# Cryptanalysis

## Discrete Log Based Systems

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# HMAC's concluded

- $HMAC(K, text) = H((K \oplus opad) || H((K \oplus ipad) || text))$
- H is a cryptographic hash like SHA-256
- ipad, the inner pad: the byte 0x36 repeated B times where B is key size
- opad, the outer pad: the byte 0x5c repeated B times
- Verification requires knowledge of K.

# Discrete log based public key systems

# Discrete Log

- If $b= a^x$, then $L_a(b)=x$. $L_a(y)$ is the discrete log function.
- If $g= b^x$, then $L_a(g)=xL_a(b)$. $L_a(b_1b_2)= L_a(b_1) + L_a(b_2)$

- **Discrete Log Problem (DLP):** Given p, prime, a: $<a>=F_p^*$. B (mod p), a, unknown, find $L_a(b)$.
- **Computational Diffie Hellman Problem (CDHP):** Given p, prime, $<a>=F_p^*$. $a^a$ (mod p), $a^b$ (mod p), find $a^{ab}$ (mod p).
- **Theorem:** CDHP $\leqq_P$ DLP. If the factorization of p-1 is known and f(p-1) is $O((\ln(p))^c)$ smooth then DLP and CDHP are equivalent.

- **Conclusion:** Exponentiation is a one way trap-door function.

# El Gamal cryptosystem

- Alice, the private keyholder, picks a large prime, p, where p-1 also has large prime divisors (say, p= 2rq+1) and a generator, g, for $F_p$*. <g>= $F_p$*. Alice also picks a random number, a (secret), and computes $A=g^a$ (mod p). Alice's public key is <A, g, p>.
- To send a message, m, Bob picks a random b (his secret) and computes B= $g^b$ (mod p). Bob transmits (B, $mA^b$)= (B, C).
- Alice decodes the message by computing $CB^{-a}$=m.
- Without knowing a, an adversary has to solve the Computational Diffie Hellman Problem to get m.

- Note: b must be random and never reused!

# El Gamal Example

- Alice chooses
  - p=919. g=7.
  - a=111, A= $7^{111}$= 461 (mod 919).
  - Alice's Public key is <919, 7, 461>

- Bob wants to send m=45, picks b= 29.
  - B=$7^{29}$ =788(mod 919), $461^{29}$= 902 (mod 919),
  - C= (45)(902)= 154(mod 919).
  - Bob transmits (788, 154).

- Alice computes $(788)^{-111}$= $902^{-1}$(mod 919).
  - (54)(902)+(-53)(919)=1. 54= $902^{-1}$ (mod 919)
  - Calculates m= (154) (54)=45 (mod 919).

# El Gamal Signature

- $< g > = \mathbb{Z}_q^*$. A picks a random as in encryption.
- Signing: Signer picks k: $1 \leq k \leq p-2$ with (k, p-1)= 1 and publishes $g^k$. k is secret.
- $Sig_K(M,k)$= (t,d)
  - t= $g^k$ (mod p)
  - d=$(M-gt)k^{-1}$ (mod p-1)
- $Ver_K(M,t,d)$ iff $g^{kt}\, t^d = g^M$ (mod p)

- Notes: It's important that M is a hash otherwise there is an existential forgery attack. It's important that k be different for every message otherwise adversary can solve for key.

# Timing

- Finding g takes about $O(\lg(p)^3)$ operations, so does primality testing and raising g to the a power mod p.
- Encryption is also $O(\lg(p)^3)$ and so is decryption.
- Note that key generation is cheap but for safety, $p>w^2$, where w is the “computational power” of the adversary.

# Finding generators (Gauss)

- Find a generator, *g,* for $F_p$*, n= (p-1)= $p_1^{e1}\ p_2^{e2} \ldots p_k^{ek}$.

```
while () {
        choose a random g∈G
        for(i=1; i<=k; k++) {
           b= g^(n/pi)
           if (b==1)
           break;
           }
         if(i>k)
        return g
  }
```

- G has $\phi(n)$ generators. Using the lower bound for $\phi(n)$ the probability that g in line 2 is a generator is at least 1/(6 ln ln *n*)

# Attack on reused nonce

- Suppose Bob reuses b for two different messages $m_1$ and $m_2$.
- An adversary, Eve, can see $<B, C_1>$ and $<B, C_2>$ where $C_i= Bm_i$ (mod p).
- Suppose Eve discovers $m_1$.
- She can compute $m_2= m_1C_2C_1^{-1}$ (mod p).

