Bloomberg

The Bloomberg Implementation of the Gaussian Copula Model

April 23, 2015

Quantitative Analytics
Bloomberg L.P.

Abstract

We describe the Bloomberg implementation of the 1-factor and 2-factor Gaussian copula models for synthetic CDO (collateralized debt obligation) tranches of credit indices. The implementation deviates from the classical lattice-based approach and employs both analytical and numerical techniques to value the conditional expected tranche loss. We use this pricer to to imply base correlations from tranche quotes, and to construct a base correlation curve. We describe interpolation and extrapolation of the base correlation curve, as well as how an index base correlation curve is used to price tranches on bespoke portfolios resembling the index.

Keywords. Collateralized debt obligation, Gaussian copula, base correlation, bespoke tranche pricing.

First published January 17, 2007.

Contents

List of Figures

1 Introduction

This white paper describes the implementation of the Bloomberg 1- and 2-factor Gaussian copula models for synthetic collateralized debt obligation (CDO) valuation in **CDST**, with an emphasis on the aspects of our approach that are novel, namely a decomposition of the conditional expected capped portfolio loss process into "intrinsic value" and "time value" components, closed form solution for the "intrinsic value", and very efficient computational scheme for the "time value". The underlying CDO structure and the Gaussian copula framework will be described very briefly since they have been described in detail elsewhere, see e.g. the papers by Andersen, Sidenius and Basu (2003) [ASB03], and Hull and White (2004) [HW04]. The single-factor model described here is also published in Flesaker (2008) [Fle08].

2 The synthetic CDO structure

A synthetic CDO is a credit default swap characterized by a settlement currency; a portfolio, defined by a list of names and accompanying notional amounts covered; a maturity date and premium payment schedule; a deal spread (a.k.a. premium); and finally an attachment point and a detachment point, determining the beginning and end of the portfolio loss tranche covered. A tranche is known as an equity tranche if the attachment point equals zero, as a super-senior tranche if the detachment point equals the underlying portfolio notional, and as a mezzanine tranche if it is neither of the above. We can always analyze arbitrary tranches as the difference between two equity tranches, one detaching at the actual tranche's detachment point minus one detaching at the actual tranche's attachment point. This is necessary in the base correlation framework, and harmless otherwise. In the following, we will therefore focus on the modeling of equity tranches.

3 Valuation assumptions

We follow the standard industry practice in credit default swap modeling and assume that we can value all cash fows by taking their expected value under a "risk neutral", perhaps more accurately described as "risk adjusted", probability measure and discounting the resulting quantities with the yield curve implied from the interest rate swap market. This is tantamount to assuming that interest rates are deterministic or that their dynamics are statistically independent of the default processes. We will further assume that each name upon default has a known fixed recovery rate as a fraction of par, which is an input to the model. These assumptions are consistent with the modelling conventions in the single name CDS market.

4 The model

Using the same single name curve stripping routine as in the Bloomberg model in CDSW, our analysis takes as a starting point a function that provides a market implied unconditional risk neutral default probability $p_i(t)$ for each name i to each date of interest t. Given a value $x =$

$(x_1, x_2)'$ of the common market factors, the content of the two factor Gaussian copula model is that the event that name i defaults before time t is independent of the default of all other names and occurring with probability:

$$p_i(t,x) = \Phi\left(\frac{\Phi^{-1}(p_i(t)) - a_i'x}{\sqrt{1 - a_i'a_i}}\right), \tag{1}$$

where $a_i = (a_{i1}, a_{i2})'$ is the factor loading vector for name i and Φ denotes the standard normal (Gaussian) distribution function. Special cases include the 1-factor model where $a_{i2} \equiv 0$, and the constant (compound) correlation model where $a_{i1} \equiv \sqrt{\rho}$ and $a_{i2} \equiv 0$. The base correlation version of the model amounts to each attachment/detachment point having its own input level of ρ, meaning that we will analyze a mezzanine tranche as the difference between two equity tranches, where the two equity tranches are valued with different correlation parameters.

The fractional loss and recovery processes for the underlying basket are given by $L(t)$ and $R(t)$, respectively, defined as follows:

$$L(t) = \frac{\sum_i \chi_{\tau_i < t} N_i (1 - R_i)}{\sum_i N_i} \tag{2}$$

$$R(t) = \frac{\sum_i \chi_{\tau_i < t} N_i R_i}{\sum_i N_i} \tag{3}$$

where τ_i denotes the default time of name i, N_i is the notional amount of name i in the basket, R_i is the fractional recovery upon default of name i, and χ_A is an indicator variable that takes on the value 1 if A is true and 0 otherwise.

