Notes on the Centroid of a Shield

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Abstract

The region to be analyzed is defined by a rectangular area one unit high and three units wide, surmounting an approximately triangular bottom area delineated by a pair of circular arcs of radius three. The centroid is a point along the axis of symmetry corresponding to the center of mass of a planar lamina of uniform density.

1 Introduction

The classical shape of the heraldic shield (called a "heater" shield because the shape resembles an old-fashioned flatiron[1]) is defined by a rectangular top region one unit high by three wide, surmounting an approximately triangular bottom region delineated by a pair of circular arcs of radius three [1].

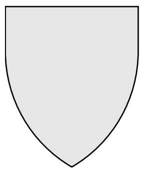

Figure 1: EPS figure.

To properly place charges on the field, it is necessary to locate the "middle" of the shield. Clearly, this lies along the axis of symmetry, but where, exactly, along a top-to-bottom line? Arguably the best choice is the centroid, corresponding to the center of gravity of a physical piece of armor [2].

2 Constructing the Shape

To draw a shield R units wide, begin with a horizontal line segment from $(-\frac{1}{2}R, -\frac{1}{3}R)$ to $(\frac{1}{2}R, \frac{1}{3}R)$. Draw a vertical line segment of length $\frac{1}{3}R$ from $(-\frac{1}{2}R, 0)$ to $(-\frac{1}{2}R, \frac{1}{3}R)$ and another from $(\frac{1}{2}R, 0)$ to $(\frac{1}{2}R, \frac{1}{3}R)$. Draw an arc

[1]The name is said to date from Victorian times[?], but I have not yet found a primary source to back up this assertion.

centered at $(-\frac{1}{2}R, 0)$ with radius R through an angle from 0 to $-\frac{\pi}{3}$, and another arc centered at $(\frac{1}{2}R, 0)$ with radius R from π to $\frac{3\pi}{2}$ (Figure 2).

The angle $\alpha = \frac{\pi}{3}$. The width of the shield is R and the bottom is the point $(0, -\frac{1}{2}\sqrt{3R^2})$.

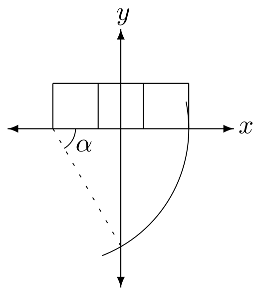

Figure 2: Construction

The arcs intersect at a point $(0, -\frac{1}{2}\sqrt{3R^2})$ which can be found by the circle formula $(x-h)^2 + (y-k)^2 = R^2$ with $h = -\frac{1}{2}R$ and $k = 0$. Then,

$$(x + \frac{1}{2}R)^2 = y^2 = R^2$$

$$x^2 + Rx + \frac{1}{4}R^2 + y^2 = R^2$$

$$y^2 = R^2 - x^2 - Rx - \frac{1}{4}R^2$$

$$y^2 = -z^2 - Rx + \frac{3}{4}R^2$$

and by the quadratic formula...

The y-intercept is $-\frac{1}{2}\sqrt{3R^2}$.

The width of the shield is R. The height of the shield is $\frac{1}{3}R + \frac{1}{2}\sqrt{3R^2}$.

3 Finding the Centroid

The centroid can be found most easily by decomposing the shield into pieces: the top rectangle, two triangles, and two circular segments (themselves decomposed into a circular sector and its inscribed triangle). For each piece, we need to find the centroid and area of that piece.

3.0.1 Rectangle

The centroid of the rectangular top section is located at the intersection of the diagonals: $(\bar{x}_r, \bar{y}_r) = (0, \frac{R}{2})$. The area of the rectangle $A_r = R$.

3.0.2 Triangles

The centroid of a right triangle lies one-third of way along the sides from the central vertex of the right angle (cite!). The height of the triangle is $h = \frac{\sqrt{3R^2}}{2}$ and its base is $b = \frac{R}{2}$; therefore $(\bar{x}_t, \bar{y}_t) = (\frac{R}{6}, \frac{\sqrt{3R^2}}{6})$ and $A_t = \frac{1}{2}R \cdot \frac{\sqrt{3R^2}}{2} = \frac{R\sqrt{3R^2}}{4} = \frac{R^2\sqrt{3}}{4} = \frac{1}{4}R^2\sqrt{3} = \frac{\sqrt{3}}{4}R^2$.

