Time-Varying Volatility as a Source of Error Predictability*

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Abstract

Among professional forecasters, the covariance between forecast errors and revisions can be negative or positive depending on the macroeconomic variable in question. I show that these apparent over- and underreactions can arise simultaneously in a noisy information environment with unobserved volatility and heterogeneity in forecasting models. Under such dynamics, the use of different models among forecasters can be rationalized by costly adoption of more sophisticated methods. The scope for overreactions is found to be decreasing in the signal-to-noise ratio. I provide empirical evidence and calibration results consistent with this mechanism and conclude that modeling relevant nonlinearities can serve as a way of making sense of the observed heterogeneity in reported forecasts.

Keywords: Rational expectations. Noisy information. Forecasting. Stochastic volatility. Nonlinear filtering.
JEL Codes: C11, C13, C15 D83, D84

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1 Introduction

Professional forecasts exhibit error predictability. At the forecaster-level, the sign of the covariance between errors and revisions varies by variables. A negative covariance is interpreted as an overreaction whereas a positive covariance is interpreted as an underreaction. At the same time, macroeconomic and financial time series have been found to exhibit stochastic volatility, structural breaks, and regime switching. Based on these complex dynamics, forecasters may choose different forecasting models for distinct variables. I show that these apparent over- and underreactions can arise in an otherwise standard noisy information model that incorporates unobserved time-varying volatility and heterogeneity in use of forecasting models.

The model is compatible with rationality in the sense that reported forecasts are optimal outcomes. In a more complex environment, the exact optimal prediction is not exactly known to the forecaster. At the same time, producing forecasts is costly in that it includes (computing) time and cognitive effort. If macroeconomic dynamics vary in their complexity, then it stands to reason that forecasters may select different models for different time series subject to these costs. Therefore, with time-varying volatility and heterogeneity in use of forecasting models, revisions can hold predictive power over errors.

Survey data is predominantly used to test theories of expectation formation. In this paper, I make use of the Survey of Professional Forecasters (SPF) which provides a panel of multi-horizon forecasts across several macroeconomic variables.[1] In the data, over- and underreactions arise along different dimensions. First, as previously documented in the literature, consensus forecasts broadly exhibit underreactions while forecaster-level predictions tend to imply overreactions. However even when pooling across respondents at the forecaster-level, overreactions arise for some SPF variables while underreactions prevail for others. I contribute to these previously documented facts by documenting the share of respondents that appear to over- and underreact to distinct macroeconomic variables.

[1] Examples other than the SPF include the Livingston survey, the Michigan Survey of Consumers, the NY Fed Survey of Consumer Expectations, Blue Chip forecasts, the ECB Survey of Professional Forecasters, and the daily Focus Survey from the Central Bank of Brazil, among others.

Simultaneous over- and underreaction prompts several fundamental questions about expectations formation. Are professional forecasters (presumably the most informed private agents in the economy) rational? Does a behavioral bias govern the manner in which expectations are formed? As policymakers increasingly pursue expectations-based policies such as forward guidance, taking a step toward reconciling theories of expectations formation with the data is of first-order importance.

Against this backdrop, I present a nonlinear noisy information model. Rather than obtaining an exact solution to the optimal inference problem, forecasters must approximate the posterior distribution. They may choose from a finite set of approximation methods. The available methods vary in complexity, and adopting a given method is subject to a cost that is increasing in sophistication. Forecasters generate a prediction that minimizes the sum of their mean squared errors and model adoption costs.

Importantly, forecasters adopt simpler, suboptimal models for macroeconomic variables that are easier to predict. The use of these models generates error predictability. On the other hand, forecasters have an incentive to adopt more sophisticated predictor functions for complex dynamics. This implies that the underlying signal-to-noise ratio governs the extent to which over- and underreactions arise. Put another way, there are certain features inherent to a given macroeconomic time series that explain why forecasters appear to either over- or underreact to that particular variable.

After providing simulation results that confirm the above intuition, I examine these implications in the data by exploiting the cross-section of macroeconomic variables for which forecasters report predictions in the SPF. I then consider a stylized version of the model to show that it can match quantitatively relevant shares of simultaneous over- and underreactions. For simplicity, I assume that forecasters can either approximate the state via the Kalman filter or the particle filter. The latter approximation method is more costly than the former.[2] Taken together, my findings show time-varying volatility, coupled with costly model adoption, can rationalize important features of survey data.

[2]The particel filter is an asymptotically efficient approximation method. I provide further details on the filter in Appendix D.

In their seminal paper, Coibion and Gorodnichenko (2015), henceforth CG, make sense of forecast error inefficiency while preserving the assumption of rationality. Using consensus-level data, CG show that projecting *ex-post* forecast errors on *ex-ante* forecast revisions delivers an estimate of information rigidity. More recently, many studies have used forecaster-level data to test for rationality.[3] In doing so, much of this literature preserves the linearity assumption made in CG, and ultimately rejects rational expectations even under imperfect information. To make sense error predictability at the forecaster-level while also matching the CG finding of underreactions at the aggregate level, several theories of non-rational expectations have been proposed.[4] My paper relates to this strand of the literature in many ways, and provides an alternate interpretation of the errors-on-revisions coefficient.

Moreover, the nonlinear noisy information model in this paper is in the spirit of Branch (2004), Evans and Ramey (1992) and Brock and Hommes (1997) who define adaptively rational equilibrium dynamics (ARED). Branch (2004) was the first to introduce this concept to expectations formation. This paper adds to those insights in key ways. First, I present more complex dynamics for the state variable. Introducing nonlinearities, such as stochastic volatility, provides a stronger justification for the use of different predictor (approximation) functions. Second, I allow for heterogenous expectations whereas Branch (2004) assumes homogeneous predictions among all who adopt a specific predictor function.[5] Taken together, my model is able to reproduce the empirical facts that are present in survey data both at the individual and aggregate levels.

While a discussion of nonlinearities has generally been absent in the survey expectations literature, the finance literature has previously tied nonlinear dynamics to error predictability. For instance, Lewis (1989) considers error predictability concerning dollar forecasts in the context of a structural break. Veronesi (1999) finds that over- and underreactions arise in a regime switching model of asset pricing. More recently, Lansing et al. (2019) attribute the predictability of excess

[3]Examples include Bordalo et al. (2020), Fuhrer (2018), Dovern et al. (2015), Andrade and Le Bihan (2013), Broer and Kohlhas (2018), Bürgi (2016).

[4]For instance Bordalo et al. (2020) rule out rationality in favor of diagnostic expectations. Other studies such as Fuster et al. (2010) argue in favor of models featuring misperception at long horizons. Daniel et al. (2003) argues for a model of overconfidence while Broer and Kohlhas (2018) present a model of relative overconfidence.

[5]Heterogeneity in that model comes from idiosyncratic “trembles" in the reported prediction.

returns to either volatility or deviations from rationality. To this end, my paper also relates to the literature on volatility in macroeconomics.[6]

Finally, due to the assumptions imposed on the state dynamics, this paper relates to the literature on nonlinear filtering. Several approximation methods have been devised in order to deal with nonlinearities in the evolution of a state variable. These methods include generalizations to Kalman filtering as well as importance sampling algorithms, among others.[7] A strand of this literature has recently formalized some basic efficiency properties of particle filtering.[8]

Below, Section 2 presents previously documented facts about error predictability at the forecaster and consensus levels, as well as a more novel fact on simultaneous over- and underreactions. Section 3, presents the nonlinear noisy information model. Section 4 introduces a stylized version of the model and provides simulation results. Section 5 documents empirical evidence consistent with a noisy nonlinear expectations model. Section 6 parameterizes the stylized model to show that it can generate within forecaster over- and underreactions. Finally, Section 7 concludes.

2 Evidence from Survey Data

The SPF is a quarterly survey provided by the Federal Reserve Bank of Philadelphia. The survey began in 1968Q4 and provides forecasts from several forecasters across a number of macroeconomic variables over many horizons, h. The variables of interest in this paper are the forecast error and the forecast revision. To construct forecast errors from forecaster i about variable x, $FE^i_{t+h,t} = x_{t+h} - x^i_{t+h|t}$, I take the difference between the realized real-time value for x at $t+h$ and the forecaster's h-step ahead prediction generated at time t. To compute forecast revisions, I exploit the term structure of forecasts generated by the survey respondents. Specifically, $FR^i_{t,t-1} = x^i_{t+h|t} - x^i_{t+h|t-1}$. This requires making use of the h-step ahead forecasts formulated in periods t and $t-1$. In other words,

[6]See for instance, Justiniano and Primiceri (2007), Kim and Nelson (1998), McConnell and Perez-Quiroz (2000), and Stock and Watson (2002).

[7]Julier and Uhlman (2004) develop a Kalman filter for nonlinear settings while Doucet and Johansen (2009) discuss particle filtering methods.