Given a fractional detachment point D, we are concerned with the risk neutral expectations of the capped loss and recovery processes:

$$\hat{L}(t) = \min[L(t), D] \tag{4}$$

$$\hat{R}(t) = \max[R(t) - (1 - D), 0] \tag{5}$$

With precise knowledge of these quantities for each date between the current time and maturity, the present value of the default leg and of the unit spread premium leg of a synthetic CDO equity tranche detaching at D can be estimated by taking the expecation of the discounted loss and premium cash flows under the risk neutral pricing measure as follows:

$$V_d = \mathbb{E}\int_0^T P(t)d\hat{L}(t) \tag{6}$$
$$= P(T)\mathbb{E}\hat{L}(T) + \int_0^T f(t)P(t)\mathbb{E}\hat{L}(t)dt$$

$$V_p = \sum_{j=1}^{J} P(t_j)\delta(t_{j-1}, t_j)\left[D - \frac{1}{t_j - t_{j-1}}\int_{t_{j-1}}^{t_j}\left(\mathbb{E}\hat{L}(t) + \mathbb{E}\hat{R}(t)\right)dt\right] \tag{7}$$

where $P(t)$ denotes the discount factor to time t, $f(t)$ is the instantaneous forward interest rate for time t, t_j denotes premium payment dates, with $t_J = T$, and $\delta(t_{j-1}, t_j)$ is the daycount fraction between consecutive premium payment dates. The second line in equation (6) follows from integration by parts of the integral in the first line and by exchanging the order of integration and expectation (which is admissible by Fubini's theorem). The integral term subtracted from the initial tranche notional in equation (7) represents a slight approximation to the reduction in notional by tranche losses and write-down from above (the representation would be exact if discount rates were zero and/or all tranche losses occurred at the end of each premium period). The model value of the tranche, aside from premium accrual, is given by $V_d - cV_p$, where c is the contractual deal spread, and the breakeven premium (replacement deal spread) on a tranche with no upfront payment is found as the ratio V_d/V_p.

The capped recovery process is only of interest for super-senior tranches, and given the assumption of known recovery per name, $E\hat{R}(t)$ can be found by a minor variation of the routine used to solve for the capped loss process, $E\hat{L}(t)$. Our approach to solving the time integrals above is to solve for the expected capped losses (and, if necessary, recoveries) for a discrete set of points in time that typically include the premium payment dates, fit a cubic spline to the resulting function of time, and evaluate the integrals analytically under the mild assumption of piecewise constant forward interest rates between the knot points of the spline.

5 Two factor pricing

Before we turn to a detailed discussion of the implementation of the single factor model, we briefly address the solution of the two-factor model. We note that, conditional on the value of the second factor, we have a regular single factor pricing problem to solve, the result of which can then be integrated back against the Gaussian distribution of the second factor. We carry out this "outer" integral by a Gauss-Hermite quadrature, which is generally the preferred scheme for numerical integration of smooth functions against a Gaussian kernel over the real line. To make the approach efficient, we first ensure that only "real" two-factor models are treated as such, meaning that if the two factor loading vectors a_1 and a_2 are collinear, the two-factor model will be replaced with an equivalent one-factor model with $a_{i1} = \sqrt{a_{i1}^2 + a_{i2}^2}$ and $a_{i2} \equiv 0$. Also, we ensure that the "less important" factor is treated as the second one, based on a comparison of the inner product of the vector of factor loadings with the loss given default per name.

6 Single factor pricing

In the remainder of the paper, we will specialize the analysis to a single state variable, X, realizations of which will be denoted by x. For each $\{x, t\}$, $L(t)$ is then the sum of a set of independent heterogeneous binary random variables. Note that, in principle, we need to know the expected capped loss for each point in time until maturity for each possible value of the Gaussian state variable, in order to calculate the unconditional expected capped loss required in (6) and (7).

$$\mathbb{E}\left[\hat{L}(t)\right] = \int_{-\infty}^{\infty} \mathbb{E}\left[\min\left(L(t), D\right)|x\right]\phi\left(x\right)dx \tag{8}$$

where $\phi\left(x\right)$ is the standard Gaussian probability density function. Figure 1 shows the conditional expected capped loss profile for an equity tranche of a typical deal.

