Quantum Computing

A brief introduction

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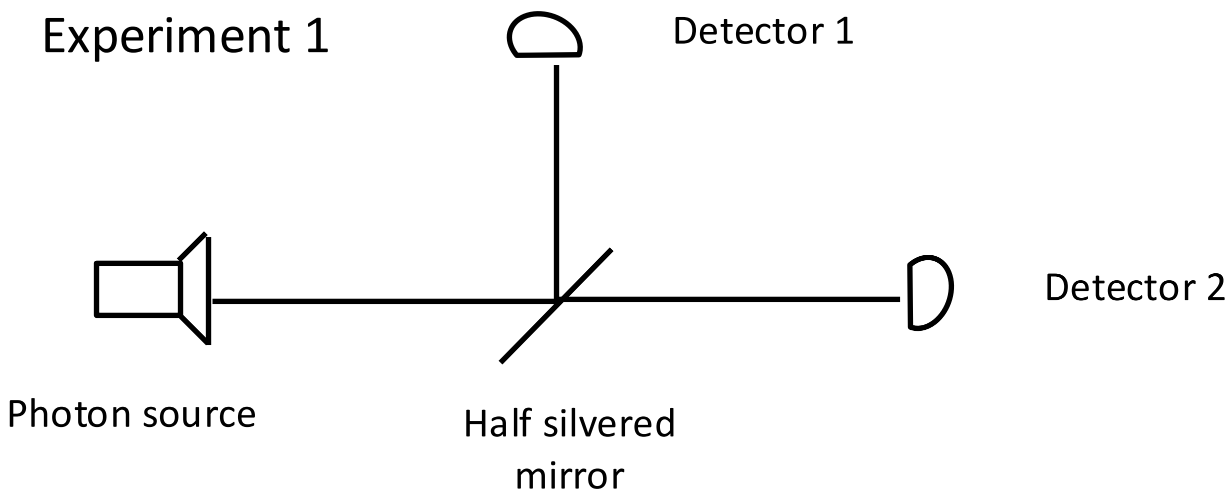
Beam splitters and QM

I can safely say that no one understands Quantum Mechanics - Feynman

Experiment 1



Photon source emits stream of photons.

P(photon arrives at Detector 1)= .5

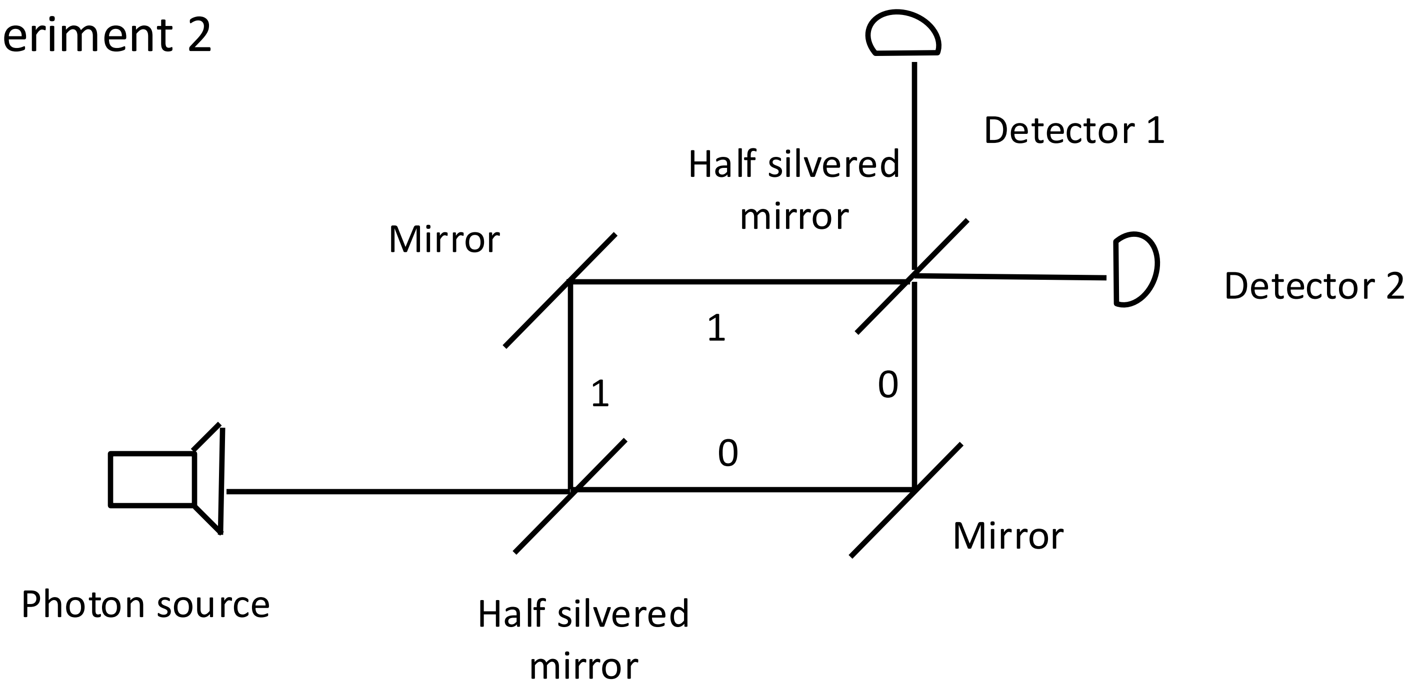
P(photon arrives at Detector 2)= .5

So far, so good

Beam splitters and QM

Mach-Zender Interferometer

Experiment 2



Photon source emits stream of photons.

P(photon arrives at Detector 1)= 0

P(photon arrives at Detector 2)= 1

Huh?

According to QM

Analysis

Beam splitter causes the photon to go into superposition:

$\alpha_1|0> + \alpha_2|1>$, $|\alpha_1|^2 = \frac{1}{2}$, $|\alpha_2|^2 = \frac{1}{2}$. $|0>$ state is right, $|1>$ is up.

$|0> = \binom{1}{0}$, $|1> = \binom{0}{1}$.

Beam splitter acts on incoming state via the matrix $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i \\ i & 1\end{pmatrix}$.

In experiment 1, if all photons leave source in state $\binom{1}{0}$, after the splitter they are in state $\frac{1}{\sqrt{2}}\begin{pmatrix}1 \\ i\end{pmatrix}$. So, they arrive at detector 1 with probability $\frac{1}{2}$ and detector 2 with probability $\frac{1}{2}$.

However, going through another beam splitter, in experiment 2, yields the output state:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i \\ i & 1\end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i \\ i & 1\end{pmatrix} \binom{1}{0} = \begin{pmatrix}0 & i \\ i & 0\end{pmatrix}\begin{pmatrix}1 \\ 0\end{pmatrix} = \begin{pmatrix}0 \\ i\end{pmatrix}.$$

So, they always arrive at detector 2.

Postulates

1. State of a system is a unit vector over $\mathbb{C}$ in Hilbert space ($\mathcal{H}$) of dimension 2^n
   – A qubit is a quantum system, with $n = 1$. A one qubit system is in general state $|\psi >= a|0> +b|1>$, $a\bar{a} + b\bar{b} = 1$
2. A system, with state, $|\psi(t)>$, evolves according to a unitary operator, namely, $U(|\psi(0)>)$
   – U is unitary if $(x, y) = (Ux, Uy)$. Note $U\overline{U^T} = I$
   – Example is a Hamiltonian: $H(t)|\psi(t)> = \mathrm{i}\hbar \frac{d|\psi(t)>|}{dt}$
   – $|\varphi(t_2)> = e^{-i\hbar H(t_2-t_1)}|\varphi(t_1)>$
3. Each observable is represented by a Hermitian operator, $\hat{Q}$, the expectation value of $\hat{Q}$ is $<\psi|\hat{Q}\psi>$. The outcome of a measurement of the operator is an eigenvalue of the operator. The probability of getting a particular eigenvalue, λ, is the square of the λ-component of $|\psi>$

Postulates

4. Two physical systems $\mathcal{H}_1$ and $\mathcal{H}_2$ can be treated as a single system, $\mathcal{H}_1 \otimes \mathcal{H}_2$. If $\mathcal{H}_1$ is in state, $|\psi_1>$ and $\mathcal{H}_2$ is in state, $|\psi_2>$, the joint state is $|\psi_1> \otimes |\psi_2>$
5. Given an orthonormal basis $\mathcal{B} = \{\varphi_i\}$, one can perform a von-Neuman measurement $\mathcal{H}_A$ on $|\psi> = \sum_i \alpha_i|\varphi_i>$ that outputs i with probability $|\alpha_i|^2$. It is projective. Further, if $|\psi> = \sum_i \alpha_i|\varphi_i>|\gamma_i>$ $\mathcal{H}_A \otimes \mathcal{H}_B$ measurement yields i with probability $|\alpha_i|^2$ and leaves state in $|\varphi_i>|\gamma_i>$. $M = \sum m_i P_i = \sum m_i \,|i><i|$

Linear Algebra

- Dirac Notation: Element in Hilbert space of dimension 2^n is represented by n-entry symbol. $|000\ldots00> \leftrightarrow (1,0,\ldots,0)^T$, $|000\ldots01> \leftrightarrow (0,1,\ldots,0)^T$, …, $|111\ldots1> \leftrightarrow (0,0,\ldots,1)^T$ where column vectors have 2^n coordinates.
- Notation: $|0> \otimes |0> \otimes \ldots \otimes |0> = |000\ldots0>$
- A is normal if $A\bar{A}^T = \bar{A}^T A$
- Spectral Theorem: If T is a normal operator in the Hilbert space $\mathcal{H}$, there is an orthonormal basis v_i; each is an eigenvector of T. For every such , there is a unitary matrix, P, $T = P\Lambda P^*$, and Λ is diagonal.
- Dual basis
- Inner product: $(v_1, v_2, \ldots, v_n) \cdot (w_1, w_2, \ldots, w_n) = \sum_{i=0}^{n} \bar{v_i}\, w_i$
- Outer product: $(|\psi><\phi|)|\gamma> = |\psi>(<\phi|\gamma>)$
- Theorem: Every linear operator can be written as $T = T_{m,n}|b_m><b_n|$,
- $T_{m,n} = <b_m|T|b_n>$

