Cryptanalysis

Block Ciphers 1

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Block ciphers

- Complicated keyed invertible functions constructed from iterated elementary rounds.
  - Confusion: non-linear functions (ROM lookup)
  - Diffusion: permute round output bits

Characteristics:

- Fast
- Data encrypted in fixed “block sizes” (64,128,256 bit blocks are common).
- Key and message bits non-linearly mixed in cipher-text

Mathematical view of block ciphers

- $E(k, x)=y$.
- E: $GF(2^m) \times GF(2^n) \rightarrow GF(2^n)$, often $m=n$.
- $E(k,x)$ is a bijection in second variable.
- $E(k, x)$ in S_N, $N= 2^n$.
- Each bit position is a balanced boolean function.
- E is easy to compute but inverse function (with k fixed) is hard to compute without knowledge of k.
- Implicit function hard to compute.
- Intersection of algebraic varieties.

A (very bad) block cipher

- Let M be an invertible nxn matrix over GF(2).
- Suppose k is an n-bit vector representing the key and p is an n bit vector representing the plaintext block
- Put c= M(p+k). C is the plaintext
- Example:
    - M= $\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix}$, k= $(1,1,1)^T$, p= $(1,0,1)^T$, c= $(1,1,0)^T$.
    -
- Why is this so bad?
- Better (but still bad)
- Let R(k) be a rule that selects an invertible matrix from $GF(2)^n$ x $GF(2)^n$. Put c=R(k)p.

- Lesson: linear is bad

Guiding Theorems

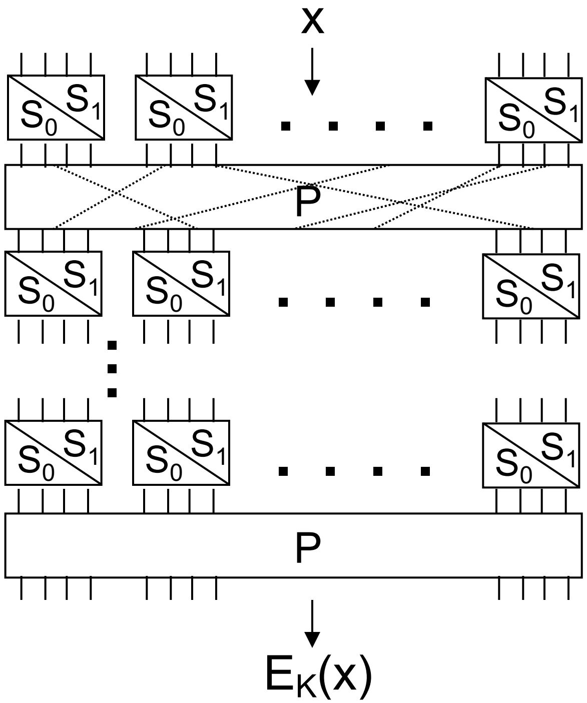
- Implicit Function Theorem: If f(x,y)= c, is a continuously differentiable function from F^n x F^m into F^m and the mxm Jacobian in the y variables is non zero in a region, there is a function g from R^n to R^m such that F(x, g(x))=c. When F is linear, this function is very easy to compute. Think of g as mapping the plaintext to the key (for fixed ciphertext).
- Functions in over finite fields are polynomials: If f is a function from k^n to k, where k is a finite field, f can be written as a polynomial in the n variables.
- Reduction in dimension: Generally (pathological exceptions aside), if f is a function from k^n to k, where k is a finite field, and f(x)=c, one variable can be written as a function of the other n-1 variables. In other words, if g is a function from k^n to k subject to the constraint f(x)=c, then g can be rewritten as a function of n-1 variables.

Data Encryption Standard

- Federal History
  - 1972 study.
  - RFP: 5/73, 8/74.
  - NSA: S-Box influence, key size reduction.
  - Published in Federal Register: 3/75.
  - FIPS 46: January, 1976.
- DES
  - Descendant of Feistel’s Lucifer.
  - Designers: Horst Feistel, Walter Tuchman, Don Coppersmith, Alan Konheim, Edna Grossman, Bill Notz, Lynn Smith, and Bryant Tuckerman.
- Brute Force Cracking
  - Key size controversy: USG wanted 48 bit keys, IBM wanted 64 bit keys. Result: 56-bit keys.
  - EFS DES Cracker: $250K, 1998. 1,536 custom chips. Can brute force a DES key in days.
  - Deep Crack and distributed net break a DES key in 22.25 hours.

Horst Feistel: Lucifer

- First serious needs for civilian encryption (in electronic banking), 1970’s
- IBM’s response: Lucifer, an iterated SP cipher
- Lucifer (v0):
  - Two fixed, 4x4 s-boxes, S_0 & S_1
  - A fixed permutation P
  - Key bits determine which s-box is to be used at each position
  - 8 x 64/4 = 128 key bits (for 64-bit block, 8 rounds)

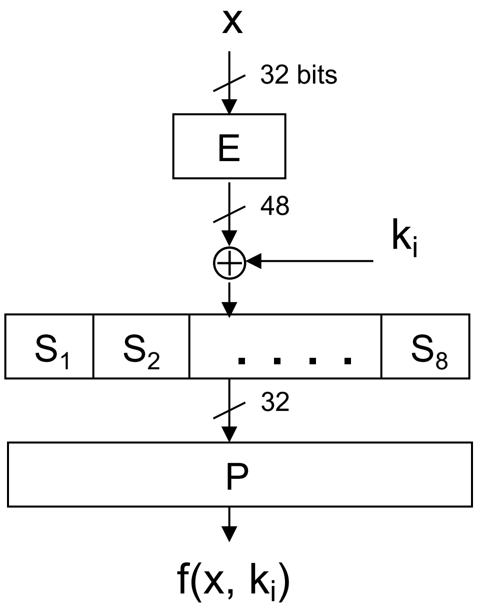


Graphic by cschen@cc.nctu.edu.tw

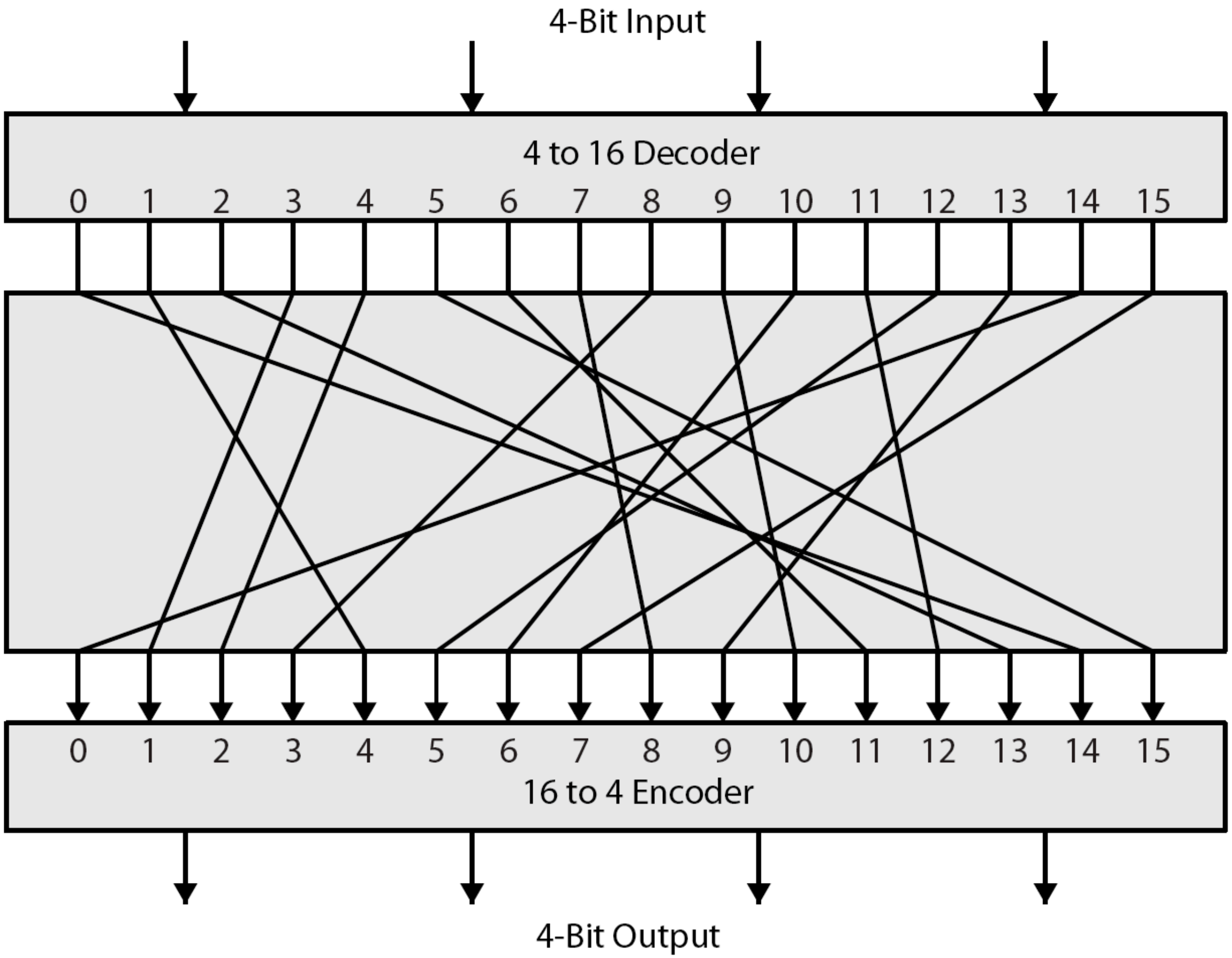
From Lucifer to DES

- 8 fixed, 6x4 s-boxes (non-invertible)
- Expansion, E, (simple duplication of 16 bits)
- Round keys are used only for xor with the input

- 56-bit key size
- 16 x 48 round key bits are selected from the 56-bit master key by the “key schedule”.

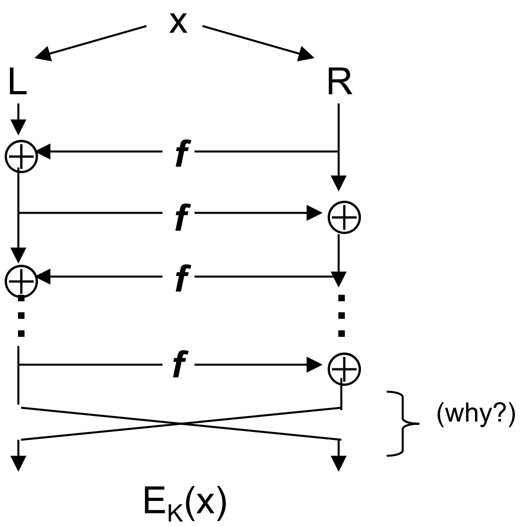


What is a “safe” block cipher

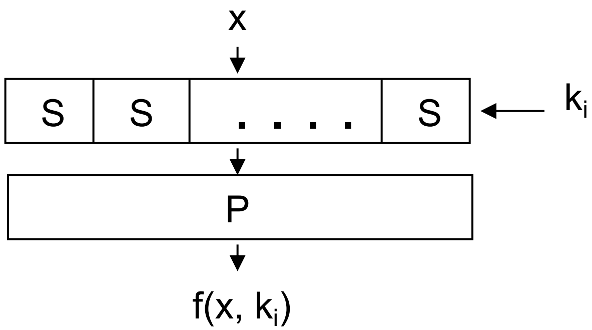


Feistel Ciphers

- A straightforward SP cipher needs twice the hardware: one for encryption (S, P), one for decryption (S^{-1}, P^{-1}).
- Feistel's solution:

