- $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$
- $\text{Tr}(A) = \langle b_n|A|b_n\rangle$
- $\rho_{in} \rightarrow \rho_{out} = \text{Tr}_b(U(\rho_{in} \otimes |000\dots\rangle\langle 000\dots 0|U^\dagger)$
- $\rho_{in} \rightarrow \sum A_i \rho_{in} A_i^\dagger$ , where  $A_i$  are Kraus operators with  $\sum A_i^\dagger A_i = I$

# No Cloning Theorem

- Theorem: Qubits can't be copied

- Proof

Suppose they can be. Then there is an operator,  $U$ , such that for any state  $|\varphi\rangle$ ,  $U(|\varphi\rangle|0\rangle) = |\varphi\rangle|\varphi\rangle$ . Now let  $|\psi\rangle$  and  $|\phi\rangle$  be non-orthogonal, different pure states.

$$(|\psi\rangle|0\rangle, |\phi\rangle|0\rangle) = \langle\psi|\phi\rangle\langle 0|0\rangle = \langle\psi|\phi\rangle.$$

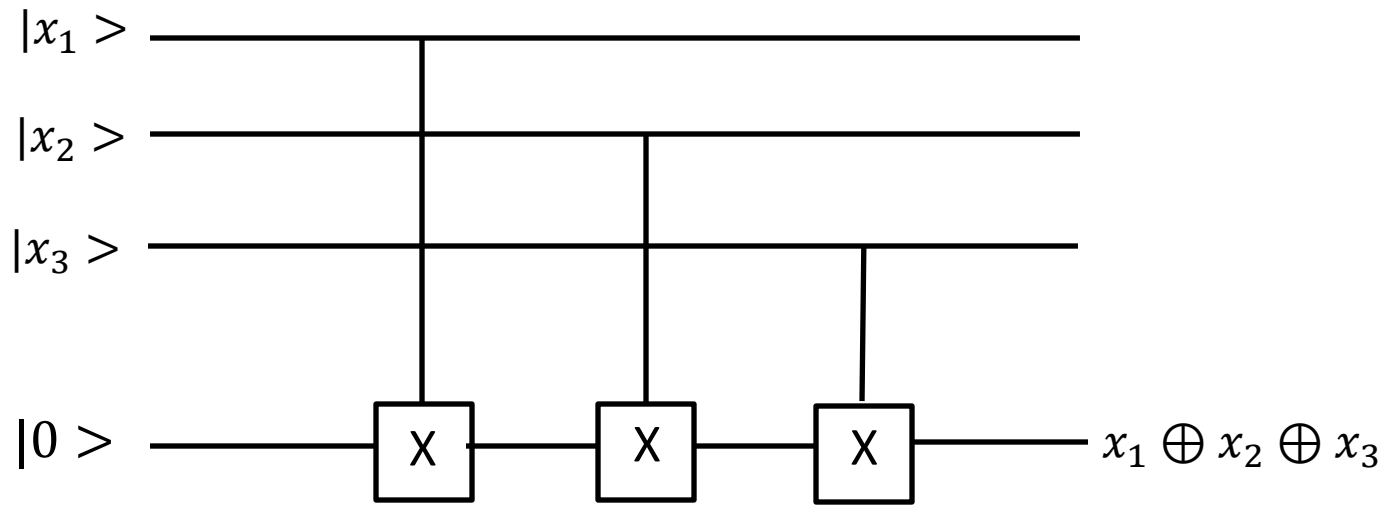
Since  $U$  is unitary,

$$\begin{aligned} \langle\psi|\phi\rangle &= (|\psi\rangle|0\rangle, |\phi\rangle|0\rangle) = (U|\psi\rangle|0\rangle, U|\phi\rangle|0\rangle) = \\ &(|\psi\rangle|\psi\rangle, |\phi\rangle|\phi\rangle) = \langle\psi|\phi\rangle^2. \end{aligned}$$

So,  $\langle\psi|\phi\rangle = 1$ . This is a contradiction.

- No checkpointing

# Parity Circuit

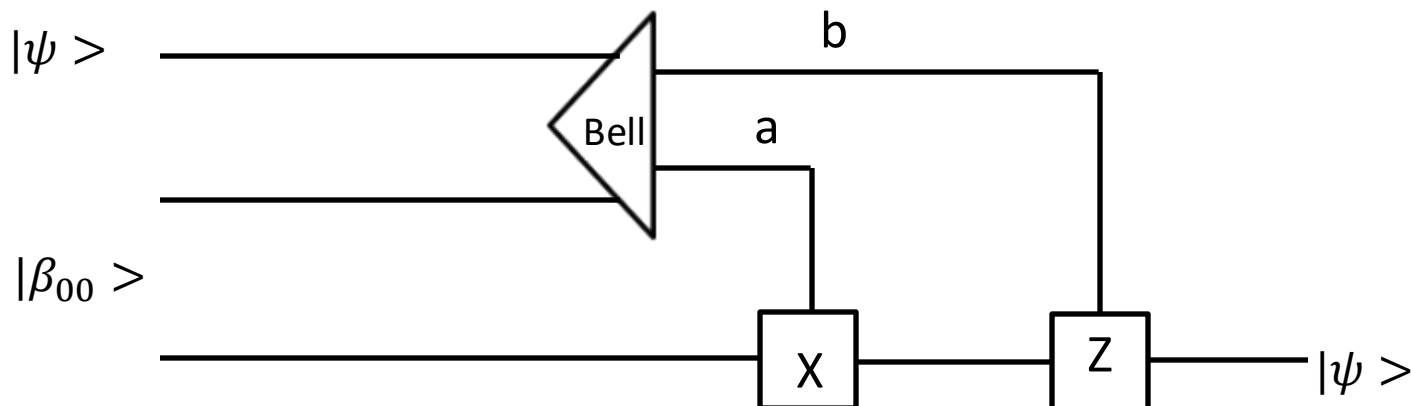


# Superdense coding

- Alice and Bob share  $|\beta_{00}\rangle$ , Alice has first bit, Bob second bit
- Alice performs one of  $I, X, Y, Z$  producing  $I \otimes I$  (to send 00),  $X \otimes I$  (to send 01),  $Y \otimes I$  (to send 10) or  $Z \otimes I$  (to send 11).
- Bob measures joint state qubit measurement

