

Cryptanalysis

Lattices

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Lattices

- The set $\Lambda = \mathbb{Z}b_1 + \mathbb{Z}b_2 + \dots + \mathbb{Z}b_n$, where b_1, b_2, \dots, b_n are linearly independent is called a lattice.
- $\Lambda^* = \{y \in \mathbb{Z}^n : (x, y) \in \mathbb{Z}, \forall x \in \Lambda\}$
- $\text{vol}(\Lambda) = \det(b_1, b_2, \dots, b_n)$, where b_1, b_2, \dots, b_n are the generators of Λ . Note that any set of generators will do since they are related by unimodular transformations.
- Let Λ be a lattice
 - The CVP problem is: Find $v \in \Lambda$: $\|v\| = \min_{w \in \Lambda, w \neq 0} (\|w\|)$
 - The CVP_γ problem is: Find $v \in \Lambda$: $\|v\| \leq \gamma \cdot \min_{w \in \Lambda, w \neq 0} (\|w\|)$
- Volume of n -dimensional sphere: $V_n(r) \approx \frac{1}{\sqrt{n\pi}} \left(\sqrt{\frac{2\pi e}{n}} r \right)^n$

Definitions

- Hermite Normal Form (HNF)

$$\begin{bmatrix} > 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \geq 0 & > 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \geq 0 & \vdots & > 0 & \ddots & \vdots & 0 & \dots & 0 \\ \geq 0 & \geq 0 & \geq 0 & \dots & 0 & 0 & \dots & 0 \\ \geq 0 & \geq 0 & \geq 0 & \dots & > 0 & 0 & \dots & 0 \end{bmatrix}$$

Minkowski's Theorem

- Let Λ be a lattice in \mathbb{R}^n and suppose $S \subseteq \mathbb{R}^n$ is a convex, centrally symmetric region. If $\text{vol}(S) > 2^n \det(\Lambda)$ then S has a non-zero lattice point of Λ .

Suppose first that Λ' is the simple lattice generated by e_1, e_2, \dots, e_n . Represent a point $r \in S$ as $r = (\alpha_1 + x_1, \alpha_2 + x_2, \dots, \alpha_n + x_n)$ with $\alpha_i \in \mathbb{Z}$ and $|x_i| \leq 1$, for $1 \leq i \leq n$. Define $T(r) = (x_1, x_2, \dots, x_n)$. If $S_1 \cap S_2 = \emptyset$, $\text{vol}(S_1 \cup S_2) = \text{vol}(S_1) + \text{vol}(S_2)$. So, if S has the property that $T(t) \neq T(s)$, $\forall s \neq t \in S$, then $\text{vol}(S) = \text{vol}(T(S))$. Note that $\text{vol}(T(S)) \leq 1$. So, if $\text{vol}(S) > 1$, there are at least two points $r^{(1)} = (\alpha_1^{(1)} + x_1, \alpha_2^{(1)} + x_2, \dots, \alpha_n^{(1)} + x_n)$, $r^{(2)} = (\alpha_1^{(2)} + x_1, \alpha_2^{(2)} + x_2, \dots, \alpha_n^{(2)} + x_n)$, where $\alpha_i^{(1)} \neq \alpha_i^{(2)}$ for some i . Since S is centrally symmetric, $-r^{(1)}, -r^{(2)} \in S$; finally, note that $0 \neq r^{(1)} - r^{(2)} \in \mathbb{Z}^n$. Similarly, if $\text{vol}(S) > 2^n$, there are at least $2^n + 1$ points $r^{(i)}$, $1 \leq i \leq 2^n + 1$ with $0 \neq r^{(i)} - r^{(j)} \in \mathbb{Z}^n$, $i \neq j$ for at least two, say $r^{(i)}$ and $r^{(j)}$, all corresponding coordinates in $r^{(i)} - T(r^{(i)})$ and $r^{(j)} - T(r^{(j)})$ are equal (mod 2). Thus, $0 \neq \frac{r^{(i)} - r^{(j)}}{2} \in \mathbb{Z}^n$. But since S is convex, $\frac{r^{(i)} - r^{(j)}}{2} \in S$. So, the result holds for the simple lattice. Suppose now that Λ is generated by a_1, a_2, \dots, a_n and put $A = [a_1, a_2, \dots, a_n]$. $e_i = A^{-1}(a_i)$, so $\text{vol}(\Lambda') = \frac{\text{vol}(\Lambda)}{\det(\Lambda)}$ and the simple lattice result thus implies the general theorem.

q-ary lattices and other definitions

- Definition: If $q \in \mathbb{Z}$, a lattice, Λ , is called q -ary if $q\mathbb{Z}^n \subseteq \Lambda \subseteq \mathbb{Z}^n$.
- Suppose $A \in \mathbb{Z}^{m \times n}$, $\Lambda_q(A) = \{y \in \mathbb{Z}^n : y = A^T x \pmod{q}, x \in \mathbb{Z}_q^m\}$.
Note $\Lambda_q(A)$ is q -ary.
- $\Lambda_q^\perp(A) = \{y \in \mathbb{Z}^n : Ay = 0 \pmod{q}\}$
- $\lambda_1(\Lambda) = \min_{v \in \Lambda} \|v\|$
- $\lambda_n(\Lambda) = \min_S (\max_{v \in S} \|v\|)$, where $S \subseteq \Lambda$ is a set of linearly independent vectors, $|S| = n$
- Solving CVP in $\Lambda_q^\perp(A)$ when A is chosen uniformly at random is as hard as worst case CVP.

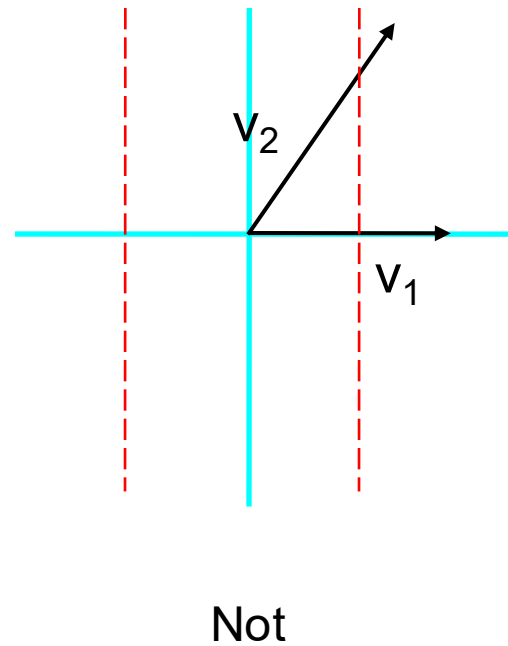
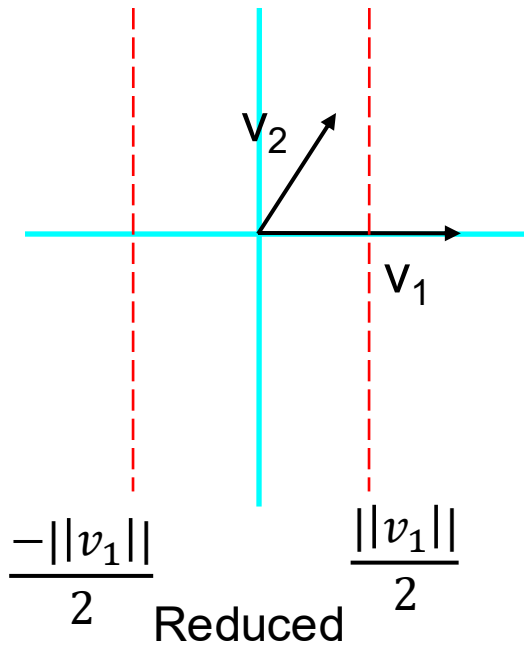
Some simple results

- Remember S is centrally symmetric if $s \in S$ implies $-s \in S$, and S is convex if $s, t \in S$ implies $us + (1 - u)t \in S, u \in [0,1]$. We used this in proving Minkowski's Theorem.
- Theorem: $\lambda_1(\Lambda) \leq \sqrt{n} \det(\Lambda)^{\frac{1}{n}}$

Let B_r be a ball centered at 0 having radius $r = \sqrt{n} \det(\Lambda)^{\frac{1}{n}}$. Let (x_1, x_2, \dots, x_n) be the coordinates of a vector v , with respect to the basis generating the lattice Λ , if $|x_i| \leq 1$ for $1 \leq i \leq n$, $v \in B_r$. So $-\det(\Lambda)^{\frac{1}{n}} (1, 1, \dots, 1)$ and $\det(\Lambda)^{\frac{1}{n}} (1, 1, \dots, 1)$ as well as the line joining them are in B_r so $\text{vol}(B_r) \geq 2^n \det(\Lambda)$ and the result follows from Minkowski's theorem.

Reduced Basis

- $\langle v_1, v_2 \rangle$ is reduced if
 - $\|v_2\| \leq \|v_1\|$; and,
 - $-1/2\|v_1\|^2 \leq (v_1, v_2) \leq 1/2\|v_1\|^2$.



Good basis and Gram-Schmidt Orthogonalization

- Good basis for lattices are orthonormal when that is possible. If a basis, b_1, b_2, \dots, b_n for Λ , is orthonormal, then, for example, $\text{vol}(\Lambda) = ||b_1|| \cdot ||b_2|| \cdot \dots \cdot ||b_n||$
- The orthogonality defect of a basis b_1, b_2, \dots, b_n is $\frac{||b_1|| \cdot ||b_2|| \cdot \dots \cdot ||b_n||}{\det(b_1, b_2, \dots, b_n)}$
- Given a space generated by b_1, b_2, \dots, b_n can also be generated by a set of vectors, $b_1^*, b_2^*, \dots, b_n^*$ with the property that $(b_i^*, b_j^*) = 0, i \neq j$. The Gram-Schmidt orthogonalization procedure computes this.