3.1 Circular Segments

The area of the circular segment is equal to the area of the circular sector minus the area of the inscribed triangle. The sector angle $\alpha = \frac{\pi}{3}$ and the area of the sector is $\pi R^2 \cdot \frac{\alpha}{2\pi} = \frac{1}{6}\pi R^2$.

The area of a triangle inscribed in the sector is $2 \cdot \frac{1}{2} \cdot \frac{R}{2} \cdot \ldots = \frac{\sqrt{3}R}{2}$, so the area of the circular segment is $A_s = \frac{1}{6}\pi R^2 - \frac{\sqrt{3}}{2}R^2 = (\frac{\pi}{6} - \frac{\sqrt{3}}{4})R^2$.

The centroid of a circular segment is found by

$$\langle y \rangle = \frac{2}{3}R^3 \sin^3 \frac{\theta}{2}$$

which leads to

$$\bar{y}_{cs} = \frac{\langle y \rangle}{A_c}$$

.

Transforming to polar coordinates, $(\rho, \theta) = (\bar{y}_{cs}, \frac{\pi}{6})$; transforming back to cartesian coordinates gives us:

$$\bar{x}_s = \rho \cos\theta = ?$$

and

$$\bar{y}_s = \rho \sin\theta = ?$$

.

3.2 Centroid

The centroid is computed from a weighted sum of the centroids of the components:

$$\bar{x} = \frac{\sum_n A_n \bar{x}_n}{\sum A_n} = 0$$

and

$$\bar{y} = \frac{\sum_n A_n \bar{y}_n}{\sum A_n} = \text{a\_big\_hairy\_expression}$$

.

As expected, the centroid lies on the axis of symmetry.

3.3 Specifying by Height

Often it is more convenient to specify the height of the shield. This is easy to transform. To construct a shield of height H, simply use the following relations:

$$A, B, C$$

4 Placing Charges on the Field

In Postscript, we have

```
newpath
0 120 moveto
0 86.67 lineto
100 86.67 100 180 240 arc
0 86.67 100 300 360 arc
100 120 lineto
closepath
gsave
  0.9 setgray fill
grestore
0 setgray stroke
showpage
```

References

[1] Classical heraldry: The shape of the shield. `http://www.baronage.co.uk/1999/herart01.html`.

[2] Tom A. Apostol and Mamikon A. Mnatsakanian. Finding centroids the easy way. *Math Horizons*, pages 7–12, September 2000.

