Cryptanalysis

Discrete Log Based Systems

John Manferdelli
JohnManferdelli@hotmail.com

HMAC's concluded

- HMAC(K, text)= H((K$\oplus$opad)||H((K$\oplus$ipad)||text)))
- H is a cryptographic hash like SHA-256
- ipad, the inner pad: the byte 0x36 repeated B times where B is key size
- opad, the outer pad: the byte 0x5c repeated B times
- Verification requires knowledge of K.

Discrete log based public key systems

Discrete Log

- If b= a^x, then $L_a(b)$=x. $L_a(y)$ is the discrete log function.
- If g= b^x, then $L_a(g)=xL_a(b)$. $L_a(b_1b_2)= L_a(b_1) + L_a(b_2)$

- **Discrete Log Problem (DLP):** Given p, prime, a: <a>=F_p*. B (mod p), a, unknown, find $L_a(b)$.
- **Computational Diffie Hellman Problem (CDHP):** Given p, prime, <a>=F_p*. a^a (mod p), a^b (mod p), find a^{ab} (mod p).
- **Theorem:** CDHP $\leqq_P$ DLP. If the factorization of p-1 is known and f(p-1) is $O((\ln(p))^c)$ smooth then DLP and CDHP are equivalent.

- **Conclusion:** Exponentiation is a one way trap-door function.

El Gamal cryptosystem

- Alice, the private keyholder, picks a large prime, p, where p-1 also has large prime divisors (say, p= 2rq+1) and a generator, g, for F_p*. <g>= F_p*. Alice also picks a random number, a (secret), and computes $A=g^a$ (mod p). Alice's public key is <A, g, p>.
- To send a message, m, Bob picks a random b (his secret) and computes $B= g^b$ (mod p). Bob transmits $(B, mA^b)= (B, C)$.
- Alice decodes the message by computing $CB^{-a}=m$.
- Without knowing a, an adversary has to solve the Computational Diffie Hellman Problem to get m.

- Note: b must be random and never reused!

El Gamal Example

- Alice chooses
  - p=919. g=7.
  - a=111, A= 7^{111}= 461 (mod 919).
  - Alice's Public key is <919, 7, 461>

- Bob wants to send m=45, picks b= 29.
  - B=7^{29} =788(mod 919), 461^{29}= 902 (mod 919),
  - C= (45)(902)= 154(mod 919).
  - Bob transmits (788, 154).

- Alice computes $(788)^{-111}$= 902^{-1}(mod 919).
  - (54)(902)+(-53)(919)=1. 54= 902^{-1} (mod 919)
  - Calculates m= (154) (54)=45 (mod 919).

El Gamal Signature

- $< g >= \mathbb{Z}_q^*$. A picks a random as in encryption.
- Signing: Signer picks k: 1≤k≤p-2 with (k, p-1)= 1 and publishes g^k. k is secret.
- $Sig_K(M,k)= (t,d)$
  - $t= g^k \pmod p$
  - $d=(M-gt)k^{-1} \pmod{p-1}$
- $Ver_K(M,t,d)$ iff $g^{kt}\, t^d=g^M \pmod p$

- Notes: It's important that M is a hash otherwise there is an existential forgery attack. It's important that k be different for every message otherwise adversary can solve for key.

Timing

- Finding g takes about $O(\lg(p)^3)$ operations, so does primality testing and raising g to the a power mod p.
- Encryption is also $O(\lg(p)^3)$ and so is decryption.
- Note that key generation is cheap but for safety, $p>w^2$, where w is the "computational power" of the adversary.

Finding generators (Gauss)

- Find a generator, *g,* for F_p*, n= (p-1)= $p_1^{e1}\ p_2^{e2} \ldots p_k^{ek}$.

```
while () {
        choose a random g∈G
        for(i=1; i<=k; k++) {
           b= g^(n/pi)
           if (b==1)
           break;
            }
         if(i>k)
        return g
  }
```

- G has $\phi(n)$ generators. Using the lower bound for $\phi(n)$ the probability that g in line 2 is a generator is at least 1/(6 ln ln *n*)

Attack on reused nonce

- Suppose Bob reuses b for two different messages m_1 and m_2.
- An adversary, Eve, can see <B, C_1> and <B, C_2> where $C_i = Bm_i \pmod p$.
- Suppose Eve discovers m_1.
- She can compute $m_2 = m_1 C_2 C_1^{-1} \pmod p$.

