Theory of Interest Rates

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1 Basic Setup

1.1 Consumer's preference

- Consider economy with $i = 1, \ldots I$ consumers with identical preferences:
- All consumers, labeled by the i, maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i), \text{ where } u' > 0, u'' < 0, \beta \in (0,1) \tag{1}$$

- All $i = 1, \ldots I$ consumers have the same utility function and discount factor.
- Consumers may have different deterministic endowments, $\{y_t^i\}_{t=0}^{\infty}$ for $i = 1, \cdots I$. There is no uncertainty.
- At time 0, the consumer can buy or sell a claim to one unit of consumption at date $t \geq 1$ at price q_t^0 (where t is the delivery date of 1 unit of consumption and 0 is date of trade)
- In finance terminology: q_t^0 is the time 0 price of a zero coupon bond maturing at time t with a face value of 1 unit of consumption.

1.2 Consumer's endowment

- Assume that the consumer owns an endowment stream $\{y_t^i\}_{t=0}^{\infty}$. At time 0, the consumer can <u>sell</u> this endowment stream for

$$w_0^i \equiv \sum_{t=0}^{\infty} \underbrace{q_t^0}_{\text{price}} \underbrace{y_t^i}_{\text{quantity}} \tag{2}$$

- Think of consumer as <u>price taker</u> given that she faces $\{q_t^0\}_{t=0}^{\infty}$ prices at time 0, given ownership of endowment $\{y_t^i\}_{t=0}^{\infty}$

1.3 Consumer's problem

- Consumer's problem: Given $\{q_t^0\}_{t=0}^{\infty}$,

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t^i) \tag{3}$$

$$\text{s.t.} \quad \underbrace{\sum_{t=0}^{\infty} q_t^0 c_t^i}_{\substack{\text{Buying delivery} \\ \text{in time } t \\ \text{of } c_t^i \text{ units}}} \leq \underbrace{\sum_{t=0}^{\infty} q_t^0 y_t^i}_{\substack{\text{selling endowments} \\ \text{for all } t \text{ at time } 0}} \tag{4}$$

(Note: The q_t^0 will contain the present value, but that will come endogenously)

- Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t^i) + \lambda^i \left[\sum_{t=0}^{\infty} q_t^0 (y_t^i - c_t^i) \right], \text{where } \lambda^i \text{ is the LM on the lifetime budget constraint} \tag{5}$$

- FONCs:

$$[c_t^i] : \boxed{\beta^t u'(c_t^i) = \lambda^i q_t^0}, t = 0, 1 \cdots \tag{6}$$

$$\text{B.C.} : \sum_{t=0}^{\infty} q_t^0 (y_t^i - c_t^i) = 0 \tag{7}$$

2 Competitive Equilibrium

The following are used to define a CE in this economy:

- A feasible allocation is a set of $\{c_t^i\}_{t=0}^{\infty}$ that satisfies $\sum_{i=1}^{I} c_t^i \leq \sum_{i=1}^{I} y_t^i$, for $t = 0, 1, \cdots$
- A price system is a $\{q_t^0\}_{t=0}^{\infty}$
- A competitive equilibrium is a feasible allocation and price system such that:
  - Taking $\{q_t^0\}_{t=0}^{\infty}$ as given, $\{c_t^i\}_{t=0}^{\infty}$ solves the household's problem for all consumers $i = 1, \ldots I$.

Reminder: Lagrange multipliers can be be rescaled:

$$x^* = \text{argmax}_x f(x) \quad \text{s.t. } g(x) = 0 \tag{8}$$

$$= \text{argmax}_x f(x) \quad \text{s.t. } A\, g(x) = 0 \tag{9}$$

For all $A \neq 0$ (which, crucially, cannot independent on x). In practice this means that the lagrange multiplier on the $g(x) = 0$ constraint can be rescaled as you like (which, we can interpret as normalizing a price level). As we will see below, though, if you have multiple constraints then it only means that one can be rescaled - and the relative level of constraints is fully determined.

We used this trick before in the move to the present value lagrange multiplier in the permanent income model.

3 Example

3.1 Example 1

- Consider an economy with I consumers having identical endowment sequences $y_t^i = y_t$ for $i = \{2 \cdots I\}$, $t = \{0, 1, \cdots\}$. Construct a competitive equilibrium with "guess and verify".