GSO, given, b_1, b_2, \dots, b_n , compute $b_1^*, b_2^*, \dots, b_n^*$

1. put $b_1^* = b_1$.

2. for $i = 2, i \leq n$

$$b_i^* = b_i - \sum_{j=1}^{i-1} \mu_{i,j} b_j, \mu_{i,j} = \frac{(b_j^*, b_i)}{(b_j^*, b_j^*)}$$

Size Reduction

- Definition: A basis b_1, b_2, \dots, b_n is *size reduced* if $|\mu_{i,j}| \leq \frac{1}{2}$, in the Gram-Schmidt orthogonalization procedure.
- If b_1, b_2, \dots, b_n is a basis for Λ , in general, $b_1^*, b_2^*, \dots, b_n^*$ is not also a lattice basis because $\mu_{i,j}$ is generally not an integer. We can find a “nearly” orthogonal set of vectors b'_1, b'_2, \dots, b'_n in Λ , by rounding the $\mu_{i,j}$. b'_1, b'_2, \dots, b'_n is also a basis for the lattice and has the same gram Schmidt basis, $b_1^*, b_2^*, \dots, b_n^*$. When performing GSO on this *reduced* basis, $|\mu_{i,j}| \leq \frac{1}{2}$.

Size-reduction

for $i = 2, i \leq n$

for $j = i - 1, j \geq 1$

$b_i \leftarrow b_i - \lceil \mu_{ij} \rceil b_j$

for $k = 1, k \leq j$

$\mu_{ik} \leftarrow \mu_{ik} - \lceil \mu_{ij} \rceil \mu_{jk}$

Size reduction and basis reordering

- Let b_1, b_2, \dots, b_n be a basis for Λ , and $b_1^*, b_2^*, \dots, b_n^*$ the resulting GSO basis. Let $B_i = ||b_i||^2$. Then b_1, b_2, \dots, b_n satisfies the *Lovasz condition* with factor δ if it is size reduced and $(\delta - \mu_{i+1,i}^2)B_i \leq B_{i+1}$. The LLL algorithm calculates such a basis.

LLL Algorithm

Given b_1, b_2, \dots, b_n generating Λ , calculate the LLL reduced basis

1. Reduce the basis b_1, b_2, \dots, b_n with the size reduction algorithm and calculate $b_1^*, b_2^*, \dots, b_n^*$ and μ_{ij}
2. Compute $B_i = ||b_i^*||^2, i = 1, 2, \dots, n$
3. for $i = 1, i < n$
 4. If $((\delta - \mu_{i+1,i}^2)B_i > B_{i+1})$
 5. Swap b_i and b_{i+1}
 6. Go to 1
7. return b_1, b_2, \dots, b_n

Example (LLL including GSO)

- LLL ($\delta = \frac{3}{4}$)
- $b_1 = (2,3,14)^T, b_2 = (0,7,11)^T, b_3 = (0,0,23)^T$.
 - GSO: $b_1^* = b_1, b_2^* = b_2 - \mu_{21}b_1, \mu_{21} = \frac{(b_1^*, b_2)}{(b_1^*, b_1^*)} = \frac{21+154}{4+9+196} = \frac{175}{209}, \mu_{31} = \frac{322}{209}, \mu_{31} = \frac{3473}{4905}. b_2^* = (-\frac{350}{209}, \frac{938}{209}, -\frac{151}{209})^T$
 - Size reduction: $b_2 = b_2 - \lceil \mu_{21} \rceil b_1 = (-2,4,-3)^T, \mu_{21} = \mu_{21} - \lceil \mu_{21} \rceil = -\frac{34}{209}; b_3 = b_3 - \lceil \mu_{32} \rceil b_2 = (-2,4,20)^T, \mu_{31} = \mu_{31} - \lceil \mu_{31} \rceil = -\frac{1432}{4905};$ last change is $b_3 = b_3 - \lceil \mu_{31} \rceil b_1 = (-4,1,6)^T, \mu_{31} = \mu_{31} - \lceil \mu_{31} \rceil = -\frac{79}{209}.$
 - Now, $b_1 = (2,3,14)^T, b_2 = (-2,4,-3)^T, b_3 = (-4,1,6)^T$.
 - $B_1 = 209, B_2 = \frac{4905}{209}, B_3 = \frac{103684}{4905}$. Lovasz condition is not satisfied for $i = 1$: since $(\delta - \mu_{21}^2)B_1 > B_2$. So swap b_1 and b_2 .
 - Applying GSO we get $\mu_{21} = \frac{-34}{29}, \mu_{31} = \frac{-6}{29},$ and $\mu_{32} = \frac{2087}{4905}.$
 - Size reduction produces: $b_2 = b_2 - \lceil \mu_{21} \rceil b_1 = (0,7,11)^T$ and $\mu_{21} = \frac{-6}{29}.$ μ_{31} and μ_{32} don't change. μ_{32}

Example (LLL including GSO) - continued

- Now Lovasz condition is satisfied for $i = 1$ since $(\delta - \mu_{21}^2)B_1 < B_2$. but not $i = 2$ since $(\delta - \mu_{32}^2)B_2 < B_3$. swap b_2 and b_3 .
 - Now, $b_1 = (-2, 4, -3)^T$, $b_2 = (-4, 1, 6)^T$, $b_3 = (0, 7, 11)^T$. $B_1 = 29$, $B_2 = \frac{1501}{29}$, $B_3 = \frac{103684}{1501}$. GSO coefficients are $\mu_{21} = \frac{-6}{29}$, $\mu_{31} = \frac{-5}{29}$, and $\mu_{32} = \frac{2087}{1501}$. Applying size reduction does not affect b_2 or μ_{21} . $b_3 = b_3 - \lceil \mu_{32} \rceil b_2 = (4, 6, 5)^T$, $\mu_{31} = \mu_{31} - \lceil \mu_{32} \rceil \mu_{21} = \frac{1}{29}$, $\mu_{31} = \frac{586}{1501}$. Both Lovasz conditions now hold.
 - LLL basis is thus $b_1 = (-2, 4, -3)^T$, $b_2 = (-4, 1, 6)^T$, $b_3 = (4, 6, 5)^T$. Notice $\|b_1\|$ is actually the shortest vector in Λ .

LLL Properties

- Suppose we apply LLL to b_1, b_2, \dots, b_n , with $b_1^*, b_2^*, \dots, b_n^*$ and B_1, B_2, \dots, B_n defined as above. With $X = \min_{v \in \Lambda} (||b_i||)$ and $\frac{1}{4} < \delta < 1$, LLL runs in $O(n^6 \ln(x)^3)$.
 - $B_i \leq ||b_i||^2 \leq (\frac{1}{2} + 2^{i-2})B_i$
 - $||b_i|| \leq 2^{\frac{i-1}{2}} ||b_i^*||$
 - $\lambda_1(\Lambda) \geq \min_i (||b_i^*||)$
 - $||b_1|| \leq 2^{\frac{n-1}{2}} \lambda_1(\Lambda)$
 - $\det(\Lambda) \leq \prod_{i=1}^n ||b_i|| \leq 2^{\frac{n(n-1)}{4}} \det(\Lambda)$
 - $||b_i|| \leq 2^{\frac{(n-1)}{4}} \det(\Lambda)^{\frac{1}{n}}$
- If w is a vector in \mathbb{R}^n and the lattice basis for Λ is b_1, b_2, \dots, b_n with $B = [b_1, b_2, \dots, b_n]$, the coefficients for w are $u = B^{-1}(w)$. w is not necessarily in the lattice but if we take each element in u and round it, $B \lfloor B^{-1}(w) \rfloor \in \Lambda$. This is *Babai rounding*.

Attack on RSA using LLL

- Attack applies to messages of the form "M xxx" where only "xxx" varies (e.g.- "The key is xxx") and xxx is small.
- From now on, assume $M(x) = B + x$ where B is fixed
 - $|x| < Y$.
 - Not that $E(M(x)) = c = (B + x)^3 \pmod n$
 - $f(x) = (B+x)^3 - c = x^3 + a_2x^2 + a_1x + a_0 \pmod n$.
- We want to find x : $f(x) = 0 \pmod n$, a solution to this, m, will be the corresponding plaintext.

Attack on RSA using LLL

- To apply LLL, let:
 - $v_1 = (n, 0, 0, 0),$
 - $v_2 = (0, Yn, 0, 0),$
 - $v_3 = (0, 0, Y^2n, 0),$
 - $v_4 = (a_0, a_1Y, a_2Y^2, a_3Y^3)$
- When we apply LLL, we get a vector, b_1 :
 - $||b_1|| \leq 2^{3/4} |\det(v_1, v_2, v_3, v_4)| = 2^{3/4} n^{3/4} Y^{3/2}$ *Equation 1.*
- Let $b_1 = c_1v_1 + \dots + c_4v_4 = (e_0, Ye_1, Y^2e_2, Y^3e_3).$ Then:

Attack on RSA using LLL

- Now set $g(x) = e_3x^3 + e_2x^2 + e_1x + e_0$.
- From the definition of the e_i , $c_4 f(x) = g(x) \pmod{n}$, so if m is a solution of $f(x) \pmod{n}$, $g(m) = c_4 f(m) = 0 \pmod{n}$.
- The trick is to regard g as being defined over the real numbers, then the solution can be calculated using an iterative solver.
- If $Y < 2^{(7/6)}n^{(1/6)}$, $|g(x)| \leq 2||b_1||$.
- So, using the Cauchy-Schwartz inequality, $||b_1|| \leq 2^{-1}n$.
- Thus $|g(x)| < n$ and $g(x) = 0$ yielding 3 candidates for x .
- Coppersmith extended this to small solutions of polynomials of degree d using a $d+1$ dimensional lattice by examining the monic polynomial $f(T) = 0 \pmod{n}$ of degree d when $|x| \leq n^{1/d}$.