- Don't reuse b's!

DSA

- Alice
  - $2^{159}<q<2^{160}$, $2^{511+64t}<p<2^{512+64t}$, $1 \leqq t \leqq 8$, $q|p-1$
  - Select primitive root x (mod p); compute: $g=x^{(p-1)/q}$ (mod p)
  - Picks a random, 1cacq-1. $A= g^a$ (mod p)
  - Public Key: (p, q, g, A). Private Key: a.
- Signature Generation
  - Pick random k, r= (g^k (mod p)) (mod q). Note : **k must be different for each signature**.
  - s= k^{-1}(h(M)+ar) (mod q). Signature is (r,s)
- Verification
  - u= s^{-1}h(x)(mod q), v= (rs^{-1})(mod q)
  - Is $g^u A^v$= r (mod p)?
- Advantages over straight El Gamal
  - Verification is more efficient (2 exponentiations rather than 3)
  - Exponent is 160 bits not 768

Baby Step Giant Step --- Shanks

- g^x=y (mod p) .
- m ~ √p.
- Compute g^{mj}, $0 \leqq j < m$.
- Sort (j, g^{mj}) by second coordinate.
- Pick i at random, compute yg^{-i} (mod p).
- If there is a match in the tables $yg^{-i} = g^{mj}$ (mod p).
- x= mj+i is the discrete log.

Baby Step Giant Step Example

- $p=193. \lfloor\sqrt{(p)}\rfloor=13.$ $m= 14.$ $a= 5.$ $b=41.$
- $2 \times 193 + (-77) \times 5 = 1$, $a^{-1}= 116.$ $a^{-14}= 189 \pmod{193}$.

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14
a^j	5	25	125	46	37	185	153	186	158	18	90	64	127	56
ba^{-mj}	26	77	78	74	90	26	89	30	73	94	10	153	160	132

- So $ba^{-(14x5)}= 90 = a^{11} \pmod{193}$.
- Thus $b= a^{14x5+11}= a^{81} \pmod{193}$.
- $L_5(41)= 193$.

Discrete log Pollard r

- $x_{i+1} = f(x_i)$
  - $f(x_i) = bx_i$, if $x_i \ \boldsymbol{\varepsilon}\ S_1$.
  - $f(x_i) = x_i^2$, if $x_i \ \boldsymbol{\varepsilon}\ S_2$.
  - $f(x_i) = ax_i$, if $x_i \ \boldsymbol{\varepsilon}\ S_3$.
- $x_i = a^{a[i]} b^{b[i]}$.
  - $a[i] = a[i]$, if $x_i \ \boldsymbol{\varepsilon}\ S_1$.
  - $a[i] = 2a[i]$, if $x_i \ \boldsymbol{\varepsilon}\ S_2$.
  - $a[i] = a[i]+1$, if $x_i \ \boldsymbol{\varepsilon}\ S_3$.
  - $b[i] = b[i]+1$, if $x_i \ \boldsymbol{\varepsilon}\ S_1$.
  - $b[i] = 2b[i]$, if $x_i \ \boldsymbol{\varepsilon}\ S_2$.
  - $b[i] = b[i]$, if $x_i \ \boldsymbol{\varepsilon}\ S_3$.
- $x_{2i} = x_i \rightarrow a_{2i} - a_i = L_a(b)(b_{2i} - b_i)$

Pollard ρ example

- p=229, n=191, b=228, a=2. $L_2(228)=110$

i	x_i	a_i	b_i
1	228	0	1
2	279	0	2
3	92	0	4
4	184	1	4
5	205	1	5
6	14	1	6
7	28	2	6
8	256	2	7
9	152	2	8
10	304	3	8
11	372	3	9
12	121	6	18
13	12	6	19
14	144	12	38

i	x_{2i}	a_{2i}	b_{2i}
1	279	0	2
2	184	1	4
3	14	1	6
4	256	2	7
5	304	3	8
6	121	6	38
7	144	12	152
8	235	48	154
9	72	48	118
10	14	96	119
11	256	97	120
12	304	98	51
13	121	5	104
14	144	10	163

- $x_{14}= x_{28}$, $(b_{14}-b_{28})= 125 \pmod{191}$, $L_2(228)=125^{-1} (a_{28}-a_{14})= 110$.

