# Quantum Computing

## A brief introduction

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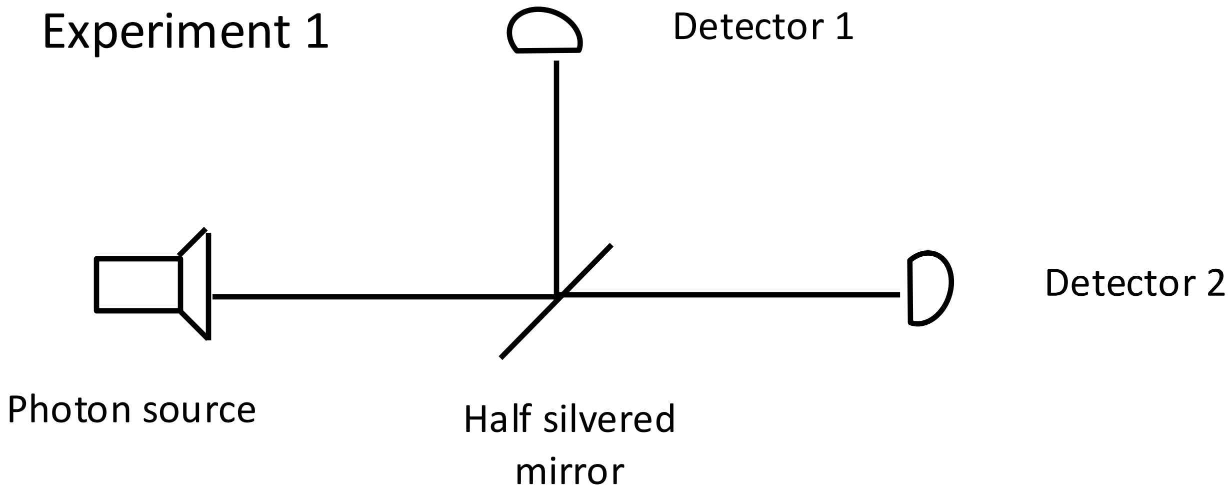
# Beam splitters and QM

I can safely say that no one understands Quantum Mechanics - Feynman

Experiment 1




Photon source emits stream of photons.

P(photon arrives at Detector 1)= .5

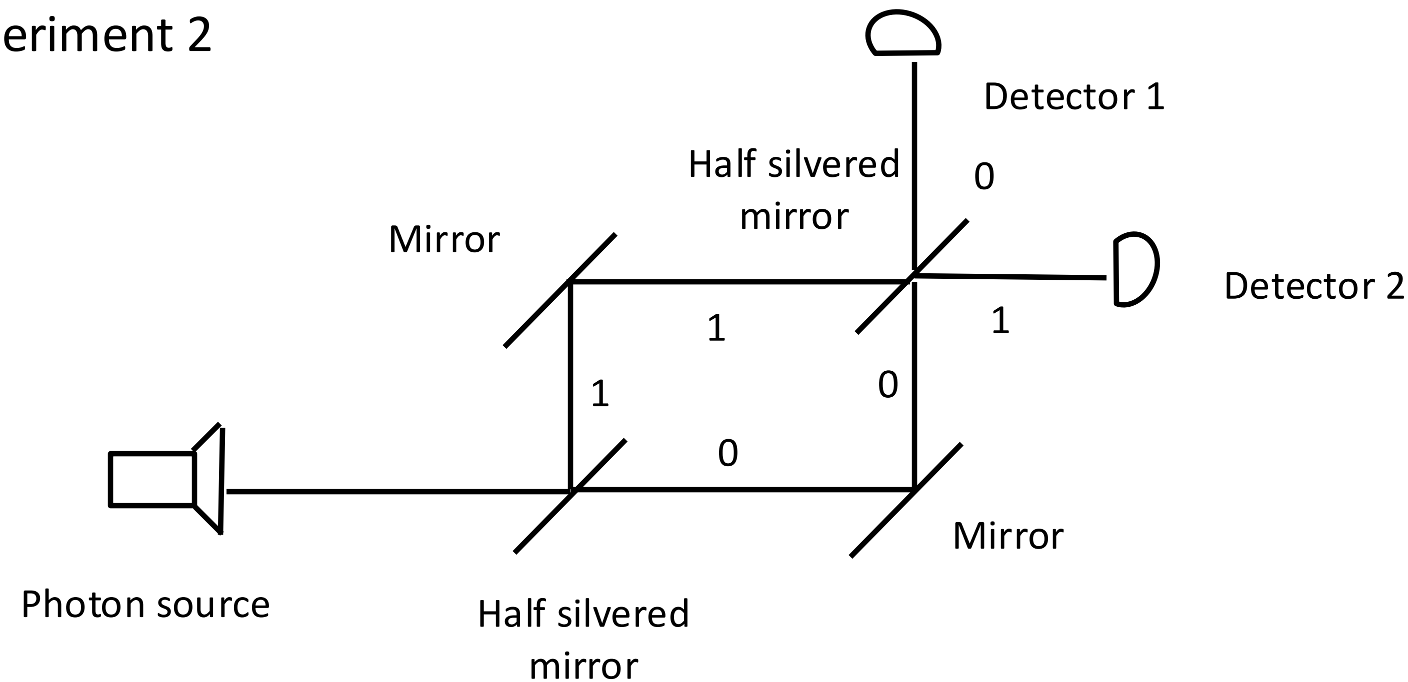
P(photon arrives at Detector 2)= .5

So far, so good

# Beam splitters and QM

## Mach-Zender Interferometer

Experiment 2




Photon source emits stream of photons.

P(photon arrives at Detector 1)= 0

P(photon arrives at Detector 2)= 1

Huh?

# According to QM

## Analysis

Beam splitter causes the photon to go into superposition:

$\alpha_1|0> + \alpha_2|1>$, $|\alpha_1|^2 = \frac{1}{2}$, $|\alpha_2|^2 = \frac{1}{2}$. $|0>$ state is right, $|1>$ is up.

$|0> = \binom{1}{0}$, $|1> = \binom{0}{1}$.

Beam splitter acts on incoming state via the matrix $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i \\ i & 1\end{pmatrix}$.

In experiment 1, if all photons leave source in state $\binom{1}{0}$, after the splitter they are in state $\frac{1}{\sqrt{2}}\binom{1}{i}$. So, they arrive at detector 1 with probability $\frac{1}{2}$ and detector 2 with probability $\frac{1}{2}$.

However, going through another beam splitter, in experiment 2, yields the output state:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i \\ i & 1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i \\ i & 1\end{pmatrix}\binom{1}{0} = \begin{pmatrix}0 & i \\ i & 0\end{pmatrix}\binom{1}{0} = \binom{0}{i}.$$

So, they always arrive at detector 2.

# Postulates

1. State of a system is a unit vector over $\mathbb{C}$ in Hilbert space ($\mathcal{H}$) of dimension $2^n$
   - A qubit is a quantum system, with $n = 1$ . A one qubit system is in general state $|\psi> = a|0> + b|1>$, $a\bar{a} + b\bar{b} = 1$
2. A system, with state, $|\psi(t)>$, evolves according to a unitary operator, namely, $U(|\psi(0)>)$
   - $U$ is unitary if $(x, y) = (Ux, Uy)$. Note $U\overline{U^T} = I$
   - Example is a Hamiltonian: $H(t)|\psi(t)> = \mathrm{i}\hbar\frac{d|\psi(t)>|}{dt}$
   - $|\varphi(t_2)> = e^{-i\hbar H(t_2-t_1)}|\varphi(t_1)>$
3. Each observable is represented by a Hermitian operator, $\hat{Q}$, the expectation value of $\hat{Q}$ is $<\psi|\hat{Q}|\psi>$. The outcome of a measurement of the operator is an eigenvalue of the operator. The probability of getting a particular eigenvalue, $\lambda$, is the square of the $\lambda$-component of $|\psi>$

# Postulates

4. Two physical systems $\mathcal{H}_1$ and $\mathcal{H}_2$ can be treated as a single system, $\mathcal{H}_1 \otimes \mathcal{H}_2$. If $\mathcal{H}_1$ is in state, $|\psi_1 >$ and $\mathcal{H}_2$ is in state, $|\psi_2 >$, the joint state is $|\psi_1 > \otimes |\psi_2 >$
5. Given an orthonormal basis $\mathcal{B} = \{\varphi_i\}$, one can perform a von-Neuman measurement $\mathcal{H}_A$ on $|\psi >= \sum_i \alpha_i|\varphi_i >$ that outputs $i$ with probability $|\alpha_i|^2$. It is projective.

   Further, if $|\psi >= \sum_i \alpha_i|\varphi_i >|\gamma_i >$ $\mathcal{H}_A \otimes \mathcal{H}_B$ measurement yields $i$ with probability $|\alpha_i|^2$ and leaves state in $|\varphi_i > |\gamma_i >$. $M = \sum m_i P_i = \sum m_i \, |i >< i|$

# Linear Algebra

- <u>Dirac Notation</u>: Element in Hilbert space of dimension $2^n$ is represented by n-entry symbol. $|000 \ldots 00 > \leftrightarrow (1, 0, \ldots, 0)^T$, $|000 \ldots 01 > \leftrightarrow (0, 1, \ldots, 0)^T$, …, $|\, 111 \ldots 1 > \leftrightarrow (0,0, \ldots, 1)^T$ where column vectors have $2^n$ coordinates.
- <u>Notation:</u> $|0 > \otimes |0 > \otimes \ldots \otimes |0 > = |000 \ldots 0 >$
- $A$ is normal if $A\bar{A}^T = \bar{A}^T A$
- <u>Spectral Theorem:</u> If $T$ is a normal operator in the Hilbert space $\mathcal{H}$, there is an orthonormal basis $v_i$; each is an eigenvector of $T$. For every such , there is a unitary matrix, $P$, $T = P\Lambda P^*$, and $\Lambda$ is diagonal.
- <u>Dual basis</u>
- <u>Inner product</u>: $(v_1, v_2, \ldots, v_n) \cdot (w_1, w_2, \ldots, w_n) = \sum_{i=0}^{n} \overline{v_i}\, w_i$
- <u>Outer product</u>: $(|\psi >< \phi|)|\gamma > = |\psi >(< \phi|\gamma>)$
- <u>Theorem</u>: Every linear operator can be written as $T = T_{m,n}|b_m >< b_n|$,
- $T_{m,n} =< b_m|T|b_n >$

# Linear Algebra (continued)

Tensor product: If $|\varphi_i> = \binom{\alpha_0}{\alpha_1}$ is a basis for $\mathcal{H}_1$ and $|\phi_i >= \binom{\beta_0}{\beta_1}$is a basis for $\mathcal{H}_2$ ,

$|\varphi_i >\otimes |\phi_i >$ is a basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$. $\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} \otimes \begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix} =$ $(\alpha_0\beta_0, \alpha_0\beta_1, \alpha_1\beta_0, \alpha_1\beta_1)^T$. $A\otimes B = \begin{Bmatrix} a_{11}B & \dots & a_{1n}B \\ \dots & \dots & \dots \\ a_{n1}B & \dots & a_{nn}B \end{Bmatrix}$

Schmidt decomposition: If $|\psi > \in \mathcal{H}_1 \otimes \mathcal{H}_2$, there is an orthonormal basis $|\varphi_i >$ for $\mathcal{H}_1$ and an orthonormal basis $|\phi_i >$ for $\mathcal{H}_2$ and $p_i \geq 0$ such that $|\psi > = \sum_i \sqrt{p_i}\, |\varphi_i > |\phi_i >$

