

# Electronics of Radio, Part 2

Notes on David Rutledge's book

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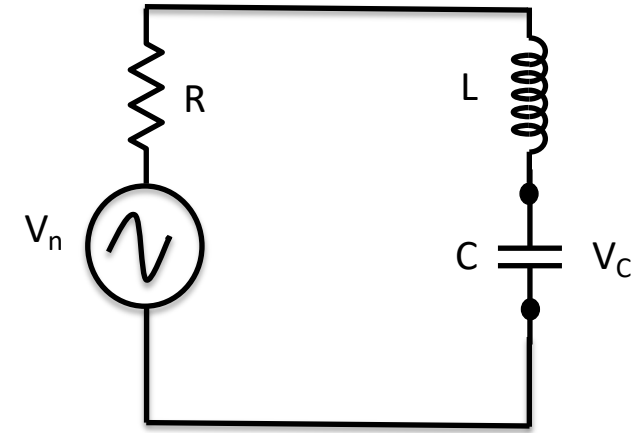
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# Noise

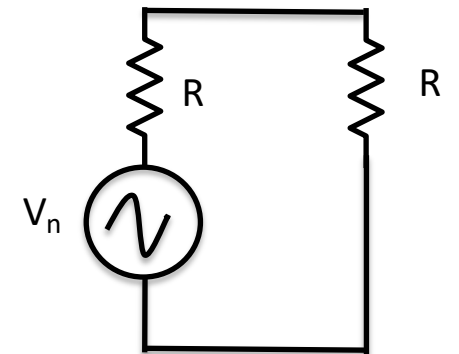
- $V_{n(rms)} = \sqrt{\frac{1}{\tau} \int_0^\tau V(t)^2 dt}$  so the average noise power is  $P_n = \frac{V_{n(rms)}^2}{R}$ , where  $R$  is the load resistance
- Define  $SNR = \frac{P}{P_n}$  and  $MDS = \frac{P_n}{G}$ .
- $NEP = \frac{N}{G}$  (Noise equivalent power)
- Noise phasors
  - $\overline{|V_n|^2} = \int |V_n|^2 p dA = E(|V_n|^2)$
  - $N = \frac{\overline{|V_n|^2}}{2R} = \frac{E(|V_n|^2)}{2R}$
  - $P_n = NB$ ,  $N$  is noise power density,  $B$  is bandwidth.
  - The units for  $B$  are  $\text{volts}/\sqrt{\text{Hz}}$
  - $E(|V_1 + V_2|^2) = E(|V_1|^2) + E(|V_2|^2) + E(V_1 \bar{V}_2) + E(V_2 \bar{V}_1)$
  - $E(V_1 \bar{V}_2) = 0$ , if they are uncorrelated. The  $N = N_1 + N_2$ .

# Nyquist

- $V_C = \frac{1}{j\omega C} \frac{V_n}{R + j\omega L + \frac{1}{j\omega C}}$
- $\overline{|V_C|^2} = \frac{\overline{|V_n|^2}}{|1 - \omega^2 LC + j\omega RC|^2}$
- Expected energy stored in C, at resonance, is  $kT = \frac{C}{2} \int_0^\infty |V_C|^2 df$ , by equipartition theorem
- $\int_0^\infty \frac{1}{|1 - \omega^2 LC + j\omega RC|^2} d\omega = \frac{1}{4RC}$
- So,  $\overline{|V_n|^2} = 8kTR$ ,  $V_{n(rms)} = \sqrt{4kTR}$
- $N = kT = \frac{|\frac{V_n}{2}|^2}{2R} = \frac{\overline{|V_n|^2}}{8R}$  (Noise density)
- $T_e = \frac{N}{k}$
- $T_n = \frac{NEP}{k}$



Noise model for Nyquist



Matched load

# System noise

- Attenuator (and filter) noise

- $$N_a = kT \left(1 - \frac{1}{L}\right), T_a = T(L - 1)$$

- $$N = G_3 G_2 G_1 kT_a + G_2 G_3 N_1 + G_3 N_2 + N_3$$

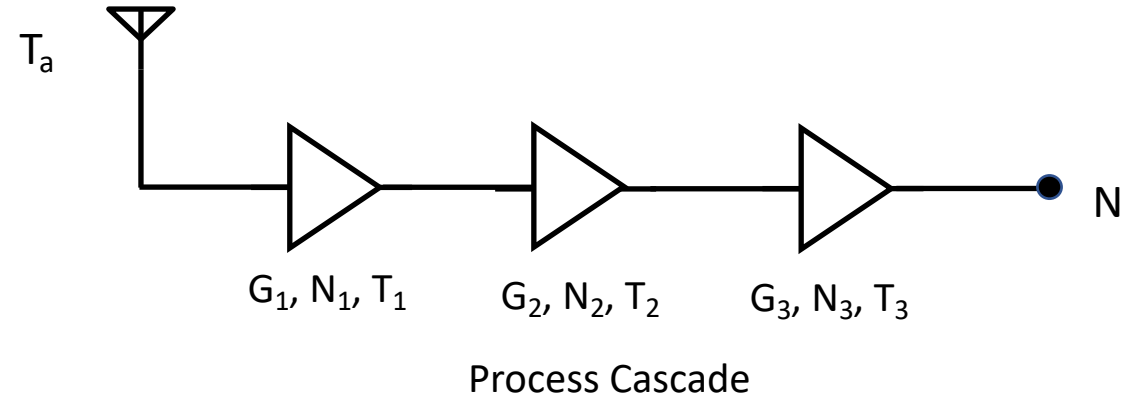
- $$T_r = T_a + T_1 + \frac{T_2}{G_1} + \frac{T_3}{G_1 G_2}$$

- $$\frac{T_n}{T_0} = F - 1, T_0 = 290 K$$

- For NorCal,  $L = 3.2$ , each filter has  $T_f = 290(L - 1) = 630K$

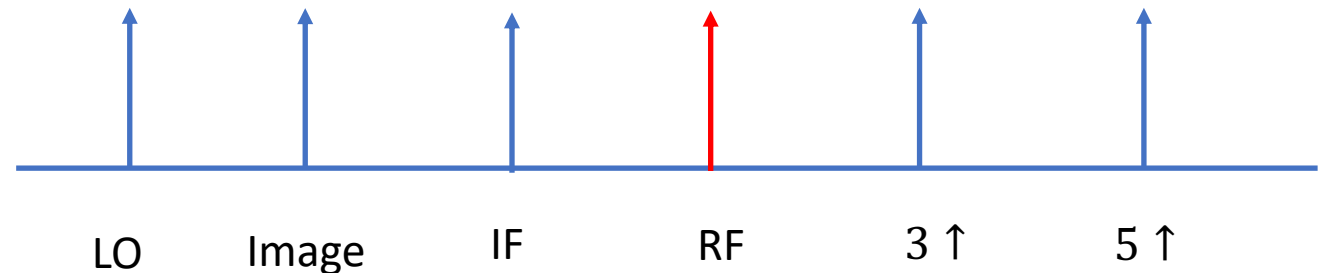
- $$T_r = T_f + LT_m + \frac{T_f L}{G} + T_m \frac{L^2}{G} = 2780K$$

- Want receiver noise  $\ll$  Antenna noise
- For NorCal, antenna noise is at least 30 dB bigger than receiver noise
- To measure noise, inject a signal that doubles the Noise power



# Mixers

- $V_{lo}(t)$  is a square wave with period  $\omega_{lo}$ . Expanding this in a Fourier series, we get:
- $V_{lo}(t) = \frac{4}{\pi}(\cos(\omega_{lo}t) - \frac{\cos(3\omega_{lo}t)}{3} + \frac{\cos(5\omega_{lo}t)}{5} \dots)$ ,  $V_{rf}(t) = V_{rf}\cos(\omega_{rf}t)$
- $V_{lo}(t)V_{rf}(t) = \frac{2V_{rf}}{\pi}(\cos(\omega_{-}t) - \frac{\cos(3\omega_{-}t)}{3} + \frac{\cos(5\omega_{-}t)}{5} \dots) + \frac{2V_{rf}}{\pi}(\cos(\omega_{+}t) - \frac{\cos(3\omega_{+}t)}{3} + \frac{\cos(5\omega_{+}t)}{5} \dots)$
- $\omega_{+} = \omega_{lo} + \omega_{rf}$  and  $\omega_{-} = |\omega_{lo} - \omega_{rf}|$
- We define  $\omega_{k+} = (k\omega_{lo} + \omega_{rf})$  and  $\omega_{k-} = |k\omega_{lo} - \omega_{rf}|$  and  $V_{k+}(t) = \frac{2V_{rf}}{k\pi}\cos(\omega_{k+}t)$  and  $V_{k-}(t) = \frac{2V_{rf}}{k\pi}\cos(\omega_{k-}t)$
- $\omega_i = \omega_{if} - \omega_{lo}$  and  $\omega_{if} = \omega_{if} + \omega_i$ ,  $\omega_i$  is a spurious signal.  $\omega_{k+}$  and  $\omega_{k-}$  are the spurs from the  $k$ th harmonic



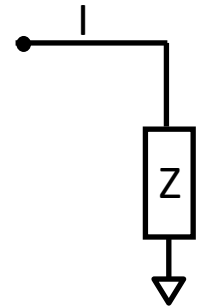
# Intermodulation

- If frequency components are  $k_1 f_1 + k_2 f_2 + \dots + k_n f_n$ , modulation order is  $|k_1| + |k_2| + \dots + |k_n|$
- Intermodulation response
  - $V = G_v V_i + G_2 V_i^2 + G_3 V_i^3 + \dots$
  - $V_i = V_1 \cos(\omega_1 t) + V_2 \cos(\omega_2 t)$
- MDS is where input power ( $P_i$ ) is  $2P_n$ . MDI is where intermodulation power is  $2P_n$ .
- Dynamic range = MDI-MDS.
- Good receivers have DR  $\sim 100dB$

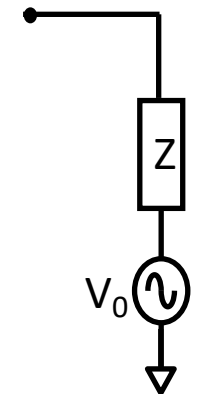


# Antennas

- Antennas characterized by impedance and pattern
- From Maxwell, for a plane wave (E in x direction, H in y direction), wave is of form  $\exp(j\omega t - j\beta z)$ 
  - $\nabla \times E = -j\mu_0\omega H, \nabla \times B = j\epsilon_0\omega E$
  - Linear polarization:  $\beta \hat{z} \times E = \mu_0\omega H, \beta E_x \hat{y} = -\mu_0\omega H$
- Substituting and taking the cross product, we get:
  - $\beta E_x = -\omega\mu_0 H_y$  and  $\beta H_y = -\omega\epsilon_0 E_x$
  - Impedance is  $\eta_0 = \frac{E_x}{H_y} = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377\Omega$
- Write  $\beta = \omega\sqrt{\mu_0\epsilon_0}$
- Power density
  - $S = \text{Re} \left( \frac{E_x \overline{H_y}}{2} \right) = \frac{(|E_x|)^2}{2\eta_0}$  (Poynting vector)
- Impedance:
  - Power delivered to antenna:  $P_t = \frac{R|I|^2}{2}$ , R is real part of Z,
  - $R = R_r + R_l, \eta = \frac{R_r}{R}, R_r$  is radiation resistance,  $R_l$  is loss.
  - Transmitter impedance = Receiver impedance.



Transmitting Antenna



Receiving Antenna

# Transmitting Antennas

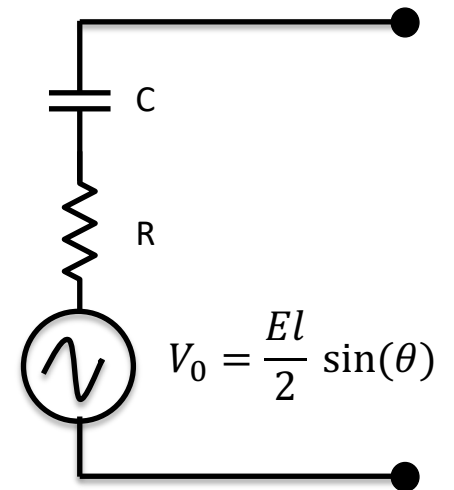
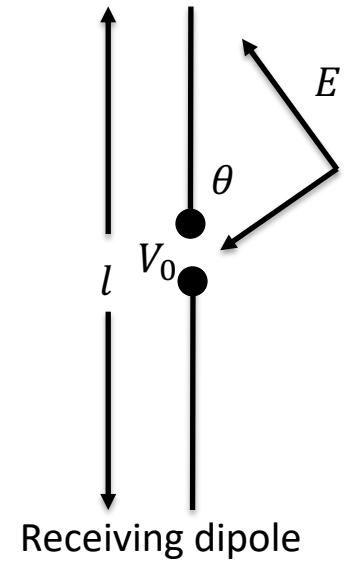
- Power density for isotropic antenna:  $S_i = \frac{P_t}{4\pi r^2}$
- Define  $G(\theta, \phi) = \frac{S(\theta, \phi)}{S_i}$ ,  $S(\theta, \phi)$  is the Poynting vector.
- $G(\theta, \phi) \equiv \frac{S(\theta, \phi)}{S_i} = \frac{4\pi r^2 S}{P_t}$ .
- Since  $S_\Omega(\theta, \phi) = r^2 S(\theta, \phi)$
- $\int G(\theta, \phi) d\Omega = \int \frac{4\pi r^2 S}{P_t} d\Omega = \frac{4\pi}{P_t} \int r^2 S d\Omega = 4\pi$





# Receiving Antenna

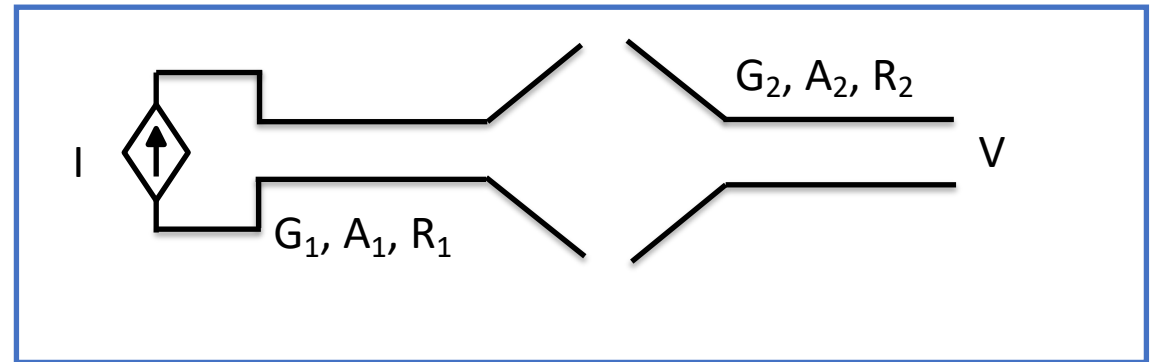
- $V_o = hE$ ,  $h$  is effective antenna length ( $h = \frac{l}{2}$  for short antenna),  $V_o$  is open circuit Thevenin voltage.
- For dipole:  $V_o = \frac{l}{2} E \sin(\theta)$ , so  $h = \frac{l}{2} \sin(\theta)$ .
- The effective area is  $A(\theta, \phi) \equiv \frac{P_r}{S(\theta, \phi)}$
- By reciprocity,  $A(\theta, \phi) = \frac{\lambda^2}{4\pi} G(\theta, \phi)$
- Thevenin equivalent relates effective area to effective length
- $P_r = \frac{|V_o|^2}{8R_a} = \frac{|hE|^2}{8R_a}$ , so  $P_r = \frac{h^2 \eta_0}{4R} S(\theta, \phi)$ , so  $A = \frac{h^2 \eta_0}{4R}$



Dipole Thevenin equivalent circuit

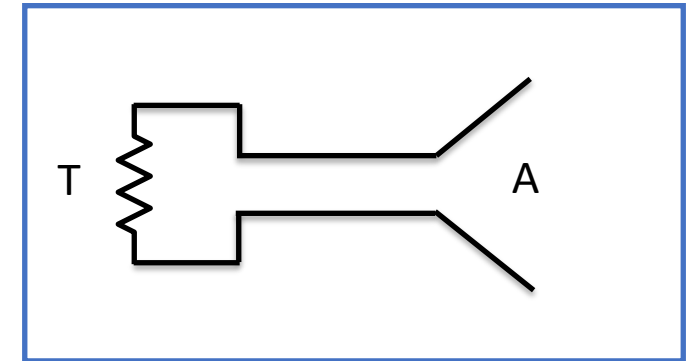
# Friis formula

- Far field:  $S = \frac{P_t G_1}{4\pi r^2}$ ,  $P_r = SA_2$ , so:
- Friis radiation formula:  $P_r = \frac{P_t}{4\pi r^2} G_1 A_2$
- For us,  $G_1 = 1$ ,  $A_2 = 150 \text{ m}^2$ ,  $r = 2000 \text{ km}$ ,  $P_t = 2 \text{ W}$
- $P_r = 6 \text{ pW}$



# Antenna Theorem

- Antenna theorem:  $\oint A(\theta, \phi) d\Omega = \lambda^2$
- Proof:
  - For cavity on right, T is constant at thermodynamic equilibrium and the same power is emitted and absorbed, the Johnson noise is  $kT$ .
  - The energy received is
    - $E = \frac{4\pi kT}{c\lambda^2}$ .
    - Set  $B = \frac{kT}{\lambda^2}$ .
    - $kT = \oint BA d\Omega = \oint A \frac{kT}{\lambda^2} d\Omega$ , giving the antenna theorem



Insulated cavity

# Antenna aperture

- Deriving antenna aperture uses thermodynamic argument: black body equilibrium

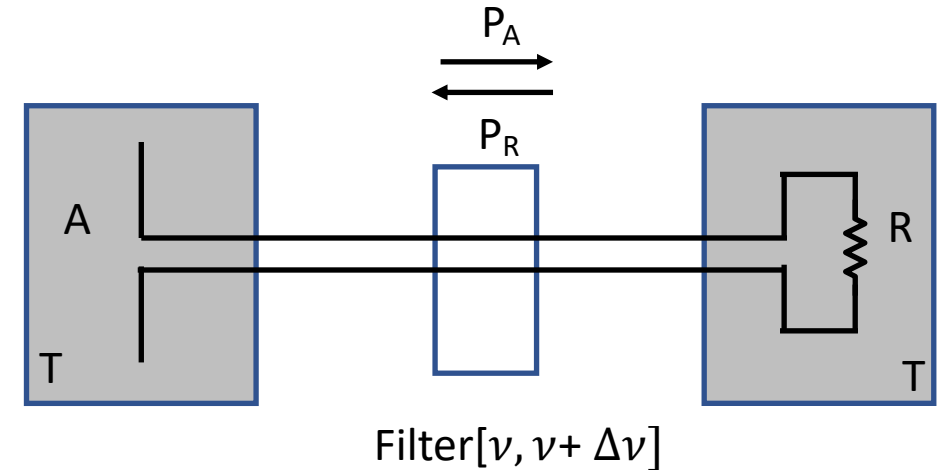
- $$P_A = \frac{A_e}{2} B_\nu \Delta\nu \int_{[0,4\pi]} d\Omega = 2\pi A_e B_\nu \Delta\nu$$

- $$B_\nu = \frac{2\nu^2 kT}{c^2} = \frac{2kT}{\lambda^2} \text{ (Rayleigh-Jeans)}$$

- $$P_R = kT\Delta\nu$$

- $$P_A = P_R$$

- $$A_e = \frac{\lambda^2}{4\pi}$$

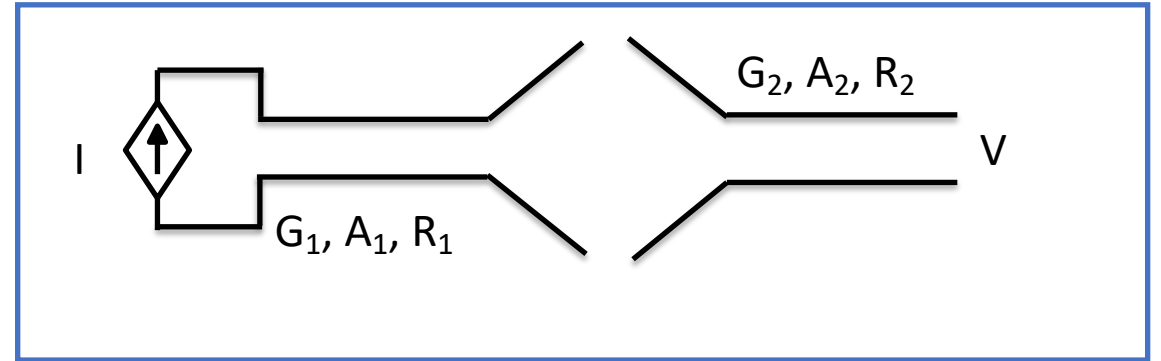


# Radiation loss and antenna aperture

- Spreading loss:  $L_s = 32 + \log(d) + 20 \log(f)$ 
  - d in kilometers
  - F in megahertz
- $W = A_e P_e, A_e = \frac{\lambda^2}{4\pi}$

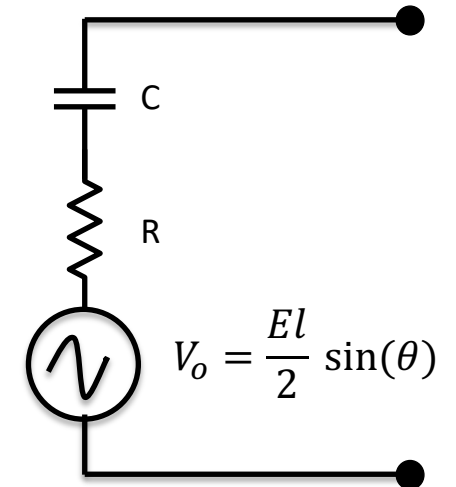
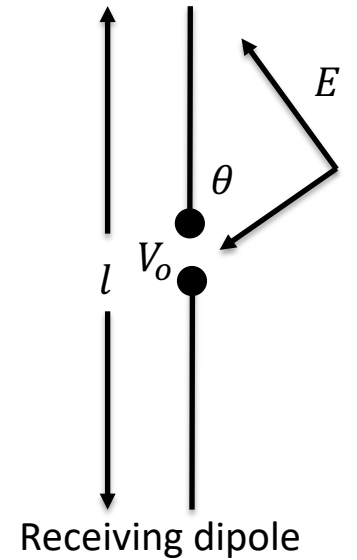
# Reciprocity

- *Reciprocity*: The position of an ideal voltmeter and ideal current source can be interchanged without changing the voltmeter reading.
- $P_t = |I|^2 \frac{R_1}{2}$
- $P_r = \frac{P_t G_1 A_2}{4\pi r^2}$ , by Friis
- $P_r = \frac{|V|^2}{8R_2}$
- $\frac{|V|^2}{8R_2} = \frac{|I|^2 R_1 G_1 A_2}{8\pi r^2}$
- $\frac{|V|^2 \pi r^2}{|I|^2 R_1 R_2} = G_1 A_2$ , so
- $G_1 A_2 = G_2 A_1$  or  $\frac{G_1}{A_1} = \frac{G_2}{A_2} = \frac{G}{A}$
- $\int G(\theta, \phi) d\Omega = 4\pi$  and by the antenna theorem,  $\int A(\theta, \phi) d\Omega = \lambda^2$ , and  $\frac{G}{A} = \frac{4\pi}{\lambda^2}$