- Don’t reuse b’s!

# DSA

- Alice
  - $2^{159}<q<2^{160}$, $2^{511+64t}<p<2^{512+64t}$, $1 \leqq t \leqq 8$, $q|p-1$
  - Select primitive root x (mod p); compute: $g=x^{(p-1)/q}$ (mod p)
  - Picks a random, 1cacq-1. A= $g^a$ (mod p)
  - Public Key: (p, q, g, A). Private Key: a.
- Signature Generation
  - Pick random k, r= ($g^k$ (mod p)) (mod q). Note : **k must be different for each signature**.
  - s= $k^{-1}$(h(M)+ar) (mod q). Signature is (r,s)
- Verification
  - u= $s^{-1}$h(x)(mod q), v= ($rs^{-1}$)(mod q)
  - Is $g^u A^v$= r (mod p)?
- Advantages over straight El Gamal
  - Verification is more efficient (2 exponentiations rather than 3)
  - Exponent is 160 bits not 768

# Baby Step Giant Step --- Shanks

- $g^x=y \pmod p$ .
- $m \sim \sqrt{p}$.
- Compute $g^{mj}$, $0 \leqq j<m$.
- Sort $(j, g^{mj})$ by second coordinate.
- Pick i at random, compute $yg^{-i} \pmod p$.
- If there is a match in the tables $yg^{-i}= g^{mj} \pmod p$.
- $x= mj+i$ is the discrete log.

# Baby Step Giant Step Example

- $p=193$. $\lfloor\sqrt{(p)}\rfloor=13$. $m= 14$. $a= 5$. $b=41$.
- $2 \times 193 + (-77) \times 5 = 1$, $a^{-1}= 116$. $a^{-14}= 189 \pmod{193}$.

| j | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $a^j$ | 5 | 25 | 125 | 46 | 37 | 185 | 153 | 186 | 158 | 18 | 90 | 64 | 127 | 56 |
| $ba^{-mj}$ | 26 | 77 | 78 | 74 | 90 | 26 | 89 | 30 | 73 | 94 | 10 | 153 | 160 | 132 |

- So $ba^{-(14x5)}= 90 = a^{11} \pmod{193}$.
- Thus $b= a^{14x5+11}= a^{81} \pmod{193}$.
- $L_5(41)= 193$.

# Discrete log Pollard r

- $x_{i+1} = f(x_i)$
  - $f(x_i) = bx_i$, if $x_i \in S_1$.
  - $f(x_i) = x_i^2$, if $x_i \in S_2$.
  - $f(x_i) = ax_i$, if $x_i \in S_3$.
- $x_i = a^{a[i]} b^{b[i]}$.
  - $a[i] = a[i]$, if $x_i \in S_1$.
  - $a[i] = 2a[i]$, if $x_i \in S_2$.
  - $a[i] = a[i]+1$, if $x_i \in S_3$.
  - $b[i] = b[i]+1$, if $x_i \in S_1$.
  - $b[i] = 2b[i]$, if $x_i \in S_2$.
  - $b[i] = b[i]$, if $x_i \in S_3$.
- $x_{2i} = x_i \rightarrow a_{2i} - a_i = L_a(b)(b_{2i} - b_i)$

# Pollard $\rho$ example

- p=229, n=191, b=228, a=2. $L_2(228)=110$

| i | $x_i$ | $a_i$ | $b_i$ |
|---:|---:|---:|---:|
| 1 | 228 | 0 | 1 |
| 2 | 279 | 0 | 2 |
| 3 | 92 | 0 | 4 |
| 4 | 184 | 1 | 4 |
| 5 | 205 | 1 | 5 |
| 6 | 14 | 1 | 6 |
| 7 | 28 | 2 | 6 |
| 8 | 256 | 2 | 7 |
| 9 | 152 | 2 | 8 |
| 10 | 304 | 3 | 8 |
| 11 | 372 | 3 | 9 |
| 12 | 121 | 6 | 18 |
| 13 | 12 | 6 | 19 |
| 14 | 144 | 12 | 38 |

| i | $x_{2i}$ | $a_{2i}$ | $b_{2i}$ |
|---:|---:|---:|---:|
| 1 | 279 | 0 | 2 |
| 2 | 184 | 1 | 4 |
| 3 | 14 | 1 | 6 |
| 4 | 256 | 2 | 7 |
| 5 | 304 | 3 | 8 |
| 6 | 121 | 6 | 38 |
| 7 | 144 | 12 | 152 |
| 8 | 235 | 48 | 154 |
| 9 | 72 | 48 | 118 |
| 10 | 14 | 96 | 119 |
| 11 | 256 | 97 | 120 |
| 12 | 304 | 98 | 51 |
| 13 | 121 | 5 | 104 |
| 14 | 144 | 10 | 163 |

- $x_{14}= x_{28}$, $(b_{14}-b_{28})= 125 \pmod{191}$, $L_2(228)=125^{-1}(a_{28}-a_{14})= 110$.