$Tr(A) = < b_n|A|b_n >$

Eigenvector: $T|\psi > = c|\psi >$

# More notation

- $A \otimes B = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}$, $x \otimes \mathrm{y} = (x_1 y_1, x_1 y_2, \dots, x_n y_n)^T$
- $|v> = (\ v_1,\ \ v_2,\ \ \dots,\ \ v_n\ )^T$, $<w| = (\ w_1,\ \ w_2,\ \ \dots,\ \ w_n\ )$ then

$|v><w| = \begin{matrix} v_1\overline{w_1} & v_1\overline{w_2} & \dots & v_1\overline{w_n} \\ \dots & \dots & \dots & \dots \\ v_n\overline{w_1} & v_n\overline{w_2} & \dots & v_n\overline{w_n} \end{matrix}$, so $I = \sum |\mathrm{i} >< i|$ and $M = \sum M_{ij} |\mathrm{i} >< j|$

- Pauli matrices
  - $\sigma_0 = I$, $\sigma_X = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_Y = Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $\sigma_Z = Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
  - $[X, Y] = iZ$, $[Y, Z] = iX$, $[Z, X] = iY$

# Mixed states and density

- For pure states, $|\psi>$, density is $\rho = |\psi >< \psi|$
- Mixed states: $\{(p_1, |\psi_1 >), (p_2, |\psi_2 >), \ldots, (p_n, |\psi_n >)\}$, where the probability that the system is in pure state $|\psi_i >$ is $p_i$ and $\sum p_i = 1$
- Density operator for mixed state is $\sum p_i |\psi_i >< \psi_i|$
- Bloch Sphere
  - Pure state in general position is $|\psi > = \cos\left(\frac{\theta}{2}\right) \Big|0 > \; + e^{i\varphi} \sin\left(\frac{\theta}{2}\right)\Big| 1 >$.
  - For mixed state $|\psi_i > = \; p_i(\alpha_{X,i}, \alpha_{Y,i}, \alpha_{Z,i})$ on interior of Block sphere
  - $\rho = \sum p_i \; |\psi_i > < \psi_i|$ evolves as $\rho = \sum p_i \; U|\psi_i > < \psi_i|U^\dagger$
  - $\rho = \; \frac{1}{2} I + \alpha_X X + \alpha_Y Y + \alpha_Z Z$

- $P(|0 >) = < 0|\psi > < \psi|0 > = Tr < 0|\psi > < \psi|0) = Tr(|0 >< 0| \; |\psi >< \psi|$

# Mixed states and density

- Partial trace: Consider composite system $AB$.
  - $\rho^A = Tr_B(\rho^{AB})$
  - $Tr_B(|a_1><a_2| \otimes <b_1|) < b_2|\ ) = |a_1><a_2| Tr(|b_1><b_2|\ ) = |a_1><a_2| <b_2|b_1>$
  - Example
    - $\rho = \frac{1}{2}(|00><00| + |00><11| + |11><00| + |11><11|)$
      $= \frac{1}{2} Tr(|0><0| \otimes |0><0| + |0><1| \otimes |0><1| + |1><0| \otimes |1><0| + |1><1| \otimes |1><1|) = \frac{1}{2}(|0><0| + |1><1|)$

# Circuits and gates

- <u>Universal gate set</u>: A gate set is universal if $\forall n > 0$, any $n$-bit unitary operator can be approximated to arbitrary accuracy by a quantum circuit from this set
- An entangling gate is on that for an input product state $|\alpha > |\beta >$, the output state is not a product state (e.g.-CNOT).
  - Example: $|\psi > = \frac{1}{\sqrt{2}}(|00 > \ +|11 >)$
- <u>Theorem:</u> A set of states with an entangling 2-qubit gate together with all 1-qubit gates is universal.
- <u>Theorem:</u> If $U$ is a 1-qubit gate, $U = e^{ix}R_z(\beta)R_y(\gamma)R_z(\delta)$

# Gates and states

- General position on Bloch sphere: $|\psi> = \cos\left(\frac{\theta}{2}\right)\Big|0> \; + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)\Big|1>$
- Measurement: $I = \sum|i><i|$, $M = \sum m_i P_i$ , $M$ is Hermitian, $P_i = |i><i|$.
- Controlled gates:
  - $c - U|0>|\psi> = \; |0> \; |\psi>$
  - $c - U|1>|\psi> = \; |1> U|\psi>$

# Common gates

- Pauli gates

  $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

  $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

  $\mathrm{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

  $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

  Note: $X^2 = Y^2 = Z^2 = I$

- Hadamard

  $$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

- Rotation

  $$R_X(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{iX\theta} \end{pmatrix} = \begin{pmatrix} e^{-iX\theta/2} & 0 \\ 0 & e^{iX\theta/2} \end{pmatrix}$$

- 2 qubit gate

  $CNOT(|xy>) = |x, x \oplus y>$

  $$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- If $A^2 = 1$, $e^{i\theta X} = \mathrm{I}\cos(\theta) + iXsin(\theta)$

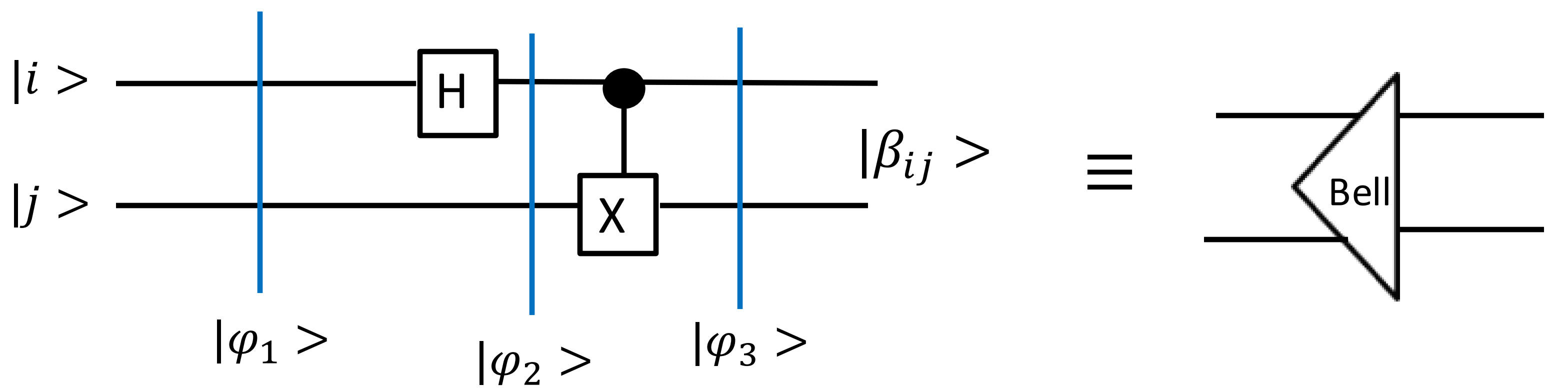
$$H^{\otimes n}(|0000\ldots 0>) = \frac{1}{\sqrt{2}}(|0> + |1>) \otimes \frac{1}{\sqrt{2}}(|0> + |1>) \otimes \ldots \otimes \frac{1}{\sqrt{2}}(|0> + |1>)$$

# Measurement in alternate basis

- Computational basis is $|i>$. $U|\varphi_j> = |j>$
- Suppose we want to measure $|\psi>$ with respect to basis $B = \{|\varphi_j>\}$
- $|\psi> = \sum \alpha_j |\varphi_j>$
- To measure wrt $B = \{|\varphi_j>\}$, Project $|\psi>$ onto $|\varphi_j><\varphi_j|$
- $\left(Tr\left(|\psi><\psi||\varphi_j><\varphi_j|\right)\right) = Tr\left(<\varphi_j|\psi><\psi|\varphi_j>\right) = \alpha_j{}^2$
- $\rho = |\psi><\psi|$ is density operator for the pure state $|\psi>$.
- $\rho = \sum p_i |\psi_i><\psi_i|$ is the density operator for mixed states $\{(p_i, |\psi_i>)$

# Converting to Bell Basis

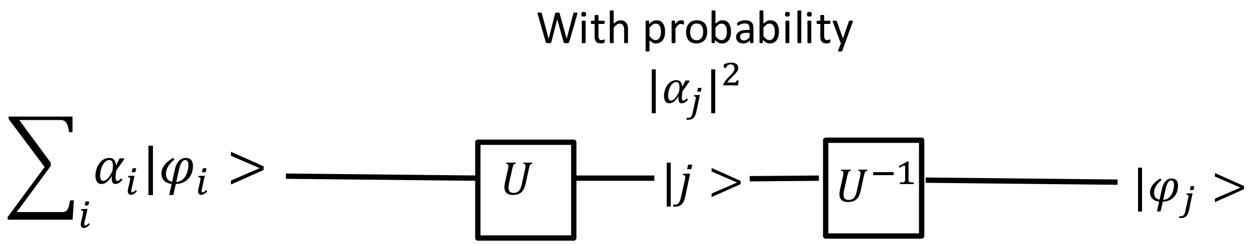
- Computational basis is $|i>$, $U|\varphi_j>=|j>$
- $|\beta_{00}>=\frac{1}{\sqrt{2}}(|00>\ +|11>)$, $|\beta_{01}>=\frac{1}{\sqrt{2}}(|01>\ +|10>)$
- $|\beta_{10}>=\frac{1}{\sqrt{2}}(|00>-\ |11>)$, $|\beta_{11}>=\frac{1}{\sqrt{2}}(|01>\ -|10>)$

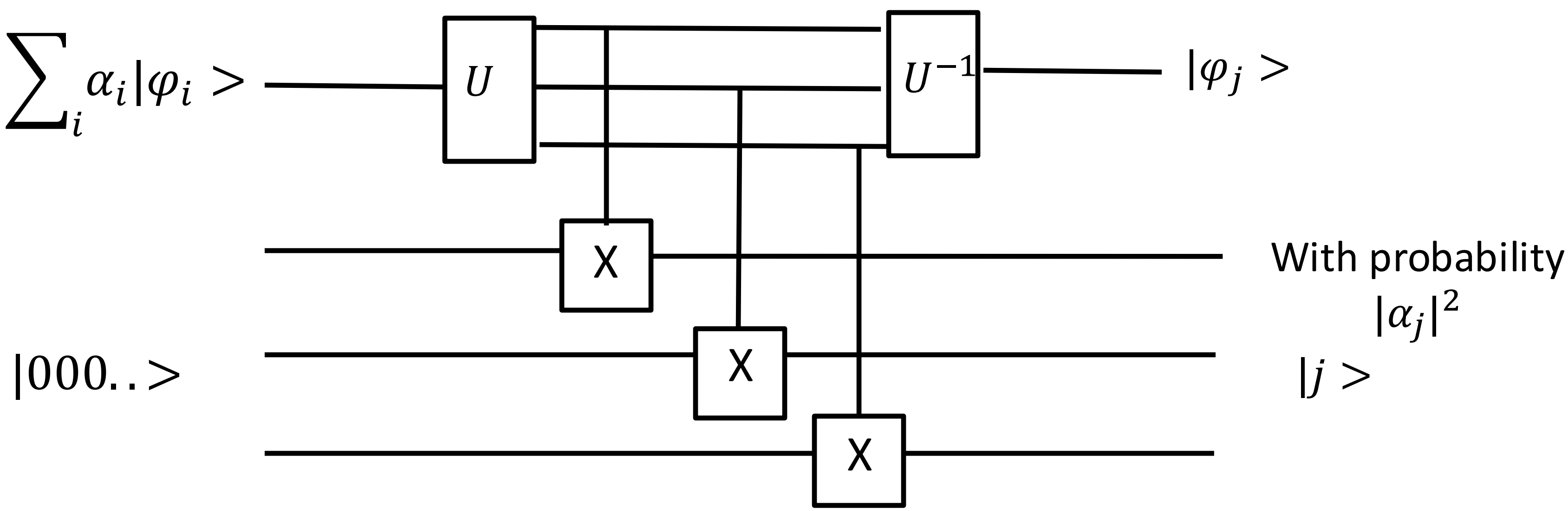



- $|\varphi_1>=|00>$
- $|\varphi_2>=\frac{1}{\sqrt{2}}(|0>+|1>)|0>=\frac{1}{\sqrt{2}}(|00>+|10>)|$
- $|\varphi_3>=|\beta_{00}>$