Linear Algebra (continued)

<u>Tensor product</u>: If $|\varphi_i >= \binom{\alpha_0}{\alpha_1}$ is a basis for $\mathcal{H}_1$ and $|\phi_i >= \binom{\beta_0}{\beta_1}$is a basis for $\mathcal{H}_2$,

$|\varphi_i > \otimes |\phi_i >$ is a basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$. $\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} \otimes \begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix} = (\alpha_0\beta_0, \alpha_0\beta_1, \alpha_1\beta_0, \alpha_1\beta_1)^T$.

$$A \otimes B = \begin{matrix} a_{11}B & \dots & a_{1n}B \\ \dots & \dots & \dots \\ a_{n1}B & \dots & a_{nn}B \end{matrix}$$

<u>Schmidt decomposition</u>: If $|\psi > \in \mathcal{H}_1 \otimes \mathcal{H}_2$, there is an orthonormal basis $|\varphi_i >$ for $\mathcal{H}_1$ and an orthonormal basis $|\phi_i >$ for $\mathcal{H}_2$ and $p_i \geq 0$ such that $|\psi > = \sum_i \sqrt{p_i}\, |\varphi_i > |\phi_i >$

$Tr(A) = < b_n|A|b_n >$

<u>Eigenvector</u>: $T|\psi > = c|\psi >$

More notation

- $A \otimes B = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}$, $x \otimes \mathrm{y} = (x_1y_1, x_1y_2 , \dots , x_ny_n)^T$
- $|v> = (\ v_1,\ \ v_2,\ \dots ,\ \ v_n\ )^T$, $< w| = (\ w_1,\ \ w_2,\ \dots ,\ \ w_n\ )$ then

$|v >< w| = \begin{matrix} v_1\overline{w_1} & v_1\overline{w_2} & \dots & v_1\overline{w_n} \\ \dots & \dots & \dots & \dots \\ v_n\overline{w_1} & v_n\overline{w_2} & \dots & v_n\overline{w_n} \end{matrix}$, so $I = \sum |\mathrm{i} >< i|$ and $M = \sum M_{ij} |\mathrm{i} >< j|$

- Pauli matricies
  - $\sigma_0 = I$, $\sigma_1 = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_2 = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $\sigma_3 = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
  - $[X, Y] = iZ$, $[Y, Z] = iX$, $[Z, X] = iY$
- $| +> = \frac{1}{\sqrt{2}}(|0 >\ + |1 >)$, $| -> = \frac{1}{\sqrt{2}}(|0 > -\ |1 >)$, $|i > = \frac{1}{\sqrt{2}}(|0 > +i|1 >)$, $| -$ $i > = \frac{1}{\sqrt{2}}(|0 > -\ i|1 >)$

Mixed states and density

- For pure states, $|\psi >$, density is $\rho = |\psi >< \psi|$
- Mixed states: $\{(p_1, |\psi_1 >), (p_2, |\psi_2 >), \dots, (p_n, |\psi_n >)\}$, where the probability that the system is in pure state $|\psi_i >$ is p_i and $\sum p_i = 1$
- Density operator for mixed state is $\sum p_i |\psi_i >< \psi_i|$
- Bloch Sphere
  - Pure state in general position is $|\psi > = \cos\left(\frac{\theta}{2}\right) \Big|0 > + e^{i\varphi} \sin\left(\frac{\theta}{2}\right)\Big| 1 >$.
  - For mixed state $|\psi_i > = p_i(\alpha_{X,i}, \alpha_{Y,i}, \alpha_{Z,i})$ on interior of Block sphere
  - $\rho = \sum p_i |\psi_i > < \psi_i|$ evolves as $\rho = \sum p_i U|\psi_i > < \psi_i|U^\dagger$
  - $\rho = \frac{1}{2} I + \alpha_X X + \alpha_Y Y + \alpha_Z Z$

- $P(|0 >) = < 0|\psi > < \psi|0 > = Tr < 0|\psi > < \psi|0) = Tr(|0 >< 0| \, |\psi >< \psi|$

Mixed states and density

- Partial trace: Consider composite system AB.
  - $\rho^A = Tr_B(\rho^{AB})$
  - $Tr_B(|a_1 >< a_2| \otimes < b_1|) < b_2|) = |a_1 >< a_2| Tr(|b_1 >< b_2|) = |a_1 >< a_2| < b_2|b_1 >$
  - Example
    - $\rho = \frac{1}{2}(|00 >< 00| + |00 >< 11| + |11 >< 00| + |11 >< 11|)$
      $= \frac{1}{2} Tr(|0 >< 0| \otimes |0 >< 0| + |0 >< 1| \otimes |0 >< 1| + |1 >< 0| \otimes |1 >< 0| + |1 >< 1| \otimes |1 >< 1|)$
      $= \frac{1}{2}(|0 >< 0| + |1 >< 1|)$

Circuits and gates

- Universal gate set: A gate set is universal if $\forall n > 0$, any n-bit unitary operator can be approximated to arbitrary accuracy by a quantum circuit from this set
- An entangling gate is on that for an input product state $|\alpha > |\beta >$, the output state is not a product state (e.g.-CNOT).
  - Example: $|\psi >= \frac{1}{\sqrt{2}}(|00 > \ +|11 >)$
- Theorem: A set of states with an entangling 2-qubit gate together with all 1-qubit gates is universal.
- Theorem: If U is a 1-qubit gate, $U = e^{ix} R_z(\beta) R_y(\gamma) R_z(\delta)$

Gates and states

- General position on Bloch sphere: $|\psi> = \cos\left(\frac{\theta}{2}\right)\Big|0> \; + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)\Big|\,1>$
- Measurement: $I = \sum|i >< i|,\; M = \sum m_i P_i$, M is Hermitian, $P_i = |i >< i|$.
- Controlled gates:
  - $c - U|0>|\psi> = \;|0> \;|\psi>$
  - $c - U|1>|\psi> = \;|1> U|\psi>$

Common gates

- Pauli gates

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\mathrm{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Note: $X^2 = Y^2 = Z^2 = I$

- Hadamard

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

$$H^{\otimes n}(|0000 \ldots 0>) = \frac{1}{\sqrt{2}}(|0> + |1>) \otimes \frac{1}{\sqrt{2}}(|0> + |1>) \otimes \ldots \otimes \frac{1}{\sqrt{2}}(|0> + |1>)$$

- Rotation

$$R_X(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{iX\theta} \end{pmatrix} = \begin{pmatrix} e^{-iX\theta/2} & 0 \\ 0 & e^{iX\theta/2} \end{pmatrix}$$

- 2 qubit gate

$CNOT(|xy>) = |x, x \oplus y>$

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

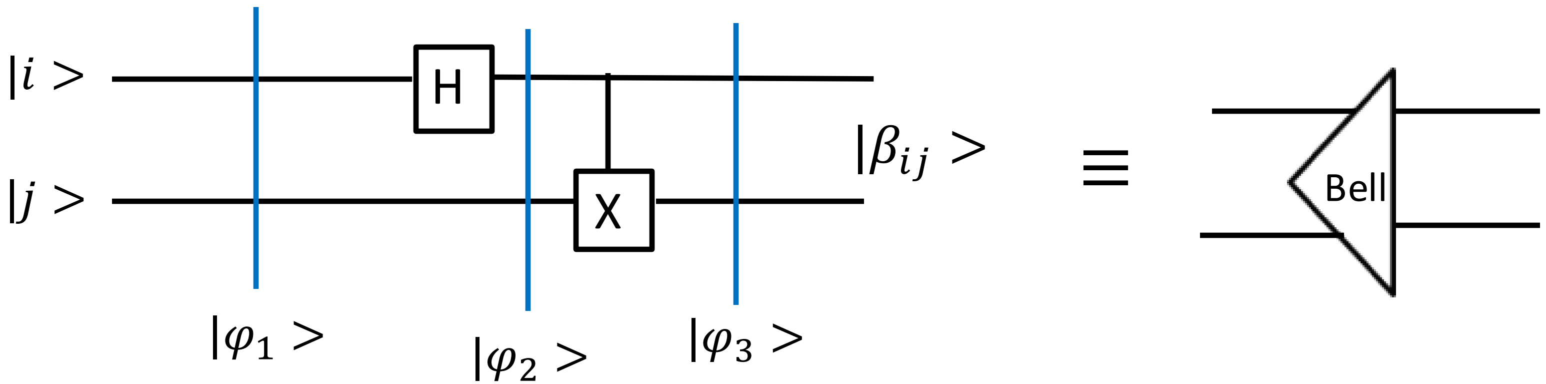
- If $A^2 = 1$, $e^{i\theta X} = \mathrm{I}\cos(\theta) + iXsin(\theta)$

Measurement in alternate basis

- Computational basis is $|i>$. $U|\varphi_j>=|j>$
- Suppose we want to measure $|\psi>$ with respect to basis $B=\{|\varphi_j>\}$
- $|\psi>= \sum \alpha_j\,|\varphi_j>$
- To measure wrt $B=\{|\varphi_j>\}$, Project $|\psi>$ onto $|\varphi_j><\varphi_j|$
- $\left(Tr\left(|\psi><\psi||\varphi_j><\varphi_j|\right)=Tr\left(<\varphi_j|\psi><\psi|\varphi_j>\right)= \alpha_j{}^2\right.$
- $\rho=|\psi><\psi|$ is density operator for the pure state $|\psi>$.
- $\rho= \sum p_i\,|\psi_i><\psi_i|$ is the density operator for mixed states $\{(p_i,|\psi_i>)$