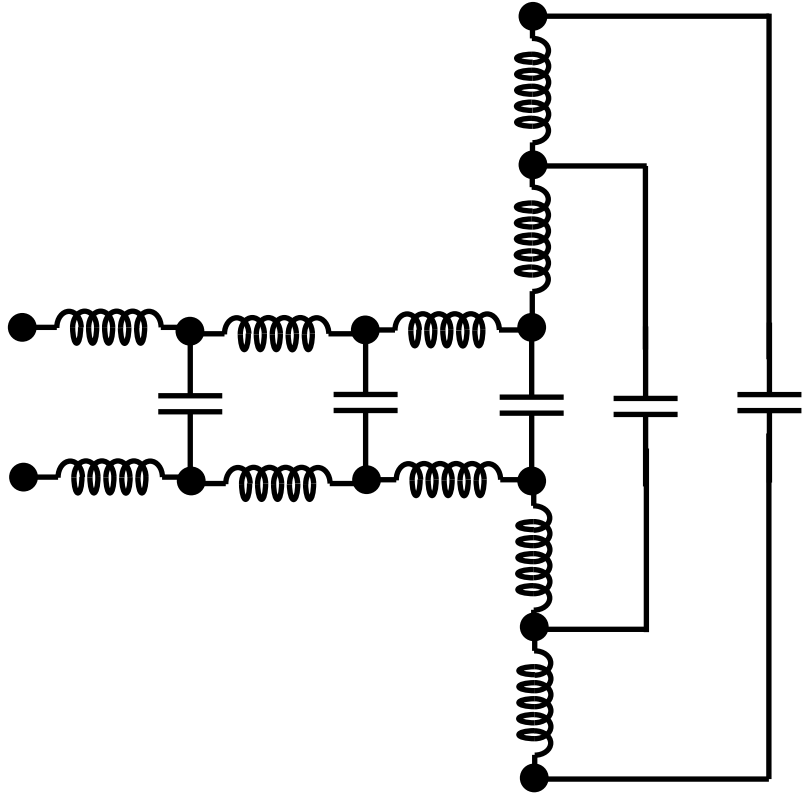
# Dipoles

- For dipole,  $h = \frac{l}{2} \sin(\theta)$ . Assume dipole is lossless so  $R = R_r$ ,  $\eta = 1$ .
- Remember  $\eta = \frac{R_r}{R}$  ( $= 1$ ),  $d\Omega = \sin(\theta) d\theta d\phi$  and  $A = \frac{h^2 \eta_0}{4R}$
- $\lambda^2 = \int A(\theta, \phi) d\Omega = \int \frac{h^2 \eta_0}{4R_r} d\Omega$ , so
- $R_r = \frac{l^2 \eta_0}{16\lambda^2} \int \sin^2(\theta) d\Omega = \eta_0 \frac{\pi}{6} \left(\frac{l}{\lambda}\right)^2$
- $\eta_0 = 120\pi$  so for half-wave dipole,  $R_r \approx 20\pi^2 \approx 49\Omega$  (It's closer to  $73\Omega$ )
  - This explains the customary  $73\Omega$  or  $73\Omega$  coax impedance
- Since  $A = \frac{h^2 \eta_0}{4R}$ ,  $A = \frac{3\lambda^2}{8\pi} \sin^2(\theta)$
- Since  $\frac{G}{A} = \frac{4\pi}{\lambda^2}$ ,  $G = \frac{3}{2} \sin^2(\theta)$ .
- For loaded whip:  $R_r = 160\pi^2 \left(\frac{l}{\lambda}\right)^2$
- For Yagi:  $G = 12dB$ , peak pattern is at 27 degrees for 7MHz



Dipole Thevenin equivalent circuit

# Another antenna model

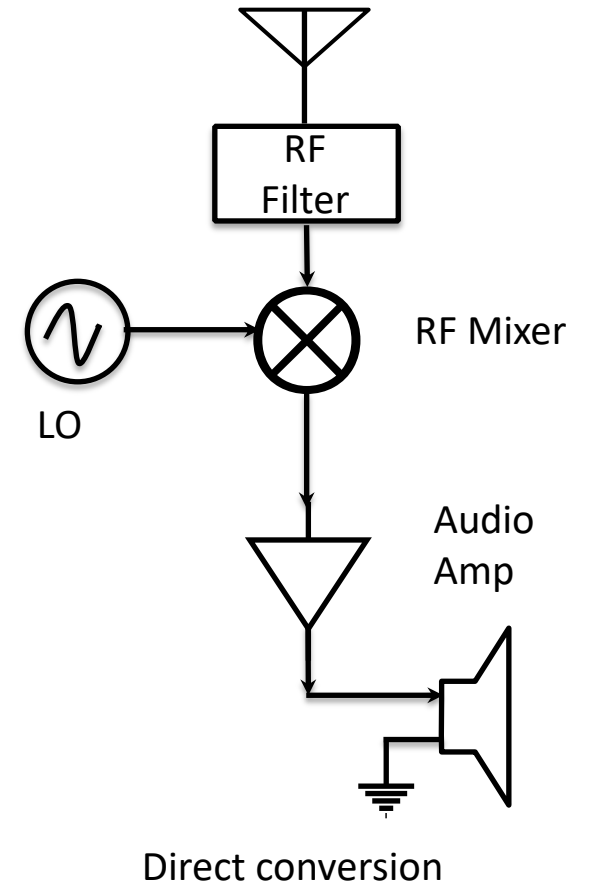


- $E = \frac{j\omega\mu}{4\pi} I_m \frac{e^{j\beta r}}{r}$
- $P_R = P_B G_A G_B \left(\frac{\lambda}{4\pi r_{AB}}\right)^2$



# Direct conversion and superhet receivers

- Image frequency
  - $\omega_{rf} = \omega_{LO} - \omega_a$
  - $\omega_i = \omega_{LO} + \omega_a$
- Superheterodyne designs
  - $\omega_{rf} = \omega_{IF} + \omega_{VFO}$
  - $\omega_{vi} = \omega_{IF} - \omega_{VFO}$
  - $\omega_{IF} = \omega_{BFO} - \omega_a$
  - $\omega_{bi} = \omega_{BFO} + \omega_a$
  - $\omega_{usb} = \omega_{VFO} + \omega_{BFO} + \omega_a$
  - $\omega_{lsb} = \omega_{VFO} + \omega_{BFO} - \omega_a$



# Exercise 35: Intermodulation

- Only  $f_{3\uparrow} = 2f_1 - f_2$ ,  $f_{3\downarrow} = 2f_2 - f_1$ ,  $f_{5\uparrow} = 3f_1 - 2f_2$  and  $f_{5\downarrow} = 3f_2 + 2f_1$  are close enough to the rf frequency to matter for intermodulation
- 1. Find coefficients and frequencies for  $[\cos(\omega_1 t) + \cos(\omega_2 t)]^5$
- 2. Find  $f_{3\uparrow}$ ,  $f_{3\downarrow}$ ,  $f_{5\uparrow}$  and  $f_{5\downarrow}$
- 3. Find the MDS and the antenna limited MDR
- $MDS = \frac{P_n}{G}$ ,  $MDI$  is input that gives output tone +  $2P_n$
- $MDR = MDI - MDS$
- This needs measurements to find the minimum detectable signal.

# Exercise 37: Antennas

1. Use the relation between gain and effective area to rewrite the Friis transmission formula in terms of gain only. Consider UHF for airplanes. If the frequency makes the quarter length stub antenna have gain 2, find the maximum possible LOS at 10km height. Required receiver power is  $-90$  dBm. Find the minimum transmission power.

- Friis radiation formula is  $P_r = \frac{P_t G A}{4\pi r^2}$ ,  $\lambda = \frac{c}{f}$  and  $\frac{G}{A} = \frac{4\pi}{\lambda^2}$ , so  $P_r = P_t \left(\frac{Gc}{4\pi r f}\right)^2$
- $P_r$  requires  $-90$  dBm power.  $-90 = 10 \log\left(\frac{P_r}{1mW}\right)$ , so  $P_r$  must be at least  $10^{-12}$  W.
- $P_t(f) = \left(\frac{4\pi r f}{Gc}\right)^2 \times 10^{-12} W$ .  $P_t(10^8) = 4.4mW$ ,  $P_t(2 \times 10^8) = 19.24mW$ ,  $P_t(3 \times 10^8) = 40mW$  and  $P_t(10^9) = 440mW$ .

2. Find the inductance to resonate with a 3m whip. Assuming the Q of the coil is 200, find the turns ratio required to give a transceiver a 50 ohm load. What is the radiation efficiency?

- For whip,  $R_r = 160\pi^2 \left(\frac{l}{\lambda}\right)^2 \approx 7.8\Omega$ .  $Q = 200 = \frac{L}{7.8}$ .  $L = 1551\Omega$ .
- $Z_s = 7.8 + 1551j$ .  $L = 3.6\mu H$
- $\frac{1551}{50} = \frac{Z_p}{Z_s} = \left(\frac{N_p}{N_s}\right)^2$ ,  $\frac{N_p}{N_s} = \sqrt{31} = 5.2$ .
- Radiation efficiency is  $\frac{P_{radiated}}{P_{input}} = \frac{R_r}{R_r + R_L} = \frac{7.8}{57.8} \approx .14$

# Exercise 17: Tuned Speaker

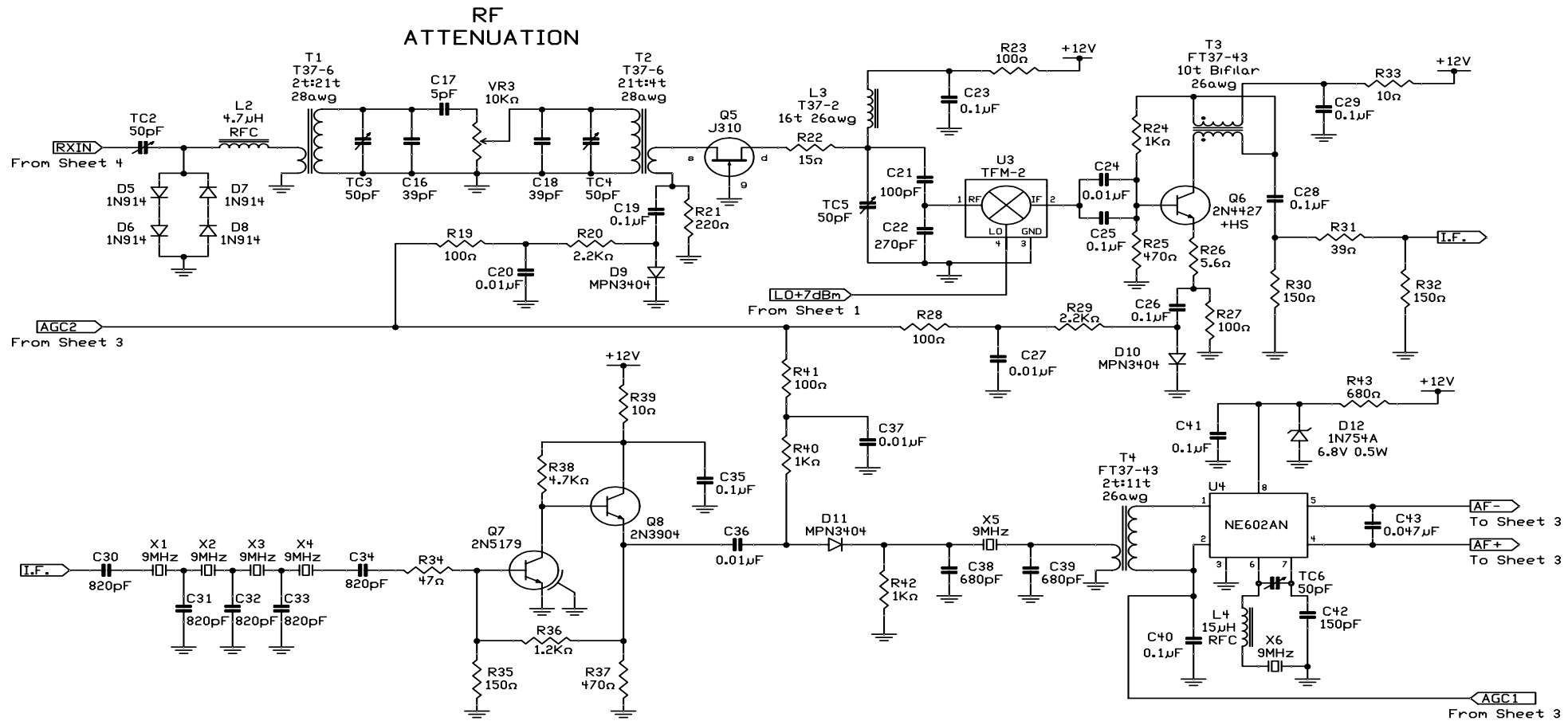
- Connect speaker to function generator 600Hz, 25mVrms.
  1. Sound peaks at resonance. Find resonant frequency  $L_p$ .
  2. Measure  $f_l$ ,  $f_u$  by noting the 3dB loss. Calculate Q.
  3. Use voltmeter to find resonance with speaker (nominally 8ohm) to calculate impedance
  4. Calculate the resonant frequency from a transmission line equivalent circuit.

# Norcal circuit diagram, 1



Copyright 1999 Dave Fifield, AD6A			
Project:	Norcal 20 CW Transceiver		
Title:	VFO	Revision:	D
Author:	Dave Fifield, AD6A	Size:	A
Date:	January 2, 1999	Sheet	1 of 5

# Norcal circuit diagram, 2



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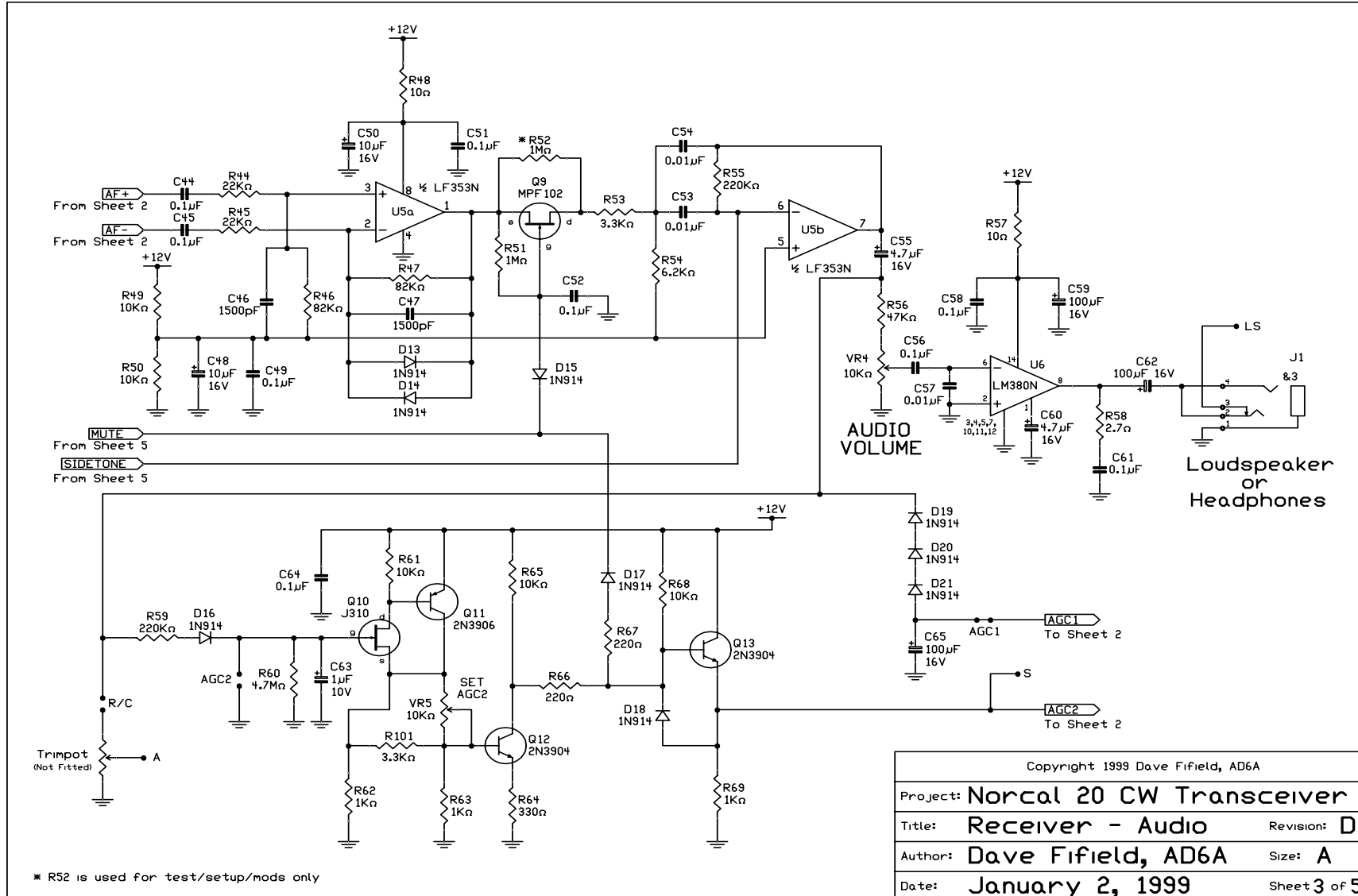
Project: Norcal 20 CW Transceiver

Title: Receiver - RF Revision: D

Author: Dave Fifield, AD6A      Size: A

Date: January 2, 1999 Sheet 2 of 5

# Norcal circuit diagram, 3



# Norcal circuit diagram, 4



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Project:	Norcal 20 CW Transceiver		
Title:	Transmitter	Revision:	D
Author:	Dave Fifield, AD6A	Size:	A
Date:	January 2, 1999	Sheet	4 of 5



# Norcal circuit diagram, 5



# In the beginning...

- The laws of EM according to Clerk Maxwell are:

$$1. \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$2. \quad \nabla \cdot \mathbf{B} = 0$$

$$3. \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$4. \quad c^2 \nabla \times \mathbf{B} = \frac{\mathbf{j}}{\epsilon_0} + \frac{\partial \mathbf{E}}{\partial t}, \quad \epsilon_0 = 8.854 \times 10^{-12} \frac{C^2}{N \cdot m^2}, \quad \frac{1}{c^2} = \epsilon_0 \mu_0$$

$$5. \quad \nabla \cdot \mathbf{j} = -\frac{\partial \rho}{\partial t}$$

- Here  $\mathbf{E}$  is the electric field,  $\mathbf{B}$  is the magnetic field,  $\mathbf{j}$  is the current density through a closed surface,  $c$  is the speed of light and  $\rho$  is the charge density at a point.
- In non-dispersive matter,  $B = \mu H = \mu_0(H + M)$ ,  $\mu = \mu_0(1 + \chi_m)$ ,
- $D = \epsilon E = \epsilon_0 E + P$ ,  $\epsilon = \kappa \epsilon_0$  and (1) becomes  $\nabla \cdot \mathbf{D} = \rho_f$ , (4) becomes  $\nabla \times \mathbf{H} = \mathbf{j}_f + \frac{\partial \mathbf{D}}{\partial t}$

- The rest of classical physics, including special relativity, is:

- Newton-Einstein:  $\mathbf{p} = m\mathbf{v}$ ,  $m = \frac{m_0}{\sqrt{1 - (\frac{v}{c})^2}}$ ,  $\mathbf{F} = m \frac{d\mathbf{p}}{dt}$ .

- Gravity:  $\mathbf{F} = -\frac{Gm_1m_2}{r^2} \mathbf{u}_r$  where  $\mathbf{u}_r$  is the unit vector from  $m_1$  to  $m_2$  and  $\mathbf{F}$  is the force on  $m_2$ .

# Solutions to the wave equation

- The solution of  $\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = -s$  is  $\psi(x, y, z, t) = \frac{S(t - \frac{r}{c})}{4\pi r}$  where  $S = \int_V s \, dV$
- Later, we will use this to find the "general" solution to Maxwell's equations
  - $\phi(r_1, t) = \int_{V_2} \frac{\rho(r_2, t - \frac{|r_1 - r_2|}{c})}{4\pi\epsilon_0 |r_1 - r_2|} dV_2$  and  $\mathbf{A}(r_1, t) = \int_{V_2} \frac{\mathbf{j}(r_2, t - \frac{|r_1 - r_2|}{c})}{4\pi\epsilon_0 c^2 |r_1 - r_2|} dV_2$ , where
  - $\mathbf{B} = \nabla \times \mathbf{A}$ ,  $\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$ , and  $c^2 \nabla \cdot \mathbf{A} = -\frac{\partial \phi}{\partial t}$
- You are not expected to have guessed this answer
- To do this, we'll need the "BAC-CAB" identity:  $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$
- When we apply this to  $\nabla \times (\nabla \times \mathbf{A})$ , we get  $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

# General solution to Maxwell's equations

- Returning to the general Maxwell equations, from  $\nabla \cdot \mathbf{B} = 0$ , we get  $\mathbf{B} = \nabla \times \mathbf{A}$
- Substituting into  $c^2 \nabla \times \mathbf{B} = \frac{\mathbf{j}}{\epsilon_0} + \frac{\partial \mathbf{E}}{\partial t}$ , we get  $c^2 \nabla \times (\nabla \times \mathbf{A}) = \frac{\mathbf{j}}{\epsilon_0} + \frac{\partial \mathbf{E}}{\partial t}$
- Applying "BAC-CAB", we get  $\nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \frac{\mathbf{j}}{c^2 \epsilon_0} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$  (Equation 1)
- Now,  $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ , so substituting for  $\mathbf{B}$ , we get  $\nabla \times \left( \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$  and so  $\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$
- Substituting into equation 1,  $\nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \frac{\mathbf{j}}{c^2 \epsilon_0} + \frac{1}{c^2} \frac{\partial}{\partial t} \left( -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \right)$ , or
- $\nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \frac{\mathbf{j}}{c^2 \epsilon_0} - \frac{1}{c^2} \nabla \frac{\partial \phi}{\partial t} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2}$ .
- $\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{\mathbf{j}}{c^2 \epsilon_0} + \nabla \left[ \frac{1}{c^2} \frac{\partial \phi}{\partial t} + (\nabla \cdot \mathbf{A}) \right]$
- Now if  $\mathbf{A}$  and  $\phi$  give  $\mathbf{B} = \nabla \times \mathbf{A}$  and  $\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$ , then  $\mathbf{A}' = \mathbf{A} + \nabla \varphi$  and  $\phi' = \phi - \frac{\partial \varphi}{\partial t}$  give  $\mathbf{B} = \nabla \times \mathbf{A}'$  and  $\mathbf{E} = -\nabla \phi' - \frac{\partial \mathbf{A}'}{\partial t}$ , for any function  $\varphi$

# General solution to Maxwell's equations

- Thus, we can pick a solution  $(A, \phi)$  with  $\nabla \cdot A = -\frac{1}{c^2} \frac{\partial \phi}{\partial t}$ . Then we get
- $\nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = -\frac{j}{c^2 \epsilon_0}$
- Substituting for  $E = -\nabla \phi - \frac{\partial A}{\partial t}$  into  $\nabla \cdot E = \frac{\rho}{\epsilon_0}$ , we get
- $\nabla \cdot (\nabla \phi) + \frac{\partial \nabla \cdot A}{\partial t} = -\frac{\rho}{\epsilon_0}$ , or  $\nabla \cdot (\nabla \phi) - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$ , or  $\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$
- The solutions are  $\phi(r_1, t) = \int_{V_2} \frac{\rho(r_2, t - \frac{|r_1 - r_2|}{c})}{4\pi\epsilon_0 |r_1 - r_2|} dV_2$  and
- $A(r_1, t) = \int_{V_2} \frac{j(r_2, t - \frac{|r_1 - r_2|}{c})}{4\pi\epsilon_0 c^2 |r_1 - r_2|} dV_2$  with  $\nabla \cdot A = -\frac{1}{c^2} \frac{\partial \phi}{\partial t}$ , with  $\nabla \cdot A = -\frac{1}{c^2} \frac{\partial \phi}{\partial t}$ .

# Solution to Maxwell's equations in free space

- Free space is defined by  $\rho = 0$  and  $j = 0$ , so our potentials satisfy
- $\nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = 0$  and  $\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0$
- These have the usual wave equation solutions  $\phi(x, y, z, t) = f(k \cdot r - \omega t)$ , etc
- Thus, in free space  $\phi$  and  $\mathbf{A}$  and hence E and B propagate as waves.

# Solution to Maxwell's equations in conductors

- In conductors,  $\mathbf{j} = \sigma \mathbf{E}$ 
  - $c^2 \nabla \times \mathbf{B} = \frac{\mathbf{j}}{\epsilon} + \frac{\partial \mathbf{E}}{\partial t} = \frac{\sigma}{\epsilon} \mathbf{E} + \frac{\partial \mathbf{E}}{\partial t}$
  - This becomes  $c^2 \frac{\partial(\nabla \times \mathbf{B})}{\partial t} = \frac{\sigma}{\epsilon} \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial^2 \mathbf{E}}{\partial t^2}$
  - $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ , so we get  $c^2 \nabla \times \frac{\partial \mathbf{B}}{\partial t} = -c^2 \nabla \times (\nabla \times \mathbf{E}) = \frac{\sigma}{\epsilon} \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial^2 \mathbf{E}}{\partial t^2} = -c^2 [\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}] = c^2 \nabla^2 \mathbf{E}$   
(since  $\rho = 0$  in a conductor)
  - Applying the trial solution  $\mathbf{E} = \mathbf{E}_0 \exp(\omega t - \mathbf{k} \cdot \mathbf{r})$ , we get  $-k^2 - i\omega\mu\sigma + \omega^2\mu\epsilon = 0$ .
  - Putting  $k = \alpha - \beta i$ ,  $\alpha = \frac{\omega}{2} \sqrt{\mu\epsilon} \left(1 + \sqrt{1 + \frac{\sigma^2}{\omega^2 \epsilon^2}}\right)$  and  $\beta = \frac{\omega\mu\sigma}{2\alpha}$ .
  - For copper,  $\sigma = 5.78 \times 10^7 \, \Omega\text{-m}$ . This explains the “skin effect” in conductors.

# Radiation, antennas

- Accelerating charges radiate energy in the form of electromagnetic waves (companion E and B fields).
- The radiation from accelerating charge  $q$  is  $\mathbf{E}_{rad} = -\frac{1}{4\pi\epsilon_0 c^2} \frac{q}{r} \mathbf{a}_\perp(t - \frac{r_{12}}{c})$ .
- Here,  $\mathbf{a}_\perp$  is the acceleration  $\perp$  to the line from  $r_1$  to  $r_2$ .
- For example, applying a time varying potential  $V_0 \sin(\omega t)$  to an antenna will cause the antenna to radiate power since the voltage and hence charges affected accelerate within the antenna, that is, their positions have a non-zero second derivative. That's how a transmitter "couples" to the antenna of a receiver. In the receiver, the radiated wave accelerates charges in the antenna replicating the original wave (at much reduced power).
- These simple radio waves are carrier waves of frequency  $\frac{\omega}{2\pi}$ . To transfer information (voice, images, binary data), we modulate carrier waves combining them with an "information source" signal. Receivers demodulate the incoming wave and recreate the original "information source" signal.