[8]See Crisan and Doucet (2002) and Hu et al. (2011)

Table 1: Pooled OLS Forecast Error Predictability Regressions

	Nowcast		One-Quarter Ahead		Two-Quarters Ahead	
	β_1	α_1	β_1	α_1	β_1	α_1
Estimate	-0.317***	0.569***	-0.231**	1.011***	-0.344***	0.565**
	(0.050)	(0.128)	(0.067)	(0.201)	(0.058)	(0.272)
Observations	65,070	2,323	54,067	2,309	52,220	2,295

Note: The table reports the estimated coefficients of forecast error predictability at the current, one-, and two-quarter ahead horizons. Across all horizons, column (1), refers to the forecaster-level errors-on-revisions regression. Column (2) refers to the consensus-level errors-on-revisions regression. Standard errors for forecaster-level regressions are as in Driscoll and Kraay (1998), while Newey-West standard errors are used for aggregate-level specifications. Data used for estimation come from SPF. *** denotes 1% significance, ** denotes 5% significance, and * denotes 10% significance.

I consider the fixed horizon, h and take the difference between two adjacent forecasts.

In the data, simultaneous over- and underreactions arise along different dimensions: across level of aggregation, across SPF variables for pooled forecasters, and across SPF variables within forecaster. CG present the following testable implication at the consensus-level which holds for an arbitrary horizon

$$FE_{t+h,t} = \alpha_0 + \alpha_1 FR_{t+h,t} + \epsilon_t \tag{1}$$

CG find that in the data, $\alpha_1 > 0$ for most variables which indicates that consensus forecasts underreact to new information. More recently, Bordalo et al. (2020) estimate the same regression at the forecaster-level

$$FE^i_{t+h,t} = \beta_0 + \beta_1 FR^i_{t+h,t} + \varepsilon^i_t \tag{2}$$

and find that $\beta_1 < 0$ for most macroeconomic series. The interpretation is that forecasters overreact to new information.

Table 1 reports estimates of β_1 and α_1, using data from the SPF. The estimates are obtained via OLS regressions, pooling across both forecasters and macroeconomic variables. Estimates are reported for three different horizons. Across all horizons considered, it is clear that overreactions dominate at the individual-level, while underreactions arise at the aggregate-level.

However even at the forecaster-level there evidence of simultaneous over- and underreaction

Table 2: Pooled OLS Regressions at $h = 0$, by Variable

Variable	Mnemonic	β_1	α_1
Consumer price index	CPI	-0.085	0.868***
Employment	EMP	-0.123	0.564***
Housing starts	HOUSING	0.063	0.359***
Industrial production	IP	-0.147*	0.513***
Nominal GDP	NGDP	-0.310***	0.421**
GDP Deflator	PGDP	-0.363***	0.350**
Real consumption	RCONSUM	-0.401***	0.098
Real federal government spending	RFEDGOV	-0.483***	0.377
Real GDP	RGDP	-0.264***	0.350**
Real nonresidential investment	RNRESIN	-0.499**	0.362
Real residential investment	RRESINV	-0.234***	0.925***
Real state/local government spending	RSLGOV	-0.660***	-0.381
3-month Treasury bill	TBILL	0.010	0.178***
10-year Treasury bond	TBOND	0.020	0.154***
Unemployment rate	UNEMP	0.082**	0.247***

Note: The table reports the OLS coefficients from errors-on-revisions regressions across 15 macroeconomic variables reported in the Survey of Professional Forecaters. Column (3) reports the coefficient in front of the revision at the forecaster-level while column (4) reports the analogous coefficient using consensus-level data. The errors and revisions are for current period forecasts ($h = 0$). *** denotes 1% significance, ** denotes 5% significance, and * denotes 10% significance.

across macroeconomic variables. Table 2 reports variable-by-variable results for nowcasts ($h = 0$) at the forecaster- and consensus-levels. The results point to individual overreactions for most variables but underreactions for some variables such as the unemployment rate.

These findings may be due to entry and exit in the SPF, or different types of forecasters systematically selecting to report predictions for certain macroeconomic variables. The data, however, suggests that the same respondent simultaneously over- and underreacts to distinct macroeconomic variables. To show this, I estimate regression (2) for each forecaster i forecasting a specific variable j. This delivers an $N_i \times N_j$ matrix of estimates $\widehat{\beta}_{1,ij}$. I keep only those estimates that are significant at 5% level. I then fix a pair of SPF variables j and k, and compute the number of forecasters such that $\widehat{\beta}_{1,ij} < 0$ and $\widehat{\beta}_{1,ik} > 0$, and normalize by the number of total forecasters reporting predictions about variables j and k. More formally, I estimate a matrix P whose elements are p_{jk} with

$$p_{jk} = \frac{\sum_i \mathbb{1}\left(\widehat{\beta}_{1,ij} < 0 \text{ and } \widehat{\beta}_{1,ik} > 0\right)}{\min\{N_j, N_k\}}$$

where N_x denotes the number of forecasters providing predictions of variable x and $\mathbb{1}(\cdot)$ is the indicator function. The elements of matrix P, therefore, denote the share of forecasters who simultaneously overreact to the row variable and underreact to the column variable. When p_{jk} is close to one, this means that nearly all forecasters overreact to variable j and underreact to variable k. On the other hand, when p_{jk} is close to zero, then almost no forecaster overreacts to variable j while also underreacting to variable k.

Figure 1 reports the results from this exercise. The figure verifies that a given forecaster tends to overreact to some variables and underreact to others. For instance, 84% of professional forecasters exhibit overreactions when forecasting real gross domestic product (RGDP) while simultaneously underreacting to their predictions about consumer prices (CPI).

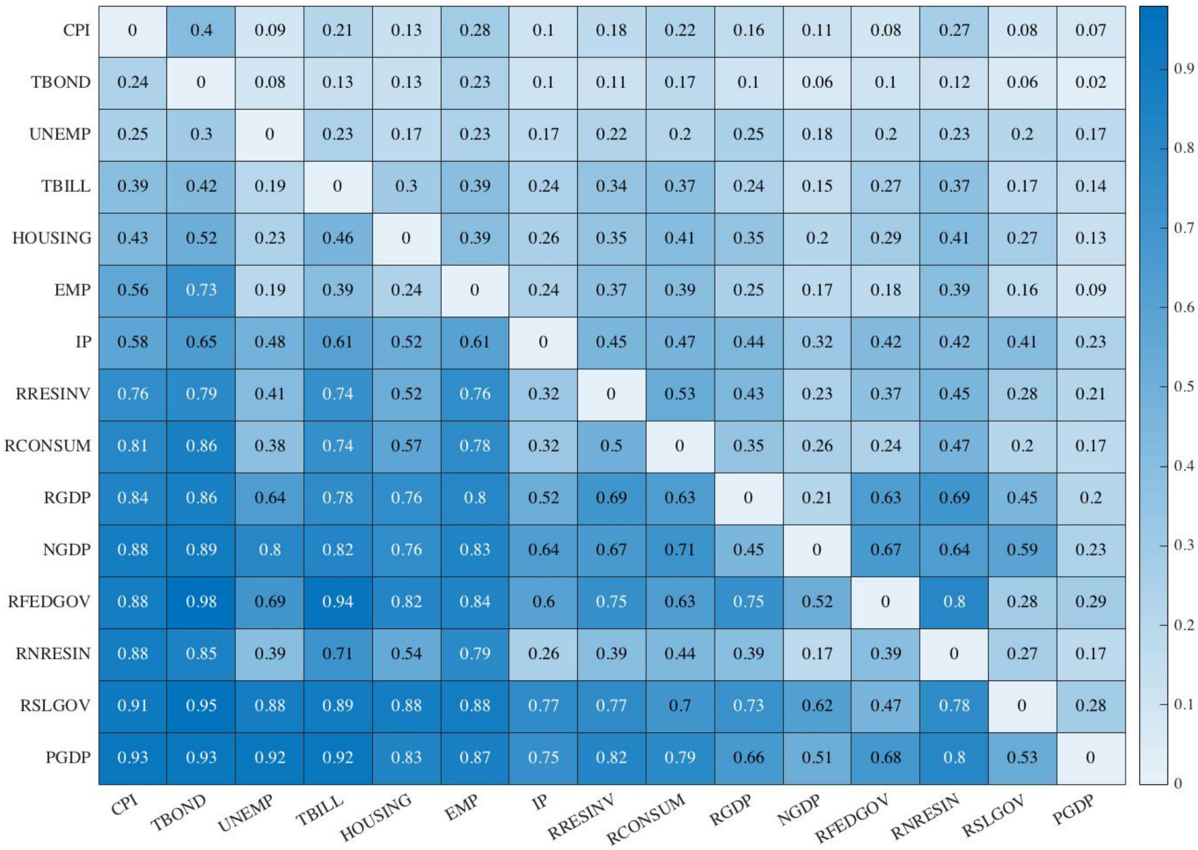
In addition to existing evidence of over- and underreactions that depend on level of aggregation as well as pooled forecasts by SPF variable, I document similar patterns *within* forecasters. In order to understand how individuals formulate these expectations, therefore, a theory of expectations formation must take into account that a single agent may overreact and underreact to different variables. I propose a model of time-varying volatility and heterogeneity in forecasting models in order to account for this fact.

