

# Cryptanalysis

## Lattices

John Manferdelli

JohnManferdelli@hotmail.com

© 2004-2025, John L. Manferdelli.

*This material is provided without warranty of any kind including, without limitation, warranty of non-infringement or suitability for any purpose. This material is not guaranteed to be error free and is intended for instructional use only.*

# Lattices

- The set  $\Lambda = \mathbb{Z}b_1 + \mathbb{Z}b_2 + \dots + \mathbb{Z}b_n$ , where  $b_1, b_2, \dots, b_n$  are linearly independent is called a lattice.
- $\Lambda^* = \{y \in \mathbb{Z}^n : (x, y) \in \mathbb{Z}, \forall x \in \Lambda\}$
- $\text{vol}(\Lambda) = \det(b_1, b_2, \dots, b_n)$ , where  $b_1, b_2, \dots, b_n$  are the generators of  $\Lambda$ . Note that any set of generators will do since they are related by unimodular transformations.
- Let  $\Lambda$  be a lattice
  - The CVP problem is: Find  $v \in \Lambda$ :  $\|v\| = \min_{w \in \Lambda, w \neq 0} (\|w\|)$
  - The  $\text{CVP}_\gamma$  problem is: Find  $v \in \Lambda$ :  $\|v\| \leq \gamma \cdot \min_{w \in \Lambda, w \neq 0} (\|w\|)$
- Volume of  $n$ -dimensional sphere:  $V_n(r) \approx \frac{1}{\sqrt{n\pi}} \left( \sqrt{\frac{2\pi e}{n}} r \right)^n$

# Definitions

- Hermite Normal Form (HNF)

$$\begin{bmatrix} > 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \geq 0 & > 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \geq 0 & \vdots & > 0 & \ddots & \vdots & 0 & \dots & 0 \\ \geq 0 & \geq 0 & \geq 0 & \dots & 0 & 0 & \dots & 0 \\ \geq 0 & \geq 0 & \geq 0 & \dots & > 0 & 0 & \dots & 0 \end{bmatrix}$$

# Minkowski's Theorem

- Let  $\Lambda$  be a lattice in  $\mathbb{R}^n$  and suppose  $S \subseteq \mathbb{R}^n$  is a convex, centrally symmetric region. If  $\text{vol}(S) > 2^n \det(\Lambda)$  then  $S$  has a non-zero lattice point of  $\Lambda$ .

Suppose first that  $\Lambda'$  is the simple lattice generated by  $e_1, e_2, \dots, e_n$ . Represent a point  $r \in S$  as  $r = (\alpha_1 + x_1, \alpha_2 + x_2, \dots, \alpha_n + x_n)$  with  $\alpha_i \in \mathbb{Z}$  and  $|x_i| \leq 1$ , for  $1 \leq i \leq n$ . Define  $T(r) = (x_1, x_2, \dots, x_n)$ . If  $S_1 \cap S_2 = \emptyset$ ,  $\text{vol}(S_1 \cup S_2) = \text{vol}(S_1) + \text{vol}(S_2)$ . So, if  $S$  has the property that  $T(t) \neq T(s)$ ,  $\forall s \neq t \in S$ , then  $\text{vol}(S) = \text{vol}(T(S))$ . Note that  $\text{vol}(T(S)) \leq 1$ . So, if  $\text{vol}(S) > 1$ , there are at least two points  $r^{(1)} = (\alpha_1^{(1)} + x_1, \alpha_2^{(1)} + x_2, \dots, \alpha_n^{(1)} + x_n)$ ,  $r^{(2)} = (\alpha_1^{(2)} + x_1, \alpha_2^{(2)} + x_2, \dots, \alpha_n^{(2)} + x_n)$ , where  $\alpha_i^{(1)} \neq \alpha_i^{(2)}$  for some  $i$ . Since  $S$  is centrally symmetric,  $-r^{(1)}, -r^{(2)} \in S$ ; finally, note that  $0 \neq r^{(1)} - r^{(2)} \in \mathbb{Z}^n$ . Similarly, if  $\text{vol}(S) > 2^n$ , there are at least  $2^n + 1$  points  $r^{(i)}$ ,  $1 \leq i \leq 2^n + 1$  with  $0 \neq r^{(i)} - r^{(j)} \in \mathbb{Z}^n$ ,  $i \neq j$  for at least two, say  $r^{(i)}$  and  $r^{(j)}$ , all corresponding coordinates in  $r^{(i)} - T(r^{(i)})$  and  $r^{(j)} - T(r^{(j)})$  are equal (mod 2). Thus,  $0 \neq \frac{r^{(i)} - r^{(j)}}{2} \in \mathbb{Z}^n$ . But since  $S$  is convex,  $\frac{r^{(i)} - r^{(j)}}{2} \in S$ . So, the result holds for the simple lattice. Suppose now that  $\Lambda$  is generated by  $a_1, a_2, \dots, a_n$  and put  $A = [a_1, a_2, \dots, a_n]$ .  $e_i = A^{-1}(a_i)$ , so  $\text{vol}(\Lambda') = \frac{\text{vol}(\Lambda)}{\det(\Lambda)}$  and the simple lattice result thus implies the general theorem.

# q-ary lattices and other definitions

- Definition: If  $q \in \mathbb{Z}$ , a lattice,  $\Lambda$ , is called  $q$ -ary if  $q\mathbb{Z}^n \subseteq \Lambda \subseteq \mathbb{Z}^n$ .
- Suppose  $A \in \mathbb{Z}^{m \times n}$ ,  $\Lambda_q(A) = \{y \in \mathbb{Z}^n : y = A^T x \pmod{q}, x \in \mathbb{Z}_q^m\}$ .  
Note  $\Lambda_q(A)$  is  $q$ -ary.
- $\Lambda_q^\perp(A) = \{y \in \mathbb{Z}^n : Ay = 0 \pmod{q}\}$
- $\lambda_1(\Lambda) = \min_{v \in \Lambda} \|v\|$
- $\lambda_n(\Lambda) = \min_S (\max_{v \in S} \|v\|)$ , where  $S \subseteq \Lambda$  is a set of linearly independent vectors,  $|S| = n$
- Solving CVP in  $\Lambda_q^\perp(A)$  when  $A$  is chosen uniformly at random is as hard as worst case CVP.

# Some simple results

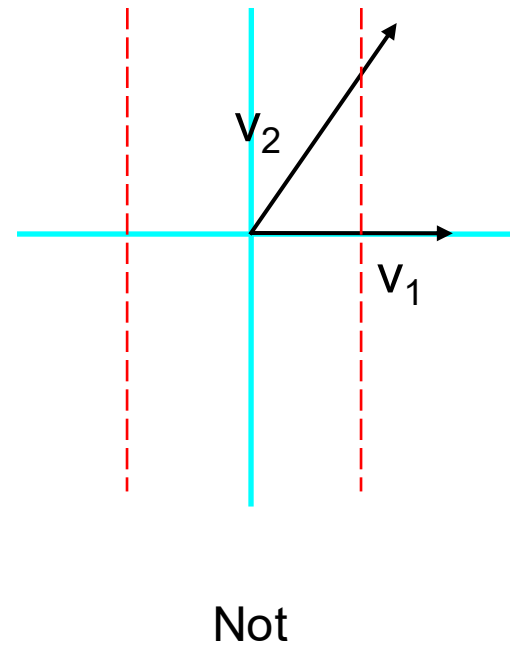
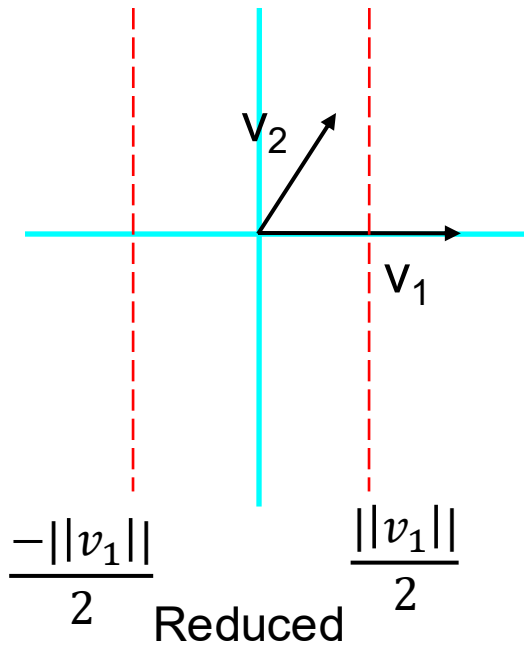
- Remember  $S$  is centrally symmetric if  $s \in S$  implies  $-s \in S$ , and  $S$  is convex if  $s, t \in S$  implies  $us + (1 - u)t \in S, u \in [0,1]$ . We used this in proving Minkowski's Theorem.

- Theorem:  $\lambda_1(\Lambda) \leq \sqrt{n} \det(\Lambda)^{\frac{1}{n}}$

Let  $B_r$  be a ball centered at 0 having radius  $r = \sqrt{n} \det(\Lambda)^{\frac{1}{n}}$ . Let  $(x_1, x_2, \dots, x_n)$  be the coordinates of a vector  $v$ , with respect to the basis generating the lattice  $\Lambda$ , if  $|x_i| \leq 1$  for  $1 \leq i \leq n$ ,  $v \in B_r$ . So  $-\det(\Lambda)^{\frac{1}{n}} (1, 1, \dots, 1)$  and  $\det(\Lambda)^{\frac{1}{n}} (1, 1, \dots, 1)$  as well as the line joining them are in  $B_r$  so  $\text{vol}(B_r) \geq 2^n \det(\Lambda)$  and the result follows from Minkowski's theorem.

