Table 4.1

Node	a_W	a_E	S_u	S_P	$a_P = a_W + a_E - S_P$
1	0	100	$200T_A$	−200	300
2	100	100	0	0	200
3	100	100	0	0	200
4	100	100	0	0	200
5	100	0	$200T_B$	−200	300

This set of equations can be rearranged as

$$\begin{bmatrix} 300 & -100 & 0 & 0 & 0 \\ -100 & 200 & -100 & 0 & 0 \\ 0 & -100 & 200 & -100 & 0 \\ 0 & 0 & -100 & 200 & -100 \\ 0 & 0 & 0 & -100 & 300 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 200T_A \\ 0 \\ 0 \\ 0 \\ 200T_B \end{bmatrix} \tag{4.23}$$

The above set of equations yields the steady state temperature distribution of the given situation. For simple problems involving a small number of nodes the resulting matrix equation can easily be solved with a software package such as MATLAB (1992). For $T_A = 100$ and $T_B = 500$ the solution of (4.23) can obtained by using, for example, Gaussian elimination:

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 140 \\ 220 \\ 300 \\ 380 \\ 460 \end{bmatrix} \tag{4.24}$$

The exact solution is a linear distribution between the specified boundary temperatures: $T = 800x + 100$. Figure 4.5 shows that the exact solution and the numerical results coincide.

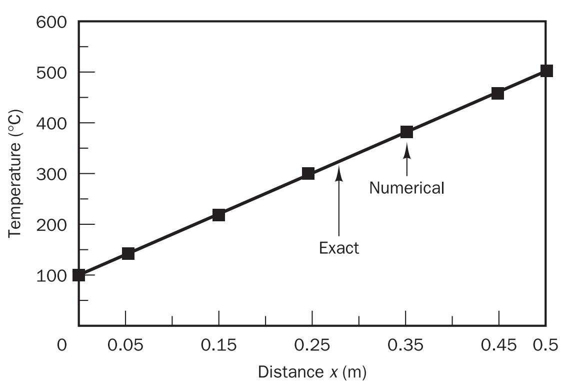


Figure 4.5 Comparison of the numerical result with the analytical solution

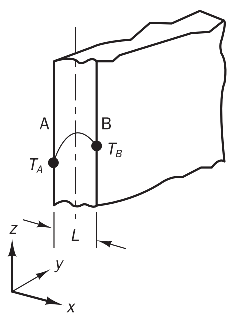
Example 4.2

Now we discuss a problem that includes sources other than those arising from boundary conditions. Figure 4.6 shows a large plate of thickness $L = 2$ cm with constant thermal conductivity $k = 0.5$ W/m.K and uniform heat generation $q = 1000$ kW/m^3. The faces A and B are at temperatures of 100°C and 200°C respectively. Assuming that the dimensions in the y- and

z-directions are so large that temperature gradients are significant in the x-direction only, calculate the steady state temperature distribution. Compare the numerical result with the analytical solution. The governing equation is

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + q = 0 \tag{4.25}$$

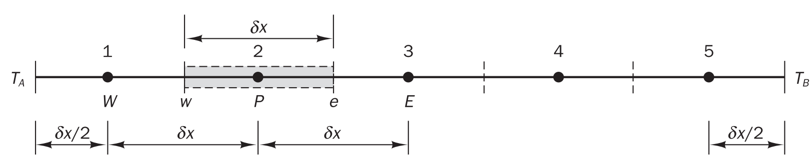
Figure 4.6



Solution

As before, the method of solution is demonstrated using a simple grid. The domain is divided into five control volumes (see Figure 4.7), giving $\delta x = 0.004$ m; a unit area is considered in the y–z plane.

