Quantum Computing

A brief introduction

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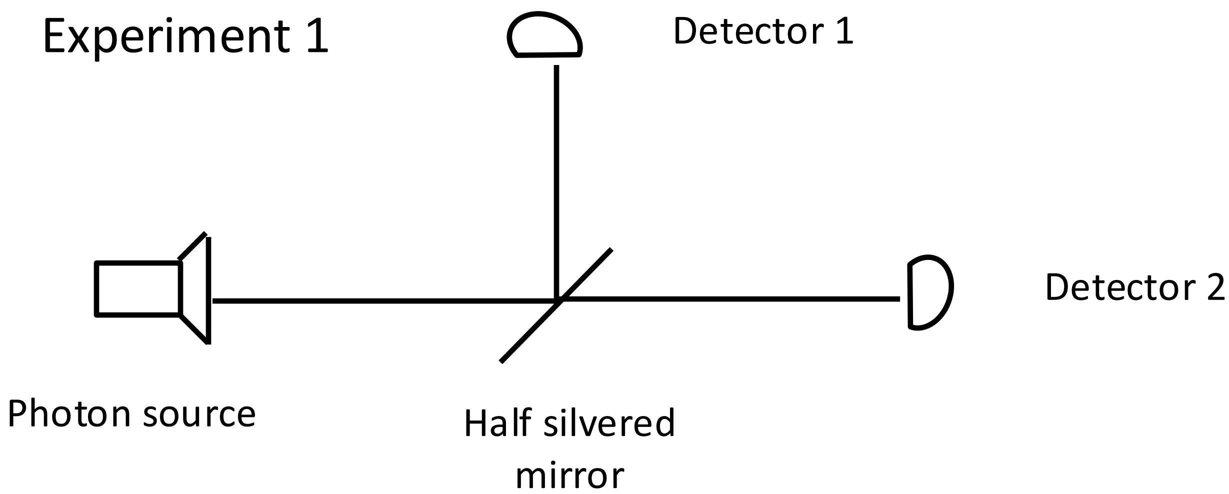
Beam splitters and QM

I can safely say that no one understands Quantum Mechanics - Feynman



Photon source emits stream of photons.

P(photon arrives at Detector 1)= .5

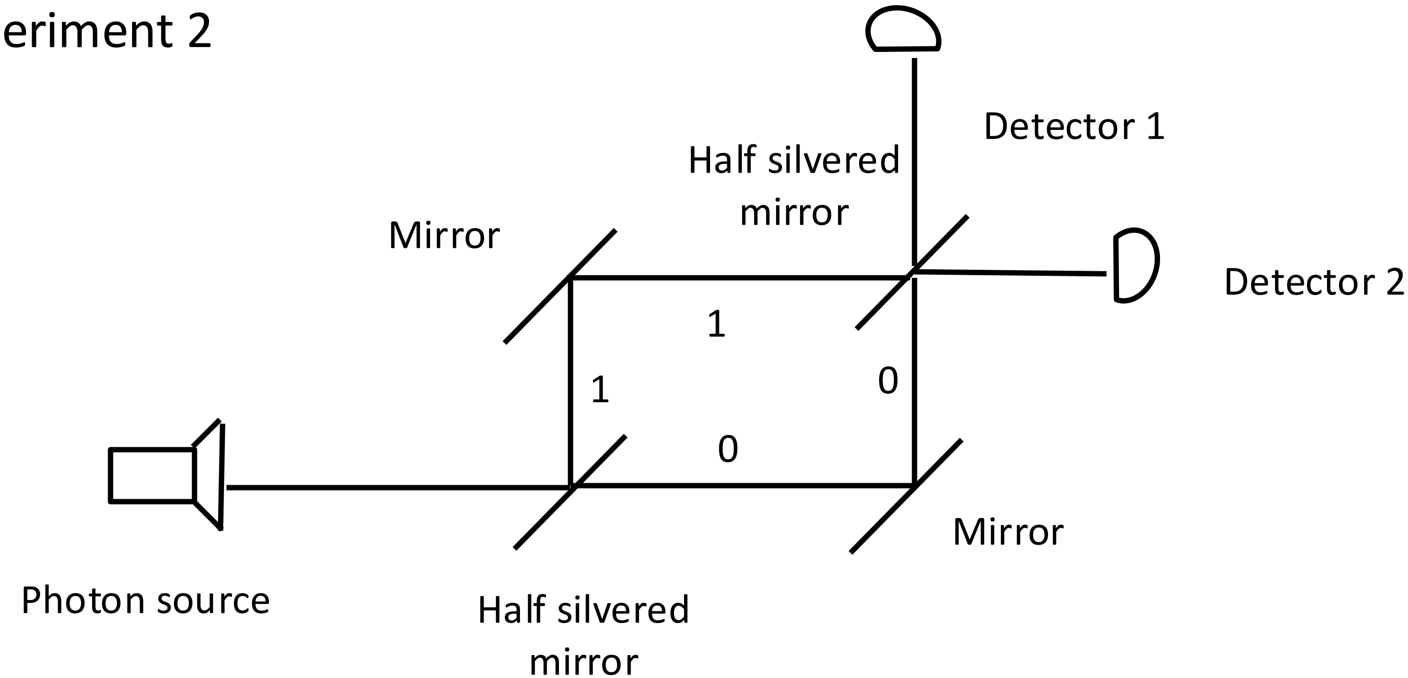
P(photon arrives at Detector 2)= .5

So far, so good

Beam splitters and QM

Mach-Zender Interferometer

Experiment 2



Photon source emits stream of photons.

P(photon arrives at Detector 1)= 0

P(photon arrives at Detector 2)= 1

Huh?

According to QM

Analysis

Beam splitter causes the photon to go into superposition:

$\alpha_1|0> + \alpha_2|1>$, $|\alpha_1|^2 = \frac{1}{2}$, $|\alpha_2|^2 = \frac{1}{2}$. $|0>$ state is right, $|1>$ is up.

$|0> = \binom{1}{0}$, $|1> = \binom{0}{1}$.

Beam splitter acts on incoming state via the matrix $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$.

In experiment 1, if all photons leave source in state $\binom{1}{0}$, after the splitter they are in state $\frac{1}{\sqrt{2}}\binom{1}{i}$. So, they arrive at detector 1 with probability $\frac{1}{2}$ and detector 2 with probability $\frac{1}{2}$.

However, going through another beam splitter, in experiment 2, yields the output state:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \binom{1}{0} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \end{pmatrix}.$$

So, they always arrive at detector 2.

Postulates

1. State of a system is a unit vector over $\mathbb{C}$ in Hilbert space ($\mathcal{H}$) of dimension 2^n
   - A qubit is a quantum system, with $n = 1$. A one qubit system is in general state $|\psi> = a|0> + b|1>$, $a\bar{a} + b\bar{b} = 1$
2. A system, with state, $|\psi(t)>$, evolves according to a unitary operator, namely, $U(|\psi(0)>)$
   - U is unitary if $(x, y) = (Ux, Uy)$. Note $U\overline{U^T} = I$
   - Example is a Hamiltonian: $H(t)|\psi(t)> = \mathrm{i}\hbar \frac{d|\psi(t)>|}{dt}$
   - $|\varphi(t_2)> = e^{-i\hbar H(t_2 - t_1)}|\varphi(t_1)>$
3. Two physical systems $\mathcal{H}_1$ and $\mathcal{H}_2$ can be treated as a single system, $\mathcal{H}_1 \otimes \mathcal{H}_2$. If $\mathcal{H}_1$ is in state, $|\psi_1>$ and $\mathcal{H}_2$ is in state, $|\psi_2>$, the joint state is $|\psi_1> \otimes |\psi_2>$
4. Given an orthonormal basis $\mathcal{B} = \{\varphi_i\}$, one can perform a von-Neuman measurement $\mathcal{H}_A$ on $|\psi> = \sum_i \alpha_i|\varphi_i>$ that outputs i with probability $|\alpha_i|^2$. It is projective. Further, if $|\psi> = \sum_i \alpha_i|\varphi_i>|\gamma_i>$ $\mathcal{H}_A \otimes \mathcal{H}_B$ measurement yields i with probability $|\alpha_i|^2$ and leaves state in $|\varphi_i>|\gamma_i>$. $M = \sum m_i P_i = \sum m_i\ |i><i|$