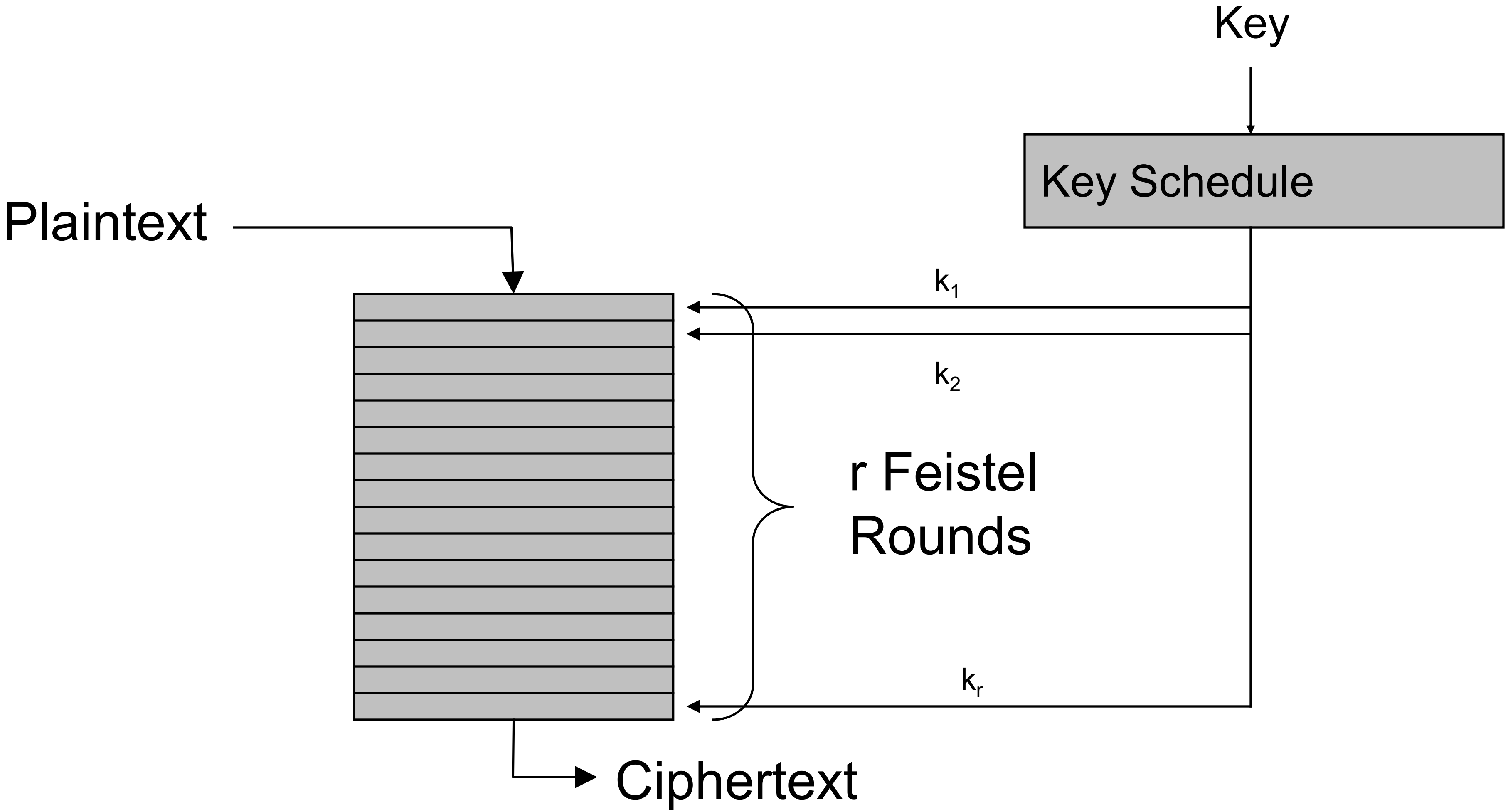


where the *f* function is SP:

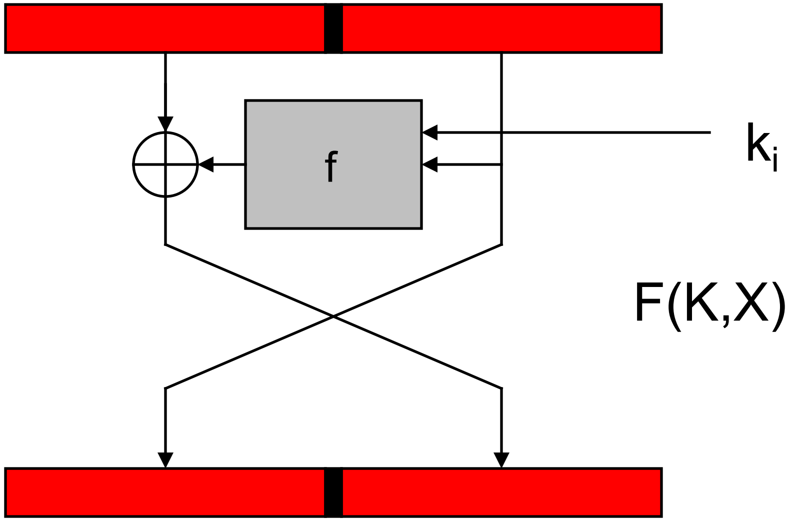


- Lucifer v1: Feistel SP cipher; 64-bit block, 128-bit key, 16 rounds.

Iterated Feistel Cipher



Feistel Round

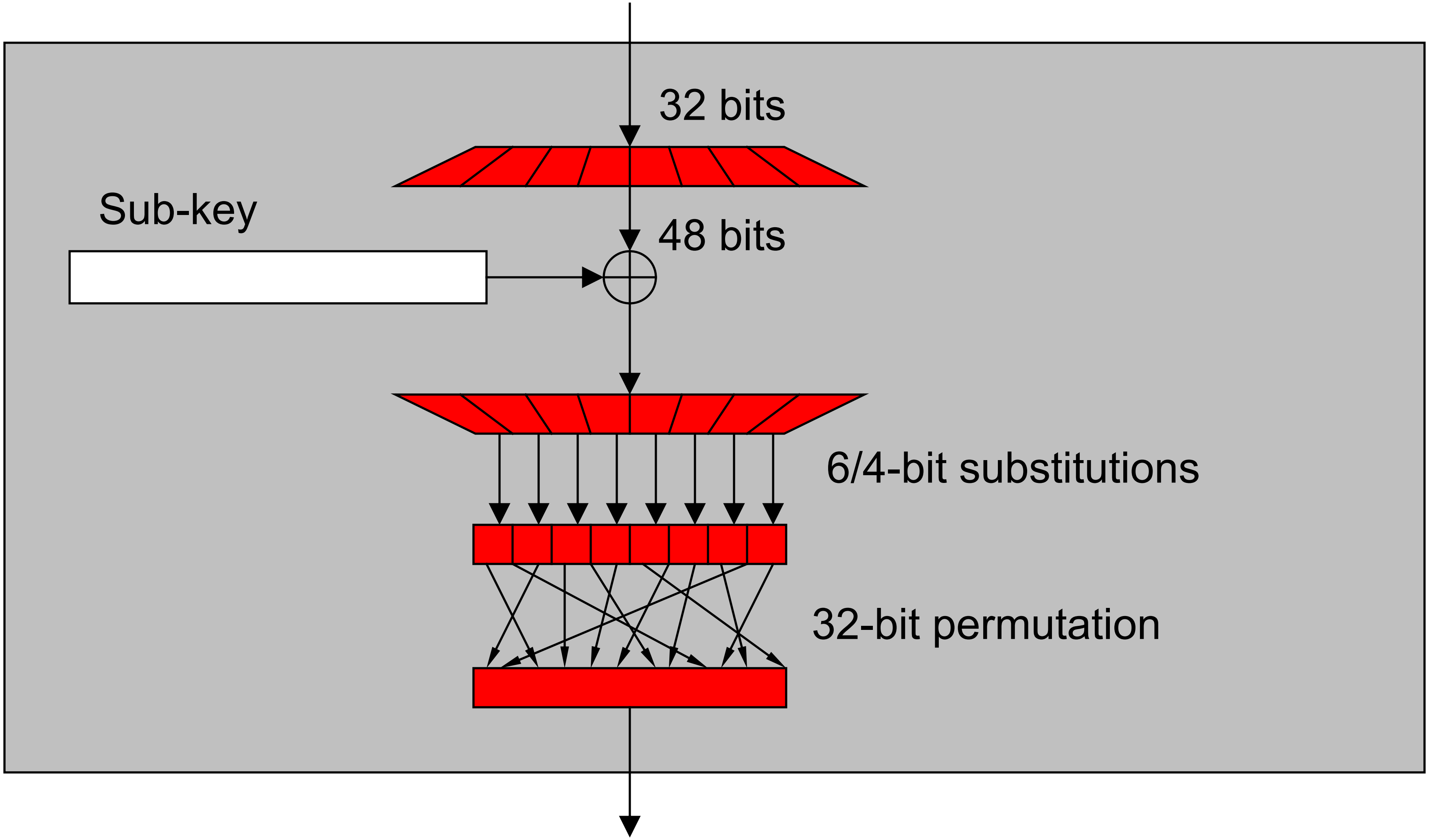


Graphic courtesy of Josh Benaloh

F(K,X)= non-linear function

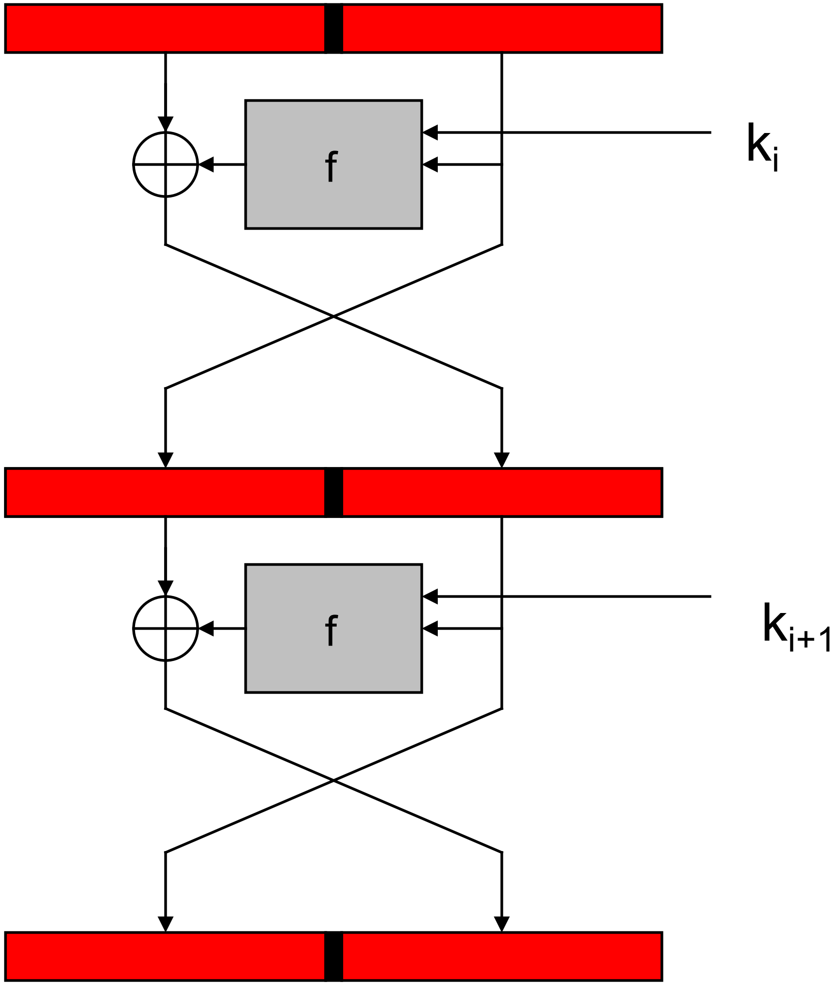
Note: If $\boldsymbol{\sigma}_i(L,R)= (L\oplus f(E(R)\oplus k_i), R)$ and $\boldsymbol{\tau}(L, R)= (R, L)$, this round is $\boldsymbol{\tau}\,\boldsymbol{\sigma}_i(L, R)$.
To invert: swap halves and apply same transform with same key: $\boldsymbol{\sigma}_i\boldsymbol{\tau}\boldsymbol{\tau}\boldsymbol{\sigma}_i(L,R)= (L,R)$.

