Time-Varying Volatility, Underreaction, and Overreaction

APPENDIX

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Appendix A Empirics

A.1 SPF: Variable Descriptions

While the paper focuses on inflation forecasts based on the GDP deflator, in this subsection I report additional results that make use of several other variables. Before presenting these results, I provide the variable descriptions below:

- NGDP–Quarterly nominal GDP growth forecast (seasonally adjusted, annual rate). Prior to 1992, these are forecasts for nominal GNP.
- RGDP–Quarterly real GDP growth forecast (seasonally adjusted, annual rate).
- PGDP–Quarterly GDP price index growth forecast (seasonally adjusted, annual rate). From 1992 - 1995, GDP implicit deflator is used, and prior to 1992, GNP implicit deflator.
- UNEMP–Forecasts for the quarterly average unemployment rate (seasonally adjusted, average of underlying monthly levels).
- EMP–Quarterly average growth of nonfarm payroll employment (seasonally adjusted, average of underlying monthly levels).
- RNRESIN–Quarterly growth forecast of real nonresidential fixed investment. Also known as business fixed investment (seasonally adjusted, annual rate).
- RRESINV–Quarterly growth forecast of real residential fixed investment (seasonally adjusted, annual rate).
- TBILL–Quarterly forecast of average three-month Treasury bill rate (percentage points, average of underlying daily levels).
- HOUSING– Quarterly growth forecast of average housing starts (seasonally adjusted, annual rate, average of underlying monthly levels).
- CPI–Quarterly forecasts of the headline CPI inflation rate (percentage points, seasonally adjusted, annual rate). Quarterly forecasts are annualized q/q percent changes of quarterly average price index level (average of underlying monthly levels).

- RCONSUM – Quarterly growth forecast of real personal consumption expenditures (seasonally adjusted, annual rate).

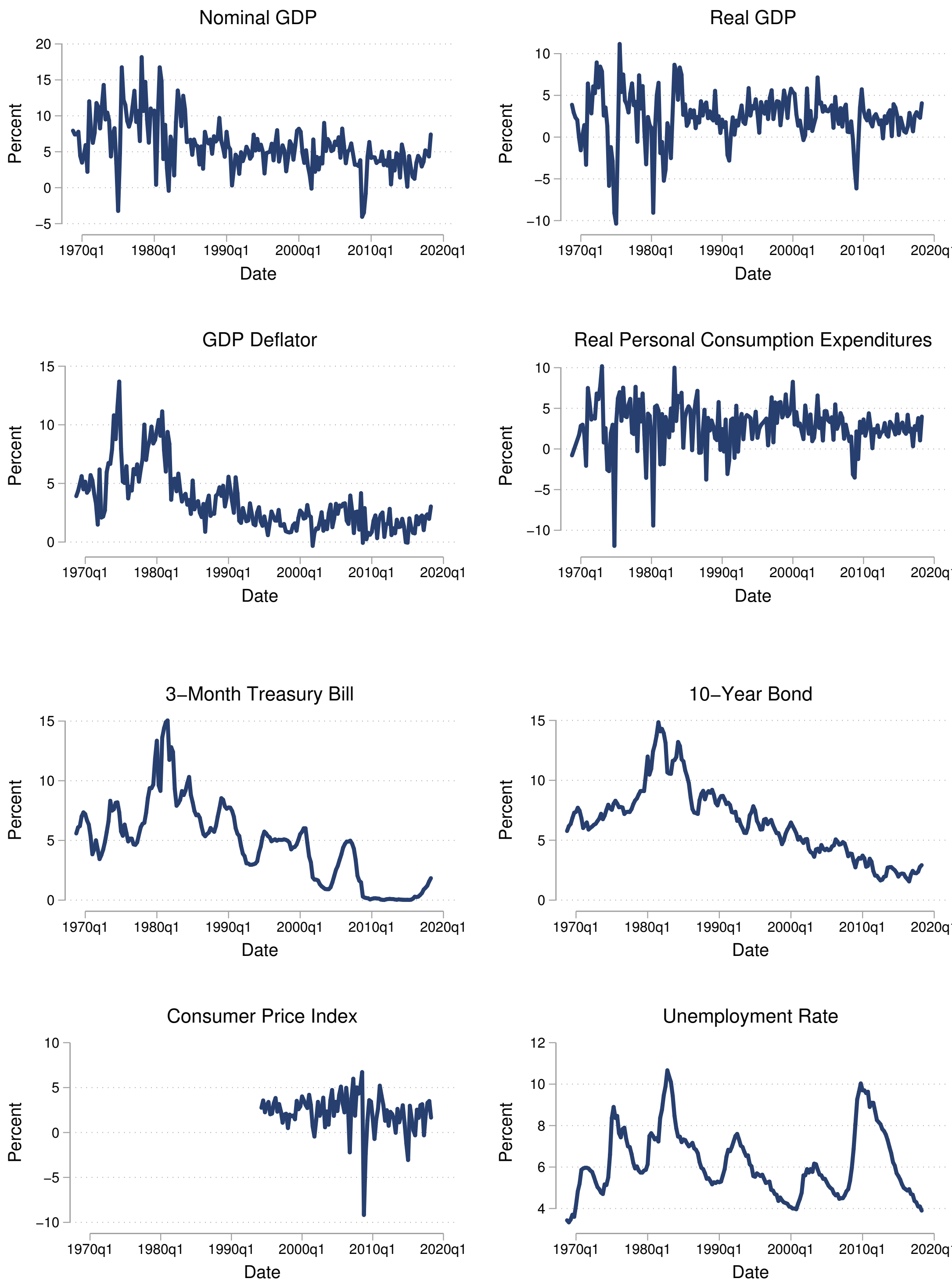
- RFEDGOV –Quarterly growth forecast of real federal government consumption and gross investment (seasonally adjusted, annual rate).