Linear Algebra

- Dirac Notation: Element in Hilbert space of dimension 2^n is represented by n-entry symbol. $|000 \dots 00 > \leftrightarrow (1, 0, \dots, 0)^T$, $|000 \dots 01 > \leftrightarrow (0, 1, \dots, 0)^T$, …, $| 111 \dots 1 > \leftrightarrow (0,0, \dots, 1)^T$ where column vectors have 2^n coordinates.
- Notation: $|0 > \otimes |0 > \otimes \dots \otimes |0 > = |000 \dots 0 >$
- A is normal if $A\bar{A}^T = \bar{A}^T A$
- Spectral Theorem: If T is a normal operator in the Hilbert space $\mathcal{H}$, there is an orthonormal basis v_i; each is an eigenvector of T. For every such , there is a unitary matrix, P, $T = P\Lambda P^*$, and Λ is diagonal.
- Dual basis
- Inner product: $(v_1, v_2, \dots, v_n) \cdot (w_1, w_2, \dots, w_n) = \sum_{i=0}^{n} \overline{v_i}\, w_i$
- Outer product: $(|\psi >< \phi|)|\gamma > = |\psi >(< \phi|\gamma>)$
- Theorem: Every linear operator can be written as $T = T_{m,n}|b_m >< b_n|$,
- $T_{m,n} =< b_m|T|b_n >$

Linear Algebra (continued)

<u>Tensor product</u>: If $|\varphi_i> = \binom{\alpha_0}{\alpha_1}$ is a basis for $\mathcal{H}_1$ and $|\phi_i> = \binom{\beta_0}{\beta_1}$ is a basis for $\mathcal{H}_2$,

$|\varphi_i> \otimes |\phi_i>$ is a basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$. $\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} \otimes \begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix} = (\alpha_0\beta_0, \alpha_0\beta_1, \alpha_1\beta_0, \alpha_1\beta_1)^T$.

$$A \otimes B = \begin{matrix} a_{11}B & \dots & a_{1n}B \\ \dots & \dots & \dots \\ a_{n1}B & \dots & a_{nn}B \end{matrix}$$

<u>Schmidt decomposition</u>: If $|\psi> \in \mathcal{H}_1 \otimes \mathcal{H}_2$, there is an orthonormal basis $|\varphi_i>$ for $\mathcal{H}_1$ and an orthonormal basis $|\phi_i>$ for $\mathcal{H}_2$ and $p_i \geq 0$ such that $|\psi> = \sum_i \sqrt{p_i}\, |\varphi_i> |\phi_i>$

$Tr(A) = < b_n|A|b_n >$

<u>Eigenvector</u>: $T|\psi> = c|\psi>$

More notation

- $A \otimes B = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}$, $x \otimes \mathrm{y} = (x_1y_1, x_1y_2, \dots, x_ny_n)^T$
- $|v> = (\ v_1,\ \ v_2,\ \ \dots,\ \ v_n\ )^T$, $<w| = (\ w_1,\ \ w_2,\ \ \dots,\ \ w_n\ )$ then

$|v><w| = \begin{matrix} v_1\overline{w_1} & v_1\overline{w_2} & \dots & v_1\overline{w_n} \\ \dots & \dots & \dots & \dots \\ v_n\overline{w_1} & v_n\overline{w_2} & \dots & v_n\overline{w_n} \end{matrix}$, so $I = \sum |\mathrm{i}><i|$ and $M = \sum M_{ij}|\mathrm{i}><j|$

- Pauli matricies
  - $\sigma_0 = I$, $\sigma_1 = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_2 = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $\sigma_3 = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
  - $[X, Y] = iZ, [Y, Z] = iX, [Z, X] = iY$
- $|+> = \frac{1}{\sqrt{2}}(|0>\ +|1>)$, $|-> = \frac{1}{\sqrt{2}}(|0> -\ |1>)$, $|i> = \frac{1}{\sqrt{2}}(|0> +i|1>)$, $|-$ $i> = \frac{1}{\sqrt{2}}(|0> -\ i|1>)$

Mixed states and density

- For pure states, $|\psi>$, density is $\rho = |\psi >< \psi|$
- Mixed states: $\{(p_1, |\psi_1>), (p_2, |\psi_2>), \ldots, (p_n, |\psi_n>)\}$, where the probability that the system is in pure state $|\psi_i>$ is p_i and $\sum p_i = 1$
- Density operator for mixed state is $\sum p_i |\psi_i >< \psi_i|$
- Bloch Sphere
  - Pure state in general position is $|\psi> = \cos\left(\frac{\theta}{2}\right) \Big|0> + e^{i\varphi} \sin\left(\frac{\theta}{2}\right)\Big|1>$.
  - For mixed state $|\psi_i> = p_i(\alpha_{X,i}, \alpha_{Y,i}, \alpha_{Z,i})$ on interior of Block sphere
  - $\rho = \sum p_i |\psi_i> < \psi_i|$ evolves as $\rho = \sum p_i U|\psi_i> < \psi_i|U^\dagger$
  - $\rho = \frac{1}{2} I + \alpha_X X + \alpha_Y Y + \alpha_Z Z$

- $P(|0>) = < 0|\psi> < \psi|0> = Tr < 0|\psi> < \psi|0) = Tr(|0 >< 0|\ |\psi >< \psi|$

Mixed states and density

- Partial trace: Consider composite system AB.
  - $\rho^A = Tr_B(\rho^{AB})$
  - $Tr_B(|a_1><a_2| \otimes <b_1|) < b_2|\ ) = |a_1><a_2| Tr(|b_1><b_2|\ ) = |a_1><a_2| < b_2|b_1>$
  - Example
    - $\rho = \frac{1}{2}(|00><00| + |00><11| + |11><00| + |11><11|)$

$$= \frac{1}{2} Tr(|0><0| \otimes |0><0| + |0><1| \otimes |0><1| + |1><0| \otimes |1><0| + |1><1| \otimes |1><1|)$$

$$= \frac{1}{2}(|0><0| + |1><1|)$$

Circuits and gates

- Universal gate set: A gate set is universal if $\forall n > 0$, any n-bit unitary operator can be approximated to arbitrary accuracy by a quantum circuit from this set
- An entangling gate is on that for an input product state $|\alpha > |\beta >$, the output state is not a product state (e.g.-CNOT).
  - Example: $|\psi >= \frac{1}{\sqrt{2}}(|00 > \ +|11 >)$
- Theorem: A set of states with an entangling 2-qubit gate together with all 1-qubit gates is universal.
- Theorem: If U is a 1-qubit gate, $U = e^{ix} R_z(\beta) R_y(\gamma) R_z(\delta)$

Gates and states

- <u>General position on Bloch sphere</u>: $|\psi> = \cos\left(\frac{\theta}{2}\right)\Big|0> \ + e^{i\varphi} \sin\left(\frac{\theta}{2}\right)\Big| 1 >$

- <u>Measurement</u>: $I = \sum |i >< i|$, $M = \sum m_i P_i$, M is Hermitian, $P_i = |i > < i|$.