# Reduced Basis

- $\langle v_1, v_2 \rangle$  is reduced if
  - $\|v_2\| \leq \|v_1\|$ ; and,
  - $-1/2\|v_1\|^2 \leq (v_1, v_2) \leq 1/2\|v_1\|^2$ .



# Good basis and Gram-Schmidt Orthogonalization

- Good basis for lattices are orthonormal when that is possible. If a basis,  $b_1, b_2, \dots, b_n$  for  $\Lambda$ , is orthonormal, then, for example,  $\text{vol}(\Lambda) = ||b_1|| \cdot ||b_2|| \cdot \dots \cdot ||b_n||$
- The orthogonality defect of a basis  $b_1, b_2, \dots, b_n$  is  $\frac{||b_1|| \cdot ||b_2|| \cdot \dots \cdot ||b_n||}{\det(b_1, b_2, \dots, b_n)}$
- Given a space generated by  $b_1, b_2, \dots, b_n$  can also be generated by a set of vectors,  $b_1^*, b_2^*, \dots, b_n^*$  with the property that  $(b_i^*, b_j^*) = 0, i \neq j$ . The Gram-Schmidt orthogonalization procedure computes this.

GSO, given,  $b_1, b_2, \dots, b_n$ , compute  $b_1^*, b_2^*, \dots, b_n^*$

1. put  $b_1^* = b_1$ .

2. for  $i = 2, i \leq n$

$$b_i^* = b_i - \sum_{j=1}^{i-1} \mu_{i,j} b_j, \mu_{i,j} = \frac{(b_j^*, b_i)}{(b_j^*, b_j^*)}$$



# Size Reduction

- Definition: A basis  $b_1, b_2, \dots, b_n$  is *size reduced* if  $|\mu_{i,j}| \leq \frac{1}{2}$ , in the Gram-Schmidt orthogonalization procedure.
- If  $b_1, b_2, \dots, b_n$  is a basis for  $\Lambda$ , in general,  $b_1^*, b_2^*, \dots, b_n^*$  is not also a lattice basis because  $\mu_{i,j}$  is generally not an integer. We can find a “nearly” orthogonal set of vectors  $b'_1, b'_2, \dots, b'_n$  in  $\Lambda$ , by rounding the  $\mu_{i,j}$ .  $b'_1, b'_2, \dots, b'_n$  is also a basis for the lattice and has the same gram Schmidt basis,  $b_1^*, b_2^*, \dots, b_n^*$ . When performing GSO on this *reduced* basis,  $|\mu_{i,j}| \leq \frac{1}{2}$ .

## Size-reduction

for  $i = 2, i \leq n$

for  $j = i - 1, j \geq 1$

$b_i \leftarrow b_i - \lceil \mu_{ij} \rceil b_j$

for  $k = 1, k \leq j$

$\mu_{ik} \leftarrow \mu_{ik} - \lceil \mu_{ij} \rceil \mu_{jk}$

# Size reduction and basis reordering

- Let  $b_1, b_2, \dots, b_n$  be a basis for  $\Lambda$ , and  $b_1^*, b_2^*, \dots, b_n^*$  the resulting GSO basis. Let  $B_i = ||b_i||^2$ . Then  $b_1, b_2, \dots, b_n$  satisfies the *Lovasz condition* with factor  $\delta$  if it is size reduced and  $(\delta - \mu_{i+1,i}^2)B_i \leq B_{i+1}$ . The LLL algorithm calculates such a basis.

## LLL Algorithm

Given  $b_1, b_2, \dots, b_n$  generating  $\Lambda$ , calculate the LLL reduced basis

1. Reduce the basis  $b_1, b_2, \dots, b_n$  with the size reduction algorithm and calculate  $b_1^*, b_2^*, \dots, b_n^*$  and  $\mu_{ij}$
2. Compute  $B_i = ||b_i^*||^2, i = 1, 2, \dots, n$
3. for  $i = 1, i < n$ 
  4. If  $((\delta - \mu_{i+1,i}^2)B_i > B_{i+1})$ 
    5. Swap  $b_i$  and  $b_{i+1}$
    6. Go to 1
7. return  $b_1, b_2, \dots, b_n$

# Example (LLL including GSO)

- LLL ( $\delta = \frac{3}{4}$ )
- $b_1 = (2,3,14)^T, b_2 = (0,7,11)^T, b_3 = (0,0,23)^T$ .
  - GSO:  $b_1^* = b_1, b_2^* = b_2 - \mu_{21}b_1, \mu_{21} = \frac{(b_1^*, b_2)}{(b_1^*, b_1^*)} = \frac{21+154}{4+9+196} = \frac{175}{209}, \mu_{31} = \frac{322}{209}, \mu_{31} = \frac{3473}{4905}. b_2^* = (-\frac{350}{209}, \frac{938}{209}, -\frac{151}{209})^T$
  - Size reduction:  $b_2 = b_2 - \lceil \mu_{21} \rceil b_1 = (-2,4,-3)^T, \mu_{21} = \mu_{21} - \lceil \mu_{21} \rceil = -\frac{34}{209}; b_3 = b_3 - \lceil \mu_{32} \rceil b_2 = (-2,4,20)^T, \mu_{31} = \mu_{31} - \lceil \mu_{31} \rceil = -\frac{1432}{4905};$  last change is  $b_3 = b_3 - \lceil \mu_{31} \rceil b_1 = (-4,1,6)^T, \mu_{31} = \mu_{31} - \lceil \mu_{31} \rceil = -\frac{79}{209}.$
  - Now,  $b_1 = (2,3,14)^T, b_2 = (-2,4,-3)^T, b_3 = (-4,1,6)^T$ .
  - $B_1 = 209, B_2 = \frac{4905}{209}, B_3 = \frac{103684}{4905}$ . Lovasz condition is not satisfied for  $i = 1$ : since  $(\delta - \mu_{21}^2)B_1 > B_2$ . So swap  $b_1$  and  $b_2$ .
  - Applying GSO we get  $\mu_{21} = \frac{-34}{29}, \mu_{31} = \frac{-6}{29},$  and  $\mu_{32} = \frac{2087}{4905}.$
  - Size reduction produces:  $b_2 = b_2 - \lceil \mu_{21} \rceil b_1 = (0,7,11)^T$  and  $\mu_{21} = \frac{-6}{29}.$   $\mu_{31}$  and  $\mu_{32}$  don't change.  $\mu_{32}$

# Example (LLL including GSO) - continued

- Now Lovasz condition is satisfied for  $i = 1$  since  $(\delta - \mu_{21}^2)B_1 < B_2$ . but not  $i = 2$  since  $(\delta - \mu_{32}^2)B_2 < B_3$ . swap  $b_2$  and  $b_3$ .
  - Now,  $b_1 = (-2, 4, -3)^T$ ,  $b_2 = (-4, 1, 6)^T$ ,  $b_3 = (0, 7, 11)^T$ .  $B_1 = 29$ ,  $B_2 = \frac{1501}{29}$ ,  $B_3 = \frac{103684}{1501}$ . GSO coefficients are  $\mu_{21} = \frac{-6}{29}$ ,  $\mu_{31} = \frac{-5}{29}$ , and  $\mu_{32} = \frac{2087}{1501}$ . Applying size reduction does not affect  $b_2$  or  $\mu_{21}$ .  $b_3 = b_3 - \lceil \mu_{32} \rceil b_2 = (4, 6, 5)^T$ ,  $\mu_{31} = \mu_{31} - \lceil \mu_{32} \rceil \mu_{21} = \frac{1}{29}$ ,  $\mu_{31} = \frac{586}{1501}$ . Both Lovasz conditions now hold.
  - LLL basis is thus  $b_1 = (-2, 4, -3)^T$ ,  $b_2 = (-4, 1, 6)^T$ ,  $b_3 = (4, 6, 5)^T$ . Notice  $\|b_1\|$  is actually the shortest vector in  $\Lambda$ .

# LLL Properties

- Suppose we apply LLL to  $b_1, b_2, \dots, b_n$ , with  $b_1^*, b_2^*, \dots, b_n^*$  and  $B_1, B_2, \dots, B_n$  defined as above. With  $X = \min_{v \in \Lambda} (||b_i||)$  and  $\frac{1}{4} < \delta < 1$ , LLL runs in  $O(n^6 \ln(x)^3)$ .
  - $B_i \leq ||b_i||^2 \leq (\frac{1}{2} + 2^{i-2})B_i$
  - $||b_i|| \leq 2^{\frac{i-1}{2}} ||b_i^*||$
  - $\lambda_1(\Lambda) \geq \min_i (||b_i^*||)$
  - $||b_1|| \leq 2^{\frac{n-1}{2}} \lambda_1(\Lambda)$
  - $\det(\Lambda) \leq \prod_{i=1}^n ||b_i|| \leq 2^{\frac{n(n-1)}{4}} \det(\Lambda)$
  - $||b_i|| \leq 2^{\frac{(n-1)}{4}} \det(\Lambda)^{\frac{1}{n}}$
- If  $w$  is a vector in  $\mathbb{R}^n$  and the lattice basis for  $\Lambda$  is  $b_1, b_2, \dots, b_n$  with  $B = [b_1, b_2, \dots, b_n]$ , the coefficients for  $w$  are  $u = B^{-1}(w)$ .  $w$  is not necessarily in the lattice but if we take each element in  $u$  and round it,  $B \downarrow B^{-1}(w) \in \Lambda$ . This is *Babai rounding*.

