Exercise 1. Crank[1]

Let us consider the manipulator robot, or crank of Figure 1 (on the left).

https://www.youtube.com/watch?v=nLd-DyiNxLo

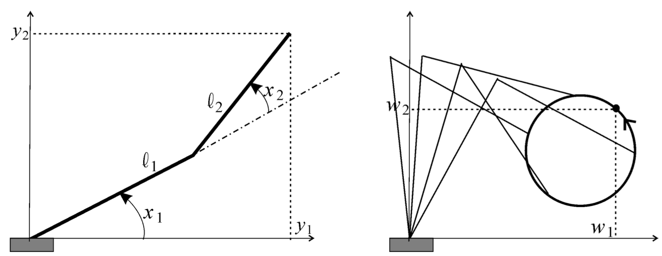

Figure 1: RR Robot manipulator.

This robot is composed of two links of length l_1 and l_2. Its two degrees of freedom denoted by x_1 and x_2 are represented in the figure. The inputs u_1, u_2 of the system are the angular speeds of the links (i.e., $u_1 = \dot{x}_1, u_2 = \dot{x}_2$). We will take as output the vector $y = (y_1, y_2)$ corresponding to the coordinates of the tip of the second link.

1) Give the state equations of the robot. We will take the state vector $\mathbf{x} = (x_1, x_2)^T$.

2) We would like y to follow a setpoint $\mathbf{w}$ describing a target circle (on the right of Figure 1). This setpoint satisfies:

$$\mathbf{w} = \mathbf{c} + r \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

   Give the expression of a control law that allows to perform this task. We will use a feedback linearization method and we will place the poles at -1.

3) Study the singularities of the control.

4) Let us consider the case $l_1 = l_2, \mathbf{c} = (3, 4)^T$ and $r = 1$. For which values of l_1 are we certain to be able to move freely on the target circle, without encountering singularities?

5) Write a program illustrating this control law.

[1]Adapted from https://www.ensta-bretagne.fr/jaulin/robmooc.pdf

Solution of Exercise 1.1

1) Give the state equations of the robot. We will take the state vector $\mathbf{x} = (x_1, x_2)^T$.

   The state equations of the crank are:

$$\begin{array}{rcl} \dot{x}_1 & = & u_1, \\ \dot{x}_2 & = & u_2, \\ y_1 & = & l_1 \cos x_1 + l_2 \cos(x_1 + x_2), \\ y_2 & = & l_1 \sin x_1 + l_2 \sin(x_1 + x_2). \end{array}$$

2) We would like y to follow a setpoint $\mathbf{w}$ describing a target circle (on the right of Figure 1). This setpoint satisfies:

$$\mathbf{w} = \mathbf{c} + r \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

   Give the expression of a control law that allows to perform this task. We will use a feedback linearization method and we will place the poles at -1.

   By differentiating the output, we obtain:

$$\begin{array}{rcl} \dot{y}_1 & = & -l_1 \dot{x}_1 \sin x_1 - l_2(\dot{x}_1 + \dot{x}_2) \sin(x_1 + x_2) \\ & = & -l_1 u_1 \sin x_1 - l_2(u_1 + u_2) \sin(x_1 + x_2), \\ \dot{y}_2 & = & l_1 \dot{x}_1 \cos x_1 + l_2(\dot{x}_1 + \dot{x}_2) \cos(x_1 + x_2) \\ & = & l_1 u_1 \cos x_1 + l_2(u_1 + u_2) \cos(x_1 + x_2). \end{array}$$

   Thus,

$$\dot{\mathbf{y}} = \begin{pmatrix} -l_1 \sin x_1 - l_2 \sin(x_1 + x_2) & -l_2 \sin(x_1 + x_2) \\ l_1 \cos x_1 + l_2 \cos(x_1 + x_2) & l_2 \cos(x_1 + x_2) \end{pmatrix} \mathbf{u} + \mathbf{0}.$$