# Pohlig-Hellman

- $p - 1 = \prod_i q_i^{r[i]}$.
- Solve $a^x = y \pmod{p}$ for $x \pmod{q_i^{r[i]}}$ and use Chinese Remainder Theorem.
- $x = x_0 + x_1 q + x_2 q^2 + \ldots + x^{r[i]-1} q^{r[i]-1}$.
- $x\,(p-1)/q = x_0(p-1)/q + (p-1)(\ldots)$
- So $b^{(p-1)/q} = a^{x[0](p-1)/q}$. Solve for $x_0$.
- The put $g = ba^{-x[0]}$ and solve $g^{(p-1)/(q \times q)} = a^{x[1](p-1)/q}$.
- This costs $O(\sum_{i=1}^{r} e_i(\lg(n) + \sqrt{q_i})$.

# Pohlig-Hellman example

- $p=251$. $a= 71$, $b=210$, $<a>=F_{251}^*$. $n=250= 2 \times 5^3$.
- $L_{71}(210)= 1 \pmod 2$.
- $x= x_0+x_1 5+x_2 5^2$.
- So $a^{n/5}= 71^{20}$. $b^{n/5}= 210^{20}= 149$.
  - $x_0= L_{20}(149)=2$.
  - $x_1= 4$
  - $x_2= 2$
- $x= 2+ 4x5 + 2x25= 72 \pmod{125}$
- Applying CRT: $L_{71}(210)= 197$.

# Index Calculus

- $g^x = y \pmod p$ .   $B = (p_1, p_2, \ldots, p_k)$.
- Precompute
  - $g^{x_j} = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$
  - $x_j = a_{1j} \log_g(p_1) + a_{2j} \log_g(p_2) + \ldots + a_{kj} \log_g(p_k)$
  - If you get enough of these, you can solve for the $\log_g(p_i)$
- Solve
  - Pick s at random and compute $y\, g^s = p_1^{c_1} p_2^{c_2} \ldots p_k^{c_k}$ then
  - $\log_g(y) + s = c_1 \log_g(p_1) + c_2 \log_g(p_2) + \ldots + c_k \log_g(p_k)$
- This takes $O(e^{(1+\ln(p)\ln(\ln(p))})$ time.
- LaMacchia and Odlyzko used Gaussian integer index calculus variant to attack discrete log.

# Index Calculus Example

- p=229. a=6. <a>= $F_{229}$*. n=228. b=13. S={2,3,5,7,11}.
- Step 1
  1. $6^{100}$ (mod 229)= 180= $2^2 \times 3^2 \times 5^1 \times 7^0 \times 11^0$.
  2. $6^{18}$ (mod 229)= 176= $2^4 \times 3^0 \times 5^0 \times 7^0 \times 11^1$.
  3. $6^{12}$ (mod 229)= 165= $2^0 \times 3^1 \times 5^1 \times 7^0 \times 11^1$.
  4. $6^{62}$ (mod 229)= 154= $2^1 \times 3^0 \times 5^0 \times 7^1 \times 11^1$.
  5. $6^{143}$ (mod 229)= 198= $2^1 \times 3^2 \times 5^0 \times 7^0 \times 11^1$.
  6. $6^{206}$ (mod 229)= 210= $2^1 \times 3^1 \times 5^1 \times 7^1 \times 11^0$.
- Taking $L_a()$ of both sides, we get:
  1. $100= 2 L_a(2)+2L_a(3)+L_a(5)$ (mod 228)
  2. $18= 4L_a(2)+L_a(11)$ (mod 228)
  3. $12= L_a(3)+L_a(5)+L_a(11)$ (mod 228)
  4. $62= L_a(2)+L_a(7)+L_a(11)$ (mod 228)
  5. $143=L_a(2)+2L_a(3)+L_a(11)$ (mod 228)
  6. $206= L_a(2)+L_a(3)+L_a(5)+L_a(7)$ (mod 228)