- INDPROD – Quarterly forecasters of level of the index of industrial production, seasonally adjusted (quarterly forecasts are for quarterly average of underlying monthly levels).

- TBOND–Quarterly average 10-year Treasury bond rate (percentage points, average of the underlying daily levels). the underlying daily levels

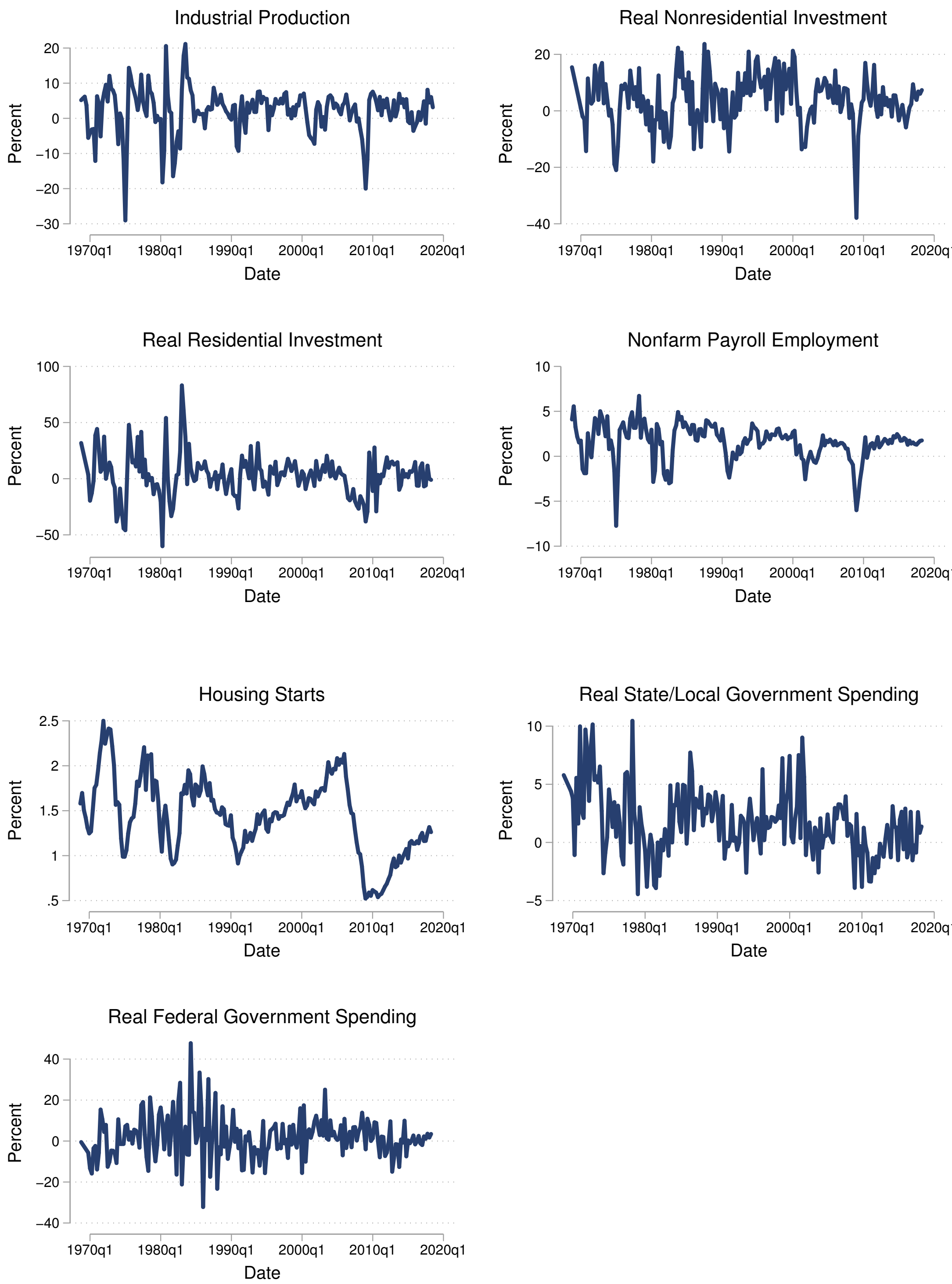
- RSLGOV–Quarterly growth forecast of real state and local government consumption and gross investment (seasonally adjusted, annual rate).