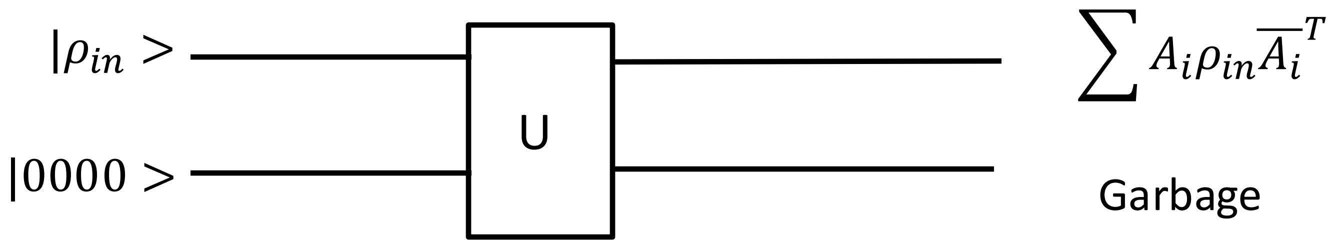
# Changing Measurement Basis

- Suppose $|\varphi_i>$ is a basis and our measurement basis is $|i>$, $U|\varphi_i> = |i>$



# Super-operator and mixed states




- $\rho = |\psi><\psi|$ , $U|\psi>$ has density ρ = $U|\psi><\psi|\overline{U}^T> = U\rho U^\dagger$
- $<0|\psi><\psi|0> = <0|\rho|0> = P(|0>)$
- $\rho = \sum_i p_i |\psi_i><\psi_i|$
- $Tr(A) = <b_n|A|b_n>$
- $\rho_{in} \rightarrow \rho_{out} = Tr_b(U(\rho_{in} \otimes |000\ldots><000\ldots0|U^\dagger)$
- $\rho_{in} \rightarrow \sum A_i \rho_{in} {A_i}^\dagger$, where $A_i$ are Kraus operators with $\sum {A_i}^\dagger A_i = I$

# No Cloning Theorem

- Theorem: Qubits can't be copied
- Proof

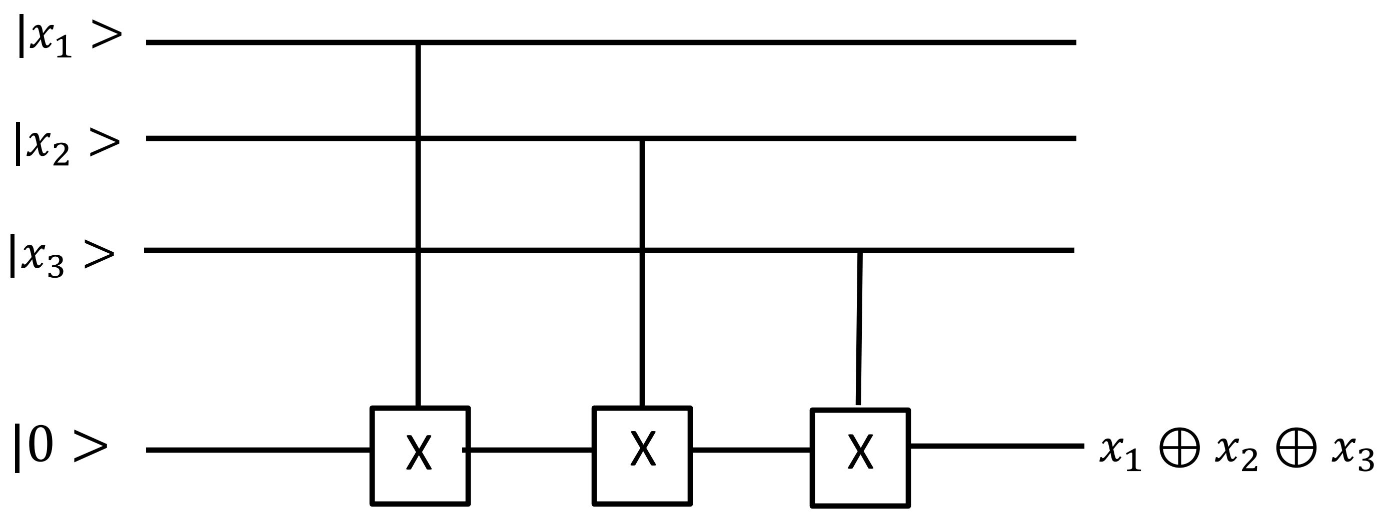
  Suppose they can be. Then there is an operator, $U$, such that for any state $|\varphi>$, $U(|\varphi>|0>) = |\varphi>|\varphi>$. Now let $|\psi>$ and $|\phi>$ be non-orthogonal, different pure states.

  $(|\psi>|0>, |\phi>|0>) = <\psi|\phi><0|0> = <\psi|\phi>$.

  Since $U$ is unitary,

  $<\psi|\phi> = (|\psi>|0>, |\phi>|0>) = (U|\psi>|0>, U|\phi>|0>) = (|\psi>|\psi>, |\phi>|\phi>) = <\psi|\phi>^2$. So, $<\psi|\phi> = 1$. This is a contradiction.
- No checkpointing

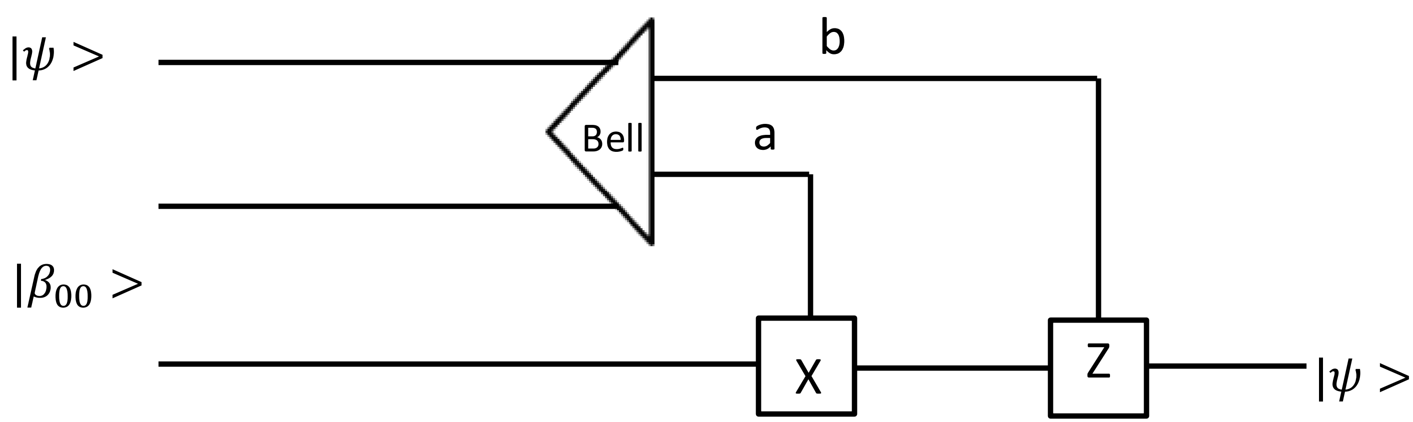
# Parity Circuit



# Superdense coding

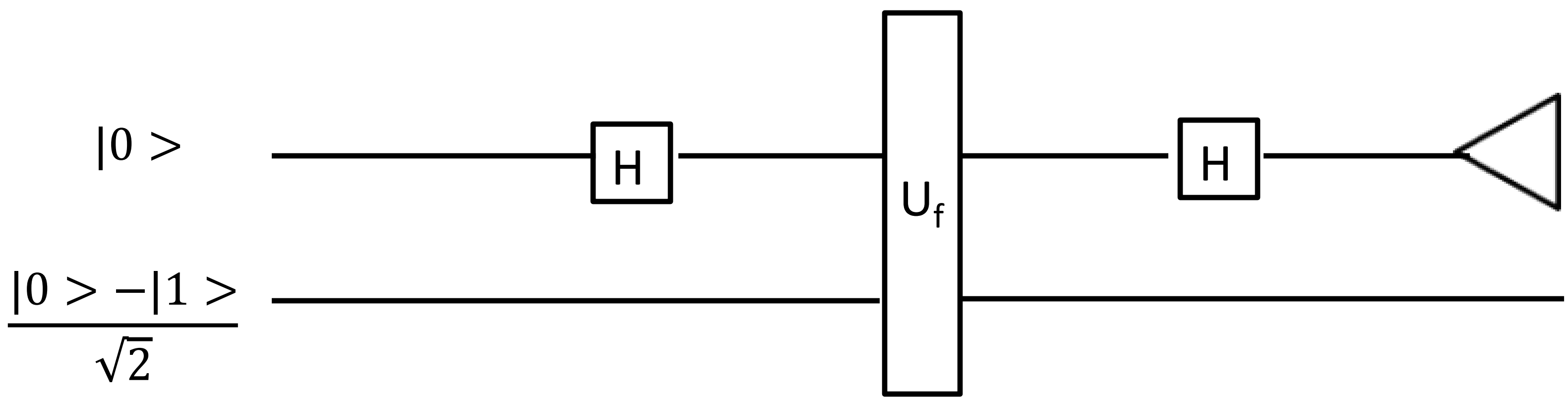
- Alice and Bob share $|\beta_{00}>$, Alice has first bit, Bob second bit
- Alice performs one of $I, X, Y, Z$ producing $I \otimes I$ (to send 00), $X \otimes I$ (to send 01), $Y \otimes I$ (to send 10) or $Z \otimes I$ (to send 11).
- Bob measures joint state qubit measurement

- Can be used to teleport $|\psi>$:
  - $I \otimes I := \frac{1}{\sqrt{2}}(|00> + |11>) \rightarrow \frac{1}{\sqrt{2}}(|00> + |11>)$
  - $X \otimes I := \frac{1}{\sqrt{2}}(|00> + |11>) \rightarrow \frac{1}{\sqrt{2}}(|01> + |10>)$
  - $Z \otimes I := \frac{1}{\sqrt{2}}(|00> + |11>) \rightarrow \frac{1}{\sqrt{2}}(|00> \; - |11>)$
  - $ZX \otimes I := \frac{1}{\sqrt{2}}(|00> + |11>) \rightarrow \frac{1}{\sqrt{2}}(|01> \; - |10>)$