- <u>Controlled gates</u>:
  - $c - U|0 > |\psi > = \ |0 > \ |\psi >$
  - $c - U|1 > |\psi > = \ |1 > U|\psi >$

Common gates

- Pauli gates

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathrm{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Note: $X^2 = Y^2 = Z^2 = I$

- Hadamard

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

$$H^{\otimes n}(|0000 \ldots 0>) = \frac{1}{\sqrt{2}}(|0> + |1>) \otimes \frac{1}{\sqrt{2}}(|0> + |1>) \otimes \ldots \otimes \frac{1}{\sqrt{2}}(|0> + |1>)$$

- Rotation

$$R_X(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{iX\theta} \end{pmatrix} = \begin{pmatrix} e^{-iX\theta/2} & 0 \\ 0 & e^{iX\theta/2} \end{pmatrix}$$

- 2 qubit gate

$$CNOT(|xy>) = |x, x \oplus y>$$

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

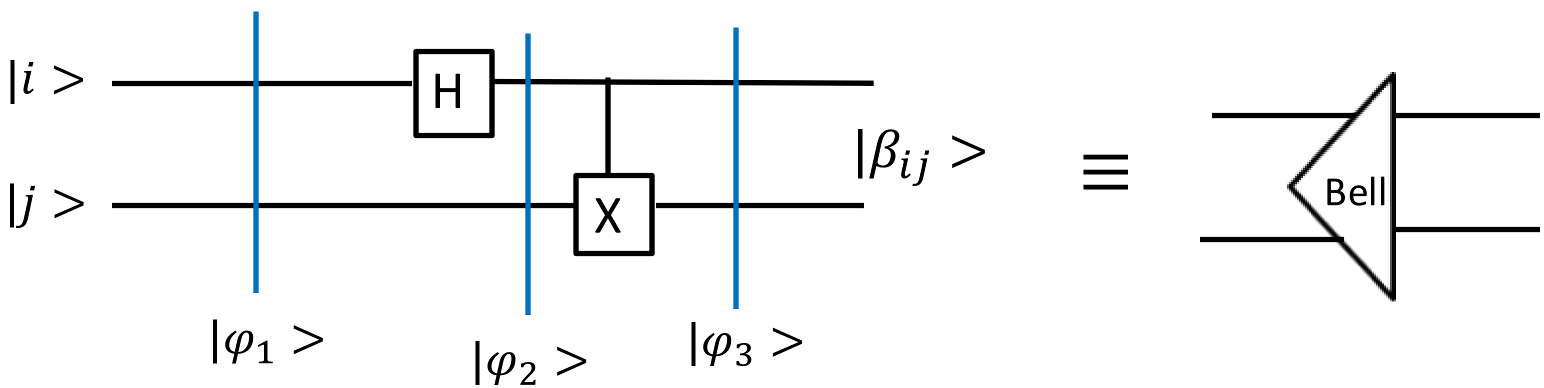
- If $A^2 = 1$, $e^{i\theta X} = \mathrm{I}\cos(\theta) + iXsin(\theta)$

Measurement in alternate basis

- Computational basis is $|i>$. $U|\varphi_j>=|j>$
- Suppose we want to measure $|\psi>$ with respect to basis $B=\{|\varphi_j>\}$
- $|\psi>= \sum \alpha_j |\varphi_j>$
- To measure wrt $B=\{|\varphi_j>\}$, Project $|\psi>$ onto $|\varphi_j><\varphi_j|$
- $\left(Tr\left(|\psi><\psi||\varphi_j><\varphi_j|\right)=Tr\left(<\varphi_j|\psi><\psi|\varphi_j>\right)= \alpha_j{}^2\right.$
- $\rho=|\psi><\psi|$ is density operator for the pure state $|\psi>$.
- $\rho= \sum p_i |\psi_i><\psi_i|$ is the density operator for mixed states $\{(p_i,|\psi_i>)$

Converting to Bell Basis

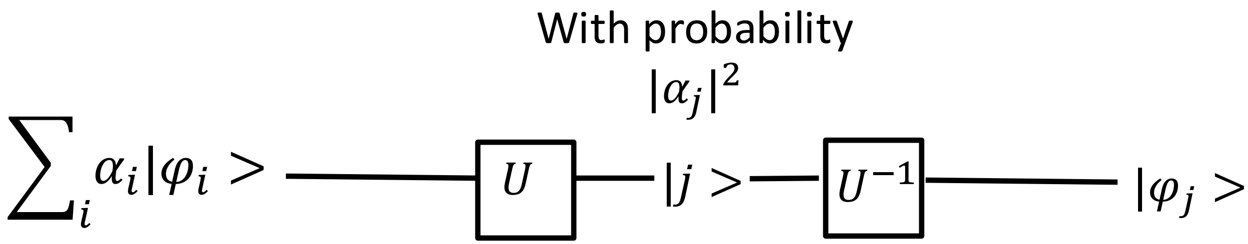
- Computational basis is $|i>$, $U|\varphi_j> = |j>$
- $|\beta_{00}> = \frac{1}{\sqrt{2}}(|00> + |11>)$, $|\beta_{01}> = \frac{1}{\sqrt{2}}(|01> + |10>)$
- $|\beta_{10}> = \frac{1}{\sqrt{2}}(|00> - |11>)$, $|\beta_{11}> = \frac{1}{\sqrt{2}}(|01> - |10>)$

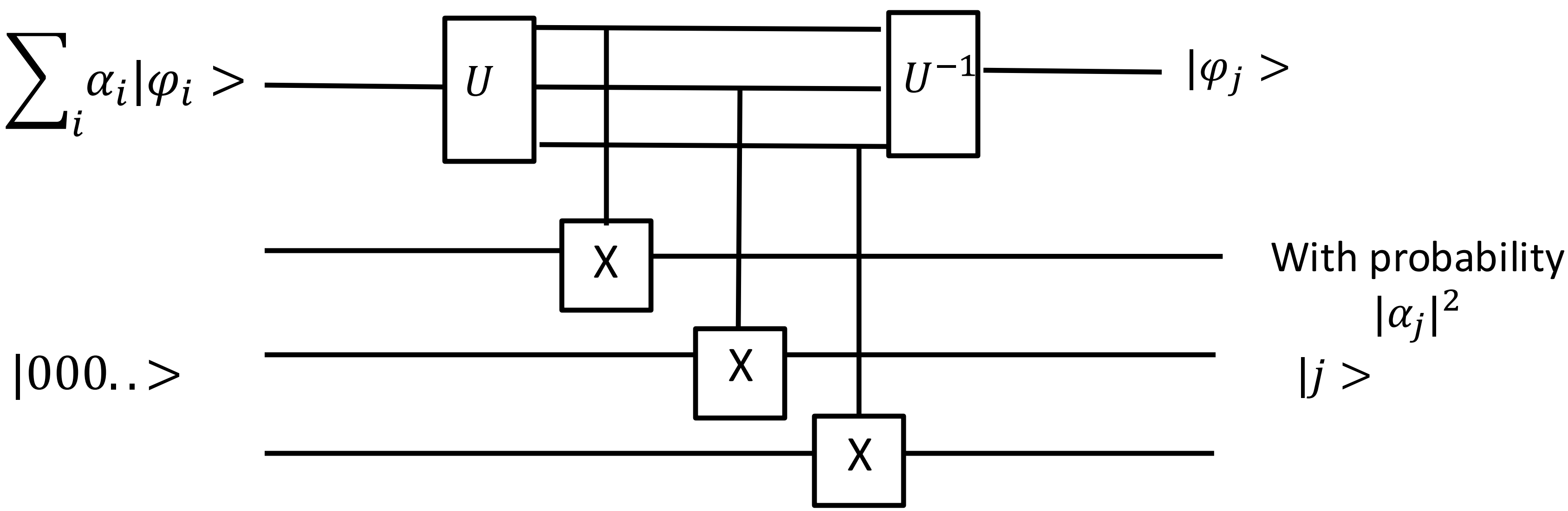


- $|\varphi_1> = |00>$
- $|\varphi_2> = \frac{1}{\sqrt{2}}(|0> + |1>)|0> = \frac{1}{\sqrt{2}}(|00> + |10>)|$
- $|\varphi_3> = |\beta_{00}>$

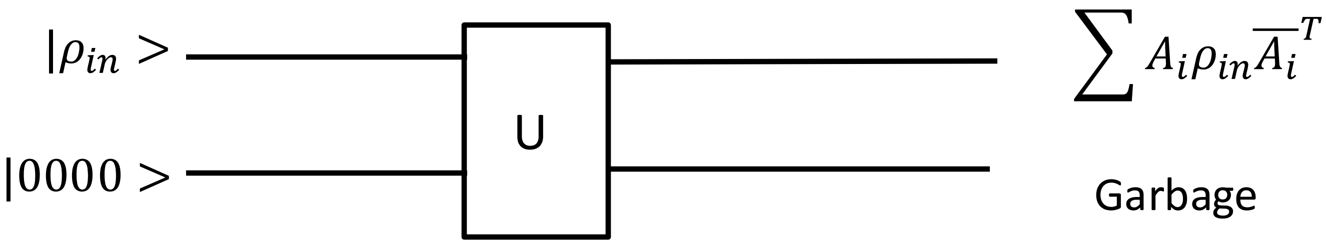
Changing Measurement Basis

- Suppose $|\varphi_i>$ is a basis and our measurement basis is $|i>$, $U|\varphi_i> = |i>$