7 A decomposition

Let x_t^* be the unique value of x where $\mathbb{E}\left[L(t)|x\right] = D$. Then we can rearrange the expression for the expected capped loss on an equity tranche as:

$$\begin{aligned}
& \mathbb{E}\left[\hat{L}(t)\right] \qquad (9) \\
= \; & \int_{-\infty}^{x_t^*} \left(D - \mathbb{E}\left[\max\left(D - L(t), 0\right)|x\right]\right)\phi\left(x\right)dx \\
& + \int_{x_t^*}^{\infty} \left(\mathbb{E}[L(t)|x] - \mathbb{E}\left[\max\left(L(t) - D, 0\right)|x\right]\right)\phi\left(x\right)dx.
\end{aligned}$$

We can further sort the integrals into "intrinsic value" minus "time value", where we have liberally borrowed terminology from the option pricing literature:

$$\begin{aligned}
& \mathbb{E}\left[\hat{L}(t)\right] \\
= \; & \int_{-\infty}^{x_t^*} D\phi\left(x\right)dx + \int_{x_t^*}^{\infty} \mathbb{E}[L(t)|x]\phi\left(x\right)dx \\
& - \int_{-\infty}^{x_t^*} \mathbb{E}\left[\max\left(D - L(t), 0\right)|x\right]\phi\left(x\right)dx \qquad (10) \\
& - \int_{x_t^*}^{\infty} \mathbb{E}\left[\max\left(L(t) - D, 0\right)|x\right]\phi\left(x\right)dx.
\end{aligned}$$

As we will see, this is useful because of the analytical tractability of the intrinsic value and the numerical tractability of the time value. Figure 2 and Figure 3 show quarterly slices of the intrinsic value and time value components of the expected capped loss. Figure 4 shows time values and intrinsic values with the time scale suppressed.

8 Intrinsic simplicity of intrinsic value

Using the Gaussian copula expression for the conditional default probabilities we can simplify the intrinsic value calculations further, to get the following closed form solution:

$$\begin{aligned} & \int_{-\infty}^{x_t^*} D\phi\,(x)\,dx + \int_{x_t^*}^{\infty} \mathbb{E}[L(t)|x]\phi\,(x)\,dx \\ = \; & D\Phi\,(x_t^*) \\ & + \int_{x_t^*}^{\infty} \sum_i (1-R_i)\,N_i\Phi\left(\frac{\Phi^{-1}\,(p_i(t))}{\sqrt{1-\rho}} - \sqrt{\frac{\rho}{1-\rho}x}\right)\phi\,(x)\,dx \\ = \; & D\Phi\,(x_t^*) + \sum_i (1-R_i)\,N_i\left[p_i(t) - \Phi_2\left(\Phi^{-1}\,(p_i(t))\,, x_t^*; \sqrt{\rho}\right)\right] \end{aligned} \tag{11}$$

where $\Phi_2(x, y; r)$ denotes the bivariate normal distribution function with correlation r. The last line follows from the properties of Gaussian integrals, see e.g. Andersen and Sidenius (2005) [AS05].

9 Time stability of time value

As illustrated in the Figure 5 and Figure 6, the time value function associated with a detachment point is nearly invariant in time, once centered around the critical value of the common factor. We are relying on this time stability as a computational heuristic, which has proven remarkably resilient in practical computations, even if the exact conditions for its validity are hard to establish. It is perhaps worth noting in this context that the popular "Large Pool Model" approximation described in the base correlation paper by McGinty and Ahluwalia (2004) [MA04] (and foreshadowed by Vasicek in 1987 [Vas87]) effectively assumes that the time value is identically equal to zero (and thus trivially satisfying time stability). For typical transaction parameters the contributions from the time values tend to be relatively small compared to the corresponding intrinsic value contributions, thus making the overall calculation fairly robust to the performance of this approximation. An exception is the case of vanishing correlations, which is quickly solved anyway, since dependence on the common factor drops out entirely.