DES Round Function

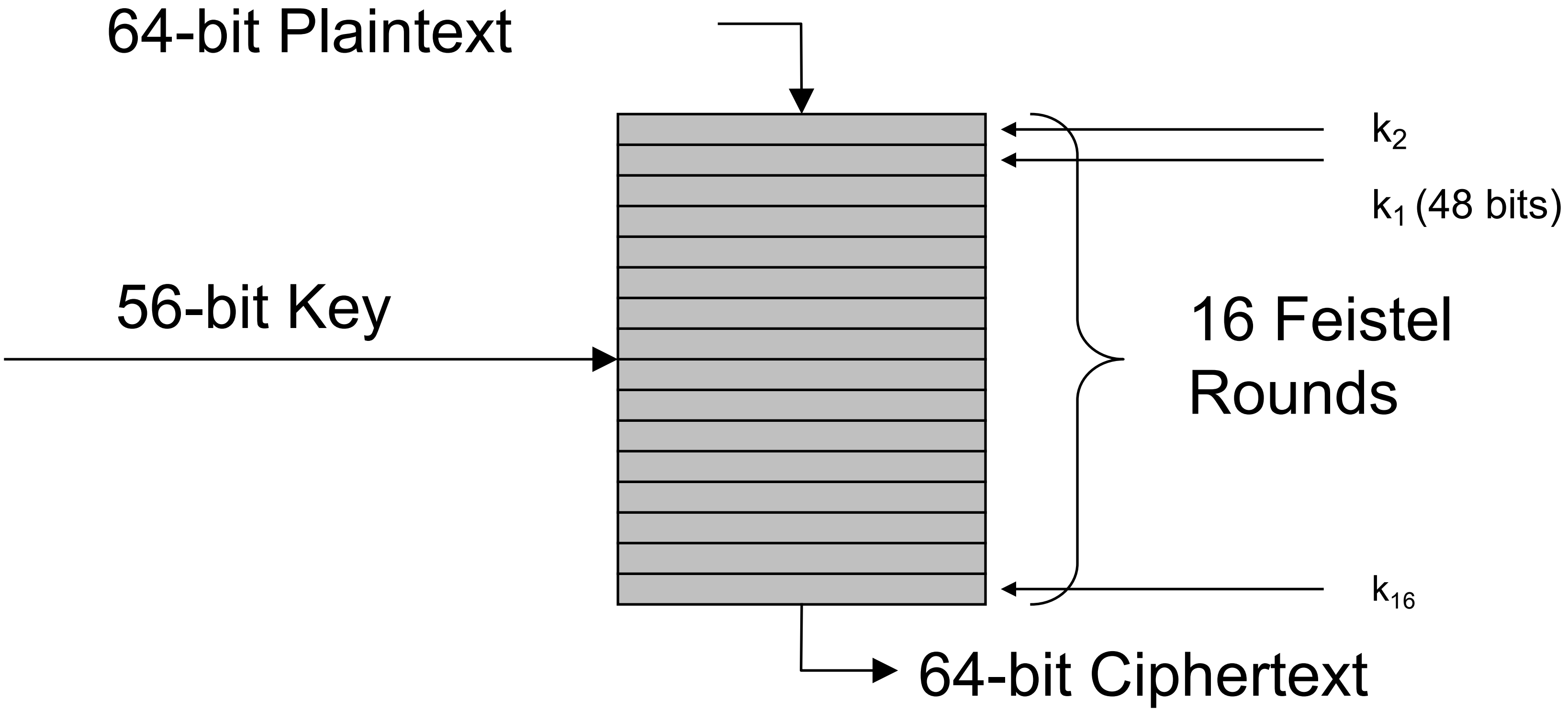


Slide courtesy of Josh Benaloh

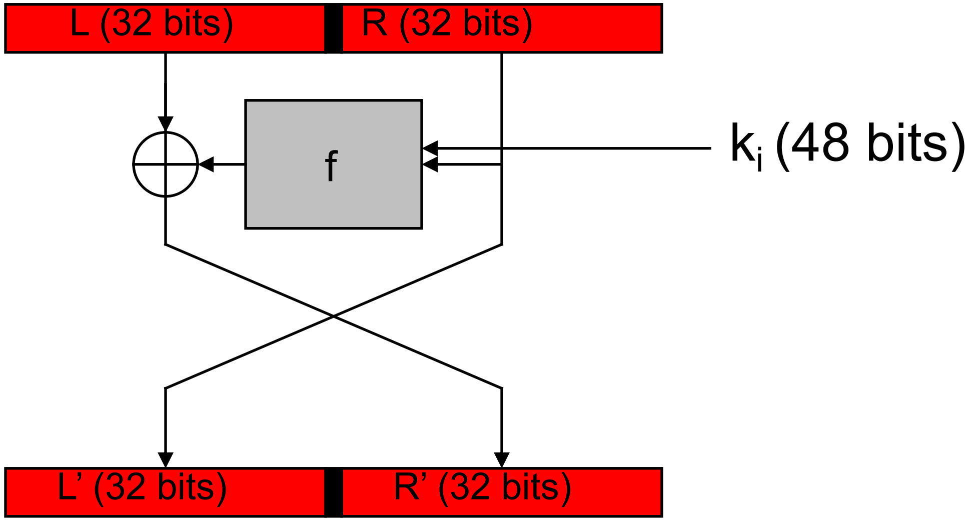
Chaining Feistel Rounds



DES


64-bit Plaintext
k_2
k_1 (48 bits)
56-bit Key
16 Feistel Rounds
k_{16}
64-bit Ciphertext

DES Round



F(K,X)= non-linear function

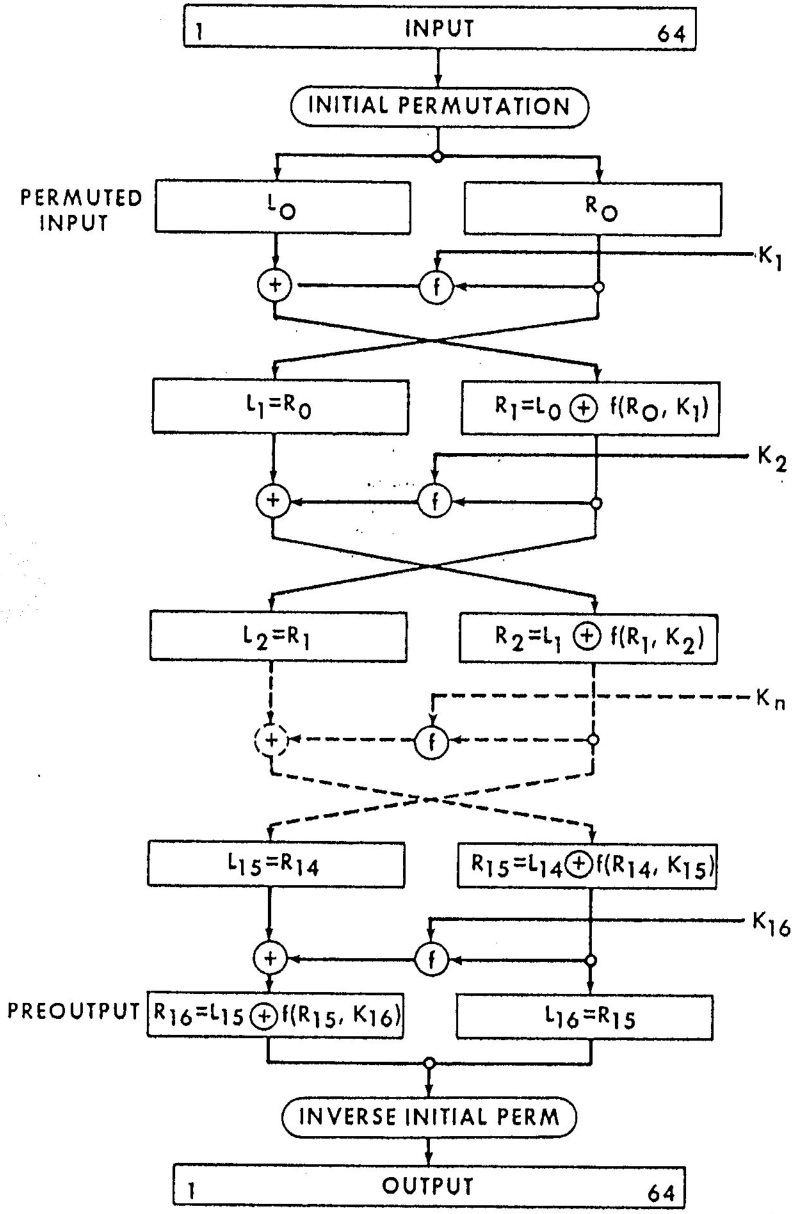
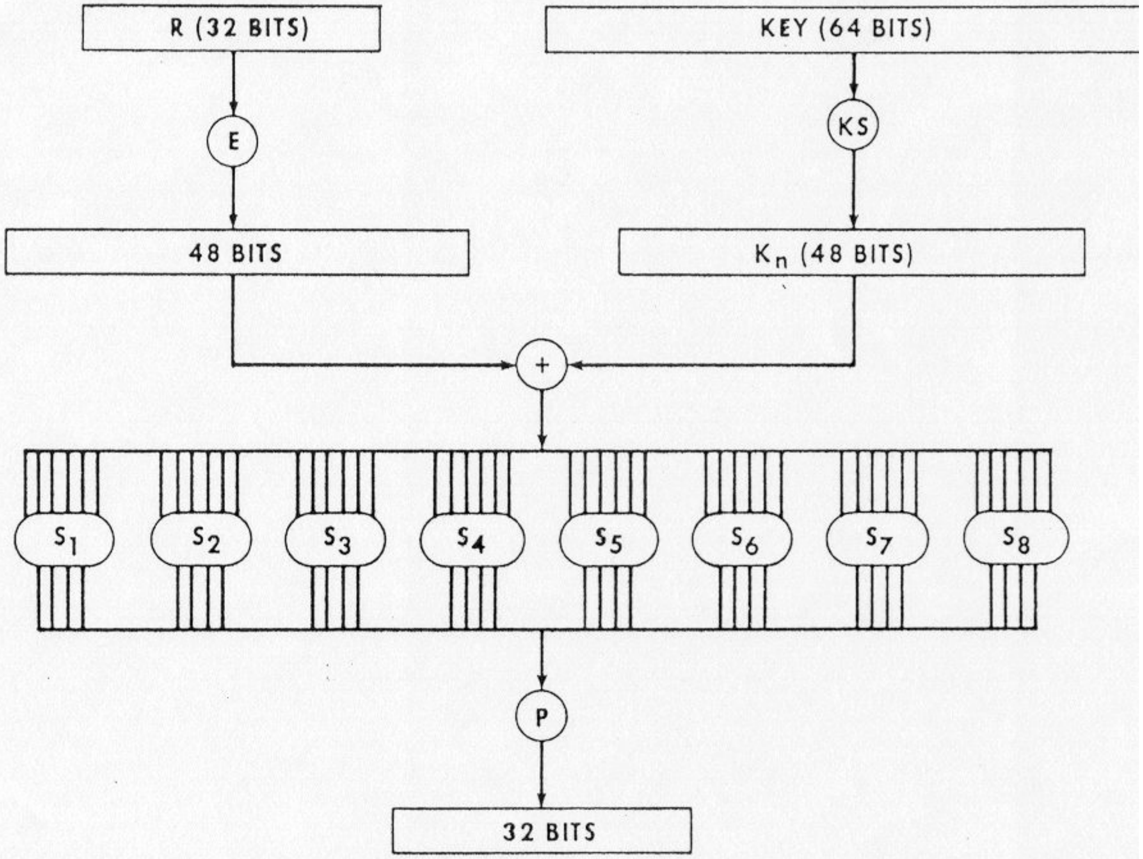


Figure 5.1. Electronic Codebook (ECB) Mode—Enciphering Computation.



K_n CHANGES FOR N=1, 2...16

E=E FUNCTION
KS=KEY SCHEDULE

Figure 5.2. Electronic Codebook (ECB) Mode—Calculation of f(R,K).

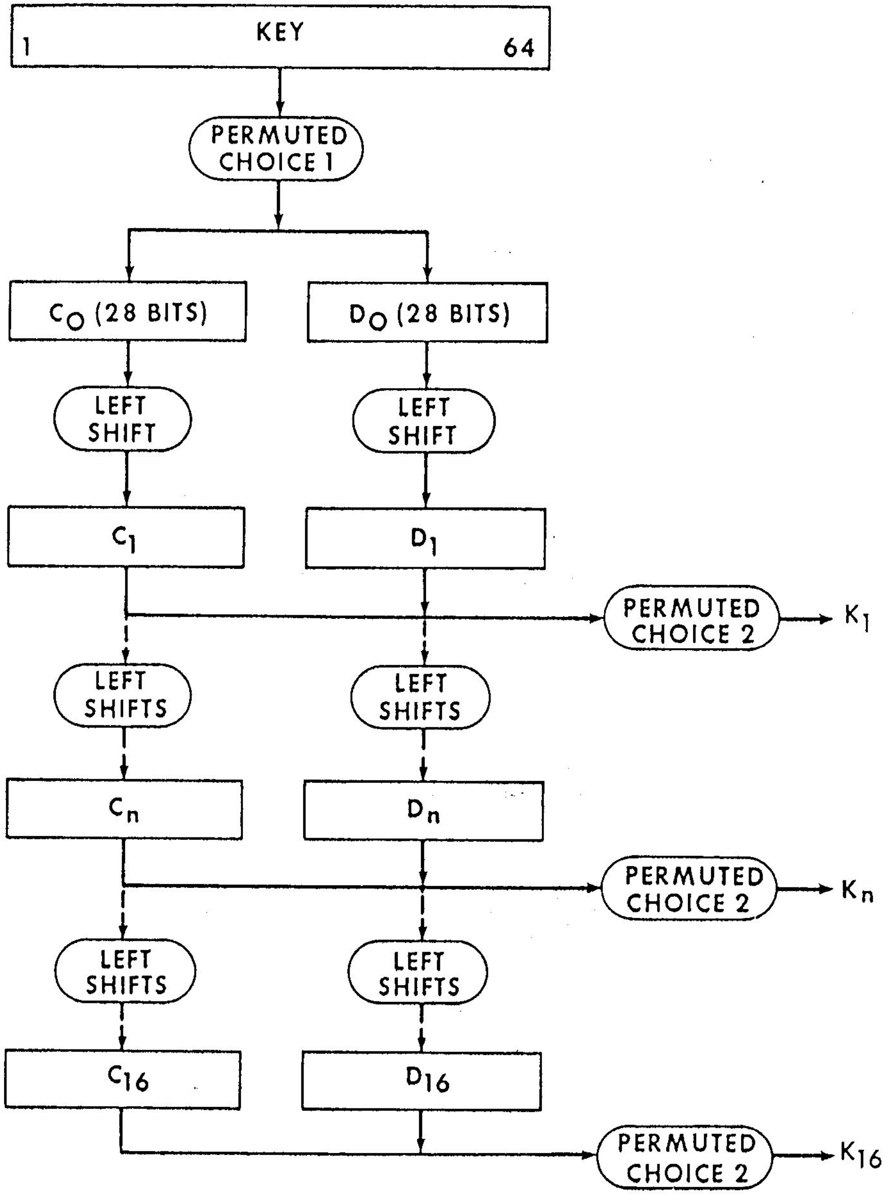


Figure 5.3. Electronic Codebook (ECB) Mode—Key Schedule (KS) Calculation.