Superoperator and mixed states



- $\rho = |\psi><\psi|$, $U|\psi>$ has density ρ = $U|\psi><\psi|\overline{U}^T> = U\rho U^\dagger$
- $<0|\psi><\psi|0> = <0|\rho|0> = P(|0>)$
- $\rho = \sum_i p_i |\psi_i><\psi_i|$
- $Tr(A) = <b_n|A|b_n>$
- $\rho_{in} \rightarrow \rho_{out} = Tr_b(U(\rho_{in} \otimes |000 \ldots><000 \ldots 0|U^\dagger)$
- $\rho_{in} \rightarrow \sum A_i \rho_{in} {A_i}^\dagger$, where A_i are Kraus operators with $\sum {A_i}^\dagger A_i = I$

No Cloning Theorem

- Qubits can't be copied

- Proof

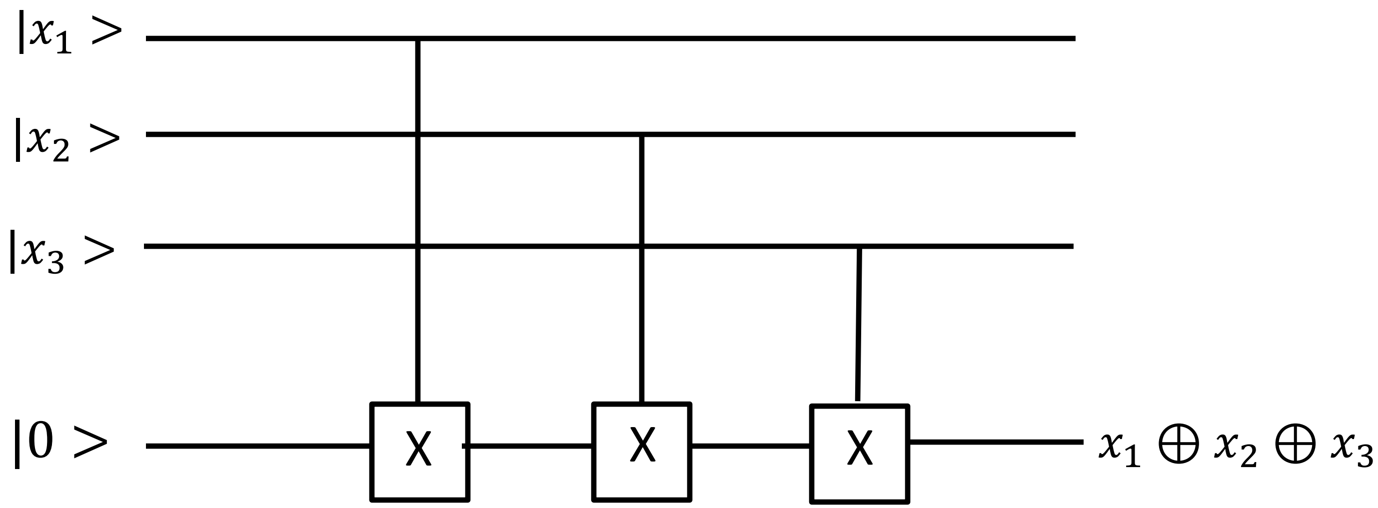
  Suppose they can be. Then there is an operator, U, such that for any state $|\varphi>$, $U(|\varphi>|0>) = |\varphi>|\varphi>$. Now let $|\psi>$ and $|\phi>$ be non-orthogonal, different pure states.

  $(|\psi>|0>, |\phi>|0>) = <\psi|\phi><0|0> = <\psi|\phi>$.

  Since U is unitary,

  $<\psi|\phi> = (|\psi>|0>, |\phi>|0>) = (U|\psi>|0>, U|\phi>|0>) = (|\psi>|\psi>, |\phi>|\phi>) = <\psi|\phi>^2$. So, $<\psi|\phi> = 1$. This is a contradiction.
- No checkpointing

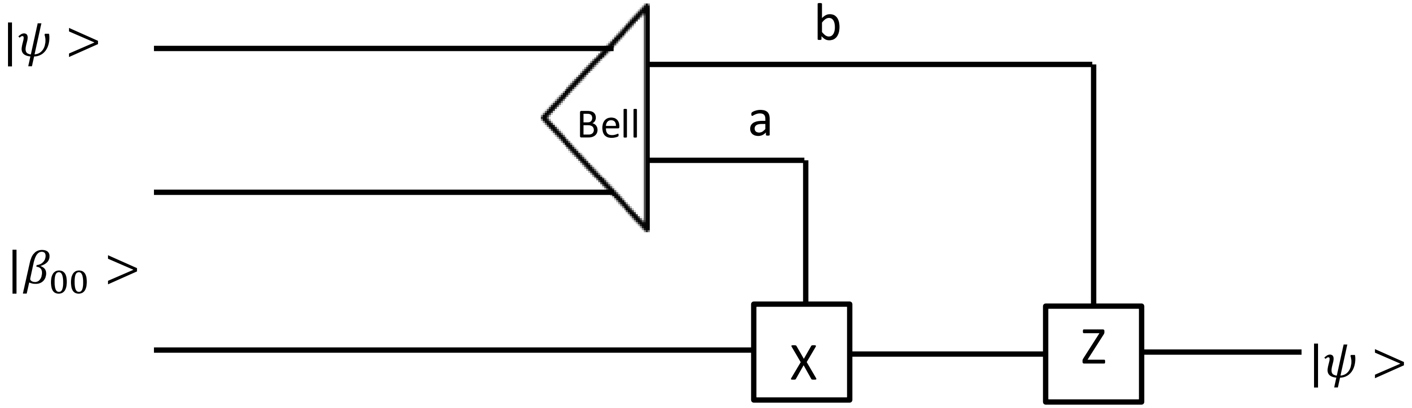
Parity Circuit



Superdense coding

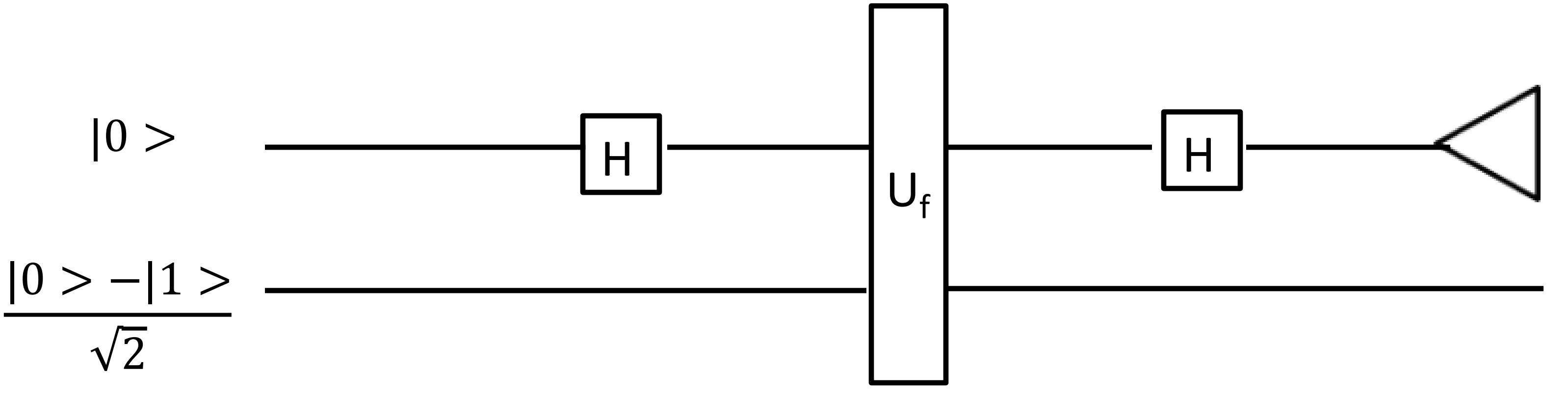
- Alice and Bob share $|\beta_{00}>$, Alice has first bit, Bob second bit
- Alice performs one of I, X, Y, Z producing $I \otimes I$ (to send 00), $X \otimes I$ (to send 01), $Y \otimes I$ (to send 10) or $Z \otimes I$ (to send 11).
- Bob measures joint state qubit measurement