# Index Calculus example - continued

- Review
  - $p=229$. $a=6$. $\langle a \rangle = F_{229}^*$. $n=228$. Solving, we got:
  - $L_a(2) = 21 \pmod{228}$
  - $L_a(3) = 208 \pmod{228}$
  - $L_a(5) = 98 \pmod{228}$
  - $L_a(7) = 107 \pmod{228}$
  - $L_a(11) = 162 \pmod{228}$

Step 2:

- Recall $b=13$. Pick $k=77$
- $13 \times 6^{77} = 147 = 3 \times 7^2 \pmod{229}$
- $L_6(13) = (L_6(3)+2L_6(7)-77) = 117 \pmod{228}$

# Diffie Hellman key exchange

Alice

Bob

A1: s= min(p size), $N_a$ in {0, … $2^{256}$-1}

$s, N_a$ →

$(p,q,g)$, $X=g^x$, $Auth_B$ ←

B1: Choose (p,q,g), x in {0, … $2^{256}$-1}

A2: Check (p,q,g) X, $Auth_B$, pick y in {0,…q-1}

$Y= g^y$, $Auth_A$ →

B2: Check Y, $Auth_A$

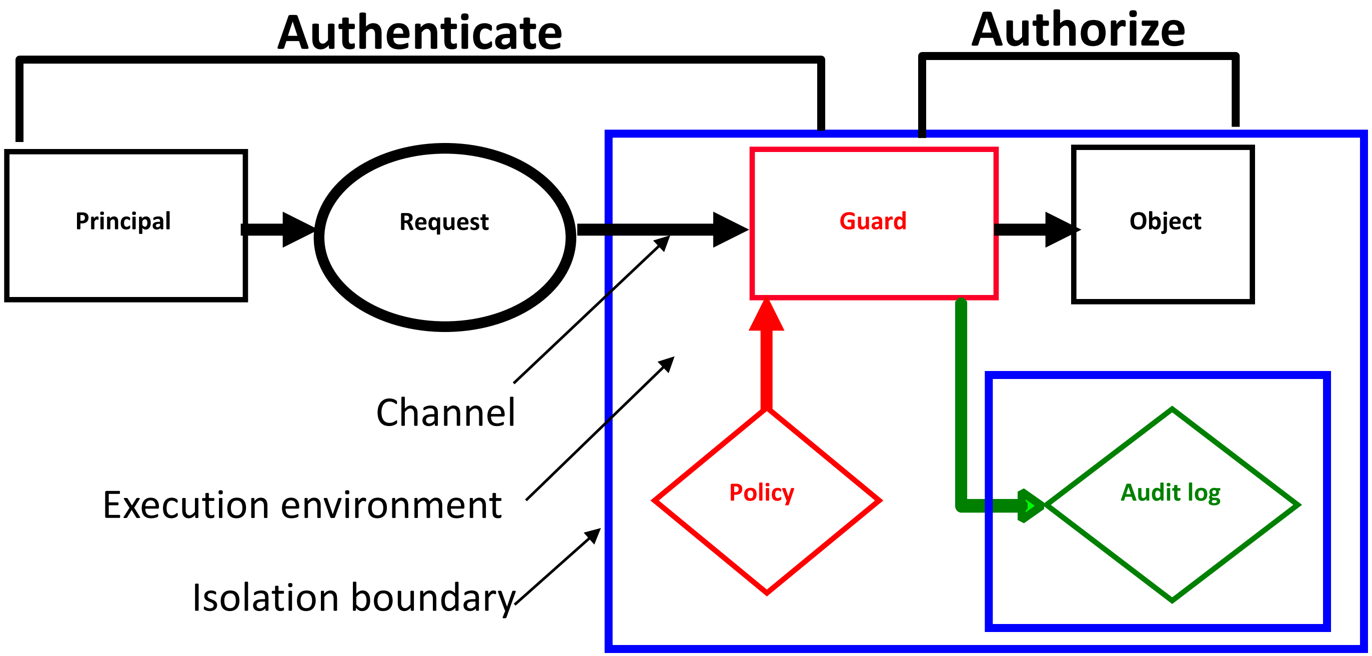
$K= X^y$

$K= Y^x$

# DH key exchange example

- p=3547, g=2.
- Alice: a= 7.
- Bob: b=17.
- $A \rightarrow B_1$: A=128 (=$2^7$), $Sign_A$(SHA-2(128||$r_1$))
- $B \rightarrow A_1$: B=3380(=$2^{17}$), $Sign_B$(SHA-2(3380||$r_2$))
- K= $128^{17}=3380^7$= 362.