DES Described Algebraically

- $\boldsymbol{\sigma}_i(L,R) = (L \oplus f(E(R) \oplus k_i), R)$
  - k_i is 48 bit sub-key for round i.
  - $f(x) = P(S_1 S_2 S_3 \ldots S_8(x))$. Each S –box operates on 6-bit quantities and outputs 4 bit quantities.
  - P permutes the resulting 32 output bits.
- $\tau(L, R) = (R, L)$.
- Each round (except last) is $\tau\boldsymbol{\sigma}_{i.}$
- Note that $\tau\tau = \tau^2 = 1 = \boldsymbol{\sigma}_i\boldsymbol{\sigma}_i = \boldsymbol{\sigma}_i^2$.
- Full DES is: $DES_K(x) = IP^{-1}\, \boldsymbol{\sigma}_{16}\, \tau \ldots \boldsymbol{\sigma}_3\, \tau\, \boldsymbol{\sigma}_2\, \tau\, \boldsymbol{\sigma}_1\, IP(x)$.
- So its inverse is: $DES_K^{-1}(x) = IP^{-1}\, \boldsymbol{\sigma}_1\, \tau \ldots \boldsymbol{\sigma}_{14}\, \tau\, \boldsymbol{\sigma}_{15}\, \tau\, \boldsymbol{\sigma}_{16}\, IP(x)$.

DES Key Schedule

$C_0D_0 = PC_1(K)$

$C_{i+1} = \text{LeftShift}(\text{Shift}_i, C_i)$, $D_{i+1} = \text{LeftShift}(\text{Shift}_i, D_i)$

$K_i = PC_2(C_i \| D_i)$

$\text{Shift}_i = <1,2,2,2,2,2,2,1,2,2,2,2,2,2,1,1>$

- Note: Irregular Key schedule protects against related key attacks. [Biham, New Types of Cryptanalytic Attacks using Related Keys, TR-753, Technion]

DES Key Schedule

```
pc1[64]
  57 49 41 33 25 17 09 01 58 50 42 34 26 18 10 02
  59 51 43 35 27 19 11 03 60 52 44 36 63 55 47 39
  31 23 15 07 62 54 46 38 30 22 14 06 61 53 45 37
  29 21 13 05 28 20 12 04 00 00 00 00 00 00 00 00

pc2[48]
  14 17 11 24 01 05 03 28 15 06 21 10 23 19 12 04
  26 08 16 07 27 20 13 02 41 52 31 37 47 55 30 40
  51 45 33 48 44 49 39 56 34 53 46 42 50 36 29 32
```

DES Key Schedule

Key schedule round 1

```
10 51 34 60 49 17 33 57  2  9 19 42
 3 35 26 25 44 58 59  1 36 27 18 41
22 28 39 54 37  4 47 30  5 53 23 29
61 21 38 63 15 20 45 14 13 62 55 31
```

Key schedule round 2

```
 2 43 26 52 41  9 25 49 59  1 11 34
60 27 18 17 36 50 51 58 57 19 10 33
14 20 31 46 29 63 39 22 28 45 15 21
53 13 30 55  7 12 37  6  5 54 47 23
```

DES Data

```
S1 (hex)
   e 4 d 1 2 f b 8 3 a 6 c 5 9 0 7
   0 f 7 4 e 2 d 1 a 6 c b 9 5 3 8
   4 1 e 8 d 6 2 b f c 9 7 3 a 5 0
   f c 8 2 4 9 1 7 5 b 3 e a 0 6 d

S2 (hex)
   f 1 8 e 6 b 3 4 9 7 2 d c 0 5 a
   3 d 4 7 f 2 8 e c 0 1 a 6 9 b 5
   0 e 7 b a 4 d 1 5 8 c 6 9 3 2 f
   d 8 a 1 3 f 4 2 b 6 7 c 0 5 e 9

S3 (hex)
   a 0 9 e 6 3 f 5 1 d c 7 b 4 2 8
   d 7 0 9 3 4 6 a 2 8 5 e c b f 1
   d 6 4 9 8 f 3 0 b 1 2 c 5 a e 7
   1 a d 0 6 9 8 7 4 f e 3 b 5 2 c
```

DES Data

```
S4 (hex)
   7 d e 3 0 6 9 a 1 2 8 5 b c 4 f
   d 8 b 5 6 f 0 3 4 7 2 c 1 a e 9
   a 6 9 0 c b 7 d f 1 3 e 5 2 8 4
   3 f 0 6 a 1 d 8 9 4 5 b c 7 2 e

S5 (hex)
   2 c 4 1 7 a b 6 8 5 3 f d 0 e 9
   e b 2 c 4 7 d 1 5 0 f a 3 9 8 6
   4 2 1 b a d 7 8 f 9 c 5 6 3 0 e
   b 8 c 7 1 e 2 d 6 f 0 9 a 4 5 3

S6 (hex)
   c 1 a f 9 2 6 8 0 d 3 4 e 7 5 b
   a f 4 2 7 c 9 5 6 1 d e 0 b 3 8
   9 e f 5 2 8 c 3 7 0 4 a 1 d b 6
   4 3 2 c 9 5 f a b e 1 7 6 0 8 d
```

DES Data

S7 (hex)

4	b	2	e	f	0	8	d	3	c	9	7	5	a	6	1
d	0	b	7	4	9	1	a	e	3	5	c	2	f	8	6
1	4	b	d	c	3	7	e	a	f	6	8	0	5	9	2
6	b	d	8	1	4	a	7	9	5	0	f	e	2	3	c

S8 (hex)

d	2	8	4	6	f	b	1	a	9	3	e	5	0	c	7
1	f	d	8	a	3	7	4	c	5	6	b	0	e	9	2
7	b	4	1	9	c	e	2	0	6	a	d	f	3	5	8
2	1	e	7	4	a	8	d	f	c	9	0	3	5	6	b

E

32	1	2	3	4	5
4	5	6	7	8	9
8	9	10	11	12	13
12	13	14	15	16	17
16	17	18	19	20	21
20	21	22	23	24	25
24	25	26	27	28	29
28	29	30	31	32	1

- Note: DES can be made more secure against linear attacks by changing the order of the S-Boxes: Matsui, On Correlation between the order of S-Boxes and the Strength of DES. Eurocrypt, 94.

DES Data

P

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
16	7	20	21	29	12	28	17	1	15	23	26	5	18	31	10
17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
2	8	24	14	32	27	3	9	19	13	30	6	22	11	4	25

- Note on applying permutations: For permutations of bit positions, like P above, the table entries consisting of two rows, the top row of which is “in order” means the following. If t is above b, the bit at b is moved into position t in the permuted bit string. For example, after applying P, above, the most significant bit of the output string was at position 16 of the input string.

Another cipher for the era: TEA

```
tea(unsigned K[4], unsigned& L, unsigned& R){
    unsigned d = 0x9e3779b9;
    unsigned s = 0;
    for(int i = 0; i < 32; i++) {
        s += d;
        L += ((R<<4)+K[0])^(R+s)^((R>>5)+K[1]);
        R += ((L<<4)+K[2])^(L+s)^((L>>5)+K[3]);
        }
    }
```

S Boxes as Polynomials over GF(2)

```
1,1:
   56+4+35+2+26+25+246+245+236+2356+16+15+156+14+146+145+13+1
   35+134+1346+1345+13456+125+1256+1245+123+12356+1234+12346

1,2:
   C+6+5+4+45+456+36+35+34+346+26+25+24+246+2456+23+236+235+2
   34+2346+1+15+156+134+13456+12+126+1256+124+1246+1245+12456
   +123+1236+1235+12356+1234+12346

1,3:
   C+6+56+46+45+3+35+356+346+3456+2+26+24+246+245+236+16+15+1
   45+13+1356+134+13456+12+126+125+12456+123+1236+1235+12356+
   1234+12346

1,4:
   C+6+5+456+3+34+346+345+2+23+234+1+15+14+146+135+134+1346+1
   345+1256+124+1246+1245+123+12356+1234+12346
```

Legend: `C+6+56+46` means $1 \oplus x_6 \oplus x_5 x_6 \oplus x_4 x_6$

Decomposable Systems

- $E_{k1||k2}(x) = E'_{k1}(x)||E''_{k2}(x)$

m	t	2^{mt}	$m2^t$
2	32	2^{64}	2^{33}
4	16	2^{64}	2^{18}

- Good mixing and avalanche condition

Feistel Ciphers defeat simple attacks

- After 4 rounds get flat statistics.
- Parallel system attack
- Even a weak round function can yield a strong Feistel cipher if iterated sufficiently.
  - Provided it’s non-linear

DES Attacks: Exhaustive Search

- Symmetry $\mathrm{DES}(\mathbf{k}\oplus\mathbf{1}, \mathbf{x}\oplus\mathbf{1})=\mathrm{DES}(\mathbf{k}, \mathbf{x})\oplus\mathbf{1}$
- Suppose we know plain/cipher text pair (p,c)

```
for(k=0;k<$2^{56}$;k++) {
 if(DES(k,p)==c) {
    printf("Key is %x\n", k);
    break;
    }
}
```

- Expected number of trials (if k was chosen at random) before success: 2^{55}

DES Attacks: Poor key hygiene

- Poor random number generator: 20 bits of entropy
  - 2^{20} vs 2^{56}
  - Second biggest real problem
  - First biggest: bad key management
- Symmetric ciphers are said to be secure in practice if no known attack works more efficiently than exhaustive search.
  - Note that the barrier is computational not information theoretic.

Suppose you decide the keyspace is too small?

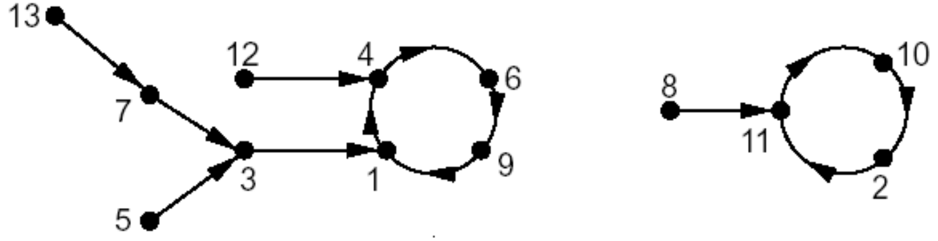
- Can you increase security by encrypting twice or more?
  - $E'(k_1||k_2, x)= E(k_1, E(k_2,x))$
- Answer: Maybe.
- Three times is the charm (triple DES).
- If you do it twice, TMTO attack reduces it to little more than one key search time (if you have a lot of memory).

What’s the complexity of breaking a Block Cipher

- Suppose there are K keys ($K=2^{56}$ for DES)
- Pick a plaintext p and sort the pairs (E(p,x), x) for x= 0,1,…, K-1)
- Ask for E(p,k)=c.
- Lookup (c,x) in the table.
- x is the key.
- O(1) after precomputation!

Random mappings

- Let F_n denote all functions (mappings) from a finite domain of size *n* to a finite co-domain of size *n*
- Every mapping is equally likely to be chosen, $|F_n| = n^n$ the probability of choosing a particular mapping is $1/ n^n$
- Example. f : {1, 2, …, 13} → {1, 2, …, 13}



Graphic by Maithili Narasimha

- As n tends to infinity, the following are expectations of some parameters associated with a random point in {1, 2, …, n} and a random function from F_n: (i) tail length: $\sqrt{\frac{\pi n}{8}}$ (ii) cycle length: $\sqrt{\frac{\pi n}{8}}$ (iii) rho-length: $\sqrt{\frac{\pi n}{2}}$.