- Can be used to teleport $|\psi>$:
  - $I \otimes I := \frac{1}{\sqrt{2}}(|00> + |11>) \rightarrow \frac{1}{\sqrt{2}}(|00> + |11>)$
  - $X \otimes I := \frac{1}{\sqrt{2}}(|00> + |11>) \rightarrow \frac{1}{\sqrt{2}}(|01> + |10>)$
  - $Z \otimes I := \frac{1}{\sqrt{2}}(|00> + |11>) \rightarrow \frac{1}{\sqrt{2}}(|00> - |11>)$
  - $ZX \otimes I := \frac{1}{\sqrt{2}}(|00> + |11>) \rightarrow \frac{1}{\sqrt{2}}(|01> - |10>)$



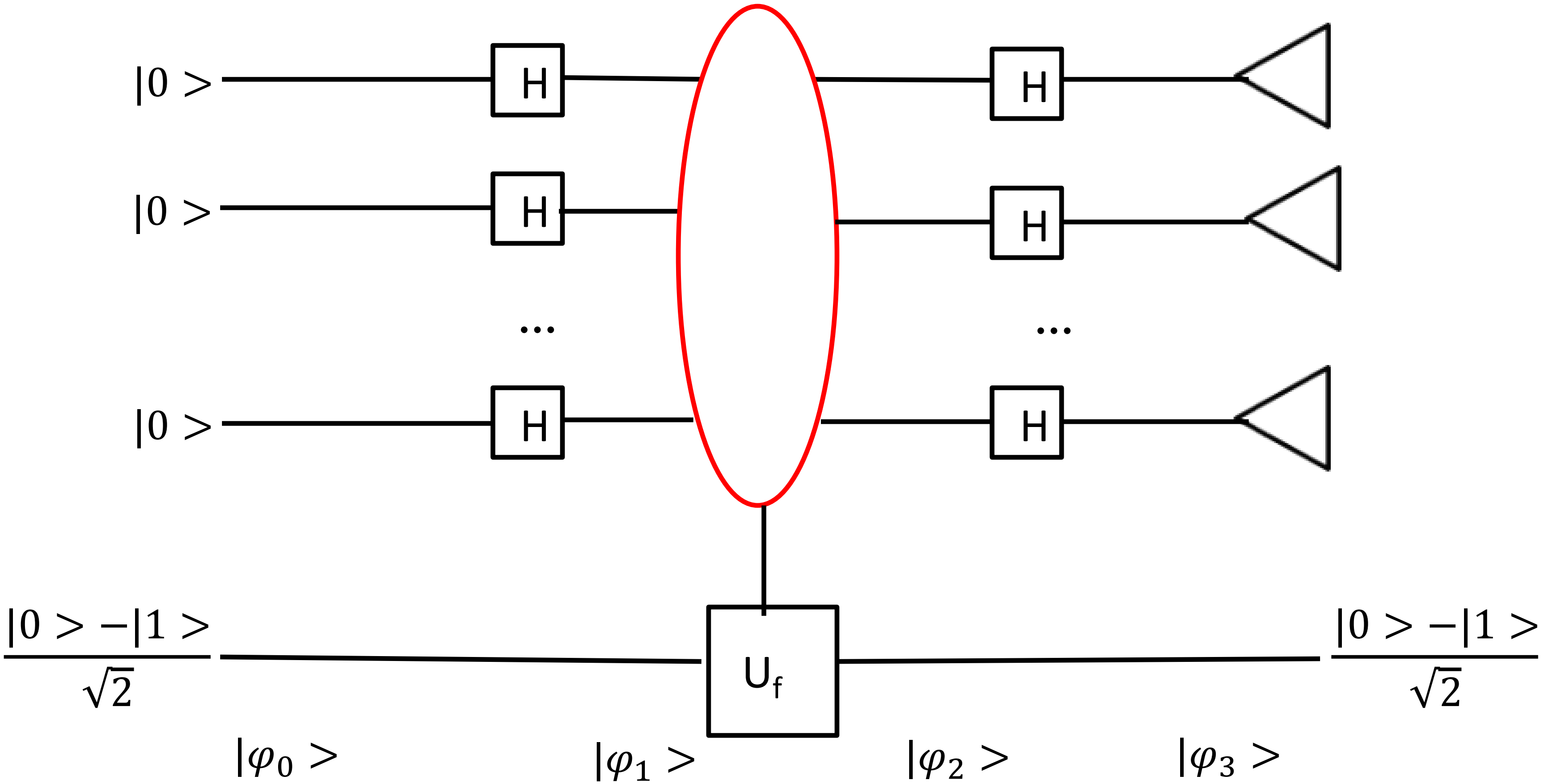
Deutch

- Problem: Determine $f(0) + f(1)$ in one measurement
- $U_f|x > |y > = |x > |y \oplus f(x) >$
- If $f(0) + f(1) = 1$, $\left|\psi_3 > = \ (-1)^{f(0)}\right|1 > \frac{|0>-|1>}{\sqrt{2}}$
- If $f(0) + f(1) = 0$, $\left|\psi_3 > = \ (-1)^{f(0)}\right|0 > \frac{|0>-|1>}{\sqrt{2}}$



Deutch-Josza

- Problem: $f: \{0,1\}^n \to \{0,1\}$, which is either constant or balanced.
- Which is it?
- Put $U_f|\boldsymbol{x}>|y> = \ |\boldsymbol{x}>|y \oplus f(\boldsymbol{x})>$, $\boldsymbol{x}$ is an n-bit quantity



DJ

- $|\varphi_0> = \ |0>^{\otimes n}\frac{|0>-|1>}{\sqrt{2}}$
- $|\varphi_1> = \ \frac{1}{\sqrt{2^n}}\sum_{\boldsymbol{x}}|\boldsymbol{x}>\frac{|0>-|1>}{\sqrt{2}}$
- $|\varphi_2> = \ \frac{1}{\sqrt{2^n}}\sum_{\boldsymbol{x}}(-1)^{f(\boldsymbol{x})}|\boldsymbol{x}>\frac{|0>-|1>}{\sqrt{2}}$
- $|\varphi_3> = \ \frac{1}{2^n}\sum_{\boldsymbol{x}}\sum_{\boldsymbol{z}}|(-1)^{f(\boldsymbol{x})+\boldsymbol{x}\cdot\boldsymbol{z}}\boldsymbol{z}>\frac{|0>-|1>}{\sqrt{2}}$

Simon

- $f\colon \{0,1\}^n \rightarrow X, \exists\, \vec{s} = s_1, s_2, \ldots, s_n$: $f(x) = f(y)$ iff $x = y$ or $x = y + \vec{s}$
- $U_f\colon |x>|b> = |x>|b \oplus f(x)>$
- $H^{\otimes n}(|x>) = \frac{1}{\sqrt{2^n}} \sum_z (-1)^{x \cdot z}\, |z>$

1. $i = 1$
2. Prepare $\frac{1}{\sqrt{2^n}} \sum_x |x>|0>$
3. Apply U_f to get $\frac{1}{\sqrt{2^n}} \sum_x |x>|f(x)>$
4. Measure second bit
5. Apply $H^{\otimes n}$ to first register
6. Measure first register to get w_i
7. If $din(w_i) \neq n$ −1, go to 2
8. Output s: $w^t s^t = 0$

Phase kick back

- $CNOT\left(\frac{|0> + |1>}{\sqrt{2}} \frac{|0> - |1>}{\sqrt{2}}\right) = \frac{|0> - |1>}{\sqrt{2}} \frac{|0> - |1>}{\sqrt{2}}$