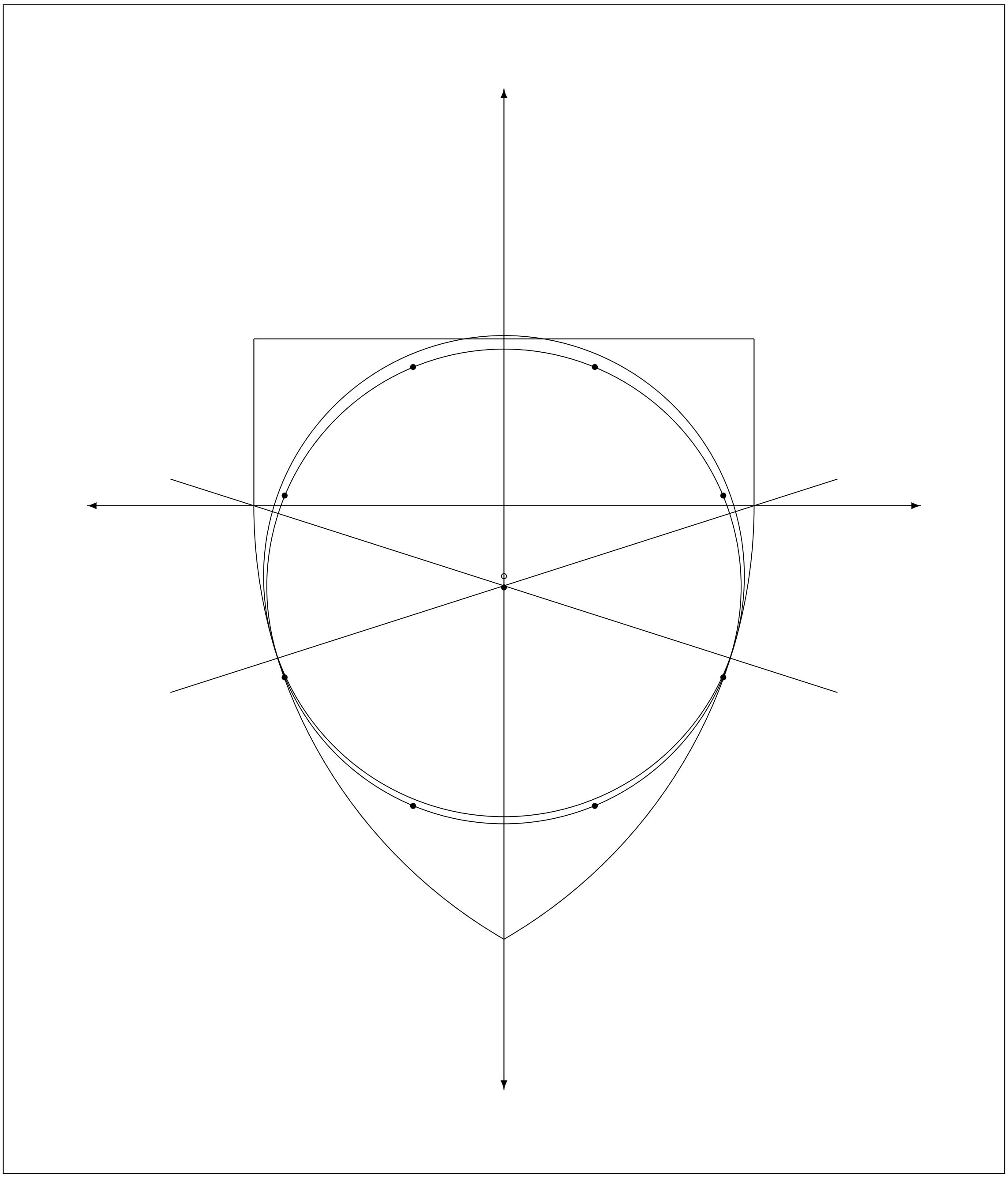

Figure 3: Centered on the origin. The diagonal lines indicate the position of the mutual tangents.