Converting to Bell Basis

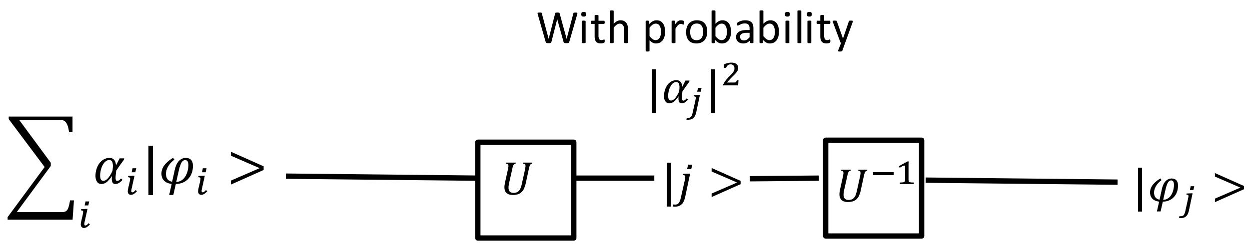
- Computational basis is $|i>$, $U|\varphi_j> = |j>$
- $|\beta_{00}> = \frac{1}{\sqrt{2}}(|00> + |11>)$, $|\beta_{01}> = \frac{1}{\sqrt{2}}(|01> + |10>)$
- $|\beta_{10}> = \frac{1}{\sqrt{2}}(|00> - |11>)$, $|\beta_{11}> = \frac{1}{\sqrt{2}}(|01> - |10>)$

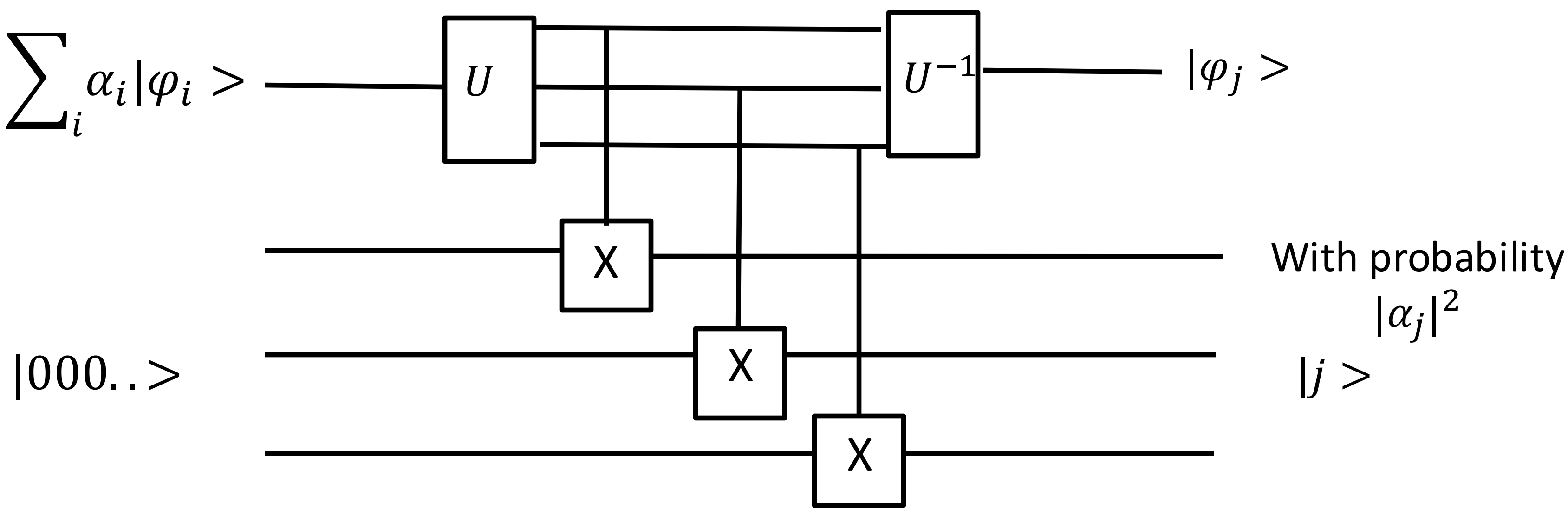


- $|\varphi_1> = |00>$
- $|\varphi_2> = \frac{1}{\sqrt{2}}(|0> +|1>)|0> = \frac{1}{\sqrt{2}}(|00> +|10>)|$
- $|\varphi_3> = |\beta_{00}>$

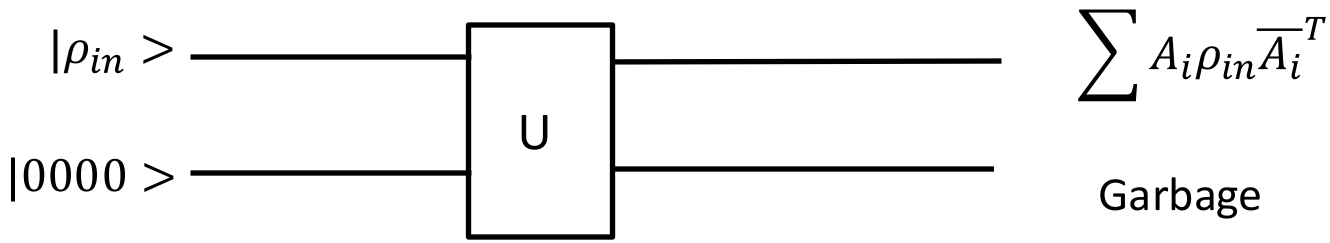
Changing Measurement Basis

- Suppose $|\varphi_i>$ is a basis and our measurement basis is $|i>$, $U|\varphi_i> = |i>$





Superoperator and mixed states



- $\rho = |\psi><\psi|$, $U|\psi>$ has density ρ = $U|\psi><\psi|\overline{U}^T> = U\rho U^\dagger$
- $<0|\psi><\psi|0> = <0|\rho|0> = P(|0>)$
- $\rho = \sum_i p_i |\psi_i><\psi_i|$
- $Tr(A) = <b_n|A|b_n>$
- $\rho_{in} \rightarrow \rho_{out} = Tr_b(U(\rho_{in} \otimes |000\ldots><000\ldots0|U^\dagger)$
- $\rho_{in} \rightarrow \sum A_i \rho_{in} A_i{}^\dagger$, where A_i are Kraus operators with $\sum A_i{}^\dagger A_i = I$

No Cloning Theorem

- Qubits can't be copied

- Proof

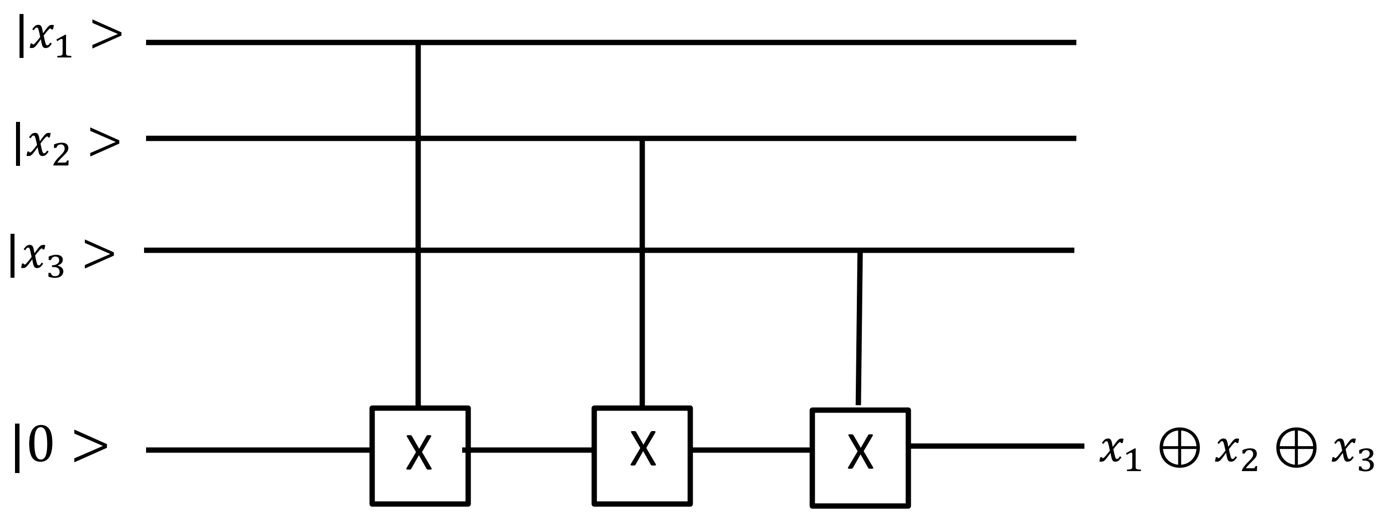
  Suppose they can be. Then there is an operator, U, such that for any state $|\varphi>$, $U(|\varphi>|0>) = |\varphi>|\varphi>$. Now let $|\psi>$ and $|\phi>$ be non-orthogonal, different pure states.

  $(|\psi>|0>, |\phi>|0>) = <\psi|\phi><0|0> = <\psi|\phi>$.