Figure 1: Real-Time Macroeconomic Time Series



Source: Survey of Professional Forecasters

Figure 2: Real-Time Macroeconomic Time Series



Source: Survey of Professional Forecasters

Appendix B Model

B.1 General Linear Noisy Information RE Model

Theories of linear rational expectations are unable to account for over- and underreactions. I show this by considering a general linear rational expectations model and providing analytical results about error predictability.

Consider a linear Gaussian state space model. Suppose there are n latent state variables and m exogenous signals.

$$\mathbf{s_t} = \mathbf{A}\mathbf{s_{t-1}} + \mathbf{B}\mathbf{w_t} \tag{B.1}$$
$$\mathbf{z_t^i} = \mathbf{C}\mathbf{s_t} + \mathbf{D}\mathbf{v_t^i}$$

Note that $\mathbf{s_t}$ is an $n \times 1$ vector, $\mathbf{A}$ is $n \times n$, $\mathbf{B}$ is $n \times n$ and $\mathbf{w_t}$ is $n \times 1$. Furthermore, $\mathbf{z_t}$ is $m \times 1$, $\mathbf{C}$ is $m \times n$, $\mathbf{D}$ is $m \times m$ and $\mathbf{v_t^i}$ is $m \times 1$. There are no other restrictions placed on the model. In particular, $\mathbf{s_t}$ can be a vector of many different state variables, or lags of itself. $\mathbf{B}$ need not be a diagonal matrix. Furthermore, $\mathbf{z_t^i}$ can include an arbitrary finite number of observed signals. The noise vector $\mathbf{v_t^i}$ can include private or public noise.[1]

From the Kalman filter, the optimal state estimate is defined as

$$\mathbf{s^i_{t|t}} = \mathbf{s^i_{t|t-1}} + \kappa(\mathbf{z^i_t} - \mathbf{z^i_{t|t-1}}) \tag{B.2}$$

where κ is the (constant) Kalman gain. Since the state is unobservable, forecasters can only formulate predictions of the signals and assess the mistakes made with regard to these observables. The optimal forecast of the signal vector $\mathbf{z_t^i}$ is

$$\mathbf{z^i_{t+1|t}} = \mathbf{z^i_{t+1|t-1}} + \mathbf{C}\mathbf{A}\kappa(\mathbf{z^i_t} - \mathbf{z^i_{t|t-1}}) \tag{B.3}$$

Forecast errors for the generalized linear model can be expressed as follows

$$\mathbf{z^i_{t+1}} - \mathbf{z^i_{t+1|t}} = (\mathbf{z^i_{t+1}} - \mathbf{z^i_{t+1|t-1}}) - \mathbf{C}\mathbf{A}\kappa(\mathbf{z^i_t} - \mathbf{z^i_{t|t-1}}) \tag{B.4}$$

[1] I index this vector by i in general to allow for forecaster-specific signals.

