

Stochastic Permanent Income Model and Government Fiscal Policy

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1 Stochastic Permanent Income

1.1 Basic setup.

Linear State Space + Normal Shock:

- Let

$$x_{t+1} = Ax_t + Cw_{t+1} \tag{1}$$

$$y_t = G \cdot x_t \tag{2}$$

where A is $n \times n$ matrix, x is $n \times 1$ vector, C is $n \times m$ matrix, $w_{t+1} \sim N(0, I_{n \times m})$, i.i.d. normal shocks; G is $1 \times n$ vector, y_t is a scalar, which means "labor income"

- Consumer's Budget Constraint (assuming $\beta R = 1$):

$$F_{t+1} = \underbrace{\frac{1}{\beta}}_{\text{gross interest rate}} \left(\underbrace{F_t}_{\text{Financial wealth}} + y_t - c_t \right) \tag{3}$$

- Recall if $\{y_t\}$ is deterministic, and $R = 1/\beta$, then for any strictly concave $u(c)$ they achieved perfect consumption smoothing:

$$c_t = (1 - \beta) \left(\underbrace{F_t}_{\text{Financial wealth}} + \underbrace{\sum_{j=0}^{\infty} \beta^j y_{t+j}}_{\text{PDV of human wealth}} \right) \tag{4}$$

- If y_t is stochastic, can we just replace the above equation with expected value?:

$$c_t = (1 - \beta) \underbrace{\left(F_t + \mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j y_{t+j} \right] \right)}_{\substack{\text{expected PDV} \\ \text{of human wealth} \\ \text{with information at} \\ \text{time } t}} \quad (5)$$

Note: if $u'(c)$ is not linear, then this is only an approximation

- Combine (3) and (5):

$$F_{t+1} = \frac{1}{\beta} \left[F_t + y_t - (1 - \beta) \left(F_t + \mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j y_{t+j} \right] \right) \right] \quad (6)$$

$$= F_{t+1} = \frac{1}{\beta} \left[\beta F_t + y_t - (1 - \beta) \mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j y_{t+j} \right] \right] \quad (7)$$

$$\Rightarrow F_{t+1} - F_t = \frac{1}{\beta} \left[y_t - (1 - \beta) \mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j y_{t+j} \right] \right] \quad (8)$$

i.e. agents adds difference between y_t and permanent income. Now use (5) at t and $t + 1$,

$$c_{t+1} = (1 - \beta) \left[F_{t+1} + \mathbb{E}_{t+1} \left[\sum_{j=0}^{\infty} \beta^j y_{t+j+1} \right] \right] \quad (9)$$

$$c_t = (1 - \beta) \left[F_t + \mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j y_{t+j} \right] \right] \quad (10)$$

$$\Rightarrow c_{t+1} - c_t = (1 - \beta)(F_{t+1} - F_t) + (1 - \beta) \left[\mathbb{E}_{t+1} \left[\sum_{j=0}^{\infty} \beta^j y_{t+j+1} \right] - \mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j y_{t+j} \right] \right] \quad (11)$$

Use (8) to find (after many steps):

Proposition:

$$c_{t+1} - c_t = (1 - \beta) \sum_{j=0}^{\infty} \beta^j \left(\underbrace{\mathbb{E}_{t+1}(y_{t+j+1})}_{\substack{\text{Forecast of } t+1, t+2, \dots \\ \text{with time } t+1 \\ \text{information}}} - \underbrace{\mathbb{E}_t(y_{t+j+1})}_{\substack{\text{With time } t \\ \text{information}}} \right) \quad (12)$$

- Consumption only changes due to “surprise” of new information changing expected value