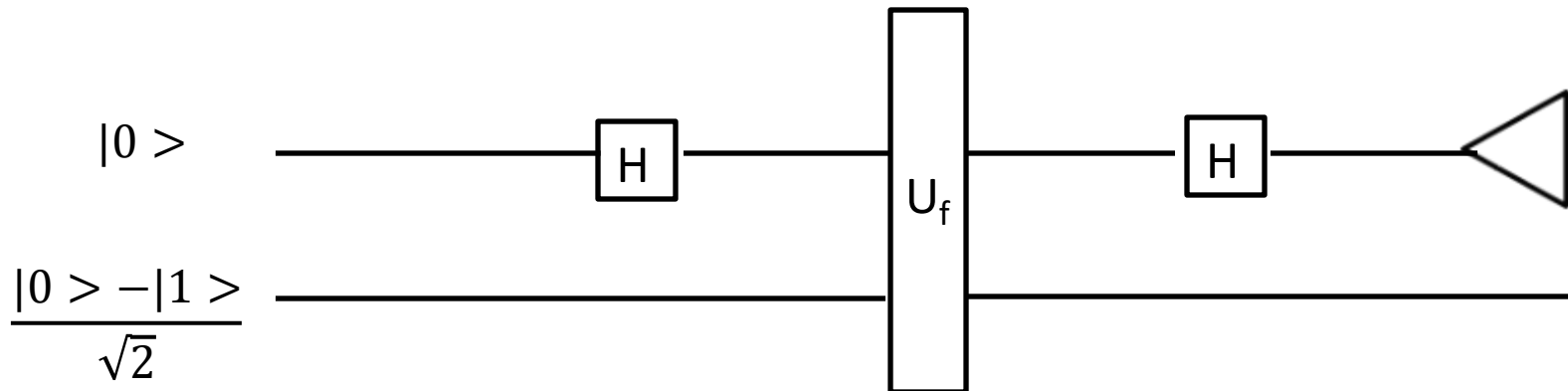
- Can be used to teleport  $|\psi\rangle$ :

- $I \otimes I := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \rightarrow \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
- $X \otimes I := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \rightarrow \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$
- $Z \otimes I := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \rightarrow \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$
- $ZX \otimes I := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \rightarrow \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$



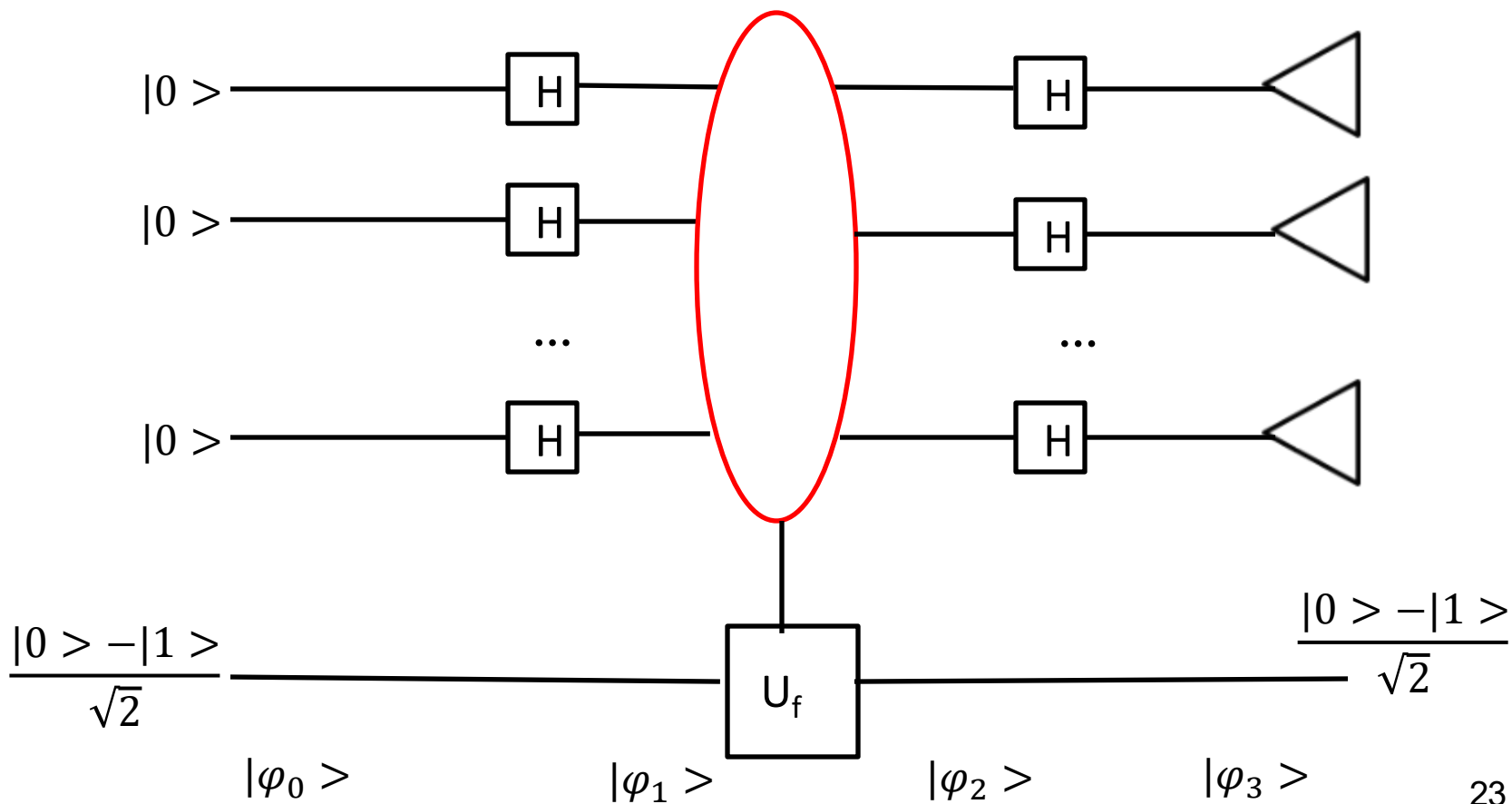
# Deutsch

- Problem: Determine  $f(0) + f(1)$  in one measurement
- $U_f |x\rangle |y\rangle = |x\rangle |y \oplus f(x)\rangle$
- If  $f(0) + f(1) = 1$ ,  $|\psi_3\rangle = (-1)^{f(0)} |1\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}}$
- If  $f(0) + f(1) = 0$ ,  $|\psi_3\rangle = (-1)^{f(0)} |0\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}}$



# Deutsch-Josza

- Problem:  $f: \{0,1\}^n \rightarrow \{0,1\}$ , which is either constant or balanced.
- Which is it?
- Put  $U_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle$ ,  $x$  is an  $n$ -bit quantity



# DJ

- $|\varphi_0\rangle = |0\rangle^{\otimes n} \frac{|0\rangle - |1\rangle}{\sqrt{2}}$
- $|\varphi_1\rangle = \frac{1}{\sqrt{2^n}} \sum_{\mathbf{x}} |\mathbf{x}\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}}$
- $|\varphi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{\mathbf{x}} (-1)^{f(\mathbf{x})} |\mathbf{x}\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}}$
- $|\varphi_3\rangle = \frac{1}{2^n} \sum_{\mathbf{x}} \sum_{\mathbf{z}} |(-1)^{f(\mathbf{x}) + \mathbf{x} \cdot \mathbf{z}} \mathbf{z}\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}}$