Furthermore, the forecast revision is

$$\mathbf{z^i_{t+1|t}} - \mathbf{z^i_{t+1|t-1}} = \mathbf{CA}\kappa(\mathbf{z^i_t} - \mathbf{z^i_{t|t-1}}) \tag{B.5}$$

Using these expressions, one can derive the two testable implications presented in the previous section.

Proposition 1. *The generalized linear model implies the following:*

(i) $\beta_1 = 0$

(ii) $\alpha_1 = \mathbf{CA(I - C}\kappa)(\kappa\mathbf{C})^{-1}(\mathbf{CA})^{-1} > 0$

Proof. We have the following expressions for forecast errors and revisions, respectively:

$$\mathbf{FE^i} = \mathbf{CA(I} - \kappa\mathbf{C})(\mathbf{s_t} - \mathbf{s^i_{t|t-1}}) + \mathbf{CBw_{t+1}} + \mathbf{Dv^i_{t+1}} - \mathbf{CA}\kappa\mathbf{Dv^i_t}$$

$$\mathbf{FR^i} = \mathbf{CA}\kappa\mathbf{Dv^i_t} + \mathbf{CA}\kappa\mathbf{C}(\mathbf{s_t} - \mathbf{s^i_{t|t-1}})$$

Then,

(a) $\beta_1 \propto \mathrm{Cov}(FE^i, FR^i) = \mathbf{CA(I} - \kappa\mathbf{C})\mathbf{\Psi}(\mathbf{CAKC})^\top - \mathbf{CAK}(\mathbf{Dv^i_t v^i_t D})(\mathbf{CA}\kappa)^\top$

where $\mathbf{\Psi}$ denotes the state estimation error variance. This becomes

$$\begin{aligned} \beta_1 \quad &\propto \quad CA(I - \kappa C)\Psi(CA\kappa C)^\top - CAK(Dv^i_t v^i_t D)(CA\kappa)^\top \\ &= CA\Big\{(I - \kappa C)\Psi C - \kappa D v^i_t v^i_t D\Big\}(CA\kappa)^\top \\ \beta_1 \quad &= 0 \end{aligned}$$

because the term in brackets is zero by the definition of the Kalman gain.

(b) Denoting $\overline{FE}$ and $\overline{FR}$ as the cross-sectional mean of the forecast error and revision, respectively, we have

$$\alpha_1 \propto \mathrm{Cov}(\overline{\mathbf{FE}}, \overline{\mathbf{FR}}) = CA(I - \kappa C)\overline{\Psi}(CA\kappa C)^\top$$

The variance of the average revision is $\text{Var}(\overline{FR}) = CA\kappa C\Psi(CA\kappa C)^\top$ Thus, we have

$$\alpha_1 = CA(I - \kappa C)(CA\kappa C)^{-1}$$

$\square$

The proofs are straightforward: (a) holds given the orthogonality condition that must be satisfied at the individual-level under rational expectations. Forecast error orthogonality implies that $\mathbb{E}\big[(\mathbf{z^i_t} - \mathbf{z^i_{t|t}})\mu\big] = \mathbf{0}$ for any μ residing in the forecaster's information set.[2] Put another way, rationality implies the optimal use of information so that no variable residing in one's information set may predict the forecast error. This very general model precludes the predictability of forecast errors at the individual-level. As a result, any such linear Gaussian model with mean square loss cannot generate error predictability, regardless of the signal structure.

Moreover, (b) is a generalization of the CG result. The extent to which the mean revision predicts mean errors is determined by the Kalman gain matrix and the matrix $\mathbf{C}$ which maps the underlying state to the observed signal vector. The generalized linear model nests the CG result. Letting $\mathbf{C} = 1$, $\mathbf{D} = \sigma_v$, $\mathbf{A} = \rho$ and $\mathbf{B} = \sigma_w$, it follows that $\alpha_1 = \frac{1-\kappa}{\kappa}$. In this limiting case, one can recover an estimate of information rigidity by projecting consensus errors on consensus revisions. Importantly, the signal structure must be such that $\mathbf{C} = 1$. If, instead, the elements of $\mathbf{C}$ include additional parameters, or there is common noise in the signal vector, then it is no longer possible to cleanly extract an the Kalman gain from a standard OLS regression.[3]

As a result, a highly generalized linear rational expectations model is unable to explain the patterns in the data.