  Since U is unitary,

  $<\psi|\phi> = (|\psi>|0>, |\phi>|0>) = (U|\psi>|0>, U|\phi>|0>) = (|\psi>|\psi>, |\phi>|\phi>) = <\psi|\phi>^2$. So, $<\psi|\phi> = 1$. This is a contradiction.
- No checkpointing

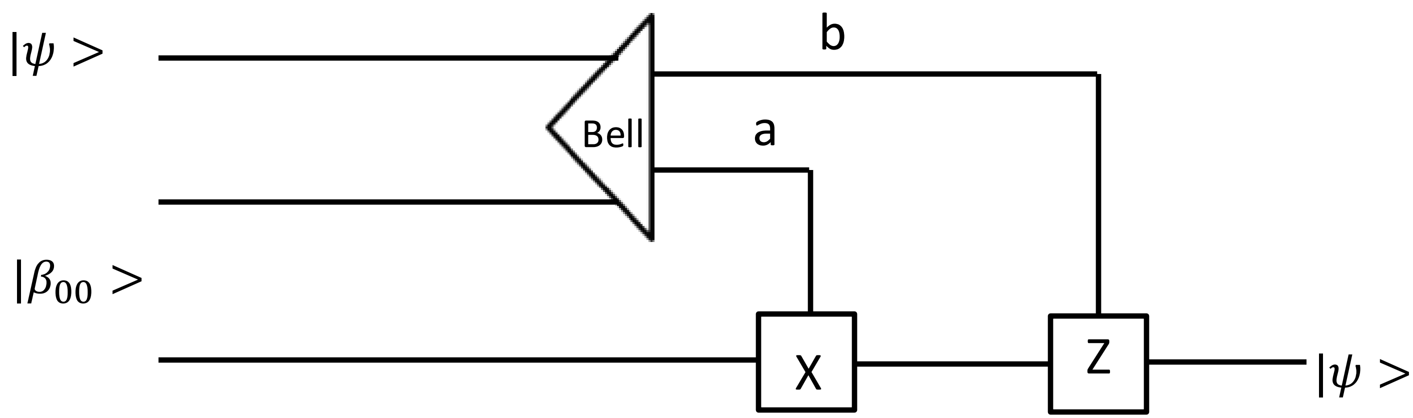
Parity Circuit



Superdense coding

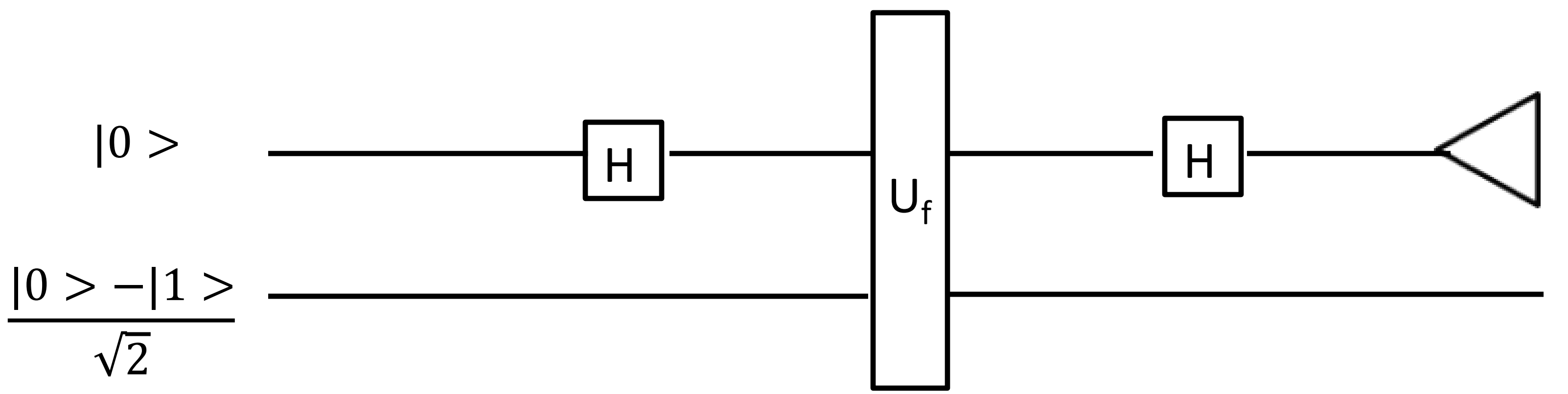
- Alice and Bob share $|\beta_{00}>$, Alice has first bit, Bob second bit
- Alice performs one of I, X, Y, Z producing $I \otimes I$ (to send 00), $X \otimes I$ (to send 01), $Y \otimes I$ (to send 10) or $Z \otimes I$ (to send 11).
- Bob measures joint state qubit measurement

- Can be used to teleport $|\psi>$:
  - $I \otimes I := \frac{1}{\sqrt{2}}(|00> + |11>) \rightarrow \frac{1}{\sqrt{2}}(|00> + |11>)$
  - $X \otimes I := \frac{1}{\sqrt{2}}(|00> + |11>) \rightarrow \frac{1}{\sqrt{2}}(|01> + |10>)$
  - $Z \otimes I := \frac{1}{\sqrt{2}}(|00> + |11>) \rightarrow \frac{1}{\sqrt{2}}(|00> \ - |11>)$
  - $ZX \otimes I := \frac{1}{\sqrt{2}}(|00> + |11>) \rightarrow \frac{1}{\sqrt{2}}(|01> \ - |10>)$



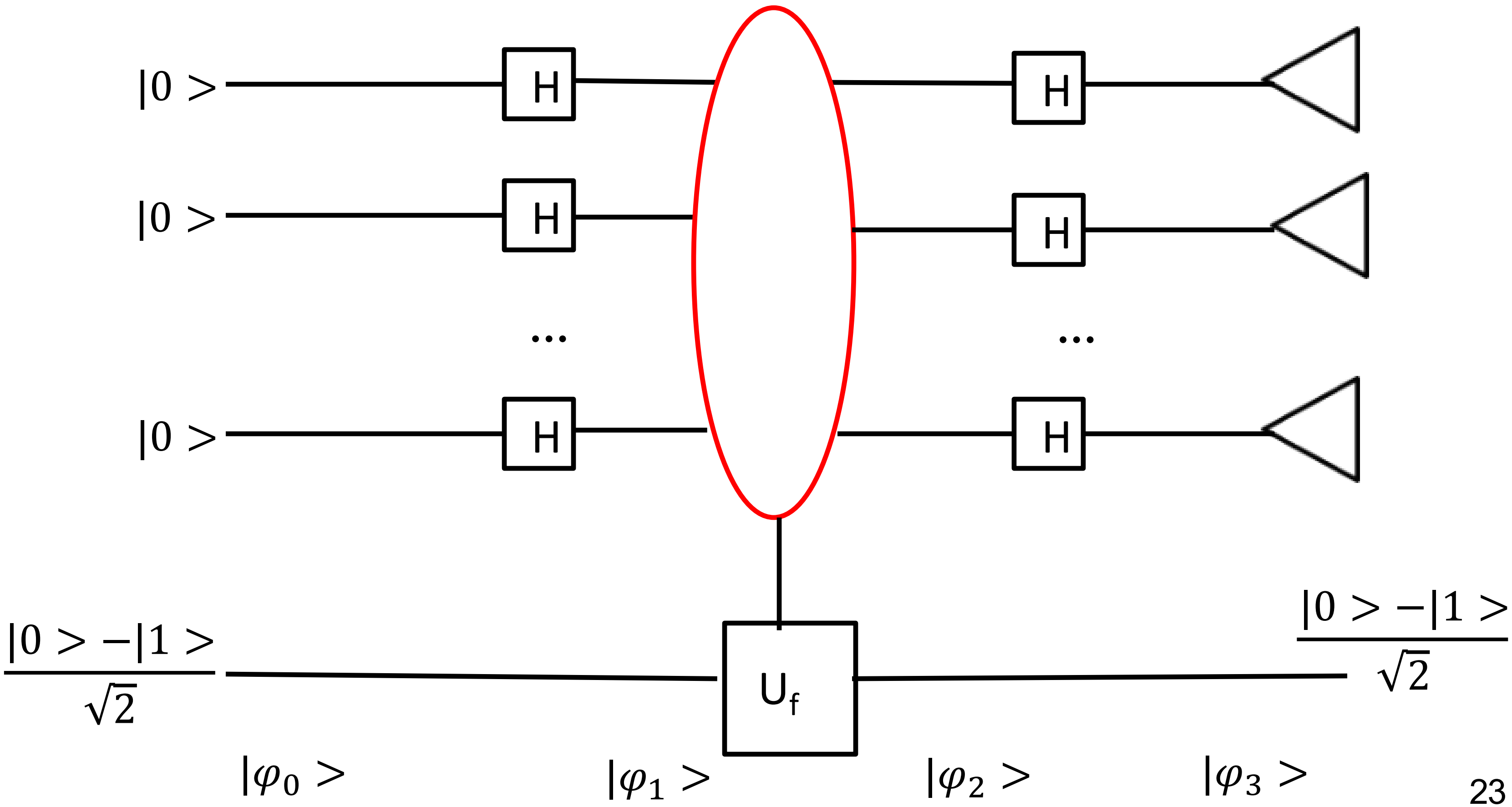
Deutch

- Problem: Determine $f(0) + f(1)$ in one measurement
- $U_f|x > |y >= |x > |y \oplus f(x) >$
- If $f(0) + f(1) = 1$, $\left|\psi_3 > = \ (-1)^{f(0)}\right|1 > \frac{|0>-|1>}{\sqrt{2}}$
- If $f(0) + f(1) = 0$, $\left|\psi_3 > = \ (-1)^{f(0)}\right|0 > \frac{|0>-|1>}{\sqrt{2}}$



Deutch-Josza

- Problem: $f:\{0,1\}^n \to \{0,1\}$, which is either constant or balanced.
- Which is it?
- Put $U_f|\boldsymbol{x}>|y> = \ |\boldsymbol{x}>|y \oplus f(\boldsymbol{x})>$, $\boldsymbol{x}$ is an n-bit quantity



DJ

- $|\varphi_0> = \ |0>^{\otimes n} \frac{|0> - |1>}{\sqrt{2}}$
- $|\varphi_1> = \frac{1}{\sqrt{2^n}} \sum_{\boldsymbol{x}} |\boldsymbol{x}> \frac{|0> - |1>}{\sqrt{2}}$
- $|\varphi_2> = \frac{1}{\sqrt{2^n}} \sum_{\boldsymbol{x}} (-1)^{f(\boldsymbol{x})} |\boldsymbol{x}> \frac{|0> - |1>}{\sqrt{2}}$
- $|\varphi_3> = \frac{1}{2^n} \sum_{\boldsymbol{x}} \sum_{\boldsymbol{z}} |(-1)^{f(\boldsymbol{x}) + \boldsymbol{x} \cdot \boldsymbol{z}} \boldsymbol{z}> \frac{|0> - |1>}{\sqrt{2}}$