# Maxwell's equations in a non-dispersive media

- $B = \mu H, D = \epsilon E$
- $\nabla \cdot D = \rho$
- $\nabla \cdot B = 0$
- $\nabla \times E = -\frac{\partial B}{\partial t}$
- $\nabla \times H = j + \frac{\partial E}{\partial t}$
- $\nabla \cdot j = -\frac{\partial \rho}{\partial t}$

# Radiation from a small dipole

- $A(1, t) = \int \frac{\vec{j}(2, t - \frac{r_{12}}{c})}{4\pi\epsilon_0 c^2 r_{12}} dV_2$ . From figure on right,  $j_z dV_2 = I dz'$ .
- Note  $\vec{p}(t) = q(t)l\vec{k}$ ,  $k = \hat{z}$  and  $\dot{q}(t) = I$ .  $\phi$  is the angle from the x axis to the projection of  $\vec{r}$  on the x-y plane.
- $\vec{r}' + z'\vec{k} = \vec{r}$  and  $|\vec{r} - z'\vec{k}| = r - z'\cos(\theta)$
- If  $l \ll cT = \lambda$ ,  $I(\vec{z}', t - \frac{r'}{c}) \approx I(0, t - \frac{r'}{c})$  and we get:
- $A_z(r, t) = \frac{\mu_0}{4\pi} \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{I(\vec{z}', t - \frac{r'}{c})}{4\pi\epsilon_0 c^2 r'} dz' = \frac{\mu_0}{4\pi} \frac{l}{r} I(0, t - \frac{r'}{c})$
- Choosing gauge,  $\nabla \cdot \vec{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0$
- $\frac{\partial \phi}{\partial t} = -\frac{l}{4\pi\epsilon_0} \frac{\partial}{\partial z} \left( \frac{1}{r} I\left(t - \frac{r}{c}\right) \right) = \frac{z}{r^2} \left( \frac{q(t - \frac{r}{c})}{r} - \frac{I(t - \frac{r}{c})}{c} \right)$
- Pick an oscillating dipole:  $q\left(t - \frac{r}{c}\right) = q_0 \cos\left(\omega\left[t - \frac{r}{c}\right]\right)$
- $I\left(t - \frac{r}{c}\right) = I_0 \sin\left(\omega\left[t - \frac{r}{c}\right]\right) = -\omega q_0 \sin\left(\omega\left[t - \frac{r}{c}\right]\right)$



# Radiation from a small dipole

- $\nabla^2 H - \epsilon\mu \frac{\partial^2 H}{\partial t^2} - \sigma\mu \frac{\partial H}{\partial t} = 0$
- $\nabla^2 E - \epsilon\mu \frac{\partial^2 E}{\partial t^2} - \sigma\mu \frac{\partial E}{\partial t} = 0$
- $A_r = \frac{\mu_0}{4\pi} \frac{I_0 l}{r} \cos(\theta) \sin(\omega [t - \frac{r}{c}])$
- $A_\phi = 0, A_\theta = -\frac{\mu_0}{4\pi} \frac{I_0 l}{r} \cos(\theta) \sin(\omega [t - \frac{r}{c}])$
- $B_\phi = \frac{1}{r} \frac{\partial}{\partial r} (r A_\theta) - \frac{1}{r} \frac{\partial A_r}{\partial \theta} = \frac{\mu_0}{4\pi} \frac{I_0 l}{r} \sin(\theta) [\frac{\omega}{r} \cos(\omega [t - \frac{r}{c}]) + \frac{1}{r} \sin(\omega [t - \frac{r}{c}])]$
- $E_r = -\frac{\partial \phi}{\partial t} - \frac{\partial A_r}{\partial t} = \frac{2I_0 \cos(\theta)}{4\pi\epsilon_0} [\frac{\sin(\omega [t - \frac{r}{c}])}{r^2 c} - \frac{\cos(\omega [t - \frac{r}{c}])}{\omega r^3}]$
- $E_\theta = \frac{-I_0 l \sin(\theta)}{4\pi\epsilon_0} (\left[\frac{1}{r^3 \omega} - \frac{\omega}{rc^2}\right] \cos(\omega [t - \frac{r}{c}]) - \frac{1}{cr^2} \sin(\omega [t - \frac{r}{c}]))$
- $E_\phi = -\frac{1}{r \sin(\theta)} \frac{\partial \phi}{\partial \phi} - \frac{\partial A_\phi}{\partial t} = 0$

# Radiation from a small dipole

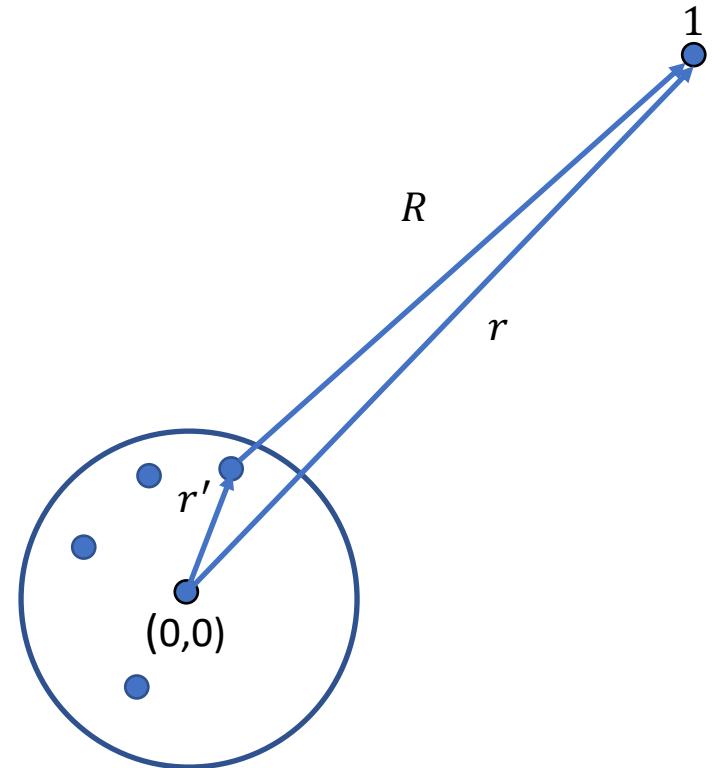
- $\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\mathbf{H} \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \frac{\partial \mathbf{D}}{\partial t} - \mathbf{E} \cdot \mathbf{j}$
- $\mathbf{S} = \mathbf{E} \times \mathbf{H}, \nabla \cdot \mathbf{S} + \frac{\partial u}{\partial t} = -\mathbf{E} \cdot \mathbf{j}$  (u is energy density)
- $\int S \cdot d\mathbf{A} = \frac{(l I_0 \omega)^2}{6\pi\epsilon_0 c^3} \cos(\omega [t - \frac{r}{c}])^2$
- $P_{av} = \frac{(l\omega)^2}{6\pi\epsilon_0 c^3} \frac{I_0^2}{2} = \frac{2\pi}{3} \sqrt{\frac{\mu_0}{\epsilon_0}} \left(\frac{l}{\lambda}\right)^2 \frac{I_0^2}{2}$
- $R_r = \frac{2\pi}{3} \sqrt{\frac{\mu_0}{\epsilon_0}} \left(\frac{l}{\lambda}\right)^2$

# Large half wave dipole

- For large half wave, add small dipoles to produce half wave antenna.
- $dE_{\theta} = I_0 \frac{\sin(\theta)}{4\pi\epsilon_0 R c^2} \omega \cos(\omega) \cos\left(\frac{2\pi z'}{\lambda}\right) dz'$
- $dB_{\phi} = I_0 \frac{\mu_0 \omega}{4\pi R c} \omega \cos\left(\omega \left[t - \frac{r}{c}\right]\right) \cos\left(\frac{2\pi z'}{\lambda}\right) dz'$
- Suffices to find:  $K = \int_{[-\frac{\pi}{2}, \frac{\pi}{2}]} \frac{1}{R} \cos\left(t - \frac{R}{c}\right) \cos(u) du = \frac{1}{2\pi\epsilon_0 r c} \cos\left(\omega \left[t - \frac{r}{c}\right]\right) \frac{\cos\left(\frac{\pi}{2} \cos(\theta)\right)}{\sin^2(\theta)}, u = \frac{2\pi z'}{\lambda}$
- $E_{\theta} = I_0 \frac{1}{2\pi\epsilon_0 r c} \cos\left(\omega \left[t - \frac{r}{c}\right]\right) \frac{\cos\left(\frac{\pi}{2} \cos(\theta)\right)}{\sin(\theta)}$
- $B_{\phi} = I_0 \frac{\mu_0}{2\pi r} \omega \cos\left(\omega \left[t - \frac{r}{c}\right]\right) \frac{\cos\left(\frac{\pi}{2} \cos(\theta)\right)}{\sin(\theta)}$
- $P_{av} = \frac{1}{4\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} I_0^2 \int_{[0, \pi]} \frac{\cos^2\left(\frac{\pi}{2} \cos(\theta)\right)}{\sin^2(\theta)} \sin(\theta) d\theta = 73.1 \Omega \frac{I_0^2}{2}$

# Radiation from charge distribution

- $r' + R = r, R = |r - r'|$
- $$\varphi(r, t) = \frac{1}{4\pi\epsilon_0} \int_{V_1} \frac{\rho(r', t - \frac{R}{c})}{|r - r'|} dv' = \frac{1}{4\pi\epsilon_0} \left[ \frac{Q}{r} + \frac{r \cdot p(t - \frac{r}{c})}{r^3} + \frac{r \cdot \frac{dp}{dt}(t - \frac{r}{c})}{cr^2} \right]$$
- $$\mathbf{A}(r, t) = \frac{\mu_0}{4\pi} \int_{V_1} \frac{\mathbf{j}(r', t - \frac{R}{c})}{|r - r'|} dv' = \frac{\mu_0}{4\pi r} \frac{d}{dt} \mathbf{p}(t - \frac{r}{c})$$
- $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \varphi$
- $$\mathbf{B}(r, t) = \frac{-\mu_0}{4\pi cr^2} \mathbf{r} \times \ddot{\mathbf{p}}(t - \frac{r}{c})$$
- $$\mathbf{E}(r, t) = -\frac{c}{r} \mathbf{r} \times \mathbf{B}(r, t)$$
- $$\frac{d\mathbf{p}}{dt} = q \frac{dr'}{dt} = qv, \quad \frac{d^2}{dt^2} \mathbf{p}(t - \frac{r}{c}) = q \frac{dv}{dt}$$
- $$S = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} = \frac{\ddot{p}^2 \sin(\theta)^2}{16\pi^2 \epsilon_0 c^3 r^2} \frac{\vec{r}}{r}$$
- $$P_R = -\frac{dW}{dt} = \frac{2}{3} \frac{q^2}{4\pi\epsilon_0 c^3} \left( \frac{dv}{dt} \right)^2$$
- Compare to Feynman later



# Radiation from a single accelerating charge

- Near zone

- $\varphi(r, t) = \frac{q}{4\pi\epsilon_0} \left[ \frac{1}{R'(1 + \frac{v \cdot n'}{c})} \right]$

- $R^* = R' - \frac{v}{c}(x_0 - x'_1)$

- $E(r, t) = \frac{1}{4\pi\epsilon_0} \frac{1}{R^{*3}} \left[ \left( R' - \frac{R' v'}{c} \right) \left( 1 - \frac{v^2}{c^2} \right) + \frac{R'}{c^2} \left( R' - \frac{R' v'}{c} \right) \times \frac{dv'}{dt} \right]$

- $B = \frac{R' \times E}{R' c}$

- $S = \frac{q^2}{16\pi^2 c^3 \epsilon_0} \frac{R' (R' \times v')^2}{(R')^5}$

# Feynman's Treatment - 1

- Want to derive the Heaviside-Feynman equation:

- $$E = \frac{q}{4\pi\epsilon_0} \left[ \frac{e_{r'}}{r'^2} + \frac{r'}{c} \frac{d}{dt} \frac{e_{r'}}{r'^2} + \frac{1}{c^2} \frac{d^2 e_{r'}}{dt^2} \right]$$
- $$cB = e_{r'} \times E$$

- Solution to Maxwell:

- $$E = -\nabla\phi - \frac{\partial A}{\partial t}, B = \nabla \times A$$
- $$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}, \nabla^2 A - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = -\frac{j}{c^2 \epsilon_0}$$
- With gauge  $\nabla \cdot A = -\frac{1}{c^2} \frac{\partial \phi}{\partial t}$

- Consider solution to  $\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = -s$

- First solve  $\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0$

- The solution of  $\frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = 0$  is
- $\varphi(x, t) = Af(x - ct) + Bg(x + ct)$
- $\frac{\partial \psi(r)}{\partial x} = \psi' \frac{\partial r}{\partial x}$  and  $\frac{\partial r}{\partial x} = \frac{x}{r}$
- $\frac{\partial^2 \psi}{\partial x^2} = \psi'' \left( \frac{\partial r}{\partial x} \right)^2 + \psi' \frac{\partial^2 r}{\partial x^2}$  and  $\frac{\partial^2 r}{\partial x^2} = \frac{1}{r} \left( 1 - \frac{x^2}{r^2} \right)$
- So,  $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \psi'' + \frac{2}{r} \psi'$
- $\nabla^2(r\psi) - \frac{1}{c^2} \frac{\partial^2(r\psi)}{\partial t^2} = 0$
- So,  $\psi(x, t, z, t) = \frac{f(t - \frac{r}{c})}{r}$



# Feynman's Treatment - 2

- As  $r \rightarrow 0$ ,  $\psi(x, t, z, t) = \frac{f(t)}{r}$ .
- Consider the electrostatic case where  $\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$  where  $\phi = \frac{q}{4\pi\epsilon_0 r}$ .
- By analogy:
  - $f\left(t - \frac{r}{c}\right) = \frac{S(t - \frac{r}{c})}{4\pi}$
  - $S = \int s \, dV$
- So, solution is:  $\psi(x, t, z, t) = \frac{1}{4\pi} \frac{S(t - \frac{r}{c})}{r}$ . Thus,
  - $A(1, t) = \int \frac{\vec{j}\left(2, t - \frac{r_{12}}{c}\right)}{4\pi\epsilon_0 c^2 r_{12}} \, dV_2$
  - $\phi(1, t) = \int \frac{\rho\left(2, t - \frac{r_{12}}{c}\right)}{4\pi\epsilon_0 r_{12}} \, dV_2$
  - $E = -\nabla\phi - \frac{\partial A}{\partial t}, B = \nabla \times A$
  - With gauge  $\nabla \cdot A = -\frac{1}{c^2} \frac{\partial \phi}{\partial t}$

# Feynman's Treatment - 3

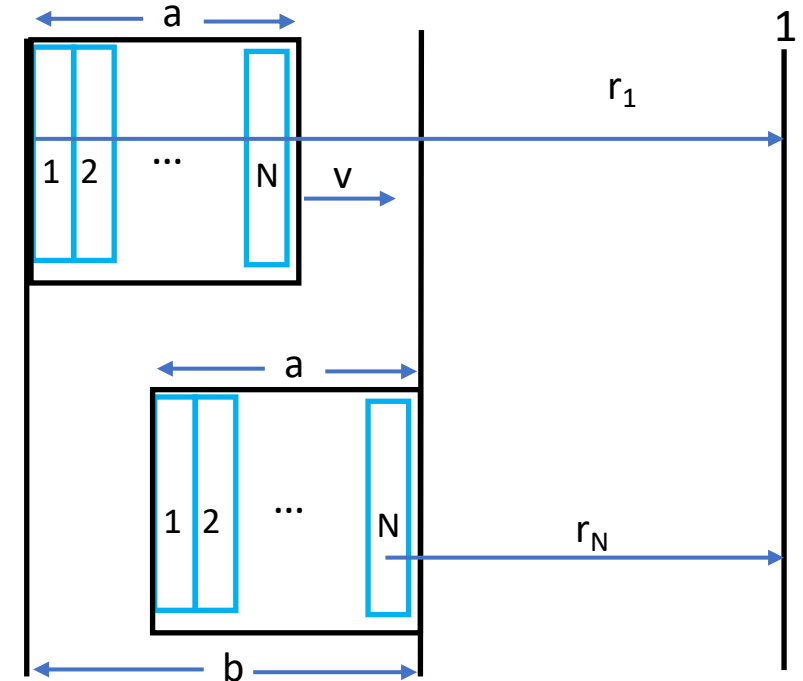
- For small blob of charge moving with velocity  $v$ ,  $j = v\rho$ .
- $A(1, t) = \int \frac{\vec{j}(2, t - \frac{r_{12}}{c})}{4\pi\epsilon_0 c^2 r_{12}} dV_2 = \frac{1}{r} \int v\rho dV_2 = \frac{qv}{r}$
- So, for a dipole with  $\vec{p}(r, t) = qd\vec{k}$ ,  $\frac{qv}{r} = \frac{\partial \vec{p}}{\partial t}$
- $A(1, t) = \frac{\dot{p}(t - \frac{r}{c})}{4\pi\epsilon_0 c^2 r}$ ,  $B = \nabla \times A$  and  $B_z = 0$ .  $A$  is due to dipole current. If the dipole charges oscillate  $p(t) = p_0 \sin(\omega t)$ ,  $A_z(1, t) = \frac{1}{4\pi\epsilon_0 c^2 r} p_0 \omega \cos(\omega[t - \frac{r}{c}])$
- $B_x = \frac{\partial A_z}{\partial y} = \frac{1}{4\pi\epsilon_0 c^2 r} \frac{\partial \dot{p}(t - \frac{r}{c})}{\partial y}$ .
- $B_y = -\frac{\partial A_z}{\partial x} = -\frac{1}{4\pi\epsilon_0 c^2 r} \frac{\partial \dot{p}(t - \frac{r}{c})}{\partial x}$
- $B_x = \frac{1}{4\pi\epsilon_0 c^2 r} \left[ -\frac{y(t - \frac{r}{c})}{r^3} - \frac{y\ddot{p}(t - \frac{r}{c})}{cr^2} \right]$ ,  $B_y = \frac{1}{4\pi\epsilon_0 c^2 r} \left[ \frac{x\dot{p}(t - \frac{r}{c})}{r^3} + \frac{x\ddot{p}(t - \frac{r}{c})}{cr^2} \right]$

# Feynman's Treatment - 4

- Rewrite previous equation as
  - $\vec{B} = \frac{1}{4\pi\epsilon_0 c^2 r^3} \left[ \dot{p} \left( t - \frac{r}{c} \right) + \frac{r}{c} \ddot{p} \left( t - \frac{r}{c} \right) \right] \times r$
  - Compare this to Biot Savart:  $dB = \frac{1}{4\pi\epsilon_0 c^2} \frac{j \times r}{r^3} dV$
- Now suppose the dipole charges oscillate  $q(t) = q_0 \sin(\omega t)$
- When  $\frac{r}{c}$  is small,  $\dot{p} \left( t - \frac{r}{c} \right) = \dot{p}(t) - \frac{r}{c} \ddot{p}(t) + \dots$
- $E \perp B$ ,  $E = cB$  and from  $\frac{\partial \phi}{\partial t} = -\nabla \cdot A$ , we get
- $\phi = \frac{1}{4\pi\epsilon_0 r^3} \left[ p \left( t - \frac{r}{c} \right) + \frac{r}{c} \dot{p} \left( t - \frac{r}{c} \right) \right] \cdot r$ , and
- $E = -\nabla \phi - \frac{\partial A}{\partial t} = \frac{1}{4\pi\epsilon_0 c^2 r^3} \left[ \frac{3(p^* \cdot r)\vec{r}}{r^2} + \frac{1}{c^2} (\ddot{p} \left( t - \frac{r}{c} \right) \times r \times r) \right]$  where
- $p^* = p \left( t - \frac{r}{c} \right) + \frac{r}{c} \dot{p} \left( t - \frac{r}{c} \right)$

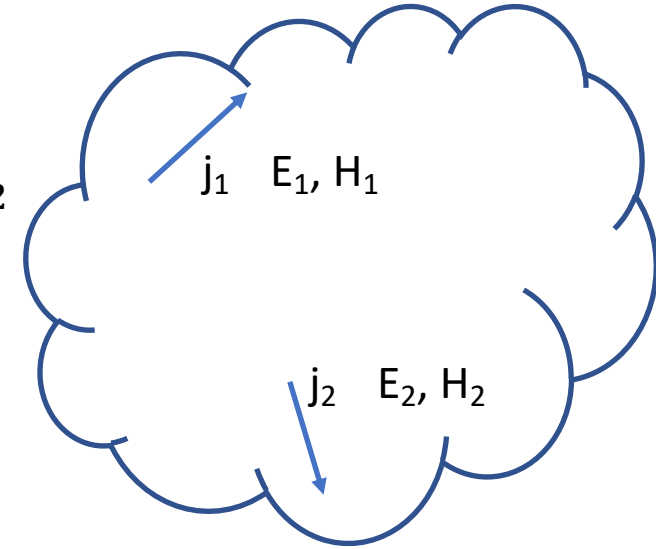
# Lienard-Wierchert

- $\phi(1, t) = \int \frac{\rho(2, t - \frac{r_{12}}{c})}{4\pi\epsilon_0 r_{12}} dV_2$
- $\phi(1, t) = \sum \frac{\rho_i \Delta V_i}{r_i'} = \sum \frac{\rho w a^2}{r'} = \frac{\rho a^3 N w}{r'} = \frac{q}{r'} \frac{b}{a}$
- $b = a + \frac{v}{c} b$
- So, we get
  - $\phi(1, t) = \frac{q}{4\pi\epsilon_0 [r - \frac{v \cdot r}{c}]_{ret}}$  and
  - $A(1, t) = \frac{q v_{ret}}{4\pi\epsilon_0 c^2 [r - \frac{v \cdot r}{c}]_{ret}}$
- Now we can compute  $E$  as above and verify the Heaviside formula



# A proof of reciprocity for transmitters and receivers

- $\nabla \cdot (E_1 \times H_2 - E_2 \times H_1) = (\nabla \times E_1) \cdot H_2 - (\nabla \times H_2) \cdot E_1 - (\nabla \times E_2) \cdot H_1 + (\nabla \times H_1) \cdot E_2$
- $\nabla \times E_1 = -j\omega\mu H_1, \nabla \times H_1 = j\omega\epsilon E_1 + j_1, \nabla \times E_2 = -j\omega\mu H_2, \nabla \times H_2 = j\omega\epsilon E_2 + j_2$
- So,  $\nabla \cdot (E_1 \times H_2 - E_2 \times H_1) = j_1 \cdot E_2 - j_2 \cdot E_1$
- $\iiint \nabla \cdot (E_1 \times H_2 - E_2 \times H_1) dV = \iiint (j_1 \cdot E_2 - j_2 \cdot E_1) dV$
- $\iiint \nabla \cdot (E_1 \times H_2 - E_2 \times H_1) dV = \oint (E_1 \times H_2 - E_2 \times H_1) \cdot dA$
- At large distances,  $E \times H$  is  $\perp$  to sphere and  $H = \frac{1}{\eta}(\hat{n} \times E)$ , so  
 $\oint (E_1 \times H_2 - E_2 \times H_1) \cdot dA = 0$
- This gives the reciprocity relation
- $\iiint (j_1 \cdot E_2) dV = \iiint (j_2 \cdot E_1) dV$

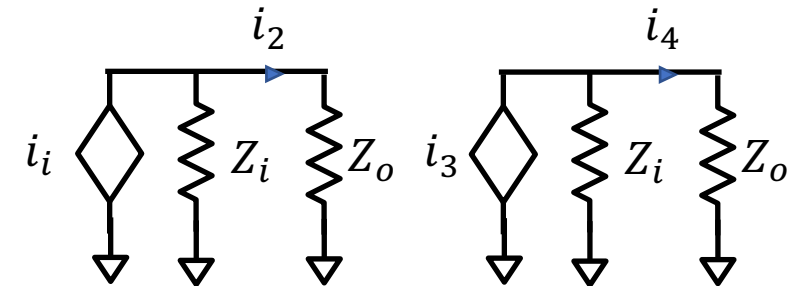
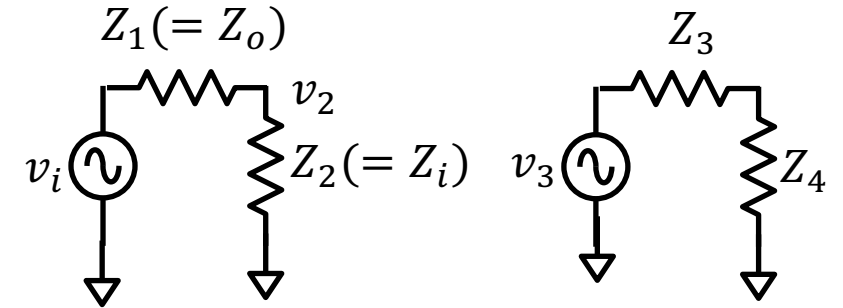


