

1. (Sequential Formulation of Neoclassical Growth with Interest Rates)

A competitive (i.e., price taking) equilibrium exists for households and firms. This is identical to the competitive equilibrium of the deterministic neoclassical growth model we did in class except:

- Consumer's can smooth consumption by investing in capital, but also have access to financial assets (with prices determined in general equilibrium).
- There will not have a set of assets traded at time 0 which provide a claim to consumption at time t (i.e., our old complete set of assets with prices q_t^0 doesn't exist).
- Instead, household's can buy and sell (at each time t rather than time 0) claims to consumption at time $t + 1$ (i.e., 1 period bonds). This single asset can be sold to each other (or to the government) on competitive markets that operate at each time period (i.e., spot markets).
- Instead of a lifetime budget constraint, households have sequential budget constraints.
- The price of the consumption good at time t is normalized to 1, so the price system will be $\{r_t, w_t, i_{t+1}\}$ where r_t and k_t are the real rental rates for capital and labor in time t goods, and i_{t+1} is the net interest rate on a bond purchased at time t .
- The gross interest rate of purchasing a unit of the bond is $1 + i_{t+1}$. Consequently, buying a claim to 1 unit of the good at time t costs $\frac{1}{1+i_{t+1}}$.¹
- The government also has a sequential budget constraint, and smooths in expenditures through the bond market and tax policy (i.e., it can also buy and sell bonds).
- Bond's are in 0 net supply. That is, if consumers hold $B_t > 0$ of the bonds, then the government would need to hold $-B_t < 0$ of the bonds.

To summarize the entire equilibrium for a representative consumer and firm,

Allocation: $\{c_t, k_t, B_t\}_{t=0}^{\infty}$. Bond holdings, B_t , are pieces of paper. k_0 and B_0 are given.

Price System: $\{r_t, w_t, i_{t+1}\}_{t=0}^{\infty}$

Government Policy: $\{\tau_{ct}, g_t, B_t^g\}_{t=0}^{\infty}$. That is, a consumption tax, government expenditures, and government bond holdings (which is negative if they owe the households money). Assume that the Government Policy is given exogenously, though it will need to be budget feasible.

Feasibility: The firm operates the same neoclassical production function (with $f'(\cdot) > 0$ and $f''(\cdot) < 0$ as before, and the bond markets clear,

$$c_t + k_{t+1} - (1 - \delta)k_t + g_t = f(k_t) \quad (1)$$

$$B_t^g = -B_t, \text{ i.e. bonds are in 0 net supply} \quad (2)$$

¹From our interest notes, consider a claim to a unit of consumption delivered at time $t + 1$ but priced in time t good, q_{t+1}^t . Then the interest rate on this 1 period claim at time t is defined as $\rho_{t,t+1}$ by $q_{t+1}^t \equiv \frac{1}{1+\rho_{t,t+1}}$. This calculation is doing the same thing for these 1 period claims.

Government Budget: The government policy is given exogenously, but it must “balance” in the long-run (i.e., when government debt is taken into account with the endogenous prices).

$$\underbrace{g_t}_{\text{Expenditures}} + \underbrace{\frac{1}{1+i_{t+1}}}_{\text{With Interest}} \underbrace{B_{t+1}^g}_{\text{New Bonds}} \leq \underbrace{\tau_{ct}c_t}_{\text{Tax Income}} + \underbrace{B_t^g}_{\text{Previous Bonds}} \quad \text{for all } t \geq 0 \quad (3)$$

For example, if $B_t^g = B_{t+1}^g$ then the government is rolling over their bonds and paying (or getting) the interest. There will also be a no-ponzi scheme condition (e.g. $\lim_{t \rightarrow \infty} |B_t^g| < \infty$)

Households's Problem: A large number of identical consumers have a typical strictly concave utility function ($u'(\cdot) > 0, u''(\cdot) < 0, u'(0) = \infty$), and provide 1 unit of labor inelastically. Taking B_0, k_0 , prices, and government policies as given

$$\max_{\{c_t, k_{t+1}, B_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad (4)$$

$$\text{s.t. } (1 + \tau_{ct})c_t + k_{t+1} + \frac{1}{1+i_{t+1}}B_{t+1} \leq B_t + (1 - \delta)k_t + r_t k_t + w_t, \quad \text{for } t \geq 0 \quad (5)$$

$$(+ \text{ a no-ponzi scheme transversality condition}) \quad (6)$$

Note that the period by period budget constraint is written with the price of the consumption good normalized to 1.