Example attack on RSA using LLL

- $p = 757285757575769$, $q = 2545724696579693$.
- $n = 1927841055428697487157594258917$.
- $B = 200805000114192305180009190000$.
- $c = (B + m)^3, 0 \leq m < 100$.
- $f(x) = (B + x)^3 - c = x^3 + a_2x^2 + a_1x + a_0 \pmod{n}$.
 - $a_2 = 602415000342576915540027570000$
 - $a_1 = 1123549124004247469362171467964$
 - $a_0 = 587324114445679876954457927616$
 - $v_1 = (n, 0, 0, 0)$
 - $v_2 = (0, 100n, 0, 0)$
 - $v_3 = (0, 0, 10^4n, 0)$
 - $v_4 = (a_0, a_1100, a_210^4, 10^6)$

Example attack on RSA using LLL

- Apply LLL, $b_1 =$
 - $308331465484476402v_1 + 589837092377839611v_2 +$
 - $316253828707108264v_3 + (-1012071602751202635)v_4 =$
 - $(246073430665887186108474, -577816087453534232385300,$
 $405848565585194400880000, -1012071602751202635000000)$
- $g(x) = (-1012071602751202635) t^3 + 40584856558519440088 t^2 +$
 $(-57781608745353442323853) t + 246073430665887186108474.$
- Roots of $g(x)$ are $42.00000000, (-.9496 \pm 76.0796i)$
- The answer is 42.

GGH Public Key System

- Pick $n, M \in \mathbb{N}$ and σ is “small”, say $\sigma = 4$
- Plaintext: $\mathcal{M} = \{x: -M \leq x \leq M\}$, Cipher-space: $\mathcal{C} \in \mathbb{Z}^n$.
- Gen:
 1. Choose $B \in \mathbb{Z}^{n \times n}$ with small entries $|B_{ij}| \leq \sigma$
 2. Check B is invertible. B is the secret key.
 3. $H = \text{HNF}(B)$
- Enc
 1. For $\vec{m} \in \mathcal{M}^n$, choose $\vec{r} \in (-\sigma, \sigma)^n$ uniformly at random
 2. $\vec{c} = H\vec{m} + \vec{r}$
- Dec
 1. Babai round $\vec{m} = H^{-1}B \downarrow ((B^{-1}(\vec{c}))) \uparrow$
- Works if $\downarrow B^{-1}(r) \uparrow = 0$.

GGH Example

- $B = \begin{bmatrix} 2 & -3 & 1 & -4 \\ -2 & 1 & 0 & 4 \\ -1 & 3 & 2 & 1 \\ -1 & -4 & 3 & -2 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 44 & 18 & 4 & 49 \end{bmatrix}$
- $B^{-1} = \frac{1}{49} \begin{bmatrix} 61 & 45 & 10 & -27 \\ -10 & -13 & 8 & -2 \\ 29 & 23 & 16 & -4 \\ 33 & 38 & 3 & -13 \end{bmatrix}, H^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-44}{49} & \frac{-18}{49} & \frac{-4}{49} & \frac{1}{49} \end{bmatrix}$
- $m = (3, -4, 1, 3)^T, r = (-1, 1, 1, -1)^T, c = Hm + r = (2, -3, 2, 210)^T$
- $B^{-1}c = \frac{1}{7}(-809, -55, -117, -396)^T, \downarrow B^{-1}c \uparrow = (-116, -8, -17, -57)^T$
- $B \downarrow B^{-1}c \uparrow = (3, -4, 1, 211)^T$
- $m = H^{-1}B \downarrow B^{-1}c \uparrow = (3, -4, 1, 3)^T$

Learning with Errors (LWE)

- Based on solving noisy linear equations $\text{mod } q$. Choose $\vec{a}_i \in \mathbb{Z}_q^n$ uniformly at random. $\vec{s} \in \mathbb{Z}_q^n$ is a secret and $m \geq n$ approximate equations $\vec{a}_i \cdot \vec{s} = b_i \pmod{q}$. Errors, e_1, e_2, \dots, e_n are chosen from distribution χ .
- Reduces to LWE:
 - Search LWE problem: Given $a_{ij}, (\vec{b} + \vec{e})$ find \vec{s} .
 - Decision LWE: Distinguish, with non-negligible probability, between $\vec{b} = A\vec{s} + \vec{e}$ and $\vec{b} \in \mathbb{Z}_q^m$ chosen uniformly at random given A, \vec{b}
- Errors chosen from distribution, $p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{x^2}{2\sigma^2})$. Often use $s = \sigma\sqrt{2\pi}$ as parameter specifying distribution.
- Regev's showed it is possible to pick parameters so that solving an LWE cipher is equivalent to solving worst-case LWE .
 - Theorem (Regev): Let $n \in \mathbb{N}$ be a security parameter, $m, q \in \mathbb{N}$, polynomial in n and $\chi = D_{\mathbb{Z},s}$ a discrete Gaussian distribution with $s = \alpha q > 2\sqrt{n}$, $0 < \alpha < 1$. Then solving the LWE decision problem is at least as hard as quantumly solving $SIVP_\gamma$ on an arbitrary n -dimensional lattice where $\gamma = \tilde{O}(n/\alpha)$.

LWE cryptosystem

- Given $(n \geq m, l, t, r, q, \chi)$ where χ is a probability distribution \mathbb{Z}_q , message space is \mathbb{Z}_2^l and cipher space is $\mathbb{Z}_q^n \times \mathbb{Z}_q^l$.
- Key Gen
 1. Choose $S \in \mathbb{Z}_q^{n \times l}$, uniformly from the distribution χ .
 2. Choose $A \in \mathbb{Z}_q^{m \times n}$, and $E \in \mathbb{Z}_q^{m \times l}$ uniformly from the distribution χ .
 3. Private key is S , public key is $(A, P = AS + E)$
- Enc
 1. For $\vec{v} \in \mathbb{Z}_2^l$, choose $\vec{a} \in \{0,1\}^m$, uniformly at random
 2. $\vec{CT} = (\vec{u} = A^T \vec{a}, \vec{c} = P^T \vec{a} + \uparrow \frac{q}{2} \downarrow \vec{v})$
- Dec
 1. Compute $\uparrow (\uparrow \frac{q}{2} \downarrow)^{-1} (\vec{c} - S^T \vec{u}) \uparrow \pmod{2}$
- Decryption may have errors. Suppose χ is a discrete Gaussian $D_{\mathbb{Z},s}$. Then $E^T \vec{a}$ has magnitude $\leq \sqrt{m}s$ with high probability. Error occurs if $E^T \vec{a} \geq \frac{q}{4}$. One can show that for any $n, \exists q, m, s$ such that the error is small and the underlying LWE problem is hard.

LWE example

- $n = 4, q = 23, m = 8, \alpha = \frac{5}{23}, s = 5, \sigma = \frac{s}{\sqrt{2\pi}}, l = 4$

- $A^{m \times n} = A = \begin{bmatrix} 9 & 5 & 11 & 13 \\ 13 & 6 & 6 & 2 \\ 6 & 21 & 17 & 18 \\ 22 & 19 & 20 & 8 \\ 2 & 17 & 10 & 21 \\ 10 & 8 & 17 & 11 \\ 5 & 16 & 12 & 2 \\ 5 & 7 & 11 & 7 \end{bmatrix}, S^{n \times l} = S = \begin{bmatrix} 5 & 2 & 9 & 1 \\ 6 & 8 & 19 & 1 \\ 19 & 18 & 9 & 18 \\ 9 & 2 & 14 & 18 \end{bmatrix}$

LWE example

- $E^{m \times l} = E = \begin{bmatrix} 0 & 22 & 1 & 21 \\ 0 & 22 & 22 & 22 \\ 6 & 21 & 17 & 18 \\ 22 & 22 & 22 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 22 & 1 & 22 \\ 22 & 0 & 0 & 1 \end{bmatrix}, P^{m \times l} = P = \begin{bmatrix} 10 & 5 & 21 & 7 \\ 3 & 1 & 13 & 1 \\ 19 & 15 & 6 & 13 \\ 22 & 22 & 22 & 0 \\ 9 & 20 & 20 & 17 \\ 15 & 21 & 1 & 2 \\ 0 & 12 & 3 & 19 \\ 16 & 2 & 7 & 15 \end{bmatrix},$

LWE example

- Encrypt $\vec{v} = (1,0,1,1)^T$, using $a = (1,1,0,1,0,0,0,1)^T$
 - $\lfloor \frac{23}{2} \vec{v} \rfloor = (12,0,12,12)^T$,
 - $(u, c) = \left(A^T a, P^T a + \lfloor \frac{23}{2} m \rfloor \right) = ((3,14,2,7)^T, (4,5,7,5)^T) \pmod{23}$
- Decrypt:
 - $\vec{v}' = c - S^T u = (11,21,12,10)^T \pmod{23}$,
 - $\lfloor \frac{1}{12} \vec{v}' \rfloor \pmod{2} = (1,0,1,1)^T$

LWE example

- Encrypt $m = (1,0,1,1)^T$, using $a = (1,1,0,1,0,0,0,1)^T$
 - $\lfloor \frac{23}{2} m \rfloor = (12,0,12,12)^T$,
 - $(u, c) = \left(A^T a, P^T a + \lfloor \frac{23}{2} m \rfloor \right) = ((3,14,2,7)^T, (4,5,7,5)^T) \pmod{23}$
- Decrypt:
 - $m' = c - S^T u = (11,21,12,10)^T \pmod{23}$,
 - $\lfloor \frac{1}{12} m' \rfloor \pmod{2} = (1,0,1,1)^T$