# Rayleigh-Jeans

- $E(x, y, z, t) = E_0 \sin(k_x x + k_y y + k_z z - \omega t)$ ,  $\nabla^2 E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0$ ,  $k = \frac{2\pi}{\lambda}$
- For cube of side length  $L$  with reflecting inner walls, modes are  $\lambda_n = \frac{2L}{n}$ .  $k_x = \frac{2\pi}{2L/n} = \frac{\pi}{L} n$
- $(\frac{n_x \pi}{L})^2 + (\frac{n_y \pi}{L})^2 + (\frac{n_z \pi}{L})^2 = \frac{1}{c^2} (\frac{2\pi c}{\lambda})^2 = \frac{4\pi^2}{\lambda^2}$ , or  $n_x^2 + n_y^2 + n_z^2 = \frac{4L^2}{\lambda^2}$
- $n = \sqrt{n_x^2 + n_y^2 + n_z^2}$ ,  $N = \frac{4\pi}{3} n^3$ .  $N$  is number of allowed nodes.
- $N = \frac{4\pi}{3} (\frac{4L^2}{\lambda^2})^{3/2} = \frac{8\pi L^3}{3\lambda^3}$ , so  $\frac{dN}{d\lambda} = -8\pi \frac{L^3}{\lambda^4}$
- Each mode contributes  $kT$  in equilibrium, so  $8\pi \frac{L^3}{\lambda^4} kT d\lambda = dN$ . Per unit volume  $8\pi \frac{kT}{\lambda^4} d\lambda = d\left(\frac{N}{V}\right)$ .
- Radiated power is  $P = \frac{1}{2} \frac{dN}{d\lambda} \frac{Ax}{t}$  and  $x = ct$ , so  $P = \frac{dN}{d\lambda} \frac{Ac}{2}$ . At  $\theta$ ,  $P = \frac{4\pi kT}{\lambda^4} Ac \cos^2(\theta)$ .
- $\frac{P}{A} = \frac{4\pi kTc}{\lambda^4} \cos^2(\theta)$ . Integrating over  $\theta$ , we get  $I(\lambda, T) = \frac{P}{A} = \frac{2\pi kTc}{\lambda^4}$ , (Rayleigh-Jeans intensity distribution)
- By contrast, Planck's distribution is  $I(\lambda, T) = \frac{2\pi hc^2}{\lambda^5} \left( \exp\left(\frac{hc}{\lambda kT}\right) - 1 \right)^{-1}$

# Coupling

- Voltage gain,  $A$ 
  - $v_o = v_3 = A \frac{Z_o}{Z_i + Z_o} v_i$
  - For two stages gain is  $(A \frac{Z_o}{Z_i + Z_o})^2$
- Current gain,  $A$ 
  - $i_o = i_3 = A \frac{Z_i}{Z_i + Z_o} i_i$
  - For two stages, gain is  $(A \frac{Z_i}{Z_i + Z_o})^2$



# Power transfer

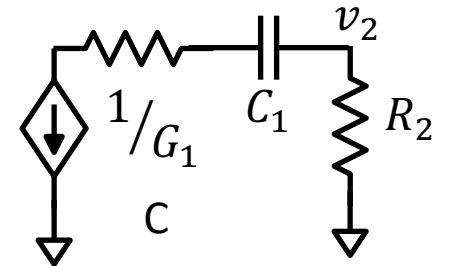
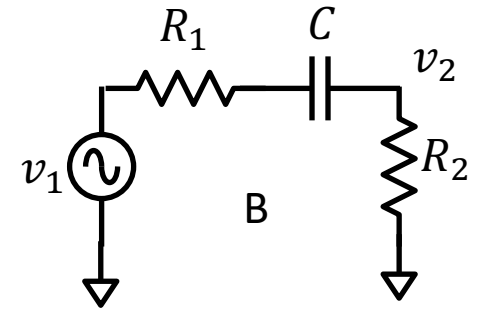
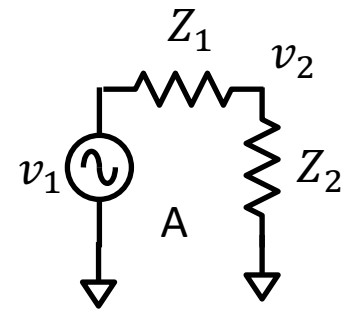
- Power transfer (A)
  - $Z_1 = R_1 + jX_1, Z_2 = R_2 + jX_2$
  - $P = \left| \frac{v_1}{R_1 + R_2 + j(X_1 + X_2)} \right|^2 R_2$

- Power transfer (B)
  - $v_2 = \frac{v_1 \left( \frac{R_1}{R_1 + R_2} \right)}{1 + \frac{1}{j\omega C(R_1 + R_2)}}$
  - $P = i_2 v_2 \cos(\phi)$
  - Max power:  $R_1 = R_2$
  - $v_2 = v_1 \frac{R_2 / (R_1 + R_2)}{1 + 1/[j\omega C(R_1 + R_2)]}$

1.  $\omega$ , large:  $G = \frac{R_2}{R_1 + R_2} = B$
2.  $\omega = \frac{1}{C(R_1 + R_2)}$ , large:  $G = B \frac{1+j}{2} = \sqrt{2}B \angle 45^\circ$
3.  $\omega$ , small: Let  $\omega_1 = C(R_1 + R_2)$ ,  $G = B \frac{\omega}{\omega_1}$

- For C
  - $v_2 = \frac{i_1}{G_1} \frac{R_2}{R_2 + 1/G_1}$

- For voltage amplifiers, we want  $R_2 \gg R_1$
- For current amplifiers, we want  $R_2 \ll R_1$





# High frequency coupling

- Equivalent circuit on right

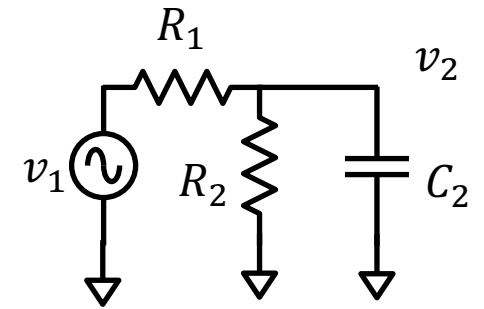
- $\frac{v_1 - v_2}{R_1} = \frac{v_2}{R_2} + jv_2\omega C_2$

- $G = \frac{v_2}{v_1} = \frac{R_2 / (R_1 + R_2)}{1 + j\omega C_2 R_1 R_2 / (R_1 + R_2)}$

1.  $\omega$  low:  $G = \frac{v_2}{v_1} = R_2 / (R_1 + R_2) = B$

2. For  $\omega_2 = \frac{R_1 + R_2}{C_2 R_1 R_2}$ ,  $G = \frac{v_2}{v_1} = \frac{B}{1 + j} = \frac{\sqrt{2}}{2} B \angle -45$

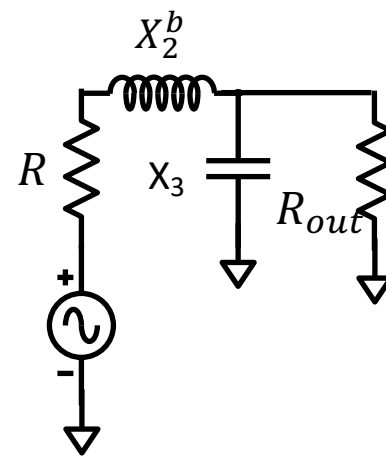
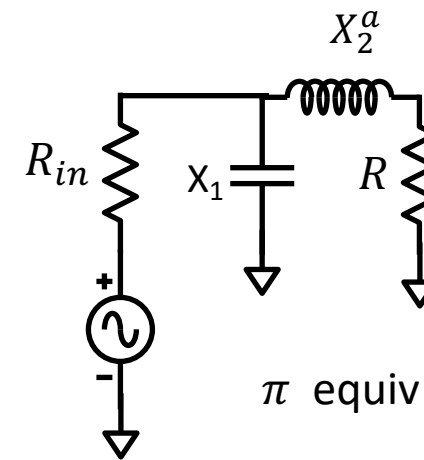
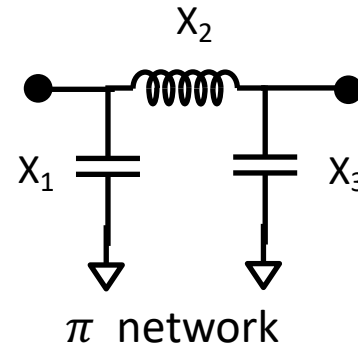
3.  $\omega$  high:  $G = \frac{v_2}{v_1} = \frac{R_2 / (R_1 + R_2)}{j\omega / \omega_2} = B \frac{\omega_2}{\omega} \angle -90$



# Impedance matching

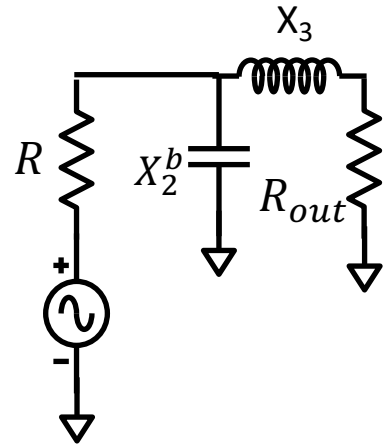
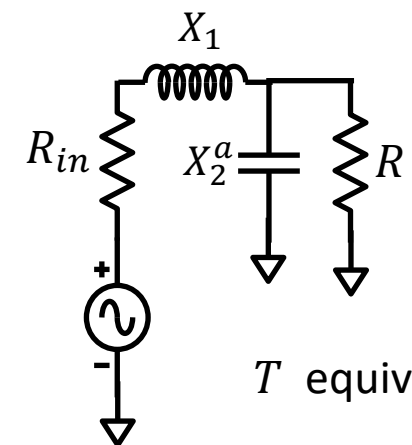
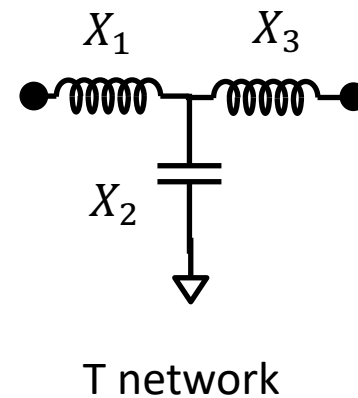
- $\pi$  network

- $Q = \sqrt{\max(R_{in}, R_{out})/R - 1}$ , choose  $R$  for BW
- $X_2^a + X_2^b = X_2$
- $R < \min(R_{in}, R_{out})$



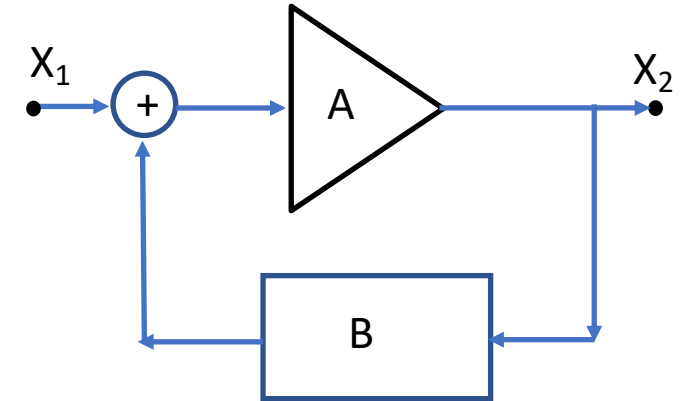
- T networks

- $Q = \sqrt{R/\min(R_{in}, R_{out}) - 1}$ , choose  $R$  for BW
- $X_2^a || X_2^b = X_2$
- $R > \max(R_{in}, R_{out})$



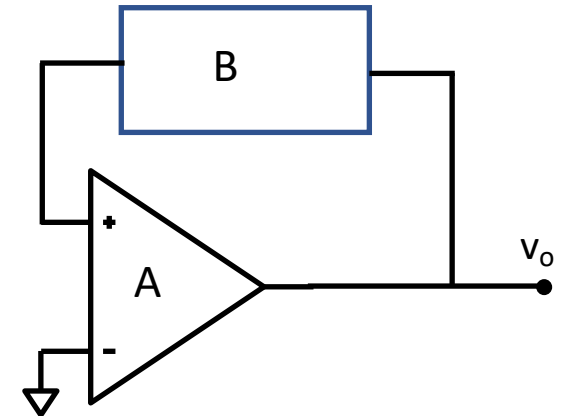
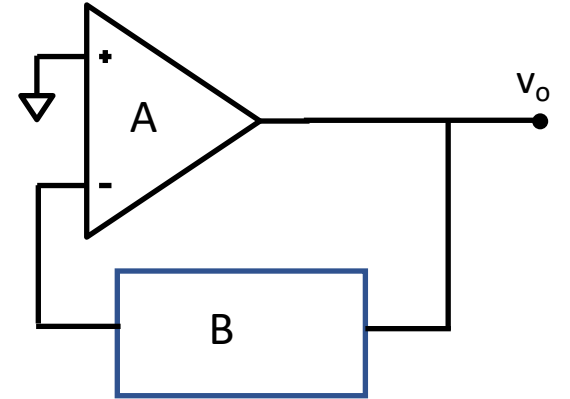
# Negative feedback

- $\frac{X_2}{A} = X_1 - BX_2$ ,  $\frac{X_2}{X_1} = \frac{A}{1+AB} \equiv K$
- No feedback ( $B = 0$ ):
  - $K = A$
- $AB < 0, 1 + AB < 1$ 
  - $K \gg A$
- $AB = -1$ 
  - $K = \infty$
- $AB \gg 0$ 
  - $K \approx \frac{1}{B}$
- $\frac{dK}{dA} = \frac{1}{(1+AB)^2}$  measures stability
- If forward gain is  $\frac{A}{1+j\omega\tau_1}$ ,  $K_{feedback} = \frac{(1+AB)K_{no-feedback}}{1+j\omega\tau_1}$ 
  - This increases frequency response
- Effect on impedance
  - $Z_1 \rightarrow Z_1(1 + AB)$ ,  $Z_2 \rightarrow \frac{Z_2}{1+AB}$



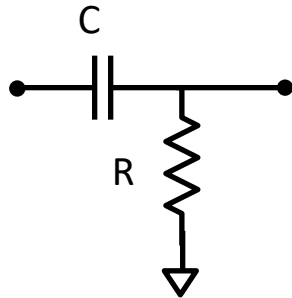
# Oscillators and feedback

- $K = \frac{A}{1+AB}$
- On upper right, oscillation conditions is  $AB = 1 \angle 180$
- For lower right,  $AB = 1 \angle 0$



# Some filter transfer functions

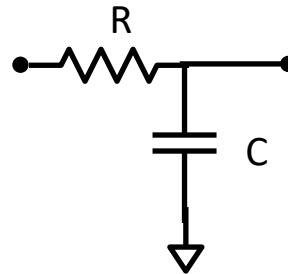
High pass



$$G_{HP} = \frac{1}{1 + \frac{\omega_0}{j\omega}}$$

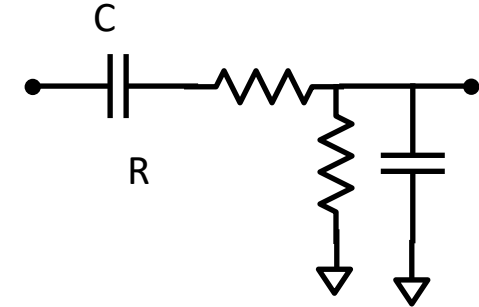
- For CR ladder filter of length 3
- $\frac{v_2}{v_1} = \frac{1}{1 - 5\omega^2 C^2 R^2 + j((6\omega CR - \omega^3 C^3 R^3))}$
- Phase shift is 0 when  $6 = \omega^2 C^2 R^2$

Low pass



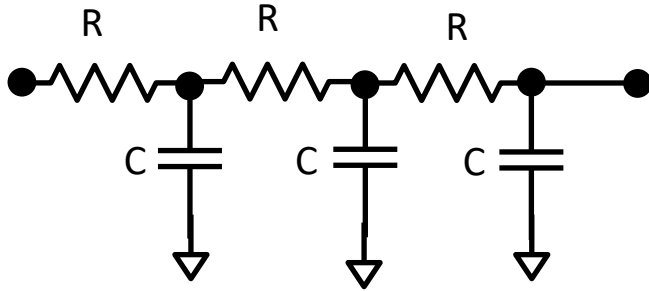
$$G_{LP} = \frac{1}{1 + j \frac{\omega}{\omega_0}}$$

Wein



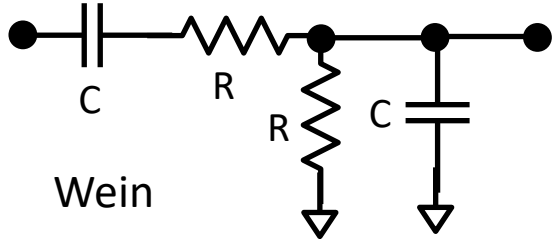
- $\frac{v_2}{v_1} = \frac{1}{3 + j(\omega CR - \frac{1}{\omega CR})}$
- Oscillation when  $\omega CR = 1$
- $\tan(\phi) = \frac{-R_1 R_2 (\omega CR - \frac{1}{\omega CR})}{R_1 + R_2}$
- $\frac{d\phi}{d\omega} = -\frac{2Q}{\omega}$  so high Q improved stability

# Feedback networks for oscillators



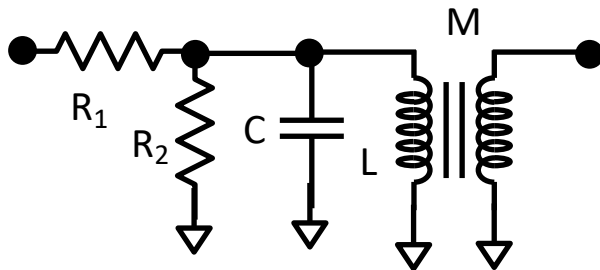
Phase Shift Ladder

- $\frac{v_2}{v_1} = \frac{1}{(1-5\omega^2 C^2 R^2) + j(6\omega CR - \omega^3 C^3 R^3)}$



Wein

- $\frac{v_2}{v_1} = \frac{1}{3}$



Tuned circuit

- $\frac{v_2}{v_1} = \left(\frac{M}{L}\right) \left( \frac{1}{(R_1/R_2 + 1) + j(R_1\omega C - R_1/(\omega L))} \right)$

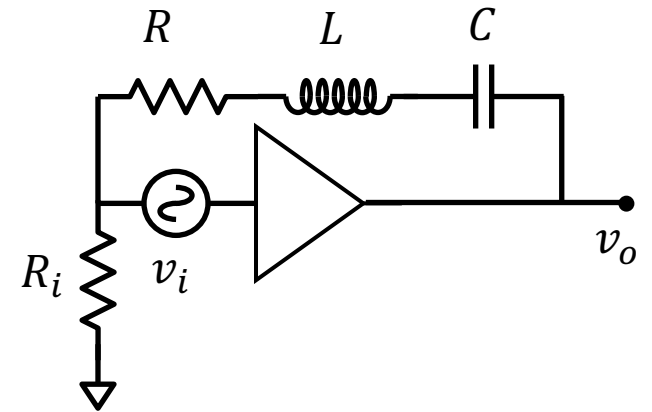
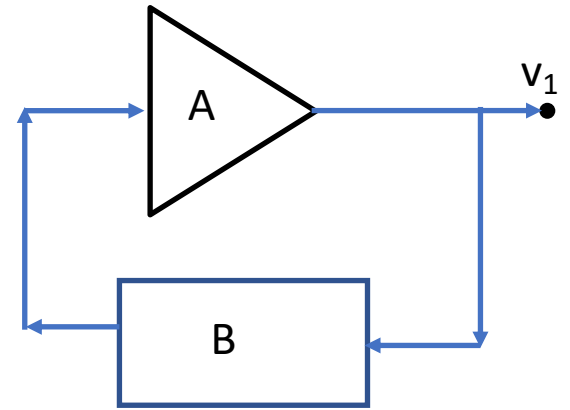
# More on oscillation

- For sine waves,  $|v_1| = 1, AB > 1$
- $v_1 = ABv_1$
- You can use the feedback networks on the previous slide for B

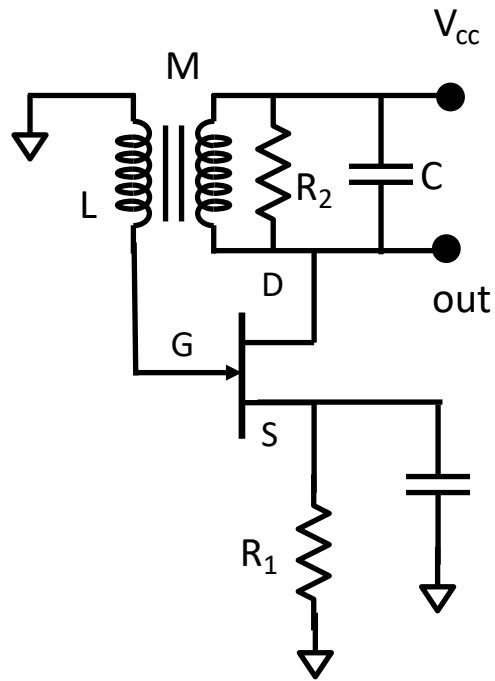
- Want hi Q for stability

- $$v_o = \frac{A}{1 - \frac{R_i A}{R_i + R + j(\omega L - 1/\omega C)}} v_i$$

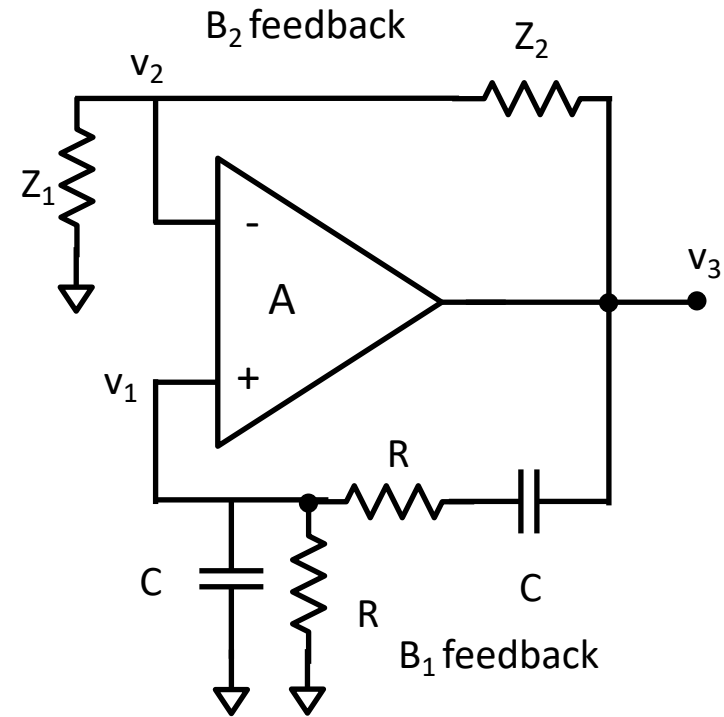
- $$Q = \frac{\omega_0 L}{R_i + R}$$



# Some oscillators



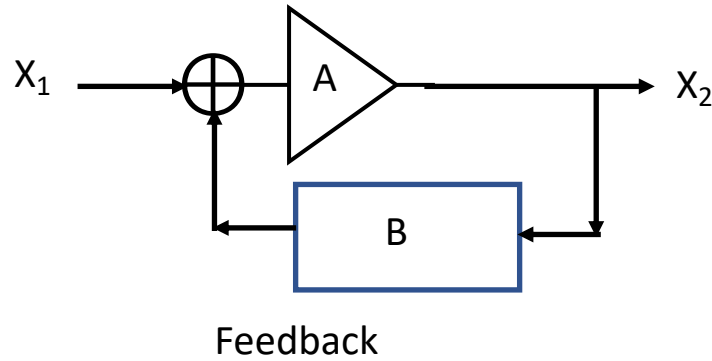
- $B = \frac{M}{L} \left( \frac{R_2}{R_1 + R_2} \right)$
- $\omega = \frac{1}{\sqrt{LC}}$



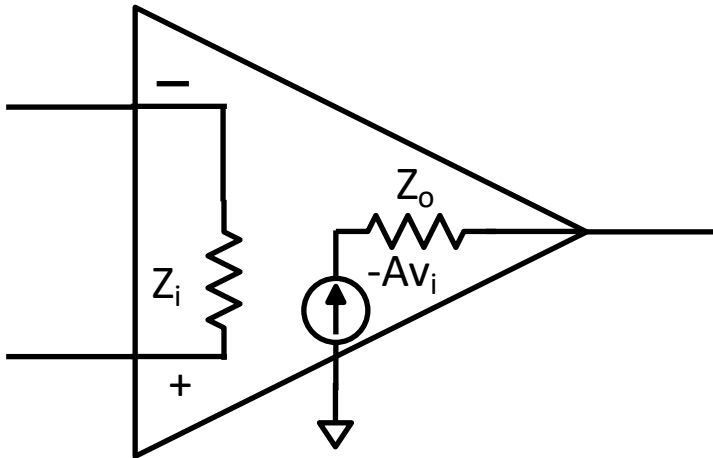
- $v_3 = A(v_1 - v_2)$
- $K = \frac{1}{B_1}$