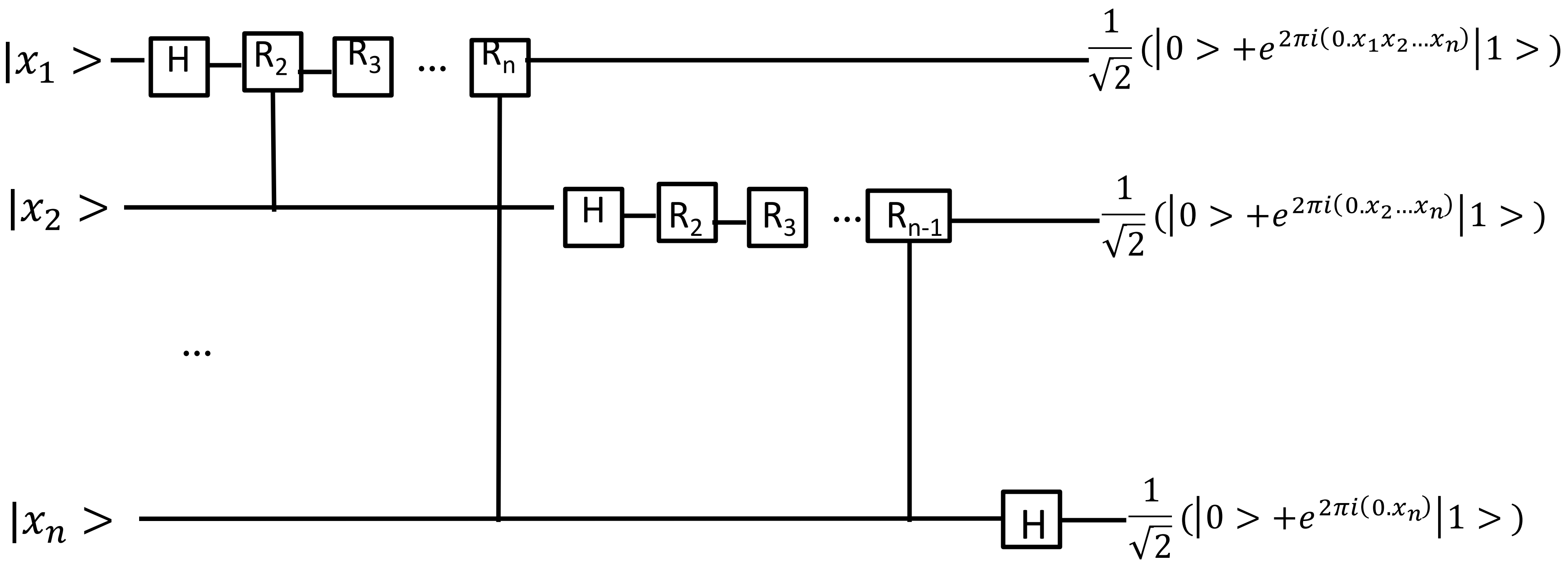
Phase Estimation

- Phase estimation problem: Given $|\psi> = \frac{1}{\sqrt{2^n}} \sum_y e^{2\pi i \omega y} |\mathrm{y}>$, estimate ω
- <u>Theorem:</u> $\frac{x}{2^n} \leq \omega \leq \frac{x+1}{2^n}$ with probability $\geq \frac{8}{\pi^2}$
- $e^{2\pi i 2^k .x_1 x_2 \ldots} = e^{2\pi i (x_{k+1} x_{k+2} \ldots)}$
- Suppose $\omega = .x_1$, $|\psi> = \frac{1}{\sqrt{2}} \sum_{|y>} e^{2\pi i \omega |y>} = \frac{1}{\sqrt{2}} (|0> + (-1)^{x_1} |1>$ and $H(|\psi>) = |x_1>$
- In general, $H^{\otimes n}(|\boldsymbol{x}>) = \frac{1}{\sqrt{2^n}} \sum_{\boldsymbol{y}} (-1)^{\boldsymbol{x} \cdot \boldsymbol{y}} |\boldsymbol{y}>$ and $H^{\otimes n}(H^{\otimes n}(|\boldsymbol{x}>)) = |\boldsymbol{x}>$
- So, $|\psi> = \frac{1}{\sqrt{2^n}} \sum_{|y>} e^{2\pi i \omega y>} |y> = \frac{1}{\sqrt{2}} (\left|0> + e^{2\pi i 2^{n-1} \omega}\right|1>) \otimes \frac{1}{\sqrt{2}} (\left|0> + e^{2\pi i 2^{n-2} \omega}\right|1>) \otimes \cdots \otimes \frac{1}{\sqrt{2}} (\left|0> + e^{2\pi i \omega}\right|1>)$
- Denote $R_n = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i 2^{-n}} \end{pmatrix}$

Quantum Fourier Transform

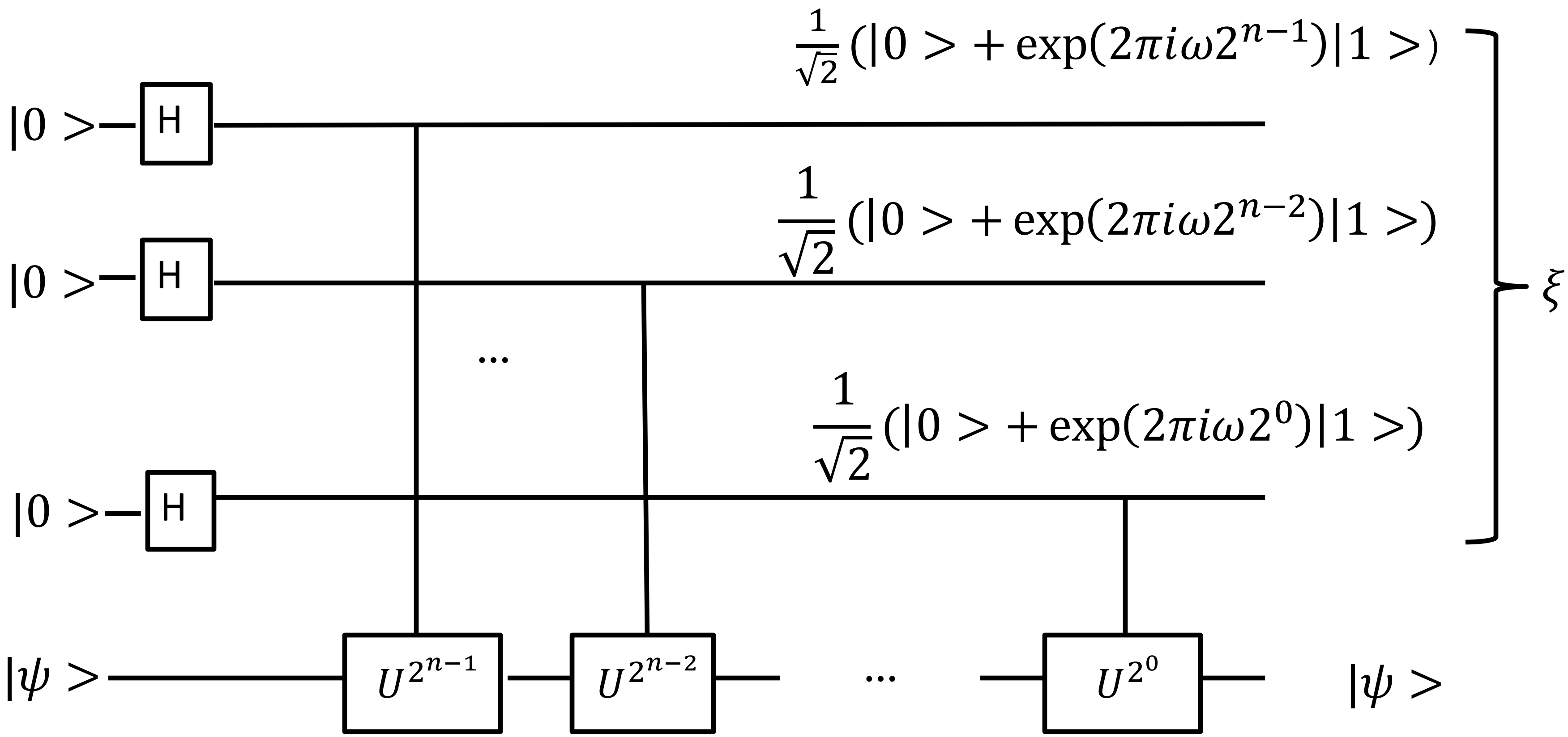
- $H^{\otimes n}(|\boldsymbol{x}>) = \frac{1}{\sqrt{2^n}} \sum_{\boldsymbol{y}} (-1)^{\boldsymbol{x} \cdot \boldsymbol{y}} |\boldsymbol{y}>$
- $H^{\otimes n}(H^{\otimes n}(|\boldsymbol{x}>)) = |\boldsymbol{x}>$
- $QFT_m(|x>) = \frac{1}{\sqrt{m}} \sum_{y=0}^{m-1} e^{2\pi i^{x}/_{m} y} |y>$
- $QFT_m{}^{-1}(|x>) = \frac{1}{\sqrt{m}} \sum_{y=0}^{m-1} e^{-2\pi i^{x}/_{m} y} |y>$