# Access Control: authentication and authorization




- Authentication is process of identifying a security principal. Here are some ways:
  - Login/password or smart card/pin (user)
  - Cryptographic Hash (program)
  - Ability to decrypt (channel)

# Authentication

- When logging on to a computer you enter
  - user name and
  - password
- The first step is called identification. You announce who you are.
- The second step is called authentication. You prove that you are who you claim to be.
- To distinguish this type of 'authentication' from other interpretations, we may refer specifically to entity authentication: The process of verifying a claimed identity.

Slide from Dieter Gollmann

# Authentication

```
Login: jlm
Password: ********

Welcome John Manferdelli
>
```

# Problems with Passwords

- Authentication by password is widely accepted and not too difficult to implement.
- Managing password security can be quite expensive; obtaining a valid password is a common way of gaining unauthorised access to a computer system.
- Typical issues
  - how to get the password to the user,
  - forgotten passwords,
  - password guessing,
  - password spoofing,
  - compromise of the password file.

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# Guessing Passwords

- Exhaustive search (brute force): Try all possible combinations of valid symbols up to a certain length.
- Intelligent search: search through a restricted name space, e.g. passwords that are somehow associated with a user like name, names of friends and relatives, car brand, car registration number, phone number,…, or try passwords that are generally popular.
- Typical example for the second approach: dictionary attack trying all passwords from an on-line dictionary.
- You cannot prevent an attacker from accidentally guessing a valid password, but you can try to reduce the probability of a password compromise.

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# Password Salting

- To slow down dictionary attacks, a salt can be appended to the password before encryption and stored with the encrypted password.
  - If two users have the same password, they will now have different entries in the file of encrypted passwords.
  - Example: Unix uses a 12 bit salt.

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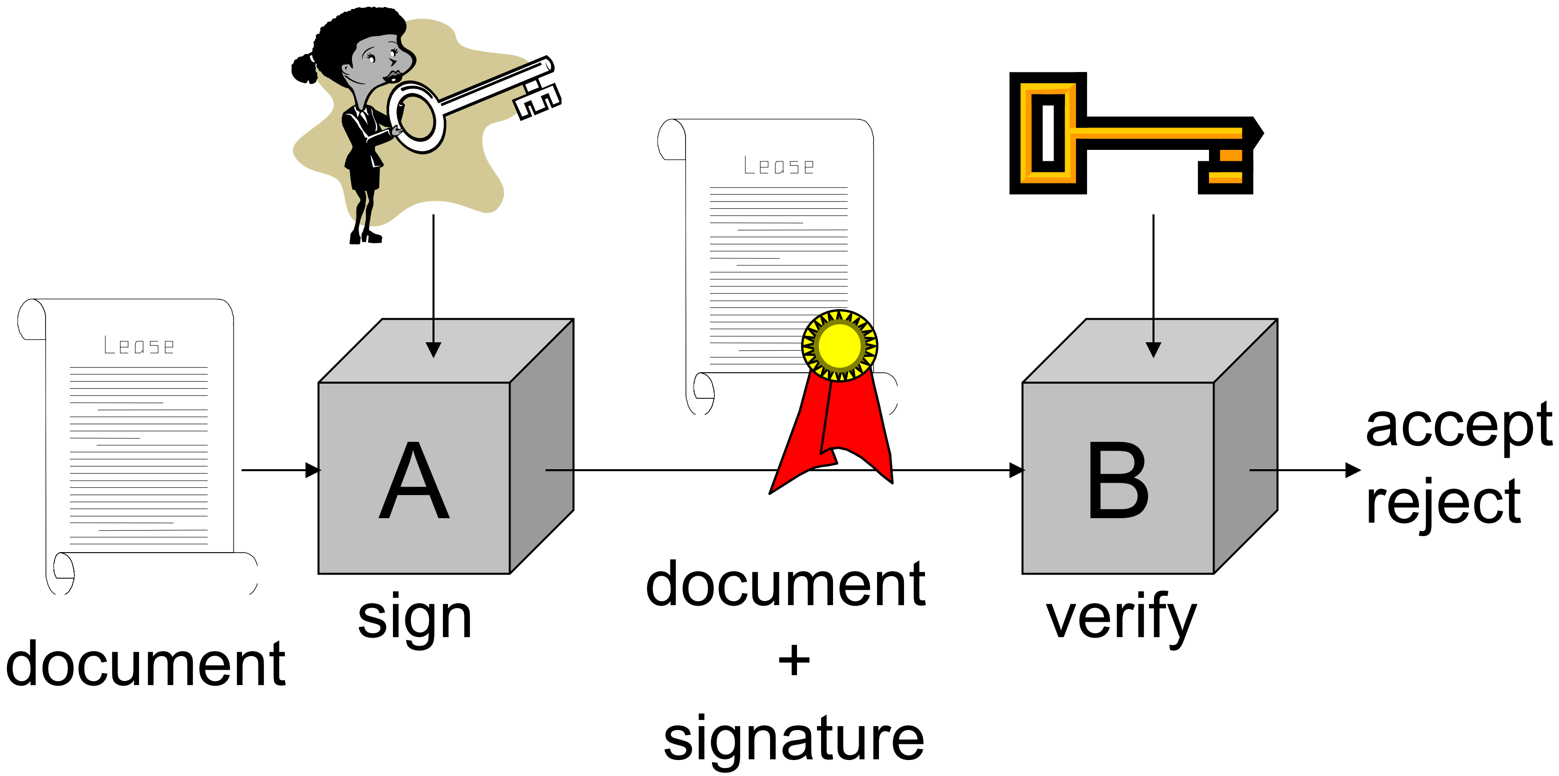
# Access Control Matrix