- Only unanticipated changes in y_{t+j}, \dots or other information which changes forecasts.
- Could be unanticipated changes in government policy or shock realizations.
- Finally, for a shock between $t \rightarrow t+1$ with our linear state space model:

$$c_{t+1} - c_t = (1 - \beta) \left[\sum_{j=0}^{\infty} \beta^j (\mathbb{E}_{t+1}(y_{t+j+1}) - \mathbb{E}_t(y_{t+j+1})) \right] \quad (13)$$

$$= (1 - \beta) \left[G(I - \beta A)^{-1} x_{t+1} - G(I - \beta A)^{-1} A x_t \right] \quad (14)$$

$$= (1 - \beta) G(I - \beta A)^{-1} \left[\underbrace{A x_t + C w_{t+1}}_{x_{t+1}} - A x_t \right] \quad (15)$$

Solution with Linear Gaussian State Space

$$\boxed{c_{t+1} - c_t = \underbrace{(1 - \beta)}_{\text{Propensity to Consume}} \underbrace{G(I - \beta A)^{-1} \cdot C w_{t+1}}_{\text{PDV of impulse response a shock to } x_{t+1}}} \quad (16)$$

i.e PDV of changes to forecasts from the realized shock.

1.2 Special case of Quadratic Preferences

- Recall Euler equation for Permanent Income Model:

$$u'(c_t) = \beta(1 + r)u'(c_{t+1}), \forall t = 0, \dots, T - 1 \quad (17)$$

If stochastic consumption and $\beta = \frac{1}{1+r}$, just replace with expectation?

$$\underbrace{u'(c_t)}_{\text{Marginal utility this period}} = \underbrace{\mathbb{E}_t[u'(c_{t+1})]}_{\text{Expectation of marginal utility next period}} \quad (18)$$

- Let $u(c) = \frac{a_1}{2}c^2 + a_2c + a_3 \Rightarrow u'(c) = a_1c + a_2$

In Euler equation:

$$a_1c_t + a_2 = \mathbb{E}_t(a_1c_{t+1} + a_2) \quad (19)$$

$$c_t = \mathbb{E}_t(c_{t+1}) \quad (20)$$

i.e., Euler equation implying perfect consumption smoothing with a deterministic process translates to consumption being a martingale if stochastic!

- Note:
 - In general, $\mathbb{E}_t(u(c)) \neq u(\mathbb{E}_t(c))$
 - Then, we can use the linear-stochastic state space model for forecasting $E_t c_{t+1}$ - Due to linearity, it simply forecasts mean.
 - This is a general result called "Certainty Equivalence" of optimizing a quadratic objective subject to a linear-gaussian state space model.
 - The decision is identical in a model with or without the certainty
 - However, the realized sequence contingent on the sequence, and utility, are not the same.

2 Examples

2.1 Pre-announced Tax Cut

- This will use a shock to knowledge about deterministic income processes, rather than a constant stream of shocks to income.
- Setup:
 - Government announce at $t = 0$ that at $t = 1$ it will borrow α from international markets at interest rate $(1 + r)$ per period and give it to each consumer.
 - They also announce that to eventually balance the budget, they will pay it back at $t = 2$ for simplicity by increasing taxation that period.
 - Assume consumers had deterministic y_{t+j} , which happens to consumption?

$$c_{t+1} - c_t = (1 - \beta) \sum_{j=0}^{\infty} \beta^j [\mathbb{E}_{t+1}(y_{t+j+1}) - \mathbb{E}_t(y_{t+j+1})]$$

$$\text{Define: } \{\hat{y}_{t+1}\}_{j=0}^{\infty} = \left\{ y_t, \underbrace{y_{t+1} + \alpha, y_{t+2} - \alpha(1+r)\beta}_{\text{Only difference}}, y_{t+3}, \dots, y_{t+j} \dots \right\}$$

