Appendix: A Note on Rationalizing Over-Reactions

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Contents

Appendix A Derivations and Proofs

A.1 Proof of Proposition 1

$$
\begin{aligned}
\beta_1^{OC} &= \frac{\mathbb{C}(x_t - \breve{x}^i_{t|t}, \breve{x}^i_{t|t} - \breve{x}^i_{t|t-1})}{\mathbb{V}(\breve{x}^i_{t|t} - \breve{x}^i_{t|t-1})} \\
&= \frac{(1-\breve{\kappa})\breve{\kappa}\mathbb{V}(x_t - x^i_{t|t-1}) - \breve{\kappa}^2\sigma_v^2}{\breve{\kappa}^2[\mathbb{V}(x_t - \breve{x}^i_{t|t-1}) + \sigma_v^2]} \\
&= \frac{(1-\breve{\kappa})\mathbb{V}(x_t - x^i_{t|t-1}) - \breve{\kappa}\sigma_v^2}{\breve{\kappa}[\mathbb{V}(x_t - \breve{x}^i_{t|t-1}) + \sigma_v^2]} \\
&= \frac{\frac{\breve{\sigma}_v^2\mathbb{V}(x_t - \breve{x}^i_{t|t-1})}{\mathbb{V}(x_t - \breve{x}^i_{t|t-1}) + \breve{\sigma}_v^2} - \frac{\mathbb{V}(x_t - \breve{x}^i_{t|t-1})\sigma_v^2}{\mathbb{V}(x_t - \breve{x}^i_{t|t-1}) + \breve{\sigma}_v^2}}{\breve{\kappa}[\mathbb{V}(x_t - \breve{x}^i_{t|t-1}) + \sigma_v^2]} \\
&= \frac{(\alpha^2 - 1)\sigma_v^2\breve{\kappa}}{\breve{\kappa}[\mathbb{V}(x_t - \breve{x}^i_{t|t-1}) + \sigma_v^2]} \\
&= \frac{(\alpha^2 - 1)\sigma_v^2}{\mathbb{V}(x_t - \breve{x}^i_{t|t-1}) + \sigma_v^2}
\end{aligned}
$$

A.2 Deriving Errors and Revisions for SI Model

In state space form, we have

$$
\begin{aligned}
\begin{bmatrix} x_t \\ F_t \end{bmatrix} &= \begin{bmatrix} \rho & 0 \\ \lambda\rho & (1-\lambda)\rho \end{bmatrix} \begin{bmatrix} x_{t-1} \\ F_{t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ \lambda \end{bmatrix} w_t \\
y_t^i &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_t \\ F_t \end{bmatrix} + v_t^i
\end{aligned}
$$

Or more compactly,

$$
\begin{aligned}
\mathbf{s_t} &= \mathbf{A}\mathbf{s_{t-1}} + \mathbf{B}w_t \\
y_t^i &= \mathbf{C}\mathbf{s_t} + v_t^i
\end{aligned}
$$

Invoking the Kalman filter

$$\begin{aligned}
\mathbf{s^i_{t|t}} &= (\mathbf{I} - \kappa\mathbf{C})\mathbf{A}\mathbf{s^i_{t-1|t-1}} + \kappa\mathbf{C}\mathbf{A}\mathbf{s_{t-1}} + \kappa\mathbf{C}\mathbf{B}w_t + \kappa v^i_t \\
\mathbf{s^i_{t|t-1}} &= \mathbf{A}\mathbf{s^i_{t-1|t-1}}
\end{aligned}$$

which implies that

$$\begin{aligned}
x^i_{t|t} &= (1-\kappa_1)\rho x^i_{t-1|t-1} + \kappa_1 y^i_t \\
F^i_{t|t} &= (\lambda-\kappa_2)\rho x^i_{t-1|t-1} + (1-\lambda)\rho F^i_{t-1|t-1} + \kappa_2 y^i_t
\end{aligned}$$