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Time memory trade off ("TMTO")

- If we can pre-compute a table of (k, $E_k(x)$) for a fixed x, then given corresponding (x,c) we can find the key in O(1) time.
- Trying random keys takes O(N) time (where N, usually, 2^k, is the number of possible keys)
- Can we balance "memory" and "time" resources?
- It is not a 50-50 proposition. Hellman showed we could cut the search time to $O(N^{(1/2)})$ by pre-computing and storing $O(N^{(1/2)})$ values.

Chain of Encryptions

- Assume block length n and key length k are equal: $n = k$
- Construct chain of encryptions:

$$SP = K_0$$

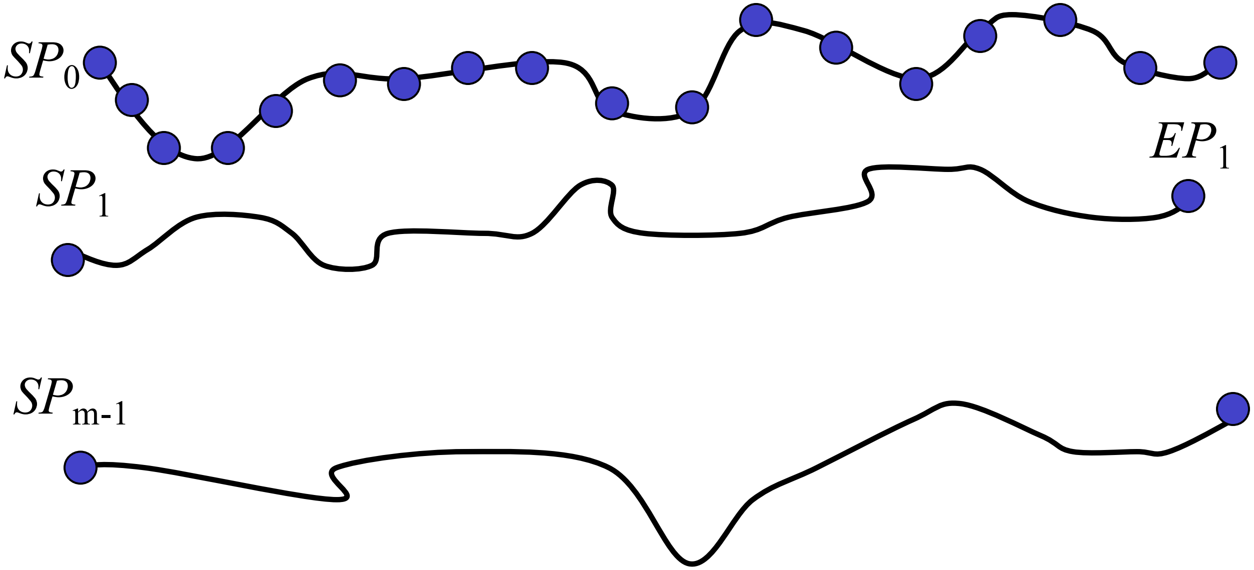
$$K_1 = E(P, SP)$$

$$K_2 = E(P, K_1)$$

$$\vdots$$

$$\vdots$$

$$EP = K_t = E(P, K_{t-1})$$



- Pre-compute m encryption chains, each of length $t + 1$
- Save only the start and end points

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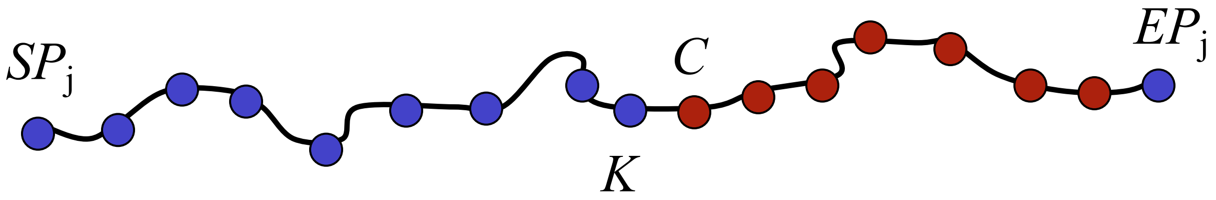
TMTO Attack

- To attack a particular unknown key K
  - For the same chosen P used to find chains, we know C where $C = E(P, K)$ and K is unknown key
  - Compute the chain (maximum of t steps)

$$X_0 = C,\ X_1 = E(P, X_0),\ X_2 = E(P, X_1), \ldots$$

- Suppose for some i we find $X_i = Ep_j$
- Since $C = E(P, K)$ key K should lie before ciphertext C in chain!



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DES TMTO

- Suppose block cipher has $k = 56$
- Suppose we find $m= 2^{28}$ chains each of length $t= 2^{28}$ and no chains overlap (unrealistic)
- Memory: 2^{28} pairs (SP_j, EP_i)
- Time: about 2^{28} (per attack)
  - Start at C, find some EP_j in about 2^{27} steps
  - Find K with about 2^{27} more steps
- Attack never fails!

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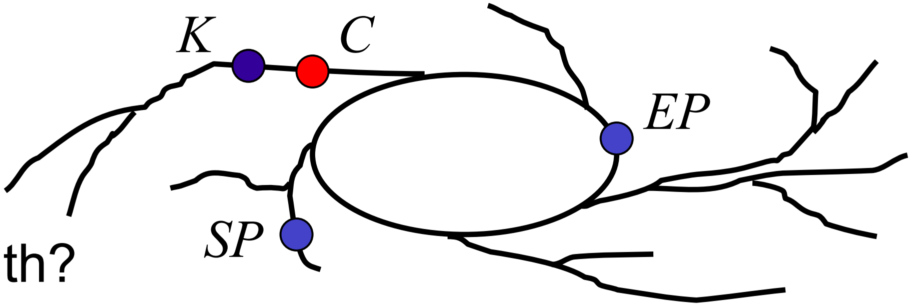
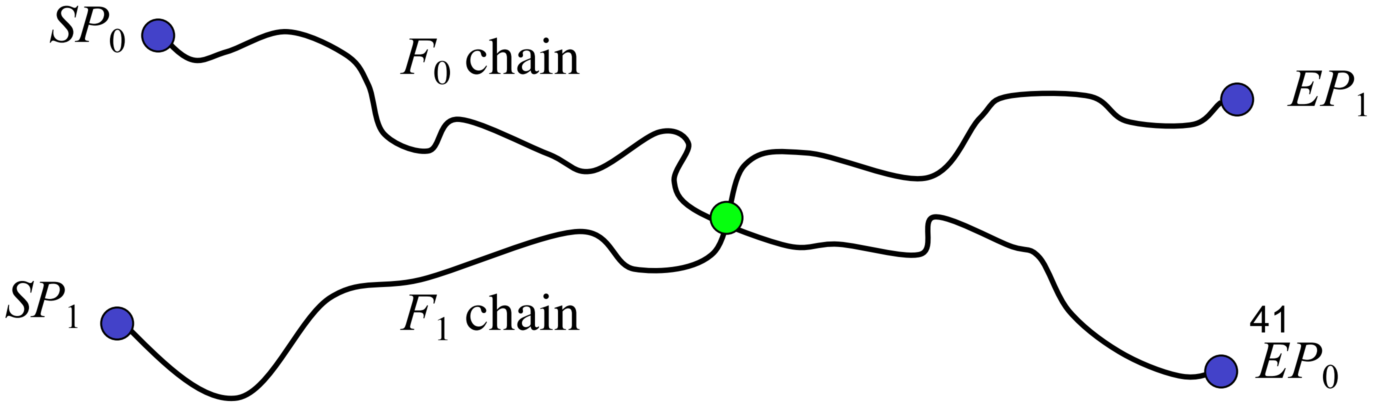
But things are a little more complicated



- Chains can cycle and merge
- False alarms, etc.
- What if block size not equal key length?
    - This is easy to deal with
- To reduce merging
    - Compute chain as $F(E(P, K_{i-1}))$ where F permutes the bits
    - Chains computed using different functions can intersect, but they will **not** merge



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TMTO in Practice

- Let
  - m= random starting points for each F(# chains/table)
  - t = Length of each chain
  - r= number of “tables”, i.e., random functions
- Then mtr = total pre-computed chain elements
- Pre-computation is about mtr work
- Each TMTO attack requires
  - About mr “memory” and about tr “time”
- Choose $m= t= r= 2^{k/3}$, mtr= 2^k.

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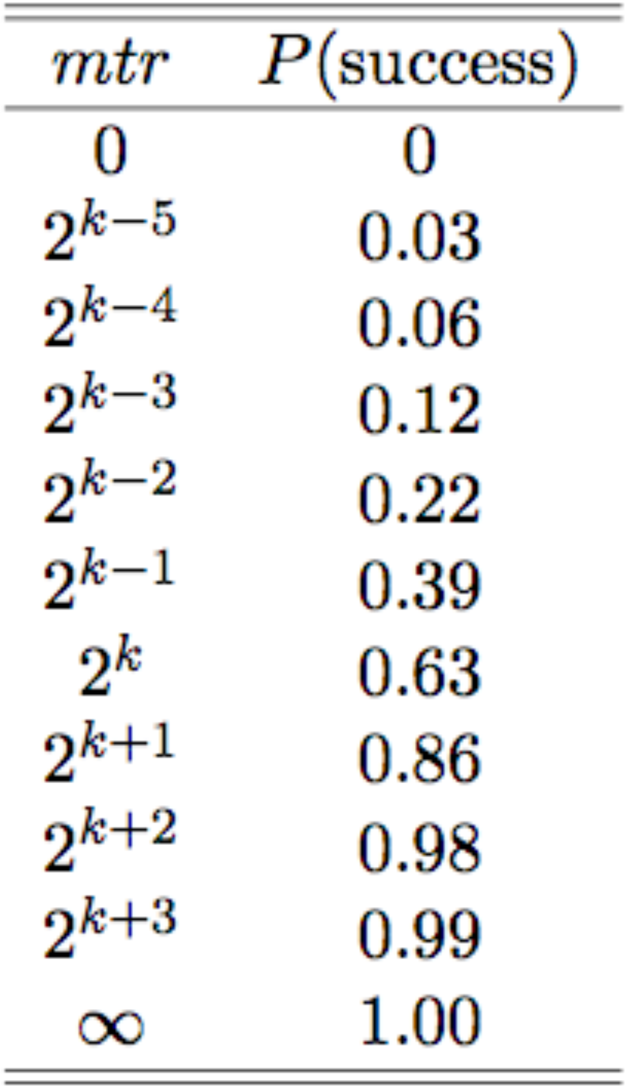
Success Probability

- Throw *n* balls into *m* urns
- What is expected number of urns that have at least one ball?
  - See Feller, *Intro. to Probability Theory*
- Why is this relevant to TMTO attack?
  - "Urns" correspond to keys
  - "Balls" correspond to constructing chains
- Assuming *k*-bit key and *m, t, r* defined as previously discussed
- Then, approximately,

  $P(\text{success}) = 1 - e^{-mtr/k}$



mtr	$P(\text{success})$
0	0
2^{k-5}	0.03
2^{k-4}	0.06
2^{k-3}	0.12
2^{k-2}	0.22
2^{k-1}	0.39
2^{k}	0.63
2^{k+1}	0.86
2^{k+2}	0.98
2^{k+3}	0.99
∞	1.00

Slide adapted from
Mark Stamp

Group theory and DES

- What is the minimum length of a product of involutions from a fixed set required to generate S_n?
- What does this have to do with the number of rounds in a cipher?
- How does this affect the increased security by “enciphering twice” with different keys?

- **Theorem** (Coppersmith and Grossman): If σ_K(L,R)= (L$\oplus$f(E(R)$\oplus$K , R), < τ, σ_K>= A_N, N= 2^n.
- Note (Netto): If a and b are chosen at random from S_n there is a good chance (~¾) that <a,b>= A_n or S_n .

DES is not a group

- Set $E_1(x)$= $DES_{0xffffffffffffffff}(x)$, $E_0(x)$= $DES_{0x0000000000000000}(x)$.
- $F(x)$= $E_1(E_0(x))$.
- There is an x: $F^m(x)$= x, $m \sim 2^{32}$, a cycle length.
- If $|F|=n$, $m|n$.
- Suppose DES is closed under composition so $F=E_k=DES_k$.
- E_k^i= E_k^j, $E_k^{(j-i)}$= I. $0 \leqq i<j \leqq 2^{56}$.
- Coppersmith found lengths of cycles for 33 plaintexts and the LCM of these cycle lengths $>2^{277}$.