Simon

- f: $\{0,1\}^n \rightarrow X, \exists\, \vec{s} = s_1, s_2, \dots, s_n$: $f(x) = f(y)$ iff $x = y$ or $x = y + \vec{s}$
- U_f: $|x>|b> = |x>|b \oplus f(x)>$
- $H^{\otimes n}(|x>) = \frac{1}{\sqrt{2^n}} \sum_z (-1)^{x \cdot z} |z>$

1. $i = 1$
2. Prepare $\frac{1}{\sqrt{2^n}} \sum_x |x>|0>$
3. Apply U_f to get $\frac{1}{\sqrt{2^n}} \sum_x |x>|f(x)>$
4. Measure second bit
5. Apply $H^{\otimes n}$ to first register
6. Measure first register to get w_i
7. If $din(w_i) \neq n - 1$, go to 2
8. Output s: $w^t s^t = 0$

Phase kick back

- $CNOT\left(\frac{|0> + |1>}{\sqrt{2}} \frac{|0> \ - |1>}{\sqrt{2}}\right) = \frac{|0> \ - |1>}{\sqrt{2}} \frac{|0> \ - |1>}{\sqrt{2}}$

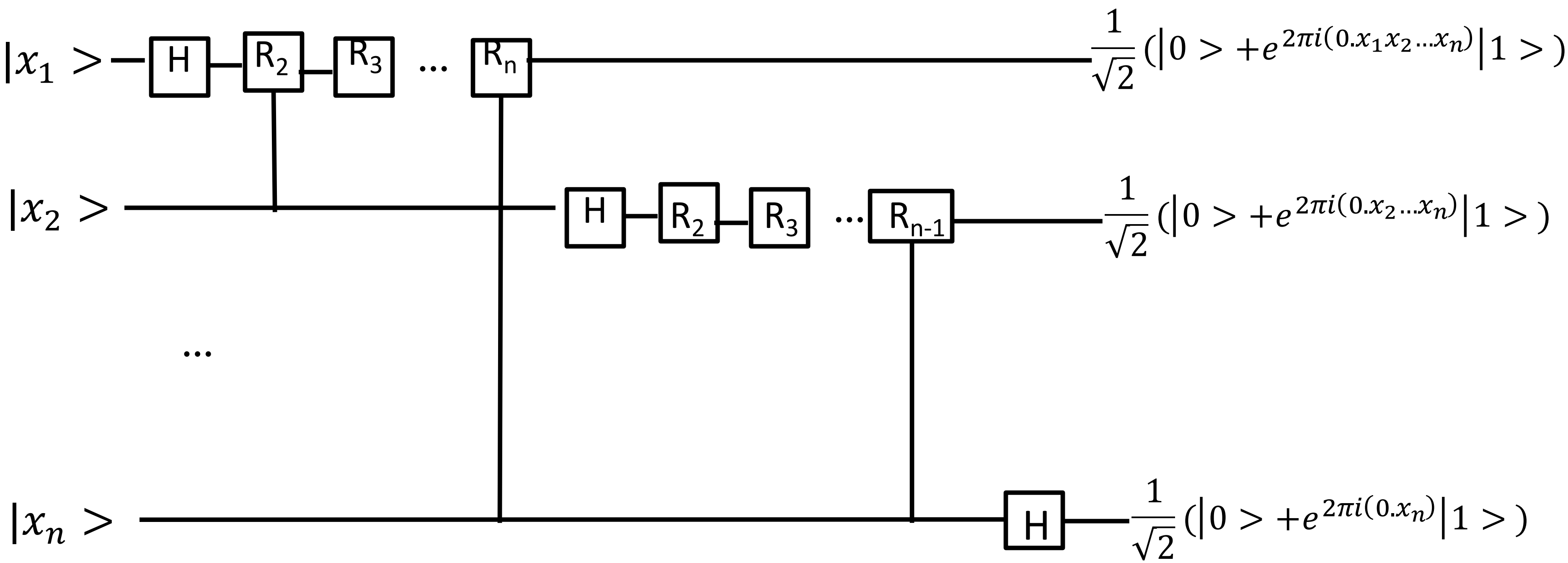
Phase Estimation

- Phase estimation problem: Given $|\psi> = \frac{1}{\sqrt{2^n}} \sum_y e^{2\pi i \omega y} |\text{y}>$, estimate ω
- <u>Theorem:</u> $\frac{x}{2^n} \leq \omega \leq \frac{x+1}{2^n}$ with probability $\geq \frac{8}{\pi^2}$
- $e^{2\pi i 2^k . x_1 x_2 \dots} = e^{2\pi i (x_{k+1} x_{k+2} \dots)}$
- Suppose $\omega = .x_1$, $|\psi> = \frac{1}{\sqrt{2}} \sum_{|y>} e^{2\pi i \omega |y>} = \frac{1}{\sqrt{2}} (|0> + (-1)^{x_1} |1>$ and $H(|\psi>) = |x_1>$
- In general, $H^{\otimes n}(|\boldsymbol{x}>) = \frac{1}{\sqrt{2^n}} \sum_{\boldsymbol{y}} (-1)^{\boldsymbol{x} \cdot \boldsymbol{y}} |\boldsymbol{y}>$ and $H^{\otimes n}(H^{\otimes n}(|\boldsymbol{x}>)) = |\boldsymbol{x}>$
- So, $|\psi> = \frac{1}{\sqrt{2^n}} \sum_{|y>} e^{2\pi i \omega y>} |y> = \frac{1}{\sqrt{2}} \left(\middle| 0> + e^{2\pi i 2^{n-1} \omega} \middle| 1> \right) \otimes \frac{1}{\sqrt{2}} \left(\middle| 0> + e^{2\pi i 2^{n-2} \omega} \middle| 1> \right) \otimes \cdots \otimes \frac{1}{\sqrt{2}} \left(\middle| 0> + e^{2\pi i \omega} \middle| 1> \right)$
- Denote $R_n = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i 2^{-n}} \end{pmatrix}$

Quantum Fourier Transform

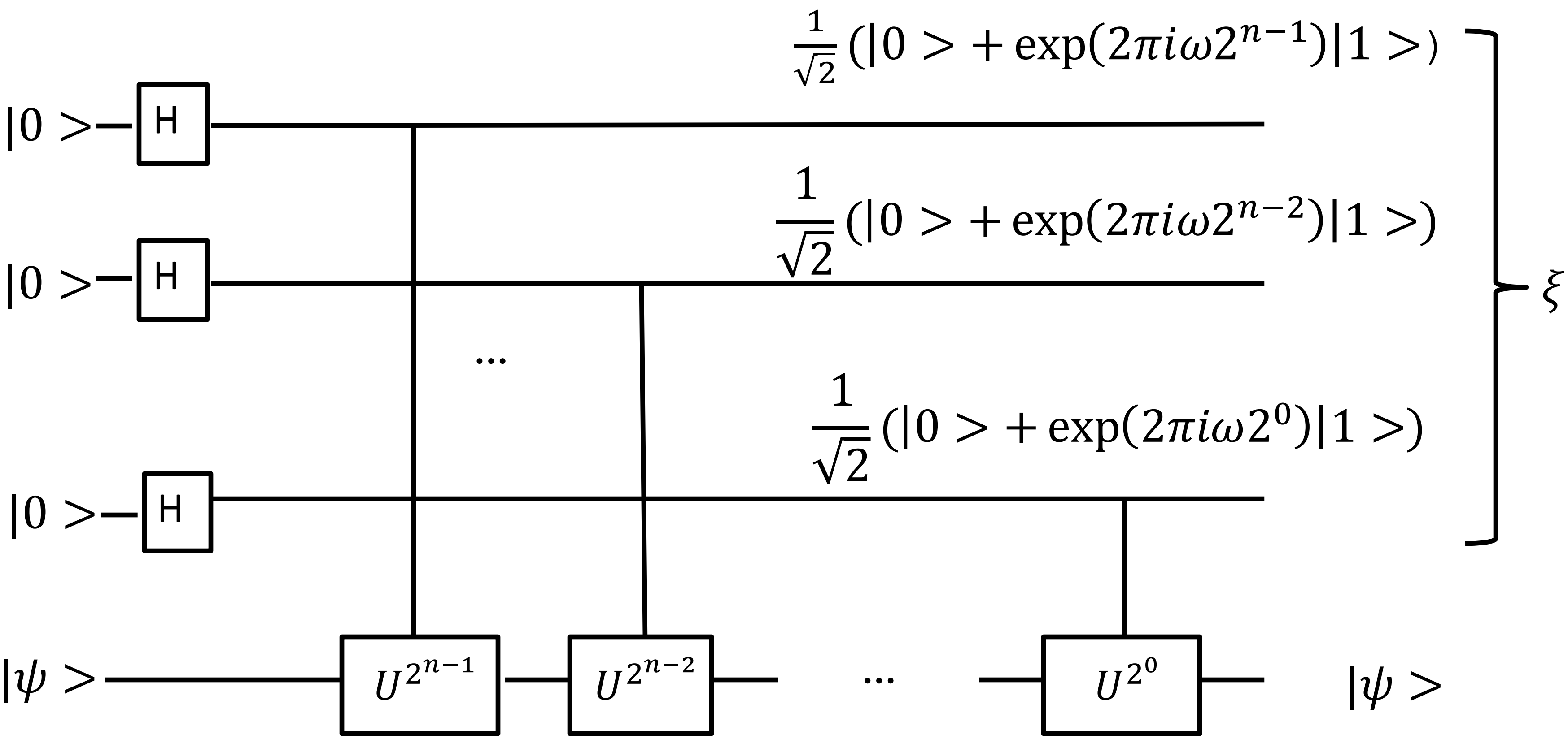
- $H^{\otimes n}(|\boldsymbol{x}>) = \frac{1}{\sqrt{2^n}} \sum_{\boldsymbol{y}} (-1)^{\boldsymbol{x} \cdot \boldsymbol{y}} |\boldsymbol{y}>$
- $H^{\otimes n}(H^{\otimes n}(|\boldsymbol{x}>)) = |\boldsymbol{x}>$
- $QFT_m(|x>) = \frac{1}{\sqrt{m}} \sum_{y=0}^{m-1} e^{2\pi i/_{m}(x \cdot y)} |y>$
- $QFT_m{}^{-1}(|x>) = \frac{1}{\sqrt{m}} \sum_{y=0}^{m-1} e^{-2\pi i/_{m}(x \cdot y)} |y>$

