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Extension Problems Related to the Higher Order Fractional Laplacian

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Abstract Caffarelli and Silvestre [*Comm. Part. Diff. Eqs.*, **32**, 1245–1260 (2007)] characterized the fractional Laplacian $(-\Delta)^s$ as an operator maps Dirichlet boundary condition to Neumann condition via the harmonic extension problem to the upper half space for $0 < s < 1$. In this paper, we extend this result to all $s > 0$. We also give a new proof to the dissipative *a priori* estimate of quasi-geostrophic equations in the framework of L^p norm using the Caffarelli–Silvestre's extension technique.

Keywords Fractional Laplacian, quasi-geostrophic equations, energy equality

MR(2010) Subject Classification 42B37, 35P30

1 Introduction

As is well known, the fractional Laplacian $(-\Delta)^s$ is a nonlocal operator, the non-locality of $(-\Delta)^s$ makes it difficult to investigate. One best way to characterize it is introduced by Caffarelli and Silvestre [3] for the case $0 < s < 1$, in which they obtained the characterizations for fractional Laplacian $(-\Delta)^s$ from solving a harmonic extension problem to the upper half space as the weighted operator that maps the Dirichlet boundary condition to the Neumann condition. More clearly, the extension function u introduced by them solves the following equation:

$$\begin{cases} \mathrm{div}(y^{1-2s}\nabla u) = 0, & (x,y) \in \mathbb{R}^{n+1}_+, \\ u(x,0) = f(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

and satisfies

$$-C_s \lim_{y\to 0^+} y^{1-2s}\partial_y u = (-\Delta)^s f.$$

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Moreover, it holds that

$$C_s \int_0^\infty \int_{\mathbb{R}^n} |\nabla u|^2 y^{1-2s} dxdy = \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi \tag{1.2}$$

for some positive constant C_s.

The significant contribution of above extension problem is to provide a way of applying the classical analysis method to the partial differential equations containing fractional Laplacian terms. And a series of important results have been obtained based on this extension technique. For instance, Caffarelli and Vasseur [4] showed that the solutions of drift diffusion equations with L^2 initial data are locally Hölder continuous under some minimal assumptions on the drift. Especially, as an application, the solutions of quasi-geostrophic equations are locally smooth for any space dimensions. Also the characterizations for fractional Laplacian operator [3] allow us to define the suitable weak solutions similar to that in 3D classical Navier–Stokes equations. Hence, the result of partial regularity obtained by Caffarelli, Kohn and Nirenberg [2] can be generalized to the fractional case. More precisely, Tang and Yu [15] studied the subcritical case with $\frac{3}{4} < s < 1$. And the critical case of $s = \frac{3}{4}$ was settled down in [7] by the authors. For MHD equations, we refer to [13] for the details.

When $0 < s < 1$, Caffarelli–Silvestre's extension method is successfully applied to many important fluid dynamic models, just as described above. While their harmonic extension function can not deal with the case $s > 1$. Thus, it is natural to consider the higher order case. As far as we know, there are a few results concerning this problem. At first, Chang and González in [6] showed some interesting connections between the extension and scattering theory in conformal geometry. And they generalized Caffarelli–Silvestre's result to $s \in (0, \frac{n}{2}]$ when n is odd and to all non-integer orders when n is even. Later on, Yang [16] generalized Chang and González's result to all non-integer s without the constraint assumption on the dimensions. The extension equations of Yang are a class of high order elliptic equations and the proof mainly depends on Caffarelli–Silvestre's energy equality (1.2).

In this paper, we try to generalize the result of Caffarelli and Silvestre [3] to the case of any positive orders including integers by solving a more general class of partial differential equations. Different from Yang's extension [16], we only use higher order derivatives with respect to variables x, while for the one extra dimension, we only use second order derivatives. Our extension method can be applied to the study of the fluid equations with higher order fractional Laplacian $(-\Delta)^s$ when $s > 1$ naturally. For example, we may use this extension technique to improve the regularity of suitable weak solutions for the Navier–Stokes equations by local energy estimate instead of the Littlewood–Paley method, which is investigated by Katz and Pavlović in [11]. Based on the scaling property of the constructed equations, we can just follow the same strategy of Cafferelli–Silvestre to prove our main result.