3 Model

I begin with a simple regime switching example. Suppose that the state is described as follows

$$\begin{aligned} s_t &= \rho s_{t-1} + \sigma_t w_t, \qquad\qquad w_t \sim \mathcal{N}(0, \sigma_w^2) \\ \sigma_t &= \chi \sigma_L + (1-\chi)\sigma_H \end{aligned}$$

Figure 1: Frequency of Over- and Underreaction



Note: The heatmap displays the share of forecasters who overreact to the row variable and simultaneously underreact to the column variable.

where $\chi = 1$ with probability q and $\chi = 0$ with probability $1 - q$. Forecasters receive private signals $y_t^i = s_t + v_t^i$ each period, where $v_t^i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_v^2)$. Forecasters do not know the value of s_t or q, and can only observe the sequence of private signals $\{y_k^i\}_{k=0}^t$.[9]

To see how overreactions arise, suppose that forecasters assess the volatility of the state to be $\sigma_t = \frac{1}{2}(\sigma_L + \sigma_H)$, and forecasts according to the Kalman filter subject to this naive approximation of the underlying state volatility. When q is closer to one, the signal-to-noise ratio is low and forecasters believe the state to be more variable than it truly is. As a result, forecasters place a larger weight

[9]The assumption that forecasters are unable to observe q is for simplicity. Alternatively, one could suppose that the probability is stochastic ($q + \epsilon$) and that forecasters cannot observe the innovation ϵ.

on new information. This undue importance to news will generate overreactions. According to this simple example, the magnitude of the overreaction depends importantly on the signal-to-noise ratio. Noisier environments deliver more negative β_1 coefficients.

On the other hand, an underreaction could arise under slightly different dynamics. One can verify this by simply allowing q to be closer to zero so that the signal-to-noise ratio rises. Now, when forecasters generate approximations, they believe the state to be less variable than it truly is thereby placing less weight on news. This mutes the effects of signal noise and creates more inertia in expectation formation than is optimal. The result is, $\beta_1 > 0$.

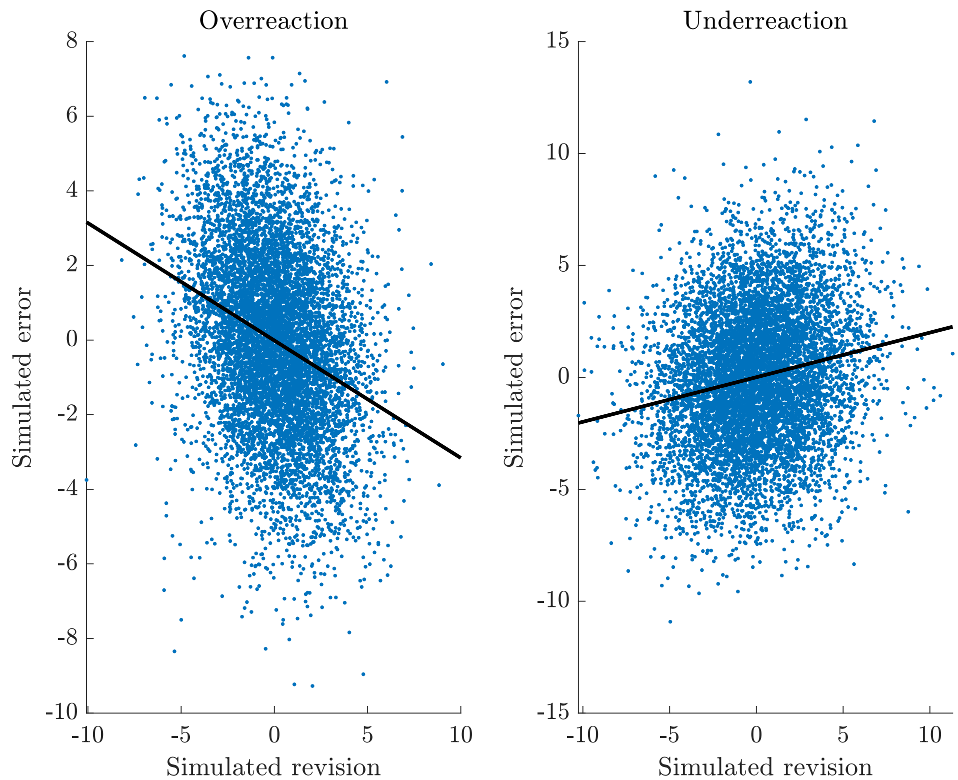
Though this is another very simple case, it illustrates how a different approximation strategy can deliver underreactions. In particular, this updating procedure delivers smoother revisions. Moreover, the extent of observed underreaction is increasing in the variance of the state. If forecasters could simply observe q, however, then there would be no need to approximate the variance of the state and error orthogonality would hold. Of course, a naive approximation will deliver a bias in predictions. However, when approximation functions come at a cost (be it cognitive, timing, or otherwise), forecasters may find it optimal to make use of such approximations. Figure 2 below plots simulated errors against simulated revisions for each of these examples. The left panel shows that overreactions prevail when the revision is noisier whereas underreactions are observed when revisions are smoother.

3.1 A Model of Unobserved Time-Varying Volatility

Nonlinearities such as time-varying volatility in the underlying state complicates the forecaster's problem as he must now formulate expectations about levels and the volatilities. Supposing that there are n latent state variables and m exogenous signals, one can generally formulate such a model as follows

$$\begin{aligned} \mathbf{\bar{s}_t} &= F(\mathbf{\bar{s}_{t-1}}, \mathbf{\overline{w}_t}) \qquad (3) \\ \mathbf{z_t^i} &= \mathbf{C}\mathbf{\bar{s}_t} + \mathbf{D}\mathbf{v_t^i} \end{aligned}$$

Figure 2: Volatility-Generated Overreactions and Underreactions



Note: The figures display simulated scatters plots from the examples described in the text. The panel titled "Over-reaction" assumes that $q = 0.80$ while the panel titled "Under-reaction" supposes that $q = 0.20$. The simulation is done by specifying one single forecaster for 10,000 periods, and fixing the approximation $\widehat{\sigma}_t = \frac{1}{2}(\sigma_L + \sigma_H)$

where $\mathbf{\bar{s}_t}$ is an $n \times 1$ vector, $\mathbf{z_t}$ is $m \times 1$, $\mathbf{C}$ is $m \times n$, $\mathbf{D}$ is $m \times m$ and $\mathbf{v_t^i}$ is $m \times 1$. There are no other restrictions placed on the model. In particular, $\mathbf{s_t}$ can be a vector of many different state variables, or lags of itself. Furthermore, $\mathbf{z_t^i}$ can include an arbitrary finite number of observed signals. The noise vector $\mathbf{v_t^i}$ can include private or public noise.[10]

In a linear model, $\mathbf{s_t} = \mathbf{A s_{t-1}} + \mathbf{B w_t}$. The crucial difference between a linear model and this one is the unobserved time-varying covariance matrix $\mathbf{B}_t$ which implies an expanded state space: $\mathbf{\bar{s}_t} = \begin{pmatrix} \mathbf{s_t} & diag(\mathbf{B_t}) \end{pmatrix}'$. As a result, the error now enters *multiplicatively* into the state. This nonlinearity is modeled by the function $F(\cdot)$ which governs the evolution of the state. While the state now exhibits stochastic volatility, the shocks remain normal, and the signal structure is

[10]I index this vector by i in general to allow for forecaster-specific signals.

unchanged. Hence, the measurement equation remains linear.[11] [12]

Whereas Kalman filtering delivers an exact optimal solution in a linear Gaussian environment, it is no longer optimal in this context. The reason for this is that the Kalman filter requires one to evaluate the expected value of $\mathbf{\bar{s}_t}$ conditional on the history of signals $\mathcal{Z}_t^i = \{\mathbf{z_1^i}, \ldots, \mathbf{z_t^i}\}$. This is made intractable due to the lack of knowledge about the underlying conditional distribution. To see this more clearly, consider the scalar case where the state is s_t and there is only a private signal available to the forecaster, z_t^i. The observation equation can be expressed as a conditional likelihood, $p(z_t^i|s_t)$ and the state evolution as $p(s_{t+1}|s_t)$. The optimal filter computes $p(s_t|\mathcal{Z}_t^i)$ from a predict-update procedure

$$p(s_t|\mathcal{Z}_{t-1}^i) = \int p(s_t|s_{t-1})p(s_{t-1}|\mathcal{Z}_{t-1}^i)ds_{t-1} \quad \text{(Predict)}$$

$$p(s_t|\mathcal{Z}_t^i) = \frac{p(z_t^i|s_t)p(s_t|\mathcal{Z}_{t-1}^i)}{p(z_t^i|\mathcal{Z}_{t-1}^i)} \quad \text{(Update)}$$

where $p(z_t|Z_{t-1}^i) = \int p(z_t^i|s_t)p(s_t|Z_{t-1}^i)ds_t$.