We estimate the time value and its single name sensitivities by a slightly modified version of Hull and White's (2004) [HW04] bucketing algorithm, as explained further below, for a set of Gauss-Laguerre quadrature points in each direction from x_T^* for the common factor at maturity. We numerically integrate these time values and their derivatives against the Gaussian density for each calculation date t, taking care to re-center the quadrature points around the corresponding values of x_t^*. We use Gauss-Laguerre quadrature in each direction to integrate the smooth function against a Gaussian density on each half-line around the kink in the time value function at x_t^*.

10 The time value computation

We compute the conditional time value for each value of the state variable x under consideration by dividing the positive part of the conditional loss distribution into a set of ranges that we will refer to as "buckets". Specifically, we have a zero loss point, followed by equally spaced boundaries for portfolio losses up to a point slightly above the detachment point. The last loss bucket is a "trapping state" that will catch the probability of all losses exceeding the detachment point. We keep track of two quantities per bucket: the probability of the portfolio loss being in the bucket, and the expected portfolio loss given that the loss is in the bucket. The initial conditions are set as the entire probability mass being located at the zero loss point and the expected loss in each bucket being equal to the bucket's mid-point. We then procede to iterate over the names in the portfolio. For each name, we iterate over the loss buckets, and from those "source" buckets that already have attained positive probability, we re-allocate the probability of a default to the bucket that contains the sum of the name's loss given default (LGD) and the expected loss in the source bucket. Once we have gone through all the names, we compute the conditional time value of the tranche as the capped expected portfolio loss minus the expected capped portfolio loss.

An important feature of the algorithm is that we can compute a full set of single name spread sensitivities at relatively little extra cost. This is done by passing in a set of perturbed conditional default probabilities along with the original ones, corresponding to the effect of the desired spread perturbations (typically, a flat 1 basis point spread increase, but this is a user specified input). From the fully built up conditional loss distribution, i.e. after having iterated over all names in the portfolio, we calculate perturbed values of the expected capped loss by treating the losses as being constrained to lie exactly on the points defined by the expected value of each loss bucket. Each name's loss will in turn be de-convolved and re-convolved with the remaining loss distribution, where we split the loss probability between neighboring points when the distance between loss points is not commensurate with the name's LGD. This requires the solution of a triangular and highly banded system of simultaneous linear equations for each name, which is usually extremely fast and accurate. A certain amount of care is taken to avoid problems with numerical instability arising from the transition matrix being near-singular when the conditional single name default probability is close to 1.

This concludes the description of the pricing techniques.

11 Base Correlations and Bespoke Pricing

Standard tranches (of standard maturities, with fixed attachment and detachment points, on standard credit indices) are quoted in the market, but a nonstandard tranche may differ in terms of maturity, detachment point or portfolio composition.

To price index tranches with nonstandard detachment points, we need to construct a base correlation curve. For bespoke portfolios, we further need a way to infer a correlation, possibly one obtained by somehow modifying the base correlation curve of a reference index closely resembling the bespoke portfolio. The non-standard tranches may differ from the standard index tranches in terms of maturity dates, attachment/detachment points, and/or the underlying portfolio. We divide the problem into four major components:

Implied Base Correlation Obtaining standard index tranche base correlations implied from market data;

Base Correlation Extrapolation Extrapolation of index base correlations outside the grid of observable detachment points and maturities;

Base Correlation Interpolation Interpolation of index base correlations between detachment points and maturities; and

Bespoke Mapping Mapping of detachment points from a bespoke portfolio to an index. We assume that the choice of reference index is made by the user for the transaction under consideration, and we do not allow mapping from multiple indices in the current version.

We describe these procedures in the sequel.

12 Calculating implied base correlations

We use the specific model implementation in **CDST** to imply base correlations from quoted market spreads and prices, rather than directly relying on standard tranche base correlations quotes provided by some dealers. Different dealers use different implementations of the Gaussian copula model, as well as different data adjustments, potentially resulting in materially different mappings between base correlations and quoted prices.

Before implying base correlations from market quotes on tranches, we need to adjust the single name CDS curves so that we can reproduce the quoted index level. The tranches are priced in the model relative to the single name curves, wihout explicit reference to the index itself, and we would argue that a failure to match the index level puts us in a poor position to price the tranches. The standard market quotes for CDX and iTraxx investment grade indices come in the form of a spread. To convert this spread to an actual market price, the index, which has a contractually specified premium level, is by convention valued in **CDSW** using a flat CDS curve given by the quoted spread and a 40% recovery rate. The resulting market value can then be compared to the corresponding quantity in CSDT for a 0-100 tranche on the underlying portfolio. We then find the unique single scale factor that, when applied to all underlying single name CDS curve points simultaneously, makes the CDST value match the CDSW value. In keeping with the general spirit of lack of time consistency in the standard usage of the model, we carry out all spread adjustments and implied correlation calculations globally for each transaction maturity date under consideration, i.e. when we analyze a 10 year contract we will make adjustments to the 5 year curve points independent of what we make when we analyze a 7 year contract.