Before stating our main result, we firstly define the weighted Sobolev Space X^s_β as the set of functions $u \in \mathcal{S}'(\mathbb{R}^{n+1}_+)/\mathcal{P}(\mathbb{R}^{n+1}_+)$ with finite norm

$$\|u\|^2_{X^s_\beta} = \int_0^\infty \int_{\mathbb{R}^n} (|\partial_y u|^2 + |D^\beta u|^2) y^{1-\frac{2s}{\beta}} dxdy.$$

Here, $D \triangleq \sqrt{(-\Delta)}$.

With above preparations, we are arriving at the main results.

Theorem 1.1 *Suppose that $f \in \dot{H}^s(\mathbb{R}^n)$ with $s > 0$, we can find a function $u \in X^s_\beta$ with $\beta > s$ by solving the following Dirichlet problem*

$$\begin{cases} \partial_y(y^{1-\frac{2s}{\beta}}\partial_y u) = y^{1-\frac{2s}{\beta}} D^{2\beta} u, & (x,y) \in \mathbb{R}^{n+1}_+, \\ u(x,0) = f(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.3}$$

such that

$$-C_{\beta,s} \lim_{y\to 0^+} y^{1-\frac{2s}{\beta}} \partial_y u = (-\Delta)^s f \tag{1.4}$$

for some positive constant $C_{\beta,s}$.

Moreover, the extension function u minimizes the functional

$$J(v) = \int_0^\infty \int_{\mathbb{R}^n} (|\partial_y v|^2 + |D^\beta v|^2) y^{1-\frac{2s}{\beta}} dx dy, \tag{1.5}$$

and it holds that

$$C_{\beta,s} J(u) = \|f\|^2_{\dot{H}^s}. \tag{1.6}$$

Remark 1.2 The norm $\|\cdot\|_{X^s_\beta}$ is compatible with the boundary conditon in (1.3) under the condition $1 - \frac{2s}{\beta} > -1$. Otherwise, $\int_{\mathbb{R}^n}\int_0^\delta |D^\beta v|^2 y^{1-\frac{2s}{\beta}} dy dx$ is not well defined for any $\delta > 0$. Especially, when s is non-integer and $\beta = [s] + 1$ (here we use the Gauss symbol, i.e., $[s]$ is the greatest integer that is less than or equal to s), from (1.4), we can characterize the nonlocal fractional Laplacian via the normal derivative of the extension to the upper half space.

Remark 1.3 Obviously, when $\beta = 1$, our extension (1.3) is nothing but Caffarelli–Silvestre's harmonic extension in [3].

This extension method is expected to have more applications. Using Caffarelli–Silvestre's extension technique, we show a new proof to the important dissipative *a priori* estimate of the following quasi-geostrophic equations:

$$\begin{cases} \theta_t + u \cdot \nabla\theta + (-\Delta)^s \theta = 0, & x \in \mathbb{R}^2, t > 0, \\ \theta(0,x) = \theta_0(x), \end{cases} \tag{1.7}$$

where s is a fixed real number satisfying $0 \le s \le 1$. $\theta(t,x)$ is a real-valued function representing the potential temperature and u the velocity formulated by $u = \nabla \times (-\Delta)^{-\frac{1}{2}}\theta = (-R_2\theta, R_1\theta)$, in which R_i, $i = 1, 2$ is the Riesz operator.