# Attack on RSA using LLL

- Attack applies to messages of the form "M xxx" where only "xxx" varies (e.g.- "The key is xxx") and xxx is small.
- From now on, assume  $M(x) = B + x$  where B is fixed
  - $|x| < Y$ .
  - Not that  $E(M(x)) = c = (B + x)^3 \pmod n$
  - $f(x) = (B+x)^3 - c = x^3 + a_2x^2 + a_1x + a_0 \pmod n$ .
- We want to find  $x$ :  $f(x) = 0 \pmod n$ , a solution to this, m, will be the corresponding plaintext.

# Attack on RSA using LLL

- To apply LLL, let:
  - $v_1 = (n, 0, 0, 0)$ ,
  - $v_2 = (0, Yn, 0, 0)$ ,
  - $v_3 = (0, 0, Y^2n, 0)$ ,
  - $v_4 = (a_0, a_1Y, a_2Y^2, a_3Y^3)$
- When we apply LLL, we get a vector,  $b_1$ :
  - $\|b_1\| \leq 2^{(3/4)} |\det(v_1, v_2, v_3, v_4)| = 2^{(3/4)} n^{(3/4)} Y^{(3/2)} \dots \text{Equation 1.}$
- Let  $b_1 = c_1v_1 + \dots + c_4v_4 = (e_0, Ye_1, Y^2e_2, Y^3e_3)$ . Then:
  - $e_0 = c_1n + c_4a_0$
  - $e_1 = c_2n + c_4a_1$
  - $e_2 = c_3n + c_4a_2$
  - $e_3 = c_4$

# Attack on RSA using LLL

- Now set  $g(x) = e_3x^3 + e_2x^2 + e_1x + e_0$ .
- From the definition of the  $e_i$ ,  $c_4 f(x) = g(x) \pmod{n}$ , so if  $m$  is a solution of  $f(x) \pmod{n}$ ,  $g(m) = c_4 f(m) = 0 \pmod{n}$ .
- The trick is to regard  $g$  as being defined over the real numbers, then the solution can be calculated using an iterative solver.
- If  $|y| < 2^{(7/6)}n^{(1/6)}$ ,  $|g(x)| \leq 2||b_1||$ .
- So, using the Cauchy-Schwartz inequality,  $||b_1|| \leq 2^{-1}n$ .
- Thus  $|g(x)| < n$  and  $g(x) = 0$  yielding 3 candidates for  $x$ .
- Coppersmith extended this to small solutions of polynomials of degree  $d$  using a  $d+1$  dimensional lattice by examining the monic polynomial  $f(T) = 0 \pmod{n}$  of degree  $d$  when  $|x| \leq n^{1/d}$ .



# Example attack on RSA using LLL

- $p = 757285757575769$ ,  $q = 2545724696579693$ .
- $n = 1927841055428697487157594258917$ .
- $B = 200805000114192305180009190000$ .
- $c = (B + m)^3, 0 \leq m < 100$ .
- $f(x) = (B + x)^3 - c = x^3 + a_2x^2 + a_1x + a_0 \pmod{n}$ .
  - $a_2 = 602415000342576915540027570000$
  - $a_1 = 1123549124004247469362171467964$
  - $a_0 = 587324114445679876954457927616$
  - $v_1 = (n, 0, 0, 0)$
  - $v_2 = (0, 100n, 0, 0)$
  - $v_3 = (0, 0, 10^4n, 0)$
  - $v_4 = (a_0, a_1100, a_210^4, 10^6)$

# Example attack on RSA using LLL

- Apply LLL,  $b_1 =$ 
  - $308331465484476402v_1 + 589837092377839611v_2 +$
  - $316253828707108264v_3 + (-1012071602751202635)v_4 =$
  - $(246073430665887186108474, -577816087453534232385300,$   
 $405848565585194400880000, -1012071602751202635000000)$
- $g(x) = (-1012071602751202635) t^3 + 40584856558519440088 t^2 +$   
 $(-57781608745353442323853) t + 246073430665887186108474.$
- Roots of  $g(x)$  are  $42.00000000, (-.9496 \pm 76.0796i)$
- The answer is 42.

# GGH Public Key System

- Pick  $n, M \in \mathbb{N}$  and  $\sigma$  is “small”, say  $\sigma = 4$
- Plaintext:  $\mathcal{M} = \{x: -M \leq x \leq M\}$ , Cipher-space:  $\mathcal{C} \in \mathbb{Z}^n$ .
- Gen:
  1. Choose  $B \in \mathbb{Z}^{n \times n}$  with small entries  $|B_{ij}| \leq \sigma$
  2. Check  $B$  is invertible.  $B$  is the secret key.
  3.  $H = \text{HNF}(B)$
- Enc
  1. For  $\vec{m} \in \mathcal{M}^n$ , choose  $\vec{r} \in (-\sigma, \sigma)^n$  uniformly at random
  2.  $\vec{c} = H\vec{m} + \vec{r}$
- Dec
  1. Babai round  $\vec{m} = H^{-1}B \downarrow ((B^{-1}(\vec{c}))) \uparrow$
- Works if  $\downarrow B^{-1}(r) \uparrow = 0$ .

# GGH Example

- $B = \begin{bmatrix} 2 & -3 & 1 & -4 \\ -2 & 1 & 0 & 4 \\ -1 & 3 & 2 & 1 \\ -1 & -4 & 3 & -2 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 44 & 18 & 4 & 49 \end{bmatrix}$
- $B^{-1} = \frac{1}{49} \begin{bmatrix} 61 & 45 & 10 & -27 \\ -10 & -13 & 8 & -2 \\ 29 & 23 & 16 & -4 \\ 33 & 38 & 3 & -13 \end{bmatrix}, H^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-44}{49} & \frac{-18}{49} & \frac{-4}{49} & \frac{1}{49} \end{bmatrix}$
- $m = (3, -4, 1, 3)^T, r = (-1, 1, 1, -1)^T, c = Hm + r = (2, -3, 2, 210)^T$
- $B^{-1}c = \frac{1}{7}(-809, -55, -117, -396)^T, \downarrow B^{-1}c \uparrow = (-116, -8, -17, -57)^T$
- $B \downarrow B^{-1}c \uparrow = (3, -4, 1, 211)^T$
- $m = H^{-1}B \downarrow B^{-1}c \uparrow = (3, -4, 1, 3)^T$

# Learning with Errors (LWE)

- Based on solving noisy linear equations  $\text{mod } q$ . Choose  $\vec{a}_i \in \mathbb{Z}_q^n$  uniformly at random.  $\vec{s} \in \mathbb{Z}_q^n$  is a secret and  $m \geq n$  approximate equations  $\vec{a}_i \cdot \vec{s} = b_i \pmod{q}$ . Errors,  $e_1, e_2, \dots, e_n$  are chosen from distribution  $\chi$ .
- Reduces to LWE:
  - Search LWE problem: Given  $a_{ij}, (\vec{b} + \vec{e})$  find  $\vec{s}$ .
  - Decision LWE: Distinguish, with non-negligible probability, between  $\vec{b} = A\vec{s} + \vec{e}$  and  $\vec{b} \in \mathbb{Z}_q^m$  chosen uniformly at random given  $A, \vec{b}$
- Errors chosen from distribution,  $p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{x^2}{2\sigma^2})$ . Often use  $s = \sigma\sqrt{2\pi}$  as parameter specifying distribution.
- Regev's showed it is possible to pick parameters so that solving an LWE cipher is equivalent to solving worst-case LWE .
  - Theorem (Regev): Let  $n \in \mathbb{N}$  be a security parameter,  $m, q \in \mathbb{N}$ , polynomial in  $n$  and  $\chi = D_{\mathbb{Z},s}$  a discrete Gaussian distribution with  $s = \alpha q > 2\sqrt{n}$ ,  $0 < \alpha < 1$ . Then solving the LWE decision problem is at least as hard as quantumly solving  $SIVP_\gamma$  on an arbitrary  $n$ -dimensional lattice where  $\gamma = \tilde{O}(n/\alpha)$ .

# LWE cryptosystem

- Given  $(n \geq m, l, t, r, q, \chi)$  where  $\chi$  is a probability distribution  $\mathbb{Z}_q$ , message space is  $\mathbb{Z}_2^l$  and cipher space is  $\mathbb{Z}_q^n \times \mathbb{Z}_q^l$ .
- Key Gen
  1. Choose  $S \in \mathbb{Z}_q^{n \times l}$ , uniformly from the distribution  $\chi$ .
  2. Choose  $A \in \mathbb{Z}_q^{m \times n}$ , and  $E \in \mathbb{Z}_q^{m \times l}$  uniformly from the distribution  $\chi$ .
  3. Private key is  $S$ , public key is  $(A, P = AS + E)$
- Enc
  1. For  $\vec{v} \in \mathbb{Z}_2^l$ , choose  $\vec{a} \in \{0,1\}^m$ , uniformly at random
  2.  $\vec{CT} = (\vec{u} = A^T \vec{a}, \vec{c} = P^T \vec{a} + \uparrow \frac{q}{2} \downarrow \vec{v})$
- Dec
  1. Compute  $\uparrow (\uparrow \frac{q}{2} \downarrow)^{-1} (\vec{c} - S^T \vec{u}) \uparrow \pmod{2}$
- Decryption may have errors. Suppose  $\chi$  is a discrete Gaussian  $D_{\mathbb{Z},s}$ . Then  $E^T \vec{a}$  has magnitude  $\leq \sqrt{m}s$  with high probability. Error occurs if  $E^T \vec{a} \geq \frac{q}{4}$ . One can show that for any  $n, \exists q, m, s$  such that the error is small and the underlying LWE problem is hard.