Quantum Fourier Circuit



Eigenvalue Estimation

- Suppose $|\psi>$ is an eigenstate of a unitary operator, U, so $U\,|\psi> = \exp(2\pi i\phi)|\psi>$. $|\phi\phi> = .x_1 x_2 \ldots x_n$ (a binary expansion)



Eigenvalue Estimation

- Applying QFT to $|0>^{\otimes n}$ produces $\frac{1}{\sqrt{2^n}}\sum_{y=0}^{2^n-1}\exp(2\pi i(\frac{y}{2^n}))|\mathrm{y}>$ (same output as previous slide)
- $U|\psi> = \exp(2\pi i\phi)\ |\psi>$, so $U^{2^j}|\psi> = \exp\left(2\pi i\phi 2^j\right)\ |\psi>$.
- Applying ${QFT_n}^{-1}$ to ξ, gives $< x_1,\ x_2, \ldots, x_n >$, where $|\phi> =. x_1 x_2 \ldots x_n$
- Measure χ to get ϕ
- $\frac{y}{2^n}$ is a good estimate for $\phi = \frac{j}{r}$

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Factorization using order finding (Shor)

- Suppose $N = pq$ and $a^r = 1\ (mod\ N)$ then $r = 0\ \big(mod\ \varphi(pq)\big)$
- If r is even, say, $r = 2s$, $(a^s + 1)(a^s - 1) = 0\ (mod\ pq)$.
- There is a good chance $p|(a^s - 1)$ but $\big(q, (a^s - 1)\big) = 1$.
- Then $\big((a^s - 1), \mathrm{N}\big) = \mathrm{p}$. Voila!
- Note that $|v_t> = \frac{1}{r}\sum_{k=0}^{r-1} \exp(-\frac{2\pi i k t}{r})|k(mod\ N)>$ is an eigenvalue of $U_x\ (k) = |xk\ (mod\ N)>$.
- In Shor, $|1> = \frac{1}{\sqrt{r}}\sum |v_t>$.
- Applying QFT^{-1} to control gives phase of eigenvalues
- Measurement of target gives $|\frac{s}{r}>$ with $\mathrm{Pr}(|y>) = \frac{1}{2^{2n}}\ |\frac{1-r^{2^n}}{1-r}|^2$, where $r = \exp(-2\pi i(\frac{y}{2^n} - \phi))$

Order Finding

<u>Problem</u>: Given $a, N \in \mathbb{Z}$ with $(a, N) = 1$, find r: $a^r (mod\ N) = 1$

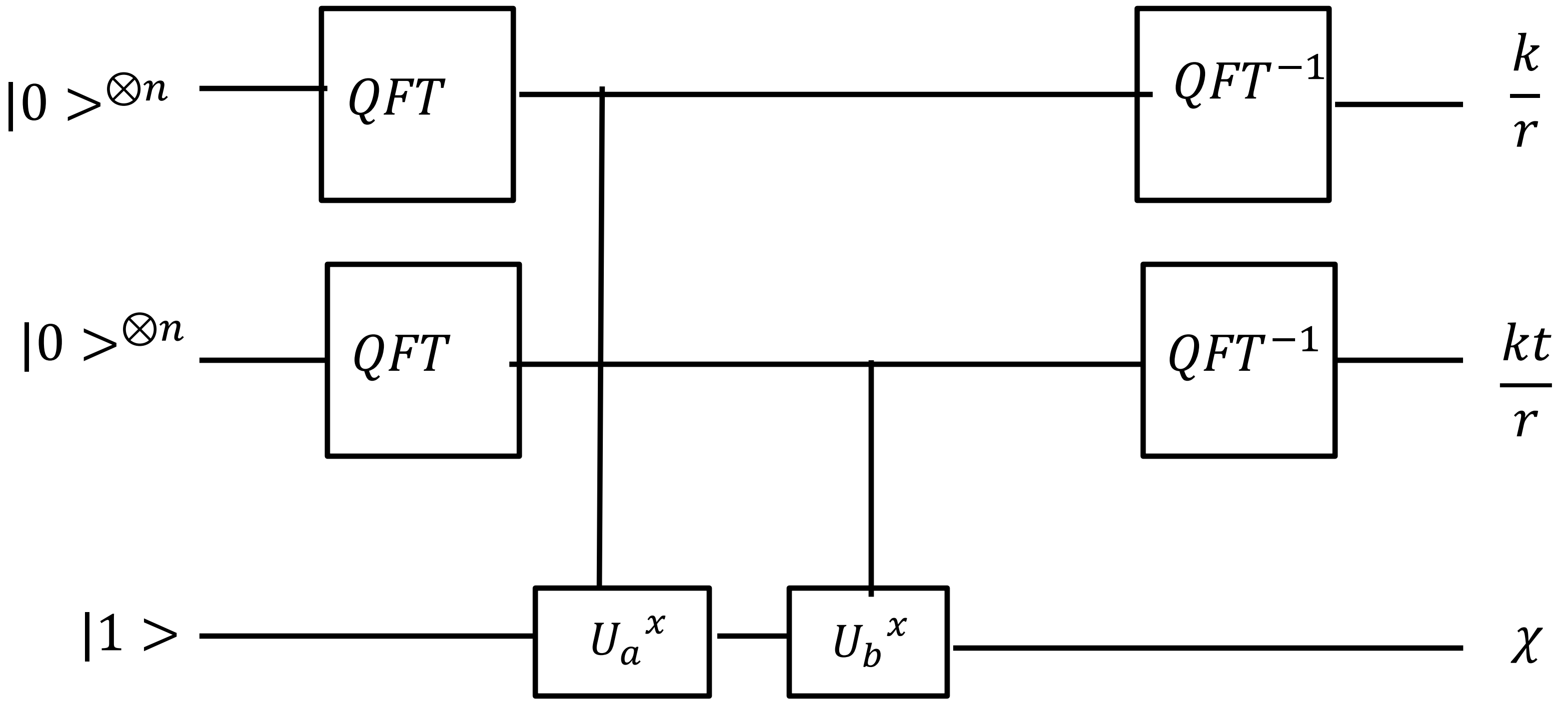
1. Choose n: $2^n \geq 2r^2$
2. Initialize control register $|000 \ldots 0 > = |0 >^{\otimes 2n}$
3. Initialize target register to $= |000 \ldots 01 > = |000 \ldots 0 > = |0 >^{\otimes 2n} \otimes |1 >$
4. Apply QFT to control register
5. Apply $c - {U_a}^x$ to control and target register
6. Apply QFT^{-1} to control register
7. Measure CR to get estimate of $\frac{x_1}{2^n}$ of multiple of $\frac{1}{r}$
8. Use continued fraction to get c_1, r_1: $\left|\frac{x_1}{2^n} - \frac{c_1}{r_1}\right| \leq 2^{-(n-1)}/_2$
9. Repeat 1-8 to get $c_2, 2$: $\left|\frac{x_2}{2^n} - \frac{c_2}{r_2}\right| \leq 2^{-(n-1)}/_2$, if none, FAIL
10. Compute $\mathrm{r} = LCM(r_1, r_2)$ and $a^r\ (mod\ N)$
11. If $a^r\ (mod\ N) = 1$, output r, otherwise FAIL

Order Finding

- Order finding has quantum complexity $O(\lg(N)^2 \lg(\lg(N)) \lg(\lg(\lg(N)))$
- Classical complexity is $\exp(O(\sqrt{\lg(N) \lg(\lg(N))}))$

Discrete log

- Suppose $a = b^x\ (mod\ p)$, b has known order. We want r: $b^r = 1(mod\ p)$
- Put $U_a(\,|x>) = |ax\ (mod\ p) >$ and $U_b(\,|x>) = |bx\ (mod\ p) >$.
- Consider the circuit below. $|1> = \frac{1}{\sqrt{r}}\sum |v_t>$. Below, $t = xy^{-1}$