# Simon

- $f: \{0,1\}^n \rightarrow X, \exists \vec{s} = s_1, s_2, \dots, s_n: f(x) = f(y)$  iff  $x = y$  or  $x = y + \vec{s}$
- $U_f: |x\rangle |b\rangle = |x\rangle |b \oplus f(x)\rangle$
- $H^{\otimes n}(|x\rangle) = \frac{1}{\sqrt{2^n}} \sum_z (-1)^{x \cdot z} |z\rangle$

1.  $i = 1$
2. Prepare  $\frac{1}{\sqrt{2^n}} \sum_x |x\rangle |0\rangle$
3. Apply  $U_f$  to get  $\frac{1}{\sqrt{2^n}} \sum_x |x\rangle |f(x)\rangle$
4. Measure second bit
5. Apply  $H^{\otimes n}$  to first register
6. Measure first register to get  $w_i$
7. If  $\text{din}(w_i) \neq n - 1$ , go to 2
8. Output  $s$ :  $w^t s^t = 0$

# Phase kick back

- $CNOT \left( \frac{|0\rangle + |1\rangle}{\sqrt{2}} \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

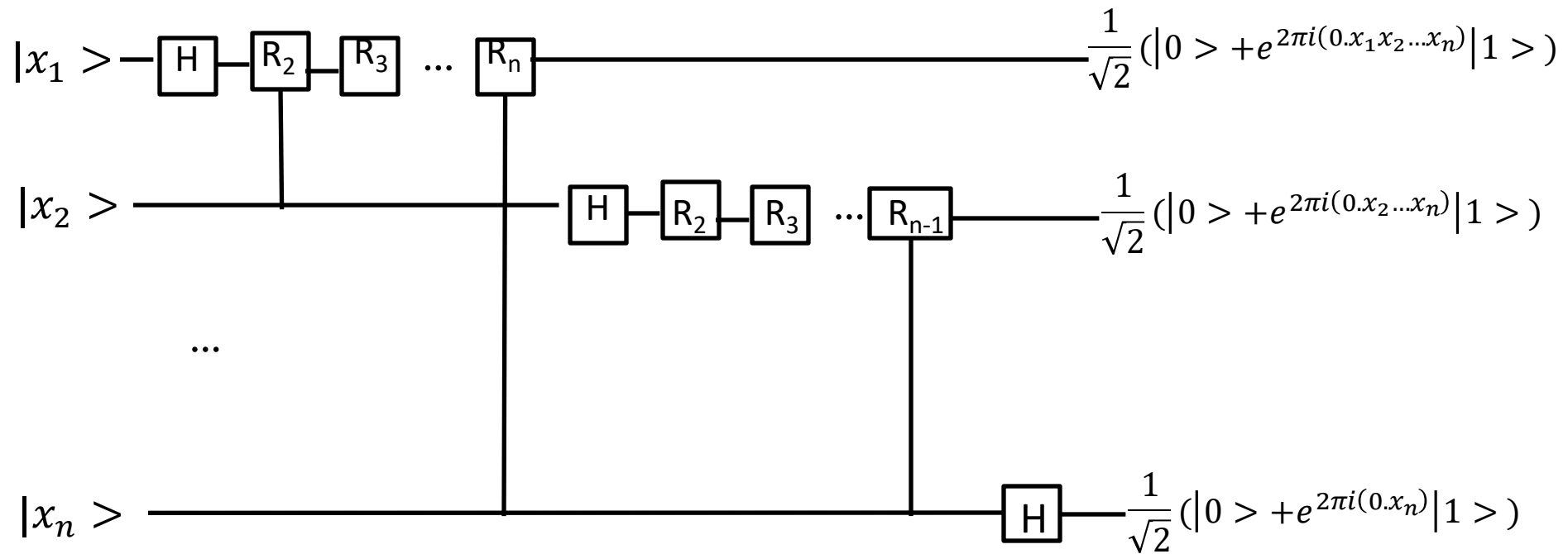
# Phase Estimation

- Phase estimation problem: Given  $|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_y e^{2\pi i \omega y} |y\rangle$ , estimate  $\omega$
- Theorem:  $\frac{x}{2^n} \leq \omega \leq \frac{x+1}{2^n}$  with probability  $\geq \frac{8}{\pi^2}$
- $e^{2\pi i 2^k \cdot x_1 x_2 \dots} = e^{2\pi i (x_{k+1} x_{k+2} \dots)}$
- Suppose  $\omega = .x_1$ ,  $|\psi\rangle = \frac{1}{\sqrt{2}} \sum_{|y\rangle} e^{2\pi i \omega |y\rangle} = \frac{1}{\sqrt{2}} (|0\rangle + (-1)^{x_1} |1\rangle)$  and  $H(|\psi\rangle) = |x_1\rangle$
- In general,  $H^{\otimes n}(|\mathbf{x}\rangle) = \frac{1}{\sqrt{2^n}} \sum_{\mathbf{y}} (-1)^{\mathbf{x} \cdot \mathbf{y}} |\mathbf{y}\rangle$  and  $H^{\otimes n}(H^{\otimes n}(|\mathbf{x}\rangle)) = |\mathbf{x}\rangle$
- So,  $|\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{|y\rangle} e^{2\pi i \omega y} |y\rangle = \frac{1}{\sqrt{2}} (|0\rangle + e^{2\pi i 2^{n-1} \omega} |1\rangle) \otimes \frac{1}{\sqrt{2}} (|0\rangle + e^{2\pi i 2^{n-2} \omega} |1\rangle) \otimes \dots \otimes \frac{1}{\sqrt{2}} (|0\rangle + e^{2\pi i \omega} |1\rangle)$
- Denote  $R_n = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i 2^{-n}} \end{pmatrix}$

