Rational and Adaptive Expectations

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Types of expectations:

- **None/Myopic**: No consideration of agent forecasts of the future in the economic model.
- **Adaptive Expectations**: Ad-hoc decisions based on previous data (i.e., looking backwards). Related to certain models of Reinforcement Learning in computer science
- **Rational Expectations**: Dynamic decisions need to be based on some expectation of the future (e.g., workers, financial markets, etc.)
- **Bounded Rationality/Behavioral**: Discipline variations on rational expectations (e.g learning, information capacity constraints, etc.). Macro people sometimes avoid the word "behavioral", because it is frequently ad-hoc.

1 Example: Inflation Forecasts

Consider a part of a macro model with agents forecasting inflation

- Let p_t be realized log price level
- Let $p^e_{t,t+1}$ be the expected price level at time $t+1$ given time t. Will consider alternatives
- Forecast error is: $FE_{t-1,t} = p_t - p^e_{t-1,t}$

1.1 Adaptive Expectations

- The first generation of taking expectations into account in macro models
- Proponents never suggested that adaptive expectation was optimal
- Possibility of systematic forecast errors, e.g. update new expectation based on a proportion of the previous forecast error

- i.e. $FE_{t-1,t} = p_t - p^e_{t-1,t}$ can be systematically non-zero even if we provide agents with the ability to forecast the future

The updating rule for Adaptive expectation is to take a fraction $\lambda \in (0,1)$ (a model parameter):

$$p^e_{t,t+1} = p^e_{t-1,t} + \lambda(p_t - p^e_{t-1,t}) \tag{1}$$

where $(p_t - p^e_{t-1,t})$ is the forecast error last year, and λ is externally given.

1.2 Rational Expectation

Recall from previous lectures that the hallmark of rational expectations is that agents simply use the mathematical expectation given all available information (i.e. where the modeller chooses the information set!)

$$p^e_{t,t+1} = \mathbb{E}\left[p_{t+1} \mid \Omega_t\right] \equiv \mathbb{E}\left[p_{t+1}\right] \tag{2}$$

where Ω_t is the information set available to agents at time t.

- Model dependent: Ω_t has present and past information
- $\mathbb{E}$ is the mathematical expectation
- Rational expectations only makes sense in the context of a specific model and information set.

1.2.1 Forecast Error

Note that the expected forecast error is given by:

$$\mathbb{E}_t\left[p_{t+1} - \mathbb{E}_t\left[p_{t+1}\right]\right] = 0 \tag{3}$$

i.e. no systematic forecast errors Of course, this is not the same as no errors. Constraining the information sets or adding uncertainty on the underlying processes can be called “bounded rationality", but it is philosophically the same:

$$p_{t+1} - \mathbb{E}_t\left[p_{t+1}\right] = \text{ (the “surprise")} \tag{4}$$

With rational expectations, market imperfections focus on information imperfections, complexity of calculating mathematical expectations, differences in the agents mental model of the variables, etc.

2 Example: Cagan-Sargent-Wallace

Some of the first criticisms of the Keynesian economics came by taking "almost" all of their assumptions and simply exploring expectations.

So take some ad-hoc money supply-demand equation without questioning too much where it might come from[1]

- This is a Keynesian-style money demand equation (LM-curve)
- p_t is the log price level
- m_t is the log money supply
- α is some elasticity of money demand
- $p^e_{t,t+1}$ is the expected price level tomorrow, which we will consider different variations.

$$\underbrace{m_t - p_t}_{\text{money supply}} = -\alpha \underbrace{\left(p^e_{t,t+1} - p_t\right)}_{\text{money demand}} \text{ (in real terms)} \tag{8}$$

Rearrange

$$p_t = \left(\frac{1}{1+\alpha}\right) m_t + \left(\frac{\alpha}{1+\alpha}\right) p^e_{t,t+1} \tag{9}$$

Where expected inflation is defined as (in terms of non-logged P),

$$\pi^e_{t,t+1} =\equiv \frac{P^e_{t,t+1}}{P_t} = p^e_{t,t+1} - p_t \tag{10}$$

This is the equilibrium price level today in terms of money supply and expected price tomorrow. The dynamics then depend on how agents form expectations.