# Op Amp models



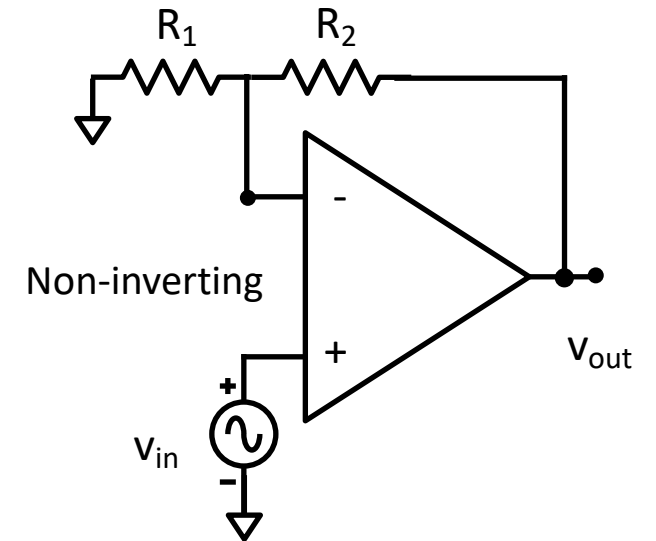
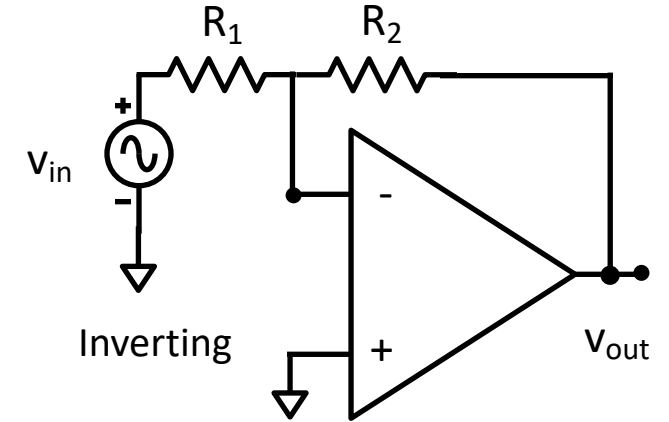
- $A(f) = \frac{A_0}{1+j(f/f_0)}$
- $A(X_1 - BX_2) = X_2$  so  $\frac{X_2}{X_1} = K = \frac{A}{1+AB}$
- $\frac{dK}{dA} = \frac{1}{(1+AB)^2}$
- $AB < 0$ , positive feedback
- $AB > 0$ , negative feedback



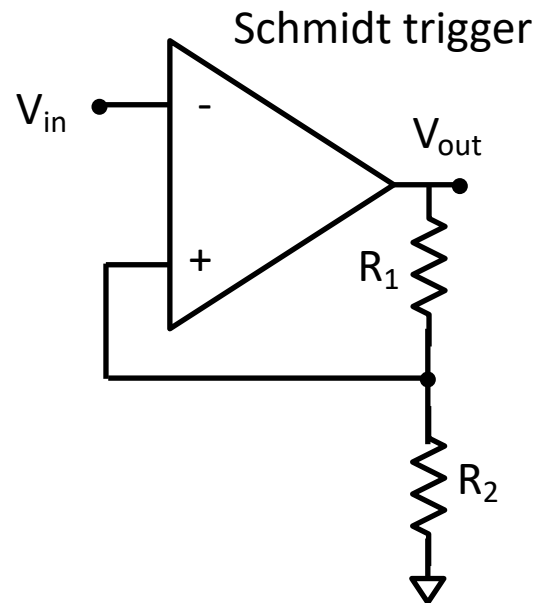
- For 741,  $Z_i \approx 2 \times 10^6 \Omega$ ,  $Z_o \approx 150 \Omega$

# Op Amps

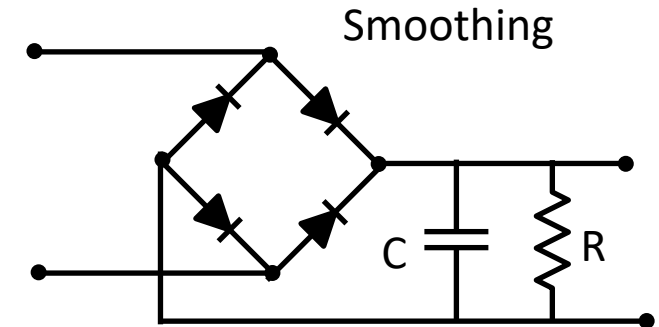
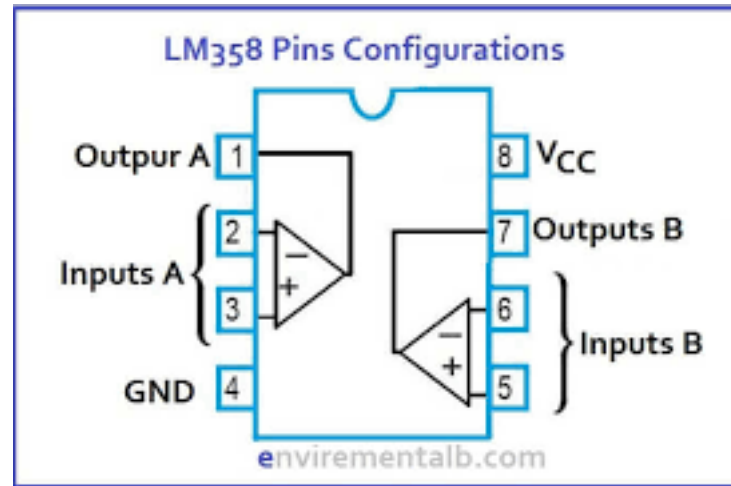
- Ideal op amp
  - $Z_{in} = \infty$
  - $A_V = \frac{V_o}{V_{i,+} - V_{i,-}}$ ,  $A_V$  is an op amp parameter between  $10^4$  and  $10^6$ .
  - $V_- = V_+$  for negative feedback
  - Output voltage increases when  $v_+ > v_-$ , decreases when  $v_+ < v_-$ .
- Example 1: Inverting amp, we'll show the gain is  $\frac{R_2}{R_1}$ 
  - $i_1 = \frac{V_{in}}{R_1}$ ,  $i_2 = \frac{V_{out}}{R_2}$  since the op amp has infinite input impedance
  - By Kirchhoff,  $i_1 = -i_2$ , so  $V_{out} = \frac{R_2}{R_1} V_{in}$
  - $Z_{in} = R_1$
  - $Z_{out}$  is same as non-inverting.
- Example 2: Non-inverting amp
  - $V_- = V_+$ , so  $V_{out} = (1 + \frac{R_2}{R_1}) V_{in}$
  - $Z_{in} > 10^6 \Omega$
  - $Z_{out} = \frac{R_{o,Th}}{1 + A_V \beta}$ ,  $\beta = \frac{R_2}{R_1 + R_2}$ ,  $R_{o,Th}$  is the Thevenin resistance of the op amp



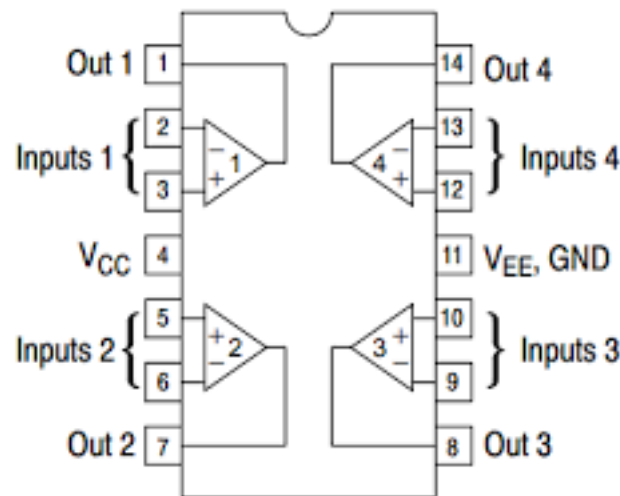
# Miscellaneous op amps



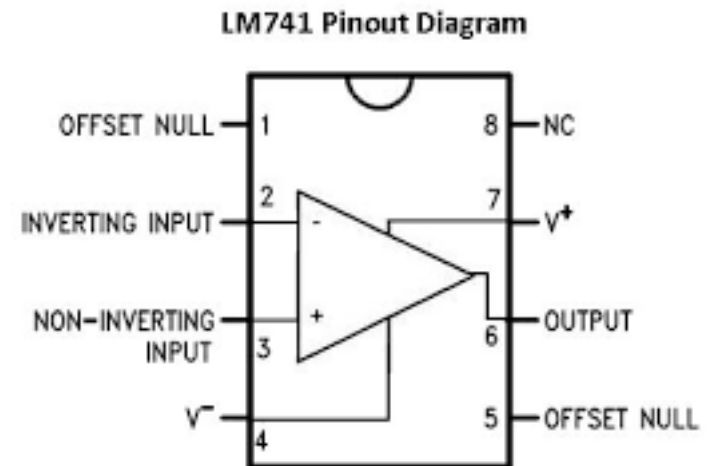
- $R_1 > R_2$
- $(V_{Th,+} - V_{Th,-}) = \frac{R_2}{R_1 + R_2} (V_{sat,+} - V_{sat,-})$
- Simple op amp model
- $\frac{1}{g(f)} = \frac{1}{g_{DC}} + j \frac{f}{f_{BW}}$



- $RC = \frac{10}{f}$

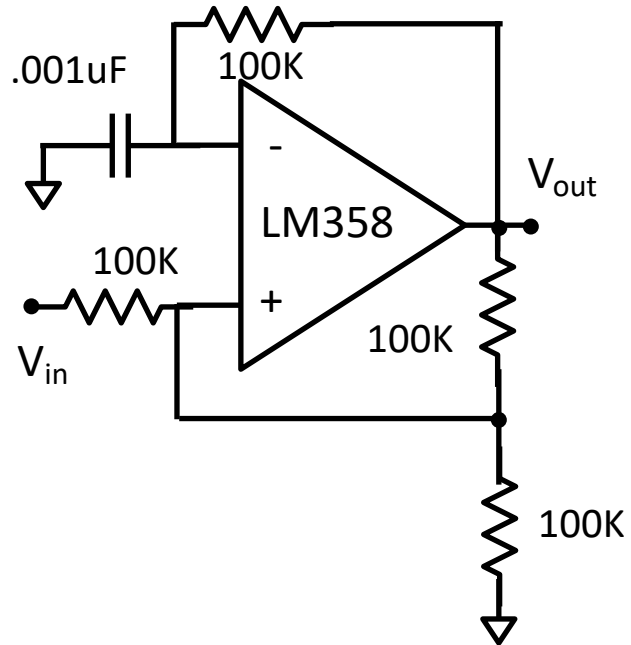


LM324

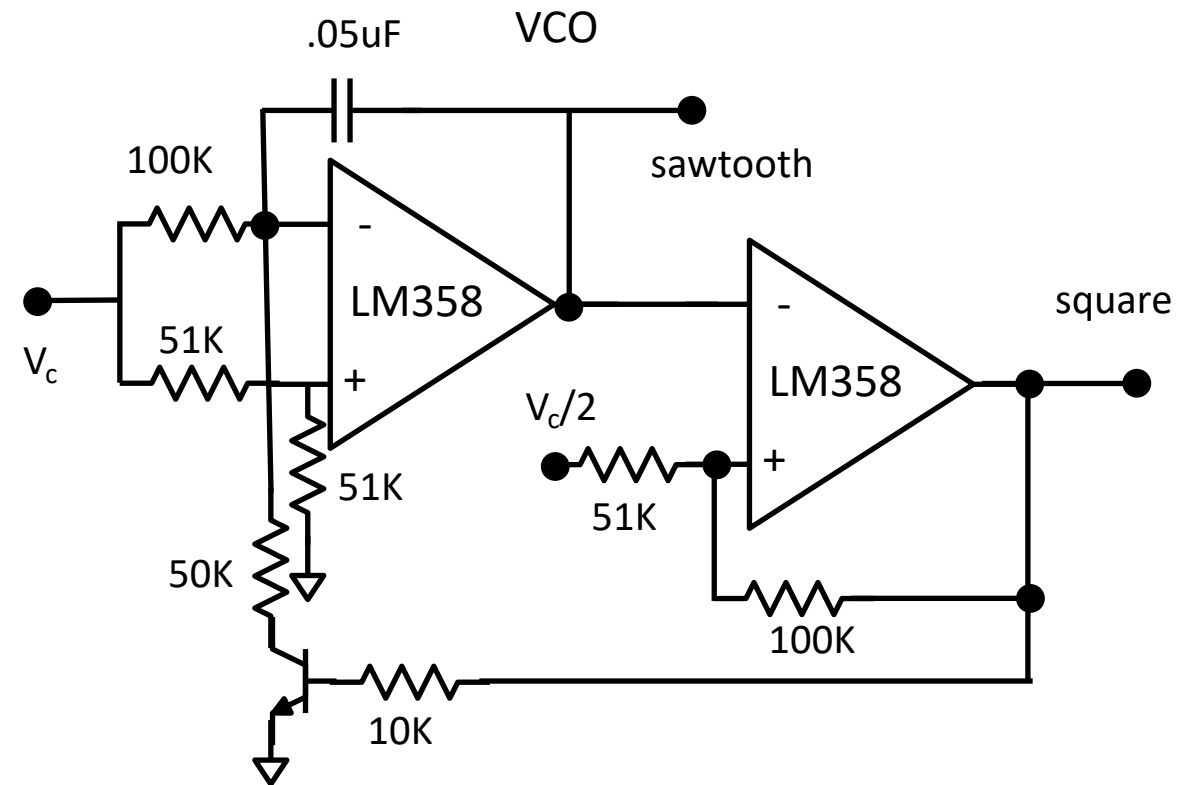
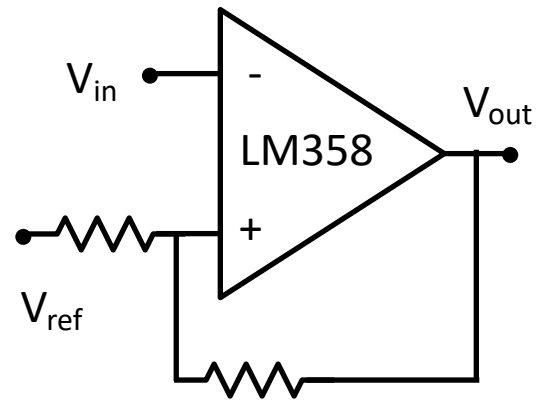


# 358 op amp

Square wave

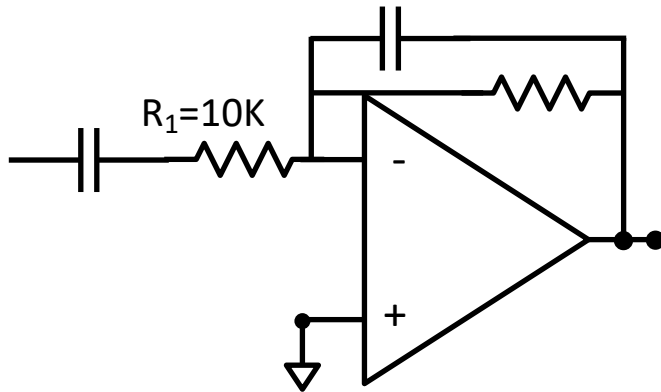


Peak

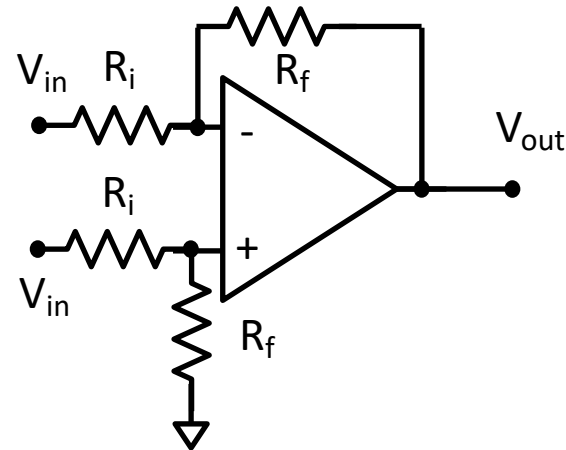
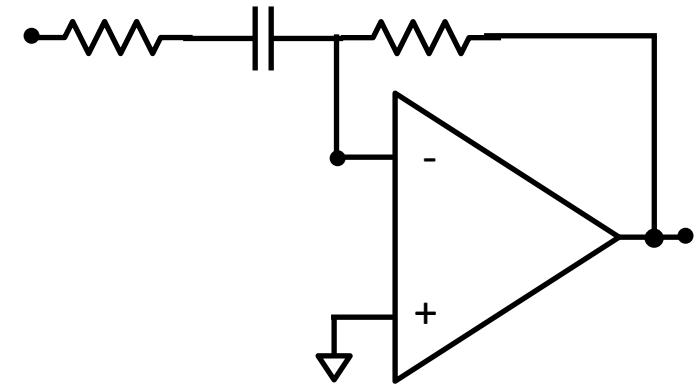


# Op integrators and differentiators and differential amp

Integrator

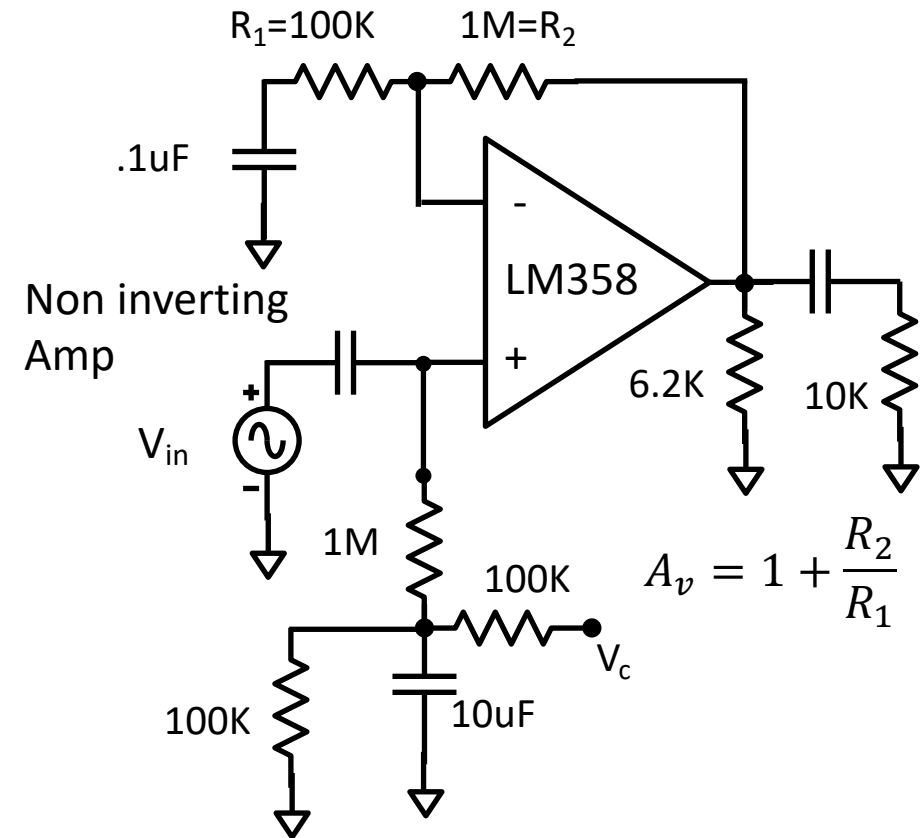
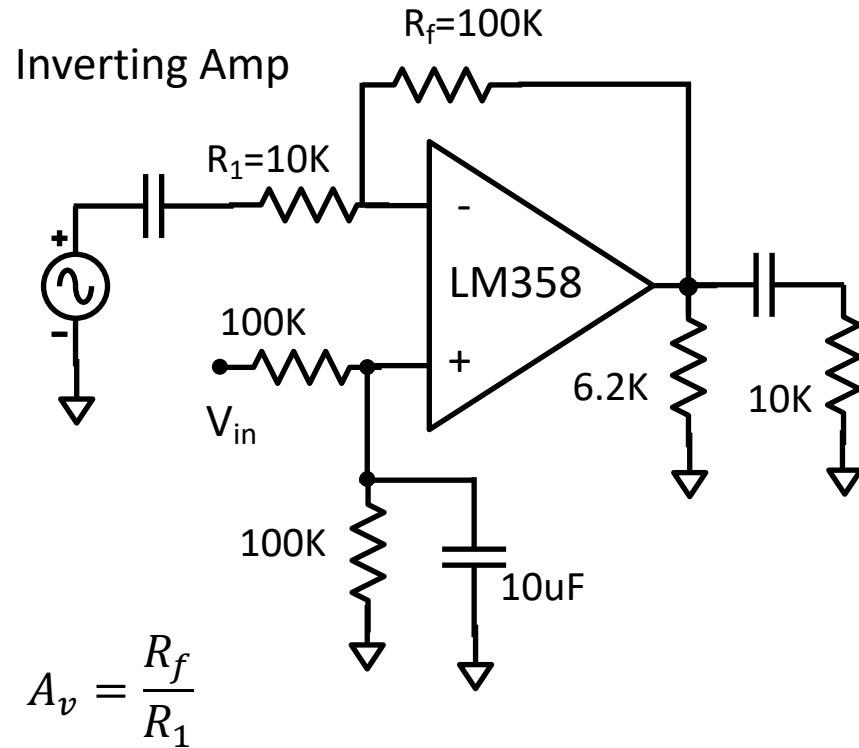


Differentiator

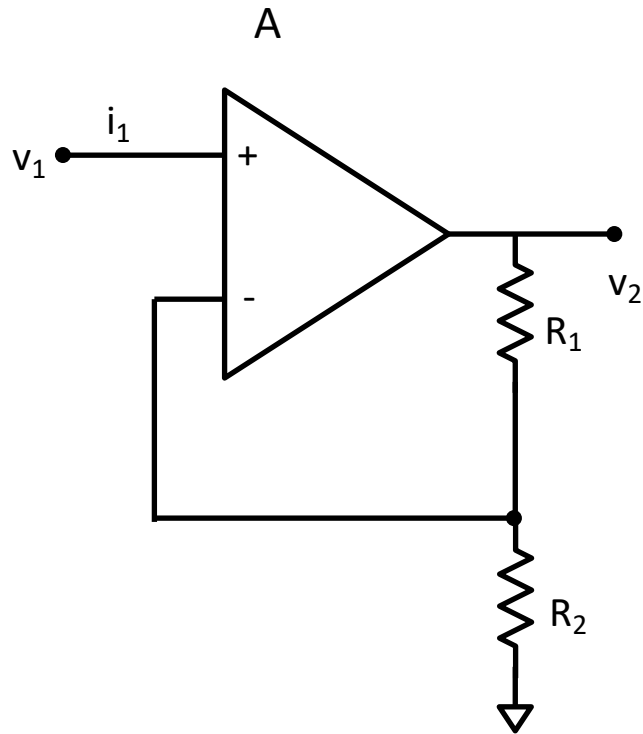


Differential amp

# Real 358 based ops



# Op amp feedback

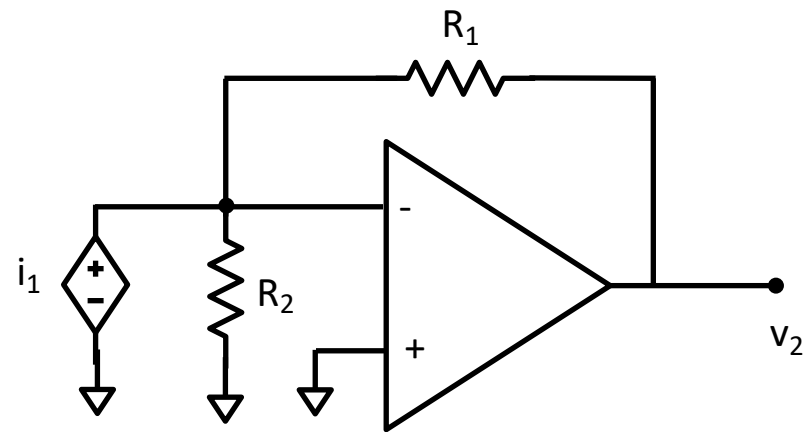


Nyquist

- Polar plot of AB

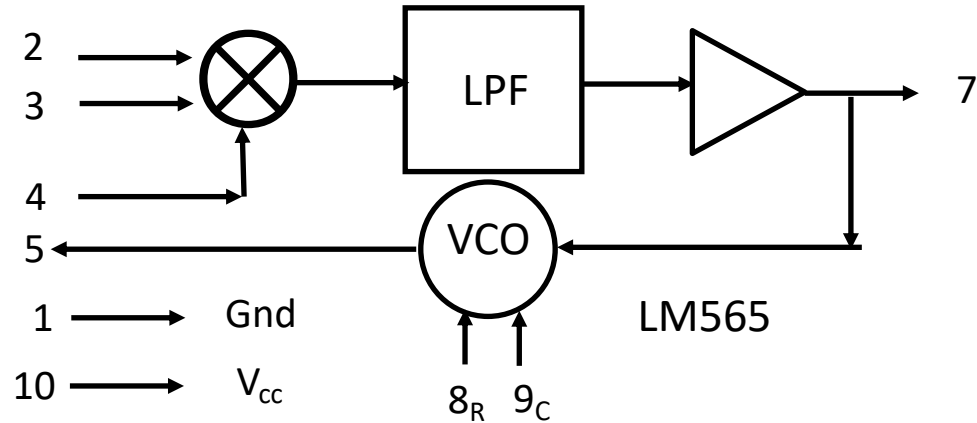
- $v_- = v_2 \frac{R_2}{R_1 + R_2}$
- $v_2 = A \left( v_1 - v_2 \frac{R_2}{R_1 + R_2} \right)$
- $\frac{v_2}{v_1} = \frac{A}{1 + A \frac{R_2}{R_1 + R_2}} = \frac{A}{1 + AB}$
- $K = \frac{A}{1 + AB}$
- $\frac{dK}{dA} = \frac{1}{(1 + AB)^2}$