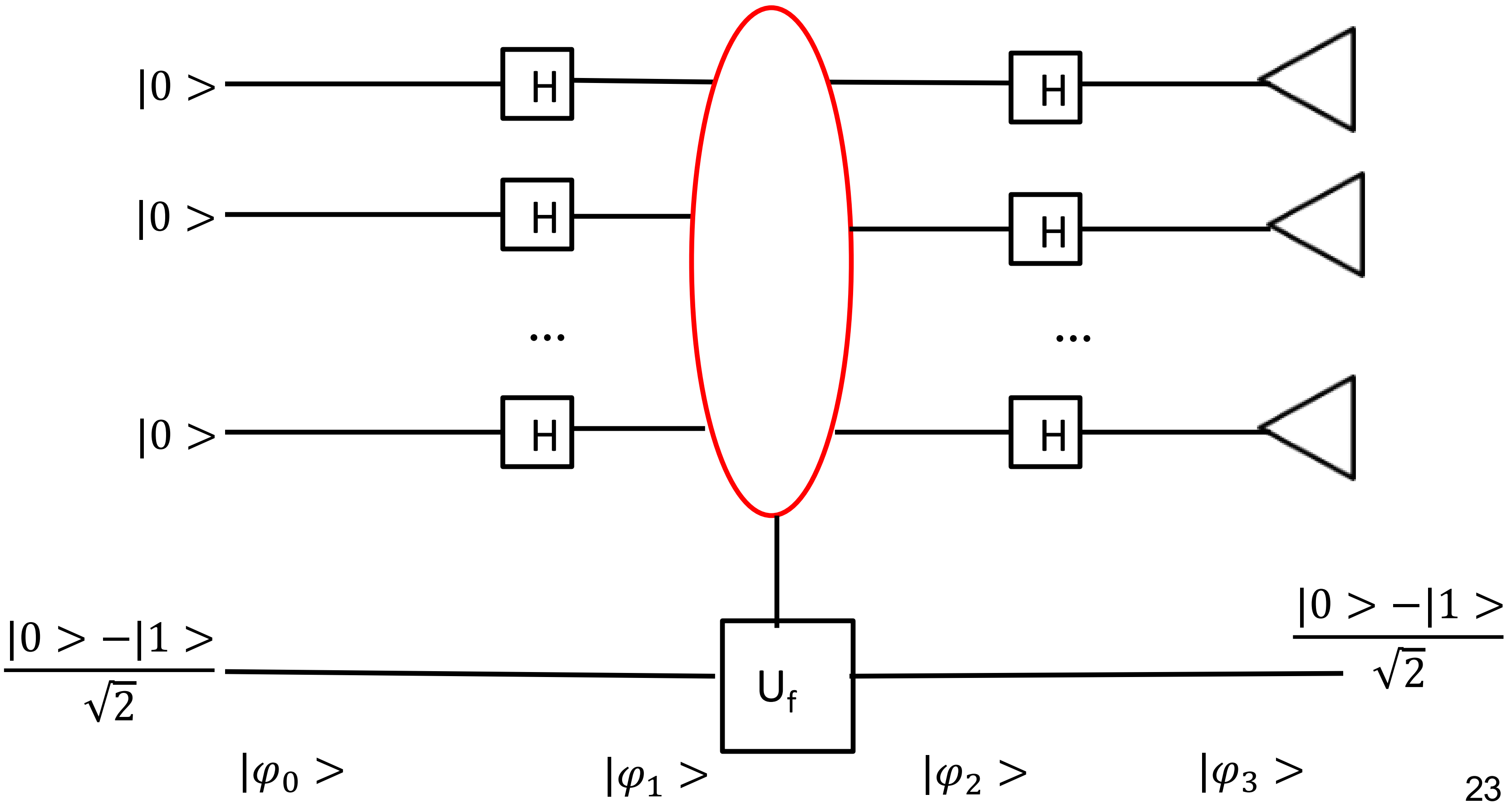
# Deutch

- Problem: Determine $f(0) + f(1)$ in one measurement
- $U_f|x > |y > = |x > |y \oplus f(x) >$
- If $f(0) + f(1) = 1$ , $\left|\psi_3 > = \ (-1)^{f(0)}\right|1 > \frac{|0>-|1>}{\sqrt{2}}$
- If $f(0) + f(1) = 0$ , $\left|\psi_3 > = \ (-1)^{f(0)}\right|0 > \frac{|0>-|1>}{\sqrt{2}}$



# Deutch-Josza

- Problem: $f: \{0,1\}^n \rightarrow \{0,1\}$, which is either constant or balanced.
- Which is it?
- Put $U_f|\boldsymbol{x}>|y> = \ |\boldsymbol{x}>|y \oplus f(\boldsymbol{x})>$, $\boldsymbol{x}$ is an n-bit quantity



# DJ

- $|\varphi_0> = \ |0>^{\otimes n} \frac{|0>-|1>}{\sqrt{2}}$
- $|\varphi_1> = \ \frac{1}{\sqrt{2^n}} \sum_{\boldsymbol{x}} |\boldsymbol{x}> \frac{|0>-|1>}{\sqrt{2}}$
- $|\varphi_2> = \ \frac{1}{\sqrt{2^n}} \sum_{\boldsymbol{x}} (-1)^{f(\boldsymbol{x})} |\boldsymbol{x}> \frac{|0>-|1>}{\sqrt{2}}$
- $|\varphi_3> = \ \frac{1}{2^n} \sum_{\boldsymbol{x}} \sum_{\boldsymbol{z}} |(-1)^{f(\boldsymbol{x})+\boldsymbol{x}\cdot\boldsymbol{z}} \boldsymbol{z}> \frac{|0>-|1>}{\sqrt{2}}$

# Simon

- $f\colon \{0,1\}^n \to X, \exists\, \vec{s} = s_1, s_2, \ldots, s_n$: $f(x) = f(y)$ iff $x = y$ or $x = y + \vec{s}$
- $U_f\colon |x> |b> = |x> |b \oplus f(x)>$
- $H^{\otimes n}(|x>) = \frac{1}{\sqrt{2^n}} \sum_z (-1)^{x \cdot z} |z>$

1. $i = 1$
2. Prepare $\frac{1}{\sqrt{2^n}} \sum_x |x> |0>$
3. Apply $U_f$ to get $\frac{1}{\sqrt{2^n}} \sum_x |x> |f(x)>$
4. Measure second bit
5. Apply $H^{\otimes n}$ to first register
6. Measure first register to get $w_i$
7. If $din(w_i) \neq n - 1$, go to 2
8. Output s: $w^t s^t = 0$

# Phase kick back

- $CNOT\left(\frac{|0> + |1>}{\sqrt{2}}\frac{|0> \; - |1>}{\sqrt{2}}\right) = \frac{|0> \; - |1>}{\sqrt{2}}\,\frac{|0> \; - |1>}{\sqrt{2}}$

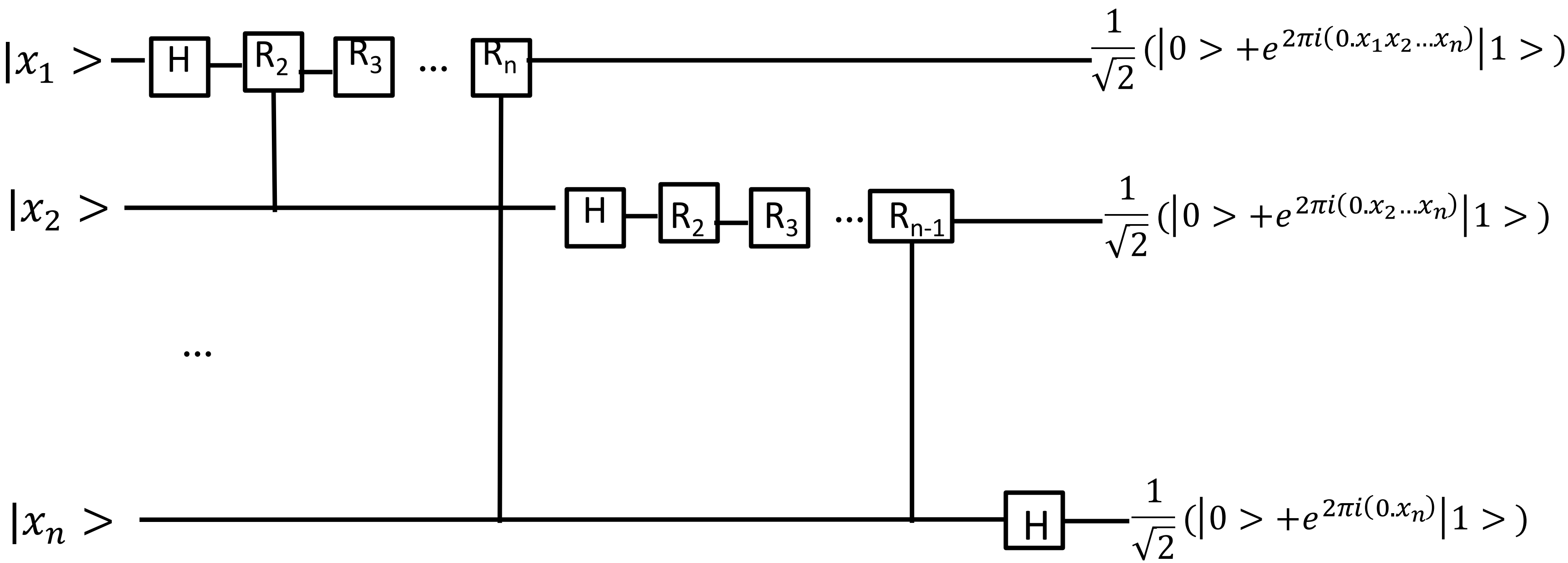
# Phase Estimation

- Phase estimation problem: Given $|\psi> = \frac{1}{\sqrt{2^n}} \sum_y e^{2\pi i \omega y} |y>$, estimate $\omega$
- <u>Theorem:</u> $\frac{x}{2^n} \leq \omega \leq \frac{x+1}{2^n}$ with probability $\geq \frac{8}{\pi^2}$
- $e^{2\pi i 2^k . x_1 x_2 \ldots} = e^{2\pi i (x_{k+1} x_{k+2} \ldots)}$
- Suppose $\omega = .x_1$, $|\psi> = \frac{1}{\sqrt{2}} \sum_{|y>} e^{2\pi i \omega |y>} = \frac{1}{\sqrt{2}} (|0> + (-1)^{x_1} |1>$ and $H(|\psi>) = |x_1>$
- In general, $H^{\otimes n}(|\boldsymbol{x}>) = \frac{1}{\sqrt{2^n}} \sum_{\boldsymbol{y}} (-1)^{\boldsymbol{x} \cdot \boldsymbol{y}} |\boldsymbol{y}>$ and $H^{\otimes n}(H^{\otimes n}(|\boldsymbol{x}>)) = |\boldsymbol{x}>$
- So, $|\psi> = \frac{1}{\sqrt{2^n}} \sum_{|y>} e^{2\pi i \omega y>} |y> = \frac{1}{\sqrt{2}} (\big|0> + e^{2\pi i 2^{n-1} \omega} \big|1>) \otimes \frac{1}{\sqrt{2}} (\big|0> + e^{2\pi i 2^{n-2} \omega} \big|1>) \otimes \cdots \otimes \frac{1}{\sqrt{2}} (\big|0> + e^{2\pi i \omega} \big|1>)$
- Denote $R_n = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i 2^{-n}} \end{pmatrix}$