Figure 4.7 The grid used



Formal integration of the governing equation over a control volume gives

$$\int_{\Delta V} \frac{d}{dx}\left(k\frac{dT}{dx}\right) dV + \int_{\Delta V} q\,dV = 0 \tag{4.26}$$

We treat the first term of the above equation as in the previous example. The second integral, the source term of the equation, is evaluated by calculating the average generation (i.e. $\bar{S}\Delta V = q\Delta V$) within each control volume. Equation (4.26) can be written as

$$\left[\left(kA\frac{dT}{dx}\right)_e - \left(kA\frac{dT}{dx}\right)_w\right] + q\Delta V = 0 \tag{4.27}$$

$$\left[k_e A\left(\frac{T_E - T_P}{\delta x}\right) - k_w A\left(\frac{T_P - T_W}{\delta x}\right)\right] + qA\delta x = 0 \tag{4.28}$$

The above equation can be rearranged as

$$\left(\frac{k_e A}{\delta x} + \frac{k_w A}{\delta x}\right) T_P = \left(\frac{k_w A}{\delta x}\right) T_W + \left(\frac{k_e A}{\delta x}\right) T_E + qA\delta x \tag{4.29}$$

This equation is written in the general form of (4.11):

$$\boxed{a_P T_P = a_W T_W + a_E T_E + S_u} \tag{4.30}$$

Since $k_e = k_w = k$ we have the following coefficients:

a_W	a_E	a_P	S_P	S_u
$\dfrac{kA}{\delta x}$	$\dfrac{kA}{\delta x}$	$a_W + a_E - S_P$	0	$qA\delta x$

Equation (4.30) is valid for control volumes at **nodal points 2, 3 and 4**.

To incorporate the boundary conditions at nodes 1 and 5 we apply the linear approximation for temperatures between a boundary point and the adjacent nodal point. At node 1 the temperature at the west boundary is known. Integration of equation (4.25) at the control volume surrounding node 1 gives

$$\left[\left(kA\frac{\mathrm{d}T}{\mathrm{d}x}\right)_e - \left(kA\frac{\mathrm{d}T}{\mathrm{d}x}\right)_w\right] + q\Delta V = 0 \tag{4.31}$$

Introduction of the linear approximation for temperatures between A and P yields

$$\left[k_e A\left(\frac{T_E - T_P}{\delta x}\right) - k_A A\left(\frac{T_P - T_A}{\delta x/2}\right)\right] + qA\delta x = 0 \tag{4.32}$$

The above equation can be rearranged, using $k_e = k_A = k$, to yield the discretised equation for **boundary node 1**:

$$\boxed{a_P T_P = a_W T_W + a_E T_E + S_u} \tag{4.33}$$

where

a_W	a_E	a_P	S_P	S_u
0	$\dfrac{kA}{\delta x}$	$a_W + a_E - S_P$	$-\dfrac{2kA}{\delta x}$	$qA\delta x + \dfrac{2kA}{\delta x}T_A$

At nodal point 5, the temperature on the east face of the control volume is known. The node is treated in a similar way to boundary node 1. At boundary point 5 we have

$$\left[\left(kA\frac{\mathrm{d}T}{\mathrm{d}x}\right)_e - \left(kA\frac{\mathrm{d}T}{\mathrm{d}x}\right)_w\right] + q\Delta V = 0 \tag{4.34}$$

$$\left[k_B A\left(\frac{T_B - T_P}{\delta x/2}\right) - k_w A\left(\frac{T_P - T_W}{\delta x}\right)\right] + qA\delta x = 0 \tag{4.35}$$

The above equation can be rearranged, noting that $k_B = k_w = k$, to give the discretised equation for **boundary node 5**:

$$\boxed{a_P T_P = a_W T_W + a_E T_E + S_u} \tag{4.36}$$

where

a_W	a_E	a_P	S_P	S_u
$\frac{kA}{\delta x}$	0	$a_W + a_E - S_P$	$-\frac{2kA}{\delta x}$	$qA\delta x + \frac{2kA}{\delta x} T_B$

Substitution of numerical values for $A = 1$, $k = 0.5$ W/m.K, $q = 1000$ kW/m^3 and $\delta x = 0.004$ m everywhere gives the coefficients of the discretised equations summarised in Table 4.2.