# Current to voltage

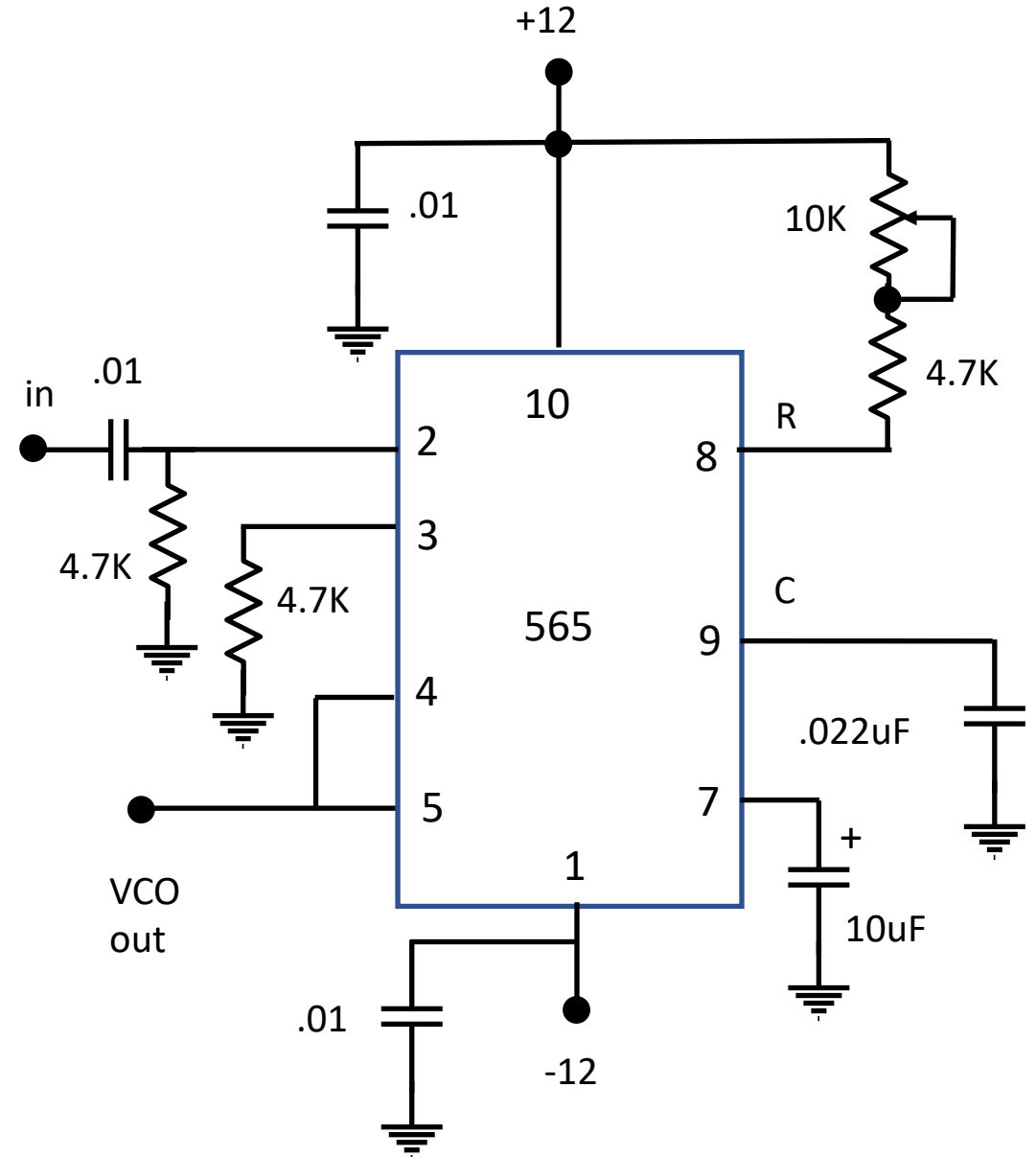




# PLL

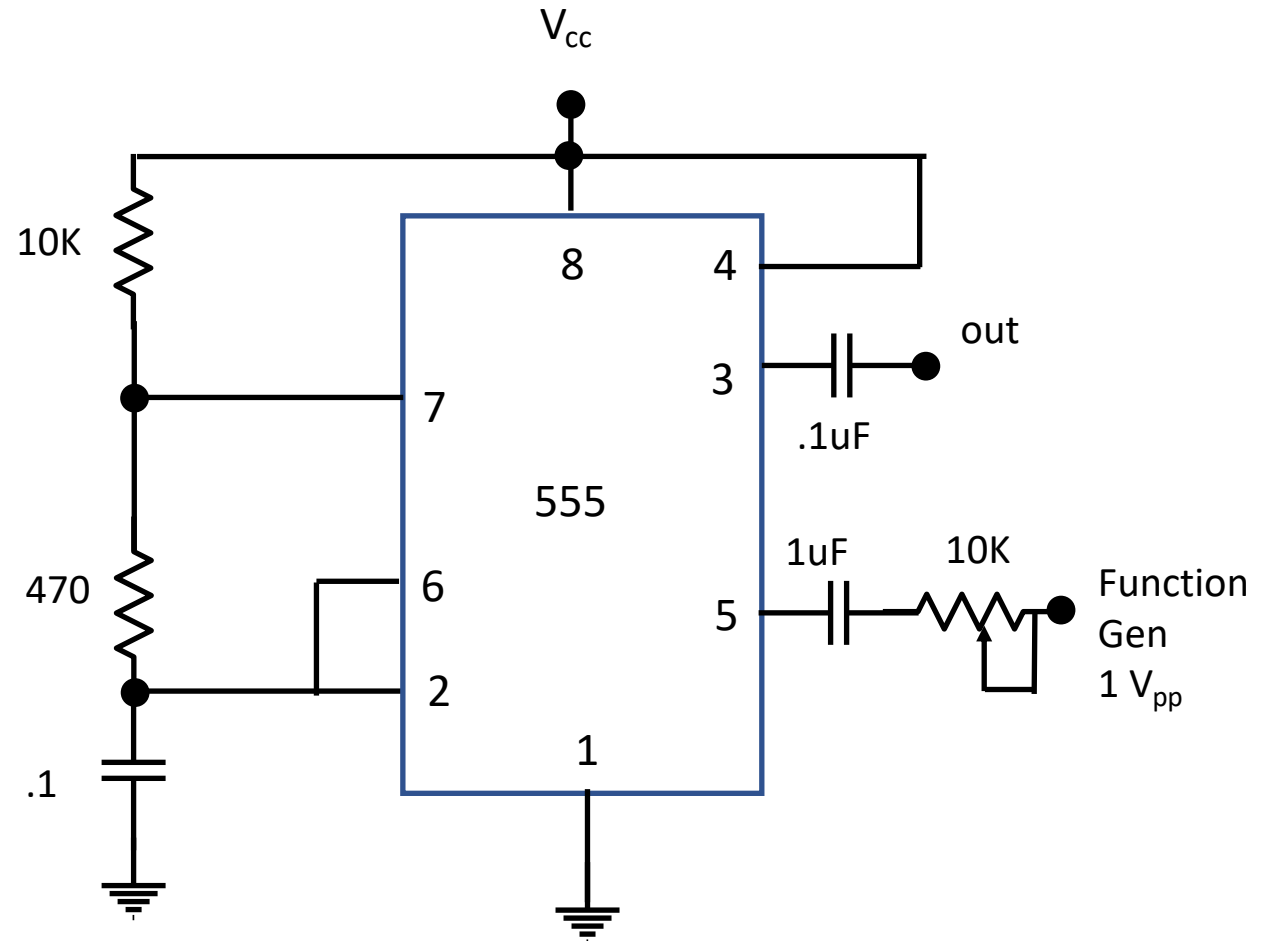


- For general PLL, if  $v_{ref}(t) = V_R \cos(\omega_0 t + \phi_R(t))$  and  $v_{VCO}(t) = V_V \cos(\omega_0 t + \phi_V(t))$ ,  $v_D = k_D(v_{ref}(t) - v_{VCO}(t))$  where  $v_D$  is the output of the phase detector.
- $f_0 = .3(RC)^{-1}$ 
  - With no input, VCO out is about 1360Hz
  - For input, set function generator to  $1V_{pp}$



# PLL FM detector

1. With no input, out is about 1360Hz
2. For input, set function generator to  $1V_{pp}$  and connect to previous 565



# Miscellaneous

- $C = Blg(1 + SNR)$
- $P_n = 4kTB$
- Diode:  $i_D = i_S[\exp(\frac{V_{diode}}{nV_T}) - 1]$ ,  $V_T = \frac{kT}{q}$

- Circuits on right

- $Y = G + jB$

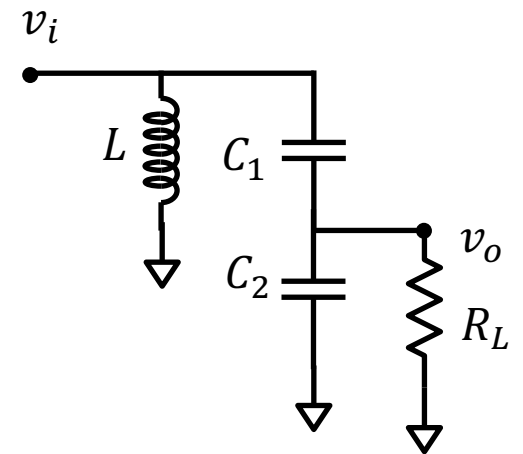
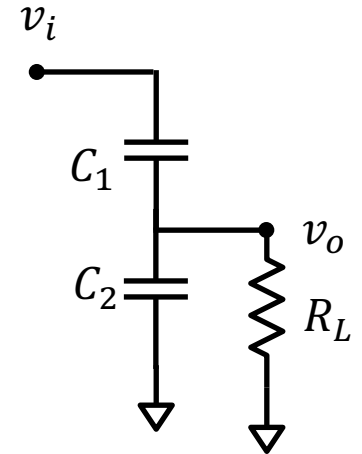
- Capacitive Divider

- $G_{in} = \frac{1}{R_L} (\frac{C_1}{C_1 + C_2})^2$ ,  $B_{in} = \frac{\omega C_1 C_2}{C_1 + C_2}$

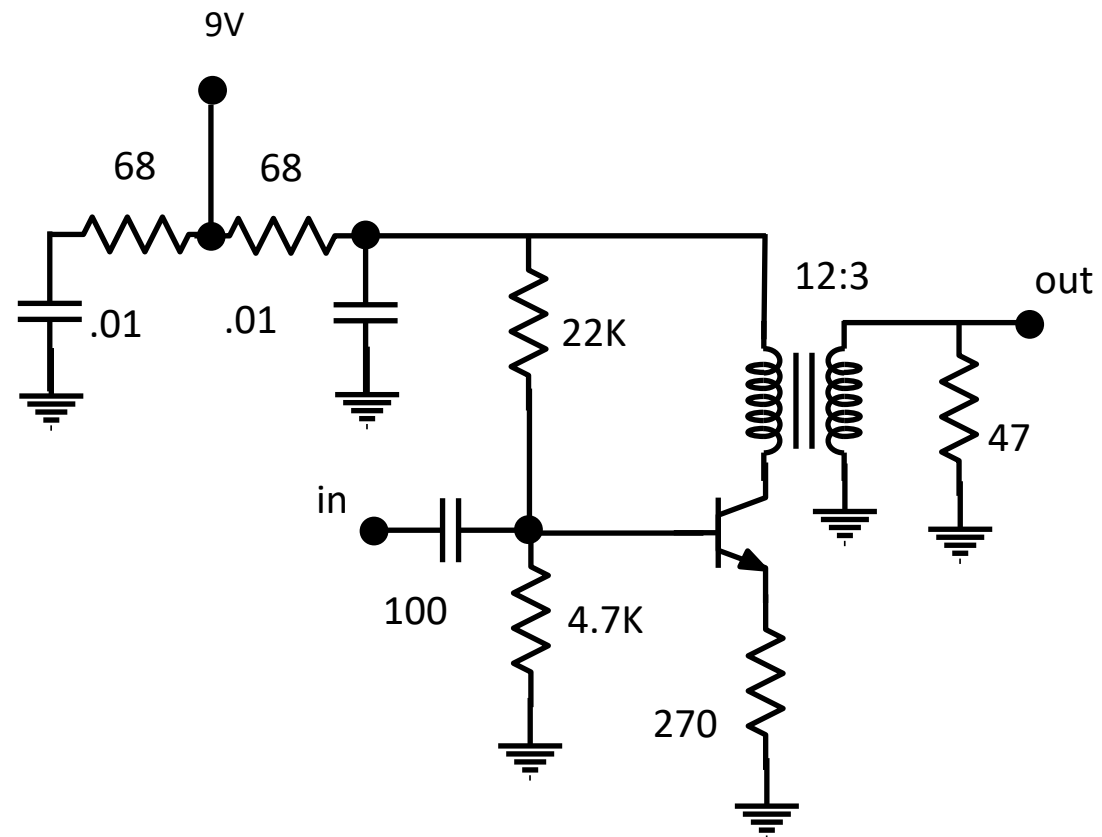
- Resonant capacitive divider

- $\omega_0^2 = \frac{C_1 + C_2}{LC_1 C_2}$

- $Q = \frac{R_{in} || R_s}{\omega_0 L}$

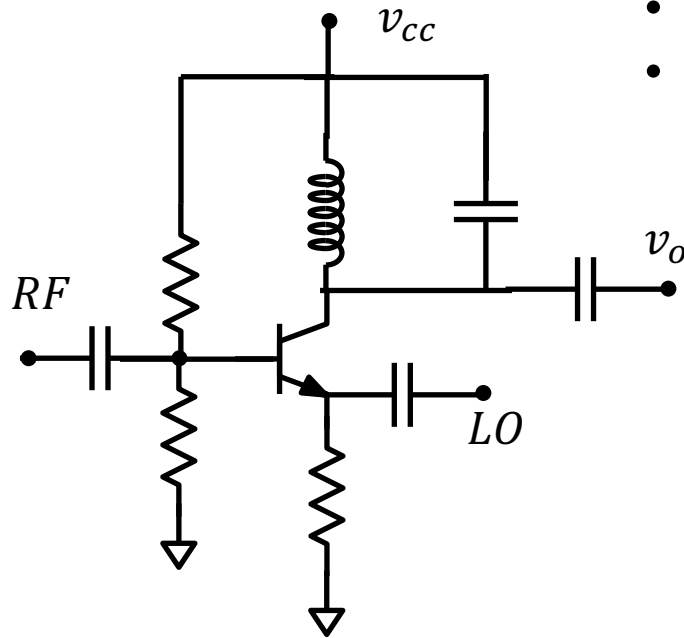


# RF Amp



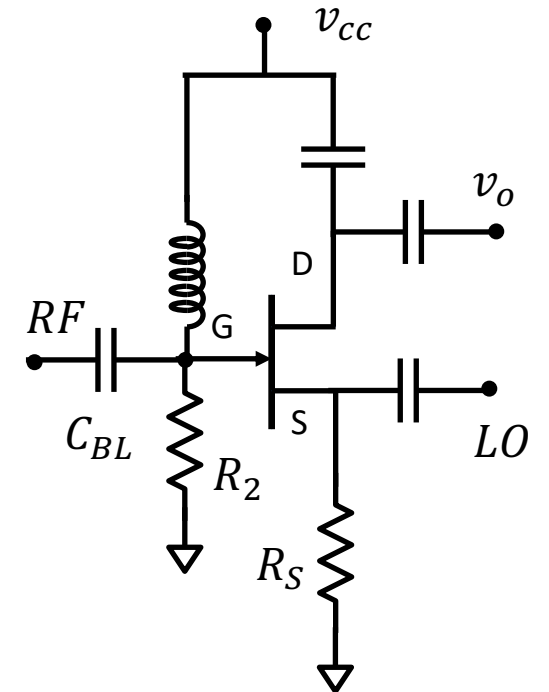
# BJT and FET mixers

BJT mixer

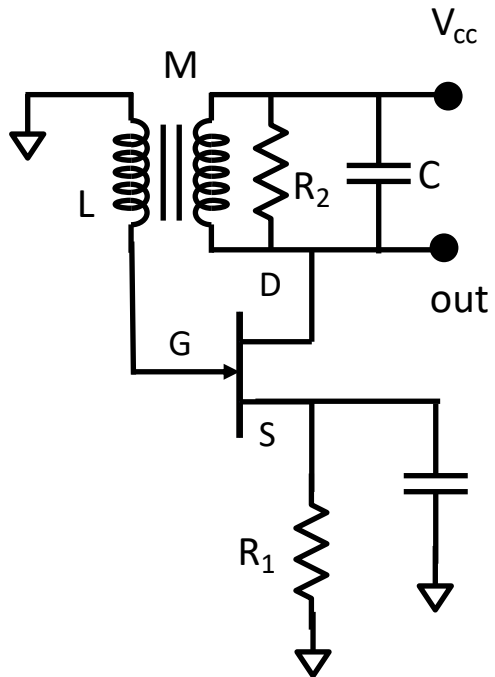


- AM:  $v(t) = [1 + ma(t)]\cos(\omega_c t)$
- Phase:  $v(t) = A\cos(\omega_c t + \phi)$ ,  $\phi = ma(t)$
- FM:  $v(t) = A\cos(\omega_c t + \phi)$ ,  $\phi = \int ma(t) dt$

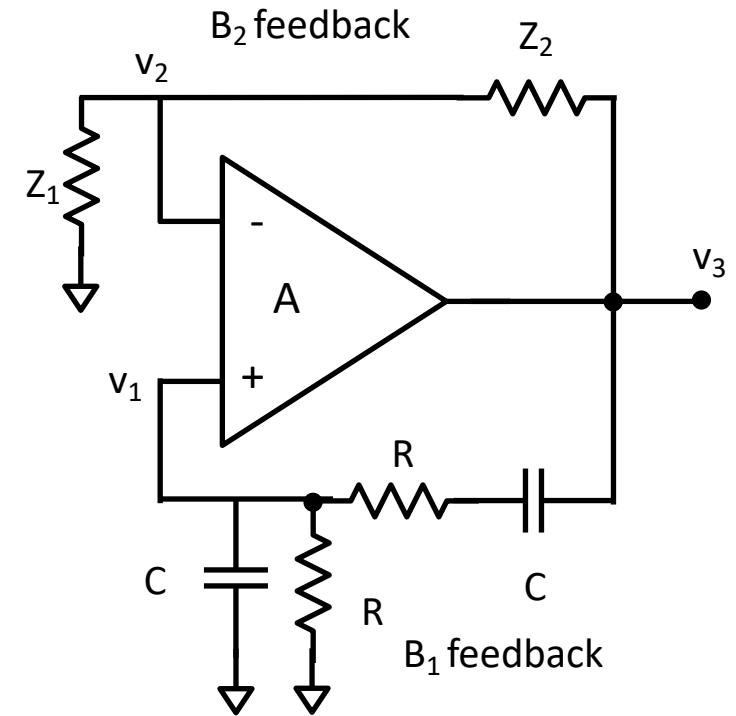
FET mixer



# Some real oscillators



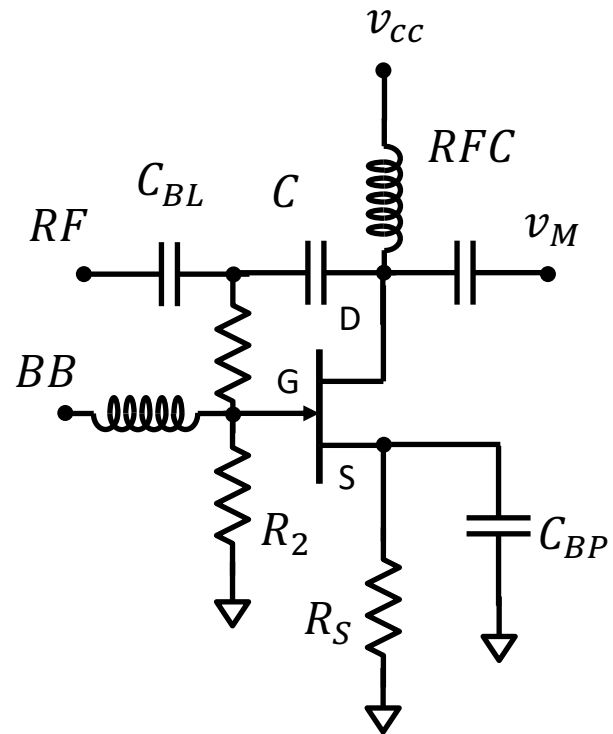
- $B = \frac{M}{L} \left( \frac{R_2}{R_1 + R_2} \right)$
- $\omega = \frac{1}{\sqrt{LC}}$



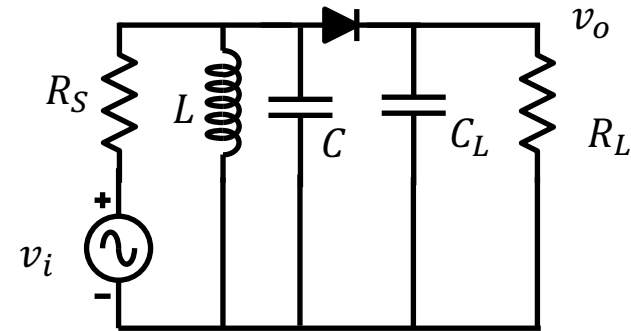
- $v_3 = A(v_1 - v_2)$
- $K = \frac{1}{B_1}$