# Quantum Fourier Transform

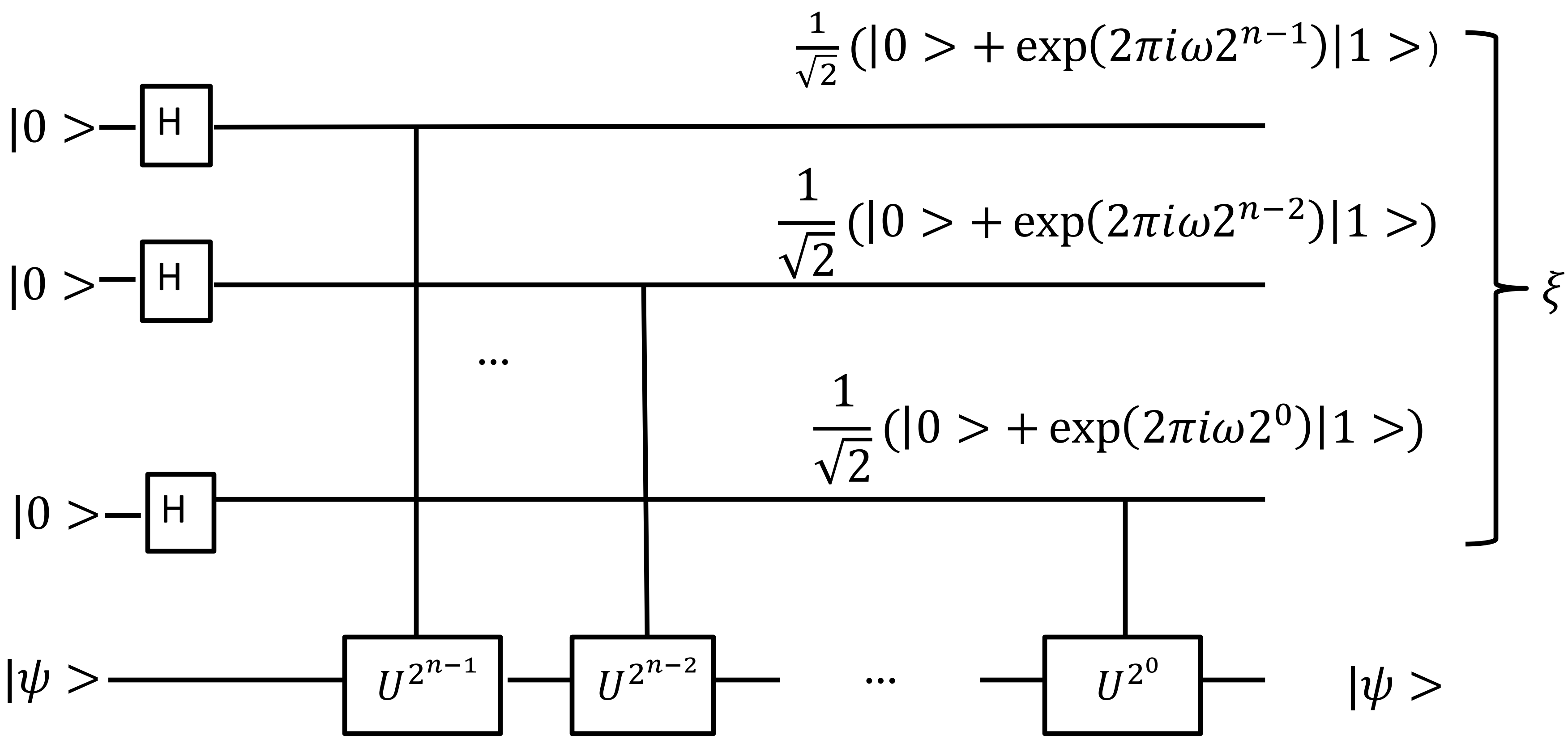
- $H^{\otimes n}(|\boldsymbol{x}>) = \frac{1}{\sqrt{2^n}} \sum_{\boldsymbol{y}} (-1)^{\boldsymbol{x} \cdot \boldsymbol{y}} |\boldsymbol{y}>$
- $H^{\otimes n}(H^{\otimes n}(|\boldsymbol{x}>)) = |\boldsymbol{x}>$
- $QFT_m(|x>) = \frac{1}{\sqrt{m}} \sum_{y=0}^{m-1} e^{2\pi i/_{m}(x \cdot y)} |y>$
- $QFT_m{}^{-1}(|x>) = \frac{1}{\sqrt{m}} \sum_{y=0}^{m-1} e^{-2\pi i/_{m}(x \cdot y)} |y>$

# Quantum Fourier Circuit



# Eigenvalue Estimation

- Suppose $|\psi>$ is an eigenstate of a unitary operator, $U$, so $U\,|\psi> =$ $\exp(2\pi i\phi)|\psi>$. $|\phi> = .x_1x_2 \ldots x_n$ (a binary expansion)



# Eigenvalue Estimation

- $U|\psi> = \exp(2\pi i\phi)\ |\psi>$, so $U^{2^j}|\psi> = \exp\left(2\pi i\phi 2^j\right)\ |\psi>$.
- Applying ${QFT_n}^{-1}$ to $\xi$, gives $< x_n,\ x_{n-1}, \ldots, x_1 >$ , where $|\phi> =. x_1 x_2 \ldots x_n$
- Measure $\chi$ to get $\phi$
- $\frac{y}{2^n}$ is a good estimate for $\phi = \frac{j}{r}$

.

# Factorization using order finding (Shor)

- Suppose $N = pq$ and $a^r = 1\ (mod\ N)$ then $r = 0\ \big(mod\ \varphi(pq)\big)$
- If $r$ is even, say, $r = 2s$, $(a^s + 1)(a^s - 1) = 0\ (mod\ pq)$.
- There is a good chance $p|(a^s - 1)$ but $\big(q, (a^s - 1)\big) = 1$.
- Then $\big((a^s - 1), \mathrm{N}\big) = \mathrm{p}$. Voila!
- Note that $|v_t> = \frac{1}{r}\sum_{k=0}^{r-1} \exp(-\frac{2\pi ikt}{r})|k(mod\ N)>$ is an eigenvalue of $U_x\ (k) = |xk\ (mod\ N)>$.
- In Shor, $|1> = \frac{1}{\sqrt{r}}\sum |v_t>$.
- Applying $QFT^{-1}$ to control gives phase of eigenvalues
- Measurement of target gives $|\frac{s}{r}>$ with $\Pr(|y>) = \frac{1}{2^{2n}}\ |\frac{1-r^{2^n}}{1-r}|^2$, where $r = \exp(-2\pi i(\frac{y}{2^n} - \phi))$

# Order Finding

Problem: Given $a, N \in \mathbb{Z}$ with $(a, N) = 1$, find $r$: $a^r (mod\ N) = 1$

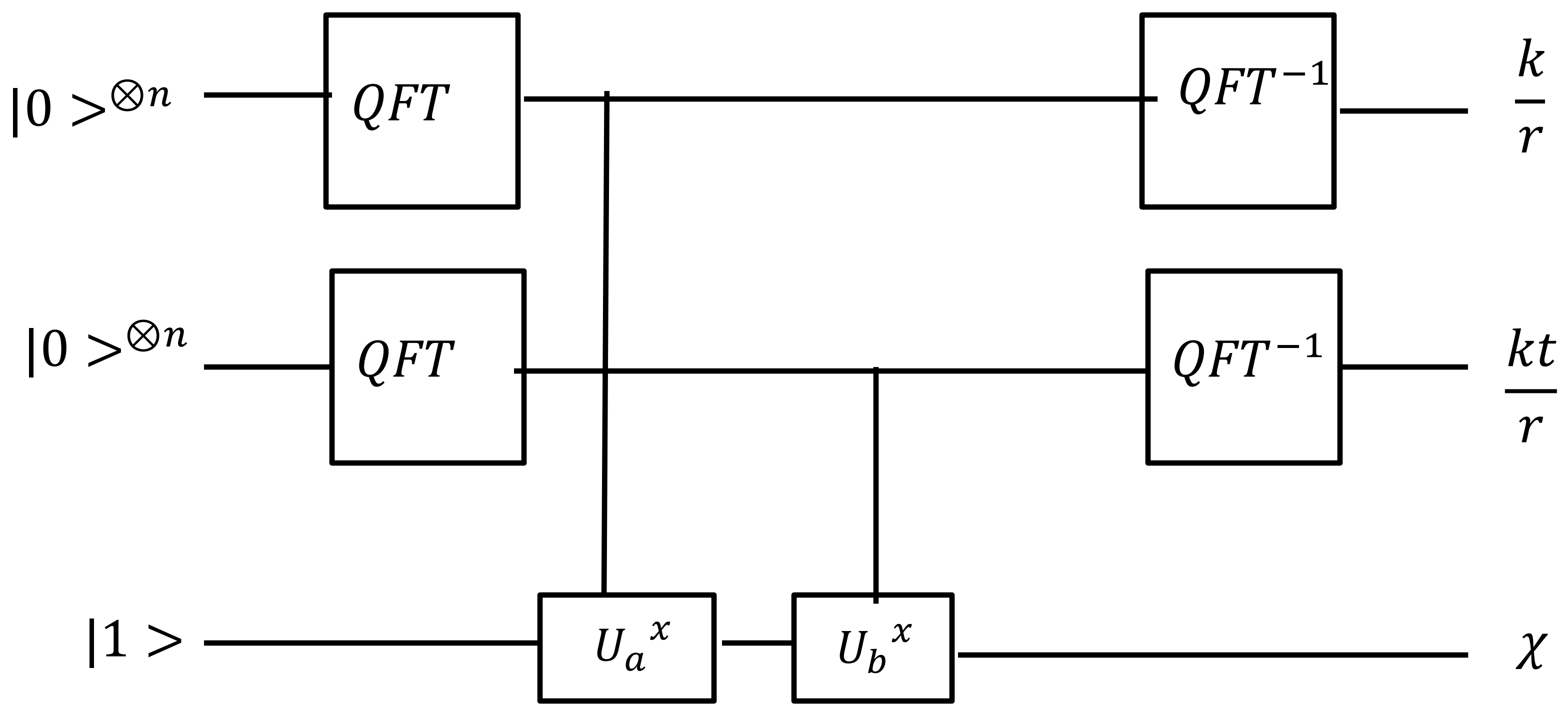
1. Choose $n$: $2^n \geq 2r^2$
2. Initialize control register $|000 \ldots 0 >= |0 >^{\otimes 2n}$
3. Initialize target register to $= |000 \ldots 01 >= |000 \ldots 0 >= |0 >^{\otimes 2n} \otimes |1 >$
4. Apply $QFT$ to control register
5. Apply $c - U_a{}^x$ to control and target register
6. Apply $QFT^{-1}$ to control register
7. Measure CR to get estimate of $\frac{x_1}{2^n}$ of multiple of $\frac{1}{r}$
8. Use continued fraction to get $c_1, r_1$: $\left|\frac{x_1}{2^n} - \frac{c_1}{r_1}\right| \leq 2^{-(n-1)}/_2$
9. Repeat 1-8 to get $c_2, 2$: $\left|\frac{x_2}{2^n} - \frac{c_2}{r_2}\right| \leq 2^{-(n-1)}/_2$, if none, FAIL
10. Compute r $= LCM(r_1, r_2)$ and $a^r\ (mod\ N)$
11. If $a^r\ (mod\ N) = 1$, output $r$, otherwise FAIL

