A nonlinear splitting algorithm for preserving asymptotic features of stochastic singular differential equations

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Introduction

The goal of this talk is to outline some preliminary results on preserving singular asymptotic structures of nonlinear stochastic problems.

These results are achieved by combining a simple technique which exploits the Lie symmetries of a differential equation (if they exist) and combines them with nonlinear operator splitting techniques.

My particular interest stems from a need to solve the so-called *quenching* problem accurately and efficiently. These problems introduce unique issues that require carefully constructed algorithms in order to guarantee the development of meaningful results.

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A Simple Motivation

The goal of this section is to simply provide a brief outline of this new method for generating adaptive time steps. This method allows for the development of an algorithm which recovers the asymptotic features of the original problem.

So, let's consider this model test problem:

$$\frac{du}{dt} = (1-u)^{-1}, \quad 0 < t < T, \tag{2.1}$$

$$u(0) = 0. \tag{2.2}$$

In order to solve (2.1)-(2.2), we are going to exploit particular symmetries of the solution and use these to construct integrators which adapt in time!

How Do We Generate These Adaptive Time-Steps‘

However, as is, we cannot find them. We first make the change of variables

$$w(t) := 1 - u(t).$$

This yields the new equation

$$\frac{dw}{dt} = -w^{-1}, \quad 0 < t < T, \tag{2.3}$$

$$w(0) = 1. \tag{2.4}$$

Note that this problem has the following solution:

$$w(t) = \sqrt{1 - 2t},$$

So, it becomes zero in finite time (singular) and it’s derivative become unbounded in this situation. This will serve as our model problem.

A Simple Motivation

Next, we let

$$t \mapsto \lambda t \qquad w \mapsto \lambda^{-\beta} w$$

and (after plugging into the differential equation) we obtain

$$\lambda^{1-\beta} w' = \lambda^{\beta}(-w)^{-1}. \tag{2.5}$$

The only way that the equation in (2.4) is invariant (under such time scaling) is if both sides of (2.5) are equal. Hence, we need $\beta = 1/2$.

> **Goal:** *Construct a numerical algorithm that respects the time-scale symmetry given by* $\beta = 1/2$.

We know that we want to create an adaptive method, so we consider the auxiliary problem

$$\frac{dt}{d\tau} = g(w), \qquad t(0) = 0. \tag{2.6}$$

A Simple Motivation

Where g (in (2.6)) satisfies $g(\lambda^{-1/2}w) = \lambda g(w)$ (further preserving the symmetry of the nonlinear term).

We have far too many choices, here, so we make the simplest one:

$$g(w) := w^2,$$

which clearly satisfies the identity. So, we now have a larger coupled system to solve:

$$\frac{dw}{d\tau} = -w, \quad 0 < t < T, \tag{2.7}$$

$$\frac{dt}{d\tau} = w^2, \quad \tau > 0, \tag{2.8}$$

$$w(0) = 1, \tag{2.9}$$

$$t(0) = 0. \tag{2.10}$$

A Simple Motivation

While the above may look tedious, it is worth noting that the original differential problem has been reduced to a linear problem. The nonlinearity is now on the time-stepping function, but this can be desirable if our primary goal is to obtain accurate solution profiles. Let's see how this works:

- We know that the solution to (2.3)-(2.4) *quenches* at time $t = 1/2$.
- The solution monotonically decreases until the value of zero is reached.
- The time derivatives goes to negative infinity as $t \to 1/2^{-}$.

So, let's see if this new approach recovers these facts.

A Simple Motivation

Let's test things by using the simplest (forward Euler) approximation to the problem and study our results.

First, since τ is our only *independent* variable, now, we discretize it into M evenly spaced intervals of uniform size. Thus, on the "τ" grid, we would have

$$\tau_m = \tau_{m-1} + \Delta\tau = m\Delta\tau$$

The "loose" assumption is that this uniform computational grid is chosen well and will not introduce further problems. So, forward Euler yields:

$$\frac{dw}{d\tau} = -w \implies w^{n+1} = w^n - \Delta\tau w^n = (1-\Delta\tau)^{n+1}$$

So, the numerical solution takes the form

$$w^{n+1} = (1-\Delta\tau)^{n+1}.$$

A Simple Motivation

Note the following:

i. $\lim_{n\to\infty} w^{n+1} = 0$, which is desired.

ii. $\Delta\tau$ affects how quickly or slowly we get there — thus, there are still decisions to be made for correct recovery of the model (it can sometimes be out of our hands).

So, with little effort, we are guaranteed to recover *at least* one of the qualitative asymptotic features of the true solution.