[1] Adapted from Bagliano 2010 notes. This often starts in absolute, rather than real terms, such as

$$\frac{M_t}{P_t} = \overline{Y} \exp(-\alpha(\bar{r} + \pi^e_{t,t+1})) \tag{5}$$

where M_t and P_t are not logged, $\overline{Y}$ is the constant real output, and $i = \bar{r} + \pi^e_{t,t+1}$. Take logs of equation (5):

$$\log M_t - \log P_t = \left(\log \overline{Y} - \alpha \bar{r}\right) - \alpha \pi^e_{t,t+1} \tag{6}$$

Let lower case be the log terms; approxmiate expected inflation as P^e:

$$\pi^e_{t,t+1} = \frac{P^e_{t,t+1}}{P_t} = p^e_{t,t+1} - p_t \tag{7}$$

Omit the constant through demeaning to get (9)

2.1 Adaptive Expectations

Assume (from equation (1)):

$$p^e_{t,t+1} = \underbrace{\lambda p_t + (1-\lambda) p^e_{t-1,t}}_{\substack{\text{weighting between realized} \\ \text{and previous forecast}}} \tag{11}$$

Iterating backwards:

$$p^e_{t,t+1} = \lambda p_t + (1-\lambda)\left[\lambda p_{t-1} + (1-\lambda) p_{t-2,t-1}\right] \tag{12}$$

$$\Rightarrow \boxed{p^e_{t,t+1} = \lambda \sum_{i=0}^{\infty} (1-\lambda)^i p_{t-i}} \quad \text{(looks backwards for all data)} \tag{13}$$

Plug into equation (9)

$$p_t = \frac{1}{1+\alpha} m_t + \frac{\alpha}{1+\alpha} \cdot \lambda \sum_{i=0}^{\infty} (1-\lambda)^i p_{t-i} \tag{14}$$

Reorganize,

$$p_t = \frac{1}{1+\alpha(1-\lambda)} m_t + \frac{\alpha\lambda}{1-\alpha(1-\lambda)} \cdot \lambda \sum_{i=1}^{\infty} (1-\lambda)^i p_{t-i} \tag{15}$$

This is an example of adaptive expectations, which depends on current money supply and previous history (information set).

The crucial thing to note: a temporary change in m_t has the same effect as a permanent one at time t.

2.2 Rational Expectations

Now consider rational expectations

$$p^e_{t,t+1} = \mathbb{E}_t\left[p_{t+1}\right] \tag{16}$$

Plug into equation (9)

$$p_t = \left(\frac{1}{1+\alpha}\right) m_t + \left(\frac{\alpha}{1+\alpha}\right) \mathbb{E}_t\left[p_{t+1}\right] \tag{17}$$

Iterate forward and take expectations of future periods:

$$\mathbb{E}_t\left[p_{t+1}\right] = \frac{1}{1+\alpha} \mathbb{E}_t\left[m_{t+1}\right] + \frac{\alpha}{1+\alpha} \mathbb{E}_t\left[p_{t+2}\right] \tag{18}$$

Plug into equation (17)

$$p_t = \left(\frac{1}{1+\alpha}\right) m_t + \left(\frac{\alpha}{1+\alpha}\right) \mathbb{E}_t \left[p_{t+1} + \frac{1}{1+\alpha} m_{t+1} + \frac{\alpha}{1+\alpha} \mathbb{E}_{t+1}\left[p_{t+2}\right]\right] \tag{19}$$

$$= \left(\frac{1}{1+\alpha}\right) m_t + \left(\frac{\alpha}{1+\alpha}\right) \mathbb{E}_t \left[p_{t+1}\right] + \frac{\alpha}{1+\alpha} \frac{1}{1+\alpha} \mathbb{E}_t \left[m_{t+1}\right] + \left(\frac{\alpha}{1+\alpha}\right)^2 \mathbb{E}_t \left[\mathbb{E}_{t+1}\left[p_{t+2}\right]\right] \tag{20}$$