Quantum Fourier Circuit



Eigenvalue Estimation

- Suppose $|\psi>$ is an eigenstate of a unitary operator, U, so $U\,|\psi> = \exp(2\pi i\phi)|\psi>$. $|\phi> = .\,x_1 x_2 \ldots x_n$ (a binary expansion)



Eigenvalue Estimation

- $U|\psi >= \exp(2\pi i\phi)\ |\psi >$, so $U^{2^j}|\psi >= \exp\left(2\pi i\phi 2^j\right)\ |\psi >$.
- Applying ${QFT_n}^{-1}$ to ξ, gives $< x_n,\ x_{n-1}, \dots, x_1 >$, where $|\phi >=. x_1 x_2 \dots x_n$
- Measure χ to get ϕ
- $\frac{y}{2^n}$ is a good estimate for $\phi = \frac{j}{r}$

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Factorization using order finding (Shor)

- Suppose $N = pq$ and $a^r = 1\ (mod\ N)$ then $r = 0\ \big(mod\ \varphi(pq)\big)$
- If r is even, say, $r = 2s$, $(a^s + 1)(a^s - 1) = 0\ (mod\ pq)$.
- There is a good chance $p|(a^s - 1)$ but $\big(q, (a^s - 1)\big) = 1$.
- Then $\big((a^s - 1), \mathrm{N}\big) = \mathrm{p}$. Voila!
- Note that $|v_t> = \frac{1}{r}\sum_{k=0}^{r-1} \exp(-\frac{2\pi ikt}{r})|k(mod\ N)>$ is an eigenvalue of $U_x\ (k) = |xk\ (mod\ N)>$.
- In Shor, $|1> = \frac{1}{\sqrt{r}}\sum |v_t>$.
- Applying QFT^{-1} to control gives phase of eigenvalues
- Measurement of target gives $|\frac{s}{r}>$ with $\Pr(|y>) = \frac{1}{2^{2n}}\ |\frac{1-r^{2^n}}{1-r}|^2$, where $r = \exp(-2\pi i(\frac{y}{2^n} - \phi))$

Order Finding

<u>Problem</u>: Given $a, N \in \mathbb{Z}$ with $(a, N) = 1$, find r: $a^r (mod\ N) = 1$

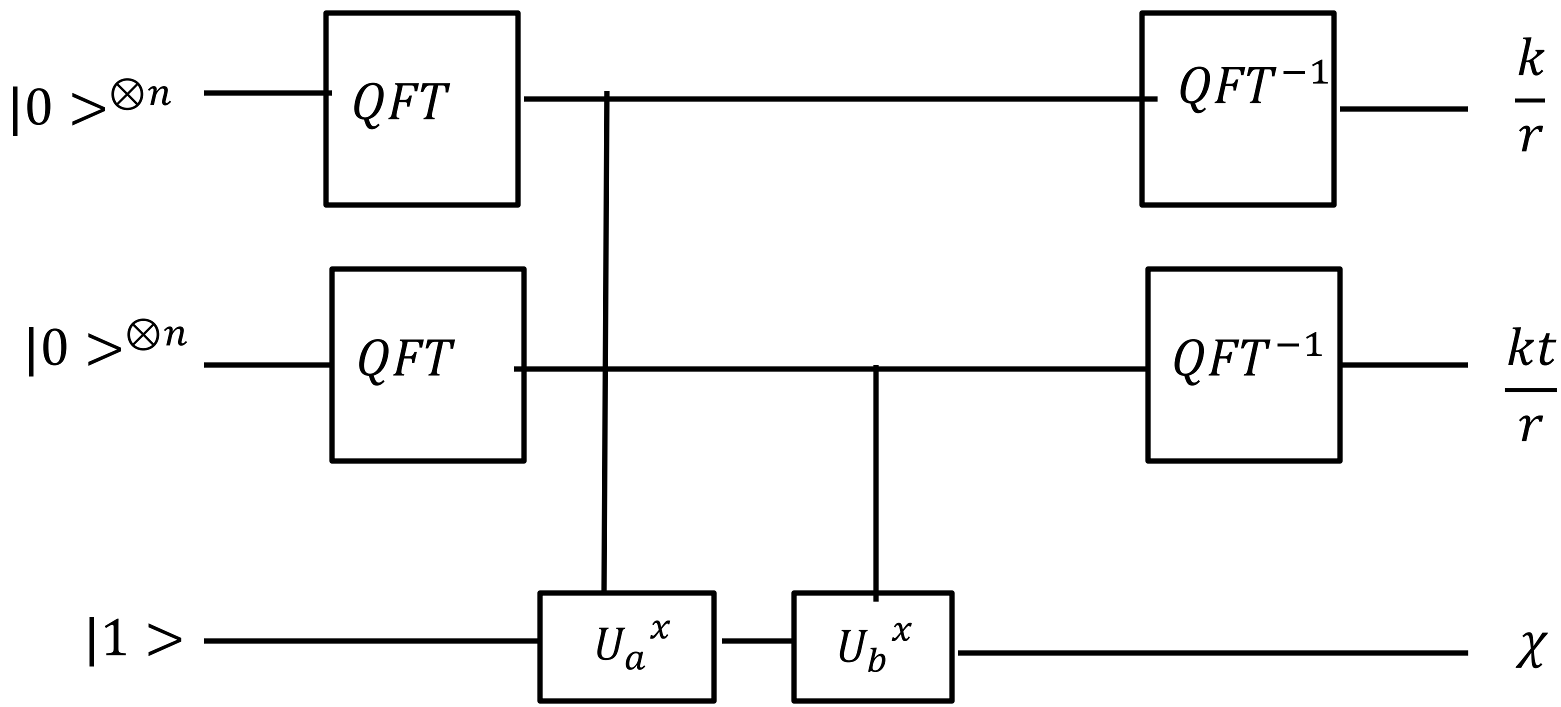
1. Choose n: $2^n \geq 2r^2$
2. Initialize control register $|000 \dots 0> = |0>^{\otimes 2n}$
3. Initialize target register to $= |000 \dots 01> = |000 \dots 0> = |0>^{\otimes 2n} \otimes |1>$
4. Apply QFT to control register
5. Apply $c - U_a{}^x$ to control and target register
6. Apply QFT^{-1} to control register
7. Measure CR to get estimate of $\frac{x_1}{2^n}$ of multiple of $\frac{1}{r}$
8. Use continued fraction to get c_1, r_1: $\left|\frac{x_1}{2^n} - \frac{c_1}{r_1}\right| \leq 2^{-(n-1)/2}$
9. Repeat 1-8 to get $c_2, 2$: $\left|\frac{x_2}{2^n} - \frac{c_2}{r_2}\right| \leq 2^{-(n-1)/2}$, if none, FAIL
10. Compute r $= LCM(r_1, r_2)$ and $a^r\ (mod\ N)$
11. If $a^r\ (mod\ N) = 1$, output r, otherwise FAIL

Order Finding

- Order finding has quantum complexity $O(\lg(N)^2 \lg(\lg(N)) \lg(\lg(\lg(N)))$
- Classical complexity is $\exp(O(\sqrt{\lg(N) \lg(\lg(N)))}$

Discrete log

- Suppose $a = b^x\ (mod\ p)$, b has known order. We want r: $b^r = 1(mod\ p)$
- Put $U_a(\,|x>) = |ax\ (mod\ p) >$ and $U_b(\,|x>) = |bx\ (mod\ p) >$.
- Consider the circuit below. $|1> = \frac{1}{\sqrt{r}}\sum |v_t>$. Below, $t = xy^{-1}$

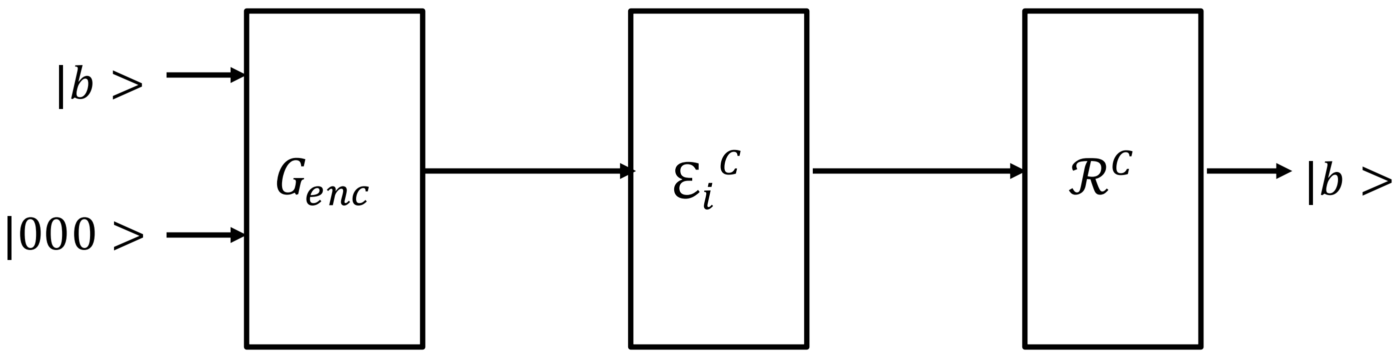


Discrete log

- Measuring first control register gives $|\frac{k}{r}>$
- Measuring first control register gives $|\frac{kt}{r}>$

- Quantum complexity is $O(\lg(p)^2 \lg(\lg(p)) \lg(\lg(\lg(p)))$
- Best known classical requires $\exp(O(\sqrt{\lg(p) \lg(\lg(p))})$