and

$$\begin{aligned}
x^i_{t|t-1} &= \rho x^i_{t-1|t-1} \\
F^i_{t|t-1} &= \lambda\rho x^i_{t-1|t-1} + (1-\lambda)\rho F^i_{t-1|t-1}
\end{aligned}$$

where $\lambda = \frac{\kappa_1 + R\kappa_2}{1+R}$ and κ_1, κ_2 are the Kalman gains belonging to the two-dimensional column vector κ. Letting $\xi = \begin{bmatrix} \frac{1}{1+R} & \frac{R}{1+R} \end{bmatrix}$, we have

$$\begin{aligned}
\widetilde{x}^i_{t|t} &= \xi\mathbf{s^i_{t|t}} \\
&= \frac{1}{1+R}x^i_{t|t} + \frac{R}{1+R}F^i_{t|t} \\
&= \frac{1}{1+R}\Big[(1-\kappa_1)\rho x^i_{t-1|t-1} + \kappa_1 y_{it}\Big] + \frac{R}{1+R}\Big[(\lambda-\kappa_2)\rho x^i_{t-1|t-1} + (1-\lambda)\rho F^i_{t-1|t-1} + \kappa_2 y_{it}\Big] \\
&= \frac{1-\kappa_1 + R(\lambda-\kappa_2)}{1+R}x^i_{t|t-1} + \lambda y_{it} + \frac{(1-\lambda)R\rho}{1+R}F^i_{t-1|t-1} \\
&= \frac{1-\lambda}{1+R}x^i_{t|t-1} + \frac{(1-\lambda)R\rho}{1+R}F^i_{t-1|t-1} + \lambda y_{it}
\end{aligned}$$

and

$$
\begin{aligned}
\widetilde{x}^{i}_{t|t-1} &= \xi \mathbf{s^{i}_{t|t-1}} \\
&= \frac{1}{1+R}\rho x^{i}_{t-1|t-1} + \frac{R}{1+R}\Big[\lambda\rho x^{i}_{t-1|t-1} + (1-\lambda)\rho F^{i}_{t-1|t-1}\Big] \\
&= \frac{1+R\lambda}{1+R}x^{i}_{t|t-1} + \frac{(1-\lambda)\rho R}{1+R}F^{i}_{t-1|t-1}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\widetilde{x}^{i}_{t|t} &= \frac{1-\lambda}{1+R}x^{i}_{t|t-1} + \frac{(1-\lambda)R\rho}{1+R}F^{i}_{t-1|t-1} + \lambda y^{i}_{t} \\
\widetilde{x}^{i}_{t|t-1} &= \frac{1+R\lambda}{1+R}x^{i}_{t|t-1} + \frac{(1-\lambda)\rho R}{1+R}F^{i}_{t-1|t-1}
\end{aligned}
$$

Furthermore,

$$
F_t = \widetilde{x}_{t|t} = (1-\lambda)\rho F_{t-1} + \lambda x_t
$$

$$
\widetilde{x}_{t|t-1} = \lambda\rho x_{t-1|t-1} + (1-\lambda)\rho F_{t-1}
$$

From here, it follows that

$$
\begin{aligned}
x_t - \widetilde{x}^{i}_{t|t} &= x_t - \frac{1-\kappa_1+(\lambda-\kappa_2)R}{1+R}x^{i}_{t|t-1} - \lambda y^{i}_{t} - \frac{(1-\lambda)R\rho}{1+R}F^{i}_{t-1|t-1} \\
&= (1-\lambda)x_t - \frac{1-\kappa_1+(\lambda-\kappa_2)R}{1+R}x^{i}_{t|t-1} - \lambda v^{i}_{t} - \frac{(1-\lambda)R\rho}{1+R}F^{i}_{t-1|t-1} \\
&= (1-\lambda)x_t - \frac{1-\lambda}{1+R}x^{i}_{t|t-1} - \frac{(1-\lambda)R\rho}{1+R}F^{i}_{t-1|t-1} - \lambda v^{i}_{t} \\
&= (1-\lambda)\Big[x_t - \frac{1}{1+R}x^{i}_{t|t-1} - \frac{R}{1+R}\rho F^{i}_{t-1|t-1}\Big] - \lambda v^{i}_{t}
\end{aligned}
$$