# Quantum Fourier Transform

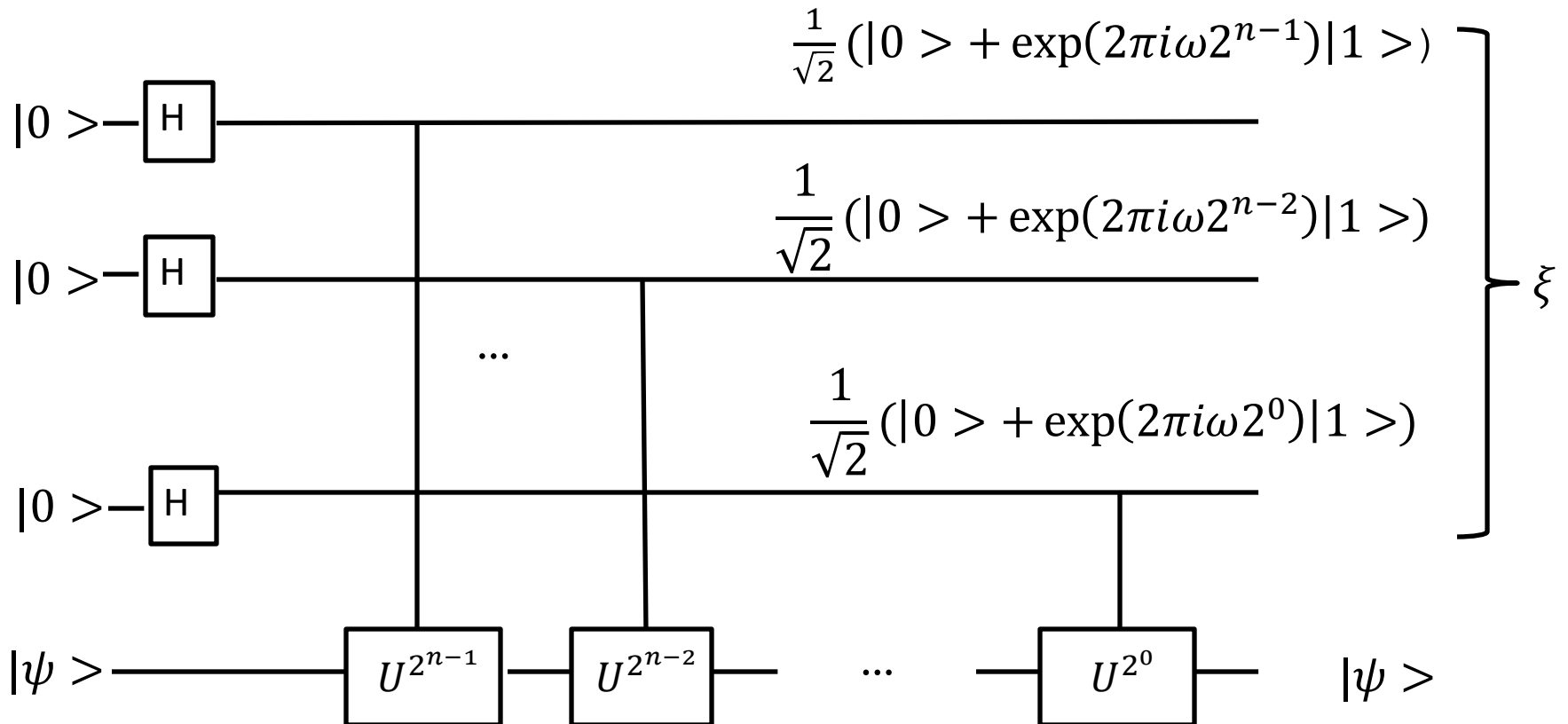
- $H^{\otimes n}(|\mathbf{x}\rangle) = \frac{1}{\sqrt{2^n}} \sum_{\mathbf{y}} (-1)^{\mathbf{x} \cdot \mathbf{y}} |\mathbf{y}\rangle$
- $H^{\otimes n}(H^{\otimes n}(|\mathbf{x}\rangle)) = |\mathbf{x}\rangle$
- $QFT_m(|x\rangle) = \frac{1}{\sqrt{m}} \sum_{y=0}^{m-1} e^{2\pi i/m(x \cdot y)} |y\rangle$
- $QFT_m^{-1}(|x\rangle) = \frac{1}{\sqrt{m}} \sum_{y=0}^{m-1} e^{-2\pi i/m(x \cdot y)} |y\rangle$

# Quantum Fourier Circuit



# Eigenvalue Estimation

- Suppose  $|\psi\rangle$  is an eigenstate of a unitary operator,  $U$ , so  $U|\psi\rangle = \exp(2\pi i\phi)|\psi\rangle$ .  $|\phi\rangle = .x_1x_2 \dots x_n$  (a binary expansion)



# Eigenvalue Estimation

- $U|\psi\rangle = \exp(2\pi i\phi) |\psi\rangle$ , so  $U^{2^j}|\psi\rangle = \exp(2\pi i\phi 2^j) |\psi\rangle$ .
- Applying  $QFT_n^{-1}$  to  $\xi$ , gives  $\langle x_n, x_{n-1}, \dots, x_1 \rangle$ , where  $|\phi\rangle = .x_1x_2 \dots x_n$
- Measure  $\chi$  to get  $\phi$
- $\frac{y}{2^n}$  is a good estimate for  $\phi = \frac{j}{r}$

# Factorization using order finding (Shor)

- Suppose  $N = pq$  and  $a^r = 1 \pmod{N}$  then  $r = 0 \pmod{\phi(pq)}$
- If  $r$  is even, say,  $r = 2s$ ,  $(a^s + 1)(a^s - 1) = 0 \pmod{pq}$ .
- There is a good chance  $p \mid (a^s - 1)$  but  $(q, (a^s - 1)) = 1$ .
- Then  $((a^s - 1), N) = p$ . Voila!
- Note that  $|v_t\rangle = \frac{1}{r} \sum_{k=0}^{r-1} \exp(-\frac{2\pi i k t}{r}) |k \pmod{N}\rangle$  is an eigenvalue of  $U_x(k) = |xk \pmod{N}\rangle$ .
- In Shor,  $|1\rangle = \frac{1}{\sqrt{r}} \sum |v_t\rangle$ .
- Applying  $QFT^{-1}$  to control gives phase of eigenvalues
- Measurement of target gives  $|\frac{s}{r}\rangle$  with  $\Pr(|y\rangle) = \frac{1}{2^{2n}} \left| \frac{1-r^{2^n}}{1-r} \right|^2$ , where  $r = \exp(-2\pi i (\frac{y}{2^n} - \phi))$



# Order Finding

Problem: Given  $a, N \in \mathbb{Z}$  with  $(a, N) = 1$ , find  $r$ :  $a^r \pmod{N} = 1$