Pohlig-Hellman

- $p - 1 = \prod_i {q_i}^{r[i]}$.
- Solve $a^x = y \pmod p$ for $x \pmod{q_i^{r[i]}}$ and use Chinese Remainder Theorem.
- $x = x_0 + x_1 q + x_2 q^2 + \ldots + x^{r[i]-1}\, q^{r[i]-1}$.
- $x\,(p-1)/q = x_0(p-1)/q + (p-1)(\ldots)$
- So $b^{(p-1)/q} = a^{x[0](p-1)/q}$. Solve for x_0.
- The put $g = ba^{-x[0]}$ and solve $g^{(p-1)/(q \times q)} = a^{x[1](p-1)/q}$.
- This costs $O(\sum_{i=1}^{r} e_i(\lg(n) + \sqrt{q_i})$.

Pohlig-Hellman example

- $p=251$. $a= 71$, $b=210$, $\langle a\rangle=F_{251}^*$. $n=250= 2 \times 5^3$.
- $L_{71}(210)= 1 \pmod 2$.
- $x= x_0+x_1 5+x_2 5^2$.
- So $a^{n/5}= 71^{20}$. $b^{n/5}= 210^{20}= 149$.
  - $x_0= L_{20}(149)=2$.
  - $x_1= 4$
  - $x_2= 2$
- $x= 2+ 4x5 + 2x25= 72 \pmod{125}$
- Applying CRT: $L_{71}(210)= 197$.

Index Calculus

- $g^x = y \pmod{p}$. B= $(p_1, p_2, \ldots, p_k)$.
- Precompute
  - $g^{x_j} = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$
  - $x_j = a_{1j} \log_g(p_1) + a_{2j} \log_g(p_2) + \ldots + a_{kj} \log_g(p_k)$
  - If you get enough of these, you can solve for the $\log_g(p_i)$
- Solve
  - Pick s at random and compute $y\,g^s = p_1^{c_1} p_2^{c_2} \ldots p_k^{c_k}$ then
  - $\log_g(y) + s = c_1 \log_g(p_1) + c_2 \log_g(p_2) + \ldots + c_k \log_g(p_k)$
- This takes $O(e^{(1+\ln(p)\ln(\ln(p))})$ time.
- LaMacchia and Odlyzko used Gaussian integer index calculus variant to attack discrete log.

Index Calculus Example

- p=229. a=6. <a>= F_{229}*. n=228. b=13. S={2,3,5,7,11}.
- Step 1
  1. 6^{100} (mod 229)= 180= $2^2 \times 3^2 \times 5^1 \times 7^0 \times 11^0$.
  2. 6^{18} (mod 229)= 176= $2^4 \times 3^0 \times 5^0 \times 7^0 \times 11^1$.
  3. 6^{12} (mod 229)= 165= $2^0 \times 3^1 \times 5^1 \times 7^0 \times 11^1$.
  4. 6^{62} (mod 229)= 154= $2^1 \times 3^0 \times 5^0 \times 7^1 \times 11^1$.
  5. 6^{143} (mod 229)= 198= $2^1 \times 3^2 \times 5^0 \times 7^0 \times 11^1$.
  6. 6^{206} (mod 229)= 210= $2^1 \times 3^1 \times 5^1 \times 7^1 \times 11^0$.
- Taking $L_a()$ of both sides, we get:
  1. 100= 2 $L_a(2)+2L_a(3)+L_a(5)$ (mod 228)
  2. 18= $4L_a(2)+L_a(11)$ (mod 228)
  3. 12= $L_a(3)+L_a(5)+L_a(11)$ (mod 228)
  4. 62= $L_a(2)+L_a(7)+L_a(11)$ (mod 228)
  5. 143=$L_a(2)+2L_a(3)+L_a(11)$ (mod 228)
  6. 206= $L_a(2)+L_a(3)+L_a(5)+L_a(7)$ (mod 228)

Index Calculus example - continued

- Review
  - $p=229$. $a=6$. $<a>= F_{229}^*$. $n=228$. Solving, we got:
  - $L_a(2)= 21 \pmod{228}$
  - $L_a(3)= 208 \pmod{228}$
  - $L_a(5) = 98 \pmod{228}$
  - $L_a(7)= 107 \pmod{228}$
  - $L_a(11)= 162 \pmod{228}$

Step 2:

- Recall $b=13$. Pick $k=77$
- $13 \times 6^{77}= 147 = 3 \times 7^2 \pmod{229}$
- $L_6(13)= (L_6(3)+2L_6(7)-77)= 117 \pmod{228}$

Diffie Hellman key exchange

Alice

Bob

A1: s= min(p size), N_a in $\{0, \dots 2^{256}-1\}$

s, N_a $\longrightarrow$

$\longleftarrow$ (p,q,g), $X=g^x$, $Auth_B$

B1: Choose (p,q,g), x in $\{0, \dots 2^{256}-1\}$

A2: Check (p,q,g) X, $Auth_B$, pick y in $\{0,\dots q-1\}$

$Y= g^y$, $Auth_A$ $\longrightarrow$

B2: Check Y, $Auth_A$

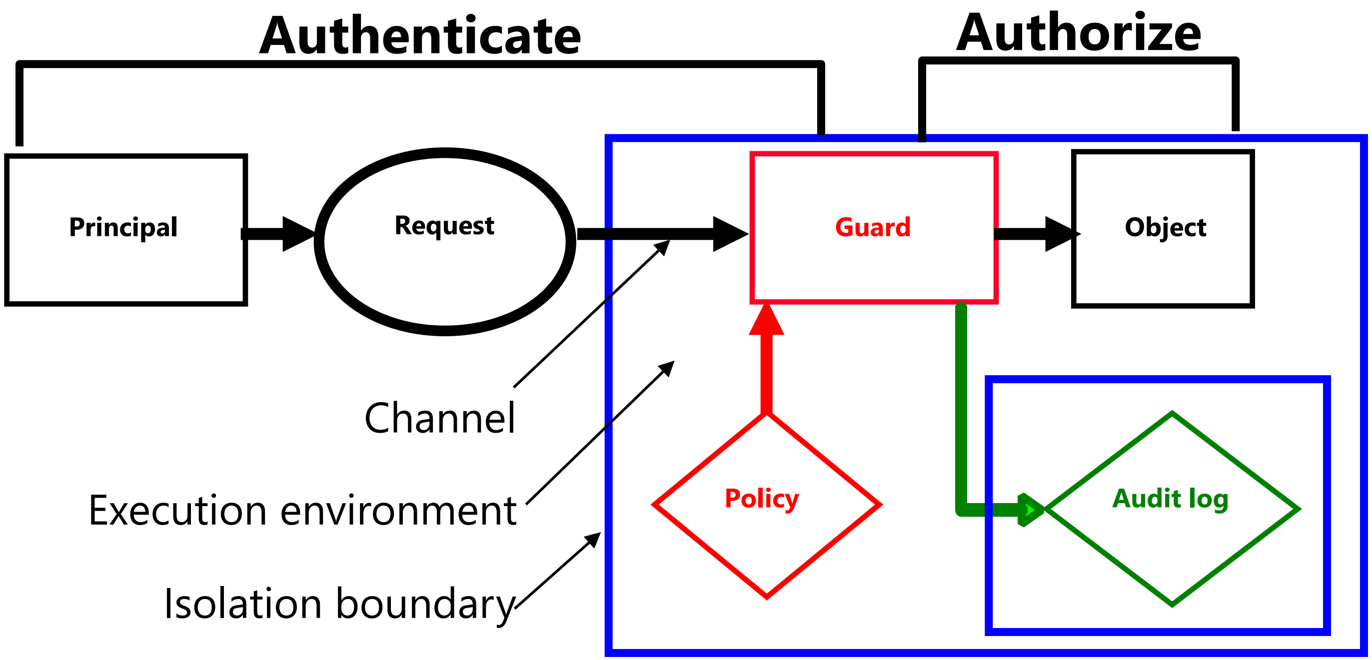
$K= X^y$

$K= Y^x$

DH key exchange example

- p=3547, g=2.
- Alice: a= 7.
- Bob: b=17.
- $A \rightarrow B_1$: A=128 (=2^7), $Sign_A$(SHA-2(128||r_1))
- $B \rightarrow A_1$: B=3380(=2^{17}), $Sign_B$(SHA-2(3380||r_2))
- K= 128^{17}=3380^7= 362.