# Order Finding

- Order finding has quantum complexity $O(\lg(N)^2 \lg(\lg(N)) \lg(\lg(\lg(N)))$
- Classical complexity is $\exp(O(\sqrt{\lg(N) \lg(\lg(N))}))$

# Discrete log

- Suppose $a = b^x \ (mod\ p)$, $b$ has known order. We want $r$: $b^r = 1(mod\ p)$
- Put $U_a(\,|x>) = |ax\ (mod\ p) >$ and $U_b(\,|x>) = |bx\ (mod\ p) >$.
- Consider the circuit below. $|1> = \frac{1}{\sqrt{r}}\sum |v_t>$. Below, $t = xy^{-1}$

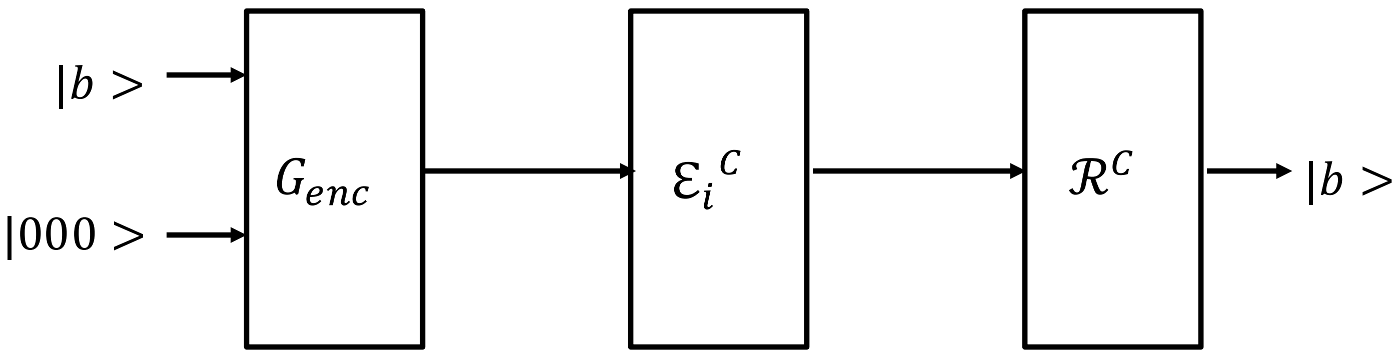


# Discrete log

- Measuring first control register gives $|\frac{k}{r}>$
- Measuring first control register gives $|\frac{kt}{r}>$

- Quantum complexity is $O(\lg(p)^2 \lg(\lg(p)) \lg(\lg(\lg(p))))$
- Best known classical requires $\exp(O(\sqrt{\lg(p) \lg(\lg(p))}))$

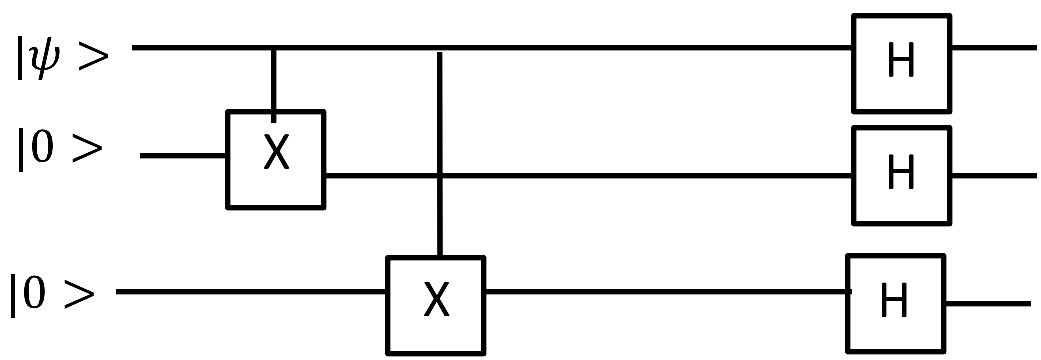
# Error Correction




- Unlike classical error correction, the no cloning theorem restricts codes
- $|0>|E> \rightarrow \ \beta_1|0>|E_1> \ +\beta_2\,|1>|E_2>$
- $|1>|E> \rightarrow \ \beta_3|0>|E_3> \ +\beta_4\,|1>|E_4>$
- $|\psi>= \ \alpha_0|0> \ +\alpha_1|1> \rightarrow . \, \alpha_0\,\beta_1|0>|E_1> \ +\alpha_0\beta_2|1>|E_2> \ +\alpha_1\beta_3\,|0>|E_3> \ +\alpha_1\beta_4\,|1>|E_4>$
- $|\psi>= \ \frac{1}{2}\,|\psi>(\beta_1|E_1> \ +\beta_3|E_3>)+\frac{1}{2}<Z|\psi>(\beta_1|E_1> \ -\beta_3|E_3>)+\frac{1}{2}<X|\psi>(\beta_2|E_2> \ +\beta_4|E_4>)+\frac{1}{2}<XZ|\psi>(\beta_2|E_2> \ +\beta_4|E_4>)$

# Error Correction

- $\rho = \ U_{err}|\psi > < E|\overline{U_{err}}^T$
- $|\psi_{enc} > = U_{enc}|\psi > |000 \ldots >$
- $\mathcal{E}_0 = I \otimes I \otimes I, \mathcal{E}_1 = X \otimes I \otimes I$
- $\mathcal{E}_2 = I \otimes X \otimes I, \mathcal{E}_3 = I \otimes I \otimes X$
- $\rho$: $|\psi >< \psi| \rightarrow (1-p)|\psi >< \psi| + p\, X|\psi >< \psi|X >$
- $\frac{1}{\sqrt{2}}(|000 > \ +|100 >) \rightarrow \frac{1}{\sqrt{2}}(|000 > \ +|111 >) \neq \frac{1}{\sqrt{2}}(|0 > \ +|1 >)^{\otimes 3}$
- 3-bit code, Shor 9-bit code



# Amplitude Amplification

- $|\psi> = A\,|00..0> = \sum_x \alpha_x |x> |junk(x)>$
- $\left|\psi> = \sum_{x,good} \alpha_x \right| x> |junk(x)> + \sum_{x,bad} \alpha_x |x> |junk(x)>$
- $|\psi_{good}> = \sum_{x,good} \alpha_x\, |x> |junk(x)>$
- $|\psi_{bad}> = \sum_{x,bad} \alpha_x\, |x> |junk(x)>$
- $|\psi> = \sqrt{p_{good}}\, |\psi_{good}> + \sqrt{p_{bad}}\, |\psi_{bad}> = \sin(\theta) |\psi_{good}> + \cos(\theta) |\psi_{bad}>$
- $p_{good} = \sin(\theta)^2$

# Grover

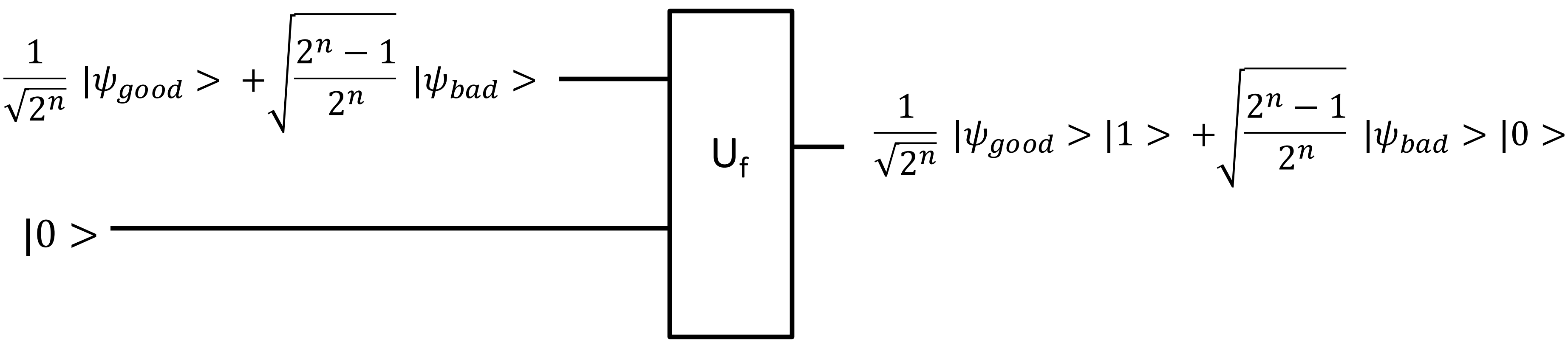
Search

Input: $U_f$: $f$: $\{0,1\}^n \rightarrow \{0,1\}$

$$f(\boldsymbol{a}) = 1,\ f(\boldsymbol{x}) = 0, \mathbf{x} \neq \boldsymbol{a}$$

$$|\psi_{good}> = \boldsymbol{w}$$

$$|\psi_{bad}> = \frac{1}{\sqrt{N-1}} \sum\nolimits_{\boldsymbol{x} \neq \boldsymbol{w}} |x>$$



# Grover

1. Initialize $n$-qubits $|0000 \dots 0>$.
2. Apply $H^{\otimes n}$ to get $\frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x$
3. Apply Grover $G$ $\frac{\pi}{4\sqrt{n}}$ times
4. Measure output

Search

Input: $U_f$: $f$: $\{0,1\}^n \rightarrow \{0,1\}$

$$f(\boldsymbol{a}) = 1,\ f(\boldsymbol{x}) = 0, \mathbf{x} \neq \boldsymbol{a}$$

$$|\psi_{good}> = \boldsymbol{w}$$

$$|\psi_{bad}> = \frac{1}{\sqrt{N-1}} \sum_{\boldsymbol{x} \neq \boldsymbol{w}} |x>$$

Algorithm $G$

1. Apply $U_f$
2. Apply $H^{\otimes n}$
3. Apply $U_{0^\perp}$
4. Apply $H^{\otimes n}$

Algorithm $U_{0^\perp}$

$$U_{0^\perp}: |\mathrm{x}> \longrightarrow -|x>, x \neq 0$$
$$U_{0^\perp}: |0> \longrightarrow 0|x>$$

# End

# Strings and thermo

- String: $dU = \frac{1}{2}\mu\omega^2 y^2 dx$, $P(t) = Z = F\frac{\partial\psi}{\partial t}$, $v_\varphi = \frac{\omega}{k}$, $Z = \frac{T}{v_\varphi}$, $\frac{T}{\mu} = \omega^2$
- Power transmitted: $P = \frac{1}{2}\mu\omega^2 A^2 v_\varphi$
- $I = I_0\left(\frac{sin(\beta/2)}{\beta/2}\right)^2$, $P_R = \frac{P_T G_R G_T \lambda^2}{(4\pi R)^2}$
- $Z = \sum_i e^{-\beta E_i}$, $\beta = \frac{1}{kT}$, $< E > = \frac{\partial(\ln(Z))}{\partial\beta}$
- $\Delta Q + \Delta W = \Delta E$, $\Delta Q$ – heat in system, $\Delta W$ - work on system
- $W = Q(1 - \frac{T}{T_0})$, $e = (1 - \frac{T_C}{T_H})$, $S = k\ln(\Omega)\Omega$
- $c_v = \frac{3}{2}R$
- $I(\lambda) = \frac{2\pi hc^2}{\lambda^5(\exp\left(\frac{hc}{k\lambda T}\right) - 1)}$