# Modulators and detectors

Phase modulator

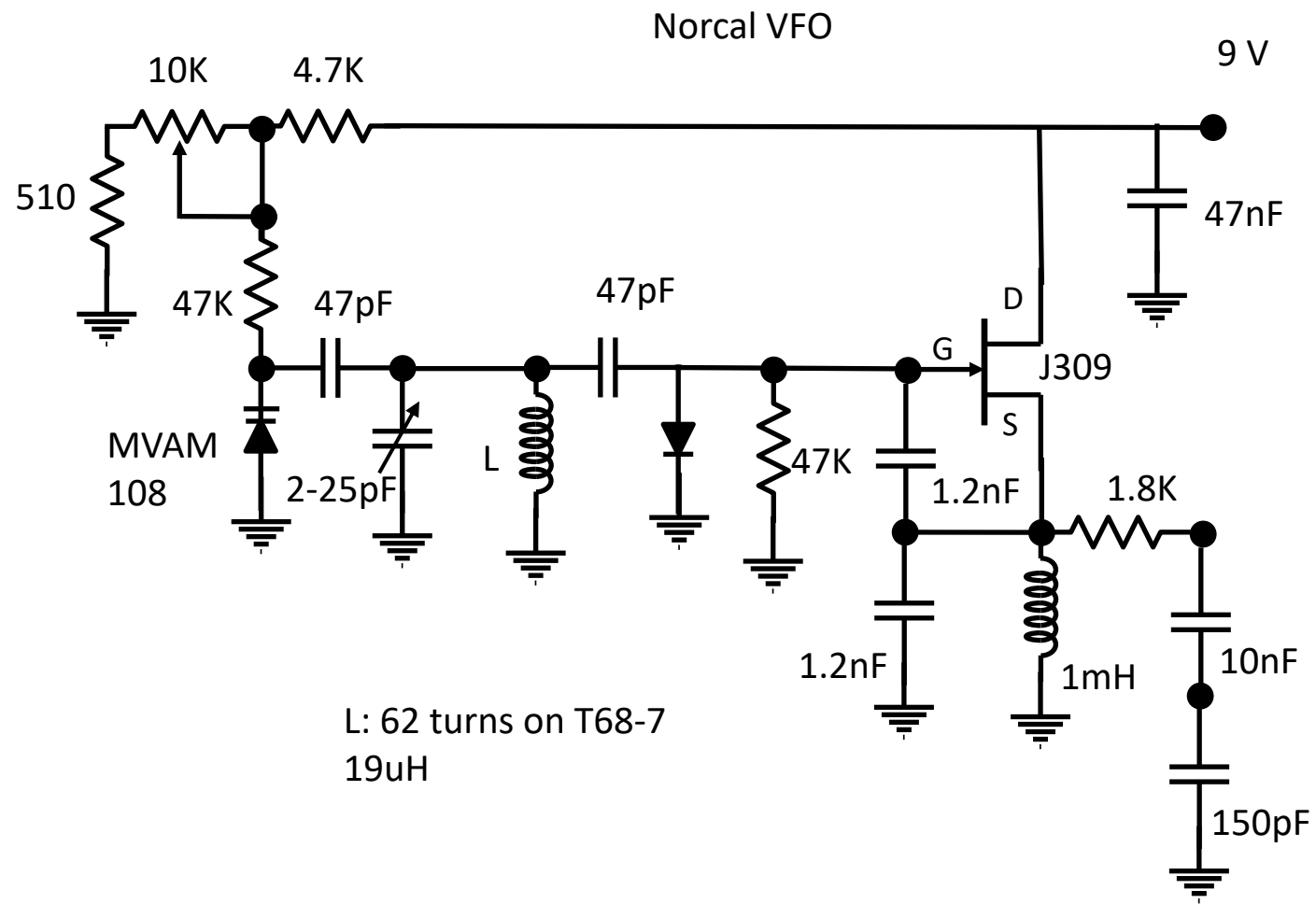
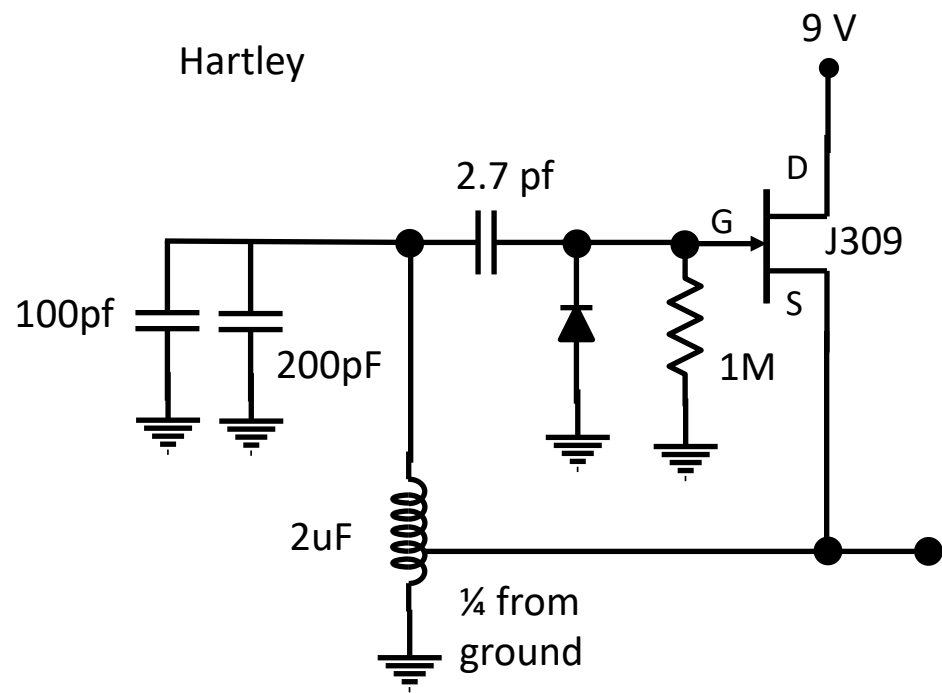


Simple FM detector



- $$v_o = \frac{R}{(R+R_S)\left[1+\frac{4Q^2}{\omega_0^2}(\omega-\omega_0)^2\right]} v_i$$
- $$Q = \frac{R||R_S}{\omega_0 L}$$

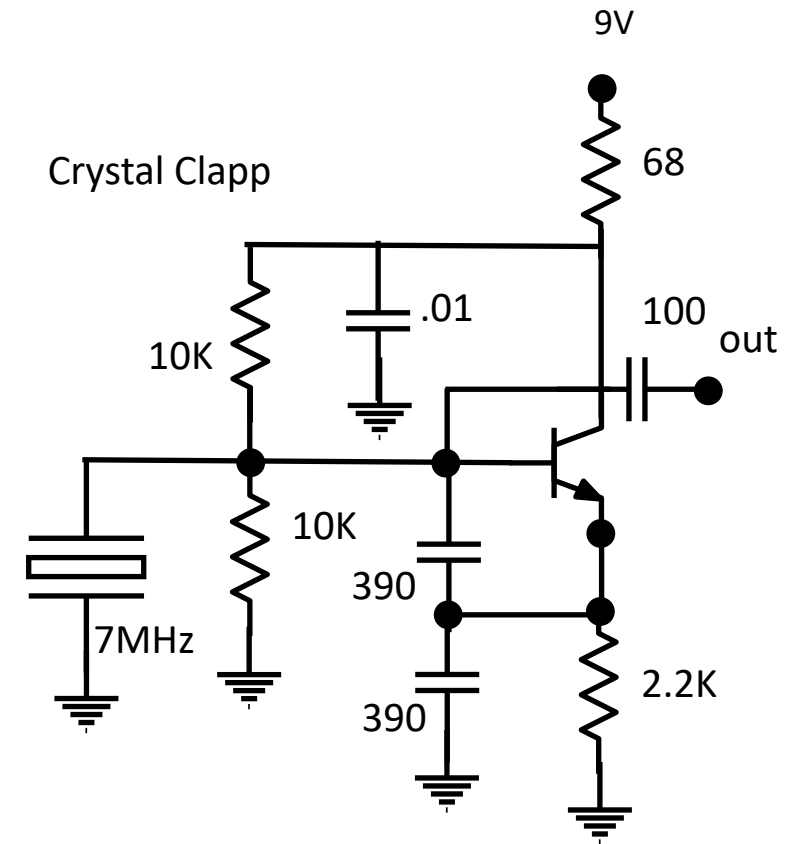
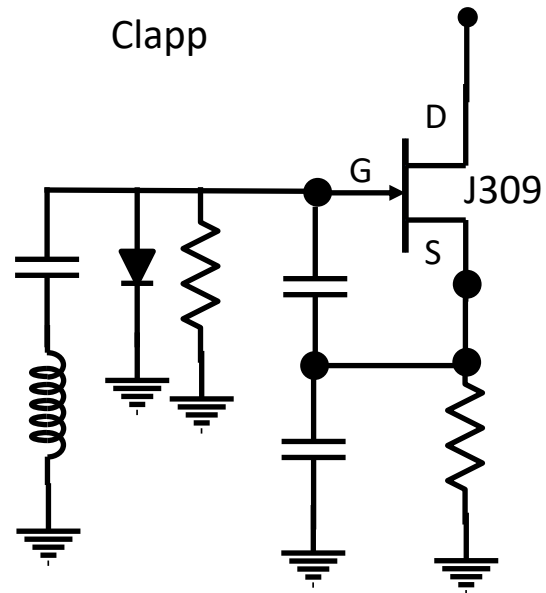
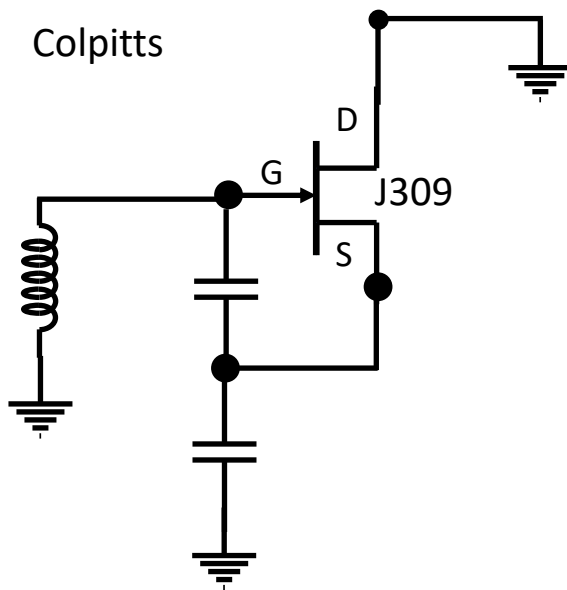
# More Oscillators



L: 62 turns on T68-7  
19uH



# More Oscillators



# GPS

- Space Segment
  - 31 operating satellites (Current generation: block IIIA – 2018)
  - 12 hour MEO (21000km) orbits, 8-12 satellites in view
  - Transmitter power: 44.8\* W,  $G_t = 12\text{dBi}$ . Clocks accurate to better than 10ns (no leap seconds).
- Control segment master at Schriever AFB (plus alternative master, 4 ground antennas and 6 ground stations). Ephemeris updates daily.
- User segment
  - Two bands: L1 (1575.42 MHz [2.046MHz BW]) civilian and military, L2 (1227.6MHz)], military only
  - Two signals: C/A (civilian) and P(Y) (military). C/A signal is Direct Sequence Spread Spectrum
  - Satellite identified by PRN (1-32).
  - L1 signal:  $s(t) = A_c C(t)D(t) \cos(\omega_c t + \theta_1) + A_p P(t)D(t) \sin(\omega_c t + \theta_2)$ 
    - $C(t)$  - coarse ranging code
    - $D(t)$  - navigation signal consists of (clock, ephemeris, almanac corrections) 37,500 bits. 50 bps transmission.
    - $P(t)$  - precision ranging (Y is encrypted)

\* This is total transmitter power, 22 W devoted to ranging code. Newer satellites have higher power and can surge power

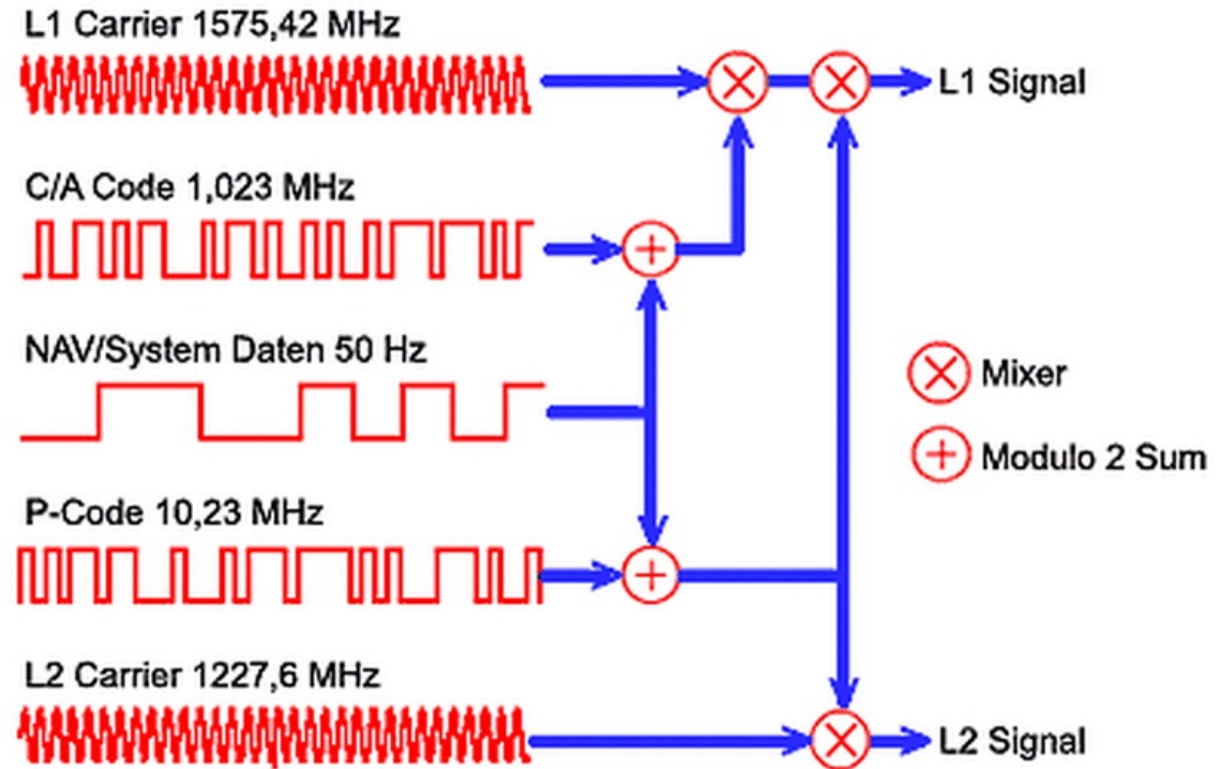
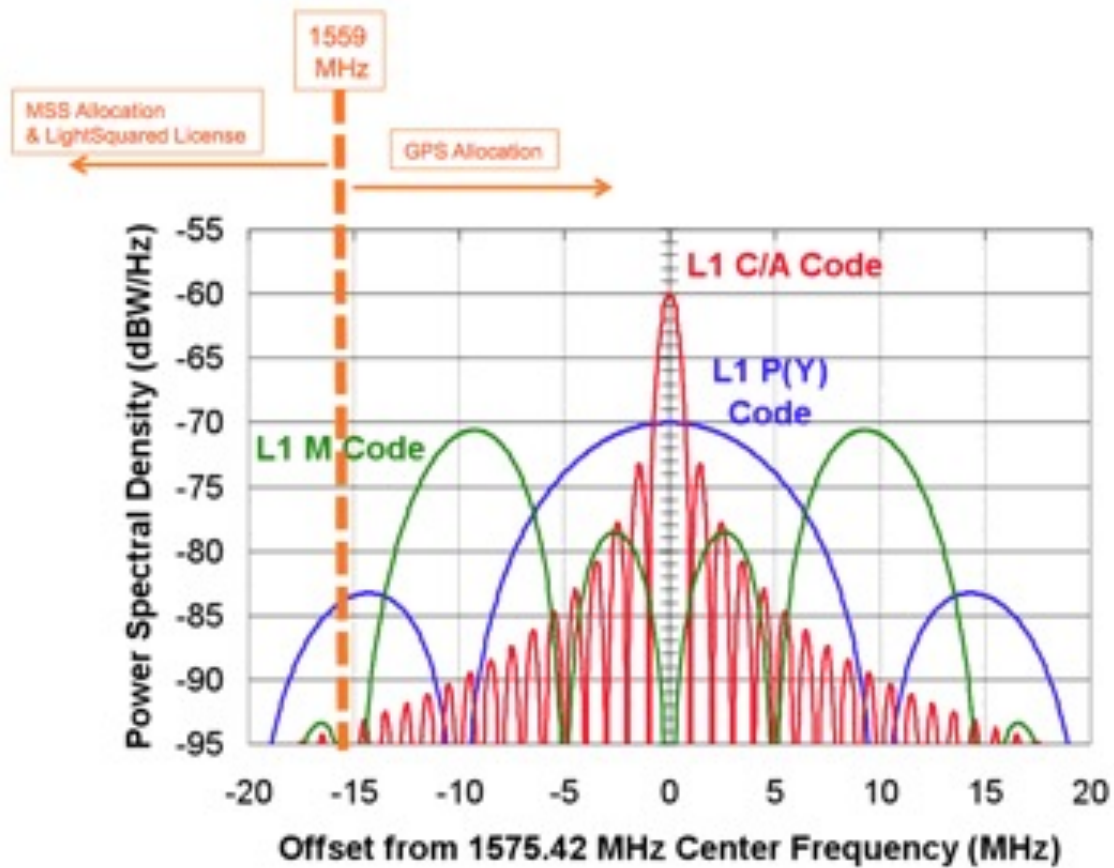
# Some data and rough calculations

- $N = kTB, k = 1.38 \times 10^{-23} \text{ J/K}$ . BW:  $2.046 \text{ MHz}$
- For comparison: laptop emission noise density is capped at  $10^{-7} \frac{\text{W}}{\text{MHz}}$
- Noise:  $1.38 \times 10^{-23} \frac{\text{J}}{\text{K}} \times 100 \times 2 \times 10^6 = 2.76 \times 10^{-15} \text{ W}, (-145.6 \text{ dB})$
- Free space loss (@100K):  $P_r = \frac{P_t G A}{4\pi r^2} \cdot \int A d\Omega = \lambda^2 \cdot P_r = \frac{(44.7)10^{1.24}}{4\pi(21 \times 10^6)^2} \times \frac{3 \times 10^8}{1.575 \times 10^9} = 4 \times 10^{-15} \text{ W} (-144 \text{ dB})$
- Final SNR:  $-144 + 145.6 + 43 = 44.6 \text{ dB}$ .
  - Lower in practice since satellite not directly overhead, and there are shadowing and multipath losses. In addition, we have receiver loss (and receiver antenna gain) so  $P_r$  could be as small as  $-170 \text{ dB}$  and final SNR is approximately  $-170 + 144 + 43 = 17 \text{ dB}$
- Receiver modeled at  $G_r = 2 \text{ dBi}$

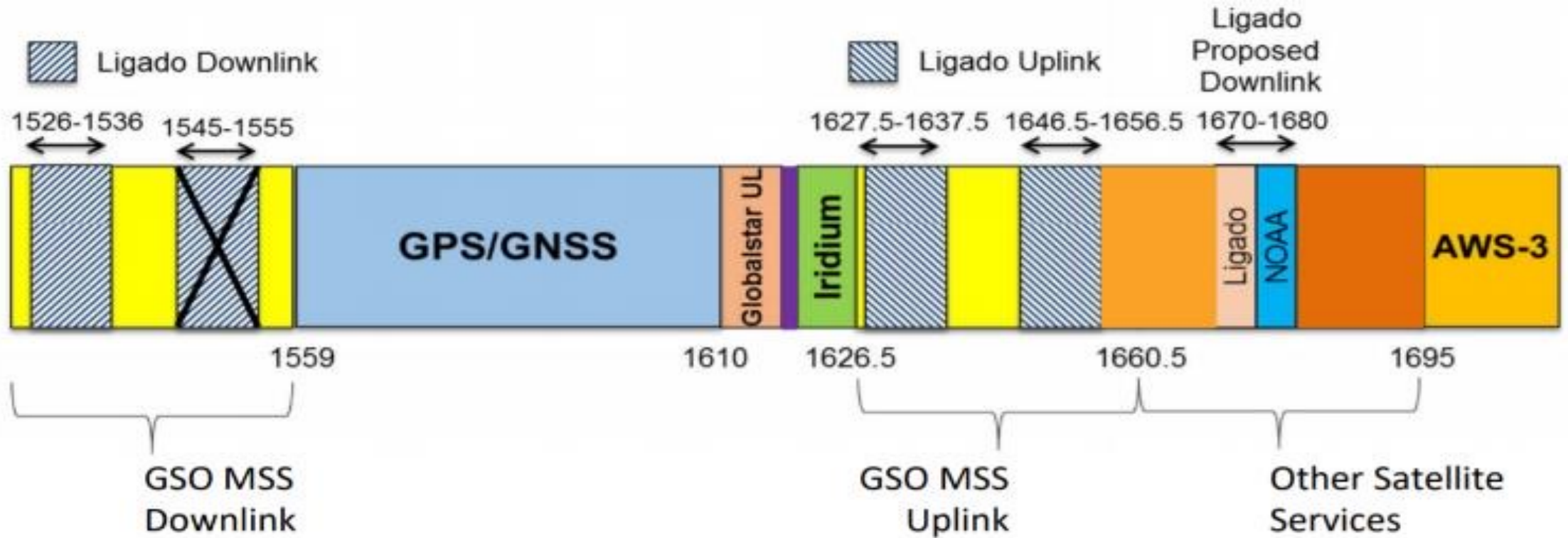
# More refined calculations

- Sat power:  $13.4\text{ dBW}$  ( $21.7\text{ W}$ )
- Sat Antenna gain:  $13.4\text{ dBW}$
- EIRP:  $26.8\text{ dBW}$
- Polarization loss:  $-3.4\text{ dBW}$
- Space loss:  $-184.4\text{ dBW}$
- Atmo loss:  $-2\text{ dBW}$
- Receiver Antenna gain:  $3\text{ dBW}$
- Total signal power at antenna base:  $-160\text{ dBW}$
- Correlated signal strength is  $-147\text{ dBW}$
- Thermal noise:  $-174\text{ dBW}$
- Noise Figure:  $F = \frac{SNR_{ideal}}{SNR_{out}}$ . Good LNA at GPS frequency has noise figure of  $2.3\text{ dBW}$
- Compression point: when amp becomes non-linear.

# Signal Structure



# GPS spectrum neighborhood



# Signal Summary

<b>GNSS System</b>	GPS	GPS		GPS	GPS
<b>Service Name</b>	C/A	L1C		P(Y) Code	M-Code
<b>Centre Frequency</b>	1575.42 MHz	1575.42 MHz		1575.42 MHz	1575.42 MHz
<b>Frequency Band</b>	L1	L1		L1	L1
<b>Access Technique</b>	CDMA	CDMA		CDMA	CDMA
<b>Signal Component</b>	Data	Data	Pilot	Data	N.A.
<b>Modulation</b>	BPSK(1)	TMBOC(6,1,1/11)		BPSK(10)	BOC <sub>sin</sub> (10,5)
<b>Sub-carrier frequency [MHz]</b>	-	1.023	1.023 & 6.138	-	10.23
<b>Code frequency</b>	1.023 MHz	1.023 MHz		10.23 MHz	5.115 MHz
<b>Primary PRN Code length</b>	1023	10230		$6.19 \cdot 10^{12}$	N.A.
<b>Code Family</b>	Gold Codes	Weil Codes		Combination and short-cycling of M-sequences	N.A.
<b>Secondary PRN Code length</b>	-	-	1800	-	N.A.
<b>Data rate</b>	50 bps / 50 sps	50 bps / 100 sps	-	50 bps / 50 sps	N.A.
<b>Minimum Received Power [dBW]</b>	-158.5	-157		-161.5	N.A.
<b>Elevation</b>	5°	5°		5°	5°

# L1 Signal Structure

- $s_m(t) = C(t) \oplus D(t)$ .  $s(t) = A_c s_m(t) \cos(\omega_c t + \theta_1)$  [bpsk]
  - $D(t)$  - 50 bps transmission (20ms).
  - $C(t)$  – 1 Mbps chipping rate. 300 meters,  $1\mu\text{s}/\text{chip}$ . Chipping code repeats 20 times for single navigation bit.
- C/A code is generated from two Gold codes and the PRN identifier (5 bits).
  - $g_1(x) = 1 + x^3 + x^{10}$ ,  $g_2(x) = 1 + x^2 + x^3 + x^8 + x^9 + x^{10}$
  - Phase selector takes PRN and uses it to select bits of the second Gold code.
- Signal Characteristics
  - Received power: -130dBm, Noise power: -111dBm
  - Spread spectrum contributes 43 dB to processing. Want SNR greater than 14 dB
  - SNR for C/A: -20dB



# Acquisition, tracking and navigation

- Acquisition
  - Receiver generates known C/A code and attempts to correlate with signal.
  - To generate C/A code, receiver needs to know which satellite it's attempting to acquire to determine code
- Tracking via delay loop to maintain C/A code alignment
- The Navigation Message includes the Ephemeris parameters, the Time parameters and Clock Corrections, the Service Parameters with satellite health information, Ionospheric parameters model, and the Almanacs, allowing the computation of the position of "all satellites in the constellation". The ephemeris and clocks parameters are usually updated every two hours, while the almanac is updated at least every six days.

# Navigation Message

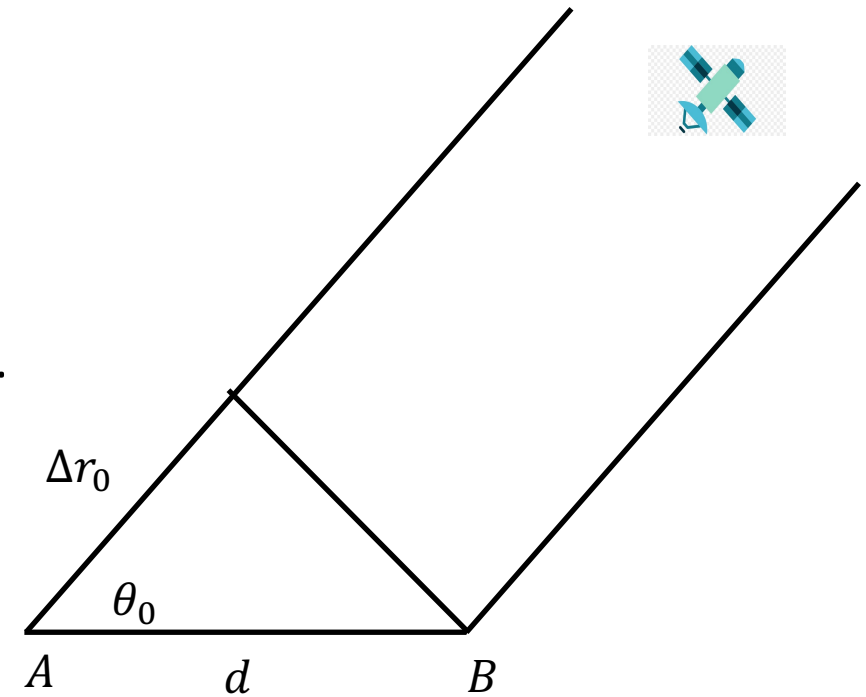
- The navigation message contains 25 pages ('frames') of 30 seconds each. Entire message takes 12.5 minutes to be transmitted. Every frame is subdivided into 5 sub-frames of 6 seconds each; every sub-frame consists of 10 words, with 30 bits per word.
- Every sub-frame starts with the telemetry word (TLM), needed for synchronism. Next, the transference word (HOW) which provides time information (seconds of the GPS week), allowing the receiver to acquire the week-long P(Y)-code segment.
  - Sub-frame 1: contains information the satellite clock. It also has information about satellite health condition.
  - Sub-frames 2 and 3: contain satellite ephemeris.
  - Sub-frame 4: provides ionospheric model parameters, UTC information (Universal Coordinate Time), part of the almanac, and indications whether the Anti-Spoofing, A/S, is activated.
  - Sub-frame 5: contains data from the almanac and the constellation status. It allows to quickly identify the satellite from which the signal comes. A total of 25 frames are needed to complete the almanac.
  - **Sub-frames 1, 2 and 3 are transmitted with each frame.** The content of sub-frames 4 and 5 is common for all satellites. So, the almanac data for all in orbit satellites can be obtained from a single tracked satellite.