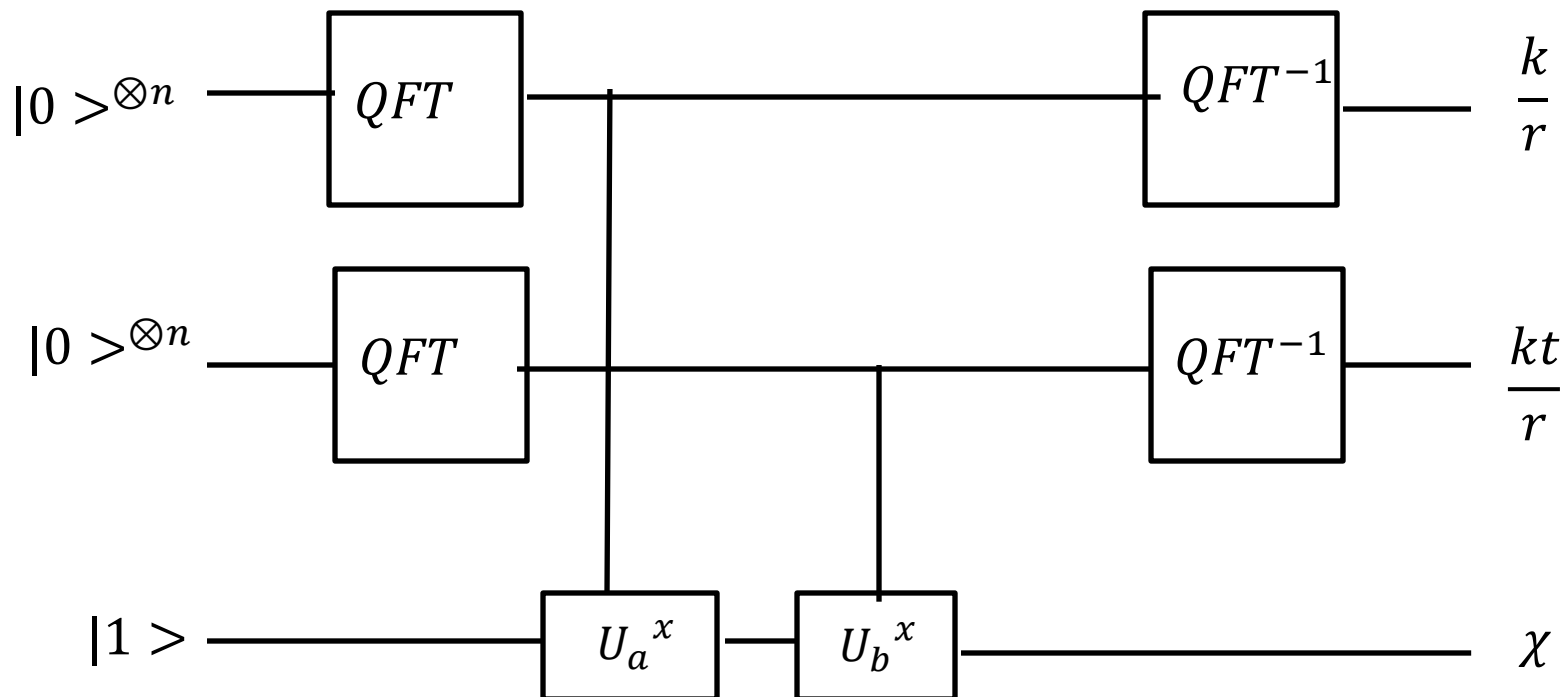
1. Choose  $n$ :  $2^n \geq 2r^2$
2. Initialize control register  $|000 \dots 0\rangle = |0\rangle^{\otimes 2n}$
3. Initialize target register to  $= |000 \dots 01\rangle = |000 \dots 0\rangle = |0\rangle^{\otimes 2n} \otimes |1\rangle$
4. Apply  $QFT$  to control register
5. Apply  $c - U_a^x$  to control and target register
6. Apply  $QFT^{-1}$  to control register
7. Measure CR to get estimate of  $\frac{x_1}{2^n}$  of multiple of  $\frac{1}{r}$
8. Use continued fraction to get  $c_1, r_1$ :  $\left| \frac{x_1}{2^n} - \frac{c_1}{r_1} \right| \leq 2^{-(n-1)/2}$
9. Repeat 1-8 to get  $c_2, r_2$ :  $\left| \frac{x_2}{2^n} - \frac{c_2}{r_2} \right| \leq 2^{-(n-1)/2}$ , if none, FAIL
10. Compute  $r = LCM(r_1, r_2)$  and  $a^r \pmod{N}$
11. If  $a^r \pmod{N} = 1$ , output  $r$ , otherwise FAIL

# Order Finding

- Order finding has quantum complexity  $O(\lg(N)^2 \lg(\lg(N)) \lg(\lg(\lg(N)))$
- Classical complexity is  $\exp(O(\sqrt{\lg(N) \lg(\lg(N))})$

# Discrete log

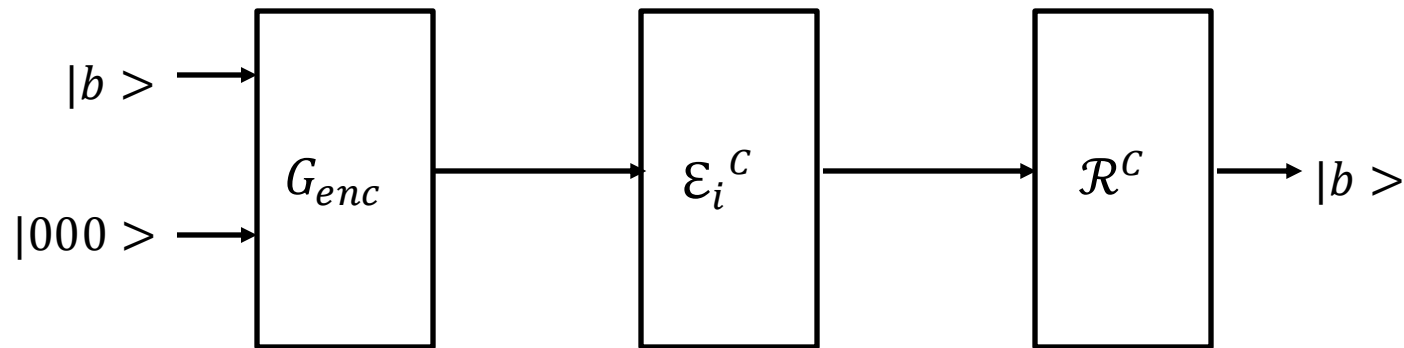
- Suppose  $a = b^x \pmod{p}$ ,  $b$  has known order. We want  $r: b^r = 1 \pmod{p}$
- Put  $U_a(|x\rangle) = |ax \pmod{p}\rangle$  and  $U_b(|x\rangle) = |bx \pmod{p}\rangle$ .
- Consider the circuit below.  $|1\rangle = \frac{1}{\sqrt{r}} \sum |v_t\rangle$ . Below,  $t = xy^{-1}$



# Discrete log

- Measuring first control register gives  $|\frac{k}{r}\rangle$
- Measuring first control register gives  $|\frac{kt}{r}\rangle$
- Quantum complexity is  $O(\lg(p)^2 \lg(\lg(p)) \lg(\lg(\lg(p))))$
- Best known classical requires  $\exp(O(\sqrt{\lg(p)} \lg(\lg(p))))$