[2] Similarly, there is a revision orthogonality condition implied by rationality which states that $\mathbb{E}(\mathbf{z^i_{t|t}} - \mathbf{z^i_{t|t-1}})|\mathcal{I}^i_t) = 0$. See Pesaran and Weale (2006).

[3] See CG for a discussion of the bias in estimated information rigidities induced by public noise.

B.2 Error Predictability Under Time-Varying Volatility

From the general nonlinear model described in the main text, the covariance of errors and revision can be signed as follows:

$$\beta_1 \propto \mathbb{C}\Bigg(\bar{\mathbf{s}}_{\mathbf{t}}, \int \bar{\mathbf{s}}_{\mathbf{t}}[\widehat{\mathbf{p}}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}^{\mathbf{i}}_{\mathbf{t}}) - \widehat{\mathbf{p}}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}^{\mathbf{i}}_{\mathbf{t-1}})]\mathbf{d}\bar{\mathbf{s}}_{\mathbf{t}}\Bigg)$$
$$- \mathbb{C}\Bigg(\int \bar{\mathbf{s}}_{\mathbf{t}}\widehat{\mathbf{p}}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}^{\mathbf{i}}_{\mathbf{t}})d\bar{\mathbf{s}}_{\mathbf{t}}, \int \bar{\mathbf{s}}_{\mathbf{t}}[\widehat{\mathbf{p}}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}^{\mathbf{i}}_{\mathbf{t}}) - \widehat{\mathbf{p}}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}^{\mathbf{i}}_{\mathbf{t-1}})]\mathbf{d}\bar{\mathbf{s}}_{\mathbf{t}}\Bigg)$$

When there are no approximation errors, error orthogonality holds and $\beta_1 = 0$. In the case of non-zero approximation errors, however, the first term is the source of observed underreaction while the second term governs the extent of overreaction. When forecast revisions are more closely related to the underlying state, then underreactions arise as the first term dominates the second. If instead, forecast revisions covary more with the current prediction than the underlying state, then overreactions result. In essence, when the approximate revision incorporates more noise than is optimally called for, then forecasters will overreact.

Similar to the approximate prediction defined above, the consensus forecast arising from approximate predictions is defined as follows

$$\alpha_1 \propto \mathbb{C}\Bigg(\bar{\mathbf{s}}_{\mathbf{t}}, \int\int \bar{\mathbf{s}}_{\mathbf{t}}\Big[\widehat{\mathbf{p}}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}^{\mathbf{i}}_{\mathbf{t}}) - \widehat{\mathbf{p}}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}^{\mathbf{i}}_{\mathbf{t-1}})\Big]d\bar{\mathbf{s}}_{\mathbf{t}}di\Bigg)$$
$$- \mathbb{C}\Bigg(\int\int \bar{\mathbf{s}}_{\mathbf{t}}\widehat{\mathbf{p}}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}^{\mathbf{i}}_{\mathbf{t}})d\bar{\mathbf{s}}_{\mathbf{t}}di, \int\int \bar{\mathbf{s}}_{\mathbf{t}}\Big[\widehat{\mathbf{p}}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}^{\mathbf{i}}_{\mathbf{t}}) - \widehat{\mathbf{p}}(\bar{\mathbf{s}}_{\mathbf{t}}|\mathcal{Z}^{\mathbf{i}}_{\mathbf{t-1}})\Big]d\bar{\mathbf{s}}_{\mathbf{t}}di\Bigg)$$

More volatile revisions increase the scope for overreaction. Upon aggregating (symmetrically) across several individual forecasts, the consensus revision will exhibit more persistence than the individual revisions. This motivates the following result

Proposition 2. *In the nonlinear noisy information model, $\alpha_1 \geq \beta_1$.*