- Note that from t to $t + 1$, the agent has the news that $\{y_{t+j}\} \rightarrow \{\hat{y}_{t+j}\}$
- This is a change in expectations:

$$c_1 - c_0 = (1 - \beta) \sum_{j=0}^{\infty} \beta^j [\mathbb{E}_1(y_{j+1}) - \mathbb{E}_0(y_{j+1})] = (1 - \beta) \sum_{j=0}^{\infty} \beta^j (\hat{y}_{j+1} - y_{j+1}) \quad (21)$$

$$\Rightarrow c_1 - c_0 = (1 - \beta) \sum_{j=0}^{\infty} \beta^j (y_{j+1} - y_{j+1}) + (1 - \beta) [\alpha - \beta(1+r)\alpha] \quad (22)$$

- Notes: If $\beta = \frac{1}{1+r}$, then $c_1 - c_0 = 0$
 i.e. tax cut has no effect because of anticipated rise in taxes. Later, we will investigate cases why $\beta = \frac{1}{1+r}$ comes out of general equilibrium.

2.2 Timing of Tax Cuts

- Setup:
 - Between time 0 and 1, government announces that it will cut taxes to give α to each individual at a deterministic time $T \geq 1$
 - Assume they do not need to pay it back and taxes will not raise to compensate.
 - What happens to consumption at time $\{0, \dots, T, T+1, \dots\}$?
 - Assume y_{t+j+1} are deterministic.

- Solve:

$$c_1 - c_0 = (1 - \beta) \sum_{j=0}^{\infty} \beta^j [\mathbb{E}_1(y_{j+1}) - \mathbb{E}_0(y_{j+1})] \quad (23)$$

$$= (1 - \beta) \sum_{j=0}^{\infty} \beta^j [y_{j+1} - y_{j+1}] + (1 - \beta) \cdot \beta^{T-1} \cdot \alpha \quad (24)$$

$$= \underbrace{(1 - \beta)}_{\text{MPC out of wealth}} \underbrace{\beta^{T-1} \cdot \alpha}_{\text{Change in permanent income}} \quad (25)$$

- For $t \geq 1$:

$$\mathbb{E}_{t+1}(y_{t+j+1}) = \mathbb{E}_t(y_{t+j+1}) \quad (26)$$

$$\Rightarrow c_{t+1} - c_t = 0, \forall t \geq 1 \quad (27)$$

- That is:
 - Changes only happen at announcement, not at tax cut, T .
 - A similar approach with stochastic income would yield the same result.

Variation: The only reason that T enters the above is that PDV of the α delivery is discounted for the T period. If instead, the government announces they will set aside α , put it in the bank at R interest, and then deliver the α with interest at time T . In that case, interest compounds for $T - 1$ period, which means that

$$c_1 - c_0 = (1 - \beta) \beta^{T-1} (R^{T-1} \alpha) = (1 - \beta) \alpha$$

i.e., the tax break (no matter when it is actually implemented) adds α to the PDV of lifetime earning.

2.3 Example from Friedman-Muth

- Setup:

$$y_t = z_t + u_t \quad (28)$$

$$z_{t+1} = z_t + \sigma_1 w_{1t+1} \quad (29)$$

$$u_{t+1} = \sigma_2 w_{2t+1} \quad (30)$$

where y_t is income, z_t is the *persistent* or "permanent income", u_t is transitory changes in income;

- Which one is a martingale (i.e., random walk here)?
- Define the vector of shocks $w_{t+1} = \begin{pmatrix} w_{1t+1} \\ w_{2t+1} \end{pmatrix} \sim N(0_2, I_{2 \times 2})$, i.e. iid normal distributed, mean 0, variance 1.