A Simple Motivation

Now, let's look at the other equation. Using forward Euler, we have

$$t^{n+1} = t^n + \Delta\tau (w^n)^2.$$

Once again, we iterate and obtain

$$t^{n+1} = t^n + \Delta\tau (w^n)^2 = t^n + \Delta\tau (1-\Delta\tau)^{2n} = \Delta\tau \sum_{k=0}^{n+1} (1-\Delta\tau)^{2k}.$$

This representation for the phase-space time is a geometric series, and we have

$$\lim_{n\to\infty} t^{n+1} = \Delta\tau \sum_{k=0}^{\infty} (1-\Delta\tau)^{2k} = \frac{\Delta\tau}{1-|1-\Delta\tau|^2} = \frac{1}{2-\Delta\tau},$$

as long as $\Delta\tau$ is reasonably small.

A Simple Motivation

Thus, we have a result that is quite nice!

Moreover, since the above (meaning original problem's) quenching time is $1/2$ we have recovered this result with order one (just like the order of our forward Euler method).

Finally, one can show that such a method preserves the positivity and monotonicity of the solution (with no extra assumptions).

- Convergence proofs based on such time-stepping algorithms open many doors for improved analysis.
- While not shown here, such methods also recover the profile of the derivative of the solution (w_t solve a *blow-up* problem, and this method recovers its qualitative features nicely).

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A Nonlinear Stochastic Problem

Due to its wide range of applications in science and engineering, I have been interested in the following stochastic differential problem,

$$du = [Au + f(u)]\,dt + g(u)\,dW, \quad 0 \le t \le T, \tag{3.11}$$
$$u(0) = u_0 \in H, \tag{3.12}$$

where H is a separable Hilbert space.

In (3.11)-(3.12):

- This equation can be ODE, PDE, coupled equations, etc.
- The operator A can be a differential operator, a coefficient matrix, or a discretization matrix (using FD, FEM, FV, etc.).
- For now, W is a *Wiener process*, but the method works for general *rough paths*.

Assumptions

Many of the results require some fairly technical assumptions, but the intention is to avoid confusion by only outlining a few important ones.

- We assume the operator A is the generator of an analytic semi-group. Loosely speaking, we assume it is some generalization of an elliptic operator.
- We assume that f is (locally) Lipschitz continuous with appropriate additional regularity.
- We assume that g is “nice” enough.
- We also further assume technical stochastic assumptions, the most important of which is a compatibility condition between the operator A and the covariance operator associated with the noise term.

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A Nonlinear Stochastic Splitting Algorithm

Fix $N \in \mathbb{N}$ and define $h = T/N$. We are concerned with developing an approximation to the true solution to (3.11)-(3.12) at time $t_n = nh$, denoted u_n, being given by

$$u_n = S^n(u_0) := \prod_{k=0}^{n-1} S_k(u_0), \tag{4.13}$$

where $S_k : H \to H$, $k = 0, 1, ..., N$, are defined as

$$S_k := e^{hA} e^{hf} e^{g\Delta W_k}, \tag{4.14}$$

with $\Delta W_k := W(t_{k+1}) - W(t_k)$.

$\Longrightarrow$ (4.14) is a stochastic version of the classical Lie-Trotter splitting operator.

Theoretical Results

Theorem (Strong Convergence)

Under appropriate assumptions we have

$$\mathbb{E}\|u_n - u(t_n)\|^2 \leq Ch^\beta,$$

where $\beta \in (0,1]$ is a regularity constant.

Theorem (Weak Convergence)

Under appropriate assumptions we have

$$\mathbb{E}\|\Phi(u_n) - \Phi(u(t_n))\| \leq Ch^\beta,$$

where $\beta \in (0,1]$ is a regularity constant and $\Phi \in C_b^2(H;\mathbb{R})$.

What is the Idea?

Why is any of this useful?

- With the operator splitting method in place, we may now consider each sub-component of the problem separately — meaning, we can resolve the linear, nonlinear, and stochastic components each with particular/different numerical methods.
- The “stiffest” component in quenching problems (and many other problems) is the nonlinear term.
- The goal is then to resolve the nonlinear component using the method presented in the first part of the talk.

More Results

Positivity?

Monotonicity?

Convergence?

Stability?

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Conclusions and Future Endeavors

In this talk we presented preliminary results relating to the preservation of asymptotic features of nonlinear stochastic problems.

- A method for preserving asymptotic features is introduced via a simple example.
- We discussed the advantages of employing nonlinear splitting in singular stochastic problems.

Future work includes:

- Running more numerical simulations to further verify the theoretical results.
- Consider the efficacy of not splitting the deterministic components — in particular, employ modified exponential integrators.

THANK YOU!

 

Questions?