The following result is well known

Proposition 1.4 *Suppose that $\theta_0 \in L^p$ with $2 \le p < \infty$ and θ is a smooth function satisfying equation* (1.7)*. Then, it holds that*

$$\|\theta(t)\|_{L^p(\mathbb{R}^2)} + \|D^s|\theta|^{\frac{p}{2}}\|_{L^2([0,t]\times\mathbb{R}^2)} \le \|\theta_0\|_{L^p(\mathbb{R}^2)}. \tag{1.8}$$

Remark 1.5 The maximum principle $\|\theta(t)\|_{L^p} \le \|\theta_0\|_{L^p}$ has been proved by Resnick [14], Córdoba and Córdoba [9]. The dissipative term $\|D^s|\theta|^{\frac{p}{2}}\|_{L^2([0,t]\times\mathbb{R}^2)}$ in (1.8) is firstly given by Córdoba and Córdoba [9] for $p = 2^n$. Later on, Ju [10] generalized it to any $p \ge 2$. By Littlewood–Paley theory, for each dyadic block, Chen et al. [8] characterized the dissipative term more precisely, i.e., $\|D^s|\Delta_j\theta|^{\frac{p}{2}}\|_{L^2} \sim 2^{sj}\|\Delta_j\theta\|_{L^p}$. In [12], Li got the same dissipative estimate as [8] by a different method.

In fact, the spirit of Theorem 1.1 can be applied to the anisotropic differential operator $T=\prod_{i=1}^n D_i^{s_i}$. Here, we use the notation $D_i \triangleq \sqrt{-\partial_i^2}$. More precisely, denote operator $G=\prod_{i=1}^n D_i^{\beta_i}$ with $\frac{s_i}{\beta_i}=\frac{s_j}{\beta_j}, 1\le i,j\le n$, we have the following theorem:

Theorem 1.6 *Suppose that $Tf\in L^2(\mathbb{R}^n)$, $s=\sum_{i=1}^n s_i$ and $\beta=\sum_{i=1}^n \beta_i > s$, we can find a function u by solving the following Dirichlet problem*

$$\begin{cases} \partial_y(y^{1-\frac{2s}{\beta}}\partial_y u)=y^{1-\frac{2s}{\beta}}G^2u, & (x,y)\in\mathbb{R}_+^{n+1},\\ u(x,0)=f(x), & x\in\mathbb{R}^n, \end{cases} \tag{1.9}$$

such that

$$-C_{\beta,s}\lim_{y\to 0^+} y^{1-\frac{2s}{\beta}}\partial_y u=T^2f$$

for some positive constant $C_{\beta,s}$.

Moreover, the extension function u minimizes the functional

$$J(v)=\int_0^\infty\int_{\mathbb{R}^n}(|\partial_y v|^2+|Gv|^2)y^{1-\frac{2s}{\beta}}dxdy, \tag{1.10}$$

and it holds that

$$C_{\beta,s}J(u)=\|Tf\|_{L^2(\mathbb{R}^n)}^2. \tag{1.11}$$

Before ending this section, we arrange this paper as follows: in Section 2, we give the proof of Theorems 1.1 and 1.6 by the arguments of energy equality. The *a priori* estimate of quasi-geostrophic equations is given in Section 3.

2 Proofs of Theorems 1.1 and 1.6

To characterize the fractional Laplacian $(-\Delta)^s$ for arbitrary $s>0$, we must solve the higher order partial differential equations (1.3). By the scaling property of (1.3) and higher order derivatives with respect to x, we can treat it as a second order ODE by taking Fourier transform in x.

Indeed, following the strategy of Caffarelli and Silvestre [3], we only need to show the following corresponding energy functionals coincide

$$C_{\beta,s}\int_0^\infty\int_{\mathbb{R}^n}(|\partial_y u|^2+|D^\beta u|^2)y^{1-\frac{2s}{\beta}}dxdy=\int_{\mathbb{R}^n}|\xi|^{2s}|\hat{f}(\xi)|^2d\xi. \tag{2.1}$$