and the forecast revision is

$$
\begin{aligned}
\widetilde{x}^{i}_{t|t} - \widetilde{x}^{i}_{t|t-1} &= \frac{(1-\kappa_1)+(\lambda-\kappa_2)R-(1+R\lambda)}{1+R}x^{i}_{t|t-1} + \lambda x_t + \lambda v^{i}_{t} \\
&= -\frac{\kappa_1-\kappa_2 R}{1+R}x^{i}_{t|t-1} + \lambda x_t + \lambda v^{i}_{t} \\
&= \lambda(x_t - x^{i}_{t|t-1} + v^{i}_{t})
\end{aligned}
$$

A.3 Proof of Proposition 2

$$
\begin{aligned}
\beta_1^{SI} &= \frac{\mathrm{Cov}(x_t - \widetilde{x}^i_{t|t}, \widetilde{x}^i_{t|t} - \widetilde{x}^i_{t|t-1})}{\mathrm{Var}(\widetilde{x}^i_{t|t} - \widetilde{x}^i_{t|t-1})} \\
&= \frac{(1-\lambda)\lambda \mathrm{Cov}(x_t, x_t - x^i_{t|t-1}) - \lambda^2 \sigma_v^2}{\lambda^2 \left[\mathrm{Var}(x_t - x^i_{t|t-1}) + \sigma_v^2 \right]} \\
&= \frac{1-\lambda}{\lambda} \frac{\mathrm{Var}(x_t) - \mathrm{Cov}(x_t, x^i_{t|t-1})}{\mathrm{Var}(x_t - x^i_{t|t-1}) + \sigma_v^2} - \frac{\sigma_v^2}{\mathrm{Var}(x_t - x^i_{t|t-1}) + \sigma_v^2} \\
&= \frac{1-\lambda}{\lambda} \frac{\frac{\sigma_w^2}{1-\rho^2} - \frac{\rho^2 \kappa_1 \cdot \frac{\sigma_w^2}{1-\rho^2}}{1-(1-\kappa_1)\rho^2}}{\mathrm{Var}(x_t - x^i_{t|t-1}) + \sigma_v^2} - (1-\kappa_1) \\
&= \frac{1-\lambda}{\lambda} \frac{\frac{\sigma_w^2}{1-\rho^2} \frac{1-\rho^2}{1-(1-G)\rho^2}}{\mathrm{Var}(x_t - x^i_{t|t-1}) + \sigma_v^2} - (1-\kappa_1) \\
&= \frac{1-\lambda}{\lambda} \frac{\mathrm{Var}(x_t - x^i_{t|t-1})}{\mathrm{Var}(x_t - x^i_{t|t-1}) + \sigma_v^2} - (1-\kappa_1) \\
&= \frac{1-\lambda}{\lambda} \kappa_1 - (1-\kappa_1) \\
&= \frac{1-\kappa_1 + R(1-\kappa_2)}{\kappa_1 + R\kappa_2} \kappa_1 - (1-\kappa_1) \\
&= \frac{R(\kappa_1 - \kappa_2)}{\kappa_1 + R\kappa_2}
\end{aligned}
$$

A.4 Equating β_1 Across Models

DE → SI

$\beta_1^{DE} = \frac{-\theta(1+\theta)}{(1+\theta)^2 + \rho^2\theta^2}$ and $\beta_1^{SI} = \frac{R(\kappa_1 - \kappa_2)}{\kappa_1 + R\kappa_2}$.