Discrete log

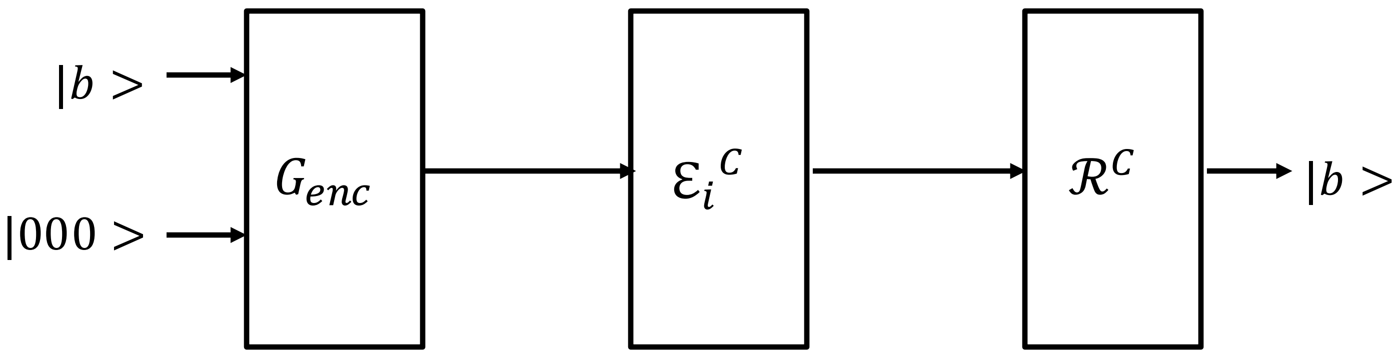
- Measuring first control register gives $|\frac{k}{r}>$
- Measuring first control register gives $|\frac{kt}{r}>$

- Quantum complexity is $O(\lg(p)^2 \lg(\lg(p)) \lg(\lg(\lg(p)))$
- Best known classical requires $\exp(O(\sqrt{\lg(p) \lg(\lg(p))})$

Hidden subgroup

- $S \leq G,\ \mathrm{f(x)} = \mathrm{f(y)}\,\mathrm{iff}\ x + S = y + S$

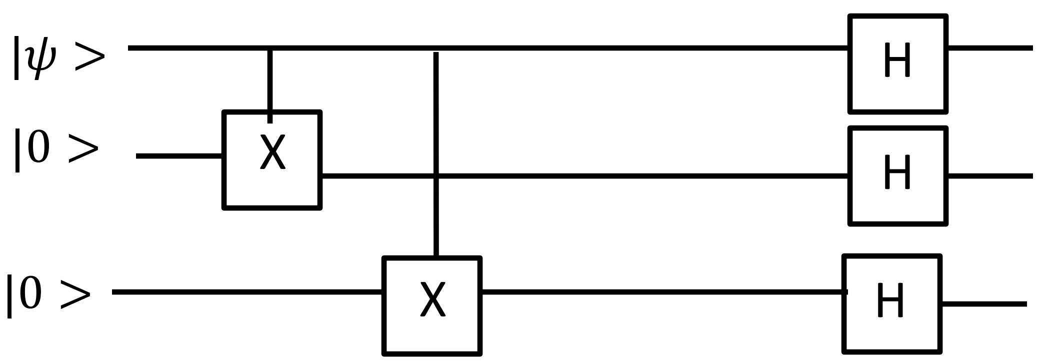
Error Correction



- Unlike classical error correction, the no cloning theorem restricts codes
- $|0>|E> \rightarrow\ \beta_1|0>|E_1>\ +\beta_2\,|1>|E_2>$
- $|1>|E> \rightarrow\ \beta_3|0>|E_3>\ +\beta_4\,|1>|E_4>$
- $|\psi> =\ \alpha_0|0>\ +\alpha_1|1> \rightarrow .\ \alpha_0\,\beta_1|0>|E_1>\ +\alpha_0\beta_2|1>|E_2>\ +\alpha_1\beta_3\,|0>$ $|E_3>\ +\alpha_1\beta_4\,|1>|E_4>$
- $|\psi> =\ \frac{1}{2}\,|\psi>(\beta_1|E_1>\ +\beta_3|E_3>) + \frac{1}{2} < Z|\psi>(\beta_1|E_1>\ \ -\beta_3|E_3>) +$ $\frac{1}{2} < X|\psi>(\beta_2|E_2>\ +\beta_4|E_4>) + \frac{1}{2} < XZ|\psi>(\beta_2|E_2>\ +\beta_4|E_4>)$

Error Correction

- $\rho = U_{err}|\psi><E|\overline{U_{err}}^T$
- $|\psi_{enc}> = U_{enc}|\psi>|000\ldots>$
- $\mathcal{E}_0 = I \otimes I \otimes I, \mathcal{E}_1 = X \otimes I \otimes I$
- $\mathcal{E}_2 = I \otimes X \otimes I, \mathcal{E}_3 = I \otimes I \otimes X$
- $\rho: |\psi><\psi| \to (1-p)|\psi><\psi| + p\,X|\psi><\psi|X>$
- $\frac{1}{\sqrt{2}}(|000> +|100>) \to \frac{1}{\sqrt{2}}(|000> +|111>) \neq \frac{1}{\sqrt{2}}(|0> +|1>)^{\otimes 3}$
- 3-bit code, Shor 9-bit code



Amplitude Amplification

- x

Grover

Search

Input: U_f: f:$\{0,1\}^n \rightarrow \{0,1\}$

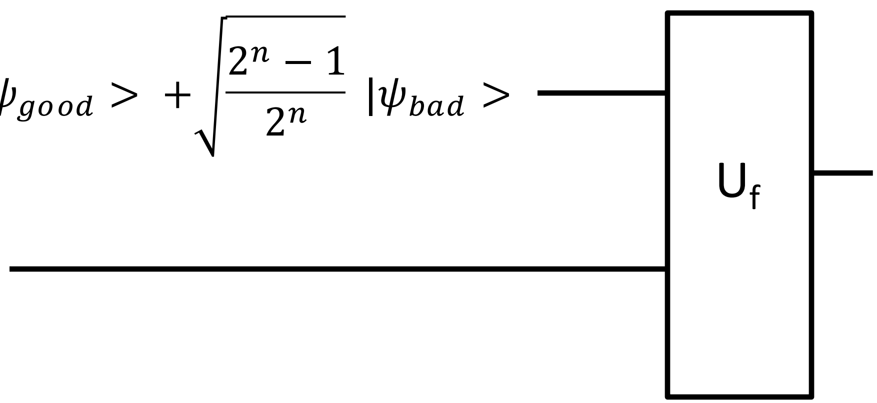
$$f(\boldsymbol{a}) = 1,\ f(\boldsymbol{x}) = 0, \mathbf{x} \neq \boldsymbol{a}$$

$$|\psi_{good}> = \boldsymbol{w}$$

$$|\psi_{bad}> = \frac{1}{\sqrt{N-1}} \sum_{\boldsymbol{x} \neq \boldsymbol{w}} |x>$$

$$\frac{1}{\sqrt{2^n}}\ |\psi_{good}>\ + \sqrt{\frac{2^n-1}{2^n}}\ |\psi_{bad}>$$

$$|0>$$



$$\frac{1}{\sqrt{2^n}}\ |\psi_{good}>|1>\ + \sqrt{\frac{2^n-1}{2^n}}\ |\psi_{bad}>|0>$$

Grover

1. Initialize n-qubits $|0000 \ldots 0>$.
2. Apply $H^{\otimes n}$ to get $\frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x$
3. Apply Grover G $\frac{\pi}{4\sqrt{n}}$ times
4. Measure output

Search

Input: U_f: $f: \{0,1\}^n \rightarrow \{0,1\}$

$$f(\boldsymbol{a}) = 1,\ f(\boldsymbol{x}) = 0, \mathbf{x} \neq \boldsymbol{a}$$

$$|\psi_{good}> = \boldsymbol{w}$$

$$|\psi_{bad}> = \frac{1}{\sqrt{N-1}} \sum_{\boldsymbol{x} \neq \boldsymbol{w}} |x>$$

Algorithm G

1. Apply U_f
2. Apply $H^{\otimes n}$
3. Apply $U_{0^\perp}$
4. Apply $H^{\otimes n}$

Algorithm $U_{0^\perp}$

$$U_{0^\perp}: |\mathrm{x}> \longrightarrow -|x>, x \neq 0$$
$$U_{0^\perp}: |0> \longrightarrow 0|x>$$

End