By law of iterated expectations: $\mathbb{E}_t \left[\mathbb{E}_{t+1}\left[p_{t+2}\right]\right] = \mathbb{E}_t \left[p_{t+2}\right]$

$$= \left(\frac{1}{1+\alpha}\right) m_t + \left(\frac{\alpha}{1+\alpha}\right) \mathbb{E}_t \left[p_{t+1}\right] + \frac{\alpha}{1+\alpha} \frac{1}{1+\alpha} \mathbb{E}_t \left[m_{t+1}\right] + \left(\frac{\alpha}{1+\alpha}\right)^2 \mathbb{E}_t \left[p_{t+2}\right] \tag{21}$$

Repeat to $t \to \infty$:

$$p_t = \frac{1}{1+\alpha} \sum_{j=0}^{\infty} \left(\frac{\alpha}{1+\alpha}\right)^j \mathbb{E}_t \left[m_{t+j}\right] \tag{22}$$

Equation (22) is forward looking with forecasts.
The past only matters since it is in the information set at time t. Hence, temporary vs. permanent changes in m_t have very different effects.

Think of the analogy to the Friedman-Muth model in the permanent income model.

Differences between adaptive and rational expectations:

1. p_{t-i} enters through conditional information set
2. p_{t-i} may <u>not</u> change p_t, unless part of expectations (pricing sequence affected only in adaptive expectations)
3. Forward versus Backward ($t-i$ and $t+j$)

2.3 Example: Permanent Increase in Money Supply

Our experiment will be to analyze the following:

- Announced at time 0, there there be a permanent increase in $\bar{m}$ to $\bar{m}+k$
- Both the announcement date and the implementation date matter a great deal for rational expectations.

2.3.1 Rational Expectations

Before change:

$$p_{t-1} = \frac{1}{1+\alpha} \cdot \frac{1}{1 - \frac{\alpha}{1+\alpha}} \bar{m} \tag{23}$$

After announcement, $\bar{m} \to \bar{m} + k$. Today:

$$p_t = \frac{1}{1+\alpha} \cdot \frac{1}{1 - \frac{\alpha}{1+\alpha}} (\bar{m} + k) \tag{24}$$

2.3.2 Adaptive Expectations

We will not solve the full dynamics in closed form, but can examine The price adjustment will be slow depending on λ. Examine (15)

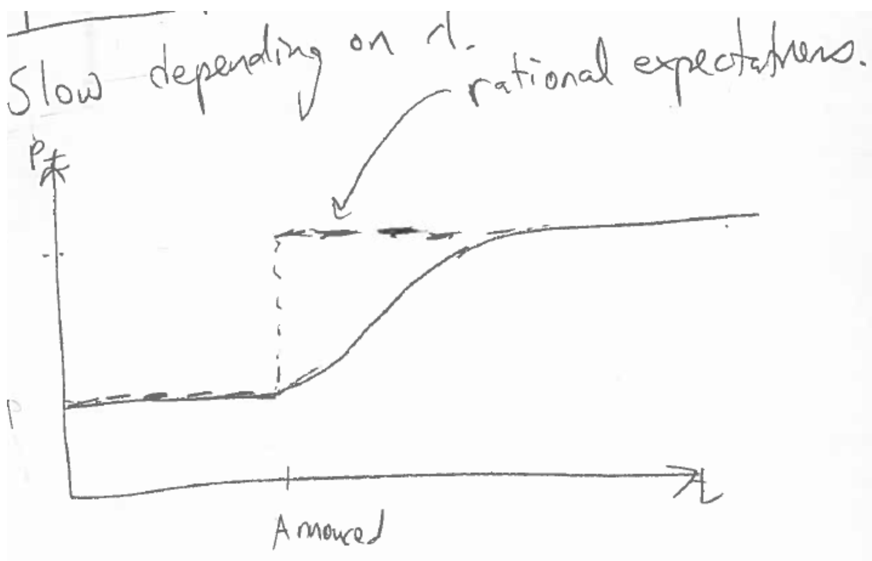

Figure 1: Adaptive Expectations Adjustment

If the general population had and adaptive expectations, investors would have an arbitrage due to the predictable increase in the price level if they used <u>all available information</u>

- They could pump enormous amounts of money out of the "unsophisticated"
- This enormous arbitrage might be seen by the individuals, but they are powerless to change anything
- You don't need many deep pocketed investors for this to be true. Asset prices reveal information

"You can fool some of the people some of the time, but not all of the people all of the time!"
"If you are so smart, why aren't you rich?"