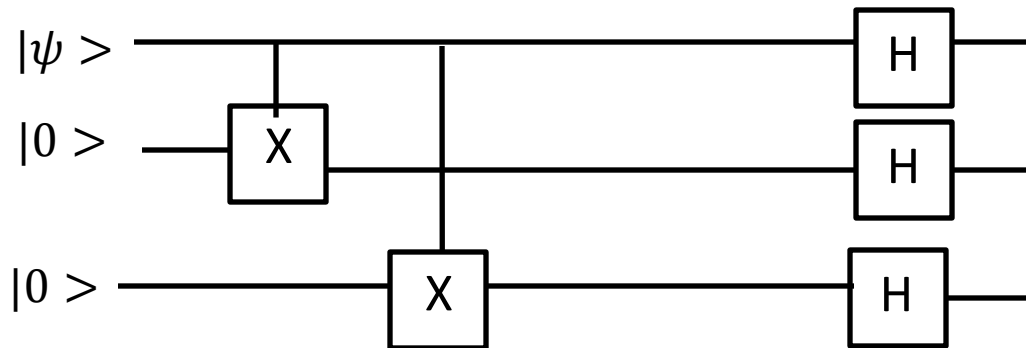
# Error Correction



- Unlike classical error correction, the no cloning theorem restricts codes
- $|0\rangle|E\rangle \rightarrow \beta_1|0\rangle|E_1\rangle + \beta_2|1\rangle|E_2\rangle$
- $|1\rangle|E\rangle \rightarrow \beta_3|0\rangle|E_3\rangle + \beta_4|1\rangle|E_4\rangle$
- $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle \rightarrow \alpha_0\beta_1|0\rangle|E_1\rangle + \alpha_0\beta_2|1\rangle|E_2\rangle + \alpha_1\beta_3|0\rangle|E_3\rangle + \alpha_1\beta_4|1\rangle|E_4\rangle$
- $|\psi\rangle = \frac{1}{2}|\psi\rangle(\beta_1|E_1\rangle + \beta_3|E_3\rangle) + \frac{1}{2}\langle Z|\psi\rangle(\beta_1|E_1\rangle - \beta_3|E_3\rangle) + \frac{1}{2}\langle X|\psi\rangle(\beta_2|E_2\rangle + \beta_4|E_4\rangle) + \frac{1}{2}\langle XZ|\psi\rangle(\beta_2|E_2\rangle - \beta_4|E_4\rangle)$

# Error Correction

- $\rho = U_{err}|\psi\rangle\langle\psi|U_{err}^\dagger$
- $|\psi_{enc}\rangle = U_{enc}|\psi\rangle|000\dots\rangle$
- $\mathcal{E}_0 = I \otimes I \otimes I, \mathcal{E}_1 = X \otimes I \otimes I$
- $\mathcal{E}_2 = I \otimes X \otimes I, \mathcal{E}_3 = I \otimes I \otimes X$
- $\rho: |\psi\rangle\langle\psi| \rightarrow (1-p)|\psi\rangle\langle\psi| + p X|\psi\rangle\langle\psi|X$
- $\frac{1}{\sqrt{2}}(|000\rangle + |100\rangle) \rightarrow \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \neq \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes^3$
- 3-bit code, Shor 9-bit code



# Amplitude Amplification

- $|\psi\rangle = A |00\dots 0\rangle = \sum_x \alpha_x |x\rangle |junk(x)\rangle$
- $|\psi\rangle = \sum_{x,good} \alpha_x |x\rangle |junk(x)\rangle + \sum_{x,bad} \alpha_x |x\rangle |junk(x)\rangle$
- $|\psi_{good}\rangle = \sum_{x,good} \alpha_x |x\rangle |junk(x)\rangle$
- $|\psi_{bad}\rangle = \sum_{x,bad} \alpha_x |x\rangle |junk(x)\rangle$
- $|\psi\rangle = \sqrt{p_{good}} |\psi_{good}\rangle + \sqrt{p_{bad}} |\psi_{bad}\rangle = \sin(\theta) |\psi_{good}\rangle + \cos(\theta) |\psi_{bad}\rangle$
- $p_{good} = \sin(\theta)^2$

# Grover

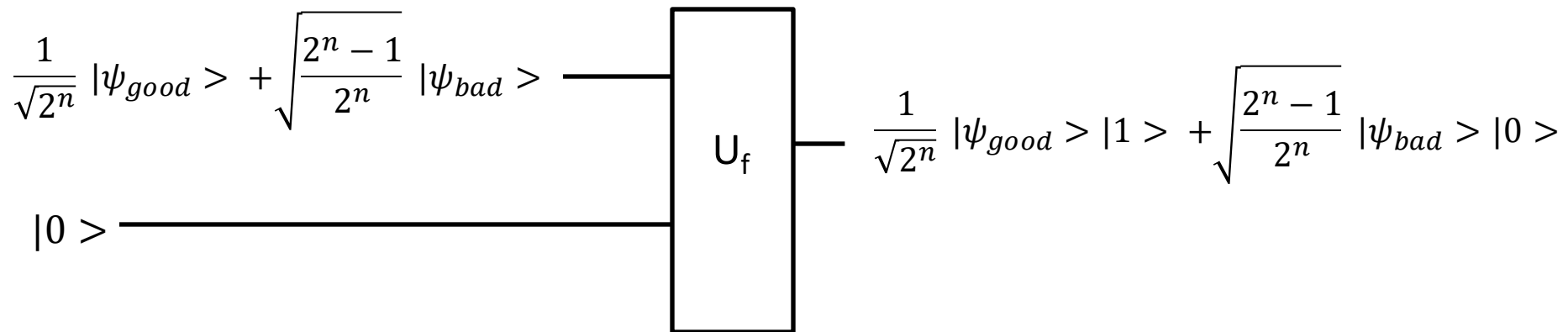
Search

Input:  $U_f: f: \{0,1\}^n \rightarrow \{0,1\}$

$$f(\mathbf{a}) = 1, f(\mathbf{x}) = 0, \mathbf{x} \neq \mathbf{a}$$

$$|\psi_{good}\rangle = \mathbf{w}$$

$$|\psi_{bad}\rangle = \frac{1}{\sqrt{N-1}} \sum_{\mathbf{x} \neq \mathbf{w}} |\mathbf{x}\rangle$$





# Grover

1. Initialize  $n$ -qubits  $|0000 \dots 0\rangle$ .
2. Apply  $H^{\otimes n}$  to get  $\frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle$
3. Apply Grover  $G$   $\frac{\pi}{4\sqrt{n}}$  times
4. Measure output

## Algorithm $G$

1. Apply  $U_f$
2. Apply  $H^{\otimes n}$
3. Apply  $U_{0\perp}$
4. Apply  $H^{\otimes n}$

## Search

Input:  $U_f: f: \{0,1\}^n \rightarrow \{0,1\}$

$$f(\mathbf{a}) = 1, f(\mathbf{x}) = 0, \mathbf{x} \neq \mathbf{a}$$

$$|\psi_{good}\rangle = |\mathbf{w}\rangle$$

$$|\psi_{bad}\rangle = \frac{1}{\sqrt{N-1}} \sum_{x \neq w} |x\rangle$$

## Algorithm $U_{0\perp}$

$$U_{0\perp}: |x\rangle \rightarrow -|x\rangle, x \neq 0$$

$$U_{0\perp}: |0\rangle \rightarrow 0|x\rangle$$

# End

# Strings and thermo

- String:  $dU = \frac{1}{2} \mu \omega^2 y^2 dx$ ,  $P(t) = Z = F \frac{\partial \psi}{\partial t}$ ,  $v_\phi = \frac{\omega}{k}$ ,  $Z = \frac{T}{v_\phi}$ ,  $\frac{T}{\mu} = \omega^2$
- Power transmitted:  $P = \frac{1}{2} \mu \omega^2 A^2 v_\phi$
- $I = I_0 \left( \frac{\sin(\beta/2)}{\beta/2} \right)^2$ ,  $P_R = \frac{P_T G_R G_T \lambda^2}{(4\pi R)^2}$
- $Z = \sum_i e^{-\beta E_i}$ ,  $\beta = \frac{1}{kT}$ ,  $\langle E \rangle = \frac{\partial(\ln(Z))}{\partial \beta}$
- $\Delta Q + \Delta W = \Delta E$ ,  $\Delta Q$  – heat in system,  $\Delta W$  – work on system
- $W = Q(1 - \frac{T}{T_0})$ ,  $e = (1 - \frac{T_C}{T_H})$ ,  $S = k \ln(\Omega) \Omega$
- $c_v = \frac{3}{2} R$
- $I(\lambda) = \frac{2\pi h c^2}{\lambda^5 (\exp(\frac{hc}{k\lambda T}) - 1)}$