Firm's Problem: A large number of identical firms operate a constant returns to scale (CRS) production function $F(K, N)$ with the usual result that $F(\frac{K}{N}, 1) = \frac{1}{N}f(k)$ when $k \equiv \frac{K}{N}$.² Taking prices as given they maximize,

$$\max_{K_t, N_t} \{F(K_t, N_t) - w_t N_t - r_t K_t\} \quad (7)$$

With this complete specification of the equilibrium,

- Define a competitive equilibrium
- Solve for the first-order-necessary conditions (FONC) of the firms to get expressions for the real rental rate of capital in terms of production function $f(\cdot)$ and the aggregate capital k_t (which is also the aggregate capital to labor ratio here since $N_t = 1$). Why are we able to use a representative firm?
- Solve for the FONC of the household's choice of $\{c_t, k_{t+1}, B_{t+1}\}_{t=0}^{\infty}$. There will now be 2 Euler equations: one for capital investment, and another for bond investment. Substitute from the FONC of the firm's problems to express in terms of allocations where possible.
- Define the real return on capital, as in the class notes: $R_t \equiv 1 - \delta + f'(k_{t+1})$. What is i_{t+1} in terms of R_t ? Interpret this relationship.³

²Note that if there is a representative firm, then $F(\frac{K}{N}) = f(k)$ and $k = K$.

³Hint: stare at the two Euler equations and combine. Is there a way to think of arbitrage between using these two different methods to smooth consumption?

- (e) If the government policy has $B_t^g = 0$ for all t , then note that $B_t = 0$ from the feasibility condition. Is there any trading of bonds between consumers, and if not, why not? If so, then why is there still an interest rate?
- (f) Let $B_t^g = B_t = 0$ forever. You can either assume that $g_t = \tau_{ct} = 0$, or that these are fixed such that $\bar{\tau}_c, \bar{g}$ balance the government budget. Calculate the steady state $\{\bar{k}, \bar{c}, \bar{B}, \bar{r}, \bar{i}\}$.
2. Consider an economy with a large number of identical, price-taking agents. Endowments, $\{y_t\}_{t=0}^\infty$, are stochastic with $y_t \in \{L, H\}$. The evolution of endowments follows a Markov chain between these L and H states with transition matrix

$$P \equiv \begin{bmatrix} \pi_{LL} & \pi_{LH} \\ \pi_{HL} & \pi_{HH} \end{bmatrix}$$

where $\pi_{ij} > 0$, $\pi_{LL} + \pi_{LH} = 1$, and $\pi_{HL} + \pi_{HH} = 1$

Consider a 2 period model (which is equivalent to comparing two sequential periods of an infinite horizon model). Consumers are able to buy and sell contingent claims for delivery of goods next period. Given a current state of the economy, S , paying $q(L|S)$ delivers 1 unit of the good if L is the state next period, and 0 units of the good if H is the state next period. Similarly, $q(H|S)$ delivers 1 unit of the good next period if the state is H and L otherwise. (Hint: Note that I have put $S \in \{L, H\}$ in the price since the prices may depend on the current state of the economy)

- (a) The consumer will maximize utility by choosing consumption today c_0 and consumption tomorrow depending on the state of the world: $c_1(L)$ and $c_1(H)$.
Note that c_0 is deterministic because they know the current state of the world. The endowment today depends on the current state S , i.e. $y_0(S)$ and the endowment tomorrow is stochastic $y_1(L|S)$ or $y_1(H|S)$.
Normalizing the price of a consumption good today to 1 (i.e., $q_0^0 = 1$), and facing prices $q(L|S)$ and $q(H|S)$, write out the optimization problem of the consumer given their endowment.
- (b) What is the feasibility condition in this economy (i.e., relate c and y in the various states of the world and time periods. Hint: you can use a representative agent)?
- (c) Use your optimization problem for the consumer and the feasibility condition to write out an asset pricing equation (the Euler equation) to relate the price to the probabilities, utilities, and endowments in various states of the world. Solve for $q(S'|S)$ for the various $S, S' \in \{L, H\}$. This is the price of a one-period state contingent bond.
- (d) Derive the risk-free price of a consumption good next period using your $q(S'|S)$. This is the price of a bond that delivers one unit of consumption regardless of the realized state of the economy. Denote this as $q(S)$.
- (e) Use this to derive the net interest rate on a 1 period risk-free bond, $r_{01}(S)$