From Heiko Knopse

LWE/Ring-LWE parameters

Level	n	q	s	P	P&A	c	Exp
Low	128	4093	8.87	2.9×10^5	7.4×10^5	3.8×10^3	30
High	320	4093	8	4.9×10^5	17.7×10^5	17.4×10^3	42

- Ring-LWE cuts ciphertext by factor of n

Ring-LWE

- Put $R = R_{n,q} = \frac{\mathbb{Z}_q[x]}{x^{n+1}+1}$, $n = 2^k$, $R \approx \mathbb{Z}_q^n$. $a \in R$, generates ideal (a) corresponding to a q -ary ideal lattice.
- Ring LWE: Given $a \in R$, and $b = as + e$, for $s, e \in R$, find s .
- Solving R-LWE is at least as hard as solving CVP_γ on arbitrary ideal lattices

NTRU Public Key System

- NTRU is a ring lattice-based system.
- $R = \frac{\mathbb{Z}[x]}{x^N - 1}$, $R_p = \frac{\mathbb{Z}_p[x]}{x^N - 1}$, $R_q = \frac{\mathbb{Z}_q[x]}{x^N - 1}$
- $(c_0 + c_1x + \dots + c_{N-1}) = (a_0 + a_1x + \dots + a_{N-1}) \otimes (b_0 + b_1x + \dots + b_{N-1})$, where $c_k = \sum_{i+j=k \pmod N} a_i b_j$
- $\mathcal{T}(d_1, d_2)$ is the set of “ternary” polynomials of degree $< N$, having d_1 coefficients equal to 1, having d_2 coefficients equal to -1 , and remaining coefficients equal to 0.
- Pick N, p prime and $q, d \in \mathbb{N}$, $(p, q) = (N, q) = 1$, $q > (6d + 1)p$.

NTRU Public Key System

- KeyGen
 1. Pick $f, g \in R, f \in \mathcal{T}(d+1, d), g \in \mathcal{T}(d, d)$.
 2. Find $f_p, f_q: f \cdot f_p = 1 \pmod{p}, f \cdot f_q = 1 \pmod{q}, h = f_q \cdot g \pmod{q}$.
 3. Public key is (N, p, q, h) , private key is f .
- Plaintext is $m \in R_p$, ciphertext is $c \in R_q$
- Encryption
 1. Chose random $r \in R, r \in \mathcal{T}(d, d)$.
 2. $c = prh + m \pmod{q}$.
- Decryption
 1. Compute $a = fc \pmod{q}$
 2. Plaintext is $f_p a$.
 3. Verify that $a = fc = f(prh + m) \pmod{q} = pfrf_qg + fm \pmod{q} = prg + fm \pmod{q}$.

NTRU Example

- $N = 5, p = 3, q = 29, d = 1, f = x^4 + x^3 - 1, g = x^3 - x^2$
- $f_p = -x^3 - x^2 + x - 1, f_q = -5x^4 + 8x^3 + 3x^2 + 11x + 13$
- $h = f_q g = 8x^4 + 2x^3 + 11x^2 + 13x - 5 \pmod{29}$
- $r = x^4 - x$
- $c = prh + m = 8x^4 + 21x^3 + 25x^2 + 20x + 15 \pmod{29}$
- $a = fc = -2x^4 + 2x^3 + 4x^2 - 3x + 1 \pmod{29}$
- We check $a = prg + fm$ in R
- $m = x^3 + x$

Some NIST Round 3 entries

- Public-Key Encryption/KEMs
 - Classic McEliece
 - CRYSTALS-KYBER
 - NTRU
 - SABER
- Digital Signatures
 - CRYSTALS-DILITHIUM
 - FALCON
 - Rainbow
- Public-Key Encryption/KEMs (Alternates)
 - BIKE;
 - FrodoKEM
 - HQC
 - NTRU Prime
 - SIKE
- Digital Signatures
 - GeMSS
 - Picnic
 - SPHINCS+

Winner: Dilithium (signing), Kyber (key-encapsulation)

Common features of Dilithium and Kyber

- Ring is $\mathbb{Z}_p[x]/(x^{256} + 1)$ in both cases
 - $p = 3329$ for Kyber
 - $p = 2^{23} - 2^{13} + 1 = 8380417$ for Dilithium
 - So, the same modular arithmetic we all grew up with.
- For $p = 3329$, there is a primitive (and hence 128 primitive) 256th roots of unity (You are not expected to understand this).
 - As a result, $x^{256} + 1$ factors into coprime 128 quadratics
 - Allows us to perform a “Number Theory Transform” that turns convolution into pointwise multiplication for ring operations giving a nice speedup
- For $p = 8380417$, there is a primitive (and hence 256 primitive) 512th roots of unity
 - As a result, $x^{256} + 1$ factors into coprime 256 linear polys
 - Allows us to perform a “Number Theory Transform” that turns convolution into pointwise multiplication for ring operations giving a nice speedup

Useful definitions

- $r^+ = r \pmod{q}, q > r^+ \geq 0$
- $r' = r \pmod{\pm}(m)$ means $r' = r \pmod{m}$ and $-\frac{m}{2} \leq r' \leq \frac{m}{2}$, if m is even;
 $-\frac{m}{2} < r' \leq \frac{m}{2}$, if m is odd
- $decompose(r, \alpha, q)$
 - $r_0 = r^+ \pmod{\pm}(\alpha)$
 - if $r^+ - r_0 == (q - 1)$
 - $r_1 = 0, r_0 = q - 1$
 - else
 - $r_1 = \frac{r^+ - r_0}{\alpha}$
 - return (r_1, r_0)

Useful definitions

- $lowbits(x, \alpha, q)$
 - $(r_1, r_0) = decompose(x, \alpha, q)$
 - return r_0
- $highbits(x, \alpha, q)$
 - $(r_1, r_0) = decompose(x, \alpha, q)$
 - return r_1
- $power2round(r, d, q)$
 - $r^+ = r \bmod(q)$
 - $r_0 = r^+ \bmod^{\pm}(2^d)$
 - return $(\frac{r^+ - r_0}{2^d}, r_0)$

Examples

- $decompose(r, \alpha, q)$ examples (second shows roundoff edge case)

q	α	r	$r \bmod^{\pm}(\alpha)$	$r - r \bmod^{\pm}(\alpha)$	r_0	r_1
17	8	5	-3	8	-3	1
17	8	15	-1	16	-2	0
3329	104	50	50	0	50	0
3329	104	100	-4	104	-4	1

SHAKE-256/SHAKE-128

- $H(v, d) = \text{SHAKE256}(v, d)$
- $H_{128}(v, d) = \text{SHAKE128}(v, d)$
- $\text{RAWSHAKE256}(J, d) = \text{KECCAK}[512](J||11, d)$
- $\text{SHAKE256}(M, d) = \text{RAWSHAKE256}(M||11, d)$
- $\text{RAWSHAKE128}(J, d) = \text{KECCAK}[256](J||11, d)$
- $\text{SHAKE128}(M, d) = \text{RAWSHAKE128}(M||11, d)$
- Note
 - $\text{SHA3}_{256}(M) = \text{KECCAK}[512](M||11, 256)$
 - $\text{SHA3}_{512}(M) = \text{KECCAK}[1024](M||11, 1024)$

Number Theory Transform (NTT)

- $p = 3329, p - 1 = 2^8 \cdot 13$.
- \mathbb{Z}_p has a primitive 256th root of unity ($\zeta = 17$ is a primitive root) but no 512 root of unity, so $x^{256} + 1$ factors into 128 coprime quadratic factors of the form $(x^2 - \xi)$, $17^{128} = -1$.
- $x^{256} + 1 = \prod_{k=0}^{127} (x^2 - \zeta^{2 \cdot \text{bitrev}_7(k)+1})$.
- $\text{bitrev}_7(k)$ reverses the bit order in a 7-bit byte, k .
 - $x^{256} + 1 = (x^2 - 17) \cdot (x^2 - 17^{129}) \cdot \dots \cdot (x^2 - 17^{255})$
- For $p = 8380417$, $\zeta = 1753$ is a primitive 512th root of unity,
 - $p - 1 = 2^{13}(2^{10} - 1) = 2^{13} \cdot 3 \cdot 11 \cdot 31$.
- Because of this, an analog of the Chinese remainder theorem holds in

$$R_p = \frac{\mathbb{Z}_p(x)}{x^{256} + 1}.$$

NTT for Dilithium

- $NTT: R_p \rightarrow T_p, f \mapsto \hat{f}, T_q = \prod_{i=0}^{255} \mathbb{Z}_q$
- For $f \in R_p$
 - $\hat{f} = \prod_{i=0}^{511} f(\text{mod } x - \zeta^{2i+1}), \zeta = 1753$, so each element in the vector is just an element of \mathbb{Z}_p
 - If $a(x) = a_0 + a_1x + a_2x^2 + \dots + a_{511}x^{255}$
 - $\hat{a} = (a(r_0), a(-r_0), \dots, a(r_{127}), a(-r_{127})), r_i = \zeta^{i+128}$
- Multiplication is then pointwise

Dilithium template

- Gen
 - $A \leftarrow R_q^{k \times l}$
 - $(s_1, s_2) \leftarrow S_\eta^l \times S_\eta^k$
 - $t = As_1 + s_2$
 - return $(pk = (A, t), sk = (A, t, s_1, s_2))$
- Verify($pk, M, \sigma = (z, c)$)
 - $w'_1 = \text{highbits}(Az - ct, 2\gamma_2)$
 - return $\|z\|_\infty < \gamma_1 - \beta \wedge c == H(M || w'_1)$
- Sign(sk, m)
 - $z = \perp$
 - while $z == \perp$
 - $y \leftarrow S_{\gamma_1-1}^l$
 - $w_1 = \text{highbits}(Ay, 2\gamma_2)$
 - $c \in B_{60}, c = H(M || w_1)$
 - // view c as polynomial in R_q
 - $z = y + cs_1$
 - if $\|z\|_\infty < \gamma_1 - \beta \vee \|\text{lowbits}(Ay - cs_2, 2\gamma_2)\|_\infty \geq \gamma_2 - \beta$
 - $z = \perp$
 - return $\sigma = (z, c)$