Proof. Recall that

$$\alpha_1 = [(\widehat{\mathbf{z}}_{t+h|t} - \widehat{\mathbf{z}}_{t+h|t-1})'(\widehat{\mathbf{z}}_{t+h|t} - \widehat{\mathbf{z}}_{t+h|t-1})]^{-1}(\widehat{\mathbf{z}}_{t+h|t} - \widehat{\mathbf{z}}_{t+h|t-1})'(\mathbf{z}_{t+h} - \widehat{\mathbf{z}}^{i}_{t+h|t})$$

and

$$\beta_1 = [(\widehat{\mathbf{z}}^i_{t+h|t} - \widehat{\mathbf{z}}^i_{t+h|t-1})'(\widehat{\mathbf{z}}^i_{t+h|t} - \widehat{\mathbf{z}}^i_{t+h|t-1})]^{-1}(\widehat{\mathbf{z}}^i_{t+h|t} - \widehat{\mathbf{z}}^i_{t+h|t-1})'(\mathbf{z}_{t+h} - \widehat{\mathbf{z}}^i_{t+h|t})$$

To prove the proposition, I will show that the covariance between consensus errors and revisions is weakly greater than that for pooled errors and revisions. I will then show that the variance of the consensus revision is weakly smaller than the variance of the pooled variance.

We can express the covariance between errors and revisions as

$$\begin{aligned}\mathbb{C}(\mathbf{z}_{t+h} - \widehat{\mathbf{z}}^i_{t+h|t}, \widehat{\mathbf{z}}^i_{t+h|t} - \widehat{\mathbf{z}}^i_{t+h|t-1}) = &\int\int(\mathbf{z}_{t+h} - \widehat{\mathbf{z}}^i_{t+h|t})(\widehat{\mathbf{z}}^i_{t+h|t} - \widehat{\mathbf{z}}^i_{t+h|t-1})didt \\ &- \int\int(\mathbf{z}_{t+h} - \widehat{\mathbf{z}}^i_{t+h|t})didt - \int\int(\widehat{\mathbf{z}}^i_{t+h|t} - \widehat{\mathbf{z}}^i_{t+h|t-1})didt\end{aligned}$$

and at the consensus level

$$\begin{aligned}\mathbb{C}(\mathbf{z}_{t+h} - \widehat{\mathbf{z}}_{t+h|t}, \widehat{\mathbf{z}}_{t+h|t} - \widehat{\mathbf{z}}_{t+h|t-1}) = &\int\left(\mathbf{z}_{t+h} - \int\widehat{\mathbf{z}}^i_{t+h|t}di\right)\left(\int[\widehat{\mathbf{z}}^i_{t+h|t} - \widehat{\mathbf{z}}^i_{t+h|t-1}]di\right)dt \\ &- \int\left(\mathbf{z}_{t+h} - \int\widehat{\mathbf{z}}^i_{t+h|t}di\right)dt - \int\int(\widehat{\mathbf{z}}^i_{t+h|t} - \widehat{\mathbf{z}}_{t+h|t-1})didt\end{aligned}$$

We wish to show that the second equation is weakly greater than the first. One can note immediately that the second and third terms of both equations are equal (given the linearity of the expectations operator), and so they cancel out. The resulting inequality that we wish to verify is

$$\int\left(\mathbf{z}_{t+h} - \int\widehat{\mathbf{z}}^i_{t+h|t}di\right)\left(\int[\widehat{\mathbf{z}}^i_{t+h|t} - \widehat{\mathbf{z}}^i_{t+h|t-1}]di\right)dt \geq \int\int(\mathbf{z}_{t+h} - \widehat{\mathbf{z}}^i_{t+h|t})(\widehat{\mathbf{z}}^i_{t+h|t} - \widehat{\mathbf{z}}^i_{t+h|t-1})didt$$