   This means that

$$\begin{array}{rcl} \mathbf{A}(\mathbf{x}) & = & \begin{pmatrix} -l_1 \sin x_1 - l_2 \sin(x_1 + x_2) & -l_2 \sin(x_1 + x_2) \\ l_1 \cos x_1 + l_2 \cos(x_1 + x_2) & l_2 \cos(x_1 + x_2) \end{pmatrix}, \\ \mathbf{b}(\mathbf{x}) & = & \mathbf{0}, \end{array}$$

   and that

$$\mathbf{R} = \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix},$$

   which is balanced. We take $\mathbf{u} = \mathbf{A}^{-1}(\mathbf{x})\,\mathbf{v}$ to have two decoupled integrators. We then choose the proportional controller:

$$
\begin{aligned}
\mathbf{v} &= (\mathbf{w}-\mathbf{y})+\dot{\mathbf{w}} \\
&= \mathbf{c}+r\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}-\begin{pmatrix}l_1\cos x_1+l_2\cos(x_1+x_2)\\ l_1\sin x_1+l_2\sin(x_1+x_2)\end{pmatrix}+r\begin{pmatrix}-\sin t\\ \cos t\end{pmatrix},
\end{aligned}
$$

which places all the poles at -1.

```
clear all close all clc

syms l1 l2 x1 x2

A = [-l1 * sin(x1) - l2* sin(x1 +x2), - l2 *sin(x1 + x2); l1 *cos(x1) + l2 *cos(x1 +x2), l2 *cos(x1 + x2)];

detA = simplify(det(A))

adjA = adjoint(A)

invA = adjA/detA

invA = simplify(inv(A))
```

We have that $\mathbf{A}(\mathbf{x})^{-1}=\frac{1}{\det\mathbf{A}(\mathbf{x})}\operatorname{adj}\mathbf{A}(\mathbf{x})$ with

```
detA =

l1*l2*sin(x2)


adjA =

[              l2*cos(x1 + x2),               l2*sin(x1 + x2)]
[- l2*cos(x1 + x2) - l1*cos(x1), - l2*sin(x1 + x2) - l1*sin(x1)]


invA =

[                    cos(x1 + x2)/(l1*sin(x2)),                      sin(x1 + x2)/(l1*sin(x2))]
[-(l2*cos(x1 + x2) + l1*cos(x1))/(l1*l2*sin(x2)), -(l2*sin(x1 + x2) + l1*sin(x1))/(l1*l2*sin(x2))]


invA =

[                    cos(x1 + x2)/(l1*sin(x2)),                      sin(x1 + x2)/(l1*sin(x2))]
[-(l2*cos(x1 + x2) + l1*cos(x1))/(l1*l2*sin(x2)), -(l2*sin(x1 + x2) + l1*sin(x1))/(l1*l2*sin(x2))]
```

3) Study the singularities of the control.

Since

$$\det\mathbf{A}(\mathbf{x})=l_1l_2\sin x_2.$$

This determinant is equal to zero if $l_1l_2\sin x_2=0$. This happens if $x_2=k\pi,\ k\in\mathbb{Z}$, where $\mathbb{Z}=\{\ldots,-2,-1,0,1,2,\ldots\}$ is the set of integer numbers.

4) Let us consider the case $l_1=l_2$, $\mathbf{c}=(3,4)^T$ and $r=1$. For which values of l_1 are we certain to be able to move freely on the target circle, without encountering singularities?

Let $l_1 = l_2 = l$. We have a singularity if $\sin x_2 = 0$. Thus, either both links are folded up (and therefore y is not on the circle) or both links are stretched out. In the latter case (which is of interest to us) the point $\mathbf{y}$ is on the circle of radius $2l$ that intersects the target circle if:

$$2l \in \left[-1 + \sqrt{4^2 + 3^2}, \sqrt{4^2 + 3^2} + 1\right] = [5 - 1, 5 + 1] = [4, 6]$$

where $\sqrt{4^2 + 3^2}$ corresponds to the distance of the center of the circle to the origin. Thus, the circle is outside the workspace of the manipulator if $l < 2$. We will have a singularity on the circle if $l \in [2, 3]$. If we wish to move freely on the circle, we need to choose $l > 3$.

5) Write a program illustrating this control law.

Solve