Given $\{\rho, \sigma_w, \sigma_v, \theta\}$ we solve for R by setting $\beta_1^{DE} = \beta_1^{SI}$

$$
\begin{aligned}
\frac{R(\kappa_1 - \kappa_2)}{\kappa_1 + R\kappa_2} &= \beta_1^{DE} \\
R(\kappa_1 - \kappa_2) &= \beta_1^{DE}(\kappa_1 + R\kappa_2) \\
R(\kappa_1 - \kappa_2) &= \beta_1^{DE}\kappa_1 + \beta_1^{DE} R\kappa_2 \\
R(\kappa_1 - \kappa_2) - \beta_1^{DE} R\kappa_2 &= \beta_1^{DE}\kappa_1 \\
R &= \frac{\beta_1^{DE}\kappa_1}{\kappa_1 - (1+\beta_1^{DE})\kappa_2}
\end{aligned}
$$

SI→DE

Given $\{\rho, \sigma_w, \sigma_v, R\}$ we solve for θ by setting $\beta_1^{DE} = \beta_1^{SI}$

$$
\begin{aligned}
\frac{-\theta(1+\theta)}{(1+\theta)^2 + \rho^2\theta^2} &= \beta_1^{SI} \\
-\theta(1+\theta) &= \beta_1^{SI}(1+\theta)^2 + \beta_1^{SI}\rho^2\theta^2 \\
-\theta - \theta^2 &= \beta_1^{SI}(1 + 2\theta + \theta^2) + \beta_1^{SI}\rho^2\theta^2 \\
0 &= \beta_1^{SI} + 2\beta_1^{SI}\theta + \theta + \beta_1^{SI}\theta^2 + \theta^2 + \beta_1^{SI}\rho^2\theta^2 \\
0 &= \beta_1^{SI} + (2\beta_1^{SI} + 1)\theta + [1 + (1+\rho^2)\beta_1^{SS}]\theta^2
\end{aligned}
$$

OC→DE

Given $\{\rho, \sigma_w, \sigma_v, \theta\}$, we solve for α by setting $\beta_1^{DE} = \beta_1^{OC}$

$$
\begin{aligned}
\frac{(\alpha^2 - 1)\sigma_v^2}{\mathbb{V}(x_t - \breve{x}^i_{t|t-1}) + \sigma_v^2} &= \beta_1^{DE} \\
\alpha^2 &= \frac{\beta_1^{DE}}{\sigma_v^2}[\mathbb{V}(x_t - \breve{x}^i_{t|t-1}) + \sigma_v^2] + 1 \\
\alpha^2 - \frac{\beta_1^{DE}}{\sigma_v^2}\mathbb{V}(x_t - \breve{x}^i_{t|t-1}) &= \beta_1^{DE} + 1
\end{aligned}
$$

and solving for the α parameter for which the above equality holds. Note that due to the recursive nature of the OC model, $\mathbb{V}(x_t - \breve{x}^i_{t|t-1}) \equiv \breve{\Psi}^i_{t|t-1}(\alpha)$ is itself a function of α.

A.5 Proof of Proposition 3

It suffices to show that $\text{Cov}(\widetilde{x}^i_{t|t} - \widetilde{x}^i_{t|t-1}, \widetilde{x}^i_{t|t-1} - \widetilde{x}^i_{t|t-2}) = 0$. The current period revision is

$$
\begin{aligned}
\widetilde{x}^i_{t|t} - \widetilde{x}^i_{t|t-1} &= \frac{1 - \lambda - 1 - R\lambda}{1+R} x^i_{t|t-1} + \lambda y_{it} \\
&= \lambda(y_{it} - x^i_{t|t-1})
\end{aligned}
$$