Access Control: authentication and authorization



- Authentication is process of identifying a security principal. Here are some ways:
  - Login/password or smart card/pin (user)
  - Cryptographic Hash (program)
  - Ability to decrypt (channel)

Authentication

- When logging on to a computer you enter
  - user name and
  - password
- The first step is called identification. You announce who you are.
- The second step is called authentication. You prove that you are who you claim to be.
- To distinguish this type of 'authentication' from other interpretations, we may refer specifically to entity authentication: The process of verifying a claimed identity.

Slide from Dieter Gollmann

Authentication

```
Login: jlm
Password: ********

Welcome John Manferdelli
>
```

Problems with Passwords

- Authentication by password is widely accepted and not too difficult to implement.
- Managing password security can be quite expensive; obtaining a valid password is a common way of gaining unauthorised access to a computer system.
- Typical issues
  - how to get the password to the user,
  - forgotten passwords,
  - password guessing,
  - password spoofing,
  - compromise of the password file.

Slide from Dieter Gollmann

Guessing Passwords

- Exhaustive search (brute force): Try all possible combinations of valid symbols up to a certain length.
- Intelligent search: search through a restricted name space, e.g. passwords that are somehow associated with a user like name, names of friends and relatives, car brand, car registration number, phone number,…, or try passwords that are generally popular.
- Typical example for the second approach: dictionary attack trying all passwords from an on-line dictionary.
- You cannot prevent an attacker from accidentally guessing a valid password, but you can try to reduce the probability of a password compromise.

Slide from Dieter Gollmann

Password Salting

- To slow down dictionary attacks, a salt can be appended to the password before encryption and stored with the encrypted password.
  - If two users have the same password, they will now have different entries in the file of encrypted passwords.
  - Example: Unix uses a 12 bit salt.