For real Dilithium, $k = 5, l = 4$

Dilithium security argument -1

$$\eta = 5, \gamma_1 = \frac{q-1}{16}, \gamma_2 = \frac{\gamma_1}{2}, R_q = \frac{\mathbb{Z}_q[x]}{x^{256}+1}, q = 2^{23} - 2^{13} + 1,$$

$$q - 1 = 2^{13} \cdot 3 \cdot 11 \cdot 31. k = 5, l = 4$$

1. $A \leftarrow R_q^{k \times l}, (s_1, s_2) \leftarrow S_\eta^l \times S_\eta^k, t = As_1 + s_2, pk = (A, t), sk = (A, t, s_1, s_2), S = R_q$
2. $y \leftarrow S_{\gamma_1-1}^l, w_1 = \text{highbits}_{2\gamma_2}(Ay)$
 - Write coefficients of $w = Ay$, as $w^{[i]} = (2\gamma_1)w_1^{[i]} + w_0^{[i]}$
 - $w_1 = \text{highbits}_{2\gamma_2}(Ay)$ then $w_0^{[i]} < \gamma_2$
3. $c \in B_{60}, c = H(M || w_1)$. Set $\beta = \max_i((cs_1)^{[i]})$. Then $\beta \leq 60\eta$.
4. Set $z = y + cs_1$, if any coefficient of $z > \gamma_1 - \beta$, reject and start over.
5. If any coefficient of $\text{lowbits}_{2\gamma_2}(Az - ct) > \gamma_2 - \beta$, reject and start over.
 - Note: $Az - ct = Ay - cs_2$
 - coefficients of $z \leq \gamma_1 - \beta$, coefficients of $\text{lowbits}_{2\gamma_2}(Az - ct) \leq \gamma_2 - \beta$
6. Signature is $\sigma = (z, c)$
 - $c \in B_{60}$ is ensured by SampleInBall in the final algorithm.
 - Parameters chosen so that expected rejections in steps 4 and 5 is between 4 and 7.

Dilithium security argument - 2

Verification

- $A\mathbf{z} - c\mathbf{t} = A\mathbf{y} - c\mathbf{s}_2$
- To show $highbits_{2\gamma_2}(A\mathbf{z} - c\mathbf{t}) = highbits_{2\gamma_2}(A\mathbf{y})$, we need only show $highbits_{2\gamma_2}(A\mathbf{y}) = highbits_{2\gamma_2}(A\mathbf{y} - c\mathbf{s}_2)$.
 - This follows because $|lowbits_{2\gamma_2}(A\mathbf{y} - c\mathbf{s}_2)|_\infty < \gamma_2 - \beta$; and,
 - The coefficients of $\|c\mathbf{s}_2\|_\infty < \beta$
 - Adding $c\mathbf{s}_2$ never causes a carry of γ_2 from the lowbits and, hence, $highbits_{2\gamma_2}(A\mathbf{z} - c\mathbf{t}) = highbits_{2\gamma_2}(A\mathbf{y})$
 - Now we can compute $highbits_{2\gamma_2}(\mathbf{w}_1)$ and hence $H(M||\mathbf{w}_1)$

Template \rightarrow Dilithium

- NTT is used to speed multiplications.
- Produce hint, h to help verifier calculate w'_1 in verify.
 - $r_1 \leftarrow \text{Highbits}(r), v \leftarrow \text{Highbits}(r + z), h \leftarrow [[r_1 \neq v]]$
 - $h \leftarrow \text{MakeHint}(- \ll (ct_0) \gg, w - \ll cs_2 \gg + \ll ct_0 \gg)$
 - $\tilde{c} \leftarrow H(\mu || w_1, \lambda)$
- Drop $d = 13$, bottom bits in t . $\omega = 75$ is man number of 1's in h .
- In final algorithm, A , is generated from a seed using SHAKE-128.
- Notes:
 - Compute $w'_1 = \text{highbits}_{2\gamma_2}(Az - ct)$ from the compressed public key.
 - SampleinBall, guarantees $c \in B_{60}$ using Fisher-Yates shuffle on $H(M || w_1)$

Template → Dilithium

- $\xi \leftarrow H(\{0,1\}^{256}), (\rho, \rho', K) \leftarrow H(\xi, 512), \hat{A} \leftarrow \text{Expand}(\rho)$
- $tr \leftarrow H(pk, 512), \text{sign: } \mu \leftarrow H(tr || M, 512), (\hat{s}_1, \hat{s}_2) \leftarrow \text{ExpandS}(\rho').$
- $(t_1, t_0) \leftarrow \text{Power2Round}(t, d), sk \leftarrow (\rho, K, tr, s_1, s_2, t_0)$
- $\text{Usehint}(h, w'_{\text{appx}}), r, z \in \mathbb{Z}_q$
 - $m \leftarrow \frac{q-1}{2\gamma_2}, (r_1, r_0) \leftarrow \text{decompose}(r)$
 - If $(h == 1 \wedge r_0 > 0)$ return $(r_1 + 1) \bmod m$
 - If $(h == 1 \wedge r_0 \leq 0)$ return $(r_1 - 1) \bmod m$
 - return r_1

Algorithm	Public key	Private key size	Signature size
ML-DSA-87	4864	2592	4595

Dilithium, unedited, motivation

- Basic scheme is Fiat-Shamir MSA-DL with aborts.
- Classic version with discrete log is:
 - Prover and verifier know $(g, y = g^x)$. Prover knows x .
 1. Prover generates r , sends commitment g^r .
 2. Verifier sends c .
 3. Prover returns $s = r - cx$.
 4. Verifier can check $g^s \cdot y^c = g^r$
- Non interactive version replaces c with hash of $g^r || M$
- LWE version
 - Publish $A, t = As_1 + s_2$
 - Prover: Pick, y , commit by sending $w_{approx} = Ay + y_2$, y_2 has small coefficients.
 - Verifier: Send challenge, c .
 - Prover: $z = y + cs_1$
 - Verifier: Check z and $Az - tc \approx w_{approx}$

Dilithium (simplified)

- Remember $A^{k \times l}$ is generated randomly from $R = \mathbb{Z}_p[x]/(x^{256} + 1)$.
- s_1 is a vector of dimension l with entries from R has random coefficients $\leq \eta$
- s_2 is a vector of dimension k with entries from R has random coefficients $\leq \eta$
- $t = As_1 + s_2$

Sign

```
y := S $\gamma_1 - 1$ l  
w1 := highbits(Ay, 2 $\gamma_2$ )  
c := SH(M || w1)  
z := y + cs1  
return (z, c)
```

Verify

```
w1' := highbits(Az - ct, 2 $\gamma_2$ )  
c' := SH(M || w1')  
Check c' == c AND ||z|| $\infty$  <  
 $\gamma_1 - \beta$ 
```

Dilithium (less simplified)

Parameters: $p = 8380417$, $k = 5$, $l = 4$, $\gamma_1 = \frac{p-1}{16}$, $\gamma_2 = \frac{\gamma_1}{2}$, $\eta = 5$, $\beta = 275$

$$R_p = \frac{\mathbb{Z}_p[x]}{x^{256}+1}$$

- **KeyGen**

- $A \in R_p^{k \times l}$, selected from random distribution over R_p
- $(s_1, s_2) \in S_\eta^k \times S_\eta^l$, selected at random, S_η^k consists of elements of R_p^k with coefficients $\leq \eta$
- Set $t = As_1 + s_2$
- Public key is (A, t) , Private key is (s_1, s_2)

For the sake of compression A is generated from a seed and SHAKE-256

Dilithium

- $\text{Sign}(\text{pk}, \text{sk}, M)$ --- simplified

1. $z = \perp$
2. **while** ($z = \perp$) {
3. $y = S_{\gamma_1}^l - 1$
4. $w_1 = \text{highbits}(Ay, 2\gamma_2)$
5. $c = \text{SHAKE} - 256(M || w_1)$
6. $z = y + cs_2$
7. **if** ($||z||_\infty \geq \gamma_1 - \beta$) **OR** $\text{lowbits}(Ay - cs_2, 2\gamma_1) \geq \gamma_2 - \beta$) **then** $z = \perp$
8. }

Signature is (z, c)

- Real Dilithium uses a number of functions to generate A from a seed. It also has a hedged version and a deterministic version. The hedged version avoids some possible side channels.