# EM

- $E_n = \frac{-13.6}{n^2}$, $a_0 = \epsilon_0 \frac{h^2}{\pi m e^2}$, $d_n = \frac{(2m)^{3/2} V E^{3/2}}{3\pi h^3}$, $g(E) = \frac{(2m)^{3/2} V}{2\pi h^2}\sqrt{E}$
- $\nabla \cdot j = -\frac{\partial \rho}{\partial t}$, $\nabla \cdot E = \frac{\rho}{\epsilon_0}$, $\nabla \times E = -\frac{\partial B}{\partial t}$, $\nabla \times B = 0$, $c^2 \nabla \times B = \frac{j}{\epsilon_0} + \frac{\partial E}{\partial t}$
- $c^2 = \frac{1}{\epsilon_0 \mu_0}$, $I = \sigma T^4$, $D = \epsilon E$, $B = \mu H$
- Solution to $\nabla^2 \psi - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = -s$, $\psi(t) = \frac{1}{4\pi}\frac{S(t-\frac{r}{c})}{r}$, $S(t) = \int s(t) dV$
- $\phi(1,t) = \int \frac{\rho(2, t-\frac{r}{c})}{4\pi \epsilon_0 r_{12}}\, dV$, $A(1,t) = \int \frac{\rho(2, t-\frac{r}{c})}{4\pi c^2 \epsilon_0 r_{12}}\, dV$
- $\nabla \phi = E + \frac{\partial A}{\partial t}$, $S = \epsilon_0 c^2 E \times B$
- Oscillating dipole: $\psi = \frac{dz}{4\pi \epsilon_0}[\frac{q(t-\frac{r}{c})}{r^3} + \frac{I(t-\frac{r}{c})}{r^2 c}]$
- $x' = \gamma(c - ut)$, $t' = \gamma(t - \frac{ux}{c^2})$, $E^2 + (pc)^2 = (m_0 c^2)^2$

# Susskind

- $|\psi>=\sum_i \alpha_i|\lambda_i>$ is the state of a system, the $|\lambda_i>$ is a complete set of orthonormal vectors which are eigenvectors
- $<L>= \sum_i P(\lambda_i)\lambda_i$ is the expected value
- $<L>=<\psi|L|\psi>= \sum_i \overline{\alpha_i}\, \alpha_i <\lambda_i|\lambda_i>, \alpha_i \in \mathbb{C}$
- If $\varphi, \psi$ are states in a continuous variable, $<\varphi|\psi>=\int_{-\infty}^{\infty} \bar{\varphi}\psi\, dx$
- $I=\sum_i |i><i|, Tr(L)= \sum_i <i|L|i>,\ I=\int|x><x|\, dx$
- If $|\psi>=\sum_i \alpha_i|i>$ , $<j|\,\psi>=\sum_i \alpha_i <j|i>$
- C-S: $\overrightarrow{|x|}\ \overrightarrow{|y|} \geq \overrightarrow{|x\cdot y|}$, Triangle: $\overrightarrow{|x}| + |\vec{y}| \ \geq |\vec{x}+\vec{y}|$
- Eigenvectors of a Hermitian operator are a complete set
- Eigenvectors with different eigenvalues are orthogonal.
- $<x>= \int x|\psi|^2\ dx, P(x \leq a \leq x+\Delta x)= |\psi|^2$

# Susskind

- Observables: $M|\lambda> = \lambda\,|\lambda>$, $\lambda$ is the observed value, $M$ is projective and Hermitian
- The rules
  1. Observables are represented by linear operators. States are vectors
  2. Results of measurements are eigenvalues
  3. Distinguishable states correspond to orthogonal eigenvalues
  4. If $|\psi>$ is a state and and $L$ is an observable, $P(\lambda_i) =<\psi|\lambda_i><\lambda_i|\psi>$
  5. Evolution of states governed by a unitary operator

# Spin

- Spin field: $\mathrm{H} = \ \sigma \cdot \mathrm{B}$
    - $< \dot{\sigma}_z > = \frac{i\omega}{2} < [\sigma_z, \sigma_z] >, < \dot{\sigma}_y > = -\omega < \sigma_x >$
- $< \dot{\sigma}_x > = \frac{-i}{\hbar} < [\sigma_x, H] >, < \dot{\sigma}_y > = \frac{-i}{\hbar} < [\sigma_y, H] >, < \dot{\sigma}_z > = \frac{-i}{\hbar} < [\sigma_z, H] >$
- $|u > = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |d > = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- $|r > = \frac{1}{\sqrt{2}}(|u > +|d >, |l > = \frac{1}{\sqrt{2}}(|u > \ - |d >$
- $|i > = \frac{1}{\sqrt{2}}(|u > +i|d >, |o > = \frac{1}{\sqrt{2}}(|u > \ -i\ |d >$
- Pauli matrices: $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$
- $\sigma_n = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix}$
- $\lambda_1 = 1, |\lambda_1 > = \begin{pmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) \end{pmatrix}, \lambda_2 = -1, |\lambda_2 > = \begin{pmatrix} -\sin(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2}) \end{pmatrix}$

# Entanglement

- Composite: $|\varphi >\in H^A$, $|\psi>\in H^B$, $\varphi \otimes \psi \in H^{AB}$
- $L|\lambda_i, \mu_i > = \lambda_i|\lambda_i, \mu_i >$, $\mathrm{M}|\lambda_i, \mu_i > = \mu_i|\lambda_i, \mu_i >$ for independent action

- If $A$ state coefficients are $a$ and $B$ state coefficients are $B$,
- $|\varphi \otimes \psi >= \sum_{a,b} |\varphi(a) >\otimes |\psi(b) > \; |ab >$
- Alice and Bob spins: $(\alpha_u|u > +\alpha_d|d >) \otimes (\beta_u|u > +\beta_d|d >)$ as product state
- General entangled: $\psi_{uu}|uu > +\psi_{ud}|ud > +\psi_{du}|du > + \psi_{dd}|dd >$
- Entangled: $|sing > = \frac{1}{\sqrt{2}}(|ud > \; - |du >)$,
- $< sing|\sigma_z|sing > = 0$

# Susskind and density

- $\tau_z \sigma_z | sing > = -| sing >, \tau_x \sigma_x | sing > = -| sing >$

- $< L > = Tr\,(\rho L) =< \psi | L | \psi >$ for states prepared with probability $p_i$
- If $< L > = p_\psi < \psi | L | \psi > + p_\varphi < \varphi | L | \varphi >$, density operator is
- $P = p_\psi | \psi > < \psi | + p_\varphi \, | \varphi >< \varphi |$
- $P_{a,a'} = \overline{\psi(a)} \psi(a') \sum_b \overline{\phi(b)}\, \phi(b)$
- $L_{a'b'a,b} =< a'b' | L | ab >$
- $< \psi | L | \psi > = \sum_{a,a',b,b'} \bar{\psi}_{a',b'} L_{a',a}\, \psi_{a,b}$

# Uncertainty, Evolution and Schrodinger

- Uncertainty: From triangle, $(\Delta X)(\Delta Y) \geq \frac{1}{2} | < \psi | [A, B] | \psi > |$
- Unitary evolution: $U^{\perp}(t)U(t) = I, |\psi(t) > = U(t)|\psi(0) >$
- $U(\epsilon) = (I + \frac{i}{\hbar}\epsilon H)|\psi(0) >$, so
- $\frac{|\psi(\epsilon)> \ -|\psi(0)>}{\epsilon} = -\frac{i}{\hbar} H|\psi(0) >$, or, $\frac{\partial |\psi(t)>}{\partial t} = -\frac{i}{\hbar} H|\psi(t) >|$
- $\frac{d}{dt} < \psi|L|\psi > =< \dot{\psi}|L|\psi >+< \psi|L|\dot{\psi} > =$
  $\frac{i}{\hbar}(< \psi|HL|\psi > \ -(< \psi|LH|\psi >) = \frac{i}{\hbar} < \psi|[H, L]|\psi >$
- This gives conservation of energy since $[H, H] = 0$
- Example: $H = \frac{\hbar\omega}{2} \sigma \cdot n$

# Momentum space and Fourier

- $P|\psi> = p|\psi>$ is equivalent to $P = -i\hbar\frac{\partial}{\partial x}$
- Eigenvectors are solutions of $-i\hbar\frac{\partial\psi}{\partial x} = p\psi$, $\psi_p(x) = Aexp(i\frac{px}{\hbar})$
- Wavelength of $Aexp(i\frac{px}{\hbar})$ is $\lambda = \frac{2\pi\hbar}{p}$
- $P(p) = \ |< P|\psi > |^2$, put $\tilde{\psi}(p) =< P|\psi >$
- $\tilde{\psi}(p) = \int dx < p|x >< x|\psi >$ but $\psi(x) =< x|\psi >$ and
- $< p|x > = \frac{1}{\sqrt{2\pi}}\exp(-i\frac{px}{\hbar})$
- Similarly, $\psi(x) =< x|\psi > = \int dp < x|p >< p|\psi >$ and
- $\psi(x) = \frac{1}{\sqrt{2\pi}}\int dp\exp(i\frac{px}{\hbar})\tilde{\psi}(p)$
- $\psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(i\frac{px}{\hbar})\tilde{\psi}(p)\,dp,$
- $\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi}}\int\exp\left(-i\frac{px}{\hbar}\right)\ \psi(x)dx$

# Susskind

- $\psi(x,t) = \mathrm{Aexp}\left(i\frac{px-\frac{p^2 t}{2m}}{\hbar}\right)$ is a complete set of solutions
- $\psi(x,t) = \int dp\, \tilde{\psi}(p)\, \exp\left(i\frac{px-\frac{p^2 t}{2m}}{\hbar}\right)$
- Dynamics:
- $X|\psi > = x\ |\psi >, P\left|\psi > = i\hbar\frac{\partial}{\partial x}\right|\psi >$
- $V|\psi > \to V(x)\psi(x)\psi$, get $E\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(X)\psi$
- $< v > = \frac{d}{dt} < \psi|X|\psi >,$
- $\frac{d}{dt} < P > = \frac{i}{\hbar}[V,P] =-< \frac{\partial V}{\partial x} >,$
- $[V,P] = i\hbar\frac{dV}{dt}$

# Standard solution

- Standard method:
  1. Get $H$
  2. Prepare $|\psi(0)>$
  3. Find $H\Big|E_j >= E_j \Big|E_j >$
  4. $\alpha_j(0) =< E_j|\psi(0) >, \alpha_j(t) = \alpha_j(0)\exp(-i\frac{E_j t}{\hbar})$
  5. $|\psi(t) >= \sum_j \alpha_j(t) \; |E_j >$
  6. $P_\lambda(t) =< \lambda|\psi(t) >^2$
- Correlation and means
  - $\Delta A^2 = \sum_a (A - \bar{A})^2 \, P(a) = |< A^2|\psi >|^2$
  - $(\Delta A)(\Delta B) \geq \frac{1}{2}|< \psi|[A, B]|\psi >^2$
  - $C(A, B) =< AB > - < A >< B >$
  - $C(\sigma_x, \tau_x)$= -1