$$q(S) \equiv \frac{1}{1 + r_{01}(S)}$$

Interpret $r_{01}(L)$ and $r_{01}(H)$. In particular, explain why bond prices are a function of the current state, S . Consider the role of income growth and consumption smoothing.

3. (Variations on Financing Government Expenditures)

The consumer values consumption, and provides 1 unit of labor inelastically. The period utility be $u(c) = \log(c)$.

Take our standard neoclassical growth model, with the possibility of consumption taxes, τ_{ct} , lump-sum taxes, τ_{ht} , and labor taxes, τ_{nt} . First, assume that taxes and government expenditures are 0 (i.e., $\bar{g} = \bar{\tau}_h = \bar{\tau}_c$) and that the economy is in a steady state (i.e. \bar{k} as the steady state capital, and $\bar{c} = f(\bar{k}) - \delta\bar{k}$). Let the capital at time 0 be this steady state capital, i.e. $k_0 = \bar{k}$.

There is a sudden announcement that $g_t = \bar{g}$ for all $t \geq 0$, where $\bar{g} = \frac{1}{4}(f(k_0) - \delta k_0)$, and the government expenditures are financed entirely through lump-sum taxes, $\bar{\tau}_h$.

- (a) Calculate the new steady state \bar{c} and \bar{k} .
- (b) What is the transition path of c_t and k_t from the k_0 initial condition?
- (c) What is the behavior of c_{t+1}/c_t and $R_{t+1} \equiv f'(k_{t+1}) + 1 - \delta$ along this transition path?
- (d) Argue that the timing of the lump-sum taxes is irrelevant (i.e., any τ_{ht} fulfilling the long-run government budget constraint gives the same allocation).
- (e) Now, consider the alternative policy that the government finances its expenditures entirely through consumption taxes. First assume that consumption taxes are constant (i.e., $\tau_{ct} = \bar{\tau}_c$ for all $t \geq 0$). Find the new steady state \bar{c} and \bar{k} and the transition path from k_0 .
- (f) In this case, would the timing of the consumption tax matter (i.e., does any τ_{ct} fulfilling the long-run government budget constraint deliver the same allocations $\{c_t, k_{t+1}\}$ along the transition dynamics?) If not, why?
- (g) Without solving the full model, would financing expenditures entirely through constant labor taxes $\bar{\tau}_n$ have the same steady state as that of lump-sum taxes? What about the transition dynamics?

4. (Variation on the Search Model)

Each period, a previously unemployed worker draws two offers to work forever at wage w from the cumulative distribution function ('cdf')

$$F(w) = \left(w/B\right)^{\frac{1}{2}}, \quad 0 \leq w \leq B$$

where $F(w) = \text{Prob}(\tilde{w} \leq w)$ where \tilde{w} is a particular wage offer. Successive draws within a period and across periods are identically and independently distributed. The unemployed worker is free to inspect both offers in a period and, if he or she wants, accept the highest among offers he or she has drawn that period. Offers from past periods cannot be recalled. The offers are to work at the accepted wage forever. There is no option to quit after an offer has been accepted, and there is no prospect of being fired. The worker wants to maximize

$$\mathbb{E} \left[\sum_{t=0}^{\infty} \beta^t y_t \right], \quad 0 < \beta < 1,$$

where

$$y_t = \begin{cases} w & \text{if employed at wage } w, \\ c & \text{if unemployed} \end{cases}$$

where w is the wage, c is unemployment compensation, and $\mathbb{E}[\dots]$ is an expected value before the offers are drawn.