and the previous period revision can be expressed as

$$
\begin{aligned}
\widetilde{x}^i_{t|t-1} - \widetilde{x}^i_{t|t-2} &= \frac{1}{1+R}(x^i_{t|t-1} - x^i_{t|t-2}) + \frac{R}{1+R}(F^i_{t|t-1} - F^i_{t|t-2}) \\
&= \frac{\rho}{1+R}(x^i_{t-1|t-1} - x^i_{t-1|t-2}) + \frac{R}{1+R}\big[\lambda\rho x^i_{t-1|t-1} + (1-\lambda)\rho F^i_{t-1|t-1} - \lambda\rho x^i_{t-1|t-2} \\
&- (1-\lambda)\rho F^i_{t-1|t-2}\big] \\
&= \frac{\rho}{1+R}\big[\kappa_1(y_{it-1} - x^i_{t-1|t-2})\big] + \frac{R}{1+R}\big[\lambda\rho(x^i_{t-1|t-1} - x^i_{t-1|t-2}) + (1-\lambda)\rho(F^i_{t-1|t-1} - F^i_{t-1|t-2})\big] \\
&= \frac{\rho\kappa_1}{1+R}\big[x_{t-1} - x^i_{t-1|t-2} + v_{it-1}\big] + \frac{R}{1+R}\big[\lambda\rho\kappa_1(x_{t-1} - x^i_{t-1|t-2} + v_{it-1}) \\
&+ (1-\lambda)\rho\kappa_2(x_{t-1} - x^i_{t-1|t-2} + v_{it-1})\big] \\
&= \left[\frac{\rho\kappa_1 + R\lambda\rho\kappa_1 + R(1-\lambda)\rho\kappa_2}{1+R}\right](x_{t-1} - x^i_{t-1|t-2} + v_{it-1}) \\
&= \frac{\rho(\kappa_1 + R\kappa_2) + \rho R\lambda(\kappa_1 - \kappa_2)}{1+R}(x_{t-1} - x^i_{t-1|t-2} + v_{it-1}) \\
&= \rho\lambda\left(1 + \frac{R(\kappa_1 - \kappa_2)}{1+R}\right)(x_{t-1} - x^i_{t-1|t-2} + v_{it-1})
\end{aligned}
$$

Then,

$$
\begin{aligned}
\mathrm{Cov}(\widetilde{x}^i_{t|t} - \widetilde{x}^i_{t|t-1}, \widetilde{x}^i_{t|t-1} - \widetilde{x}^i_{t|t-2}) &= \mathrm{Cov}\left[\lambda(y_{it} - x^i_{t|t-1}), \rho\lambda\left(1 + \frac{R(\kappa_1 - \kappa_2)}{1+R}\right)(x_{t-1} - x^i_{t-1|t-2} + v_{it-1})\right] \\
&= \mathrm{Cov}\Bigg[\lambda(1-\kappa_1)\rho(x_{t-1} - x^i_{t-1|t-2}) - \lambda\kappa_1\rho v_{it-1} + \lambda(v_{it} + w_t), \\
&\rho\lambda\left(1 + \frac{R(\kappa_1 - \kappa_2)}{1+R}\right)(x_{t-1} - x^i_{t-1|t-2} + v_{it-1})\Bigg] \\
&= (\lambda\rho)^2\left[1 + \frac{R(\kappa_1 - \kappa_2)}{1+R}\right]\left[(1-\kappa_1)\Psi - \kappa_1\sigma_v^2\right] \\
&= 0
\end{aligned}
$$

By definition, $\kappa_1 = \frac{\Psi}{\Psi + \sigma_v^2}$ where Ψ is the steady state forecast error variance (i.e. the variance that solves the Ricatti equation: $\Psi = (1-\kappa_1)\rho^2\Psi + \sigma_w^2$). From this it follows that the last term in hard brackets is zero.