If DES were a group…

- Suppose $E_{K1}(E_{K2}(x))= E_{K3}(x)$, that there are N possible keys, plaintexts and ciphertexts and that for a given plaintext-ciphertext pair there is only one possible key then there is a birthday attack that finds the key in

  $O(N^{(1/2)})$.

- Construct $D_{K1}(x)$ for $O(N^{(1/2)})$ random keys, K_1 and $E_{K2}(x)$ for $O(N^{(1/2)})$ random keys, K_2. If there is a match, $c=E_{K1}(E_{K2}(x))$. This has the same effect as finding K_3.

DES Key Schedule

- C_0D_0= $PC_1(K)$
- C_{i+1}= LeftShift(Shift$_i$, C_i), D_{i+1}= LeftShift(Shift$_i$, D_i).
- K_i= $PC_2(C_i \,||D_i)$
- Shift$_i$= <1,2,2,2,2,2,2,1,2,2,2,2,2,2,1,1>

- Note: Irregular Key schedule protects against related key attacks. [Biham, New Types of Cryptanalytic Attacks using Related Keys, TR-753, Technion]

Weak Keys

- DES has:
  - Four weak keys k for which $E_k(E_k(m))= m$.
  - Twelve semi-weak keys which come in pairs k_1 and k_2 and are such that $E_{k1}(E_{k2}(m))= m$.
  - Weak keys are due to “key schedule” algorithm

How Weak Keys Arise

- A 28 bit quantity has potential symmetries of period 1, 2, 4, 7, and 14.
- Suppose each of C_0 and D_0 has a symmetry of period 1; for example C_0 =0x0000000, D_0= 0x1111111. We can easily figure out a master key (K) that produces such a C_0 and D_0.
- Then $DES_K(DES_K(x))=x$.

Interlude: Useful Math for Boolean Functions

- Algebraic Representations
- Linear Functions
- Affine approximations
- Bent Functions: functions furthest from linear
- Hadamard transforms
- MDS, linear codes, RS codes
- Random Functions
- Correlation and Correlation Immunity

- Some Notation:
  - Let $L_1(P) \oplus L_2(C) = L_3(K) \oplus c$ with probability p_i
  - $e_i = |1- p_i|$ called the “bias”

Boolean Functions

- The distance between two boolean functions f and g is $d(f,g)=\#\{X|f(X)\neq g(X)\}$.
- *Distance*: For Boolean function f(X) and g(X), $d(f,D)= \min_{[g(X)\in D]} d(f,g)$
- *Affine function*: $h(x)= a_1x_1 \oplus a_2x_2 \oplus \ldots \oplus a_nx_n \oplus c$
- *nl(f)* denotes the minimum distance between f(X) and the set of affine functions D_{affine}. *nl(f)= d(f, D_{affine})*, *D_{affine}= RM(1,n).*
- *Balance*: f(X) is balanced iff there is an equal number of 0's and 1's in the output of f(X).
- *Algebraic normal form* (ANF):
- *Degree*: deg(f),the highest degree term in ANF.
  - Example: $f(X)= x_1+x_2$, deg(f)=1,$g(X)=x_1x_2$, deg(g)=2
- **Lagrange Interpolation Theorem:** Every function in n variables can be expressed as a polynomial (hence ANF).
- Degree is not the best measure of nonlinearity. $f(x_1,\ldots,x_n)= x_1\oplus\ldots\oplus x_n\oplus x_1\ldots x_n$ has high degree but differs from a linear function at only 1 of 2^n possible arguments.

Example: polynomial representation

- If f is boolean function on n variables $x_1, x_2, \ldots, x_n$ and $\mathbf{a}=(a_1, a_2, \ldots, a_n)$ then $f(x_1, x_2, \ldots, x_n)= \sum_{\mathbf{a}} g(\mathbf{a})\, x_1^{a1}\, x_2^{a2} \ldots, x_n^{an}$ where $g(\mathbf{a}) = \sum_{\mathbf{b<a}} f(b_1, b_2, \ldots, b_n)$. Here **b<a** means the binary representation of b does not have a 1 unless there is a corresponding 1 in the representation of a.

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	1
1	0	0	0
0	1	0	1
1	1	0	0
0	0	1	1
1	0	1	0
0	1	1	0
1	1	1	1

- g(0,0,0)= f(0,0,0)=1
- g(0,1,0)=f(0,0,0)+f(0,1,0)=0
- g(1,0,0)=f(0,0,0)+f(1,0,0)=1
- g(1,1,0)=f(0,0,0)+f(1,0,0))+f(0,1,0))+f(1,1,0)=0
- g(0,0,1)=f(0,0,0)+f(0,0,1)=0
- g(0,1,1)=f(0,0,0)+f(0,0,1) +f(0,1,0)+f(0,1,1)=1
- g(0,0,1)= g(1,0,1)= g(0,1,1)= g(1,1,1)= 0

- $f(x_1, x_2, x_3)= 1+x_1+x_2\, x_3$

Best affine approximation of f_1

- f_1

```
0000 0001 0010 0011 0100 0101 0110 0111
   1    0    0    1    0    1    1    0
 1000 1001 1010 1011 1100 1101 1110 1111
    0    1    1    0    0    1    1    0
```

- $\mathcal{W}(f)(w)=F(w) = 2^{-n} \sum_x (-1)^{f(x)\oplus(w,x)}$
- As polynomial: $1+x_4+x_3+x_2+x_1+x_2x_1$
- Spectrum:

```
0000  0001  0010  0011  0100  0101  0110  0111
0.00  0.00  0.00  0.50  0.00  0.00  0.00 -0.50
1000  1001  1010  1011  1100  1101  1110  1111
0.00  0.00  0.00 -0.50  0.00  0.00  0.00 -0.50
```

- $L(x)= x_3+x_4$ is best linear approximation. $\text{dist}(f_1, L(x))= 8\ (.5+1)=12$, so they disagree on 16-12=4 values

Differential Characteristics

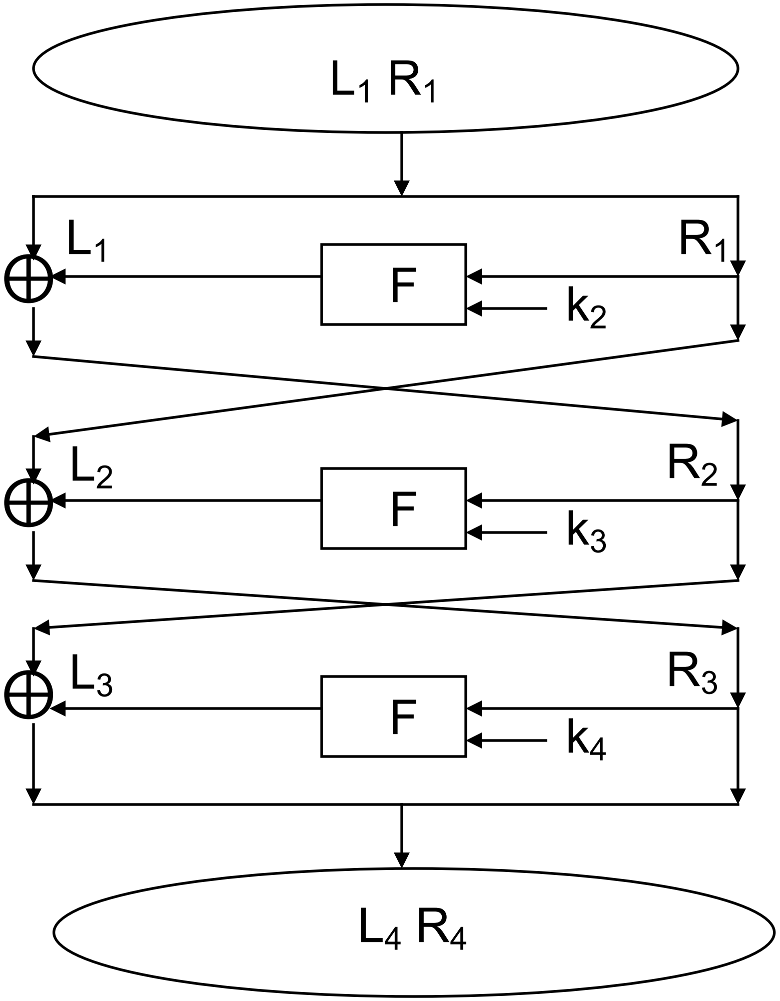
- Let E and E* be inputs to a cipher and C and C* be corresponding outputs with $E \oplus E^* = E'$ and $C \oplus C^* = C'$.
- The notation E’ → C’, p means the “input xor”, E’ produces the “output xor” C’ with probability p. Not all input/output xors and possible and the distribution is uneven. This can be used to find keys. E’ → C’, p is called a *characteristic*.
- Notation: $D_j(x',y') = \{u: S_j(u) \oplus S_j(u \oplus x') = y'\}$. $k_j \varepsilon x \oplus D_j(x',y')$
- For the characteristic 0x34→d in S-box 1 from inputs $1 \oplus 35 = 34$, $D_1(34,d) = \{06, 10, 16, 1c, 22, 24, 28, 32\}$ and $k_j \varepsilon \{7, 10, 17, 1d, 23, 25, 29, 33\} = 1 \oplus D_1(34,d)$

Differential Cryptanalysis – 3 rounds

- $L_1 \oplus L_3 = f(k_2, R_1)$. ………. (1)
- $L_4 \oplus L_3 = f(k_4, R_3)$. ………. (2)
- $R_4 = R_3$, $L_2 = R_1$, $L_3 = R_2$.

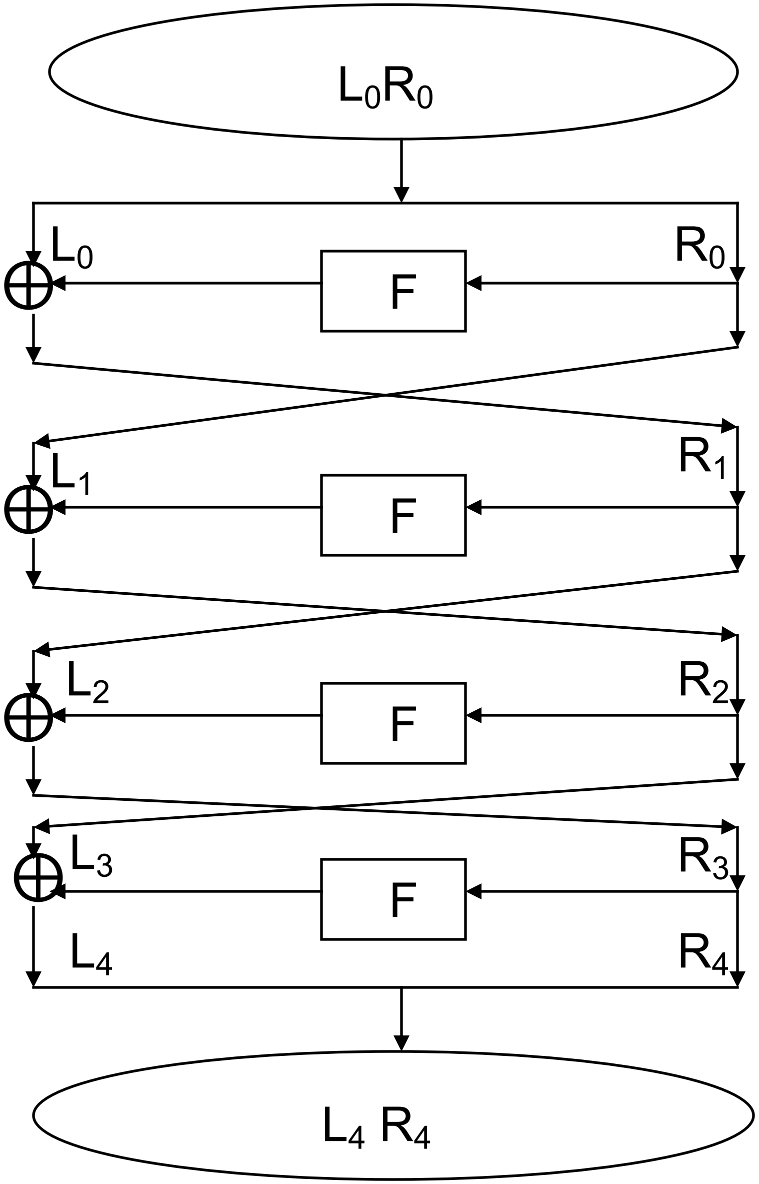
- 1&2 → $L_4 \oplus L_1 = f(k_2, R_1) \oplus f(k_4, R_3)$.