In a linear Gaussian environment, this can be exactly computed via the Kalman recursions.[13] In a nonlinear setting, however, computing $p(s_t|\mathcal{Z}_t^i)$ is not feasible as the density cannot be obtained analytically.

In light of this, forecasters must approximate the nonlinear state. I assume that this is done by selecting from a set of costly approximation functions, $A \in \mathcal{A}$. Forecasters first select an approximation function so as to obtain an estimate of the posterior density of the underlying state. Forecasters then report their predictions (i.e. first moment of this approximated density). Hence, the forecaster's loss function can be defined as

$$\mathcal{L} = \min_{A \in \mathcal{A}} \left[\big(\mathbf{z_{t+h}^i} - \mathbf{\widehat{z}_{t+h|t}^{i,A}}\big)'\big(\mathbf{z_{t+h}^i} - \mathbf{\widehat{z}_{t+h|t}^{i,A}}\big) + c_A^i \right] \tag{4}$$

[11] This could be generalized to a nonlinear measurement as well. I abstract away from this for simplicity.

[12] While I consider stochastic volatility, any nonlinearity can deliver the results presented in the paper. In particular, this model can also speak to unobserved changes in the persistence of macroeconomic time series.

[13] $p(s_t|\mathcal{Z}_t^i) \sim \mathcal{N}(s_{t|t}^i, \Psi_{t|t}^i)$, where $s_{t|t}^i$ is the expected value of the posterior density and $\Psi_{t|t}^i$ is the variance.

where the first term is the mean square error arising from individual i's forecast which makes use of approximation function A, and the second term denotes the cost associated with adopting approximation function A.[14] I assume that these forecaster-specific costs are drawn randomly $c_A^i \sim U(0, \bar{c}_A)$.[15] This cost embodies unobserved heterogeneity among forecasters that result in the use of different forecasting models.

3.2 Approximate Predictions

After applying either of the approximation methods discussed, forecasters generate a prediction of the state and an update according to the new information received. Since agents are formulating a forecast subject to an approximation of the state, I call these approximate predictions. An approximate prediction is defined as

$$\widehat{\bar{\mathbf{s}}}^{\mathbf{i}}_{\mathbf{t|t}} = \int \bar{\mathbf{s}}_{\mathbf{t}} \, \widehat{p}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}_t^i) \, d\bar{\mathbf{s}}_{\mathbf{t}} \tag{5}$$

In essence, the forecaster predicts the current state according to the approximated density $\widehat{p}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}_t^i)$. In a linear Gaussian setting, the density is obtained exactly so that $\widehat{p}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}_t^i) = p(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}_t^i)$ and errors are orthogonal.

One can express the approximate prediction as a deviation from the optimal minimum mean square error forecast

$$\widehat{\bar{\mathbf{s}}}^{i}_{t|t} = \underbrace{\mathbb{E}(\bar{\mathbf{s}}_t|\mathcal{Z}_t^i)}_{\text{Optimal}} + \underbrace{\int \bar{\mathbf{s}}_t[\widehat{\mathbf{p}}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}_{\mathbf{t}}^{\mathbf{i}}) - \mathbf{p}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}_{\mathbf{t}}^{\mathbf{i}})]d\bar{\mathbf{s}}_{\mathbf{t}}}_{\text{Approximation error}} \tag{6}$$

Whereas existing theories of expectation formation restrict the deviation from the optimal forecast to be either positive (overreactions) or negative (underreactions), the direction of the approximation error is unrestricted.

[14] Forecasters have knowledge of the mean square error associated with each A.

[15] One could alternatively assume heterogeneous signal precision. Models featuring heterogeneous signal-to-noise ratios have been proposed in the literature, particularly to explain forecast disagreement.

From the nonlinear model subject to approximation errors, the covariance of errors and revision can be signed as follows:

$$\beta_1 \propto \mathbb{C}\bigg(\bar{\mathbf{s}}_{\mathbf{t}}, \int \bar{\mathbf{s}}_{\mathbf{t}}[\widehat{\mathbf{p}}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}^{\mathbf{i}}_{\mathbf{t}}) - \widehat{\mathbf{p}}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}^{\mathbf{i}}_{\mathbf{t-1}})]\mathbf{d}\bar{\mathbf{s}}_{\mathbf{t}}\bigg)$$
$$- \mathbb{C}\bigg(\int \bar{\mathbf{s}}_{\mathbf{t}}\widehat{\mathbf{p}}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}^{\mathbf{i}}_{\mathbf{t}})d\bar{\mathbf{s}}_{\mathbf{t}}, \int \bar{\mathbf{s}}_{\mathbf{t}}[\widehat{\mathbf{p}}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}^{\mathbf{i}}_{\mathbf{t}}) - \widehat{\mathbf{p}}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}^{\mathbf{i}}_{\mathbf{t-1}})]\mathbf{d}\bar{\mathbf{s}}_{\mathbf{t}}\bigg)$$

When there are no approximation errors, error orthogonality holds and $\beta_1 = 0$. In the case of non-zero approximation errors, however, the first term is the source of observed underreaction while the second term governs the extent of overreaction. When forecast revisions are more closely related to the underlying state, then underreactions arise as the first term dominates the second. If instead, forecast revisions covary more with the current prediction than the underlying state, then overreactions result. In essence, when the approximate revision incorporates more noise than is optimally called for, then forecasters will overreact.

3.3 Aggregate Error Predictability

Similar to the approximate prediction defined above, the consensus forecast arising from approximate predictions is defined as follows

$$\alpha_1 \propto \mathbb{C}\bigg(\bar{\mathbf{s}}_{\mathbf{t}}, \int\int \bar{\mathbf{s}}_{\mathbf{t}}\Big[\widehat{\mathbf{p}}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}^{\mathbf{i}}_{\mathbf{t}}) - \widehat{\mathbf{p}}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}^{\mathbf{i}}_{\mathbf{t-1}})\Big]d\bar{\mathbf{s}}_{\mathbf{t}}di\bigg)$$
$$- \mathbb{C}\bigg(\int\int \bar{\mathbf{s}}_{\mathbf{t}}\widehat{\mathbf{p}}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}^{\mathbf{i}}_{\mathbf{t}})d\bar{\mathbf{s}}_{\mathbf{t}}di, \int\int \bar{\mathbf{s}}_{\mathbf{t}}\Big[\widehat{\mathbf{p}}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}^{\mathbf{i}}_{\mathbf{t}}) - \widehat{\mathbf{p}}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}^{\mathbf{i}}_{\mathbf{t-1}})\Big]d\bar{\mathbf{s}}_{\mathbf{t}}di\bigg)$$

In the previous subsection, it was determined that more volatile revisions increase the scope for overreaction. Upon aggregating (symmetrically) across several individual forecasts, the consensus revision will exhibit more persistence than the individual revisions. This motivates the following result

Proposition 1. *In the nonlinear noisy information model,* $\alpha_1 \geq \beta_1$.

Proof. See Appendix B. □

The model implies that the OLS coefficient estimated from an errors on revisions regression will be weakly greater than the analogous coefficient obtained from a pooled regression of individual forecasters. This result does depend on the presence of nonlinear dynamics. In fact, this holds in the linear setting as well (see Appendix B).[16]

3.4 Relation to Some Theories of Expectation Formation

Several other theories of expectations formation have been proposed in the literature. Here, I consider a few prominent theories and assess their ability to generate the empirical facts presented in Section 2.

According to diagnostic expectations, forecasters over-weight new information according to a parameter $\theta > 0$ which ultimately governs the extent of overreaction. This parameter comes from the representativeness heuristic of Kahneman and Tversky (1974). The diagnostic nowcast is defined as

$$x_{t|t}^{i,\theta} = x_{t|t}^{i} + \theta(x_{t|t}^{i} - x_{t|t-1}^{i})$$

This theory makes use of a distorted Kalman filter called the diagnostic Kalman filter. Nonetheless, as shown in Bordalo et al. (2020), diagnostic expectations are able to generate $\beta_1 < 0$ and $\alpha_1 > 0$. Because $\theta > 0$, this theory cannot accommodate underreactions. In other words, the sign restriction imposed on θ implies that diagnostic errors and revisions always covary negatively.

Models of overconfidence also distort the Kalman gain. The distortion stemming from overconfidence, however, is different. According to this theory, forecasters misperceive the precision of their own signals. Suppose that forecasters observe only one private signal

$$z_t^i = s_t + v_t^i, \qquad v_t^i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_v^2)$$

[16] This need not be the case in other economic settings in which agents take actions, and their decisions are aggregated in a manner other than by taking a simple mean. Depending on the context, it is possible for the aggregate decision to also exhibit overreaction or excess volatility. Bordalo et al. (2020) provide a discussion of this.

but they perceive $\widetilde{\sigma}_v < \sigma_v$ (they believe their own private signals to be more precise than they truly are). This results in a distorted assessment of the noise in the system. Overconfident beliefs are recursive so that the distorted gain injects a bias to the update in each period. These beliefs are then projected forward only to be further distorted by the over-weighting of new information in the subsequent period. In other words, at an arbitrary point in time, forecasters exhibit both a non-zero ex-ante forecast error as well as a weighting error. Models of overconfidence can generate individual overreaction as well as aggregate underreaction, however, overconfidence is similarly unable to generate individual underreaction.