Appendix B Empirics

Variable	Mnemonic	β_1
Consumer price index	CPI	-0.205***
		(0.067)
Employment	EMP	-0.083
		(0.197)
Housing starts	HOUSING	0.093**
		(0.041)
Industrial production	IP	-0.212***
		(0.045)
Nominal GDP	NGDP	0.113***
		(0.027)
GDP Deflator	PGDP	-0.302***
		(0.045)
Real consumption	RCONSUM	-0.303***
		(0.065)
Real federal government spending	RFEDGOV	-0.240***
		(0.060)
Real GDP	RGDP	-0.169***
		(0.038)
Real nonresidential investment	RNRESIN	-0.180***
		(0.056)
Real residential investment	RRESINV	-0.152**
		(0.074)
Real state/local government spending	RSLGOV	-0.236***
		(0.058)
3-month Treasury bill	TBILL	0.056
		(0.071)
10-year Treasury bond	TBOND	-0.004
		(0.045)
Unemployment rate	UNEMP	0.167**
		(0.068)

* p<0.10, ** p<0.05, *** p<0.01

Table B1: Pooled Revision Persistence Regressions at $h = 0$, by Variable

Note: The table reports the OLS coefficients from revision persistence regressions across 15 macroeconomic variables reported in the Survey of Professional Forecaters. Column (3) reports the coefficient in front of the revision at the forecaster-level. The revisions are for current period forecasts ($h = 0$).

Appendix C Simulation Results

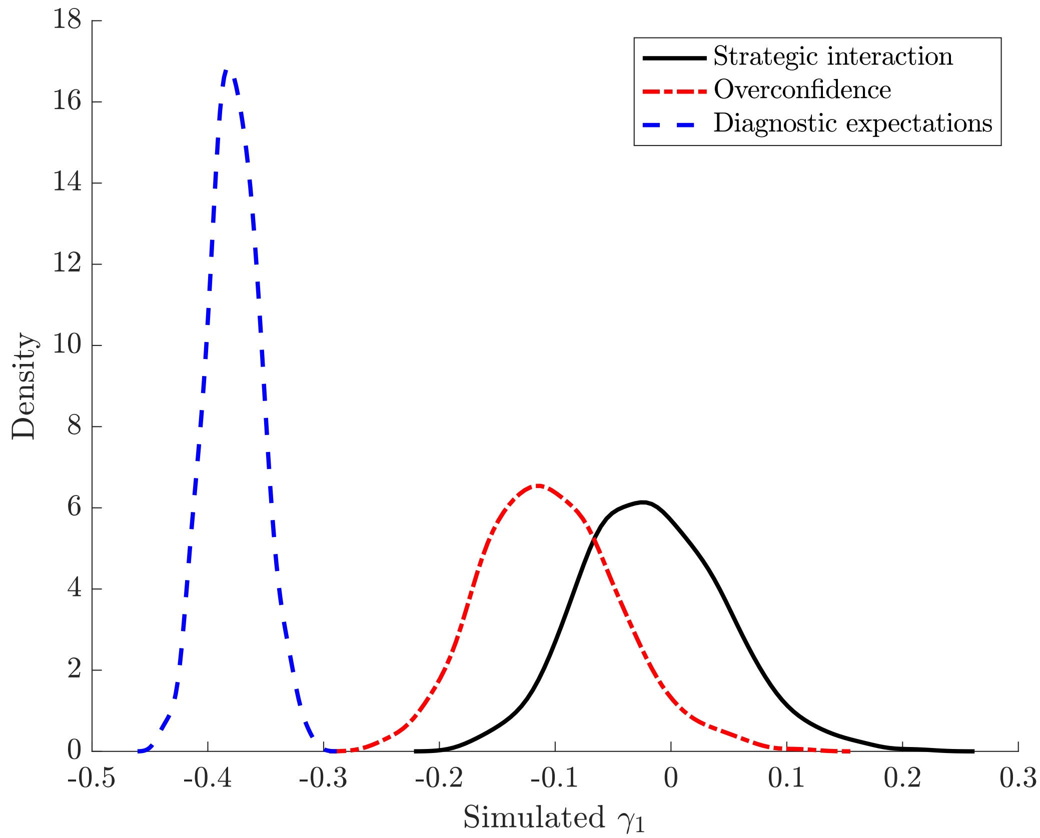

Figure C1: Mapping Over-reactions to Strategic Substitutability

Note: The figure plots the simulated densities of the various γ_1 coefficients across the three models described in the main text.