# EM

- $E_n = \frac{-13.6}{n^2}, a_0 = \epsilon_0 \frac{h^2}{\pi m e^2}, d_n = \frac{(2m)^{3/2} V E^{3/2}}{3\pi h^3}, g(E) = \frac{(2m)^{3/2} V}{2\pi h^2} \sqrt{E}$
- $\nabla \cdot j = -\frac{\partial \rho}{\partial t}, \nabla \cdot E = \frac{\rho}{\epsilon_0}, \nabla \times E = -\frac{\partial B}{\partial t}, \nabla \times B = 0, c^2 \nabla \times B = \frac{j}{\epsilon_0} + \frac{\partial E}{\partial t}$
- $c^2 = \frac{1}{\epsilon_0 \mu_0}, I = \sigma T^4, D = \epsilon E, B = \mu H$
- Solution to  $\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = -s, \psi(t) = \frac{1}{4\pi} \frac{S(t-\frac{r}{c})}{r}, S(t) = \int s(t) dV$
- $\phi(1, t) = \int \frac{\rho(2, t-\frac{r}{c})}{4\pi \epsilon_0 r_{12}} dV, A(1, t) = \int \frac{\rho(2, t-\frac{r}{c})}{4\pi c^2 \epsilon_0 r_{12}} dV$
- $\nabla \phi = E + \frac{\partial A}{\partial t}, S = \epsilon_0 c^2 E \times B$
- Oscillating dipole:  $\psi = \frac{dz}{4\pi \epsilon_0} \left[ \frac{q(t-\frac{r}{c})}{r^3} + \frac{I(t-\frac{r}{c})}{r^2 c} \right]$
- $x' = \gamma(c - ut), t' = \gamma(t - \frac{ux}{c^2}), E^2 + (pc)^2 = (m_0 c^2)^2$

# Susskind

- $|\psi\rangle = \sum_i \alpha_i |\lambda_i\rangle$  is the state of a system, the  $|\lambda_i\rangle$  is a complete set of orthonormal vectors which are eigenvectors
- $\langle L \rangle = \sum_i P(\lambda_i) \lambda_i$  is the expected value
- $\langle L \rangle = \langle \psi | L | \psi \rangle = \sum_i \bar{\alpha}_i \alpha_i \langle \lambda_i | \lambda_i \rangle, \alpha_i \in \mathbb{C}$
- If  $\varphi, \psi$  are states in a continuous variable,  $\langle \varphi | \psi \rangle = \int_{-\infty}^{\infty} \bar{\varphi} \psi dx$
- $I = \sum_i |i\rangle \langle i|, \text{Tr}(L) = \sum_i \langle i | L | i \rangle, I = \int |x\rangle \langle x| dx$
- If  $|\psi\rangle = \sum_i \alpha_i |i\rangle, \langle j | \psi \rangle = \sum_i \alpha_i \langle j | i \rangle$
- C-S:  $|\vec{x}| |\vec{y}| \geq |\vec{x} \cdot \vec{y}|$ , Triangle:  $|\vec{x}| + |\vec{y}| \geq |\vec{x} + \vec{y}|$
- Eigenvectors of a Hermitian operator are a complete set
- Eigenvectors with different eigenvalues are orthogonal.

# Susskind

- Observables:  $M|\lambda\rangle = \lambda|\lambda\rangle$ ,  $\lambda$  is the observed value,  $M$  is projective and Hermitian
- The rules
  1. Observables are represented by linear operators. States are vectors
  2. Results of measurements are eigenvalues
  3. Distinguishable states correspond to orthogonal eigenvalues
  4. If  $|\psi\rangle$  is a state and  $L$  is an observable,  
 $P(\lambda_i) = \langle \psi | \lambda_i \rangle \langle \lambda_i | \psi \rangle$
  5. Evolution of states governed by a unitary operator

# Spin

- Spin field:  $H = \sigma \cdot B$ 
  - $\langle \dot{\sigma}_z \rangle = \frac{i\omega}{2} \langle [\sigma_z, \sigma_z] \rangle, \langle \dot{\sigma}_y \rangle = -\omega \langle \sigma_x \rangle$
- $\langle \dot{\sigma}_x \rangle = \frac{-i}{\hbar} \langle [\sigma_x, H] \rangle, \langle \dot{\sigma}_y \rangle = \frac{-i}{\hbar} \langle [\sigma_y, H] \rangle, \langle \dot{\sigma}_z \rangle = \frac{-i}{\hbar} \langle [\sigma_z, H] \rangle$
- $|u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- $|r\rangle = \frac{1}{\sqrt{2}}(|u\rangle + |d\rangle), |l\rangle = \frac{1}{\sqrt{2}}(|u\rangle - |d\rangle)$
- $|i\rangle = \frac{1}{\sqrt{2}}(|u\rangle + i|d\rangle), |o\rangle = \frac{1}{\sqrt{2}}(|u\rangle - i|d\rangle)$
- Pauli matrices:  $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$
- $\sigma_n = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$
- $\lambda_1 = 1, |\lambda_1\rangle = \begin{pmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) \end{pmatrix}, \lambda_2 = -1, |\lambda_2\rangle = \begin{pmatrix} -\sin(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2}) \end{pmatrix}$

# Entanglement

- Composite:  $|\varphi\rangle \in H^A$ ,  $|\psi\rangle \in H^B$ ,  $\varphi \otimes \psi \in H^{AB}$
- $L|\lambda_i, \mu_i\rangle = \lambda_i|\lambda_i, \mu_i\rangle$ ,  $M|\lambda_i, \mu_i\rangle = \mu_i|\lambda_i, \mu_i\rangle$  for independent action
- If  $A$  state coefficients are  $a$  and  $B$  state coefficients are  $B$ ,
- $|\varphi \otimes \psi\rangle = \sum_{a,b} |\varphi(a)\rangle \otimes |\psi(b)\rangle = |ab\rangle$
- Alice and Bob spins:  $(\alpha_u|u\rangle + \alpha_d|d\rangle) \otimes (\beta_u|u\rangle + \beta_d|d\rangle)$  as product state
- General entangled:  $\psi_{uu}|uu\rangle + \psi_{ud}|ud\rangle + \psi_{du}|du\rangle + \psi_{dd}|dd\rangle$
- Entangled:  $|sing\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$ ,
- $\langle sing | \sigma_z | sing \rangle = 0$