By distributing the revision into the error on either side of the inequality, we can express each side as the sum of two terms. The first of these will drop out as we will have

$$\int\mathbf{z}_{t+h}\left(\int[\widehat{\mathbf{z}}^i_{t+h|t} - \widehat{\mathbf{z}}^i_{t+h|t-1}]di\right)dt$$

on the LHS and

$$\int \int \mathbf{z}_{t+h}(\widehat{\mathbf{z}}^i_{t+h|t} - \widehat{\mathbf{z}}^i_{t+h|t-1})didt$$

on the RHS. Again, due to the linearity of the expectations operator, these terms cancel out. The remaining inequality is therefore

$$\begin{aligned}
-\int \left(\int \widehat{\mathbf{z}}^i_{t+h|t}di\right)\left(\int [\widehat{\mathbf{z}}^i_{t+h|t} - \widehat{\mathbf{z}}^i_{t+h|t-1}]di\right)dt &\geq -\int\int \widehat{\mathbf{z}}^i_{t+h|t}(\widehat{\mathbf{z}}^i_{t+h|t} - \widehat{\mathbf{z}}^i_{t+h|t-1})didt \\
\int \left(\int \widehat{\mathbf{z}}^i_{t+h|t}di\right)\left(\int [\widehat{\mathbf{z}}^i_{t+h|t} - \widehat{\mathbf{z}}^i_{t+h|t-1}]di\right)dt &\leq \int\int \widehat{\mathbf{z}}^i_{t+h|t}(\widehat{\mathbf{z}}^i_{t+h|t} - \widehat{\mathbf{z}}^i_{t+h|t-1})didt \\
\int \left(\int \widehat{\mathbf{z}}^i_{t+h|t}di\right)^2 dt - \int \left(\int \widehat{\mathbf{z}}^i_{t+h|t}di\right)\left(\int \widehat{\mathbf{z}}^i_{t+h|t-1}di\right)dt &\leq \int\int \widehat{\mathbf{z}}^{i2}_{t+h|t}didt - \int\int \widehat{\mathbf{z}}^i_{t+h|t}\widehat{\mathbf{z}}^i_{t+h|t-1}didt \\
\int\int \widehat{\mathbf{z}}^i_{t+h|t}\widehat{\mathbf{z}}^i_{t+h|t-1}didt - \int \left(\int \widehat{\mathbf{z}}^i_{t+h|t}di\right)\left(\int \widehat{\mathbf{z}}^i_{t+h|t-1}di\right)dt &\leq \int\int \widehat{\mathbf{z}}^{i2}_{t+h|t}didt - \int \left(\int \widehat{\mathbf{z}}^i_{t+h|t}di\right)^2 dt \\
\int \left[\int \widehat{\mathbf{z}}^i_{t+h|t}\widehat{\mathbf{z}}^i_{t+h|t-1}di - \left(\int \widehat{\mathbf{z}}^i_{t+h|t}di\right)\left(\int \widehat{\mathbf{z}}^i_{t+h|t-1}di\right)\right]dt &\leq \int \left[\int \widehat{\mathbf{z}}^{i2}_{t+h|t}di - \left(\int \widehat{\mathbf{z}}^i_{t+h|t}di\right)^2\right]dt
\end{aligned}$$

which is true since the terms in hard brackets on the RHS is the cross-sectional variance of the forecast whereas the term in hard brackets on the LHS is a cross-sectional covariance. Hence, the covariance of the consensus errors with consensus revisions is weakly greater than the covariance of individual-level pooled errors and revisions.

Finally, I show that the variance of the consensus revision is weakly smaller than the variance of the pooled revision. This is simpler to verify. Note that

$$\mathbb{V}(\widehat{\mathbf{z}}^i_{t+h|t} - \widehat{\mathbf{z}}^i_{t+h|t-1}) = \int\int (\widehat{\mathbf{z}}^i_{t+h|t} - \widehat{\mathbf{z}}^i_{t+h|t-1})^2 didt - \left(\int\int [\widehat{\mathbf{z}}^i_{t+h|t} - \widehat{\mathbf{z}}^i_{t+h|t-1}]didt\right)^2$$

and

$$\mathbb{V}(\widehat{\mathbf{z}}_{t+h|t} - \widehat{\mathbf{z}}_{t+h|t-1}) = \int \left(\int [\widehat{\mathbf{z}}^i_{t+h|t} - \widehat{\mathbf{z}}^i_{t+h|t-1}]di\right)^2 dt - \left(\int \left[\int [\widehat{\mathbf{z}}^i_{t+h|t} - \widehat{\mathbf{z}}^i_{t+h|t-1}]di\right]dt\right)^2$$

