A nonlinear splitting algorithm for preserving asymptotic features of stochastic singular differential equations

Joshua Lee Padgett

Department of Mathematics and Statistics
Texas Tech University

Joint Mathematics Meeting 2020 — Denver, Colorado

Table of Contents

Table of Contents

Introduction

Table of Contents

A Simple Motivation

The goal of this section is to simply provide a brief outline of this new method for generating adaptive time steps. The goal is to develop an algorithm that allows for the recovery of the asymptotic features of the original problem. So, let's consider this model test problem:

$$\frac{du}{dt} = (1-u)^{-1}, \quad 0 < t < T, \tag{2.1}$$

$$u(0) = 0. \tag{2.2}$$

In order to solve (2.1)-(2.2), we are going to find the Lie symmetries of the solution and use these to construct integrators which adapt in time!

How Do We Generate These Adaptive Time-Steps?

However, as is, we cannot find them. We first make the change of variables

$$w(t) := 1 - u(t).$$

This yields the new equation

$$\frac{dw}{dt} = -w^{-1}, \quad 0 < t < T, \tag{2.3}$$

$$w(0) = 1. \tag{2.4}$$

Note that this problem has the following solution:

$$w(t) = \sqrt{1 - 2t},$$

So, it becomes zero in finite time (singular) and it's derivative become unbounded in this situation. This will serve as our model problem.

A Simple Motivation

Next, we let

$$t \mapsto \lambda t \qquad w \mapsto \lambda^{-\beta} w$$

and (after plugging into the differential equation) we obtain

$$\lambda^{1-\beta} w' = \lambda^{\beta}(-w)^{-1}. \tag{2.5}$$

The only way that the equation in (2.4) is invariant (under such time scaling) is if both sides of (2.5) are equal. Hence, we need $\beta = 1/2$.

> **Goal:** *Construct a numerical algorithm that respects the time-scale symmetry given by $\beta = 1/2$.*

We know that we want to create an adaptive method, so we consider the auxiliary problem

$$\frac{dt}{d\tau} = g(w), \qquad t(0) = 0. \tag{2.6}$$

A Simple Motivation

Where g (in (2.6)) satisfies $g(\lambda^{-1/2}w) = \lambda g(w)$ (further preserving the symmetry of the nonlinear term).

We have far too many choices, here, so we make the simplest one:

$$g(w) := w^2,$$

which clearly satisfies the identity. So, we now have a larger coupled system to solve:

$$\frac{dw}{d\tau} = -w, \quad 0 < t < T, \tag{2.7}$$

$$\frac{dt}{d\tau} = w^2, \quad \tau > 0, \tag{2.8}$$

$$w(0) = 1, \tag{2.9}$$

$$t(0) = 0. \tag{2.10}$$

A Simple Motivation

While the above may look tedious, it is worth noting that the original differential problem has been reduced to a linear problem. The nonlinearity is now on the time-stepping function, but this can be desirable if our primary goal is to obtain accurate solution profiles. Let's see how this works:

- We know that the solution to (2.3)-(2.4) *quenches* at time $t = 1/2$.
- The solution monotonically decreases until the value of zero is reached.
- The time derivatives goes to negative infinity as $t \to 1/2^-$.

So, let's see if this new approach recovers these facts.

A Simple Motivation

Let's test things by using the simplest (forward Euler) approximation to the problem and study our results.

First, since τ is our only *independent* variable, now, we discretize it into M evenly spaced intervals of uniform size. Thus, on the "τ" grid, we would have

$$\tau_m = \tau_{m-1} + \Delta\tau = m\Delta\tau$$

The "loose" assumption is that this uniform computational grid is chosen well and will not introduce further problems. So, forward Euler yields:

$$\frac{dw}{d\tau} = -w \implies w^{n+1} = w^n - \Delta\tau w^n = (1-\Delta\tau)^{n+1}$$

So, the numerical solution takes the form

$$w^{n+1} = (1-\Delta\tau)^{n+1}.$$

A Simple Motivation

Note the following:

i. $\lim_{n\to\infty} w^{n+1} = 0$, which is desired.

ii. $\Delta\tau$ affects how quickly or slowly we get there — thus, there are still decisions to be made for correct recovery of the model (it can sometimes be out of our hands).

So, with little effort, we are guaranteed to recover *at least* one of the qualitative asymptotic features of the true solution.

A Simple Motivation

Now, let's look at the other equation. Using forward Euler, we have

$$t^{n+1} = t^n + \Delta\tau (w^n)^2.$$

Once again, we iterate and obtain

$$t^{n+1} = t^n + \Delta\tau (w^n)^2 = t^n + \Delta\tau (1-\Delta\tau)^{2n} = \Delta\tau \sum_{k=0}^{n+1} (1-\Delta\tau)^{2k}.$$

This representation for the phase-space time is a geometric series, and we have

$$\lim_{n\to\infty} t^{n+1} = \Delta\tau \sum_{k=0}^{\infty} (1-\Delta\tau)^{2k} = \frac{\Delta\tau}{1-|1-\Delta\tau|^2} = \frac{1}{2-\Delta\tau},$$

as long as $\Delta\tau$ is reasonably small.

A Simple Motivation

Thus, we have a result that is quite nice!

Moreover, since the above (meaning original problem's) quenching time is $1/2$ we have recovered this result with order one (just like the order of our forward Euler method).

Finally, one can show that such a method preserves the positivity and monotonicity of the solution (with no extra assumptions).

- Convergence proofs based on such time-stepping algorithms open many doors for improved analysis.
- While not shown here, such methods also recover the profile of the derivative of the solution (w_t solve a *blow-up* problem, and this method recovers its qualitative features nicely).

Table of Contents

A Nonlinear Stochastic Problem

Table of Contents

A Nonlinear Stochastic Splitting Algorithm

Table of Contents

Conclusions and Future Endeavors

THANK YOU!

Questions?