Dilithium

- $\text{Verify}(\text{pk}, M, z, c)$ --- simplified
 1. $w'_1 = \text{highbits}(Az - ct, 2\gamma_2)$
 2. Return true if $\|z\|_\infty \leq \gamma_1 - \beta$ AND $c = \text{SHAKE} - 256(M \| w'_1)$, otherwise return false

Useful definitions

- *usehint*(h, r)
 - $m = \frac{p-1}{2\gamma_2}$
 - $(r_1, r_0) = \text{decompose}(r, 2\gamma_2, p)$
 - If $h == 1$ and $r_0 > 0$ then return $(r_1 + 1) \bmod(m)$
 - If $h == 1$ and $r_0 \leq 0$ then return $(r_1 - 1) \bmod(m)$
 - return r_1
- *makehint*(z, r)
 - $r_1 = \text{highbits}(r)$
 - $v_1 = \text{highbits}(r + z)$
 - return $r_1 \neq v_1$

Useful definitions

- *RejNTTPoly*(ρ) // returns NTT polynomial
 - $c = 0 ; j = 0$
 - while ($j < 256$)
 - $\hat{a}[j] = \text{coeffFromThreeBytes}(H_{128}(\rho||c), H_{128}(\rho||c + 1), \dots, H_{128}(\rho||c + 2))$
 - $c += 3$
 - If ($\hat{a}[j] \neq \perp$) then $j++$
 - return \hat{a}

- *RejBoundedPoly*(ρ)
 - $c = 0 ; j = 0$
 - while ($j < 256$)
 - $z = H(\rho)[c]$
 - $z_0 = \text{CoeffFromHalfByte}(z \bmod(16), \eta)$
 - $z_1 = \text{CoeffFromHalfByte}(\lfloor z/16 \rfloor, \eta)$
 - If ($z_0 \neq \perp$)
 - $a_j = z_0; j++$
 - If ($z_1 \neq \perp$ and $j < 256$)
 - $a_j = z_1; j++$
 - $c++$
 - return a

Useful definitions

- *ExpandA*(ρ)
 - *for* ($r = 0; r < k; k++$)
 for ($s = 0; s < l$)
 $\hat{A}[r, s] = \text{RejNTTPoly}(\rho || \text{IntegerToBits}(s, 8) || \text{IntegerToBits}(r, 8))$*return* \hat{A}
- *ExpandS*(ρ)
 - *for* ($r=0; r<l; r++$)
 - $s_1[r] = \text{RejBoundedPoly}(\rho || \text{IntegerToBits}(r, 16))$
 - *for* ($r=0; r<k; r++$)
 - $s_2[r] = \text{RejBoundedPoly}(\rho || \text{IntegerToBits}(r + l16))$*return* (s_1, s_2)
- *ExpandMask*(ρ, μ)
 - $c = 1 + \text{bitlen}(\gamma_1 - 1)$
 - *for*($r = 0; r < l; r++$)
 - $n = \text{IntegerToBits}(\mu + r, 16)$
 - $v = (H(\rho || n)[32rc], H(\rho || n)[32rc + 1], \dots, H(\rho || n)[32rc + 32c - 1])$
 - $s[r] = \text{BitUnpack}(v, \gamma_1 - 1, \gamma_1)$
 - *return* s

Useful definitions

- // Calculate $c(x)$, coefficients are 1, -1 or 0
- *SampleInBall*(ρ, τ)
 - $c(x) := 0; k=8;$
 - *for*($i = 256 - \tau; i < 256; i++$)
 - *while*($H(\rho)[[k]] > i$) // $H(\rho)[[k]]$ is k th byte
 $k++$
 - $j = H(\rho)[k]$
 - $c_i = c_j$
 - $c_j = (-1)^{H(\rho)[i+\tau-256]}$ // $[k]$ is bit position k
 - $k++$
 - *return* c

SampleInBall generates an element of B_{60} pseudorandomly; it is based on the Fisher-Yates shuffle. The first 8 bytes of $H(\rho)$ choose the signs of the nonzero entries of c ; subsequent bytes choose the positions of those nonzero entries

Here H is SHAKE256 used as an XOF.

NTT for Dilithium

- $NTT(w)$ --- outputs $\widehat{w}_j = (w(\zeta_0), w(-\zeta_0), w(\zeta_1), w(-\zeta_1), \dots, w(-\zeta_{127}))$
 - $for(j = 0; j < 256; j++) \widehat{w}[j] = w[j]$
 - $k = 0; len = 128$
 - $while(len \geq 1)$
 - $start = 0$
 - $while(start < 256)$
 - $k++$
 - $zeta = \zeta^{bitrev(k)} \bmod(q)$
 - $for(j = start; j \leq start + len - 1)$
 - $t = zeta \cdot \widehat{w}[j + len]$
 - $\widehat{w}[j + len] = \widehat{w}[j] - t$
 - $\widehat{w}[j] = \widehat{w}[j] + t$
 - $start += 2 \cdot len$
 - $len = len/2$

NTT for Dilithium

- $NTT^{-1}(\hat{w})$
 - $for(j = 0; j < 256; j++) w[j] = \hat{w}[j]$
 - $k = 256; len = 1$
 - $while(len < 256)$
 - $start = 0$
 - $while(start < 256)$
 - $k--$
 - $zeta = \zeta^{bitrev(k)} \bmod(q)$
 - $for(j = start; j \leq start + len - 1)$
 - $t = w[j]$
 - $w[j] = t + w[j + len]$
 - $w[j + len] = t - w[j + len]$
 - $w[j + len] = zeta \cdot w[j + len]$
 - $start += 2 \cdot len$
 - $len = len/2$
 - $f = 8347861$
 - $for(j = 0; j < 256; j++) w[j] = f \cdot w[j]$

Dilithium, unedited, motivation

- Preliminary lattice version is prover generates: $A \in \mathbb{Z}_q^{k \times l}$, $S_1 \in \mathbb{Z}_q^{l \times n}$, $S_2 \in \mathbb{Z}_q^{k \times n}$, with short coefficients and computes $t = AS_1 + S_2$. Public key is (A, t) . Private key is (S_1, S_2)
 1. Prover generates $y \in \mathbb{Z}_q^l$ with “small coefficients”. Sends commitment as Ay
 2. Verifiers sends challenge $c \in \mathbb{Z}_q^n$ with small coefficients
 3. Prover returns $z = y + S_1c$.
 4. Verifier checks coefficients of z are small and that $Az - tc \approx Ay$
- To avoid having z leak S_1 , signer applies rejection sampling to z .
- Dilithium
 1. Uses elements of $R_q = \frac{\mathbb{Z}_q[x]}{x^{256}+1}$ rather than \mathbb{Z}_q .
 2. Uses a seed, ρ , to generate A , compresses t by dropping low order bits.
 3. Signs a message representative, μ , which is a hash of the public key and the message
 4. Uses a rounded version of $w = Ay$, w_1 .
 5. Provides a hint, h , to help reconstruct w_1 from z

Dilithium parameters for security category 5

Parameter	Meaning	Value
q	modulus	8380417
d	# dropped bits from t	13
τ	# ± 1 s in $c(x)$	60
λ	Collision strength	256
γ_1	Coefficient range of y	2^{19}
γ_2	Low order rounding range	$\frac{q-1}{32}$
(k, l)	Dimensions of A	(8,7)
η	Private key range	2
$\beta = \tau \cdot \eta$		120
ω	Max # of 1's in hint	75

Dilithium, Keygen

- Keygen

1. $\xi := \mathbb{Z}_2^{256}$ (random)
2. $(\rho, \rho', K) := H(\xi, 1024)$, (256, 512, 256) bits respectively
3. $\hat{A} := \text{ExpandA}(\rho)$
4. $(s_1, s_2) := \text{ExpandS}(\rho')$
5. $t := NTT^{-1}(\hat{A} NTT(s_1)) + s_2$
6. $(t_1, t_0) := \text{Power2Round}(t, d)$
7. $pk := pkEncode(\rho, t_1)$
8. $tr := H(\text{BytesToBits}(pk), 512)$
9. $sk := skEncode(\rho, K, tr, s_1, s_2, t_0)$
10. return (pk, sk)

Dilithium, Sign

1. $(\rho, K, tr, s_1, s_2, t_0) := skdecode(sk)$
2. $\hat{s}_1 := NTT(s_1), \hat{s}_2 := NTT(s_2), \hat{t}_0 := NTT(t_0); \hat{A} := ExpandA(\rho)$
3. $\mu := H(tr || M, 512); rnd := \mathbb{Z}_2^{256}$
4. $\rho' := H(K || rnd || \mu, 512)$
5. $\kappa = 0$
6. **while**(1) {
 - a. $y = ExpandMask(\rho', \kappa)$
 - b. $w := NTT^{-1}(\hat{A} NTT(y)), w_1 := highbits(w, 2\gamma_2)$
 - c. $\tilde{c} := H(\mu || w_1 Encode(w_1), 2\lambda)$
 - d. $(\hat{c}_1, \hat{c}_2) := \text{first 256 and last } 256 - 2\lambda \text{ bits}$
 - e. $c := SampleBall(\hat{c}_1); \hat{c} = NTT(c)$
 - f. $cs_1 := NTT^{-1}(\hat{c} \hat{s}_1); cs_2 := NTT^{-1}(\hat{c} \hat{s}_2);$
 - g. $z := y + cs_1$
 - h. $r_0 := lowbits(w - cs_2)$
 - i. **if** $(||z||_\infty \geq \gamma_1 - \beta \text{ or } ||r_0||_\infty \geq \gamma_2 - \beta)$ **then continue**
 - j. $ct_0 := NTT^{-1}(\tilde{c} t_0); h := makehint(-ct_0, w - cs_2 + ct_0)$
 - k. **if** $(||ct_0||_\infty < \gamma_2 \text{ and } \# \text{ 1's in } h \leq \omega)$ **then break**
 - l. $\kappa += l$
9. $\sigma := sigEncode(\tilde{c}, z \bmod^\pm(q), h)$

Dilithium, Verify

- Verify

1. $(\rho, t_1) := pkdecode(pk)$
2. $(\tilde{c}, z, h) := sigdecode(\sigma)$
3. $\hat{A} := ExpandA(\rho)$
4. $tr := H(BytestoBits(pk), 512)$
5. $\mu := H(tr || M, 512)$
6. $(\tilde{c}_1, \tilde{c}_2) := \text{first 256 and last } 256 - 2\lambda \text{ bits}$
7. $c := SampleBall(\tilde{c}_1)$
8. $w'_{approx} := NTT^{-1}(\tilde{A} \cdot NTT(z) - NTT(c)NTT(t_1 2^d))$
9. $w'_1 := usehint(h, w'_{approx})$ // $w_{approx} = (Az - ct_1) \cdot 2^d$
10. $\tilde{c}' := H(\mu || w1Encode(w'_1, 2\lambda))$
11. return $\|z\|_\infty < \gamma_1 - \beta$ and $\tilde{c} == \tilde{c}'$ and $\# \text{ 1's in } h \leq \omega$

Kyber

- Kyber is a key encapsulation algorithm that uses a public key encryption algorithm similar to Dilithium in conjunction with an encapsulation mechanism (Fujisaki-Okamoto transform) which converts a conditionally secure encryption into a CCA safe encapsulation. Here are some definitions.
 - $PRF_{\eta}(s, b) = \text{shake256}(s || b, 64 \cdot \eta)$
 - $XOF(\rho, i, j) = \text{shake128}(\rho || i || j)$
 - $H(s) = \text{sha3}_{256}(s)$, $J(s) = \text{shake256}_{32}(s)$
 - $G(s) = \text{sha3}_{512}(s)$
 - NTT and NTT^{-1} are different for Kyber and Dilithium
- Fujisaki-Okamoto transform:
 - $\mathcal{E}_{pk}^{hy}(m) = \mathcal{E}_{pk}^{asym}(\sigma, H(\sigma, m)) || \mathcal{E}_{G(\sigma)}^{sym}(m)$
 - σ is random string, G, H are hash functions, $\mathcal{E}_{G(\sigma)}^{sym}$ is symmetric encryption with key $G(\sigma)$ and \mathcal{E}_{pk}^{asym} is original asymmetric encryption algorithm.