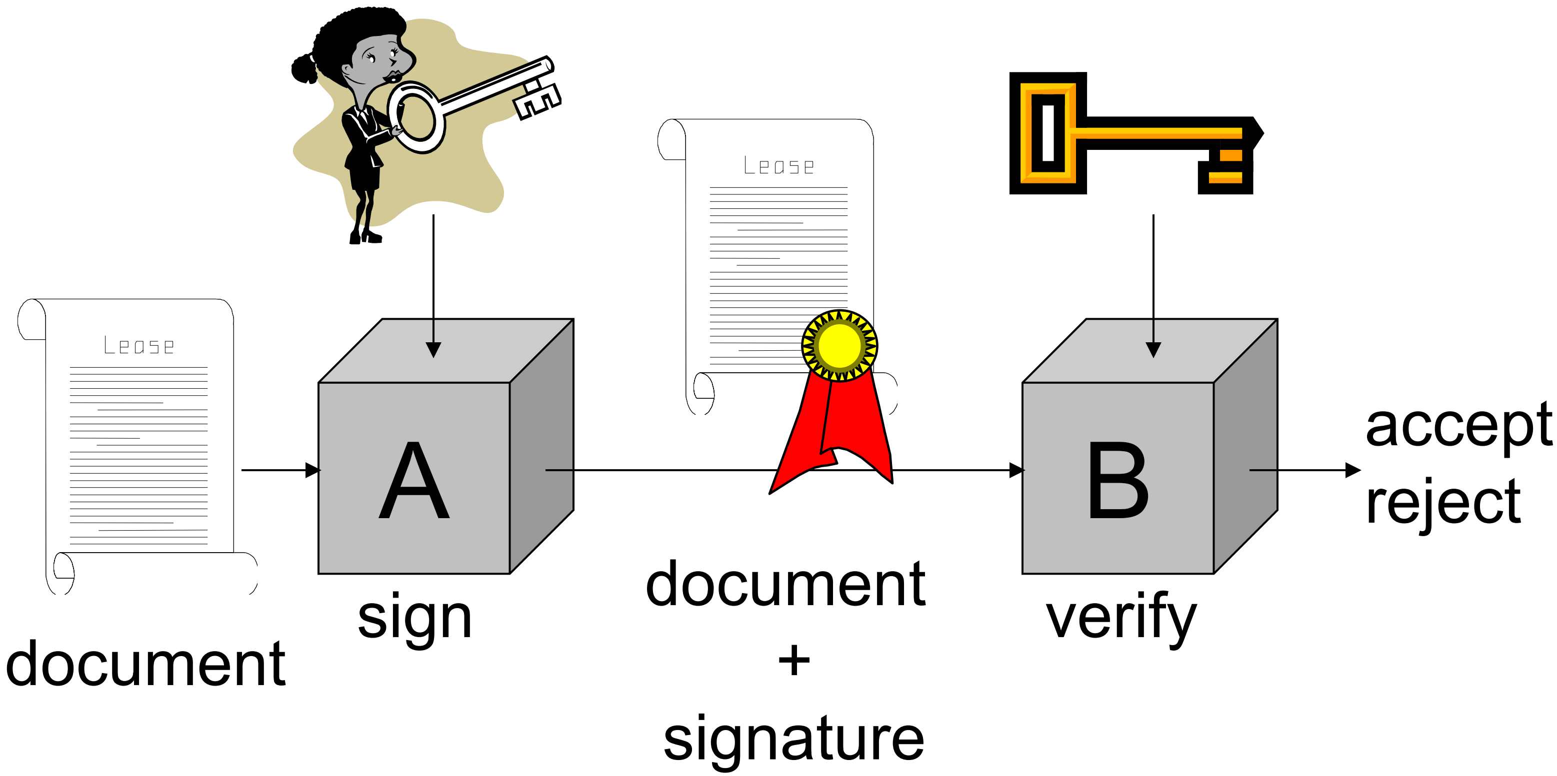
Slide from Dieter Gollmann

Access Control Matrix

- Capabilities:
  - access rights are stored with the subject
  - rows of the access control matrix
- Access Control Lists (ACLs)
  - access rights are stored with the object.
  - columns of the access control matrix.

	bill.doc	edit.exe	fun.com
Alice	-	{exec}	{exec,read}
Bob	{read,write}	{exec}	{exec,read,write}

Digital signatures



Slide from Dieter Gollmann

Digital Signatures

- *A* has a public verification key and a private signature key ($\rightarrow$ public key cryptography).
- *A* uses her private key to compute her signature on document *m*.
- *B* uses a public verification key to check the signature on a document *m* he receives.
- This provides non-repudiation.
- Signature algorithm= hash+padding+private key operation

Slide from Dieter Gollmann

Bleichenbacher Attack on PKCS1

- Chosen-ciphertext attack.
- RSA PKCS #1 v1.5 : $c = (00 \,||\, 02 \,||\, r \,||\, 0 \,||\, m)^e \bmod n$
- Attacker can test if 16 MSBs of plaintext = '02'.
- Attack: to decrypt a given ciphertext C do:
  - Pick $r \in Z_n$.
  - Compute $C' = r^e \cdot C = (r \cdot \mathrm{PKCS1}(M))^e$.
  - Send C' to oracle and use response.

Slide from Dan Boneh

Side-Channel Attacks

- Some attack vectors …
  - Fault Attacks
  - Timing Attacks
  - Cache Attacks
  - Power Analysis
  - Electromagnetic Emissions
  - Acoustic Emissions

Slide from Dieter Gollmann

End

Berlekamp factorization

- $f(x) = \prod_{i=1}^{t} f_i(x)$ over F_p, $\deg(f(x))=n$. $f_i(x)$ irreducible.

  F={f(x)};
  for($i=1$; $i<n$;i++)

  $x^{iq} = \sum_{j=0}^{n-1} q_{ij} x^j \pmod{f(x)}$, q_{ij} eF_p.

  Find basis $\langle v_1, \ldots, v_t \rangle$ of null space of $(Q-I_n)$;
  // w= $w_0, \ldots, w_{n-1}$. $w(x) = w_0+w_1 x+ \ldots +w_{n-1} x^{n-1}$
  for($i=1$; $i \leqslant t$;i++) {

  for (h(x)**ε**F, deg (h)>1;) {

  Compute (h(x), $v_i(x)$-a), a **ε** F_p;
  Replace h(x) in F with these;

  }

  return (F);
- $O(n^3+tpn^2)$, t= # irreducible factors. Can be reduced to $O(n^3+t \lg(p)n^2)$.

Berlekamp factorization example

- Factor x^7-1 over GF(2).

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \end{pmatrix} = \begin{pmatrix} 1 \\ x^2 \\ x^4 \\ x^6 \\ x^1 \\ x^3 \\ x^5 \end{pmatrix}$$

- Adding and solving get:
  - 1
  - $x^4+x^2+x = x(x^3+x+1)$
  - $x^6+x^5+x^3= x^3(x^3+x^2+1)$
  - Dividing into x^7-1, we get: $(x+1)$