# LWE example

- $n = 4, q = 23, m = 8, \alpha = \frac{5}{23}, s = 5, \sigma = \frac{s}{\sqrt{2\pi}}, l = 4$

- $A^{m \times n} = A = \begin{bmatrix} 9 & 5 & 11 & 13 \\ 13 & 6 & 6 & 2 \\ 6 & 21 & 17 & 18 \\ 22 & 19 & 20 & 8 \\ 2 & 17 & 10 & 21 \\ 10 & 8 & 17 & 11 \\ 5 & 16 & 12 & 2 \\ 5 & 7 & 11 & 7 \end{bmatrix}, S^{n \times l} = S = \begin{bmatrix} 5 & 2 & 9 & 1 \\ 6 & 8 & 19 & 1 \\ 19 & 18 & 9 & 18 \\ 9 & 2 & 14 & 18 \end{bmatrix}$

# LWE example

- $E^{m \times l} = E = \begin{bmatrix} 0 & 22 & 1 & 21 \\ 0 & 22 & 22 & 22 \\ 6 & 21 & 17 & 18 \\ 22 & 22 & 22 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 22 & 1 & 22 \\ 22 & 0 & 0 & 1 \end{bmatrix}, P^{m \times l} = P = \begin{bmatrix} 10 & 5 & 21 & 7 \\ 3 & 1 & 13 & 1 \\ 19 & 15 & 6 & 13 \\ 22 & 22 & 22 & 0 \\ 9 & 20 & 20 & 17 \\ 15 & 21 & 1 & 2 \\ 0 & 12 & 3 & 19 \\ 16 & 2 & 7 & 15 \end{bmatrix},$



# LWE example

- Encrypt  $\vec{v} = (1,0,1,1)^T$ , using  $a = (1,1,0,1,0,0,0,1)^T$ 
  - $\downarrow \frac{23}{2} \vec{v} \uparrow = (12,0,12,12)^T$ ,
  - $(u, c) = \left( A^T a, P^T a + \downarrow \frac{23}{2} m \uparrow \right) = ((3,14,2,7)^T, (4,5,7,5)^T) \pmod{23}$
- Decrypt:
  - $\vec{v}' = c - S^T u = (11,21,12,10)^T \pmod{23}$ ,
  - $\downarrow \frac{1}{12} \vec{v}' \uparrow \pmod{2} = (1,0,1,1)^T$

# LWE example

- Encrypt  $m = (1,0,1,1)^T$ , using  $a = (1,1,0,1,0,0,0,1)^T$ 
  - $\lfloor \frac{23}{2} m \rfloor = (12,0,12,12)^T$ ,
  - $(u, c) = \left( A^T a, P^T a + \lfloor \frac{23}{2} m \rfloor \right) = ((3,14,2,7)^T, (4,5,7,5)^T) \pmod{23}$
- Decrypt:
  - $m' = c - S^T u = (11,21,12,10)^T \pmod{23}$ ,
  - $\lfloor \frac{1}{12} m' \rfloor \pmod{2} = (1,0,1,1)^T$

From Heiko Knopse

# LWE/Ring-LWE parameters

Level	n	q	s	P	P&A	c	Exp
Low	128	4093	8.87	$2.9 \times 10^5$	$7.4 \times 10^5$	$3.8 \times 10^3$	30
High	320	4093	8	$4.9 \times 10^5$	$17.7 \times 10^5$	$17.4 \times 10^3$	42

- Ring-LWE cuts ciphertext by factor of n

# Ring-LWE

- Put  $R = R_{n,q} = \frac{\mathbb{Z}_q[x]}{x^{n+1}+1}$ ,  $n = 2^k$ ,  $R \approx \mathbb{Z}_q^n$ .  $a \in R$ , generates ideal  $(a)$  corresponding to a  $q$ -ary ideal lattice.
- Ring LWE: Given  $a \in R$ , and  $b = as + e$ , for  $s, e \in R$ , find  $s$ .
- Solving R-LWE is at least as hard as solving  $CVP_\gamma$  on arbitrary ideal lattices

# NTRU Public Key System

- NTRU is a ring lattice-based system.
- $R = \frac{\mathbb{Z}[x]}{x^N - 1}$ ,  $R_p = \frac{\mathbb{Z}_p[x]}{x^N - 1}$ ,  $R_q = \frac{\mathbb{Z}_q[x]}{x^N - 1}$
- $(c_0 + c_1x + \cdots + c_{N-1}) = (a_0 + a_1x + \cdots + a_{N-1}) \otimes (b_0 + b_1x + \cdots + b_{N-1})$ , where  $c_k = \sum_{i+j=k \pmod N} a_i b_j$
- $\mathcal{T}(d_1, d_2)$  is the set of “ternary” polynomials of degree  $< N$ , having  $d_1$  coefficients equal to 1, having  $d_2$  coefficients equal to  $-1$ , and remaining coefficients equal to 0.
- Pick  $N, p$  prime and  $q, d \in \mathbb{N}$ ,  $(p, q) = (N, q) = 1$ ,  $q > (6d + 1)p$ .

# NTRU Public Key System

- KeyGen
  1. Pick  $f, g \in R, f \in \mathcal{T}(d+1, d), g \in \mathcal{T}(d, d)$ .
  2. Find  $f_p, f_q: f \cdot f_p = 1 \pmod{p}, f \cdot f_q = 1 \pmod{q}, h = f_q \cdot g \pmod{q}$ .
  3. Public key is  $(N, p, q, h)$ , private key is  $f$ .
- Plaintext is  $m \in R_p$ , ciphertext is  $c \in R_q$
- Encryption
  1. Chose random  $r \in R, r \in \mathcal{T}(d, d)$ .
  2.  $c = prh + m \pmod{q}$ .
- Decryption
  1. Compute  $a = fc \pmod{q}$
  2. Plaintext is  $f_p a$ .
  3. Verify that  $a = fc = f(prh + m) \pmod{q} = pfrf_qg + fm \pmod{q} = prg + fm \pmod{q}$ .

# NTRU Example

- $N = 5, p = 3, q = 29, d = 1, f = x^4 + x^3 - 1, g = x^3 - x^2$
- $f_p = -x^3 - x^2 + x - 1, f_q = -5x^4 + 8x^3 + 3x^2 + 11x + 13$
- $h = f_q g = 8x^4 + 2x^3 + 11x^2 + 13x - 5 \pmod{29}$
- $r = x^4 - x$
- $c = prh + m = 8x^4 + 21x^3 + 25x^2 + 20x + 15 \pmod{29}$
- $a = fc = -2x^4 + 2x^3 + 4x^2 - 3x + 1 \pmod{29}$
- We check  $a = prg + fm$  in  $R$
- $m = x^3 + x$

# Some NIST Round 3 entries

- Public-Key Encryption/KEMs
  - Classic McEliece
  - CRYSTALS-KYBER
  - NTRU
  - SABER
- Digital Signatures
  - CRYSTALS-DILITHIUM
  - FALCON
  - Rainbow
- Public-Key Encryption/KEMs (Alternates)
  - BIKE;
  - FrodoKEM
  - HQC
  - NTRU Prime
  - SIKE
- Digital Signatures
  - GeMSS
  - Picnic
  - SPHINCS+

**Winner:** Dilithium (signing), Kyber (key-encapsulation)



# Some common features of Dilithium and Kyber

- Ring is  $\mathbb{Z}_p[x]/(x^{256} + 1)$  in both cases
  - $p = 3329$  for Kyber
  - $p = 2^{23} - 2^{13} + 1 = 8380417$  for Dilithium
  - So, the same modular arithmetic we all grew up with.
- For  $p = 3329$ , there is a primitive (and hence 128 primitive) 256<sup>th</sup> roots of unity (You are not expected to understand this).
  - As a result,  $x^{256} + 1$  factors into coprime 128 quadratics
  - Allows us to perform a “Number Theory Transform” that turns convolution into pointwise multiplication for ring operations giving a nice speedup
- For  $p = 8380417$ , there is a primitive (and hence 256 primitive) 512<sup>th</sup> roots of unity
  - As a result,  $x^{256} + 1$  factors into coprime 256 linear polys
  - Allows us to perform a “Number Theory Transform” that turns convolution into pointwise multiplication for ring operations giving a nice speedup

# Useful definitions

- $r^+ = r \pmod{q}, q > r^+ \geq 0$
- $r' = r \pmod{\pm}(m)$  means  $r' = r \pmod{m}$  and  $-\frac{m}{2} \leq r' \leq \frac{m}{2}$ , if  $m$  is even;  
 $-\frac{m}{2} < r' \leq \frac{m}{2}$ , if  $m$  is odd
- $decompose(r, \alpha, q)$ 
  - $r_0 = r^+ \pmod{\pm}(\alpha)$
  - if  $r^+ - r_0 == (q - 1)$ 
    - $r_1 = 0, r_0 = q - 1$
  - else
    - $r_1 = \frac{r^+ - r_0}{\alpha}$
  - return  $(r_1, r_0)$

# Useful definitions

- $lowbits(x, \alpha, q)$ 
  - $(r_1, r_0) = decompose(x, \alpha, q)$
  - return  $r_0$
- $highbits(x, \alpha, q)$ 
  - $(r_1, r_0) = decompose(x, \alpha, q)$
  - return  $r_1$
- $power2round(r, d, q)$ 
  - $r^+ = r \bmod(q)$
  - $r_0 = r^+ \bmod^{\pm}(2^d)$
  - return  $(\frac{r^+ - r_0}{2^d}, r_0)$

# Examples

- $decompose(r, \alpha, q)$  examples (second shows roundoff edge case)

$q$	$\alpha$	$r$	$r \bmod^{\pm}(\alpha)$	$r - r \bmod^{\pm}(\alpha)$	$r_0$	$r_1$
17	8	5	-3	8	-3	1
17	8	15	-1	16	-2	0
3329	104	50	50	0	50	0
3329	104	100	-4	104	-4	1