- $L_4 \oplus L_1 = f(k_2, R_1) \oplus f(k_4, R_3)$. ……..(3)
- $L_4{}^* \oplus L_1{}^* = f(k_2, R_1{}^*) \oplus f(k_4, R_3{}^*)$. …. (4)
- 3&4 → $L_4' \oplus L_1' = f(k_2, R_1{}^*) \oplus f(k_4, R_3{}^*) \oplus f(k_2, R_1{}^*) \oplus f(k_4, R_3{}^*)$.
- $R_1 = R_1{}^*$ → $L_4' \oplus L_1' = f(k_4, R_3) \oplus f(k_4, R_3{}^*)$.



Simplified DES

- L_{i+1}= R_i, each 6 bits.
- R_{i+1}= $L_i \oplus f(R_i,K_i)$
- K is 9 bits.
- $E(x)= (x_1\ x_2\ x_4\ x_3\ x_4\ x_3\ x_5\ x_6)$
- S_1
  - `101 010 001 110 011 100 111 000`
  - `001 100 110 010 000 111 101 011`
- S_2
  - `100 000 110 101 111 001 011 010`
  - `101 011 000 111 110 010 001 100`
- K_i is 8 bits of K starting at ith bit.



Differential Cryptanalysis – 3 rounds

L_1, R_1 : 000111 011011

L_1*, R_1*: 101110 011011

L_1', R_1': 101001 000000

L_4, R_4 : 100101 000011

L_4*, R_4*: 011000 100100

L_4', R_4': 111101 100111

$E(R_4)$: 0000 0011

$E(R_4')$: 1010 1011

$L_4' \oplus L_1'$: 111 101⊕101 001= 010 100.

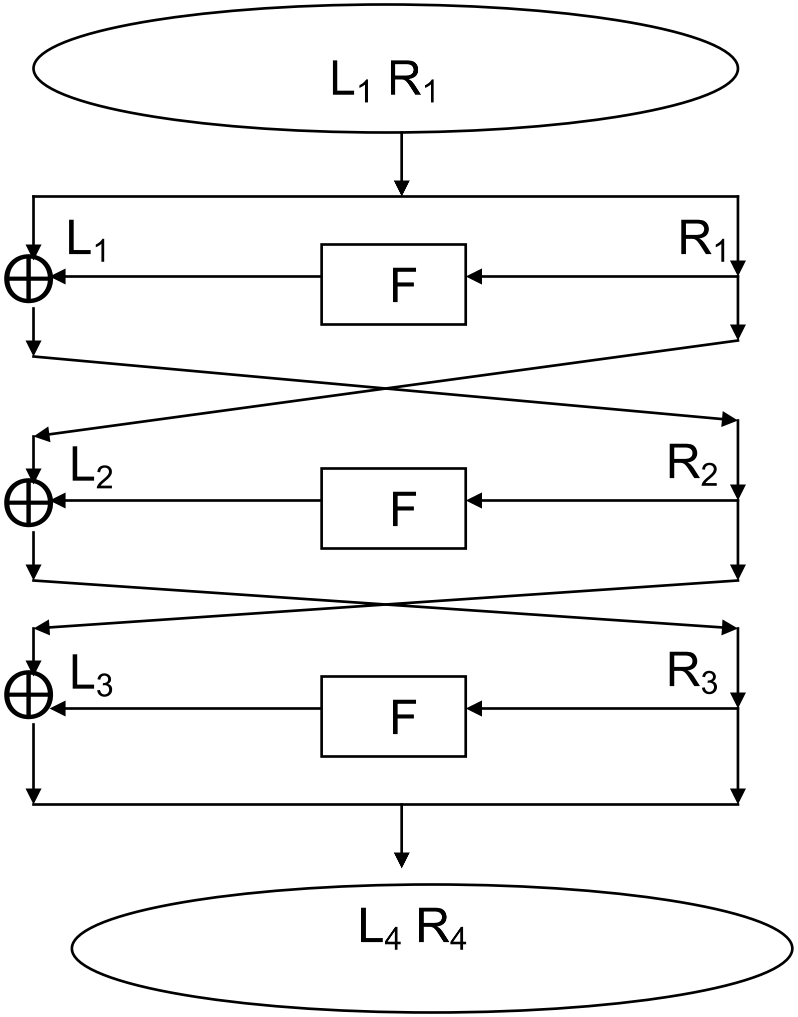
S_1': 1010 → 010(1001,0011).

S_2': 1011 → 100(1100,0111).

$(E(R_4 \oplus k_4)_{1..4}$=1001|0011, k_4= 1001|0011.

$(E(R_4) \oplus k_4)_{5..8}$= 1100|0111,$k_4$= 1111|0100.

K= 00x001101



Differential Cryptanalysis 4 rounds

Pick

```
    L0', R0': 011010 001100.
```

Then

```
    E(R0'):   0011 1100.
    0011 → 011 with p=3/4
    1100 → 010 with p=1/2
```

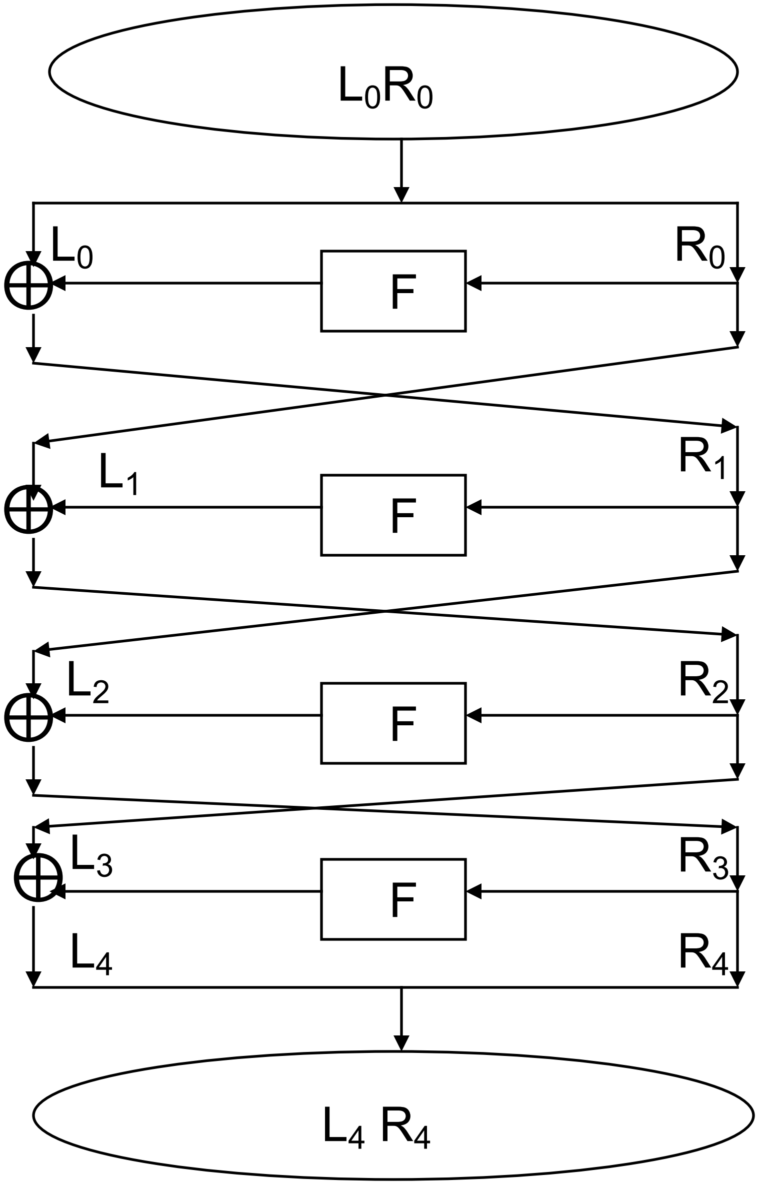
So

```
    f(R0',k1)= 011 010, p=3/8.
```

Thus

```
    L1',R1': 001100, 000000 p=3/8.
```

- 3/8 of the pairs with this differential produce this result. 5/8 scatter the output differential at random.



Estimating cost of Differential Attack

- Given m pairs of text, p the probability of a right pair, k the number of keys, γ the number of suggested keys per right pair and λ the ratio of non-discarded pairs to the total number of pairs
- Average count is $\frac{\lambda\gamma m}{k}$
- $SN = \frac{mp}{\frac{\gamma\lambda m}{k}} = \frac{kp}{\gamma\lambda}$
- Right pairs are binomially distributed and for small p can be Poisson approximated by X ~ P(m, p)

Comments on Differential Cryptanalysis of DES

# Rounds	Needed pairs	Analyzed Pairs	Bits Found	# Char rounds	Char prob	S/N	Chosen Plain
4	2^{3}	2^{3}	42	1	1	16	2^{4}
6	2^{7}	2^{7}	30	3	1/16	2^{16}	2^{8}
8	2^{15}	2^{13}	30	5	1/10486	15.6	2^{16}
16	2^{57}	2^{5}	18	15	$2^{-55.1}$	16	2^{58}

DES S-Box Design Criteria

- No S-box is linear or affine function of its input.
- Changing one bit in the input of an S-Box changes at least two output bits.
- S-boxes were chosen to minimize the difference between the number of 1’s and 0’s when any input bit is held constant.
- $S(X)$ and $S(X\oplus 001100)$ differ in at least 2 bits
- $S(X)^1S(X\oplus 11xy00)$

Comments on effect of components on Differential Cryptanalysis

- E
  - Without expansion, there is a 4 round iterative characteristic with p= 1/256
- P
  - Major influence. If P=I, there is a 10-round characteristic with p= $2^{-14.5}$ (but other attacks would be worse).
- S Box order
  - If S1, S7 and S4 were in order, there would be a 2 round iterative characteristic with p= 1/73. However, Matsui found an order (24673158) that is better and also better against Linear crypto. Optimum order for LC resistance: 27643158.
- S properties
  - S boxes are nearly optimum against differential crypto

Linear Cryptanalysis

- Basic idea:
  - Suppose $\alpha_i(P) \oplus \beta_i(C) = g_i(k)$ holds with g_i, linear, for i= 1, 2, …, m.
  - Each equation imposes a linear constraint and reduces key search by a factor of 2.
  - Guess (n-m-1) bits of key. There are $2^{(n-m-1)}$. Use the constraints to get the remaining keys.
- Can we find linear constraints in the "per round" functions and knit them together?
- No! Per Round functions do not have linear constraints.

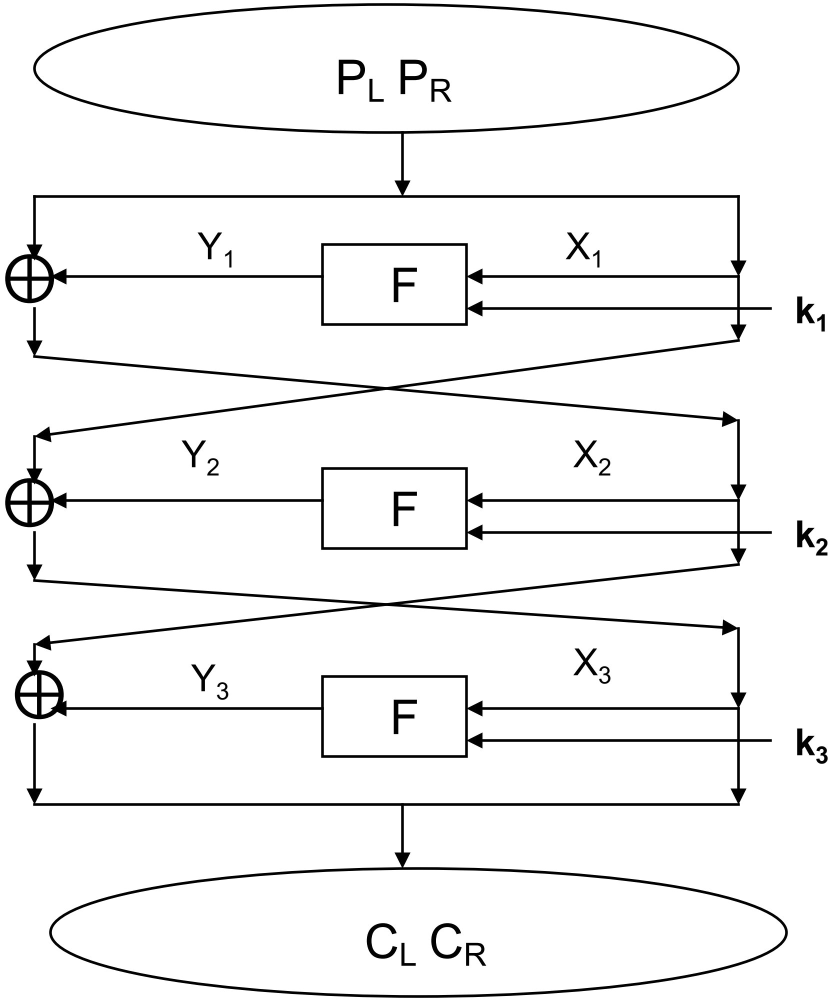
Linear Cryptanalysis

- Next idea
  - Can we find $\alpha(P) \oplus \beta(C) = L(k)$ which holds with L, linear, with probability p?
  - Suppose $\alpha(P) \oplus \beta(C) = L(k)$, with probability p>.5.
  - Collect a lot of plain/cipher pairs.
  - Each will “vote” for $L(k)=0$ or $L(k)=1$.
  - Pick the winner.