- Guess:

  - Non-trades: $c_t^i = y_t^i$, where $\forall i = 1, \cdots I, t = \{0, 1, \cdots\}$
  - From this, use the equation to reverse engineer prices: $q_t^0 = \beta^t \frac{u'(y_t^i)}{u'(y_0^i)}$. Note that since $y_i t$ is constant for all i, this can hold.
  - From this guess, $q_0^0 = u'(y_0^i)/u'(y_0^i) = 1$, and from (6), $\lambda^i = u'(y_0^i)$
  - This is setting the (indeterminate) initial price level because the budget constraint is in nominal terms.
    * Because of this indeterminacy, we could choose to normalize either the λ^i or the q_0^0, given $q_0^0 = u'(c_0^i)/\lambda^i$ from (6).
    * Another way to think of this is that the budget constraint in (4), could be divided by q_0^0 to get the equivalent $\sum\limits_{t=0}^{\infty} \frac{q_t^0}{q_0^0}(y_t - c_t) = 0$, where only the interpretation of λ changes.
    * Another common normalization is to have $\lambda = 1$ and hence $q_0^t = \beta^t u'(y_t^i)$, in which case $q_0^0 = u'(y_0^i)$

- Verify:

  - FONC: We used this to reverse engineer the prices: $\beta^t \frac{u'(c_t^i)}{u'(c_0^i)} = q_t^0$, this can hold true if $c_t^i = y_t^i$
  - Budget: $\sum\limits_{t=0}^{\infty} q_t^0(y_t^i - c_t^i) = 0$ holds since no trades occur
  - Feasibility: Trivial

3.2 Example 2

- Let $I = 1$, the strategy in previous example still works (a "representative consumer), if all have same endowment but we still assume price taking.

- Assume that the endowment is constant, $y_t^i = y_0$ (Constant!), then:

$$q_t^0 = \beta^t \frac{u'(y_0)}{u'(y_0)} \tag{10}$$

  Consequently,

$$q_t^0 = \beta^t \tag{11}$$

- The above formula is our $R\beta = 1$ specification. i.e. since $\frac{q_1^0}{q_0^0} = \beta$, $\frac{1}{\beta}$ pays for 1 unit of consumption today.

3.3 Example 3

- More generally for this representative agent, if $\{y_t\}$, then:

$$q_t^0 = \beta^t \frac{u'(c_t)}{u'(c_0)}, \forall t \tag{12}$$

3.4 Example 4

- Assume $I = 1, u(c_t) = \ln c_t$, $\Rightarrow u'(c_t) = \frac{1}{c_t}$, $y_t = y_0 \delta^t$

- Competitive equilibrium, non-trade:

$$\Rightarrow \boxed{q_t^0 = \left(\frac{\beta}{\delta}\right)^t} \tag{13}$$

- Require $\frac{\beta}{\delta} < 1$ for wealth $\sum\limits_{t=0}^{\infty} q_t^0 y_t < \infty$

3.5 Example 5

- $I = 2$, such that: $\begin{cases} y_t^1 = \{1, 0, 1, 0, \cdots\} \\ y_t^2 = \{0, 1, 0, 1, \cdots\} \end{cases}$,

  Note that $y_t^1 + y_t^2 = 1, \forall t$

- Guess:

- (1) $c_t^1 = c^1, c_t^2 = c^2$, for $t = \{0, 1, \cdots\}$ for some c^1, c^2 to be determined, where $c^1 + c^2 = 1, \forall t$. (Total consumption smoothing)
- (2) $q_t^0 = \beta^t$

  Notes: Could have chosen initial level to be any constant. Only relative prices matter. Related to lagrange multiplier

- Verify:
  - For consumer 1, use (6):

$$\lambda^1 q_t^0 = \beta^t u'(c^1) \Rightarrow \lambda^1 \beta^t = \beta^t u'(c^1) \tag{14}$$

    By cancelling out β^t, $\lambda^1 = u'(c^1)$, which is constant.

  - For consumer 2, also use (6):

$$\lambda^2 q_t^0 = \beta^t u'(c^2) \Rightarrow \lambda^2 = u'(c^2) \tag{15}$$

    By using the same method as above, which is constant

- To find c^1 and c^2, use budget:

$$\sum_{t=0}^{\infty} q_t^0 c_t^1 = \sum_{t=0}^{\infty} q_t^0 y_t^1 \Rightarrow \tag{16}$$

$$\frac{c^1}{1-\beta} = \sum_{t=0}^{\infty} \beta^t y_t^1 = 1 + \beta^2 + \beta^4 + \cdots = \frac{1}{1-\beta^2} \Rightarrow \tag{17}$$

$$\boxed{c^1 = \frac{1}{1+\beta}} \tag{18}$$

  Similarly,

$$\sum_{t=0}^{\infty} q_t^0 c_t^2 = \sum_{t=0}^{\infty} q_t^0 y_t^2 \Rightarrow \tag{19}$$

$$\frac{c^2}{1-\beta} = \sum_{t=0}^{\infty} \beta^t y_t^2 = \beta + \beta^3 + \cdots = \frac{\beta}{1-\beta^2} \Rightarrow \tag{20}$$

$$\boxed{c^2 = \frac{\beta}{1+\beta}} \tag{21}$$

- Check:

$$c^1 + c^2 = \frac{1}{1+\beta} + \frac{\beta}{1+\beta} = 1 \text{ (Feasible!)} \tag{22}$$

3.6 Example 6

- Assume $I = 2$, with:

$$y_t^1 = \begin{cases} 1 \text{ for } t = 0, \cdots 10 \\ 0 \text{ for } t \geq 11 \end{cases} \tag{23}$$

$$y_t^2 = \begin{cases} 0 \text{ for } t = 0, \cdots 10 \\ 1 \text{ for } t \geq 11 \end{cases} \tag{24}$$

where $y_t^1 + y_t^2 = 1, \forall t$.