# SHAKE-256/SHAKE-128

- $H(v, d) = \text{SHAKE256}(v, d)$
- $H_{128}(v, d) = \text{SHAKE128}(v, d)$
- $\text{RAWSHAKE256}(J, d) = \text{KECCAK}[512](J||11, d)$
- $\text{SHAKE256}(M, d) = \text{RAWSHAKE256}(M||11, d)$
- $\text{RAWSHAKE128}(J, d) = \text{KECCAK}[256](J||11, d)$
- $\text{SHAKE128}(M, d) = \text{RAWSHAKE128}(M||11, d)$
- Note
  - $\text{SHA3}_{256}(M) = \text{KECCAK}[512](M||11, 256)$
  - $\text{SHA3}_{512}(M) = \text{KECCAK}[1024](M||11, 1024)$

# Number Theory Transform (NTT)

- $p = 3329, p - 1 = 2^8 \cdot 13$ .
- $\mathbb{Z}_p$  has a primitive 256<sup>th</sup> root of unity ( $\zeta = 17$  is a primitive root) but no 512 root of unity, so  $x^{256} + 1$  factors into 128 coprime quadratic factors of the form  $(x^2 - \xi), 17^{128} = -1$ .
- $x^{256} + 1 = \prod_{k=0}^{127} (x^2 - \zeta^{2 \cdot \text{bitrev}_7(k)+1})$ .
- $\text{bitrev}_7(k)$  reverses the bit order in a 7-bit byte,  $k$ .
  - $x^{256} + 1 = (x^2 - 17) \cdot (x^2 - 17^{129}) \cdot \dots \cdot (x^2 - 17^{255})$
- For  $p = 8380417, \zeta = 1753$  is a primitive 512<sup>th</sup> root of unity,
  - $p - 1 = 2^{13}(2^{10} - 1) = 2^{13} \cdot 3 \cdot 11 \cdot 31$ .
- Because of this, an analog of the Chinese remainder theorem holds in
 
$$R_p = \frac{\mathbb{Z}_p(x)}{x^{256} + 1}.$$

# NTT

- $NTT: R_p \rightarrow T_p, f \mapsto \hat{f}$
- For  $f \in R_p$ 
  - $\hat{f} = \left( f \pmod{x^2 - \zeta^{2rev_7(0)+1}} \right)$
- $NTT(f) = (\hat{f}_0, \hat{f}_1, \dots, \hat{f}_{255})$
- For  $\hat{h} = \hat{f} \cdot \hat{g}$ ,
  - $\hat{h}_{2i} + \hat{h}_{2i+1}x = (\hat{f}_{2i} + \hat{f}_{2i+1}x) \cdot (\hat{g}_{2i} + \hat{g}_{2i+1}x) \pmod{x^2 - \zeta^{2rev_7(i)+1}}$

# Dilithium template

- Gen
  - $A \leftarrow R_q^{k \times l}$
  - $(s_1, s_2) \leftarrow S_\eta^l \times S_\eta^k$
  - $t = As_1 + s_2$
  - return  $(pk = (A, t), sk = (A, t, s_1, s_2))$
- Verify( $pk, M, \sigma = (z, c)$ )
  - $w'_1 = \text{highbits}(Az - ct, 2\gamma_2)$
  - return  $\|z\|_\infty < \gamma_1 - \beta \wedge c == H(M || w'_1)$
- Sign( $sk, m$ )
  - $z = \perp$
  - while  $z == \perp$ 
    - $y \leftarrow S_{\gamma_1-1}^l$
    - $w_1 = \text{highbits}(Ay, 2\gamma_2)$
    - $c \in B_{60}, c = H(M || w_1)$
    - $z = y + cs_1$
    - if  $\|z\|_\infty < \gamma_1 - \beta \vee \|\text{lowbits}(Ay - cs_2, 2\gamma_2)\|_\infty \geq \gamma_2 - \beta$ 
      - $z = \perp$
  - return  $\sigma = (z, c)$



# Dilithium security argument -1

Parameters:  $\eta, \gamma_1, \gamma_2$

1.  $A \leftarrow R_q^{k \times l}, (s_1, s_2) \leftarrow S_\eta^l \times S_\eta^k, t = As_1 + s_2, pk = (A, t), sk = (A, t, s_1, s_2)$
2.  $y \leftarrow S_{\gamma_1-1}^l, w_1 = \text{highbits}_{2\gamma_2}(Ay)$ 
  - Write coefficients of  $w = Ay$ , as  $w^{[i]} = (2\gamma_1)w_1^{[i]} + w_0^{[i]}$
  - $w_1 = \text{highbits}_{2\gamma_2}(Ay)$  then  $w_0^{[i]} < \gamma_2$
3.  $c \in B_{60}, c = H(M || w_1)$ . Set  $\beta = \max_i(cs_1^{[i]})$ . Then  $\beta \leq 60\eta$ .
4. Set  $z = y + cs_1$ , if any coefficient of  $z > \gamma_1 - \beta$ , reject and start over.
5. If any coefficient of  $\text{lowbits}_{2\gamma_2}(Az - ct) > \gamma_2 - \beta$ , reject and start over.
  - Note:  $Az - ct = Ay - s_2$
  - coefficients of  $z \leq \gamma_1 - \beta$ , coefficients of  $\text{lowbits}_{2\gamma_2}(Az - ct) \leq \gamma_2 - \beta$
6. Signature is  $\sigma = (z, c)$

Parameters chosen so that expected number of rejections in 4 and 5 is between 4 and 7

# Dilithium security argument - 2

When verifying

- $A\mathbf{z} - c\mathbf{t} = A\mathbf{y} - \mathbf{s}_2$
- To show  $\text{highbits}_{2\gamma_2}(A\mathbf{z} - c\mathbf{t}) = \text{highbits}_{2\gamma_2}(A\mathbf{y})$ , we need only show  $\text{highbits}_{2\gamma_2}(A\mathbf{y}) = \text{highbits}_{2\gamma_2}(A\mathbf{y} - c\mathbf{s}_2)$ .
  - This follows because  $|\text{lowbits}_{2\gamma_2}(A\mathbf{y} - c\mathbf{s}_2)|_\infty < \gamma_2 - \beta$ ; and,
  - The coefficients of  $\|c\mathbf{s}_2\|_\infty < \beta$
  - Adding  $c\mathbf{s}_2$  never causes a carry of  $\gamma_2$  from the lowbits and, hence,  $\text{highbits}_{2\gamma_2}(A\mathbf{z} - c\mathbf{t}) = \text{highbits}_{2\gamma_2}(A\mathbf{y})$

# Dilithium (simplified)

- Remember  $A^{k \times l}$  is generated randomly from  $R = \mathbb{Z}_p[x]/(x^{256} + 1)$ .
- $s_1$  is a vector of dimension  $l$  with entries from  $R$  has random coefficients  $\leq \eta$
- $s_2$  is a vector of dimension  $k$  with entries from  $R$  has random coefficients  $\leq \eta$
- $t = As_1 + s_2$

Sign

```
 $y := S_{\gamma_1-1}^l$   
 $w_1 := \text{highbits}(Ay, 2\gamma_2)$   
 $c := SH(M || w_1)$   
 $z := y + cs_1$   
return  $(z, c)$ 
```

Verify

```
 $w_1' := \text{highbits}(Az - ct, 2\gamma_2)$   
 $c' := SH(M || w_1')$   
Check  $c' == c$  AND  $\|z\|_\infty <$   
 $\gamma_1 - \beta$ 
```

# Dilithium (less simplified)

**Parameters:**  $p = 8380417$ ,  $k = 5$ ,  $l = 4$ ,  $\gamma_1 = \frac{p-1}{16}$ ,  $\gamma_2 = \frac{\gamma_1}{2}$ ,  $\eta = 5$ ,  $\beta = 275$

- KeyGen

- $A \in R^{k \times l}$ , selected from random distribution over  $R_p$
- $(s_1, s_2) \in S_\eta^k \times S_\eta^l$ , selected at random,  $S_\eta^k$  consists of elements of  $R^k$  with coefficients  $\leq \eta$
- Set  $t = As_1 + s_2$
- Public key is  $(A, t)$ , Private key is  $(s_1, s_2)$

For the sake of compression  $A$  is generated from a seed and SHAKE-256

# Dilithium

- $\text{Sign}(\text{pk}, \text{sk}, M)$  --- simplified

1.  $z = \perp$
2. **while** ( $z = \perp$ ) {
3.    $y = S_{\gamma_1}^l - 1$
4.    $w_1 = \text{highbits}(Ay, 2\gamma_2)$
5.    $c = \text{SHAKE} - 256(M || w_1)$
6.    $z = y + cs_2$
7.   **if** ( $||z||_\infty \geq \gamma_1 - \beta$ ) **OR**  $\text{lowbits}(Ay - cs_2, 2\gamma_1) \geq \gamma_2 - \beta$ ) **then**  $z = \perp$
8. }

Signature is  $(z, c)$

- Real Dilithium uses a number of functions to generate  $A$  from a seed. It also has a hedged version and a deterministic version. The hedged version avoids some possible side channels.