# Harmonic Oscillator, etc

- $H = \frac{1}{2}\dot{X}^2 + \frac{1}{2}\omega^2 x^2 = \frac{P^2+w^2x^2}{2} = \frac{1}{2}(P + i\omega X)(P - i\omega X) - \omega^2[X, P]$
- Ground state: $\psi(x) = \exp(\frac{\omega}{2\hbar}x^2)$, $E_0 = \frac{\omega\hbar}{2}$
- $a^- = \frac{i}{\sqrt{2\omega\hbar}}(P - i\omega X)$, $a^+ = \frac{-i}{\sqrt{2\omega\hbar}}(P + i\omega X)$
- $[a^-, a^+] = 1.$ If $N = a^+a^-$, $H = \omega\hbar(N + \frac{1}{2})$, $N|n >= n|n >$,
- $a^+|\mathrm{n} >= |\mathrm{n} + 1 >$
- Particle in box: $E_n = \frac{n^2h^2}{8mL^2}$, $\psi_n = Csin(\frac{n\pi x}{L})$

# Multi-particle systems

- Two particles: $\psi(r_1, r_2, t)$
- $H = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(r_1, r_2, t)$
- Product state, particle 1 in state a, particle 2 in state b: $\psi(r_1, r_2) = \psi_a(r_1) \cdot \psi_b(r_2)$
- $\psi_\pm(r_1, r_2) = A[\psi_a(r_1) \cdot \psi_b(r_2) \pm \psi_a(r_2) \cdot \psi_b(r_1)]$
  - Bosons: + sign, Fermions: - sign
- For fermions, if both have same state, $\psi_-(r_1, r_2) = 0$
- $\psi(r_1, r_2) = \pm\psi(r_2, r_1)$, + for bosons, - for fermions
- Distinguishable and indistinguishable particles
- Free electron gas in lattice $(k_x, k_y, k_z)$
- $E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3}$, $dE = \frac{\hbar^2 k^2}{2m}\frac{V}{\pi^2}k^2\, dk$

# Quantum Field Theory

- $E_n = \hbar\omega(n + \frac{1}{2})$, $p = \frac{h}{\lambda} = \hbar k$
- For a given wave number, number of particles is the exitation number (n)
- $|\psi(x)> = \int dx\, \psi(x)|x>$, $|\psi(x_1, x_2)> = \int dx_1 dx_2\, \psi(x_1, x_2)|x_1, x_2>$
- Procedure: Express field in wave number Fourier space, exitations corresponding to wave numbers correspond to particles
- $E = \int \rho\, dV$, $\rho = \frac{1}{2}(\,\partial_t \phi\,)^2 + \frac{1}{2}(\,\partial_x \phi\,)^2 + V(\phi)$, $\phi$ a free field
- $\phi(x,t) = \sum_k \alpha_k(t) e^{-ikx}$
- $\tilde{p}(k) = \frac{1}{2}(\,\partial_t a\,)^2 + \frac{1}{2} a k^2 + \frac{1}{2} a^2 k^2$, $\omega^2 = k^2 + m^2$
- Free scalar field is infinite collection of harmonic oscillators

# QFT

- $P(\{\phi(x)\}) = |\psi(\{\phi(x)\}|^2$, probability is for entire field
- Coupling constant for QED is $\sqrt{\alpha}$, $\alpha \approx \frac{1}{137}$ is fine constant
- Find Lagrangian density and integrate to get Lagrangian
- $\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 - \frac{1}{2}(m_\phi)^2 \phi^2$
- With interactions: $\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 - \frac{1}{2}(m_\phi)^2 \phi^2 + \frac{1}{2}(\partial_t \theta)^2 - \frac{1}{2}(\partial_x \theta)^2 - \frac{1}{2}(m_\theta)^2 \theta^2 - A\phi^2\theta - B\phi^2\theta^2$
  - $A, B$ are coupling constants
- $V(\phi) = \mu^2\phi^2 + \lambda\phi^4$ for “sombrero” potential

# Quantum Field Theory

- Path Integrals
- $U(\epsilon) = e^{-i\epsilon t}$,
- $C_{1,2} = \int dx \ < x_2|e^{-i\frac{H}{2}t}|x >< x|e^{-i\frac{H}{2}t}|x_1 >$
- $< x_2|U^N|x_1 >$ is evolution over small time slices
- $C_{1,2} = \int e^{iA/\hbar}$, integral is over all paths
- $\mathcal{L}_{QED} = -\sqrt{\alpha}e^- e^+ \gamma$, $\mathcal{L}_{int} = c\phi^4$

# Symmetries

- Translational symmetry: $V\psi(x-\epsilon) = \psi(x) - i\frac{\epsilon}{\hbar}\frac{\partial\psi}{\partial x}$
- $\hat{V} = \left(1 - i\frac{\epsilon}{\hbar}\frac{\partial}{\partial x}\right) = (1 + i\frac{\epsilon}{\hbar}\widehat{p_x})$. Generator of symmetry group is $\frac{p_x}{\hbar}$
- Noether: If $\phi \to \phi + D\phi$ changes $\mathcal{L}$ by $D\mathcal{L} = \partial_\mu W^\mu$ then $J_N^\mu = \prod_\mu D\phi - W^\mu(x)$ then $\partial_\mu J_N^\mu = 0$. Continuous symmetries conserve currents.
- $U: |\psi_1 >= \psi_2, V: |\psi > \to |\psi' >$
- $UV\ |\psi_1 > = |\psi_2' >\ , UV = VU$
- $V(\epsilon) = 1 - i\epsilon G$

# Symmetries

Translational symmetry: $V\psi(x-\epsilon) = \psi(x) - i\frac{\epsilon}{\hbar}\frac{\partial\psi}{\partial x}$

- $\hat{V} = \left(1 - i\frac{\epsilon}{\hbar}\frac{\partial}{\partial x}\right) = (1 + i\frac{\epsilon}{\hbar}\widehat{p_x})$. Generator of symmetry group is $\frac{p_x}{\hbar}$
- Noether: If $\phi \to \phi + D\phi$ changes $\mathcal{L}$ by $D\mathcal{L} = \partial_\mu W^\mu$ then $J_N^\mu = \prod_\mu D\phi - W^\mu(x)$ then $\partial_\mu J_N^\mu = 0$. Continuous symmetries conserve currents.
- Rotational symmetries: $\psi(\theta) \to \psi(\theta - \epsilon)$ then $\delta\psi = -\epsilon\frac{\partial\psi}{\partial\theta}$.
- Rewrite as: $\delta\psi = (-i\epsilon)(-\frac{i}{\hbar}\frac{\partial\psi}{\partial\theta})$ so $-\frac{i}{\hbar}\frac{\partial}{\partial\theta} \to L_z$. $\delta\psi = (-i\hbar\frac{\partial}{\partial\theta})\psi$ and $L_z$ is a generator
- $L_z\,|\psi> = m|\psi>$ for spin: $M\big(\psi(\theta)\big) = \psi(-\theta)$. $LM \neq ML$
- If $A, B$ are symmetries, so is $[A, B] = iC$. Group operation in the Lie group is $(A, B) \to C$

# Symmetries

- Let $(x, y) \rightarrow (x + \delta x, y + \delta y)$ by a small rotation, $\epsilon$. $\delta x = -\epsilon y,\ \ \delta y = \epsilon x.\ \ \delta\psi = -\frac{\partial\psi}{\partial x}\epsilon y + \frac{\partial\psi}{\partial y}\epsilon x = i\epsilon L_z \psi$
- $L_z\,|m >= m\,|m >$
- Define $L_+ = L_x + iL_y$ and $L_- = L_x - iL_y$
- $(L_+L_z - L_zL_+)|m >= -L_+|m >$
- $(m + \ 1)L_+\,|m\rangle = \ L_zL_+|m\rangle$
- $(m - 1)L_-|m > L_zL_-|m\rangle$
- Apply Hamiltonian: $HL_+\,|m\rangle = L_+H|m >= EL_+|m >$
- $H|m + 1 >= E|m + 1 >$

# Fundamental Forces

- Spin 0 corresponds to scalar described by scalar field dynamics modeled  by Klein Gordan equation
- Spin $\frac{1}{2}$ corresponds to spinor described by spinor field dynamics modeled  by Dirac equation
- Spin 1, massless corresponds to vector described by massive gauge field dynamics modeled  by Proca equation
- Spin 0 massive corresponds to vector described by massless gauge field dynamics modeled  by Maxwell equation

# QFT

- $\left[X,[Y,Z]\right] + \left[Z,[X,Y]\right] + \left[Y,[Z,X]\right] = 0$
- $a^{\pm} = \frac{P \pm i\omega X}{\sqrt{2\omega}}, N = a^{+}a^{-} + E_0, N$ is number operator, $E_0$ is ground state
- $H = \hbar\omega N,\ N\,|n> = n\,|n>$
- $[a^{\pm}, a^{\pm}] = 0, \left[a_i^{-}, a_j^{+}\right] = \delta_{ij}$
- $|n_1 n_2 \ldots n_k >$ $i$th state has $n_i$ particles, $E_T = \sum_i \hbar n_i \omega_i + E_0$
- $a^{+}\big|n> = \sqrt{n+1}\big|n+1>, a^{-}|n> = |n-1>$
- $a_s^{+}|n_1 n_2 \ldots n_k > = \sqrt{n_s + 1}|n_1 n_2, n_{s-1}, n_s + 1, \ldots n_k >$
- Definition: $\Psi^{\dagger} = \sum_i a_i^{+} \bar{\psi}_i(x)$ is an operator and observable. Value at every $x$ is a field

# QFT

- $|n_1 n_2 \dots n_k>$ $i$th state has $n_i$ particles
- $|000 \dots 0>$ is vacum state
- $E = \sum_i n_i \omega_i = \sum_i a_i^+ a_i^-$
- $\Psi^+(x) = \sum_i a_i^+ \bar{\psi}_i(x)$, applying $\Psi^+(x)$ creates particle at $x$
- $\overline{\psi_i}(x) =< i|x>$
- $\sum_i \overline{\psi_i}(x) a_i^+ \ |0> = |x>$
- For Bosons: $\Psi^+(x)\Psi^+(y) = \Psi^+(y)\Psi^+(x) = |xy> = |yx>$
- $\int dx \Psi^+(x)\Psi^-(x) = \sum_i N_i$
- Boson fields: photon, graviton, Higgs, gluons

# QFT

- $L_{int} = -\rho\phi, L = T - V, H = \sum_i p_i \dot{q}_i - L, p_i = \frac{\partial L}{\partial \dot{q}_i}$
- Hamilton: $\dot{q}_i = \frac{\partial H}{\partial p_i}, \dot{p}_i = -\frac{\partial H}{\partial q_i}$

- For Dirac: $E^2 = P^2 + m^2 c^4, \frac{\partial \psi}{\partial t} + c\frac{\partial \psi}{\partial x} = 0$
- $i\frac{\partial}{\partial t}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$
- $H = \alpha P + \beta m$

# Tensors and EM

- $J^\mu = \begin{pmatrix} c\rho \\ \vec{J} \end{pmatrix}$, $\partial_\mu = (\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$
- Maxwell's equations in tensor notation: $\partial_\mu F^{\mu\upsilon} = \mu_0 J^\upsilon$,$\partial_\mu\ {}^*F^{\mu\upsilon}$
- $\frac{1}{2} F_{\mu\upsilon} F^{\mu\upsilon} = -\frac{E^2}{c} + B^2$
- $${}^*F^{\mu\upsilon} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & \frac{E_z}{c} & -\frac{E_y}{c} \\ B_y & -\frac{E_z}{c} & 0 & \frac{E_x}{c} \\ B_z & \frac{E_y}{c} & -\frac{E_x}{c} & 0 \end{pmatrix}$$
- $$F^{\mu\upsilon} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$