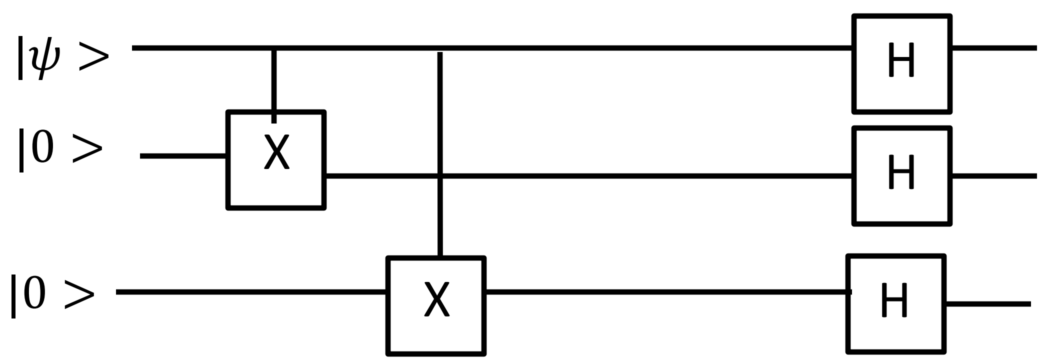
Error Correction



- Unlike classical error correction, the no cloning theorem restricts codes
- $|0>|E> \rightarrow \ \beta_1|0>|E_1> \ + \beta_2\,|1>|E_2>$
- $|1>|E> \rightarrow \ \beta_3|0>|E_3> \ + \beta_4\,|1>|E_4>$
- $|\psi> = \ \alpha_0|0> \ + \alpha_1|1> \rightarrow . \, \alpha_0\,\beta_1|0>|E_1> \ + \alpha_0\beta_2|1>|E_2> \ + \alpha_1\beta_3\,|0>$ $|E_3> \ + \alpha_1\beta_4\,|1>|E_4>$
- $|\psi> = \frac{1}{2}\,|\psi>(\beta_1|E_1> \ + \beta_3|E_3>) + \frac{1}{2} < Z|\psi>(\beta_1|E_1> \ - \beta_3|E_3>)$ + $\frac{1}{2} < X|\psi>(\beta_2|E_2> \ + \beta_4|E_4>)$ + $\frac{1}{2} < XZ|\psi>(\beta_2|E_2> \ + \beta_4|E_4>)$

Error Correction

- $\rho = \ U_{err}|\psi > < E|\overline{U_{err}}^{T}$
- $|\psi_{enc} > = U_{enc}|\psi > |000 \ldots >$
- $\mathcal{E}_0 = I \otimes I \otimes I, \mathcal{E}_1 = X \otimes I \otimes I$
- $\mathcal{E}_2 = I \otimes X \otimes I, \mathcal{E}_3 = I \otimes I \otimes X$
- $\rho\colon |\psi >< \psi| \rightarrow (1-p)|\psi >< \psi| + p\, X|\psi >< \psi|X >$
- $\frac{1}{\sqrt{2}}(|000 > \ +|100 >) \rightarrow \frac{1}{\sqrt{2}}(|000 > \ +|111 >) \neq \frac{1}{\sqrt{2}}(|0 > \ +|1 >)^{\otimes 3}$
- 3-bit code, Shor 9-bit code



Amplitude Amplification

- $|\psi> = A\,|00..0> = \sum_x \alpha_x |x> |junk(x)>$
- $|\psi> = \sum_{x,good} \alpha_x |x> |junk(x)> + \sum_{x,bad} \alpha_x |x> |junk(x)>$
- $|\psi_{good}> = \sum_{x,good} \alpha_x\,|x> |junk(x)>$
- $|\psi_{bad}> = \sum_{x,bad} \alpha_x\,|x> |junk(x)>$
- $|\psi> = \sqrt{p_{good}}\,|\psi_{good}> + \sqrt{p_{bad}}\,|\psi_{bad}> = \sin(\theta)|\psi_{good}> + \cos(\theta)|\psi_{bad}>$
- $p_{good} = \sin(\theta)^2$

Grover

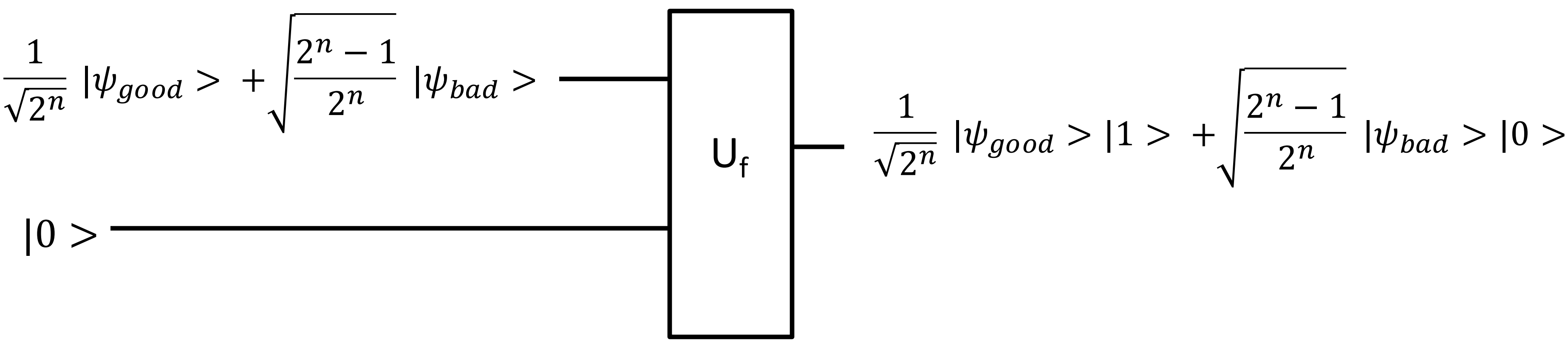
Search

Input: U_f: f: $\{0,1\}^n \rightarrow \{0,1\}$

$f(\boldsymbol{a}) = 1$, $f(\boldsymbol{x}) = 0, \mathbf{x} \neq \boldsymbol{a}$

$|\psi_{good}> = \boldsymbol{w}$

$$|\psi_{bad}> = \frac{1}{\sqrt{N-1}} \sum_{\boldsymbol{x} \neq \boldsymbol{w}} |x>$$



Grover

1. Initialize n-qubits $|0000\ldots0>$.
2. Apply $H^{\otimes n}$ to get $\frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}|x$
3. Apply Grover G $\frac{\pi}{4\sqrt{n}}$ times
4. Measure output

Search

Input: U_f: f: $\{0,1\}^n \rightarrow \{0,1\}$

$f(\boldsymbol{a}) = 1,\ f(\boldsymbol{x}) = 0, \mathbf{x} \neq \boldsymbol{a}$

$|\psi_{good}> = \boldsymbol{w}$

$$|\psi_{bad}> = \frac{1}{\sqrt{N-1}}\sum\nolimits_{\boldsymbol{x}\neq\boldsymbol{w}}|x>$$

Algorithm G

1. Apply U_f
2. Apply $H^{\otimes n}$
3. Apply $U_{0^\perp}$
4. Apply $H^{\otimes n}$

Algorithm $U_{0^\perp}$

$U_{0^\perp}$: $|\mathrm{x}> \longrightarrow -|x>, x \neq 0$
$U_{0^\perp}$: $|0> \longrightarrow 0|x>$

End

Miscellaneous (Susskind)

- $Z = \sum_i e^{-\beta E_i}$, $\beta = \frac{1}{kT}$, $< E >= \frac{\partial(\ln(Z))}{\partial\beta}$
- String: $dU = \frac{1}{2}\mu\omega^2 y^2 dx$, $P(t) = Z = F\frac{\partial\psi}{\partial t}$, $v_\varphi = \frac{\omega}{k}$, $Z = \frac{T}{v_\varphi}$, $\frac{T}{\mu} = \omega^2$
- $I = I_0(\frac{sim(\beta/_2)}{\beta/_2})^2$, $P_R = -$
- $|\psi(t) >= U(t)|\psi(0) >$, $U^\perp(t)U(t) = I$
- $U(\epsilon) = (I + \frac{i}{\hbar}\epsilon H)|\psi(0) >$, so
- $\frac{|\psi(\epsilon)> \; -|\psi(0)>}{\epsilon} = -\frac{i}{\hbar}H|\psi(0) >$ or $\frac{\partial|\psi(t)>}{\partial t} = -\frac{i}{\hbar}H|\psi(t) >|$
- $\frac{d}{dt} < \psi|L|\psi >=< \dot{\psi}|L|\psi >+< \psi|L|\dot{\psi} > =$

  $\frac{i}{\hbar}(< \psi|HL|\psi > \;\; -(< \psi|LH|\psi >) = \frac{i}{\hbar} < \psi|[H, L]|\psi >$
- This gives conservation of energy since $[H, H] = 0$

Miscellaneous (Susskind)

- Standard method:
  1. Get H
  2. Prepare $|\psi(0)>$
  3. Find $H\left|E_j>= E_j\right|E_j>$
  4. $\alpha_j(0) =< E_j|\psi(0)>$, $\alpha_j(t) = \alpha_j(0)\exp(-i\frac{E_j t}{\hbar})$
  5. $|\psi(t)>= \sum_j \alpha_j(t)\ |E_j>$
  6. $P_\lambda(t) =< \lambda|\psi(t)>^2$
- Correlation and means
  - $\Delta A^2 = \sum_a (A-\bar{A})^2\, P(a) = |< A^2|\psi>|^2$
  - $(\Delta A)(\Delta B) \geq \frac{1}{2}|< \psi|[A,B]|\psi>^2$
  - $C(A,B) =< AB > \; - < A >< B >$
  - $C(\sigma_x, \tau_x)$= -1
- Position: $X\psi = x\psi$, Momentum: $P = -\frac{i}{\hbar}\frac{\partial}{\partial x}$