# Dilithium

- $\text{Verify}(\text{pk}, M, z, c)$  --- simplified
  1.  $w'_1 = \text{highbits}(Az - ct, 2\gamma_2)$
  2. Return true if  $\|z\|_\infty \leq \gamma_1 - \beta$  AND  $c = \text{SHAKE} - 256(M \| w'_1)$ , otherwise return false

# Useful definitions

- *// Calculate  $c(x)$ , coefficients are 1, -1 or 0*
- *sampleinball( $\rho, \tau$ )*
  - *$c(x) := 0; k=8;$*
  - *for( $i = 256 - \tau; i < 256; i++$ )*
    - *while( $H(\rho)[k] > i$ )*
      - $k++$*
      - $j = H(\rho)[k]$*
      - $c_i = c_j$*
      - $c_j = (-1)^{H(\rho)[i+\tau+256]}$*
      - $k++$*
  - *return  $c$*

# Useful definitions

- $usehint(h, r)$ 
  - $m = \frac{p-1}{2\gamma_2}$
  - $(r_1, r_0) = decompose(r, 2\gamma_2, p)$
  - If  $h == 1$  and  $r_0 > 0$  then return  $(r_1 + 1)mod(m)$
  - If  $h == 1$  and  $r_0 \leq 0$  then return  $(r_1 - 1)mod(m)$
  - return  $r_1$
- $makehint(z, r)$ 
  - $r_1 = highbits(r)$
  - $v_1 = highbits(r + z)$
  - return  $r_1 \neq v_1$



# Useful definitions

- *RejNTTPoly*( $\rho$ ) // returns NTT polynomial
  - $c = 0 ; j = 0$
  - while ( $j < 256$ )
    - $\hat{a}[j] = \text{coeffFromThreeBytes}(H_{128}(\rho||c), H_{128}(\rho||c + 1), \dots, H_{128}(\rho||c + 2))$
    - $c += 3$
    - If ( $\hat{a}[j] \neq \perp$ ) **then**  $j++$
  - return  $\hat{a}$
  
- *RejBoundedPoly*( $\rho$ )
  - $c = 0 ; j = 0$
  - while ( $j < 256$ )
    - $z = H(\rho)[c]$
    - $z_0 = \text{CoeffFromHalfByte}(z \bmod(16), \eta)$
    - $z_1 = \text{CoeffFromHalfByte}(\lfloor z/16 \rfloor, \eta)$
    - If ( $z_0 \neq \perp$ )
      - $a_j = z_0; j++$
    - If ( $z_1 \neq \perp$  and  $j < 256$ )
      - $a_j = z_1; j++$
    - $c++$
  - return  $a$

# Useful definitions

- *ExpandA*( $\rho$ )
  - *for* ( $r = 0; r < k; k++$ )
    - for* ( $s = 0; s < l$ )
      - $\hat{A}[r, s] = \text{RejNTTPoly}(\rho || \text{IntegerToBits}(s, 8) || \text{IntegerToBits}(r, 8))$
  - return*  $\hat{A}$
- *ExpandS*( $\rho$ )
  - *for* ( $r=0; r<l; r++$ )
    - $s_1[r] = \text{RejBoundedPoly}(\rho || \text{IntegerToBits}(r, 16))$
  - *for* ( $r=0; r<k; r++$ )
    - $s_2[r] = \text{RejBoundedPoly}(\rho || \text{IntegerToBits}(r + l16))$
  - return* ( $s_1, s_2$ )
- *ExpandMask*( $\rho, \mu$ )
  - $c = 1 + \text{bitlen}(\gamma_1 - 1)$
  - *for*( $r = 0; r < l; r++$ )
    - $n = \text{IntegerToBits}(\mu + r, 16)$
    - $v = (H(\rho || n)[32rc], H(\rho || n)[32rc + 1], \dots, H(\rho || n)[32rc + 32c - 1])$
    - $s[r] = \text{BitUnpack}(v, \gamma_1 - 1, \gamma_1)$
  - *return*  $s$

# NTT for Dilithium

- $NTT(w)$  --- outputs  $\widehat{w}_j = (w(\zeta_0), w(-\zeta_0), w(\zeta_1), w(-\zeta_1), \dots, w(-\zeta_{127}))$ 
  - $for(j = 0; j < 256; j++) \widehat{w}[j] = w[j]$ 
    - $k = 0; len = 128$
    - $while(len \geq 1)$ 
      - $start = 0$
      - $while(start < 256)$ 
        - $k++$
        - $zeta = \zeta^{bitrev(k)} \bmod(q)$
        - $for(j = start; j \leq start + len - 1)$ 
          - $t = zeta \cdot \widehat{w}[j + len]$
          - $\widehat{w}[j + len] = \widehat{w}[j] - t$
          - $\widehat{w}[j] = \widehat{w}[j] + t$
        - $start += 2 \cdot len$
      - $len = len/2$

# NTT for Dilithium

- $NTT^{-1}(\hat{w})$ 
  - *for*( $j = 0; j < 256; j++$ )  $w[j] = \hat{w}[j]$
  - $k = 256; len = 1$
  - *while*( $len < 256$ )
    - $start = 0$
    - *while* ( $start < 256$ )
      - $k--$
      - $zeta = \zeta^{bitrev(k)} \bmod(q)$
      - *for*( $j = start; j \leq start + len - 1$ )
        - $t = w[j]$
        - $w[j] = t + w[j + len]$
        - $w[j + len] = t - w[j + len]$
        - $w[j + len] = zeta \cdot w[j + len]$
        - $start += 2 \cdot len$
    - $len = len/2$
  - $f = 8347861$
  - *for*( $j = 0; j < 256; j++$ )  $w[j] = f \cdot w[j]$

# Dilithium, unedited, motivation

- Basic scheme is Fiat-Shamir MSA-DL with aborts.
- Classic version with discrete log is:
  - Prover and verifier know  $(g, y = g^x)$ . Prover knows  $x$ .
  - 1. Prover generates  $r$ , sends commitment  $g^r$ .
  - 2. Verifier sends  $c$ .
  - 3. Prover returns  $s = r - cx$ .
  - 4. Verifier can check  $g^s \cdot y^c = g^r$
- Non interactive version replaces  $c$  with hash of  $g^r || M$

# Dilithium, unedited, motivation

- Preliminary lattice version is prover generates:  $A \in \mathbb{Z}_q^{k \times l}$ ,  $S_1 \in \mathbb{Z}_q^{l \times n}$ ,  $S_2 \in \mathbb{Z}_q^{k \times n}$ , with short coefficients and computes  $t = AS_1 + S_2$ . Public key is  $(A, t)$ . Private key is  $(S_1, S_2)$ 
  1. Prover generates  $y \in \mathbb{Z}_q^l$  with “small coefficients”. Sends commitment as  $Ay$
  2. Verifiers sends challenge  $c \in \mathbb{Z}_q^n$  with small coefficients
  3. Prover returns  $z = y + S_1c$ .
  4. Verifier checks coefficients of  $z$  are small and that  $Az - tc \approx Ay$
- To avoid having  $z$  leak  $S_1$ , signer applies rejection sampling to  $z$ .
- Dilithium
  1. Uses elements of  $R_q = \frac{\mathbb{Z}_q[x]}{x^{256}+1}$  rather than  $\mathbb{Z}_q$ .
  2. Uses a seed,  $\rho$ , to generate  $A$
  3. Compresses  $t$  by dropping low order bits
  4. Signs a message representative,  $\mu$ , which is a hash of the public key and the message
  5. Uses a rounded version of  $w = Ay$ ,  $w_1$ .
  6. Provides a hint,  $h$ , to help reconstruct  $w_1$  from  $z$

# Dilithium parameters for security category 5

Parameter	Meaning	Value
$q$	modulus	8380417
$d$	# dropped bits from $t$	13
$\tau$	# $\pm 1$ s in $c(x)$	60
$\lambda$	Collision strength	256
$\gamma_1$	Coefficient range of $y$	$2^{19}$
$\gamma_2$	Low order rounding range	$\frac{q-1}{32}$
$(k, l)$	Dimensions of $A$	(8,7)
$\eta$	Private key range	2
$\beta = \tau \cdot \eta$		120
$\omega$	Max # of 1's in hint	75

ML-DSA-87	Private	Public	Sig
Size (Bytes)	4864	2592	4595

# Dilithium, Keygen

- Keygen

1.  $\xi := \mathbb{Z}_2^{256}$  (random)
2.  $(\rho, \rho', K) := H(\xi, 1024)$ , (256, 512, 256) bits respectively
3.  $\hat{A} := \text{ExpandA}(\rho)$
4.  $(s_1, s_2) := \text{ExpandS}(\rho')$
5.  $t := NTT^{-1}(\hat{A} NTT(s_1)) + s_2$
6.  $(t_1, t_0) := \text{Power2Round}(t, d)$
7.  $pk := pkEncode(\rho, t_1)$
8.  $tr := H(\text{BytesToBits}(pk), 512)$
9.  $sk := skEncode(\rho, K, tr, s_1, s_2, t_0)$
10. return  $(pk, sk)$



# Dilithium, Sign

- Sign

1.  $(\rho, K, tr, s_1, s_2, t_0) := skdecode(sk)$
2.  $\hat{s}_1 := NTT(s_1), \hat{s}_2 := NTT(s_2), \hat{s}_1 := NTT(t_0)$
3.  $\hat{A} := ExpandA(\rho)$
4.  $\mu := H(tr || M, 512)$
5.  $rnd := \mathbb{Z}_2^{256}$
6.  $\rho' := H(K || rnd || \mu, 512)$
7.  $\kappa = 0$
8. **while**(1) {
  - a.  $y = ExpandMask(\rho', \kappa)$
  - b.  $w := NTT^{-1}(\hat{A} NTT(y)), w_1 := highbits(w, 2\gamma_2)$
  - c.  $\tilde{c} := H(\mu || w_1 Encode(w_1), 2\lambda)$
  - d.  $(\hat{c}_1, \hat{c}_2) := \text{first 256 and last } 256 - 2\lambda \text{ bits}$
  - e.  $c := SampleBall(\hat{c}_1); \hat{c} = NTT(c)$
  - f.  $cs_1 := NTT^{-1}(\hat{c} \hat{s}_1); cs_2 := NTT^{-1}(\hat{c} \hat{s}_2);$
  - g.  $z := y + cs_1$
  - h.  $r_0 := lowbits(w - cs_2)$
  - i. **If**  $(||z||_\infty \geq \gamma_1 - \beta \text{ or } ||r_0||_\infty \geq \gamma_2 - \beta)$  **then continue**
  - j.  $ct_0 := NTT^{-1}(\tilde{c} t_0); h := makehint(-ct_0, w - cs_2 + ct_0)$
  - k. **If**  $(||ct_0||_\infty < \gamma_2 \text{ and } \# \text{ 1's in } h \leq \omega)$  **then break**
  - l.  $\kappa += l$
9.  $\sigma := sigEncode(\tilde{c}, z \bmod^\pm(q), h)$