In actually solving for the base correlations, we start with the equity tranche and solve for the detachment correlation ρ which makes its theoretical up-front value equal to the market quote:

$$\frac{D_k(T,\rho) - 500P_k(T,\rho)}{k} - U^M(k,T) = 0, \tag{12}$$

where $D_k(T,\rho)$ and $P_k(T,\rho)$ denote the model value of the default and unit premium legs, respectively, for the underlying index tranche with detachment point k, maturity T and using a flat correlation of ρ, and $U^M(k,T)$ denotes the market quoted up-front payment on the tranche (with 500 basis points running premium). Since the left hand side is strictly decreasing in ρ, a solution will be unique if it exists, and we will denote it $\rho(k,T)$. There is a lower bound on the value of the market quoted upfront payment below which (12) will have no solution, but this range is of little practical interest since entering it introduces an arbitrage opportunity between the equity tranche and the index itself. There is also an upper bound on the set of market prices that will provide a solution, corresponding to the model value at zero correlation. This has yet to be breached in actual markets, but would not at all be inconsistent with no arbitrage, only with a single factor constant correlation Gaussian model for pricing the equity tranche. Note that, in the case of an equity index tranche being priced "too high", the entire base correlation approach breaks down.

We solve recursively for higher detachment point base correlations, using the result of the previous calculation each time. For a mezzanine tranche with attachment point k_1, detachment point k_2, and market spread $S^M(k_1,k_2,T)$, having already solved for $\rho(k_1,T)$ we solve for the correlation ρ such that:

$$\frac{D_{k_2}(T,\rho) - D_{k_1}(T,\rho(k_1,T))}{P_{k_2}(T,\rho) - P_{k_1}(T,\rho(k_1,T))} - S^M(k_1,k_2,T) = 0, \tag{13}$$

13 Extrapolation and interpolations

Starting with a grid of base correlations from standard tranches, typically for 5 detachment points for 3 different on-the-run maturities, we carry out all extrapolation by keeping the base correlations flat. With extrapolation, we mean in time between the valuation date and the first maturity, and we mean in detachment point between 0 and 3% (inside the equity tranche) and between the highest quoted detachment point (22% for iTraxx and 30% for CDX) and 100%. As a practical matter, the base correlation is irrelevant for detachment levels in excess of the maximum theoretical loss level, which would be 60% if we assume a flat 40% recovery across the names in the index portfolio. By keeping the base correlations flat we ensure that the valuation is monotonic in detachment point in the extrapolation region, and we also avoid the inevitable questions of what to do when the extrapolated values of the correlation hits 0 or 1, which would regularly arise if we were to try to do linear or spline based extrapolation of the base correlation itself.

For maturity and detachment points in the interior of the standard index tranche grid, we perform a bilinear interpolation of the base correlation. The interpolated base correlation for such a point will be a convex combination of the four implied base correlations corresponding to the available index tranches with maturity and detachment points immediately above and below the point of interest, with linear weights. While the order is irrelevant, we can visualize the process as first performing a linear intepolation between detachment points for each standard maturity, and then a second linear interpolation between maturities for each detachment point level.

Bilinear intepolation has the advantages of simplicity and transparency as well as locality: a change to one point in the input grid will only affect the interpolated base correlations for adjacent maturities and detachment points. Unfortunately, linear interpolation of base correlation in the detachment point dimension can result in theoretical arbitrage opportunities between tranches with slightly different detachment points around the grid points. We are not aware of any direct base correlation interpolation scheme that can guarantee the absence of such arbitrage opportunities, but it can be accomplished by applying monotonic interpolation methods to the underlying "tranche loss surface" and implying a dense set of base correlations from the resulting tranche prices. We expect to explore this angle in future versions of the model.