Spin (mostly Susskind)

- $|u>=\begin{pmatrix}1\\0\end{pmatrix}$, $|d>=\begin{pmatrix}0\\1\end{pmatrix}$
- $|r>=\frac{1}{\sqrt{2}}(|u>+|d>$, $|l>=\frac{1}{\sqrt{2}}(|u>\ -|d>$
- $|i>=\frac{1}{\sqrt{2}}(|u>+i|d>$, $|o>=\frac{1}{\sqrt{2}}(|u>\ -i\ |d>$
- Pauli matricies: $\sigma_z=\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$, $\sigma_x=\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$, $\sigma_y=\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}$
- $\sigma_n=\begin{pmatrix}\cos(\theta) & \sin(\theta)\\\sin(\theta) & -\cos(\theta)\end{pmatrix}$
- $\lambda_1=1$, $|\lambda_1>=\begin{pmatrix}\cos(\frac{\theta}{2})\\\sin(\frac{\theta}{2})\end{pmatrix}$, $\lambda_2=-1$, $|\lambda_2>=\begin{pmatrix}-\sin(\frac{\theta}{2})\\\cos(\frac{\theta}{2})\end{pmatrix}$

Spin (mostly Susskind)

- $|\psi> = \sum_i \alpha_i|\lambda_i>$ is the state of a system, the $|\lambda_i>$ is a complete set of orthonormal vectors which are eigenvectors
- $< L > = \sum_i P(\lambda_i)\lambda_i$ is the expected value
- $< L > = < \psi|L|\psi > = \sum_i \overline{\alpha_i}\, \alpha_i < \lambda_i|\lambda_i >, \alpha_i \in \mathbb{C}$
- If φ, ψ are states in a continuous variable, $< \varphi|\psi > = \int_{-\infty}^{\infty} \bar{\varphi}\psi \, dx$
- $I = \sum_i |i >< i|, Tr(L) = \sum_i < i|L|i >$
- $I = \int |x >< x| dx$
- $\psi_G(x) = \frac{1}{\sqrt{2\pi}} \exp(i\frac{px}{\hbar}), \tilde{\psi}(p) =< p|x > = \int dx < p|\psi >< x|\psi >$
- Product state: $|sing > = \frac{1}{\sqrt{2}}(|ud > \ -|du >)$
- $\int FG' = -\int GF'$

Spin (mostly Susskind)

- Observables: $M|\lambda >= \lambda\,|\lambda >$, λ is the observed value, M is projective and Hermitian
- The rules
  1. Observables are represented by linear operators. States are vectors
  2. Results of measurements are eigenvalues
  3. Distinguishable states correspond to orthogonal eigenvalues
  4. If $|\psi >$ is a state and and L is an observable, $P(\lambda_i) =< \psi|\lambda_i >< \lambda_i|\psi >$
  5. Evolution of states governed by a Unitary operator

Spin (mostly Susskind)

- $|\psi >= \sum_i \alpha_i |\lambda_i >$ is the state of a system, the $|\lambda_i >$ is a complete set of orthonormal vectors which are eigenvectors
- $< L >= \sum_i P(\lambda_i)\lambda_i$ is the expected value
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- Density: $< L >= Tr\,(\rho L)$ for states prepared with probability p_i

Spin (mostly Susskind)

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- $< L > = < \psi|L|\psi > = \sum_i \overline{\alpha_i}\, \alpha_i < \lambda_i|\lambda_i >, \alpha_i \in \mathbb{C}$
- If φ, ψ are states in a continuous variable, $< \varphi|\psi > = \int_{-\infty}^{\infty} \bar{\varphi}\psi \, dx$
- State is a unit vector: $|\psi> = \cos(\frac{\theta}{2})|0> + e^{i\varphi}\sin(\frac{\theta}{2})|1>$
- $I = \sum_i |i> < i|, Tr(L) = \sum_i < i|L|i >$
- Density: $< L > = Tr\,(\rho L)$ for states prepared with probability p_i
- Spin field: $\mathrm{H} = \sigma \cdot \mathrm{B}$
  - $< \dot{\sigma}_z > = \frac{i\omega}{2} < [\sigma_z, \sigma_z] >, < \dot{\sigma}_y > = -\omega < \sigma_x >$
- $< \dot{\sigma}_x > = \frac{-i}{\hbar} < [\sigma_x, H] >, < \dot{\sigma}_y > = \frac{-i}{\hbar} < [\sigma_y, H] >, < \dot{\sigma}_z > = \frac{-i}{\hbar} < [\sigma_z, H] >$

Spin (mostly Susskind)

- Density: $< L > = Tr\ (\rho L)$ for states prepared with probability p_i
- If $< L > = p_\psi < \psi|L|\psi > + p_\varphi < \varphi|L|\varphi >$, the density operator is
- $P = p_\psi |\psi >< \psi| p_\varphi |\varphi >< \varphi|$
- $P_{a,a'} = \overline{\psi(a)}\psi(a') \sum_b \overline{\phi(b)}\ \phi(b)$
- $< \psi|L|\psi > = \sum_{a,a',b,b'} \bar{\psi}_{a',b'} L_{a',a}\ \psi_{a,b}$
- $\psi(x,t) = \exp(i \frac{px - \frac{p^2 t}{2m}}{h})$
- $< X|\psi > = \frac{1}{\sqrt{2\pi}} \int dx \exp(i \frac{px}{\hbar}) \bar{\psi}(x)$
- Composite: $\varphi \in H^A$,ψ∈ H^B, $\varphi \otimes \psi \in H^{AB}$
- Dynamics: $< v > = \frac{d}{dt} < \psi|X|\psi >, \frac{d}{dt} < P > = \frac{i}{\hbar}[V, P], [V, P] = i\hbar \frac{dV}{dt}$

Harmonic Oscillator, etc

- $< p|x > = \frac{1}{\sqrt{2\pi}} \exp(-i\,\frac{px}{\hbar})$
- $\psi_p(x) = \frac{1}{\sqrt{2\pi}} \int \quad \exp(i\,\frac{px}{\hbar})\tilde{\psi}(p)\,dp = \frac{1}{\sqrt{2\pi}} \int dp < x|p >< p|\psi >$,
- $\widetilde{\psi}(p) = \frac{1}{\sqrt{2\pi}} \int \exp\left(-i\,\frac{px}{\hbar}\right)\,\psi(x)dx$

- $H = \frac{1}{2}\dot{X}^2 + \frac{1}{2}\omega^2 x^2 = \frac{P^2 + w^2 x^2}{2} = \frac{1}{2}(P + i\omega X)(P - i\omega X) - \omega^2[X, P]$
- Ground state: $\psi(x) = \exp(\frac{\omega}{2\hbar}x^2)$, $E_0 = \frac{\omega\hbar}{2}$
- $a^- = \frac{i}{\sqrt{2\omega\hbar}}(P - i\omega X)$, $a^+ = \frac{-i}{\sqrt{2\omega\hbar}}(P + i\omega X)$
- $[a^-, a^+] = 1.$ If $N = a^+a^-$, $H = \omega\hbar(N + \frac{1}{2})$, $N|n> = n|n>$,
- $a^+|\mathrm{n}> = |\mathrm{n} + 1>$
- Particle in box: $E_n = \frac{n^2 h^2}{8mL^2}$, $\psi_n = Csin(\frac{n\pi x}{L})$

Some physics

- $E_n = \frac{-13.6}{n^2}$, $a_0 = \epsilon_0 \frac{h^2}{\pi m e^2}$, $d_{\,n} = \frac{(2m)^{3/2} V E^{3/2}}{3\pi h^3}$, $g(E) = \frac{(2m)^{3/2} V}{2\pi h^2}\sqrt{E}$
- $\nabla \cdot j = -\frac{\partial \rho}{\partial t}$, $\nabla \cdot E = \frac{\rho}{\epsilon_0}$, $\nabla \times E = -\frac{\partial B}{\partial t}$, $\nabla \times B = 0$, $c^2 \nabla \times B = \frac{j}{\epsilon_0} + \frac{\partial E}{\partial t}$
- $c^2 = \frac{1}{\epsilon_0 \mu_0}$, $I = \sigma T^4$, $D = \epsilon E$, $B = \mu H$
- Solution to $\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = -s$, $\psi(t) = \frac{1}{4\pi}\frac{S(t-\frac{r}{c})}{r}$, $S(t) = \int s(t)\,dV$
- $\phi(1,t) = \int \frac{\rho(2,t-\frac{r}{c})}{4\pi\epsilon_0 r_{12}}\, dV$, $A(1,t) = \int \frac{\rho(2,t-\frac{r}{c})}{4\pi c^2 \epsilon_0 r_{12}}\, dV$, $\nabla\phi = E + \frac{\partial A}{\partial t}$, $S = \epsilon_0 c^2 E \times B$
- Oscillating dipole: $\psi = \frac{dz}{4\pi\epsilon_0}[\frac{q(t-\frac{r}{c})}{r^3} + \frac{I(t-\frac{r}{c})}{r^2 c}]$
- $x' = \gamma(c - ut)$, $t' = \gamma(t - \frac{ux}{c^2})$, $E^2 + (pc)^2 = (\,m_0 c^2\,)^2$

Thermo

- $\Delta Q + \Delta W = \Delta E$, ΔQ – heat in, ΔW work on
- $W = Q(1 - \frac{T}{T_0})$, $e = (1 - \frac{T_C}{T_H})$, $S = k\ln(\Omega)\Omega$
- $c_v = \frac{3}{2}R$
- $I(\lambda) = \frac{2\pi hc^2}{\lambda^5(\exp\left(\frac{hc}{k\lambda T}\right) - 1)}$

Hidden subgroup

- $S \leq G$, $\mathrm{f(x)} = \mathrm{f(y)} \mathrm{iff}\ x + S = y + S$

Hidden subgroup

- $S \leq G,\ \mathrm{f(x)} = \mathrm{f(y)}\,\mathrm{iff}\ x + S = y + S$