Strategic interaction models can also generate error predictability. For instance, strategic substitution can drive errors and revisions in opposite directions. This is because forecasters have a dual objective of not only minimizing their errors but also of distinguishing themselves from the average forecast. These models differ from the previous two in that strategic interaction models are rational. While this class of models can generate $\beta_1 > 0$ or $\beta_1 < 0$ depending on the strategic motive assumed, it is unable to jointly deliver $\beta_1 > 0$ and $\beta_1 < 0$.

Models of noisy memory, first introduced in Azeredo da Silveira and Woodford (2019) can also generate overreactions. In a noisy memory model, forecasters do not have access to their full history of signals due to finite memory capacity. The authors find that while rational inattention can explain individual underreactions, noisy memory may explain individual overreactions. Developing a hybrid rational inattention-noisy memory model could be another successful way of modeling simultaneous over- and underreactions.

Several other theories of expectations formation have been found to be inconsistent with the data. Models of reputational concerns imply smoothing which can only generate $\beta_1 > 0$. Moreover, asymmetric loss functions deliver counterfactually biased expectations, whereas the data show that professional forecasts are not unconditionally biased.

4 Stylized Model

To extract further insight as to how time-varying volatility can generate over- and underreactions, I consider next a stylized model of stochastic volatility. I provide simulation-based results that describe the source of over- and underreactions. In the subsequent section I provide cross-sectional evidence consistent with this mechanism.

For simplicity, I suppose that forecasters can choose between two models: a Kalman filter (KF) and a particle filter (PF).Forecasters utilizing KF ignore the stochastic volatility and assess only the unconditional volatility of the state when formulating predictions. The underlying state is described as follows

$$
\begin{aligned}
s_t &= \rho s_{t-1} + e^{h_t/2} w_t, \quad w_t \sim \mathcal{N}(0,1) \\
h_t &= \phi_0 + \phi_1 h_{t-1} + \sigma_\eta \eta_t, \quad \eta_t \sim \mathcal{N}(0,1)
\end{aligned}
\tag{7}
$$

Furthermore, forecasters observe a contemporaneous private signal as well as a lagged public signal[17]

$$
\begin{aligned}
y_t^i &= s_t + v_t^i, \qquad v_t^i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma_v^2) \\
x_{t-1} &= s_{t-1} + e_{t-1}, \quad e_{t-1} \sim \mathcal{N}(0, \sigma_e^2)
\end{aligned}
\tag{8}
$$

The specific assumptions on the volatility of s_t implies that this model is nonlinear. In addition to being unable to observe s_t, forecasters are also unable to observe h_t. As such, the innovation w_t now enters into the state multiplicatively. Moreover, let $\mathbf{z}_t^i = \begin{pmatrix} y_t^i & x_{t-1} \end{pmatrix}'$ denote the vector of signals observed by forecasters.

Each period consists of two stages. In the first stage, a forecaster observes $\mathbf{z_t^i}$ and selects an approximation function. The use of these models comes at a random cost, $c^i \sim U(0, \overline{c})$ such that

[17] One can alternatively envision that forecasters observe a macroeconomic variable with a transitory (e_t) and persistent (s_t) component. The persistent component is what is relevant for forecasting the target variable as it governs the long-run evolution of the random variable. Furthermore, this component is unobserved.

the PF cost distribution has a higher upper bound relative to KF. Then in the second stage, given the predictor function and the history of signals $\mathcal{Z}_t^i$, the forecaster reports a prediction of the public signal $\widehat{x}_{t|t}^i$.[18] For simplicity, I normalize $\overline{c}_{KF} = 0$. With this in mind, forecasters minimize the following loss function:

$$\mathcal{L} = \min\left[MSE_{KF}^i, MSE_{PF}^i + c_{PF}^i\right], \quad c_{PF}^i \sim U(0, \overline{c}_{PF})$$

a forecaster will choose to make use of the more sophisticated PF if and only if

$$MSE_{KF}^i - MSE_{PF}^i \geq c_{PF}^i \tag{9}$$

The lefthand side of the inequality reflects the benefit to adopting the PF when formulating a prediction whereas the righthand side denotes the relative cost to adopting the PF.

This model belongs to the class of models described in the previous section. As a result, the extent to which the forecast revision covaries with the realized state drives the scope for underreaction in the model. This implies that β_1 depends on the underlying amount of noise in the model. Fundamentally this is a noisy information environment in which forecasters infer the state subject to private and public signals. It stands to reason that as the signal-to-noise ratio (SNR) falls, forecast revisions are driven by the noise in the system. In this case, it is *as if* forecasters report their predictions with measurement error since an upward revision in the reported forecast will mechanically result in a more negative forecast error. On the other hand, when the SNR is high, then fluctuations in the underlying state drive the forecast revisions. As a result, an upward revision delivers a more positive forecast error.

The simulation results reported in Table 3 confirm that the nonlinear model is able to jointly explain the patterns observed in the data. Consider the difference in parameter values between models of individual underreactions and models of individual overreactions. As I reduce the SNR from

[18] At t, forecasters make use of their full history of signals in order to formulate a state estimate, $\widehat{\mathbf{s}}_{\mathbf{t}|\mathbf{t}}^{\mathbf{i}}$. The first element of the forecasted state vector is $\widehat{s}_{t|t}^i$. Based on the assumption that $x_t = s_t + e_t$, it follows that $\widehat{x}_{t|t}^i = \widehat{s}_{t|t}^i$.

Table 3: Signal-to-Noise Ratio and Implied OLS Coefficients

Individual	Aggregate	SNR	β_1	α_1
Underreaction	Underreaction	1.427	0.12	0.46
Overreaction	Underreaction	0.433	-0.14	0.18

Note: The table simulates the errors-on-revision coefficient at the forecaster-level (β_1) and the consensus-level (α_1) for two different simulated signal-to-noise ratios. I simulate a panel of 40 forecasters over 200 periods (of which the first 100 periods are discarded). Each simulated forecaster generates a prediction according to the model. I then run a pooled regression for 2,000 simulated panels and report the mean point estimate.

1.43 to 0.43, the sign of β_1 changes.[19] These simulations suggest that observed under- and overreactions at the individual level can be explained by different underlying data generating processes among otherwise rational forecasters.

I next plot the simulated β_1 distribution across high signal-to-noise ratio and low signal-to-noise ratio parameterizations. For each of 2,000 simulations, I generate a panel of 40 forecasters over 200 periods. I then collect the nowcast errors and revisions for these forecasters and compute β_1. Figure 3 plots the density of β_1 across the simulations. The results confirm that the model can generate error predictability, and that over- and underreactions depend on the SNR.[20] I next turn to the SPF data to document facts consistent with this mechanism.

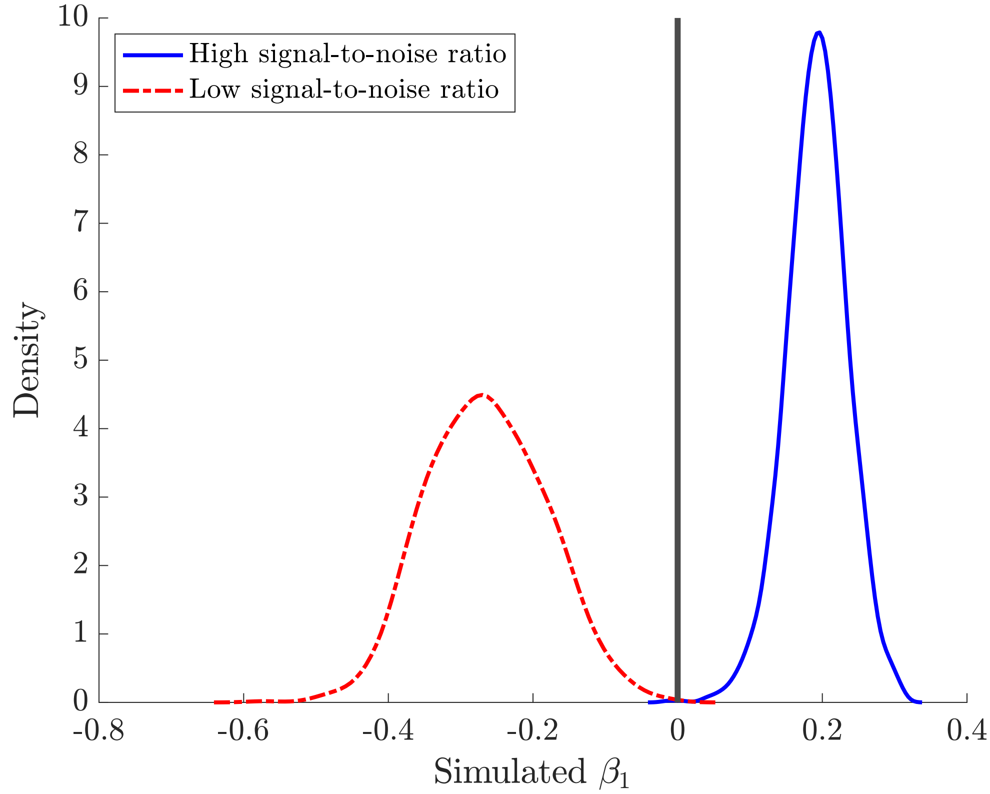
5 Evidence from the Survey of Professional Forecasters

This section exploits the variation across the macroeconomic variables reported in the SPF. Each of the variables is presumed to follow a specific data generating process. As a result, $\beta_1 = \beta_1(\rho, \sigma_v, \sigma_e, \phi_0, \phi_1, \sigma_\eta)$ will, in general, vary in the cross-section of SPF variables. I document four facts by measuring proxies for signal and noise. In general, variables that exhibit greater state volatility or less noise will tend to be variables for which we observe underreactions. On the other hand, variables that feature elevated amounts of noise will be associated with overreactions.