- (a) Verify that $F(w) = \left(w/B\right)^{\frac{1}{2}}$ is a legitimate cdf. Find the cdf for the maximum of 2 draws (i.e., find the cdf of $z \equiv \max\{z_1, z_2\} \in \tilde{F}(z)$) and verify it is a cdf.
- (b) Find the worker's optimal search strategy and show that it has a 'reservation wage' form.⁴ Draw the value function for a worker given a particular w .
- (c) Let \bar{w} be the reservation wage. Find a formula for the reservation wage as a function of B, β, c .⁵
- (d) Given this \bar{w} , find a formula for ψ = probability that an unemployed worker leaves unemployment this period as a function of \bar{w}, B, β, c (eliminating \bar{w} if you found a closed form solution in part c).

⁴Hints: (1) setup the model recursively in a way isomorphic to the model from class, (2) For any n and independent draws, z_1, \dots, z_n from cdf $F(z)$, the cdf of the maximum of these is $z \equiv \max\{z_1, \dots, z_n\} \sim F(z)^n$

⁵Hint: While you could solve for \bar{w} directly, feel free to leave it in an *implicit* form if you are finding the algebra difficult. However, you should be able to eliminate any recursive value functions

5. (Rationalizing Consumption Patterns)

Assume a completely standard neoclassical growth model where the government policy is some $\{g_t, \tau_{kt}, \tau_{ht}\}_{t=0}^{\infty}$. Define

$$R_t \equiv (1 - \delta) + (1 - \tau_{kt+1})f'(k_{t+1})$$

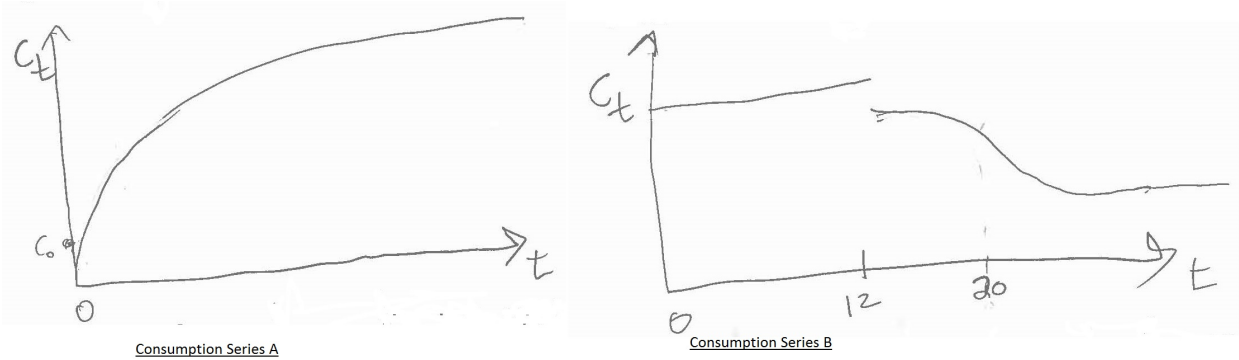


Figure 1: Two Different Consumption Patterns

- Make up any story you like on sequences of $\{g_t, \tau_{kt}, \tau_{ht}\}_{t=0}^{\infty}$ which rationalizes the path of consumption in Figure 1, Consumption Series (A). You are free to come up with any explanation you want that is consistent with the model and these observations.
- Describe the behavior of k_t, g_t, k_{t+1} and R_t for your story.
- Instead, assume that you observe the path of consumption in Figure 1, Consumption Series (B). Make up any story you like on sequences of $\{g_t, \tau_{kt}, \tau_{ht}\}_{t=0}^{\infty}$ which rationalizes this path of consumption. In doing so, please distinguish between changes in taxes and expenditures that are foreseen vs. unforeseen.
- Describe the behavior of k_t, g_t, k_{t+1} and R_t for your story.

6. (Special Permanent Income Model + Asset Pricing)

A consumer faces a time-invariant, risk-free gross interest rate of $R \equiv e^r$ with $r > 0$. The consumer can borrow or lend at this rate up to a “no-Ponzi scheme” condition. The savings (or debt) of the consumer is denoted F_t .

Let the discount factor be β , and define $\beta \equiv e^{-\rho}$ where ρ is the discount rate.⁶

Finally, assume that the consumer’s income, Y_t , is a stochastic process following:

$$Y_{t+1} = Y_t \exp(\sigma \epsilon_{t+1})$$

where $\epsilon_{t+1} \sim N(0, 1)$. The consumer chooses $\{C_t, F_{t+1}\}_{t=0}^{\infty}$ (which may now be stochastic) to maximize their expected utility, i.e.