Once again, the second term in each of the above revision variance equations will cancel out. The resulting condition that we wish to verify is

$$\int \left(\int [\widehat{\mathbf{z}}^i_{t+h|t} - \widehat{\mathbf{z}}^i_{t+h|t-1}] di \right)^2 dt \leq \int \int (\widehat{\mathbf{z}}^i_{t+h|t} - \widehat{\mathbf{z}}^i_{t+h|t-1})^2 di dt$$

which holds by Jensen's inequality. $\square$

The model implies that the OLS coefficient estimated from an errors on revisions regression will be weakly greater than the analogous coefficient obtained from a pooled regression of individual forecasters. This result does depend on the presence of nonlinear dynamics. In fact, this holds in the linear setting as well (see Appendix B).[4]

[4]This need not be the case in other economic settings in which agents take actions, and their decisions are aggregated in a manner other than by taking a simple mean. Depending on the context, it is possible for the aggregate decision to also exhibit overreaction or excess volatility. Bordalo et al. (2020) provide a discussion of this.

Appendix C Calibration

I internally calibrate four parameters: $\{\sigma_{v,1}, \sigma_{v,2}, \bar{c}_{PF,1}, \bar{c}_{PF,2}\}$ where one denotes the first variable (real GDP) and two denotes the second variable (unemployment). These parameters are calibrated to match four moments: $\{\beta_{1,1}, \beta_{1,2}, \alpha_{1,1}, \alpha_{1,2}\}$.

Based on the calibration, the two state variables must only be simulated once. From this, I then simulate a panel of forecasters who select KF or PF depending on the mean square errors and their model adoption cost draw. Following the endogenous model selection decision, I simulate a panel of errors and revisions from which I then compute model-implied errors-on-revisions coefficients. I simulate a panel of forecasters roughly 7 times the size of the size of the panel of SPF forecasters and discard the first 1,000 observations of the simulated state variables.

I then collect the targeted empirical moments in a stacked vector $m(X)$ which comes from the SPF sample. I next stack the model-based moments, which depend on $\theta = (\beta_{1,1} \;\; \alpha_{1,1} \;\; \beta_{1,2} \;\; \alpha_{1,2})'$, in the vector $m(\theta)$. Finally I search the parameter space to find the $\widehat{\theta}$ that minimizes the following objective

$$\min_{\theta} \big(m(\theta) - m(X)\big)' W \big(m(\theta) - m(X)\big)$$

where the weighting matrix is set to be the identity matrix, $W = I$.

Appendix D Details on Particle Filtering

In this section I briefly summarize the particle filter which is a popular nonlinear filter that have been devised to handle state dynamics such as unobserved stochastic volatility.

In their seminal paper, Gordon et. al. (1993) propose the bootstrap filter which is a popular variant to the particle filter. In principle, this approach makes use to mass points (particles) to approximate the underlying filtering density, $p(s_t|Z_t^i)$. This is done by defining the set of particles and associated weights: $\chi = \{s^{(n)}, \omega^{(n)}\}_{n=1}^N$.

Importantly, the filter still follows a general predict-update algorithm. For each particle n, the forecaster propagates the estimate through the nonlinear system

$$s_t^{i,(n)} = F\big(s_{t-1}^{i(n)}, w_t\big)$$

and then updates the weight,[5]

$$\widetilde{\omega}_t^{i,(n)} = \omega_{t-1}^{i,(n)} \cdot p\big(z_t^i|s_t^{i,(n)}\big)$$

The forecaster then normalizes the weights

$$\omega_t^{i,(n)} = \frac{\widetilde{w}_t^{i,(n)}}{\sum_{n=1}^N \widetilde{\omega}_t^{i,(n)}}$$

so that they sum to one. Lastly, the nowcast of the state is computed as a weighted average of the particles

$$\widehat{s}_{t|t}^i = \sum_{n=1}^N s_t^{i,(n)} \cdot \omega_t^{i,(n)}.$$

One common issue with sequential importance sampling is that the sample of particles tends to degenerate as few particles are given most of the weight. As a result, I make use of the common sequential importance resampling scheme in which I resample the particles, each with a probability equal to its weight.

Forecast errors and revisions are analogous to the formulation with the Kalman filter generalizations. The only difference is that the particle filtered estimates are not formulated by making use of the Kalman filtering equations. Nonetheless, these estimates approximate the optimal forecast.

[5] The precise manner in which the weights are updated depends on choices for the importance distribution.