Taking Fourier transform in x variable on u of (1.3), then $\widehat{u}(\xi,y)$ satisfies the following ODE

$$\partial_y(y^{1-\frac{2s}{\beta}}\partial_y\hat{u})=y^{1-\frac{2s}{\beta}}|\xi|^{2\beta}\hat{u}, \tag{2.2}$$

where $\widehat{u}(\xi,y)=\int_{\mathbb{R}^n}\mathrm{e}^{-\mathrm{i}x\cdot\xi}u(x,y)dx$. Then, denoting $\hat{u}(\xi,y)=\hat{f}(\xi)\phi(|\xi|^\beta y)$, by the scaling property of equations (2.2), ϕ satisfies

$$\begin{cases} \phi''+\dfrac{1-\frac{2s}{\beta}}{z}\phi'-\phi=0,\\ \phi(0)=1,\\ \phi(z)\to 0, \quad z\to+\infty, \end{cases} \tag{2.3}$$

where $z=|\xi|^\beta y$, and the corresponding energy functional is

$$J(\phi)=\int_0^\infty(|\phi'|^2+|\phi|^2)z^{1-\frac{2s}{\beta}}dz. \tag{2.4}$$

The existence and uniqueness of the minimizer of functional (2.4) can be easily derived from the standard arguments, see [1] for instance.

Thus, the energy of function u satisfies

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^n} (|\partial_y u|^2 + |D^\beta u|^2) y^{1-\frac{2s}{\beta}} dx dy &= \int_0^\infty \int_{\mathbb{R}^n} (|\partial_y \hat{u}|^2 + |\xi|^{2\beta} |\hat{u}|^2) y^{1-\frac{2s}{\beta}} d\xi dy \\ &= \int_0^\infty \int_{\mathbb{R}^n} |\hat{f}|^2 |\xi|^{2\beta} (|\phi'(|\xi|^\beta y)|^2 + |\phi(|\xi|^\beta y)|^2) y^{1-\frac{2s}{\beta}} d\xi dy \\ &= \int_0^\infty \int_{\mathbb{R}^n} |\hat{f}|^2 |\xi|^{2s} (|\phi'(z)|^2 + |\phi(z)|^2) z^{1-\frac{2s}{\beta}} d\xi dz \\ &= J(\phi) \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi.\end{aligned}$$

This shows the energy equality (2.1). To obtain the characterization (1.4) of fractional Laplacian, we can easily see that the left hand side of above is the Euler–Lagrange equation of (1.3) and the right hand side is the fractional harmonic equation of order s. Therefore, we complete the proof of Theorem 1.1.

For the proof of Theorem 1.6, taking Fourier transform in x variable on u of (1.9), $\hat{u}(\xi, y)$ satisfies the following ODE

$$\partial_y (y^{1-\frac{2s}{\beta}} \partial_y \hat{u}) = y^{1-\frac{2s}{\beta}} \prod_{i=1}^n |\xi_i|^{2\beta_i} \hat{u}. \tag{2.5}$$

Then, denoting $\hat{u}(\xi, y) = \hat{f}(\xi)\phi(\prod_{i=1}^n |\xi_i|^{\beta_i} y)$, by the scaling property of equation (2.5), ϕ satisfies

$$\begin{cases} \phi'' + \dfrac{1-\frac{2s}{\beta}}{z}\phi' - \phi = 0, \\ \phi(0) = 1, \\ \phi(z) \to 0, \quad z \to +\infty, \end{cases} \tag{2.6}$$

where $z = \prod_{i=1}^n |\xi_i|^{\beta_i} y$, and the corresponding energy functional is

$$J(\phi) = \int_0^\infty (|\phi'|^2 + |\phi|^2) z^{1-\frac{2s}{\beta}} dz. \tag{2.7}$$

Now, we can just perform the same procedure as that in the proof of Theorem 1.1. We omit the proof here. Thus, the proof of Theorem 1.6 is completed.

Remark 2.1 In fact, equation (2.3) is the Bessel equation. Using Frobenius's method, two independent solutions can be given by the modified Bessel functions. Then another approach to the main result can also come from the simple analysis of the modified Bessel functions, see [5] for instance.

3 Application to Quasi-geostrophic Equations

In this section, we will present a new and elementary proof to Proposition 1.4 by Caffarelli–Silvestre's extension technique.