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma} \right] \quad (8)$$

$$\text{s.t. } C_t + F_{t+1} \leq R F_t + Y_t, \text{ for } t \geq 0 \quad (9)$$

With the LOM

$$Y_{t+1} = Y_t e^{\sigma \epsilon_{t+1}} \quad (10)$$

(a) Find the first-order necessary conditions for this problem.⁷ Verify that they imply,

$$1 = \mathbb{E}_t [R M_{t+1}] \quad (11)$$

where

$$M_{t+1} \equiv \beta \frac{u'(C_{t+1})}{u'(C_t)}$$

(b) Verify that

$$M_{t+1} = \exp(-(\rho + \gamma(c_{t+1} - c_t)))$$

where $c_t \equiv \log C_t$. Then take logs to define $\log M_{t+1} \equiv m_{t+1}$ and

$$m_{t+1} = -\rho - \gamma(c_{t+1} - c_t) \quad (12)$$

(c) Assume there is a representative consumer and the consumer is observed to set $C_t = Y_t$ and $F_{t+1} = 0$ for all $t \geq 0$ (thinking of a Lucas-style asset pricing model in general equilibrium). Use (12), the log $y_t \equiv \log Y_t$ in (10) with

$$y_{t+1} - y_t = \sigma \epsilon_{t+1}$$

To show that $m_{t+1} \sim N(\mu_m, \sigma_m^2)$, i.e. an iid random normal for some mean μ_m and variance σ_m^2 functions of parameters.

Find the μ and σ_m^2 .

⁶From a Taylor Series approximation, this is approximately equal to $\beta \equiv \frac{1}{1+\rho}$ for small ρ .

⁷Be careful to keep expectations around when the information set only allows forecasts. As the law of motion in (10) is not a constraint on the choice, you don’t put it in as a Lagrange Multiplier. Instead, you should solve the problem with the binding constraint in (9) and then apply (10) as the forecast after you have the Euler Equation.

- (d) Use (11) and the previous parts to find the value of constant net interest rate, r , which rationalizes this behavior.⁸
- (e) Interpret the role of aggregate uncertainty, σ , on interest rates, r . Why/when would γ matter? If $\sigma = 0$, why/when would γ matter?

⁸Hint: Use the optimality conditions from previous parts and deduce the r necessary to clear the markets. Also, for a normal random variable $z \sim N(\mu_z, \sigma_z^2)$ note that $\mathbb{E}[e^z] = e^{\mu_z + \frac{1}{2}\sigma_z^2}$.

7. (Transition Dynamics with Risk-Neutral Consumers)

Assume that, given an initial condition $k_0 > 0$, a risk-neutral consumer has the following preferences,⁹

$$\sum_{t=0}^{\infty} \beta^t c_t \quad (13)$$

$$\text{s.t. } c_t + k_{t+1} = f(k_t) + (1 - \delta)k_t \quad (14)$$

$$c_t \geq 0 \quad (15)$$

with the standard properties, $f'(\cdot) > 0$, $f''(\cdot) < 0$, and $f'(0) = \infty$. Define $\beta \equiv \frac{1}{1+\rho}$ with $\rho > 0$.

- (a) For a given k_0 , formulate the optimal planning problem and find the first-order necessary conditions.¹⁰ Define \bar{k} as the steady state capital
- (b) For $k_0 > \bar{k}$, describe the optimal time path of $\{c_t, k_{t+1}\}_{t=0}^{\infty}$
- (c) For $k_0 < \bar{k}$, describe the optimal time path of $\{c_t, k_{t+1}\}_{t=0}^{\infty}$
- (d) Let the savings rate s_t be defined as

$$k_{t+1} = s_t f(k_t) + (1 - \delta)k_t$$

Explain how s_t varies as a function of k_t in this model using the dynamics of parts (a), (b), and (c)

⁹Alternatively, consider the limit of power-utility $u(c) \equiv \lim_{\gamma \rightarrow 0} \frac{c^{1-\gamma}}{1-\gamma}$.

¹⁰Hint: unlike our example in class, you need to keep Lagrange multipliers on some of the inequality constraints. Remember that linear objectives usually mean corner solutions.