# Processing and Accuracy

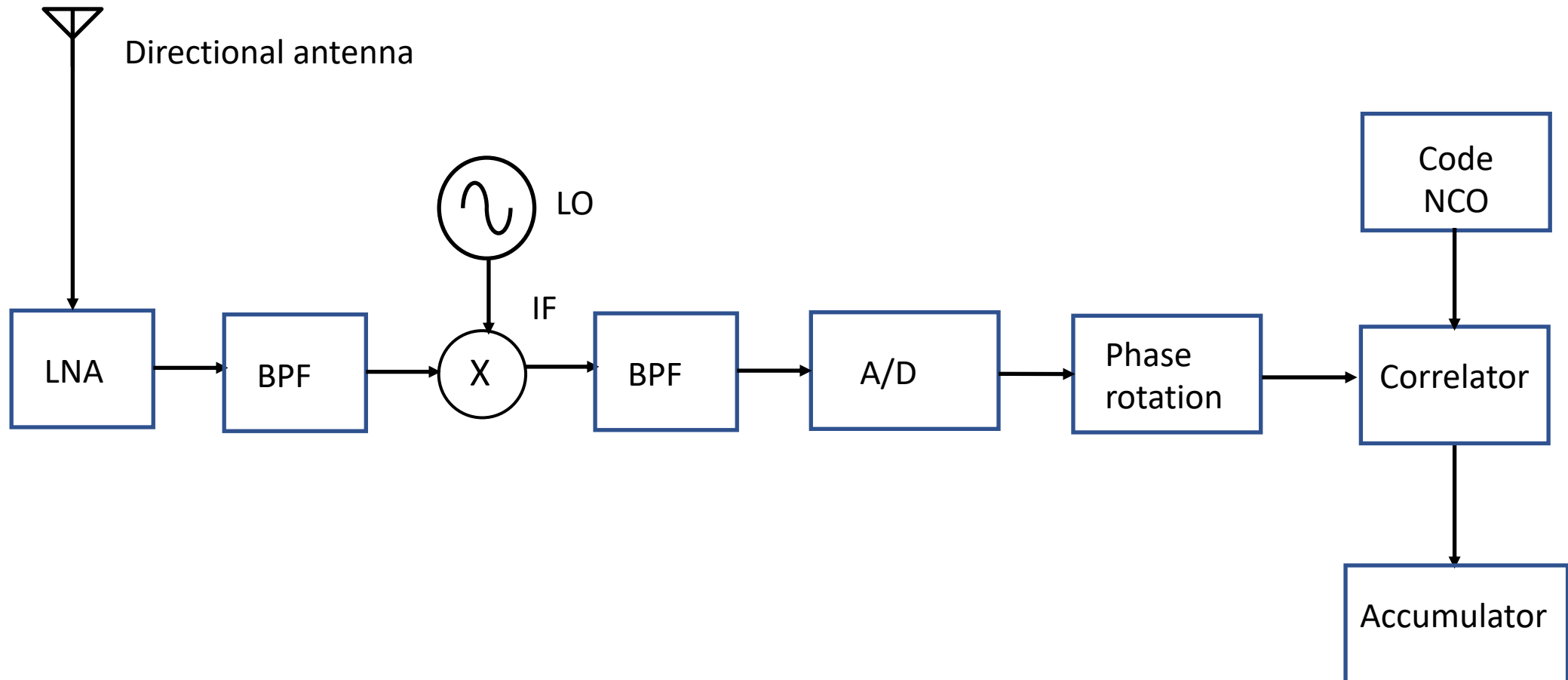
- For satellite k:  $d_k = \sqrt{(x_k - x)^2 + (y_k - y)^2 + (z_k - z)^2}$ 
  - Need to account for skew between satellite clocks and receiver clock. Four satellites required.
- Direct calculation
  - $b$  is the clock bias.  $s_i$  is  $i^{\text{th}}$  satellite time.  $t_i$  is actual reception time.  $\hat{t}_i$  is receiver reception time.
  - $d_i = (\hat{t}_i - b - s_i)c$ ,  $p_i = bc + d_i$
  - Solve equations for variables  $(x, y, z, b)$
- Least squares:  $(\hat{x}, \hat{y}, \hat{z}, \hat{b}) = \underset{(x,y,z,b)}{\operatorname{argmin}} \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 + bc + p_i}$
- *Carrier-Phase Enhancement* .
  - Receiver mixes self generated carrier with received carrier. Low frequency beat is a measure of Doppler and hence relative speed of SV to receiver.
  - Corrects timing errors caused by non-zero PRN pulse transition using the L1 carrier wave, which has a period about one-thousandth of the C/A Gold code bit providing an additional clock.
  - The phase difference error in the normal GPS amounts to 2–3 m error. Carrier-Phase Enhancement reduces this to 3 cm (1.2 in).
  - $\frac{d\varphi}{dt} = f$ .
  - $A_{ideal} = A_0 \sin(2\pi[f_0 + \varphi_0]) = A_0 \sin(2\pi f_0) \cos(2\pi \varphi_0) + A_0 \cos(2\pi f_0) A_0 \sin(2\pi \varphi_0) = A_0^S \sin(\omega_0 t) + A_0^C \cos(\omega_0 t)$

# Carrier phase differencing

- $\Delta r = d \cos(\theta_0)$
- Wave arrives at B first. Arrives at A with phase delay  $\phi_0 = \lfloor \phi_0 \rfloor + \Delta_0$ , so
- $d \cos(\theta_0) = \lambda(\phi_0)$
- Satellite moves in time  $t_1$  changing angle to  $\theta_1$
- Suppose  $\lfloor \phi_0 \rfloor = \lfloor \phi_1 \rfloor = N$ , and put  $d' = \frac{d}{\lambda}$ 
  - $d' \cos(\theta_0) - N = \Delta_0$
  - $d' \cos(\theta_1) - N = \Delta_1$
- We can solve for  $N, d'$
- Now,  $\phi = \lambda^{-1}[r - I + T] + f \cdot (\delta t_u - \delta t^s) + N + \epsilon_\phi$
- $I$  is ionospheric delay,  $T$  is tropospheric delay which we can estimate.
- $\epsilon_\phi \approx .025 \text{ cycles}$  (5mm), so we can recover  $r$  to mm precision
- Used in Differential GPS



# GPS receiver



# More careful estimates\*

Elevation	5°	40°	90°
Power at sat input	14.3dB	14.3dB	14.3dB
Sat antenna gain	12.1dB	12.9dB	10.2dB
EIRP	26.4dB	27.2dB	24.5dB
Range	25240km	22020km	20190km
Path loss	-159 dB/m <sup>2</sup>	-157.8 dB/m <sup>2</sup>	-157.1 dB/m <sup>2</sup>
Atmo loss	.5dB	.5dB	.5dB
Received power density	-133.1dB/m <sup>2</sup>	-131.1dB/m <sup>2</sup> 7.8x10 <sup>-14</sup> W/m <sup>2</sup>	-133.1dB/m <sup>2</sup> 4.9x10 <sup>-14</sup> W/m <sup>2</sup>

\*Misra and Enge, GPS

# More careful estimates\*

Elevation	5°	40°	90°
Received power density	$4.9 \times 10^{-14} \text{W/m}^2$	$7.8 \times 10^{-14} \text{W/m}^2$	$4.9 \times 10^{-14} \text{W/m}^2$
Effective area of receiver	$2.7 \times 10^{-3} \text{m}^2$ -24.5dBm <sup>2</sup>	$2.7 \times 10^{-3} \text{m}^2$	$2.7 \times 10^{-3} \text{m}^2$
Isotropic receiver power	-158.5 dB	-156.5dB	-158.5dB
$G_r$	-4 dBic	2 dBic	4dBoc
C/A received power	-162.5dB	-154.5dB	-154.5dB

\*Misra and Enge, GPS

# More careful estimates\*

- $-162.5dB \leq C \leq -154.5dB$
- $P_N = N_0(2 \times 10^7)$
- $\sigma_{\Delta\tau} = cT_c \sqrt{\frac{d}{4T_c/N_0}}$  where  $\tau$  is the noise sample interval,  
 $d$  is the correlator interval,  $T$  is the averaging time,  $d$  is measured in chips and so  $dT_c$  is the time sampling

Noise loss

Elevation	Ant cable	LNA	cable
Gain	-1dB	20dB	-10dB
F	1dB	2-3dB	10dB
T	75.4	290	2610

SNR

Elevation	5°	90°
Received C/A power (C)	-162.5dB	-154.5dB
$N_0$ for 3dB LNA	-201 W/Hz	-201W/Hz
C/ $N_0$	34.5 db-Hz	46.5 dBHz
C/ $P_n$ 20MHz	-34.6 dB	-26.5dB
C/ $P_n$ 2MHz	-24.5dB	-16.5dB

\*Misra and Enge, GPS



# Signal Processing Nomenclature

- $X(\omega) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ ,  $\mathcal{F}^{-1}[X(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$
- Output operator:  $y(t) = \mathcal{T}[x(t)]$ . LTI:  $y_1(t) + y_2(t) = \mathcal{T}[x_1(t) + x_2(t)]$
- Impulse response:  $h(t) = \mathcal{T}[\delta(t)] = \int_{-\infty}^{\infty} x(\tau)h(\tau - t)d\tau$
- Causal signal  $h(t) = 0, t < 0$ . For causal signal,  $y(t) = \int_{-\infty}^t x(\tau)h(\tau - t)d\tau$
- Frequency Response:  $H(\omega) = \mathcal{F}[h(t)]$ .  $Y(\omega) = H(\omega)X(\omega)$
- Misc
  - BPSK with SNR 10dB,  $BER < 10^{-7}$  at 1Mbps
  - QAM with SNR 20dB,  $BER < 10^{-7}$  at 4Mbps
  - QAM256 with SNR 30dB,  $BER < 10^{-7}$  at 8Mbps
  - CSMA/CD (carrier sense, multiple access, collision detect)
  - CDMA (code division multiplexing with chipping)

# Morse

Symbol	Code	Symbol	Code	Symbol	Code
a	.-	m	--	y	-.--
b	-...	n	-.	z	--..
c	-.-.	o	---	0	-----
d	-..	p	.--.	1	.----
e	.	q	--.-	2	..---
f	..-.	r	--.	3	...--
g	--.	s	...	4	....-
h	....	t	-	5	.....
i	..	u	..-	6	-....
j	.---	v	...-	7	--...
k	-.-	w	.--	8	---..
l	.-..	x	-..-	9	----.

# Pinouts

## IRF510 MOSFET Pinout

TO-220 Package

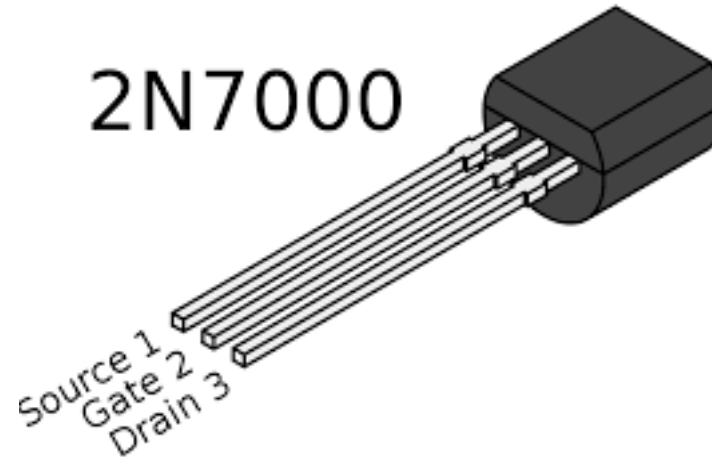


N Channel Mosfet

S = Source  
G = Gate  
D = Drain

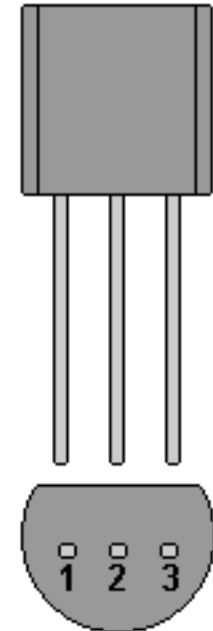
[www.componentsinfo.com](http://www.componentsinfo.com)  
Electronics Components Uses, Features, Pinouts, Equivalents, Applications & More...

## 2N7000

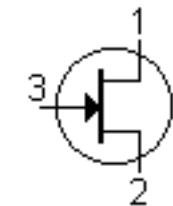


## 2N3904 pinout

1. Emitter
2. Base
3. Collector



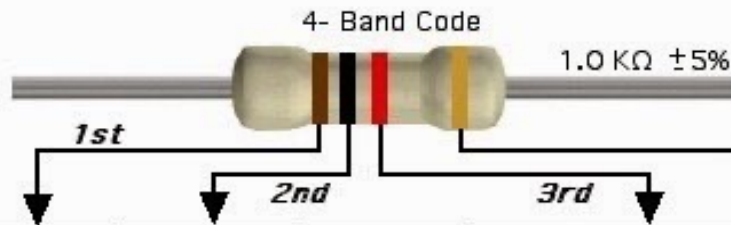
1. DRAIN
2. SOURCE
3. GATE



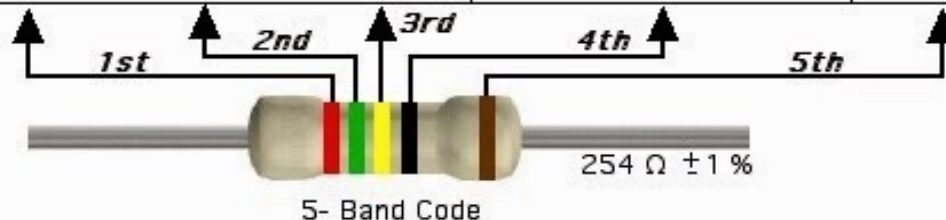
JEDEC TO-92  
J309

# Color codes

## RESISTOR COLOR CODE GUIDE



Color	1st Band	2nd Band	3rd Band	Decimal Multiplier		Tolerance
Black	0	0	0	1	1	
Brown	1	1	1	10	10	± 1 %
Red	2	2	2	100	100	± 2 %
Orange	3	3	3	1K	1,000	
Yellow	4	4	4	10K	10,000	
Green	5	5	5	100K	100,000	
Blue	6	6	6	1M	1,000,000	
Violet	7	7	7	10M	10,000,000	
Gray	8	8	8	100,000,000		
White	9	9	9	1,000,000,000		
Gold				0.1		± 5 %
Silver				0.01		± 10 %
None						± 20 %



- Resistor markings in ohms
- Capacitor markings in picoFarads
- Inductor markings in microHenries

# Component data



Core Size	26	3	15	1	2	6	10	12/17	0
T-12-( )	*	60	50	48	20	17	12	7.5	2.4
T-16-( )	145	61	55	44	22	19	13	8	3
T-20-( )	185	76	65	52	25	22	16	10	3.5
T-25-( )	245	100	85	70	34	27	19	12	4.5
T-30-( )	335	140	93	85	43	36	25	16	6
T-37-( )	285	120	90	80	40	30	25	15	4.9
T-44-( )	370	180	160	105	52	42	33	18.5	6.5
T-50-( )	330	175	135	100	49	40	31	18	6.4
T-68-( )	435	195	180	115	57	47	32	21	7.5
T-80-( )	460	180	170	115	55	45	32	22	8.5
T-94-( )	600	248	200	160	84	70	58	*	10.6
T-106-( )	930	450	345	325	135	116	*	*	19
T-130-( )	810	350	250	200	110	96	*	*	15
T-157-( )	1000	420	*	320	140	115	*	*	*
T-184-( )	1690	720	*	500	240	195	*	*	*
T-200-( )	920	425	*	250	120	100	*	*	*
T-200A-( )	1600	*	*	*	218	*	*	*	*

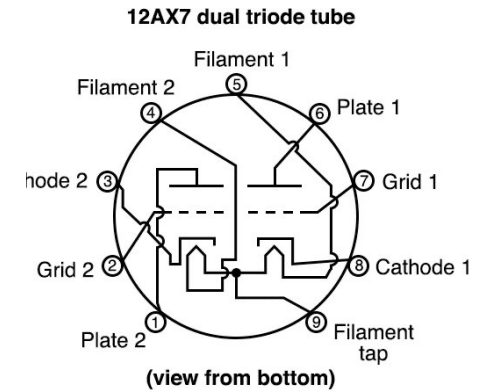
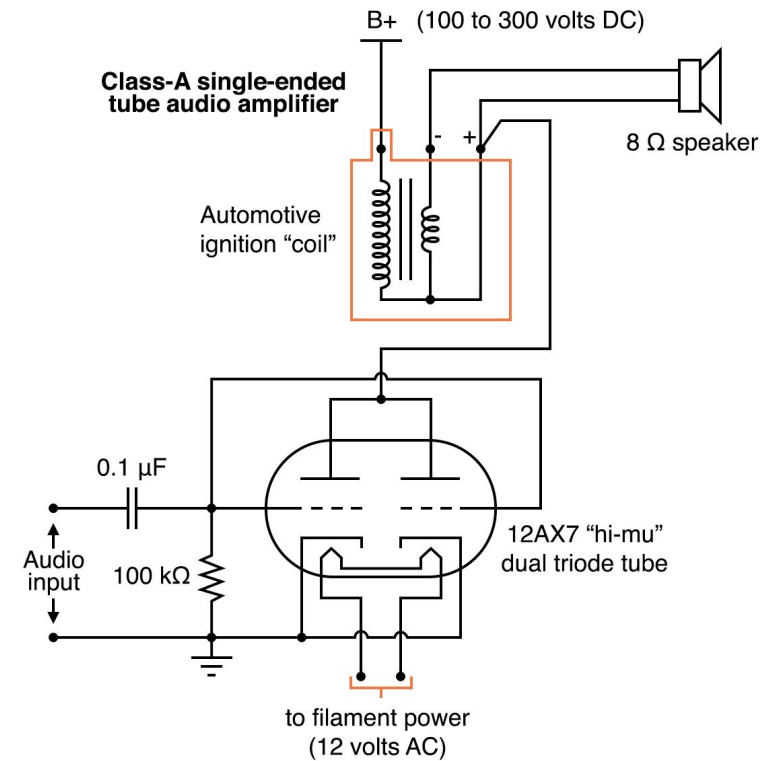
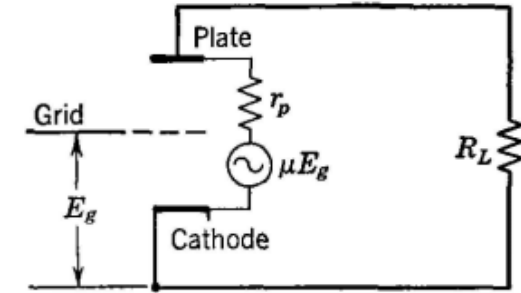
## IRON POWDER TOROIDS - $A_l$ Values \*\*

\* size not available in this material

\*\*  $L = \mu H / 100$  turns

# Vacuum tube equivalent circuits

- $\mu V_{gc} = i_p r_p + V_{pc}$
- $i_p = \frac{\mu V_{gc}}{r_p + R_L}$
- $V_L = i_p R_L$
- $A_v = \frac{V_L}{V_{gc}}$

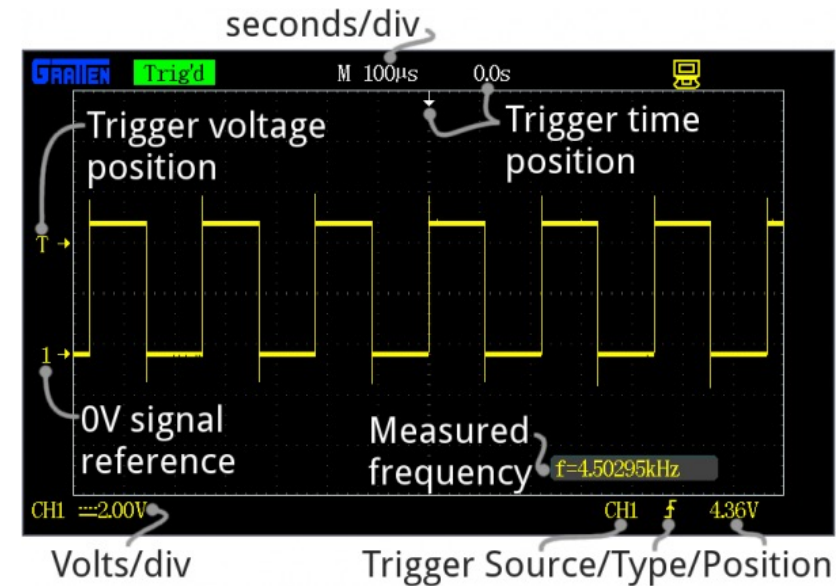


# Oscilloscopes

- Measures timing characteristics and voltage characteristics
  - Frequency and period
  - Duty cycle
  - Rise and fall time
  - Amplitude
  - Maximum and minimum voltages
  - Mean and average voltages
- Specs
  - Bandwidth – Oscilloscopes have limits as to how fast they can see a signal change. The bandwidth of a scope specifies the range of frequencies it can reliably measure.
  - Digital vs. Analog
  - Number channels
  - Sampling rate
  - Rise time - Fastest rising pulse it can measure.  $\text{Rise time} = .35 / \text{BW}$
  - Max input
  - Resolution
  - Vertical sensitivity
  - Time base
  - Input impedance

# Oscilloscopes

- Subsystems
  - Display
  - Horizontal
  - Vertical
  - Trigger
  - Inputs

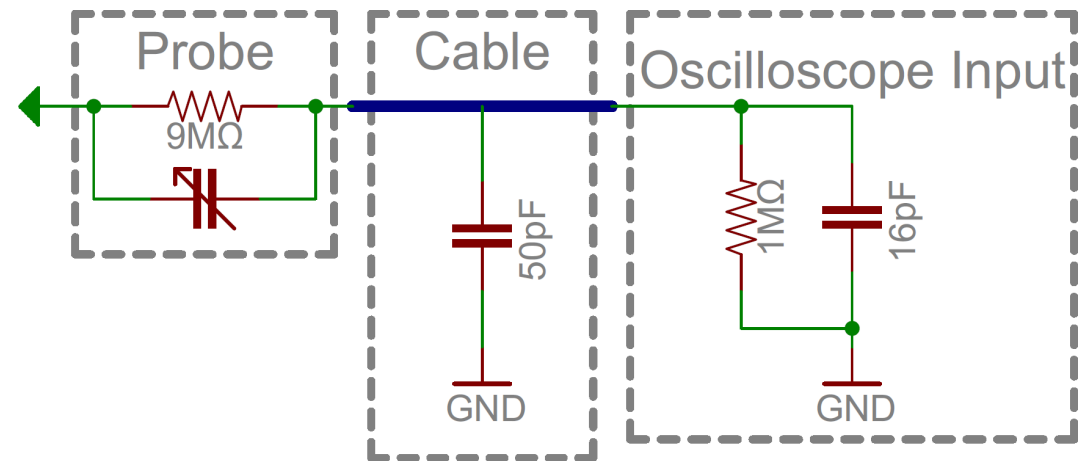


- The Trigger tells the scope what parts of the signal to start measuring. Consists of a level knob and a set of buttons to select the source and type of the trigger.
- Trigger types:
  - An edge trigger will cause the scope to start measuring when the signal voltage passes a certain level. An edge trigger can be set to catch on a rising or falling edge (or both).
  - A pulse trigger tells the scope to start measuring on a specified “pulse” of voltage. You can specify the duration and direction of the pulse. For example, it can be a tiny blip of 0V -> 5V -> 0V, or it can be a seconds-long dip from 5V to 0V, back to 5V.
  - A slope trigger will cause the scope to start based on a positive or negative slope over a specified amount of time.



# Oscilloscopes

- Probes
  - Passive probes are included with most scopes. Most passive probes are **attenuated**. Attenuating probes have a large resistance intentionally built-in and shunted by a small capacitor to minimize the effect that a long cable might have on loading the circuit.
  - Most probes have a  $9\text{M}\Omega$  resistor for attenuating, which, when combined with a standard  $1\text{M}\Omega$  input impedance on a scope, creates a  $1/10$  voltage divider. These probes are commonly called 10X attenuated probes. Many probes include a switch to select between 10X and 1X (no attenuation).



# Blank