[19] Although robust and reliable estimates of the SNR are missing in this literature, CG provide some estimates using cross-country data which are in line with the simulated SNR values here. In addition, these simulated SNR values are similar in magnitude to those that I estimate in Section 7.

[20] If all forecasters made use of the particle filter, however, then in large samples we would expect this distribution to be centered at zero.

Figure 3: Covariance of Errors and Revisions Depends on Driving Process



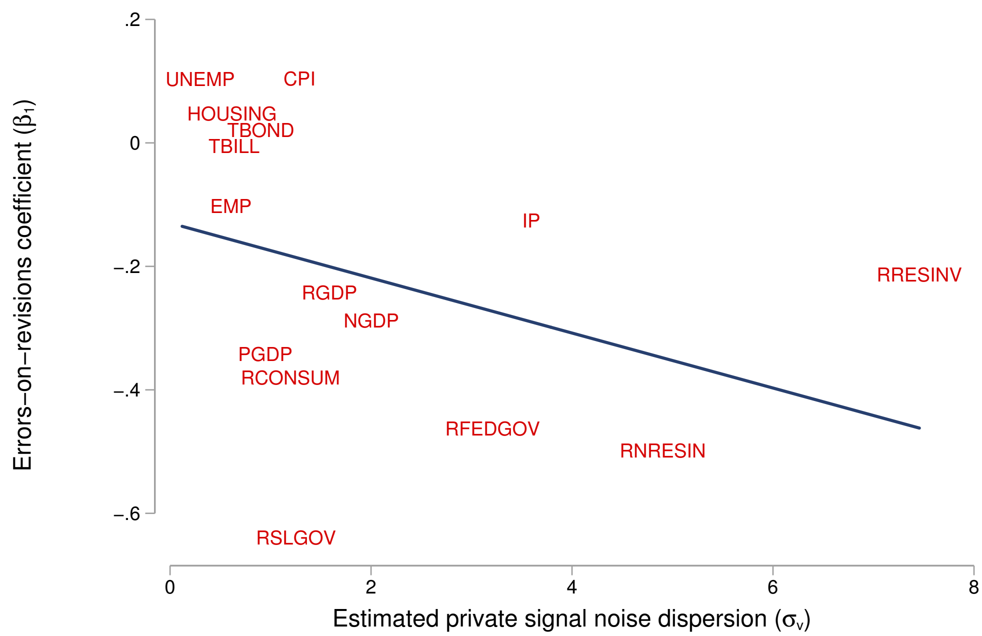
Note: The figure plots two simulated densities of β_1 arising from a pooled individual-level errors-on-revisions regression from the model described in Section 5. The red dashed line plots the simulation in which private signal noise variance is relatively high whereas the solid blue line plots the simulation in which private signal noise variance is relatively low.

Testable Prediction 1: Error Predictability and Private Noise

In this model, forecasters revise their predictions according to the realization of the lagged public signal as well as their contemporaneous private signal. The private signal noise therefore feeds into the forecast revision. From the perspective of the model, the variance of private signal noise determines the amount of dispersion in revisions across forecasters. More dispersed signal noise admits more pronounced differences in revisions.

With this insight, I collect the different β_1 coefficients across SPF variables and compute the interquartile range of revisions across forecasters for each variable. Figure 4 plots the results. As the model suggests, variables exhibiting greater dispersion in revisions tend to be those for which

Figure 4: Error Predictability and Revision Dispersion



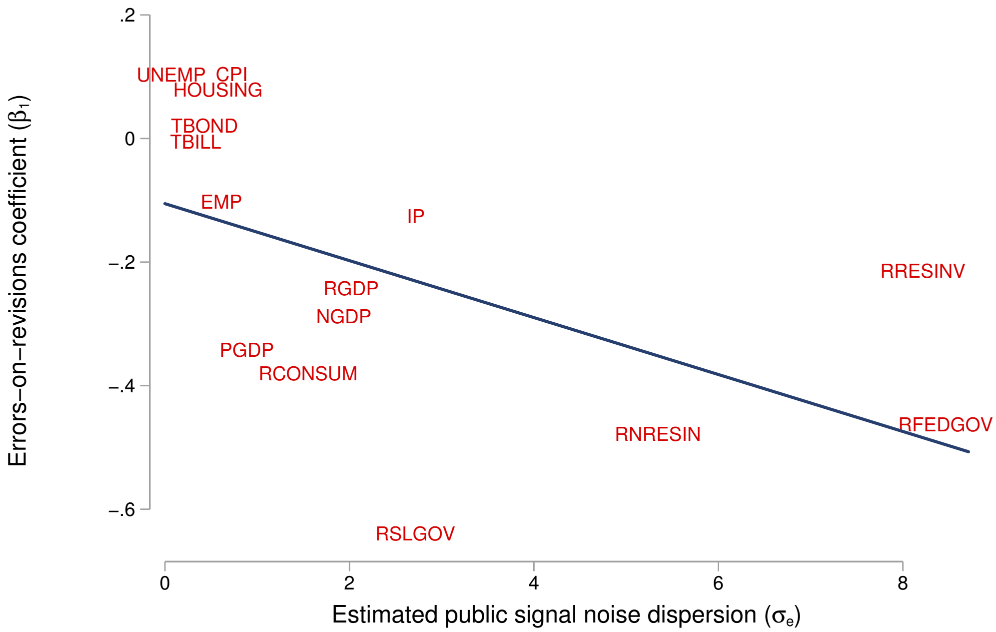
Note: For each SPF variable, the figure plots the estimated errors-on-revisions coefficient at the forecaster-level against estimated private noise dispersion, proxied by the interquartile range of forecast revisions. Slope of fitted line is -0.045.

forecasters overreact.

Testable Implication 2: Error Predictability and Public Noise

While Figure 4 relates β_1 to private signal noise, there is also common noise present in the model. I next turn to measure the noisiness of the public signal. To reiterate, while the SPF variable of interest has sometimes been modeled as the latent state in the literature, it is best thought of as a lagged public signal. This is because the SPF variables are observed by all forecasters with a lag. With this in mind, the official government revisions made to these variables across different vintages can provide a measure of public signal noise. Assuming that the vintages following the initial real-time release of the variable eliminate some of the common noise, one can quantify these revisions over time. As a matter of notation, define x_t^I as the real-time (initial) data release for a given variable, and x_t^L as the last release of the variable. Then, we can define $\text{noise}_t^{\text{public}} = \text{Var}(x_t^I - x_t^L)$.

Figure 5: Error Predictability and Public Noise



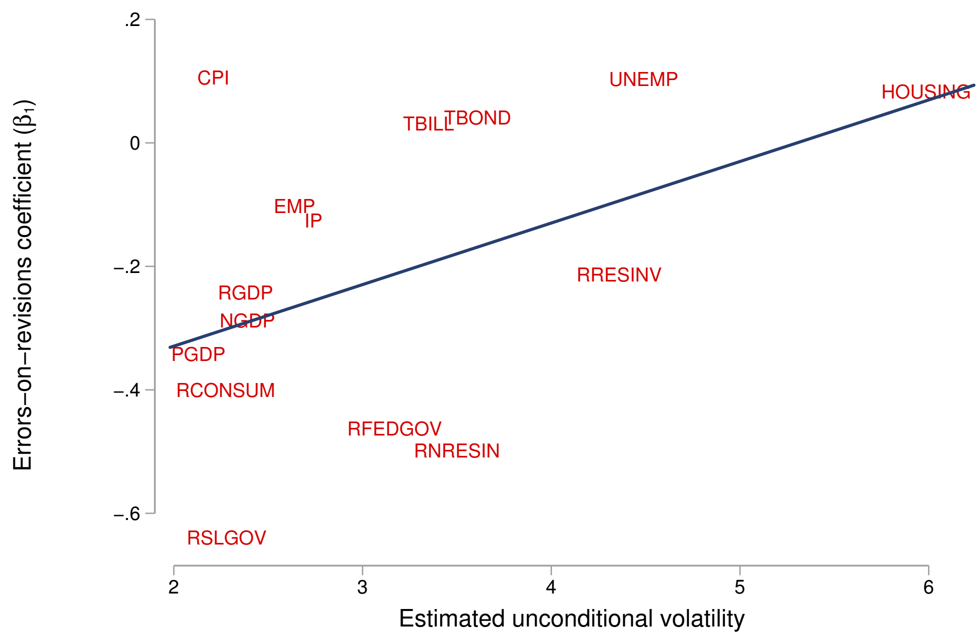
Note: For each SPF variable, the figure plots the estimated errors-on-revisions coefficient at the forecaster-level against estimated public noise dispersion, proxied by the standard deviation of government revisions to real-time data. Slope of fitted line is -0.046.