- Setup in linear state space form:

$$\text{Since } x_t = \begin{pmatrix} z_t \\ u_t \end{pmatrix}, \text{ we have:} \quad (31)$$

$$\underbrace{\begin{pmatrix} z_{t+1} \\ u_{t+1} \end{pmatrix}}_{x_{t+1}} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_A \cdot \underbrace{\begin{pmatrix} z_t \\ u_t \end{pmatrix}}_{x_t} + \underbrace{\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}}_C \underbrace{\begin{pmatrix} w_{1t+1} \\ w_{2t+1} \end{pmatrix}}_{w_{t+1}} \quad (32)$$

$$y_t = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_G \cdot \underbrace{\begin{pmatrix} z_t \\ u_t \end{pmatrix}}_{x_t} \quad (33)$$

$$I - \beta A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 - \beta & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \quad (34)$$

$$(I - \beta A)^{-1} = \begin{pmatrix} \frac{1}{1-\beta} & 0 \\ 0 & 1 \end{pmatrix}, \text{ since diagonal matrix, its inverse is just 1 over each element} \quad (35)$$

$$G(I - \beta A)^{-1} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{1-\beta} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{1-\beta} & 1 \end{pmatrix} \quad (36)$$

- Recall:

$$c_t = (1 - \beta) \left[F_t + \mathbb{E}_t \left(\sum_{j=0}^{\infty} \beta^j y_{t+j} \right) \right] \quad (37)$$

$$= (1 - \beta) \left[F_t + G(I - \beta A)^{-1} x_t \right] \quad (38)$$

$$\text{in this example} = (1 - \beta) \left[F_t + \begin{pmatrix} \frac{1}{1-\beta} & 1 \end{pmatrix} \cdot \begin{pmatrix} z_t \\ u_t \end{pmatrix} \right] \quad (39)$$

$$c_t = (1 - \beta) \left[F_t + \frac{1}{1 - \beta} z_t + u_t \right] \Rightarrow \quad (40)$$

$$c_t = (1 - \beta) F_t + z_t + (1 - \beta) u_t \quad (41)$$

Note: coefficient on u_t is $(1 - \beta)$, the marginal propensity to consumer (MPC) out of transitory income: coefficient of z_t is 1, which is the MPC out of permanent income. The marginal propensity to consumer out of financial wealth F_t is the same as before.

- Recall:

$$c_{t+1} - c_t = (1 - \beta) G(1 - \beta A)^{-1} \cdot C \cdot w_{t+1} \quad (42)$$

$$= (1 - \beta) \begin{pmatrix} \frac{1}{1-\beta} & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \cdot \begin{pmatrix} w_{1t+1} \\ w_{2t+1} \end{pmatrix} \quad (43)$$

$$= \sigma_1 w_{1t+1} + (1 - \beta) \sigma_2 w_{2t+1} \quad (44)$$

i.e. Consumes all of the permanent shock, and the MPC out of the transitory shock.

- What about savings?

Recall:

$$F_{t+1} - F_t = \frac{1}{\beta} \left[y_t - (1 - \beta) \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j y_{t+j} \right] \quad (45)$$

$$= \frac{1}{\beta} \left[G \cdot x_t - (1 - \beta) G(I - \beta A)^{-1} x_t \right] \quad (46)$$

$$= \frac{1}{\beta} G \left[I - (1 - \beta) G(I - \beta A)^{-1} \right] x_t \quad (47)$$

$$G \cdot I = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix} \quad (48)$$

$$G(I - \beta A)^{-1} = \begin{pmatrix} \frac{1}{1-\beta} & 1 \end{pmatrix} \text{ from before } \Rightarrow \quad (49)$$

$$F_{t+1} - F_t = \frac{1}{\beta} \left[\begin{pmatrix} 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 - \beta \end{pmatrix} \right] \begin{pmatrix} z_t \\ u_t \end{pmatrix} \quad (50)$$

$$= \frac{1}{\beta} \begin{pmatrix} 0 & \beta \end{pmatrix} \begin{pmatrix} z_t \\ u_t \end{pmatrix} \quad (51)$$

$$= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} z_t \\ u_t \end{pmatrix} \Rightarrow \quad (52)$$

$$\boxed{F_{t+1} - F_t = u_t} \quad (53)$$

i.e. Consumer spends all of z_t , saves nothing but a fraction of transitory income (Note returns on savings to F_{t+1})