# Dilithium, Verify

- Verify

1.  $(\rho, t_1) := pkdecode(pk)$
2.  $(\tilde{c}, z, h) := sigdecode(\sigma)$
3.  $\hat{A} := ExpandA(\rho)$
4.  $tr := H(BytestoBits(pk), 512)$
5.  $\mu := H(tr || M, 512)$
6.  $(\tilde{c}_1, \tilde{c}_2) := \text{first 256 and last } 256 - 2\lambda \text{ bits}$
7.  $c := SampleBall(\tilde{c}_1)$
8.  $w'_{approx} := NTT^{-1}(\tilde{A} \cdot NTT(z) - NTT(c)NTT(t_1 2^d))$
9.  $w'_1 := usehint(h, w'_{approx})$
10.  $\tilde{c}' := H(\mu || w1Encode(w'_1, 2\lambda))$
11. return  $\|z\|_\infty < \gamma_1 - \beta$  and  $\tilde{c} == \tilde{c}'$  and  $\# \text{ 1's in } h \leq \omega$

# Useful Kyber definitions

- $PRF_{\eta}(s, b) = \text{shake256}(s || b, 64 \cdot \eta)$
- $XOF(\rho, i, j) = \text{shake128}(\rho || i || j) \rho \rho$
- $H(s) = \text{sha3}_{256}(s)$
- $J(s) = \text{shake256}_{32}(s)$
- $G(s) = \text{sha3}_{512}(s)$
- $NTT$  and  $NTT^{-1}$  are different for Kyber and Dilithium
- Kyber is based on the Fujisak-Okamoto transform

# Useful definitions

- Parse:  $\mathcal{B}^* \rightarrow R_q^n$
- Input:  $B = b_0, b_1, \dots \in \mathcal{B}^*$
- Output:  $\hat{a} \in R_q, a \in R_q$   
     $i = 0; j = 0;$   
    while  $j < i$   
         $d = b_i + 256 \cdot b_{i+1}$   
        if  $d < 19q$   
             $\hat{a}_j = d$   
             $j++$   
         $i += 2$   
    return  $\hat{a}_0 + \hat{a}_1x + \dots + \hat{a}_{n-1}x^{n-1}$

# Useful definitions

- $SamplePolyCBD(B, \eta)$  --- samples from (Central Binomial) distribution  $D_\eta(R_q)$   
 Output:  $f \in R_q$   
 $b := ByteToBits(B)$   
 $for(i = 0; i < 256; i++)$   
 $x = \sum_{j=0}^{\eta-1} b[2i\eta + j]; y = \sum_{j=0}^{\eta-1} b[2i\eta + \eta + j]$   
 $f[i] := (x - y) \bmod(q)$   
 return  $f$
- $Sample(a_1, a_2, \dots, a_\eta, b_1, \dots, b_\eta) \leftarrow \{0,1\}^{2\eta}$ , output  $\sum_{i=1}^{\eta} (a_i - b_i)$
- For central binomial distribution with  $N = 10000, p = \frac{1}{2}, \sigma = \sqrt{Np(1-p)}$ ,
- $P(4900 \leq n_1 \leq 5100) = \sum_{j=4900}^{5100} \binom{N}{j} p^j (1-p)^{N-j} \approx \Phi\left(\frac{5100-5000}{50}\right) - \Phi\left(\frac{4900-5000}{50}\right)$ ,
- $\Phi$  is CDF for normal distribution

# Useful definitions

- $encode_d(x)$ ,  $x$  is an array of length 256,  $m = 2^d$ ,  $1 \leq d \leq 12$ 
  - for ( $i = 0; i < 256; i++$ )
    - $a = x[i]$
    - for ( $j = 0; j < d; j++$ )
      - $b[d \cdot i + j] = a \pmod{2}$
      - $a = \frac{a - b[d \cdot i + j]}{2}$
  - return bits-to-bytes( $b$ )
- $decode_d(x)$ ,  $x$  is a byte array of length  $32d$ ,  $m = 2^d$ ,  $1 \leq d \leq 12$ 
  - $b = \text{bytes-to-bits}(x)$
  - for ( $i = 0; i < 256; i++$ )
    - $out[i] = \sum_{j=0}^{d-1} b[i \cdot d + j] \cdot 2^j$
  - return  $out$

# Useful definitions

- *SampleNTT*(*b*) --- samples uniformly from  $T_q$   
     $i := 0; j := 0$   
    while ( $j < 256$ )  
         $d_1 = b[i] + 256(b[i + 1] \bmod 16)$   
         $d_2 = b[i + 1]/16 + 16(b[i + 2]$   
        if ( $d_1 < q$ )  
             $\hat{a}[j] = d_1 ; j++$   
        if ( $d_2 < q$  and  $j < 256$ )  
             $\hat{a}[j] = d_2 ; j++$   
         $i += 3$   
    return  $\hat{a}$

# Useful definitions

- $compress(x, d, q)$ 
  - $x \rightarrow \uparrow \frac{2^d}{q} \cdot x \downarrow$
- $decompress(y, d, q)$ 
  - $y \rightarrow \uparrow \frac{q}{2^d} \cdot y \downarrow$
- $compress(decompress(x, d, q), d, q) = x$
- $decompress(compress(y, d, q), d, q) = t, (t - y) \bmod^{\pm}(q) \leq \uparrow \frac{q}{2^{d+1}} \downarrow$



# NTT for Kyber

- $NTT(f)$ 
  - $\hat{f} = f; k = 1$
  - for( $len = 128; len \geq 2; len = len/2$ )
    - for( $start = 128; start < 256; start += 2len$ )
    - $zeta = \zeta^{bitrev(k)} \bmod(q); k++$
    - for( $j = start; j < start + len; j++$ )
      - $t = zeta \cdot \hat{f}[j + len] \bmod(q)$
      - $\hat{f}[j + len] = \hat{f}[j] - t \bmod(q)$
      - $\hat{f}[j] = \hat{f}[j] + t \bmod(q)$
  - return( $\hat{f}$ )

# NTT for Kyber

- $NTT^{-1}(\hat{f})$ 
  - $f = \hat{f}; k = 127$
  - for( $len = 2; len \leq 128 \leq len = 2 \cdot len$ )
    - for( $start = 0; start < 256; start += 2len$ )
    - $zeta = \zeta^{bitrev(k)} \bmod(q); k -= 1$
    - for( $j = start; j < start + len; j ++$ )
      - $t = f[j] \bmod(q)$
      - $f[j] = f[j] + f[j + len] \bmod(q)$
      - $f[j + len] = zeta \cdot (f[j + len] - t) \bmod(q)$
  - return( $f \cdot 3303 \bmod(q)$ )

# Kyber (simplified a little)

- Parameters:  $(p = 3329, R = \frac{\mathbb{Z}_p[x]}{x^{256}+1}, k = 4, \eta = 2), \hat{x} = NTT(x)$
- Make public key
  - $KeyGen_{PKE}$ , generate a Dilithium-like key (see full version)
  - $\hat{t} = \hat{A}\hat{s} + \hat{e}$ ,  $A$  is generated from seed  $\rho$
- $Enc_{PKE}(m, r)$  [ $r \in R^k$  is generated from  $CDB_{\eta_1}$ ,  $e_1$  is generated from  $CDB_{\eta_2}$ ]
  - $\hat{r} = NTT(r)$
  - $u(x) = NTT^{-1}(\hat{A}^T \hat{r}) + e_1$
  - $\mu = decompress_1(decode_1(m)), v = NTT^{-1}(\hat{t}^T \cdot \hat{r}) + e_2 + \mu$
  - $c_1 = encode_{d_u}(compress_{d_u}(u)), c_2 = encode_{d_v}(compress_{d_v}(r))$
  - return  $(c_1, c_2)$
- $Dec_{PKE}(c_1, c_2)$ 
  - $w = v - NTT^{-1}(\hat{s} \cdot NTT(u))$
  - return  $encode_1(compress_1(w))$

# Kyber simplified a little

- $KEMKeygen$ 
  - $z = \mathbb{Z}_2^{256}$  (random)
  - $(ek_{PKE}, dk_{PKE}) = KeyGen_{PKE}()$
  - $ek_{KEM} = ek_{PKE}; dk_{KEM} = dk_{PKE} || e_{PKE} || H(e_{PKE}) || z$
  - return  $(ek_{KEM}, dk_{KEM})$
- $KEMencaps(pk_{KEM})$ 
  1.  $m$  is a random 32-byte value
  2.  $(K, r) = SHA3_{512}(m || H(e_{PKE}))$
  3.  $c = Enc_{PKE}(ek, m, r)$
  4. return  $(K, c)$
- $KEMdecaps(sk_{KEM})$ 
  1.  $m' = Dec_{PKE}(dk, c)$
  2.  $(K', r') = SHA3_{512}(m' || H(e_{PKE}))$
  3.  $\bar{K} = SHAKE256(z || c, 32)$
  4.  $c' = Enc_{PKE}(e_{PKE}, m', r')$
  5. If  $(c == c')$  return  $K'$  else error (Note: failure rate is  $2^{-174}$  for ML-KEM-1024)