I construct this variable from the first and last data vintage for all SPF variables in my sample, and then measure the dispersion of this public noise over time. Figure 5 relates β_1 with this measure of public signal noise. The results are consistent with the intuition of the model: variables exhibiting higher measured noise dispersion tend to deliver observed overreactions.

Testable Prediction 3: Error Predictability and Unconditional Volatility

Moreover, the model predicts that with more unconditional variability in the state, there is less scope for overreaction. To test this, I proceed to estimate ϕ_0 for each SPF variable. I then construct

Figure 6: Error Predictability and Unconditional Volatility



Note: For each SPF variable, the figure plots the estimated errors-on-revisions coefficient at the forecaster-level against estimated unconditional volatility of the state, $\exp(\widehat{\phi}_0/2)$. Slope of fitted line is 0.100.

an estimate of unconditional volatility of the state [21]

$$\text{vol}_j = \exp\left(\frac{\widehat{\phi}_0}{2}\right)$$

Figure 6 relates β_1 to vol_j. The figure supports the hypothesis that variables exhibiting more variability in the state tend to provide greater scope for underreactions. Furthermore, note that the variance of the state is increasing in ρ. Hence, the model predicts that more persistent variables will reduce the scope for overreactions. This is consistent with Bordalo et al. (2020) who verify this empirically.

[21] I estimate the parameters of the stochastic volatiliity model, $\{\phi_0, \phi_1, \sigma_\eta\}$ using MCMC techniques (see Kastner and Fruhwirth-Schattner 2014).

Testable Prediction 4: Error Predictability and Release Frequency

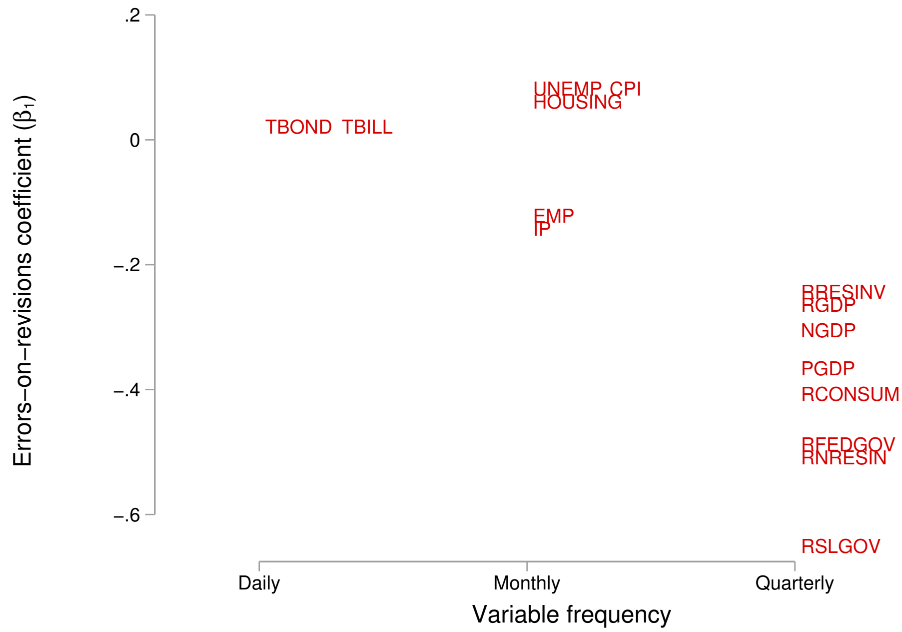
As an additional way to measure signal precision, I consider the frequency with which these different variables are made available to the public. While professional forecasters report predictions in each quarter, some variables are made available at higher frequencies. Specifically the SPF conducts its survey at roughly the middle of each quarter. However, some of the SPF variables are released at a monthly frequency. For instance employment statistics are released on the Friday of each month. The survey asks forecasters to provide a quarterly average of these series. Furthermore, the financial time series are available at a daily frequency. As a result, forecasters have more information pertaining to some x_t at the time that they report their predictions than others. This reduces the effective noise in the lagged public signal.[22] Hence, variables available at higher frequencies should raise the scope for underreaction. Note that this does not preclude overreactive behavior in financial markets as has been readily documented. Here, I simply argue that quarterly (average of daily observations) predictions of a financial variable are better informed by the presence of daily observations through the middle of the quarter when the reported forecast is requested. On the other hand, the latest information that forecasters have for quarterly variables, such as GDP, is the previous quarter's release and an advance estimate. Since there is additional information available for some variables and not others, and the existence of this additional information depends on the variable frequency, then it follows that there is more scope for underreaction among variables that are available at higher frequencies.

Jointly Testing for Overreaction and Underreaction Channels

As an additional check, I formally test for these channels jointly. For the data to accord with this theory of nonlinear expectations, it should be the case that an interaction of the forecast revision with each of these variables either raises or reduces the extent to which β_1 is negative in the pooled specification (column 1 of Table 1). To complete this exercise, I incorporate two new regressors (and

[22] Alternatively, one could suppose that forecasters receive an additional informative public signal for monthly/daily SPF variables.

Figure 7: Error Predictability and Release Frequency



Note: For each SPF variable, the figure plots the estimated errors-on-revisions coefficient at the forecaster-level against variable's release frequency $\{\text{Daily}, \text{Monthly}, \text{Quarterly}\}$.

all possible interactions), each capturing a source of either noise or state volatility. As a measure of noise, I select the release frequency explained above. For my measure of fundamental volatility, I take a factor analysis approach. Since the latent state and its volatility are unobservable, it is natural to consider an index of the shared variation among all SPF variables. From this exercise, I obtain a time-varying index of what I call fundamental volatility.[23]

Given these two new regressors, I modify the baseline errors-on-revisions regression (pooling across all SPF variables as in Table 1). In addition to projecting errors on revisions, I also specify the "Noise" variable (from the variable release frequency), the "Fundamental Volatility" (from the

[23]For this analysis, I drop nominal GDP since its components reside in my data set. Furthermore, I exclude CPI due to its shorter available history. For the remaining macroeconomic variables, I compute five-year rolling standard deviations and then estimate underlying principal factors. The results deliver two factors that explain roughly equal amounts of the common variance of the final vintage of SPF variables. Based on the factor loadings, I call the first factor a real residential factor, and the second a real non-residential factor (the residential factor loads highly on housing and real residential investment whereas the second factor does not). While both factors deliver the correct sign in my regression specification, I report the regression that specifies the real non-residential factor as it delivers statistically detectable results.

constructed index). I also include interactions of each of these with the forecast revision as well as all interactions with each other. The regression results are reported in Table 4.

The first column of the table reproduces the first column of Table 1. The second column of Table 5 reports the fully specified regression while the third column incorporates fixed effects. The relationship of interest remains the extent to which the forecast error and revision are related. Noise and fundamental volatility have no statistically detectable relation with the forecast error. In other words, these proxies cannot independently predict forecasters' mistakes. The regression also includes all possible interaction terms between the regressors. The only interaction terms that are statistically significant are those crossed with the forecast revision. Furthermore, the signs of these two interactions are consistent with the expected signs according to my theory of nonlinear expectations. In particular, noise raises the scope for overreactions as evidenced by the negative cross term between the noise proxy and the revision. On the other hand, fundamental volatility reduces the scope for overreaction as seen by the positive coefficients in front the relevant interaction terms.

6 Parameterization

The model is able to generate over- and underreactions across level of aggregation, and across variables, pooling over forecasters. In this section, I parameterize the stylized model for real GDP and unemployment in order to demonstrate that it can also generate simultaneous over- and underreactions *within* forecaster.

Specifically, I calibrate the public noise dispersion, the persistence of the latent state, and the stochastic volatility parameters $\{\sigma_e, \rho, \phi_0, \phi_1, \sigma_\eta\}$. I then find the values of $\overline{c}_A$ and σ_v that minimize the distance between the model-simulated and empirical estimates of the errors-on-revisions coefficients at the forecaster and consensus-levels (β_1 and α_1).