# Susskind and density

- $\tau_z \sigma_z |sing\rangle = -|sing\rangle, \tau_x \sigma_x |sing\rangle = -|sing\rangle$
- $\langle L \rangle = Tr(\rho L) = \langle \psi | L | \psi \rangle$  for states prepared with probability  $p_i$
- If  $\langle L \rangle = p_\psi \langle \psi | L | \psi \rangle + p_\phi \langle \phi | L | \phi \rangle$ , density operator is
- $P = p_\psi |\psi\rangle \langle \psi| + p_\phi |\phi\rangle \langle \phi|$
- $P_{a,a'} = \overline{\psi(a)} \psi(a') \sum_b \overline{\phi(b)} \phi(b)$
- $L_{a'b',a,b} = \langle a'b' | L | ab \rangle$
- $\langle \psi | L | \psi \rangle = \sum_{a,a',b,b'} \bar{\psi}_{a',b'} L_{a',a} \psi_{a,b}$

# Uncertainty, Evolution and Schrodinger

- Uncertainty: From triangle,  $(\Delta X)(\Delta Y) \geq \frac{1}{2} | \langle \psi | [A, B] | \psi \rangle |$
- Unitary evolution:  $U^\dagger(t)U(t) = I, |\psi(t)\rangle = U(t)|\psi(0)\rangle$
- $U(\epsilon) = (I + \frac{i}{\hbar}\epsilon H)|\psi(0)\rangle$ , so
- $\frac{|\psi(\epsilon)\rangle - |\psi(0)\rangle}{\epsilon} = -\frac{i}{\hbar}H|\psi(0)\rangle$ , or,  $\frac{\partial |\psi(t)\rangle}{\partial t} = -\frac{i}{\hbar}H|\psi(t)\rangle$
- $\frac{d}{dt} \langle \psi | L | \psi \rangle = \langle \dot{\psi} | L | \psi \rangle + \langle \psi | L | \dot{\psi} \rangle =$   
 $\frac{i}{\hbar} (\langle \psi | HL | \psi \rangle - \langle \psi | LH | \psi \rangle) = \frac{i}{\hbar} \langle \psi | [H, L] | \psi \rangle$
- This gives conservation of energy since  $[H, H] = 0$
- Example:  $H = \frac{\hbar\omega}{2} \sigma \cdot n$

# Momentum space and Fourier

- $P|\psi\rangle = p|\psi\rangle$  is equivalent to  $P = -i\hbar \frac{\partial}{\partial x}$
- Eigenvectors are solutions of  $-i\hbar \frac{\partial \psi}{\partial x} = p\psi$ ,  $\psi_p(x) = A \exp(i \frac{px}{\hbar})$
- Wavelength of  $A \exp(i \frac{px}{\hbar})$  is  $\lambda = \frac{2\pi\hbar}{p}$
- $P(p) = |\langle P|\psi\rangle|^2$ , put  $\tilde{\psi}(p) = \langle P|\psi\rangle$
- $\tilde{\psi}(p) = \int dx \langle p|x\rangle \langle x|\psi\rangle$  but  $\psi(x) = \langle x|\psi\rangle$  and
- $\langle p|x\rangle = \frac{1}{\sqrt{2\pi}} \exp(-i \frac{px}{\hbar})$
- Similarly,  $\psi(x) = \langle x|\psi\rangle = \int dp \langle x|p\rangle \langle p|\psi\rangle$  and
- $\psi(x) = \frac{1}{\sqrt{2\pi}} \int dp \exp(i \frac{px}{\hbar}) \tilde{\psi}(p)$
- $\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(i \frac{px}{\hbar}) \tilde{\psi}(p) dp$ ,
- $\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi}} \int \exp\left(-i \frac{px}{\hbar}\right) \psi(x) dx$

# Susskind

- $\psi(x, t) = A \exp\left(i \frac{px - \frac{p^2 t}{2m}}{\hbar}\right)$  is a complete set of solutions
- $\psi(x, t) = \int dp \tilde{\psi}(p) \exp\left(i \frac{px - \frac{p^2 t}{2m}}{\hbar}\right)$
- Dynamics:
- $X|\psi\rangle = x|\psi\rangle, P|\psi\rangle = i\hbar \frac{\partial}{\partial x}|\psi\rangle$
- $V|\psi\rangle \rightarrow V(x)\psi(x)$ , get  $E\psi = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(X)\psi$
- $\langle v \rangle = \frac{d}{dt} \langle \psi | X | \psi \rangle,$
- $\frac{d}{dt} \langle P \rangle = \frac{i}{\hbar} [V, P] = -\langle \frac{\partial V}{\partial x} \rangle,$
- $[V, P] = i\hbar \frac{dV}{dx}$

# Standard solution

- Standard method:
  1. Get  $H$
  2. Prepare  $|\psi(0)\rangle$
  3. Find  $H|E_j\rangle = E_j |E_j\rangle$
  4.  $\alpha_j(0) = \langle E_j | \psi(0) \rangle$ ,  $\alpha_j(t) = \alpha_j(0) \exp(-i \frac{E_j t}{\hbar})$
  5.  $|\psi(t)\rangle = \sum_j \alpha_j(t) |E_j\rangle$
  6.  $P_\lambda(t) = |\langle \lambda | \psi(t) \rangle|^2$
- Correlation and means
  - $\Delta A^2 = \sum_a (A - \bar{A})^2 P(a) = |\langle A^2 | \psi \rangle|^2$
  - $(\Delta A)(\Delta B) \geq \frac{1}{2} |\langle \psi | [A, B] | \psi \rangle|^2$
  - $C(A, B) = \langle AB \rangle - \langle A \rangle \langle B \rangle$
  - $C(\sigma_x, \tau_x) = -1$

# Harmonic Oscillator, etc

- $H = \frac{1}{2}\dot{X}^2 + \frac{1}{2}\omega^2 x^2 = \frac{P^2 + \omega^2 x^2}{2} = \frac{1}{2}(P + i\omega X)(P - i\omega X) - \omega^2[X, P]$
- Ground state:  $\psi(x) = \exp(-\frac{\omega}{2\hbar}x^2)$ ,  $E_0 = \frac{\omega\hbar}{2}$
- $a^- = \frac{i}{\sqrt{2\omega\hbar}}(P - i\omega X)$ ,  $a^+ = \frac{-i}{\sqrt{2\omega\hbar}}(P + i\omega X)$
- $[a^-, a^+] = 1$ . If  $N = a^+ a^-$ ,  $H = \omega\hbar(N + \frac{1}{2})$ ,  $N|n\rangle = n|n\rangle$ ,
- $a^+|n\rangle = |n+1\rangle$
- Particle in box:  $E_n = \frac{n^2\hbar^2}{8mL^2}$ ,  $\psi_n = C\sin(\frac{n\pi x}{L})$

# Hidden subgroup

- $S \leq G$ ,  $f(x) = f(y)$  iff  $x + S = y + S$