# Kyber parameters

Alg	$n$	$q$	$k$	$\eta_1$	$\eta_2$	$d_u$	$d_v$	Strength
KEM-512	256	3329	2	3	2	10	768	128
KEM-768	256	3329	3	2	2	10	1088	192
KEM-1024	256	3329	4	2	2	11	1568	256

ML-KEM-1024 is security category 5

Type	Encap-key	Decap-key	Ciphertext	Key
KEM-512	800	1632	768	32
KEM-768	1184	2400	1088	32
KEM-1024	1568	3168	1568	32

Size in bytes

# Full Kyber

- $KeyGen_{PKE}$ 
  1.  $d = \mathbb{Z}_2^{256}, \text{random}$
  2.  $(\rho, \sigma) = G(d); N = 0$
  3.  $for(i = 0; i < k; i++)$ 
    - $for(j = 0; j < k; j++)$ 
      - $\hat{A}[i, j] = SampleNTT(XOF(\rho, i, j))$
  4.  $for(i = 0; i < k; i++)$ 
    - $s[i] = SamplePolyCBD(PRF_{\eta_1}(\sigma, N)); N++$
  5.  $for(i = 0; i < k; i++)$ 
    - $e[i] = SamplePolyCDB(PRF_{\eta_1}(\sigma, N)); N++$
  6.  $\hat{s} = NTT(s); \hat{e} = NTT(e)$
  7.  $\hat{t} = \hat{A}\hat{s} + \hat{e}$
  8.  $ek_{PKE} = ByteEncode_{12}(\hat{t}) || \rho; dk_{PKE} = ByteEncode_{12}(\hat{s})$
  9.  $return(e_{PKE}, d_{PKE})$

# Full Kyber

$Enc_{PKE}(m, r)$

$N = 0; \hat{t} = ByteDecode_{12}(ek_{PKE}[0:384k]); \rho = ek_{PKE}[384k + 384k + 32]$

$for(i = 0; i < k; i++)$

$for(j = 0; j < k; j++)$

$\hat{A}[i, j] = SampleNTT(XOF(\rho, i, j))$

$for(i = 0; i < k; i++)$

$r[i] = SamplePolyCBD_{\eta_1}(PRF_{\eta_2}(r, N)); N++$

$for(i = 0; i < k; i++)$

$e_1[i] = SamplePolyCBD_{\eta_2}(PRF_{\eta_2}(r, N)); N++$

$e_2 = SamplePolyCBD_{\eta_2}(PRF_{\eta_2}(r, N))$

$\hat{r} = NTT(r)$

$\mathbf{u}(x) = NTT^{-1}(\hat{A}^T \hat{r}) + e_1$

$\mu = decompress_1(decode_1(m)), v = NTT^{-1}(\hat{t}^T \cdot \hat{r}) + e_2 + \mu$

$c_1 = encode_{d_u}(compress_{d_u}(u)), c_2 = encode_{d_v}(compress_{d_v}(r))$

return  $(c_1, c_2)$

# Full Kyber

- $Dec_{PKE}(c_1, c_2)$ 
  1.  $c_1 = c[0:32d_u k]; c_2 = c[32(d_u k + d_v)]$
  2.  $\mathbf{u} = decompress_{d_u}(ByteDecode_{d_u}(c_1))$
  3.  $v = decompress_{d_v}(ByteDecode_{d_v}(c_2))$
  4.  $\hat{s} = ByteDecode_{12}(d_{PKE})$
  5.  $w = v - NTT^{-1}(\hat{s}^T \cdot NTT(\mathbf{u}))$
  6.  $m = ByteEncode_1(compress_1(w))$
  7. return  $m$



# Full Kyber

- $Keygen_{KEM}$ 
  - $z = \mathbb{Z}_2^{256}$  (random)
  - $(ek_{PKE}, dk_{PKE}) = KeyGen_{PKE}()$
  - $ek_{KEM} = ek_{PKE}; dk_{KEM} = dk_{PKE} || ek_{PKE} || H(ek_{PKE}) || z$
  - return  $(ek_{KEM}, dk_{KEM})$
  - $\hat{t} = \hat{A}\hat{s} + \hat{e}$ ,  $A$  is generated from seed  $\rho$
  - return  $(ek_{PKE}, dk_{PKE})$

# Full Kyber

- $KEMencaps(pk_{KEM})$ 
  1.  $m$  is a random 32-byte value
  2.  $(K, r) = G(m || H(ek))$
  3.  $c = Enc_{PKE}(ek, m, r)$
  4. return  $(K, c)$

# Full Kyber

- $KEMdecaps(c, dk)$ 
  1.  $dk_{PKE} = dk[0:384k]$
  2.  $ek_{PKE} = dk[384k:768k + 32]$
  3.  $h = dk[768k + 32:768k + 64]$
  4.  $z = dk[768k + 64:768k + 96]$
  5.  $m' = Dec_{PKE}(dk, c)$
  6.  $(K', r') = G(m' || H(e_k))$
  7.  $\bar{K} = J(z || c, 32)$
  8.  $c' = Enc_{PKE}(ek, m', r')$
  9. If  $(c == c')$  return  $K'$  else error

# Kyber Notes

- Define  $\mathbf{Adv}_{m,k,\eta}^{mlwe} = |\Pr[b' = 1, \mathbf{A} \leftarrow R_q^{m \times k}; (\mathbf{s}, \mathbf{e}) \leftarrow \beta_\eta^k \times \beta_\eta^m; \mathbf{b} = \mathbf{As} + \mathbf{e}; b' = A(\mathbf{A}, \mathbf{b})] - \Pr[b' = 1, \mathbf{A} \leftarrow R_q^{m \times k}; \mathbf{b} \leftarrow R_q^m; b' = A(\mathbf{A}, \mathbf{b})]|$ .
- **Theorem:** Suppose XOF and G are random oracles. For all adversaries, A, there are adversaries, B, C:  $\mathbf{Adv}_{kyber,CPAPKE}^{cpa}(\mathbf{A}) \leq 2 \mathbf{Adv}_{k+1,k,\eta}^{mlwe}(\mathbf{B}) + \mathbf{Adv}_{PRF}^{prf}(\mathbf{C})$
- **Theorem:** Suppose XOF and G are random oracles. For any classical adversary, A, that make at most  $q_{RO}$  to random oracles XOF, H, G there are adversaries B, C of the same running time:  $\mathbf{Adv}_{kyber,CCAKEM}^{cca}(\mathbf{A}) \leq 2 \mathbf{Adv}_{k+1,k,\eta}^{mlwe}(\mathbf{B}) + \mathbf{Adv}_{PRF}^{prf}(\mathbf{C}) + 4\delta q_{RO}$
- **Theorem:** Suppose XOF and G are random oracles. For any quantum adversary, A, that make at most  $q_{RO}$  to random oracles XOF, H, G there are adversaries B, C of the same running time:  $\mathbf{Adv}_{kyber,CCAKEM}^{cca}(\mathbf{A}) \leq 4q_{RO} \sqrt{\mathbf{Adv}_{k+1,k,\eta}^{mlwe}(\mathbf{B}) + \mathbf{Adv}_{PRF}^{prf}(\mathbf{C}) + 8\delta q_{RO}^2}$

# Attacks

The Blum-Kalai-Wasserman (BKW) algorithm is a combinatorial algorithm used to solve the Learning With Errors (LWE).

The attack typically involves two main phases:

**Reduction Phase:** This phase progressively reduces the dimension of the LWE/LWR problem, essentially trying to simplify the equations involved. This is achieved by combining samples (vectors with associated 'noise' or errors) in a way that eliminates certain positions in the vectors, albeit at the cost of increasing the noise in the remaining positions.

**Solving Phase:** Once the problem is reduced to a manageable size, the remaining entries of the secret are recovered. This often involves techniques like hypothesis testing to distinguish the correct guess of the secret sub-vector from incorrect ones.

End

# LLL Theorem

- Let  $L$  be the  $n$ -dimensional lattice generated by  $\langle v_1, \dots, v_n \rangle$  and  $\lambda_1$  the length of the shortest vector in  $L$ . The LLL algorithm produces a reduced basis  $\langle b_1, \dots, b_n \rangle$  of  $L$ .
  1.  $\|b_1\| \leq 2^{(n-1)/4} D^{1/n}$ .
  2.  $\|b_1\| \leq 2^{(n-1)/2} \lambda_1$ .
  3.  $\|b_1\| \|b_2\| \dots \|b_n\| \leq 2^{n(n-1)/4} D$ .
- If  $\|b_i\|^2 \leq C$  algorithm takes  $O(n^4 \lg(C))$ .

# Gauss again

- Let  $\langle v_1, v_2 \rangle$  be a basis for a two-dimensional lattice  $L$  in  $\mathbb{R}^2$ . The following algorithm produces a reduced basis.

```
for(;;) {  
    if(||v1|| > ||v2||)  
        swap v1 and v2;  
    t = [(v1, v2) / (v1, v1)]; // [] is the "closest  
        integer" function  
    if(t == 0)  
        return;  
    v2 = v2 - tv1;  
}
```

- $\langle v_1, v_2 \rangle$  is now a reduced basis and  $v_1$  is a shortest vector in the lattice.



# Useful definitions

- *Usehint, makehint*
- *Smallball, etc*
- $\text{SamplePolyCBD}_\eta(x)$
- $\text{PRF}_\eta(x)$
- $\text{SampleNTT}(x)$
- $\text{MultiplyNTT}$
- $\text{BaseCaseMultiply}$
- $\text{NTT}$
- $\text{NTT}^{-1}$