Table 4.2

Node	a_W	a_E	S_u	S_P	$a_P = a_W + a_E - S_P$
1	0	125	$4000 + 250T_A$	−250	375
2	125	125	4000	0	250
3	125	125	4000	0	250
4	125	125	4000	0	250
5	125	0	$4000 + 250T_B$	−250	375

Given directly in matrix form the equations are

$$\begin{bmatrix} 375 & -125 & 0 & 0 & 0 \\ -125 & 250 & -125 & 0 & 0 \\ 0 & -125 & 250 & -125 & 0 \\ 0 & 0 & -125 & 250 & -125 \\ 0 & 0 & 0 & -125 & 375 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 29000 \\ 4000 \\ 4000 \\ 4000 \\ 54000 \end{bmatrix} \tag{4.37}$$

The solution to the above set of equations is

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 150 \\ 218 \\ 254 \\ 258 \\ 230 \end{bmatrix} \tag{4.38}$$

Comparison with the analytical solution

The analytical solution to this problem may be obtained by integrating equation (4.25) twice with respect to x and by subsequent application of the boundary conditions. This gives

$$T = \left[\frac{T_B - T_A}{L} + \frac{q}{2k}(L - x) \right] x + T_A \tag{4.39}$$

The comparison between the finite volume solution and the exact solution is shown in Table 4.3 and Figure 4.8 and it can be seen that, even with a coarse grid of five nodes, the agreement is very good.

Table 4.3

Node number	*1*	*2*	*3*	*4*	*5*
x (m)	0.002	0.006	0.01	0.014	0.018
Finite volume solution	150	218	254	258	230
Exact solution	146	214	250	254	226
Percentage error	2.73	1.86	1.60	1.57	1.76

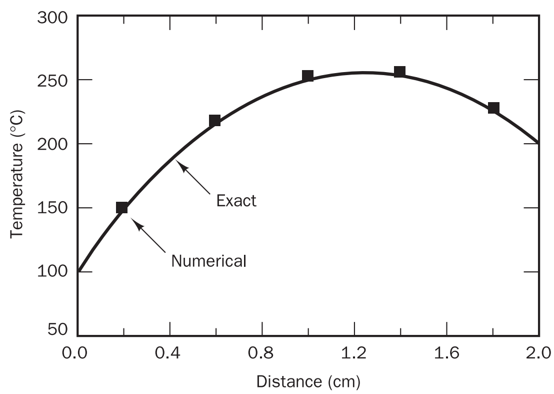


Figure 4.8 Comparison of the numerical result with the analytical solution

Example 4.3

In the final worked example of this chapter we discuss the cooling of a circular fin by means of convective heat transfer along its length. Convection gives rise to a temperature-dependent heat loss or sink term in the governing equation. Shown in Figure 4.9 is a cylindrical fin with uniform cross-sectional area A. The base is at a temperature of 100°C (T_B) and the end is insulated. The fin is exposed to an ambient temperature of 20°C. One-dimensional heat transfer in this situation is governed by

$$\frac{d}{dx}\left(kA\frac{dT}{dx}\right) - hP(T - T_\infty) = 0 \tag{4.40}$$

where h is the convective heat transfer coefficient, P the perimeter, k the thermal conductivity of the material and T_∞ the ambient temperature. Calculate the temperature distribution along the fin and compare the results with the analytical solution given by

$$\frac{T - T_\infty}{T_B - T_\infty} = \frac{\cosh[n(L - x)]}{\cosh(nL)} \tag{4.41}$$

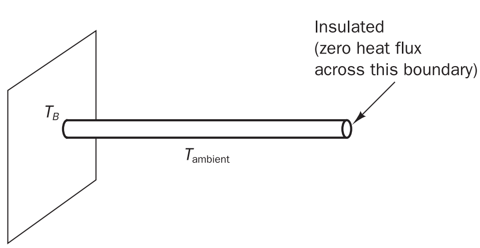


Figure 4.9 The geometry for Example 4.3