- Capabilities:
  - access rights are stored with the subject
  - rows of the access control matrix
- Access Control Lists (ACLs)
  - access rights are stored with the object.
  - columns of the access control matrix.

| | bill.doc | edit.exe | fun.com |
|---|---|---|---|
| Alice | - | {exec} | {exec,read} |
| Bob | {read,write} | {exec} | {exec,read,write} |

# Digital signatures

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# Digital Signatures

- *A* has a public verification key and a private signature key ($\rightarrow$ public key cryptography).
- *A* uses her private key to compute her signature on document *m*.
- *B* uses a public verification key to check the signature on a document *m* he receives.
- This provides non-repudiation.
- Signature algorithm= hash+padding+private key operation

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# Bleichenbacher Attack on PKCS1

- Chosen-ciphertext attack.
- RSA PKCS #1 v1.5 : $c = (00 \,||\, 02 \,||\, r \,||\, 0 \,||\, m)^e \bmod n$
- Attacker can test if 16 MSBs of plaintext = ’02’.
- Attack: to decrypt a given ciphertext C do:
  - Pick $r \in Z_n$.
  - Compute $C' = r^e \cdot C \;\; = (r \cdot \mathrm{PKCS1}(M))^e$.
  - Send C’ to oracle and use response.

Slide from Dan Boneh

# Side-Channel Attacks

- Some attack vectors …
  - Fault Attacks
  - Timing Attacks
  - Cache Attacks
  - Power Analysis
  - Electromagnetic Emissions
  - Acoustic Emissions

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# End

# Berlekamp factorization

- $f(x)= \prod_{i=1}^{t} f_i(x)$ over $F_p$, $\deg(f(x))=n$. $f_i(x)$ irreducible.

  $F=\{f(x)\}$;<br>
  for($i=1$; $i<n$;$i$++)<br>
      $x^{iq}= \sum_{j=0}^{n-1} q_{ij} x^j \pmod{f(x)}$, $q_{ij}$ e$F_p$.<br>
  Find basis $\langle v_1, \ldots, v_t \rangle$ of null space of $(Q-I_n)$;<br>
  // $w= w_0, \ldots, w_{n-1}$. $w(x) = w_0+w_1 x+ \ldots +w_{n-1} x^{n-1}$<br>
  for($i=1$; $i \leqslant t$;$i$++) {<br>
      for ($h(x)$ **ε** $F$, $\deg(h)>1$;) {<br>
          Compute $(h(x), v_i(x)-a)$, $a$ **ε** $F_p$;<br>
          Replace $h(x)$ in $F$ with these;<br>
      }<br>
  return ($F$);
- $O(n^3+tpn^2)$, $t=$ # irreducible factors. Can be reduced to $O(n^3+t \lg(p)n^2)$.

# Berlekamp factorization example

- Factor $x^7$-1 over GF(2).

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \end{pmatrix} = \begin{pmatrix} 1 \\ x^2 \\ x^4 \\ x^6 \\ x^1 \\ x^3 \\ x^5 \end{pmatrix}$$

- Adding and solving get:
  - 1
  - $x^4+x^2+x = x(x^3+x+1)$
  - $x^6+x^5+x^3= x^3(x^3+x^2+1)$
  - Dividing into $x^7$-1, we get: (x+1)