Proof When $s = 0$ or $s = 1$, the result is well known. Thus, we only focus on $0 < s < 1$. At first, multiplying both sides of the equation (1.7) by $p|\theta|^{p-2}\theta$ and using the divergence free property of u, we have

$$\frac{d}{dt}\int_{\mathbb{R}^2}|\theta|^p dx + p\int_{\mathbb{R}^2}(-\Delta)^s\theta|\theta|^{p-2}\theta dx = 0. \tag{3.1}$$

Let $\beta = 1$ in Theorem 1.1 and denote the extension function of $\theta(t,x)$ by $\tilde{\theta}(t,x,y)$. The second term on the left hand side of (3.1) becomes

$$\begin{aligned} p\int_{\mathbb{R}^2}(-\Delta)^s\theta|\theta|^{p-2}\theta dx &= -C_s p\int_{\mathbb{R}^2}|\theta|^{p-2}\theta \lim_{y\to 0^+} y^{1-2s}\partial_y\tilde{\theta}dx \\ &= -C_s p\int_{\mathbb{R}^2}\lim_{y\to 0^+}(y^{1-2s}\partial_y\tilde{\theta}|\tilde{\theta}|^{p-2}\tilde{\theta})dx \\ &= C_s p\int_0^\infty\int_{\mathbb{R}^2}\partial_y(y^{1-2s}\partial_y\tilde{\theta}|\tilde{\theta}|^{p-2}\tilde{\theta})dxdy. \end{aligned}$$

Then by (1.3), the above equals

$$\begin{aligned} &C_s p\int_0^\infty\int_{\mathbb{R}^2} y^{1-2s}(-\Delta)\tilde{\theta}|\tilde{\theta}|^{p-2}\tilde{\theta} + y^{1-2s}\partial_y\tilde{\theta}\partial_y(|\tilde{\theta}|^{p-2}\tilde{\theta})dxdy \\ &\quad = C_s p\int_0^\infty\int_{\mathbb{R}^2} y^{1-2s}(|\nabla_x\tilde{\theta}|^2|\tilde{\theta}|^{p-2} + (p-2)|\nabla_x|\tilde{\theta}||^2|\tilde{\theta}|^{p-2}) \\ &\qquad + y^{1-2s}(|\partial_y\tilde{\theta}|^2|\tilde{\theta}|^{p-2} + (p-2)|\partial_y|\tilde{\theta}||^2|\tilde{\theta}|^{p-2})dxdy \\ &\quad = C_s p(p-1)\int_0^\infty\int_{\mathbb{R}^2} y^{1-2s}(|\nabla_x|\tilde{\theta}||^2|\tilde{\theta}|^{p-2} + |\partial_y|\tilde{\theta}||^2|\tilde{\theta}|^{p-2})dxdy \\ &\quad = C_s\frac{4(p-1)}{p}\int_0^\infty\int_{\mathbb{R}^2} y^{1-2s}(|\nabla_x|\tilde{\theta}|^{\frac{p}{2}}|^2 + |\partial_y|\tilde{\theta}|^{\frac{p}{2}}|^2)dxdy. \end{aligned}$$

Let ω be the extension function of $|\theta|^{\frac{p}{2}}$. Due to the property of minimizer (1.5) and the energy equality (1.6), we have

$$\begin{aligned} p\int_{\mathbb{R}^2}(-\Delta)^s\theta|\theta|^{p-2}\theta dx &= -C_s p\int_{\mathbb{R}^2}|\theta|^{p-2}\theta\lim_{y\to 0^+} y^{1-2s}\partial_y\tilde{\theta}dx \\ &\geq C_s\int_0^\infty\int_{\mathbb{R}^2} y^{1-2s}(|\nabla_x|\tilde{\theta}|^{\frac{p}{2}}|^2 + |\partial_y|\tilde{\theta}|^{\frac{p}{2}}|^2)dxdy \\ &\geq C_s\int_0^\infty\int_{\mathbb{R}^2} y^{1-2s}|\nabla_x\omega|^2 + |\partial_y\omega|^2 dxdy \\ &= \||\theta|^{\frac{p}{2}}\|^2_{\dot{H}^s}. \end{aligned}$$

Here we use the fact $\frac{4(p-1)}{p} > 1$ for $p \geq 2$. This completes the proof of Proposition 1.4.

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