- $p = 1/2 + \boldsymbol{\varepsilon}$ requires $c\, \boldsymbol{\varepsilon}^{-2}$ texts (we’ll see why later).
- $\boldsymbol{\varepsilon}$ is called “bias”.

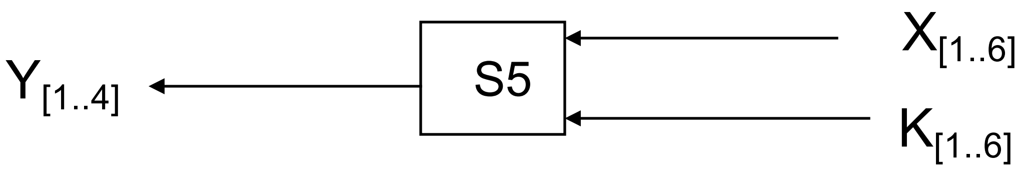
Linear Cryptanalysis Notation

- Matsui numbers bits from right to left, rightmost bit is bit 0. FIPS (and everyone else) goes from left to right starting at 1. I will use the FIPS conventions. To map Matsui positions to everyone else's:
  - M(i)= 64-EE(i). For 32 bits make the obvious change.
- Matsui also refers to the two portions of the plaintext and cipher-text as (P_H, P_L), (C_H, C_L), we'll stick with (P_L, P_R), (C_L, C_R).



Linear and near linear dependence

- Here is a linear relationship over GF(2) in S5 that holds with probability 52/64 (from $NS_5(010000,1111) = 12$:



- $X[2] \oplus Y[1] \oplus Y[2] \oplus Y[3] \oplus Y[4] = K[2] \oplus 1$.
- Sometimes written: $X[2] \oplus Y[1,2,3,4] = K[2] \oplus 1$.
- You can find relations like this using the “Boolean Function” techniques we describe a little later
- After applying P, this becomes

  $X[17] \oplus F(X,K)[3,8,14,25] = K[26] \oplus 1$

Linear Cryptanalysis of 3 round DES

$X[17] \oplus Y[3,8,14,25] = K[26] \oplus 1$, $p = 52/64$

- Round 1

$X_1[17] \oplus Y_1[3,8,14,25] = K_1[26] \oplus 1$

$P_R[17] \oplus P_L[3,8,14,25] \oplus R_1[3,8,14,25] =$
$K_1[26] \oplus 1$

- Round 3

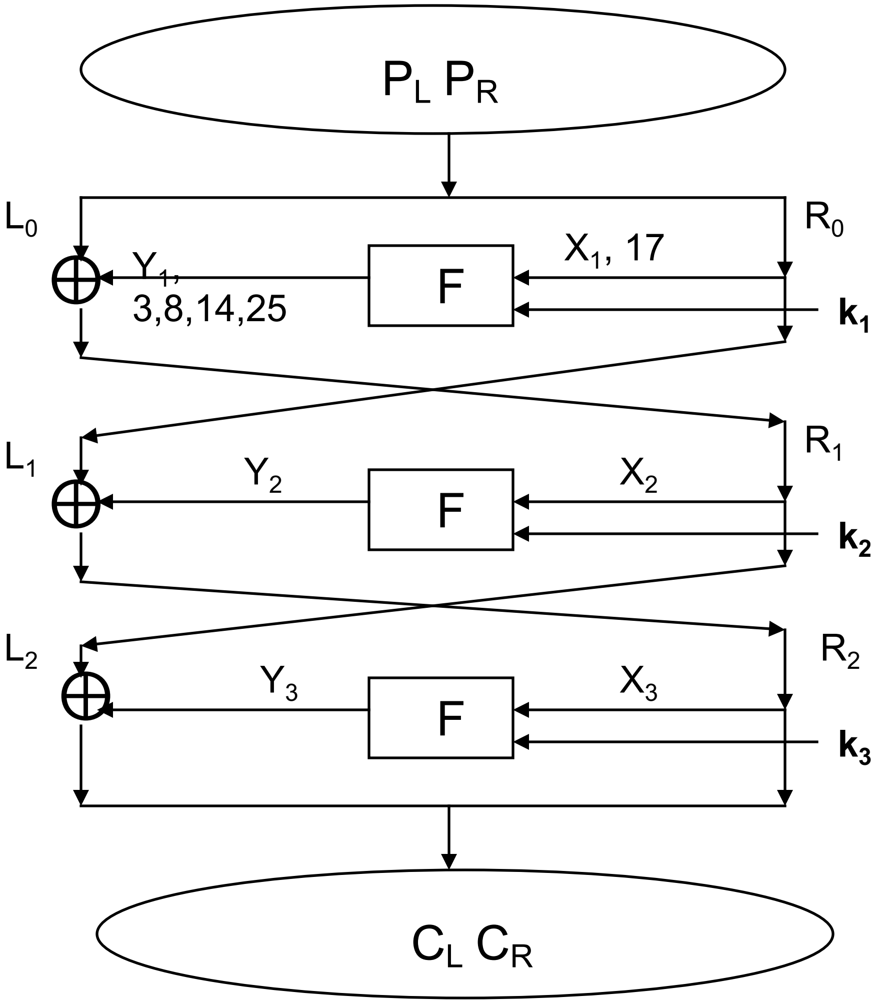
$X_3[17] \oplus Y_3[3,8,14,25] = K_3[26] \oplus 1$

$R_1[3,8,14,25] \oplus C_L[3,8,14,25] \oplus C_R[17] =$
$K_3[26] \oplus 1$

- Adding the two get:

$P_R[17] \oplus P_L[3,8,14,25] \oplus C_L[3,8,14,25] \oplus C_R[17]$
$=$
$K_1[26] \oplus K_3[26]$

Thus holds with $p = (52/64)^2 + (12/64)^2 = .66$



Piling up Lemma

- Let X_i ($1 \leqq i \leqq n$) be independent random variables whose values are 0 with probability p_i. Then the probability that $X_1 \oplus X_2 \oplus \ldots \oplus X_n = 0$ is

$$\tfrac{1}{2} + 2^{n-1} \prod_{[1,n]} (p_i - 1/2)$$

Proof:

By induction on n. It's tautological for n=1.

Suppose $\Pr[X_1 \oplus X_2 \oplus \ldots \oplus X_{n-1} = 0] = q = \tfrac{1}{2} + 2^{n-2} \prod_{[1,n-1]} (p_i - 1/2)$.

Then $\Pr[X_1 \oplus X_2 \oplus \ldots \oplus X_n = 0] = qp_n + (1-q)(1-p_n) = \tfrac{1}{2} + 2^{n-1} \prod_{[1,n]} (p_i - 1/2)$ as claimed.

Mathematics of biased voting

- Central Limit Theorem. Let $X, X_1, \dots, X_n$ be independent, identically distributed random variables and let $S_n = X_1+X_2+ \dots +X_n$. Let $m= E(X)$ and $\sigma^2=E((X-\mu)^2)$. Finally set $T_n= (S_n-n\mu)/(\sigma\sqrt{}n)$, $n(x)= 1/(\sqrt{}2\pi) \exp(-x^2/2)$ and

$$N(a,b)= \int_{[a,b]} n(x)\, dx.$$

Then

$$\Pr(a \leqq T_n \leqq b)= N(a,b).$$

- N is the Normal Distribution; it is symmetric around $x=0$.
- $N(-\infty,0)= \frac{1}{2}$.
- $N(-.5, .5)=.38$, $N(-.75,.75)= .55$, $N(-1,1)= .68$,
- $N(-2,2)=.9546$, $N(-3,3)= .9972$

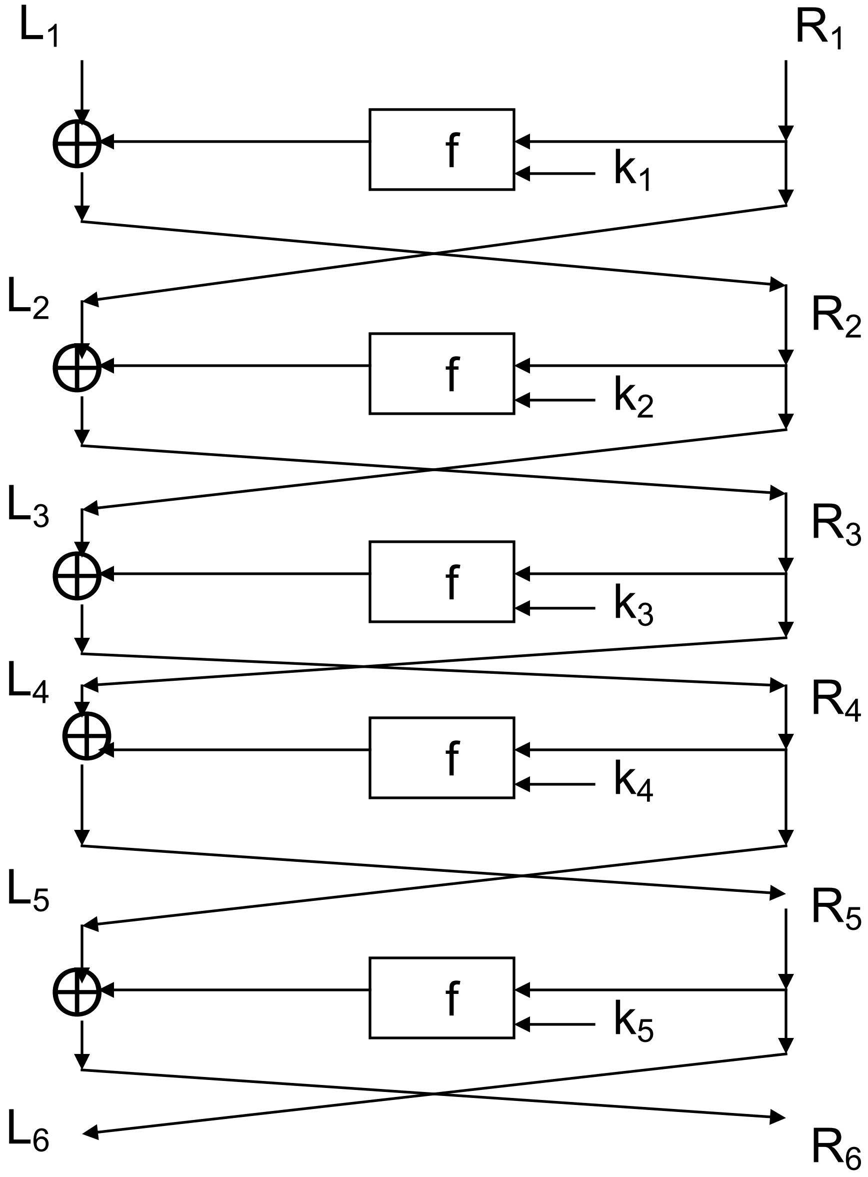
Application of CLT to LC

- $p= \frac{1}{2}+\epsilon$, $1-p= \frac{1}{2}-\epsilon$. Let $L(k,P,E_k(P))=0$ be an equation over GF(2) that holds with probability p. Let X_i be the outcome (1 if true, 0 if false) of an experiment picking P and testing whether L holds for the real k.
- $E(X_i)= p$, $E((X_i-p)^2)= p(1-p)^2 + (1-p)(0-p)^2= p(1-p)$. Let T_n be as provided in the CLT.
- Fixing n, what is the probability that more than half the X_i are 1 (i.e.- What is the probability that n random equations vote for the right key)?
- This is just $\Pr(T_n \boldsymbol{\sigma} -\epsilon\sqrt{n}/\sqrt{(1/4-\epsilon^2)})$. If $n=d^2 \epsilon^{-2}$, this is just $\Pr(T_n \boldsymbol{\sigma} -d/ \sqrt{(1/4-\epsilon^2)})$ or, if ϵ is small $\Pr(T_n \boldsymbol{\sigma}-2d)$.
- Some numerical values: d= .25, N(-.5, ∞) = .69, d= .5, N(-1, ∞) = .84,

d= 1, N(-2, ∞) = .98, d= 1.5, N(-3, ∞) = .999.

End

Thank you IBM, and collaborators, for a great cipher.


L1
R1
f
k1
L2
R2
f
k2
L3
R3
f
k3
L4
R4
f
k4
L5
R5
f
k5
L6
R6

…