For real GDP and unemployment respectively, I set σ_e equal to the standard deviation of the data revisions made to each variable over the sample period. The data revision is taken to be the

Table 4: Modified Forecast Error Predictability Panel Regressions

	Forecast Error		
	(1)	(2)	(3)
Revision	-0.317***	-0.165**	-0.245***
	(0.050)	(0.073)	(0.056)
Revision × Noise		-0.194**	-0.115**
		(0.089)	(0.055)
Revision × Fundamental Volatility		0.106**	0.095***
		(0.041)	(0.022)
Forecaster × Variable FE	N	N	Y
Quarter FE	N	N	Y
Observations	65,070	58,740	58,740

Note: The table reports estimated coefficients of forecast error predictability across three pooled specifications. The variable *Noise* is equal to 0 if the SPF variable is released at a non-quarterly frequency and 1 otherwise. The *Fundamental Volatility* variable is a time-varying index of fundamental volatility constructed as described in the text. Column (1) reports a simple regression of errors-on-revisions, columns (2) and (3) include the two proxies described, where column (3) includes fixed effects. In addition to the variables reported in the table, Columns (2) and (3) include the proxies individually as well as all of their interactions . Standard errors for forecaster-level regressions are as in Driscoll and Kraay (1998). Data used for estimation come from SPF (1964Q4-2018Q3). *** denotes 1% significance, ** denotes 5% significance, and * denotes 10% significance.

difference between the first and final release of the data series. For the remaining parameters, I consider the revised data rather than the real-time data. Intuitively, these series should be more highly correlated with the unobserved latent state. I then estimate an AR(1) on the revised series and set ρ equal to the estimated AR(1) coefficient. Finally, I collect the squared residuals from this autoregression and estimate $\{\phi_0, \phi_1, \sigma_\eta\}$. Table 5 reports the calibration for each variable.

I parameterize σ_v and $\overline{c}_A$ by minimizing the distance between the model-implied $\{\beta_1, \alpha_1\}$ from its empirical counterpart. Since I am estimating these parameters for GDP and unemployment, the procedure amounts to searching a four dimensional parameter space and matching four moments. For each simulation, I generate two state variables according to the dynamics described in Section 5. I then simulate the lagged public signal as well as the contemporaneous private signal for each variable. In every period, forecasters decide which models to use and report a forecast for each variable according to the state dynamics, signals received, and loss function described in Section 5. From this simulated panel of forecasters, I construct the errors-on-revisions coefficients. I minimize

Table 5: External Parameterization

Parameter	Description	Unemployment	Real GDP
ρ	State persistence	0.98	0.30
σ_e	Standard deviation of public noise	0.07	2.02
ϕ_0	Level of log variance	-0.76	0.14
ϕ_1	Persistence of log variance	0.73	0.92
σ_η	Volatility of log variance	0.69	0.39

Note: The table reports parameterization for unemployment and real GDP. Stochastic volatility parameters $\{\phi_0, \phi_1, \sigma_\eta\}$ are estimated according to the algorithm presented in Kastner and Fruhwirth-Schnatter (2014).

Table 6: Calibration Results

Variable	σ_v	$\overline{c}_A$	Implied SNR	Share using PF
Unemployment	0.181	0.989	0.984	0.010
Real GDP	2.092	1.683	0.583	0.685

Note: The first two columns of the top panel of the table report estimated coefficients of private signal noise dispersion and upper bound of approximation cost function. The third column reports the implied signal-to-noise ratio while the final column reports the implied share of forecasters making use of the particle filter of the naive Kalman filter. The bottom panel compares the model-implied share of over- and underreactions with the data (for real GDP and unemployment).

the distance between the simulated and empirical OLS coefficients by making use of simulated annealing, a standard global stochastic optimization routine.

The results, reported in Table 6, indicate that real GDP is characterized by more noise than the unemployment rate. The implied signal-to-noise ratio for real GDP is about 0.58 whereas it is 0.98 for unemployment. This is consistent with the intuition of the model as well as the cross-sectional evidence in the previous section: variables that exhibit higher signal-to-noise ratios tend to be the variables for which underreactions are observed.

Furthermore, the upper bound of the cost distribution is higher for real GDP than it is for unemployment. The higher upper support for real GDP is a result of the larger errors made when predicting a noisier variable. These estimates imply that nearly no forecaster opts to use the particle filter to forecast unemployment whereas almost 70% of forecasters choose the particle filter as the preferred model when predicting real GDP. This is consistent with the intuition as real GDP

Table 7: Model Fit

	Model	Data
Unemployment		
Errors-on-revisions, forecaster-level (β_1)	0.14	0.08
Errors-on-revisions, consensus (α_1)	0.28	0.25
Real GDP		
Errors-on-revisions, forecaster-level (β_1)	-0.21	-0.26
Errors-on-revisions, consensus (α_1)	0.35	0.35
Share that overreact to real GDP and underreact to unemployment		
	0.56	0.64

Note: The table reports empirical and model-implied moments. The calibration directly targets the errors-on-revisions moments for unemployment and real GDP at the forecaster and consensus-levels. The final row reports the share of forecasters that overreact to real GDP and simultaneously underreact ot unemployment.

appears to be noisier than unemployment in the data. Hence, forecasters are more likely to incur the cost to adopt a more sophisticated forecast.

Table 7 reports the model fit. The procedure was able to successfully match patterns of over- and underreaction observed in the pooled forecaster-level and consensus regressions. Lastly, the stylized model is able to successfully match the share of simultaneous over- and underreactions. Figure 1 reports that about 64% of forecasters in the sample overreact to real GDP while simultaneously underreacting to unemployment. Based on the parameterization devised here, the stylized model 56% of simulated forecasters overreact real GDP and underreact to unemployment.

6.1 Implications for Information Rigidities

What does time-varying volatility coupled with imperfect information imply about interpreting the reduced form coefficient β_1 as an information rigidity? This is a relevant question as some of the recent literature has adopted the reduced form errors-on-revisions regressions to quantify information rigidities (Baker et al. 2020, Ryngaert 2017). Based on my model, it is apparent that β_1 does not cleanly map to the Kalman gain as it does in the scalar linear context. The key intuition of

Bayesian filtering, however, still holds, and the optimal weight placed on innovation errors remains a sufficient statistic for capturing the rate of learning. This weight depends on the covariances of the state estimation error and the measurement error. Quantifying the rate of learning, however, is not readily feasible from a projection of errors on revisions.

In fact, from the perspective of this model, the coefficient coming from errors on revisions regressions can reveal misleading insights on the extent of information rigidity. β_1 is positive when the SNR is high. This would typically imply larger information rigidities, however, my model suggests that this implies a higher rate of learning. Furthermore, a negative β_1 arises when signals are less informative. This suggests that the reduced form coefficients β_1 (and α_1) may be limited in what they reveal about information frictions.

6.2 Implications for State Dependence

What does this mean for state dependence? If β_1 and α_1 were to rise in recessions, then the model would imply that the signal to noise ratio is countercyclical and information rigidities fall during economic downturns. If, on the other hand, these coefficients fall, then the signal-to-noise ratio is procyclical and information rigidities actually rise in recessions. CG document evidence indicating that α_1 falls in recessions. They interpret this as a reduction in information rigidities, however, my model would suggest that this implies a *rise* in information rigidities as the signal-to-noise ratio is falling which implies that the system experiences elevated amounts of noise (i.e. information is less precise in recessions). This is an important distinction between my model and the extant literature as it delivers an opposite answer to the question of whether individuals trust their signal more or less in recessions.

However, after performing a similar exercise to that in Table 5 by interacting a quarterly recession indicator with forecast revisions, I find no evidence that β_1 or α_1 changes with the business cycle. I also run this exercise by replacing the recession indicator with revised real GDP growth. It is possible, however, that the signal-to-noise ratio is insensitive to business cycle fluctuations because both the state and the signal experience stochastic volatility. I abstract away from volatility in signal precision, and so there is a limit to what one can glean from this model as it pertains to state dependence of information rigidities.

Nonetheless, one could distinguish between two types of uncertainty: fundamental uncertainty and information uncertainty. The first maps to time-varying volatility in the state while the latter arises when signal noise experiences stochastic volatility. There is a literature that stresses the importance of uncertainty shocks. These are often modeled as fundamental uncertainty shocks, however, shocks to information precision are also studied in the literature.[24] According to this model, the state dependence of β_1 and α_1 depend on the signal-to-noise ratio which in turn depends on how fundamental vs. information uncertainty evolve over the business cycle. If both rise in recessions, then is possible that the signal-to-noise ratio is acyclical thereby rendering β_1 and α_1 roughly constant over the cycle as well.

7 Conclusion

This paper shows that a noisy information model that incorporates time-varying volatility can generate error predictability at the forecaster-level. Several recent studies have explained error predictability with a model of non-rational expectations. Here, I show that this empirical fact can arise among rational forecasters attempting to predict a nonlinear state. Forecasters optimally select different models based on the complexity of the state dynamics. This heterogeneity in predictor functions can jointly deliver coincident over- and underreactions among forecasters. To this end, violations of error orthogonality are best understood as evidence against *linear* rational expectations.

[24] For instance, Dun Jia (2016)

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