14 Bespoke mapping

Finally, we map bespoke detachment points of interest into index detachment points by multiplying them with the ratio of portfolio default leg values. If we denote the bespoke portfolio variables with an overbar, k represents attachment and detachment points, and $D(t)$ is the theoretical present value of the default leg of the underlying portfolio, the mapping is given as:

$$k = min\left[\bar{k}\frac{D(T)}{\bar{D}(T)}, 100\%\right] \tag{14}$$

We then proceed to analyze a bespoke CDO with attachment and detachment points $(\bar{k}_1, \bar{k}_2)$ in CDST with base correlations given by $\rho(k_1, T)$ and $\rho(k_2, T)$ from the interpolated index correlation surface.

We believe that this version of "expected loss mapping" is very close to the actual practice of many market participants, and that its simplicity and transparency may outweigh some of its theoretical shortcomings. Alternative schemes, e.g. based on matching the ratio of tranche default leg value to portfolio default leg value between the index and bespoke tranches, may be added in the future. However, it should be kept in mind that no bespoke mapping approach in this general spirit can address the more fundamental problem that correlation differences between two portfolios ultimately will arise from aspects of the underlying names beyond what is reflected in their CDS curves.

15 Figures

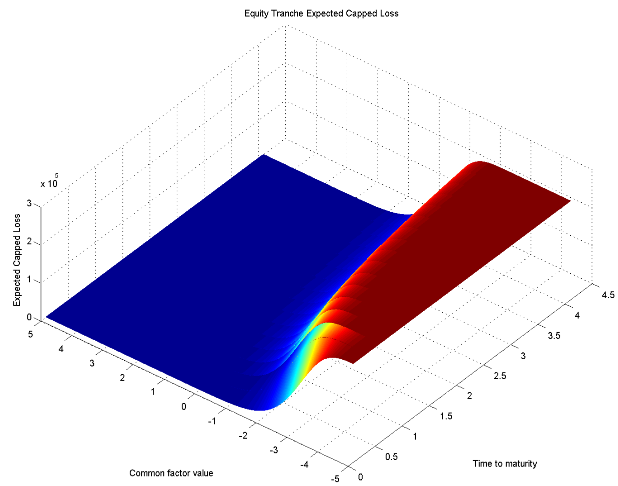

Figure 1: The conditional expected capped loss profile for an equity tranche of a typical deal.

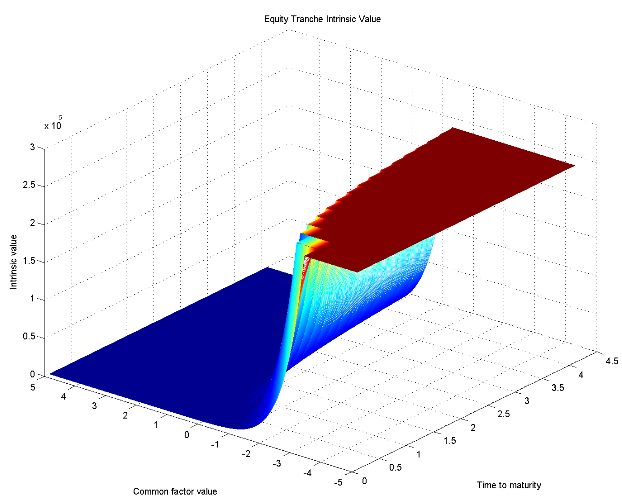

Figure 2: Quarterly slices of the intrinsic value component of the capped expected loss.

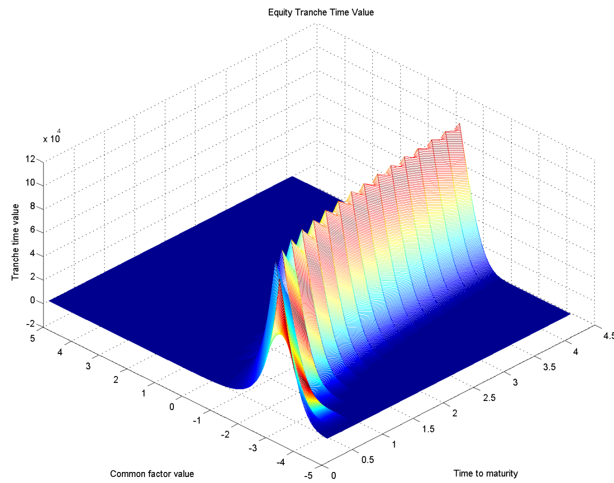

Figure 3: Quarterly slices of the time value component of the expected capped loss.