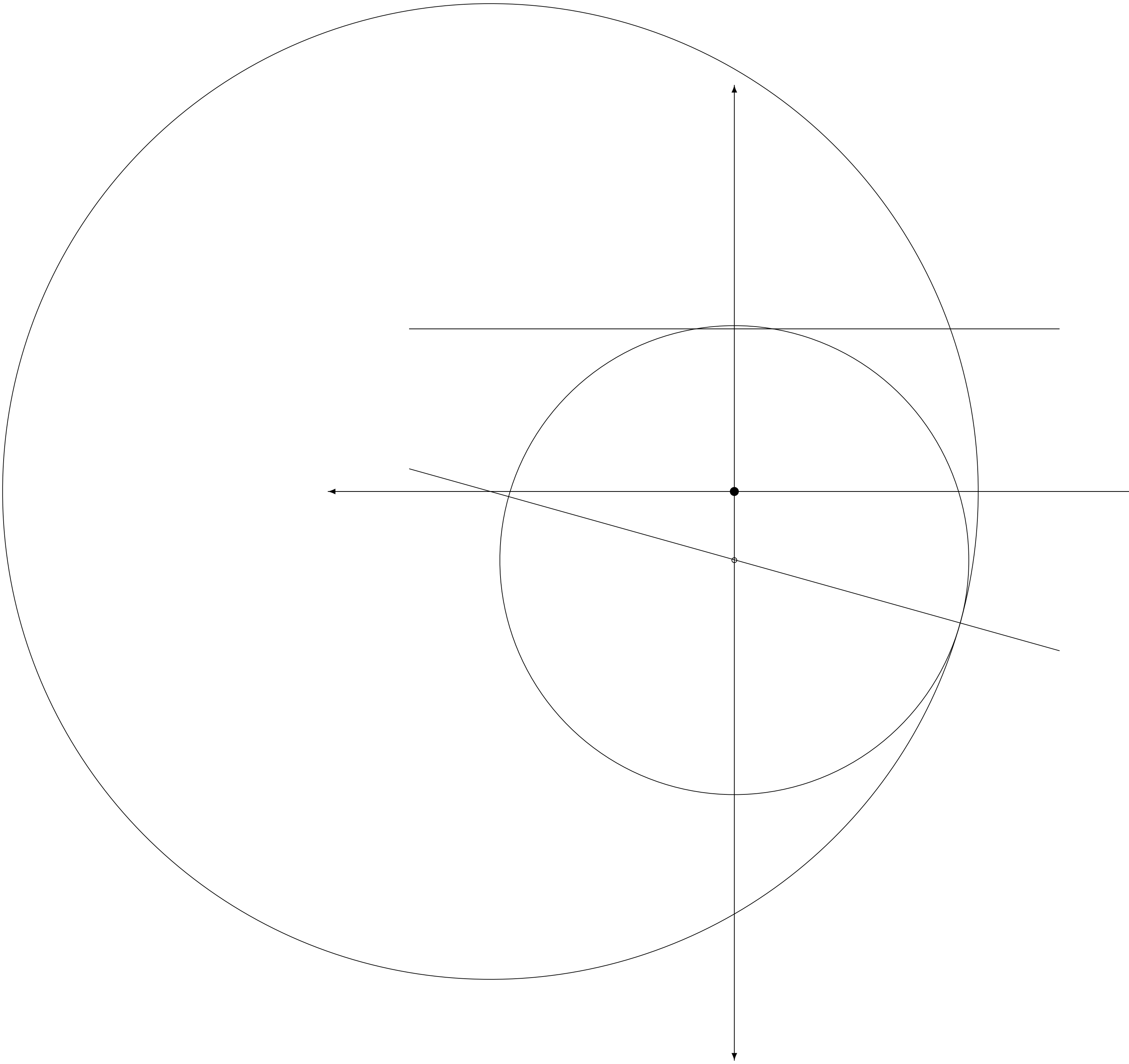

Figure 4: For figuring hypercentroid.

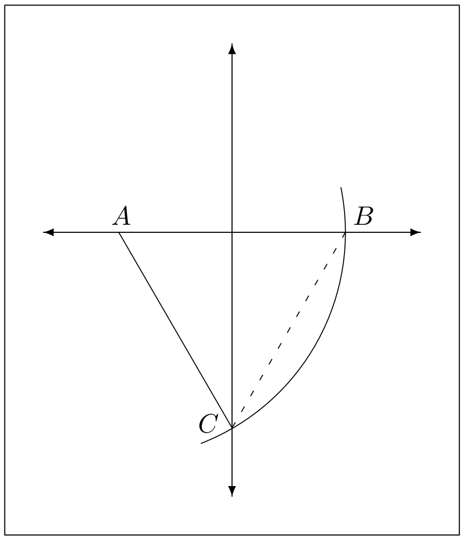

Figure 5: Circular Segment

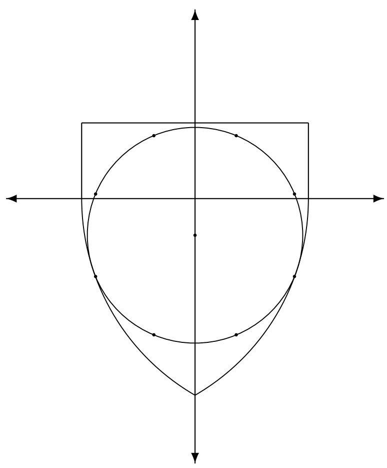

Figure 6: Location of tangent

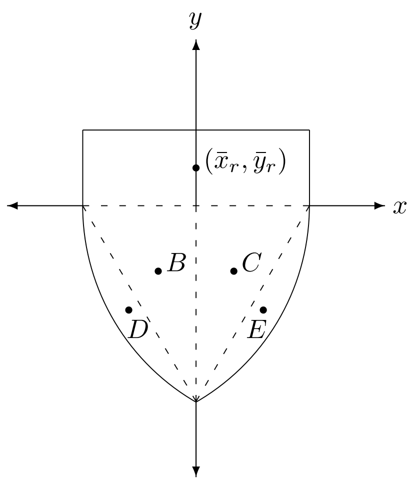

Figure 7: Finding the centroid