- Guess as before: $\begin{cases} c_t^1 = c^1 \\ c_t^2 = c^2 \\ q_t^0 = \beta^t \end{cases}$

- To find c^1:

$$\sum_{t=0}^{\infty} \beta^t c^1 = \sum_{t=0}^{10} \beta^t = \frac{1-\beta^{11}}{1-\beta} \tag{25}$$

$$\Rightarrow \boxed{c^1 = 1 - \beta^{11}} \tag{26}$$

To find c^2:

$$\sum_{t=0}^{\infty} \beta^t c^2 = \sum_{t=11}^{\infty} \beta^t = \frac{\beta^{11}}{1-\beta} \tag{27}$$

$$\Rightarrow \boxed{c^2 = \beta^{11}} \tag{28}$$

So we have $c^1 + c^2 = 1$

- Does c^1 consume more?

$$c^1 > c^2 \text{ if } 1 - \beta^{11} > \beta^{11} \tag{29}$$

$$\Rightarrow 1 > 2\beta^{11} \tag{30}$$

$$\Rightarrow \text{ only if } \beta < 2^{-11} \tag{31}$$

4 Term Structure and Interest Rates

4.1 A Special Case

Recall our old 1 period riskless investment, at a gross interest rate of $R > 0$. Consider that it could very with time and convert to bonds at time 0 for a single period investment. If it costs q_1^0 to buy a claim to a unit of the good at time 1, then spending 1 delivers $1/q_1^0$ delivered tomorrow, that is,

$$R_t \equiv 1/q_{t+1}^t \tag{32}$$

In the special case of the aggregate endowment $\sum_i y_t^i = \bar{y}$ for all t, then we know $q_t^0 = \beta^t$. That is,

$$R = 1/\beta \tag{33}$$

The punchline here is that $\beta R = 1$ is the natural interest rate for an economy with a constant endowment (regardless of ho that endowment is allocated).

4.2 Yield to Maturity

Recall, q_t^0 is the time 0 price of a zero coupon bond maturing at time t. Can convert q_t^0 to a t-period interest rate (or t-period yield to maturity) using the definition:

$$\frac{q_t^0}{q_0^0} \equiv \frac{1}{(1+r_{0,t})^t} \tag{34}$$

As an example, assume with a representative agent that $y_t = \bar{y}$ for all t, then from our previous example,

$$\frac{q_0^t}{q_0^0} = \beta^t = \frac{1}{(1+r_{0,t})^t} \tag{35}$$

Take the t'th root,

$$1 + r_{0,t} \equiv \frac{1}{\beta} \tag{36}$$

As $r_{0,1}$ is the *net* interest rate, $1 + r_{0,1}$ is the *gross* interest rate on a 1 period bond. If we let $R \equiv 1 + r_{0,1}$, then we get our standard $\beta R = 1$ case. This is a special case of a constant *aggregate* endowment, or the case of risk-neutrality with any endowment. Also note that there is a flat yield curve in this case due to (36).

More generally, given $\{q_t^0\}$ and $q_0^0 = 1$, we can plot $r_{0,t}$. One can get "yield curve" data from bond market prices online, then calculate the implied q_t^0. This gives us agent forecasts on future aggregate consumption.[1]

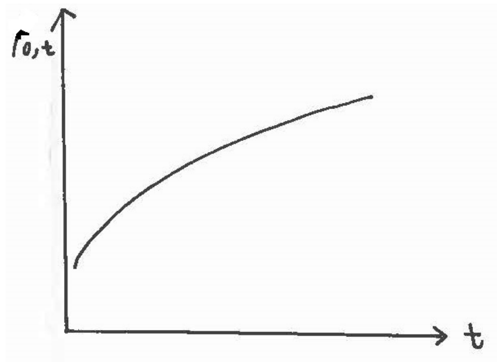

Figure 1: A typical yield curve in the data

[1]To a first order this can also be approximated by,

$$\underbrace{\frac{q_t^0}{q_0^0} \equiv \frac{1}{(1+r_{0,t})^t}}_{\text{definition}} \approx \underbrace{exp(-tr_{0,t})}_{\text{First order approximation}} \tag{37}$$

So given $\{q_t^0\}$,

$$r_{0,t} = \frac{\log(q_t^0)}{-t} \tag{38}$$