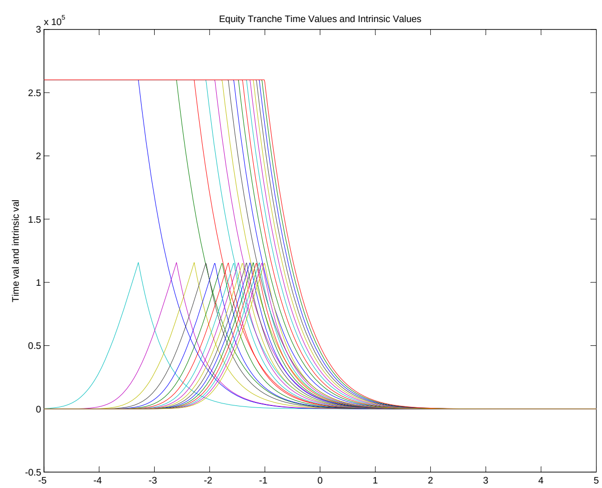

Figure 4: Time values and intrinsic values with the time scale suppressed.

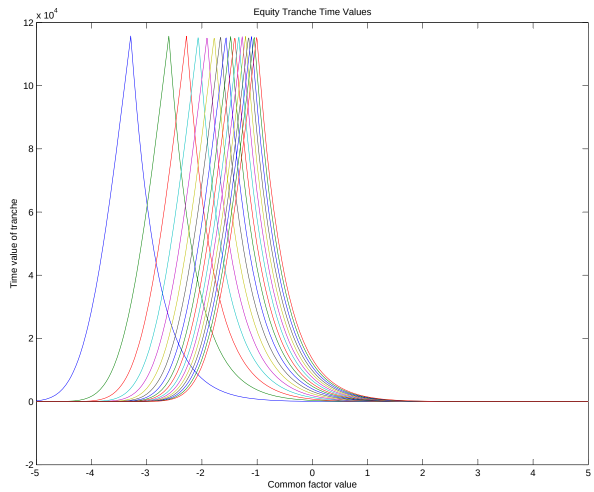

Figure 5: Quarterly time value slices from about 1M to 4Y.

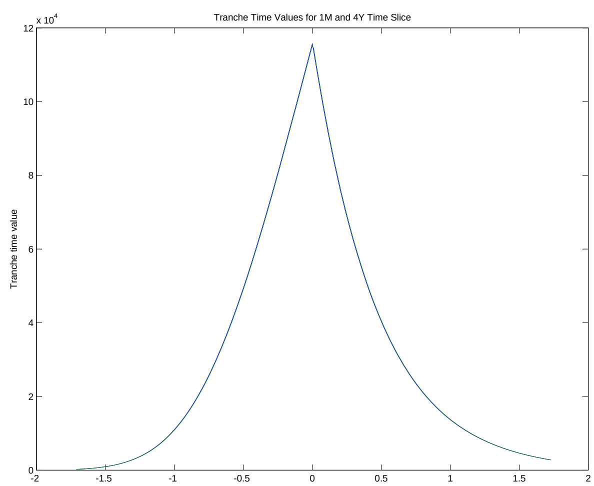

Figure 6: The shortest and longest maturity from the Figure 5 properly centered.

References

[AS05] L. Andersen and J. Sidenius. Extensions to the Gaussian Copula: Random Recovery and Random Factor Loadings. *Journal of Credit Risk*, 1(1), 2005.

[ASB03] L. Andersen, J. Sidenius, and S. Basu. All Your Hedges in One Basket. *RISK*, Nov 2003.

[Fle08] B. Flesaker. Fast Solution of the Gaussian Copula Model. In Jean-Pierre Fouque, Thomas B. Fomby, Knut Solna, editor, *Econometrics and Risk Management (Advances in Econometrics, Volume 22)*, pages 1–13. Emerald Group Publishing Limited, 2008.

[HW04] J. Hull and A. White. Valuation of a CDO and an n^{th} to Default CDS without Monte Carlo Simulation. *Journal of Derivatives*, 12(2):8–23, Winter 2004.

[MA04] L. McGinty and R. Ahluwalia. A Model for Base Correlation Calculation. JPMorgan Credit Strategy, May 2004.

[Vas87] O. Vasicek. Limiting Loan Loss probability Distribution. KMV working paper, 1987.