Useful definitions

- Parse: $\mathcal{B}^* \rightarrow R_q^n$
- Input: $B = b_0, b_1, \dots \in \mathcal{B}^*$
- Output: $\hat{a} \in R_q^n$,
 $i = 0; j = 0;$
 while $j < i$
 $d = b_i + 256 \cdot b_{i+1}$
 if $d < 19q$
 $\hat{a}_j = d$
 $j++$
 $i += 2$
 return $\hat{a}_0 + \hat{a}_1x + \dots + \hat{a}_{n-1}x^{n-1}$

Useful definitions

- $SamplePolyCBD(B, \eta)$ --- samples from (Central Binomial) distribution $D_\eta(R_q)$
 Output: $f \in R_q^{256}$
 $b := ByteToBits(B)$
 $for(i = 0; i < 256; i++)$
 $x = \sum_{j=0}^{\eta-1} b[2i\eta + j]; y = \sum_{j=0}^{\eta-1} b[2i\eta + \eta + j]$
 $f[i] := (x - y) \bmod(q)$
 $return f$
- $Sample(a_1, a_2, \dots, a_\eta, b_1, \dots, b_\eta) \leftarrow \{0,1\}^{2\eta}$, output $\sum_{i=1}^{\eta} (a_i - b_i)$
- For central binomial distribution with $N = 10000, p = \frac{1}{2}, \sigma = \sqrt{Np(1-p)}$,
- $P(4900 \leq n_1 \leq 5100) = \sum_{j=4900}^{5100} \binom{N}{j} p^j (1-p)^{N-j} \approx \Phi\left(\frac{5100-5000}{50}\right) - \Phi\left(\frac{4900-5000}{50}\right)$,
- Φ is CDF for normal distribution

Useful definitions

- $encode_d(x)$, x is an array of length 256, $m = 2^d$, $1 \leq d \leq 12$
 - for ($i = 0$; $i < 256$; $i++$)
 - $a = x[i]$
 - for ($j = 0$; $j < d$; $j++$)
 - $b[d \cdot i + j] = a \pmod{2}$
 - $a = \frac{a - b[d \cdot i + j]}{2}$
 - return bits-to-bytes(b)
- $decode_d(x)$, x is a byte array of length $32d$, $m = 2^d$, $1 \leq d \leq 12$
 - $b = \text{bytes-to-bits}(x)$
 - for ($i = 0$; $i < 256$; $i++$)
 - $out[i] = \sum_{j=0}^{d-1} b[i \cdot d + j] \cdot 2^j$
 - return out

Useful definitions

- *SampleNTT()* --- samples uniformly from T_q

$i := 0; j := 0$

while ($j < 256$)

$d_1 = b[i] + 256(b[i + 1] \bmod 16)$

$d_2 = b[i + 1]/16 + 16(b[i + 2]$

if ($d_1 < q$)

$\hat{a}[j] = d_1 ; j++$

if ($d_2 < q$ and $j < 256$

$\hat{a}[j] = d_2 ; j++$

$i += 3$

return \hat{a}

Useful definitions

- $compress(x, d, q) \dashv\dashv compress_d: \mathbb{Z}_q \rightarrow \mathbb{Z}_{2^d}, x \rightarrow \lceil \frac{2^d}{q} \cdot x \rceil$
 - $x \rightarrow \lceil \frac{2^d}{q} \cdot x \rceil$
- $decompress(y, d, q) \dashv\dashv decompress_d: \mathbb{Z}_{2^d} \rightarrow \mathbb{Z}_q, y \rightarrow \lceil \frac{q}{2^d} \cdot y \rceil$
 - $y \rightarrow \lceil \frac{q}{2^d} \cdot y \rceil$
- $compress(decompress(x, d, q), d, q) = x$
- $decompress(compress(y, d, q), d, q) = t, (t - y) \bmod^\pm(q) \leq \lceil \frac{q}{2^{d+1}} \rceil$

NTT for Kyber

- $NTT: R_p \rightarrow T_p, f \mapsto \hat{f}$
- For $f \in R_p$
 - $\hat{f} = \left(\sum_{i=0}^{127} f \left(\text{mod } x^2 - \zeta^{2\text{rev}_7(i)+1} \right) \right)$
- $NTT(f) = (\hat{f}_0, \hat{f}_1, \dots, \widehat{f_{255}})$
- For $\hat{h} = \hat{f} \cdot \hat{g}$,
 - $\hat{h}_{2i} + \hat{h}_{2i+1}x = (\hat{f}_{2i} + \hat{f}_{2i+1}x) \cdot (\hat{g}_{2i} + \hat{g}_{2i+1}x) \pmod{x^2 - \zeta^{2\text{rev}_7(i)+1}}$

NTT for Kyber

- $NTT(f)$
 - $\hat{f} = f; k = 1$
 - for($len = 128; len \geq 2; len = len/2$)
 - for($start = 128; start < 256; start += 2len$)
 - $zeta = \zeta^{bitrev(k)} \bmod(q); k++$
 - for($j = start; j < start + len; j++$)
 - $t = zeta \cdot \hat{f}[j + len] \bmod(q)$
 - $\hat{f}[j + len] = \hat{f}[j] - t \bmod(q)$
 - $\hat{f}[j] = \hat{f}[j] + t \bmod(q)$
 - return(\hat{f})

NTT for Kyber

- $NTT^{-1}(\hat{f})$
 - $f = \hat{f}; k = 127$
 - for($len = 2; len \leq 128 \leq len = 2 \cdot len$)
 - for($start = 0; start < 256; start += 2len$)
 - $zeta = \zeta^{bitrev(k)} \bmod(q); k -= 1$
 - for($j = start; j < start + len; j ++$)
 - $t = f[j] \bmod(q)$
 - $f[j] = f[j] + f[j + len] \bmod(q)$
 - $f[j + len] = zeta \cdot (f[j + len] - t) \bmod(q)$
 - return($f \cdot 3303 \bmod(q)$)

NTT for Kyber

- $\text{MultiplyNTT}(\hat{f}, \hat{g})$
- For($i = 0; i < 128; i++$)
 - $(\hat{h}_{2i}, \hat{h}_{2i+1}) \leftarrow \text{Basecasemultiply}(\hat{f}_{2i}, \hat{f}_{2i+1}, \hat{g}_{2i}, \hat{g}_{2i+1}, \zeta^{2 \cdot \text{bitrev}(i)+1})$
- $\text{Basecasemultiply}(a_0, a_1, b_0, b_1, \gamma)$
 - $c_0 \leftarrow a_0 \cdot b_0 + a_1 \cdot b_1 \cdot \gamma$
 - $c_1 \leftarrow a_0 \cdot b_1 + a_1 \cdot b_0$
 - return (c_0, c_1)

Kyber (simplified a little)

- Parameters: $(p = 3329, \zeta = 1753, R_p = \frac{\mathbb{Z}_p[x]}{x^{256}+1}, k = 4, \eta = 2), \hat{x} = NTT(x)$
- Make public key
 - $KeyGen_{PKE}$, generates a Dilithium-like key (see full version)
 - $\hat{t} = \hat{A}\hat{s} + \hat{e}$, A is generated from seed ρ . $A \in R_p^{k \times k}$, $s, e \in R_p^k$
- $Enc_{PKE}(m, r)$ [$r \in R_p^k$ is generated from CDB_{η_1} , e_1 is generated from CDB_{η_2}]
 - $\hat{r} = NTT(r)$
 - $u(x) = NTT^{-1}(\hat{A}^T \hat{r}) + e_1$
 - $\mu = decompress_1(decode_1(m)), v = NTT^{-1}(\hat{t}^T \cdot \hat{r}) + e_2 + \mu$
 - $c_1 = encode_{d_u}(compress_{d_u}(u)), c_2 = encode_{d_v}(compress_{d_v}(r))$
 - return (c_1, c_2)
- $Dec_{PKE}(c_1, c_2)$
 - $w = v - NTT^{-1}(\hat{s} \cdot NTT(u))$
 - return $encode_1(compress_1(w))$

Kyber simplified a little

- *KEMKeygen*
 - $z = \mathbb{Z}_2^{256}(\text{random})$
 - $(ek_{PKE}, dk_{PKE}) = \text{KeyGen}_{PKE}()$
 - $ek_{KEM} = ek_{PKE}; dk_{KEM} = dk_{PKE} || e_{PKE} || H(e_{PKE}) || z$
 - return (ek_{KEM}, dk_{KEM})
- *KEMencaps*(pk_{KEM})
 1. m is a random 32-byte value
 2. $(K, r) = \text{SHA3}_{512}(m || H(e_{PKE}))$
 3. $c = \text{Enc}_{PKE}(ek, m, r)$
 4. return (K, c)
- *KEMdecaps*(sk_{KEM})
 1. $m' = \text{Dec}_{PKE}(dk, c)$
 2. $(K', r') = \text{SHA3}_{512}(m' || H(e_{PKE}))$
 3. $\bar{K} = \text{SHAKE256}(z || c, 32)$
 4. $c' = \text{Enc}_{PKE}(e_{PKE}, m', r')$
 5. If $(c == c')$ return K' else error

Kyber parameters

Alg	n	q	k	η_1	η_2	d_u	d_v	Strength
KEM-512	256	3329	2	3	2	10	768	128
KEM-768	256	3329	3	2	2	10	1088	192
KEM-1024	256	3329	4	2	2	11	1568	256

ML-KEM-1024 is security category 5

Type	Encap-key	Decap-key	Ciphertext	Key
KEM-512	800	1632	768	32
KEM-768	1184	2400	1088	32
KEM-1024	1568	3168	1568	32

Size in bytes

Full Kyber

- $KeyGen_{PKE}$
 1. $d = \mathbb{Z}_2^{256}, \text{random}$
 2. $(\rho, \sigma) = G(d); N = 0$
 3. $for(i = 0; i < k; i++)$
 - $for(j = 0; j < k; j++)$
 - $\hat{A}[i, j] = SampleNTT(XOF(\rho, i, j))$
 4. $for(i = 0; i < k; i++)$
 - $s[i] = SamplePolyCBD(PRF_{\eta_1}(\sigma, N)); N++$
 5. $for(i = 0; i < k; i++)$
 - $e[i] = SamplePolyCDB(PRF_{\eta_1}(\sigma, N)); N++$
 6. $\hat{s} = NTT(s); \hat{e} = NTT(e)$
 7. $\hat{t} = \hat{A}\hat{s} + \hat{e}$
 8. $ek_{PKE} = ByteEncode_{12}(\hat{t}) || \rho; dk_{PKE} = ByteEncode_{12}(\hat{s})$
 9. $return(e_{PKE}, d_{PKE})$

Full Kyber

$Enc_{PKE}(m, r)$

$N = 0; \hat{t} = ByteDecode_{12}(ek_{PKE}[0:384k]); \rho = ek_{PKE}[384k + 384k + 32]$

$for(i = 0; i < k; i++)$

$for(j = 0; j < k; j++)$

$\hat{A}[i, j] = SampleNTT(XOF(\rho, i, j))$

$for(i = 0; i < k; i++)$

$r[i] = SamplePolyCBD_{\eta_1}(PRF_{\eta_2}(r, N)); N++$

$for(i = 0; i < k; i++)$

$e_1[i] = SamplePolyCBD_{\eta_2}(PRF_{\eta_2}(r, N)); N++$

$e_2 = SamplePolyCBD_{\eta_2}(PRF_{\eta_2}(r, N))$

$\hat{r} = NTT(r)$

$u(x) = NTT^{-1}(\hat{A}^T \hat{r}) + e_1$

$\mu = decompress_1(decode_1(m)), v = NTT^{-1}(\hat{t}^T \cdot \hat{r}) + e_2 + \mu$

$c_1 = encode_{d_u}(compress_{d_u}(u)), c_2 = encode_{d_v}(compress_{d_v}(r))$

return (c_1, c_2)

Full Kyber

- $Dec_{PKE}(c_1, c_2)$
 1. $c_1 = c[0:32d_u k]; c_2 = c[32(d_u k + d_v)]$
 2. $\mathbf{u} = decompress_{d_u}(ByteDecode_{d_u}(c_1))$
 3. $v = decompress_{d_v}(ByteDecode_{d_v}(c_2))$
 4. $\hat{s} = ByteDecode_{12}(d_{PKE})$
 5. $w = v - NTT^{-1}(\hat{s}^T \cdot NTT(\mathbf{u}))$
 6. $m = ByteEncode_1(compress_1(w))$
 7. return m

Full Kyber

- $Keygen_{KEM}$
 - $z = \mathbb{Z}_2^{256}$ (random)
 - $(ek_{PKE}, dk_{PKE}) = KeyGen_{PKE}()$
 - $ek_{KEM} = ek_{PKE}; dk_{KEM} = dk_{PKE} || ek_{PKE} || H(ek_{PKE}) || z$
 - return (ek_{KEM}, dk_{KEM})
 - $\hat{t} = \hat{A}\hat{s} + \hat{e}$, A is generated from seed ρ
 - return (ek_{PKE}, dk_{PKE})

Full Kyber

- $KEMencaps(pk_{KEM})$
 1. m is a random 32-byte value
 2. $(K, r) = G(m || H(ek))$
 3. $c = Enc_{PKE}(ek, m, r)$
 4. return (K, c)

Full Kyber

- $KEMdecaps(c, dk)$
 1. $dk_{PKE} = dk[0:384k]$
 2. $ek_{PKE} = dk[384k:768k + 32]$
 3. $h = dk[768k + 32:768k + 64]$
 4. $z = dk[768k + 64:768k + 96]$
 5. $m' = Dec_{PKE}(dk, c)$
 6. $(K', r') = G(m' || H(e_k))$
 7. $\bar{K} = J(z || c, 32)$
 8. $c' = Enc_{PKE}(ek, m', r')$
 9. If $(c == c')$ return K' else error

Kyber Notes

- Define $\mathbf{Adv}_{m,k,\eta}^{mlwe} =$
 $|\Pr[b' = 1, \mathbf{A} \leftarrow R_q^{m \times k}; (\mathbf{s}, \mathbf{e}) \leftarrow \beta_\eta^k \times \beta_\eta^m; \mathbf{b} = \mathbf{A}\mathbf{s} + \mathbf{e}; b' = A(\mathbf{A}, \mathbf{b})] -$
 $\Pr[b' = 1, \mathbf{A} \leftarrow R_q^{m \times k}; \mathbf{b} \leftarrow R_q^m; b' = A(\mathbf{A}, \mathbf{b})]|.$
- **Theorem:** Suppose XOF and G are random oracles. For all adversaries, A, there are adversaries, B, C: $\mathbf{Adv}_{kyber,CPAPKE}^{cpa}(\mathbf{A}) \leq 2 \mathbf{Adv}_{k+1,k,\eta}^{mlwe}(\mathbf{B}) + \mathbf{Adv}_{PRF}^{prf}(\mathbf{C})$
- **Theorem:** Suppose XOF and G are random oracles. For any classical adversary, A, that make at most q_{RO} to random oracles XOF, H, G there are adversaries B, C of the same running time: $\mathbf{Adv}_{kyber,CCAKEM}^{cca}(\mathbf{A}) \leq 2 \mathbf{Adv}_{k+1,k,\eta}^{mlwe}(\mathbf{B}) + \mathbf{Adv}_{PRF}^{prf}(\mathbf{C}) + 4\delta q_{RO}$
- **Theorem:** Suppose XOF and G are random oracles. For any quantum adversary, A, that make at most q_{RO} to random oracles XOF, H, G there are adversaries B, C of the same running time: $\mathbf{Adv}_{kyber,CCAKEM}^{cca}(\mathbf{A}) \leq 4q_{RO} \sqrt{\mathbf{Adv}_{k+1,k,\eta}^{mlwe}(\mathbf{B}) + \mathbf{Adv}_{PRF}^{prf}(\mathbf{C}) + 8\delta q_{RO}^2}$

Kyber Parameters

Alg	Failure rate	Alg	Failure rate	Alg	Failure rate
KEM-512	2^{-139}	KEM-768	2^{-164}	KEM-1024	2^{-174}

Failure rates

Attacks

The Blum-Kalai-Wasserman (BKW) algorithm is a combinatorial algorithm used to solve the Learning With Errors (LWE).

The attack typically involves two main phases:

Reduction Phase: This phase progressively reduces the dimension of the LWE/LWR problem, essentially trying to simplify the equations involved. This is achieved by combining samples (vectors with associated 'noise' or errors) in a way that eliminates certain positions in the vectors, albeit at the cost of increasing the noise in the remaining positions.

Solving Phase: Once the problem is reduced to a manageable size, the remaining entries of the secret are recovered. This often involves techniques like hypothesis testing to distinguish the correct guess of the secret sub-vector from incorrect ones.

End

LLL Theorem

- Let L be the n -dimensional lattice generated by $\langle v_1, \dots, v_n \rangle$ and $|$ the length of the shortest vector in L . The LLL algorithm produces a reduced basis $\langle b_1, \dots, b_n \rangle$ of L .
 1. $||b_1|| \leq 2^{(n-1)/4} D^{1/n}$.
 2. $||b_1|| \leq 2^{(n-1)/2} |$.
 3. $||b_1|| ||b_2|| \dots ||b_n|| \leq 2^{n(n-1)/4} D$.
- If $||b_i||^2 \leq C$ algorithm takes $O(n^4 \lg(C))$.

Gauss again

- Let $\langle v_1, v_2 \rangle$ be a basis for a two-dimensional lattice L in \mathbb{R}^2 . The following algorithm produces a reduced basis.

```
for(;;) {  
    if(||v1|| > ||v2||)  
        swap v1 and v2;  
    t = [(v1, v2) / (v1, v1)]; // [] is the "closest  
        integer" function  
    if(t == 0)  
        return;  
    v2 = v2 - tv1;  
}
```

- $\langle v_1, v_2 \rangle$ is now a reduced basis and v_1 is a shortest vector in the lattice.