Permanent Income Model

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1 Permanent Income Model

1.1 Basic Setup

This started with Friedman in the 1950s, and was done in rational expectations with Hall in the late 70s.

- Agent has an (exogenous) deterministic income $\{y_{t+j}\}_{j=0}^{\infty}$ and (initially exogenous) savings F_t.
- Chooses sequence of consumption to maximize the PDV of utility of consumption $u(c_t)$ for all t
- That is: at time t, solves (given exogenous $\{y_{t+j}\}$ and F_t):

$$\max_{(c_{t+j})_{j=0}^{\infty}} \left\{ \sum_{j=0}^{\infty} \beta^j u(c_{t+j}) \right\} \tag{1}$$

$$\text{s.t.} \sum_{j=0}^{\infty} \underbrace{\left(\frac{1}{R}\right)^j}_{\substack{\text{discounting with}\\ \text{interest rate}}} (\underbrace{y_{t+j}}_{\text{labor income}} - \underbrace{c_{t+j})}_{\text{consumption}} + \underbrace{F_t}_{\text{assets}} = 0 \tag{2}$$

where $\beta \in (0,1)$ is the discount factor and $R > 1$ is the gross interest rate
i.e. Future assets would pay for any difference between earnings and labor income(when discounted)

1.2 Lagrangian

$$\mathcal{L} = \sum_{j=0}^{\infty} \beta^j u(c_{t+j}) + \lambda \left[\sum_{j=0}^{\infty} \left(\frac{1}{R}\right)^j (y_{t+j} - c_{t+j}) + F_t \right] \tag{3}$$

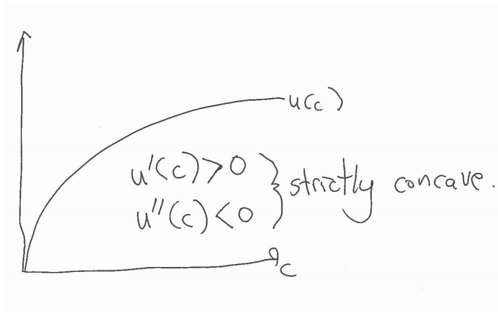

Figure 1: Stricly Concave Utility Function

Note:
- Infinite number of variables $\{c_{t+j}\}_{j=0}^{\infty}$ and one constraint
- The constraint is a lifetime budget constraint

- First Order Necessary Condition(FONC):

$$c_{t+j} : \beta^j u'(c_{t+j}) - \lambda R^{-j} = 0 \tag{4}$$

$$\Rightarrow u'(c_{t+j}) = \left(\frac{1}{\beta R}\right)^j \cdot \lambda, \forall j \tag{5}$$

- For example:
If $R = \frac{1}{\beta}$, then $u'(c_{t+j}) = \left(\frac{1}{\beta R}\right)^j \cdot \lambda = \lambda, \forall j$
i.e. slope at $c_{t+j} = \lambda$, which means the marginal utility is constant
$\Rightarrow c_{t+j}$ is constant for all j
$\Rightarrow \bar{c} = c_{t+j} = u'^{-1}(\lambda)$

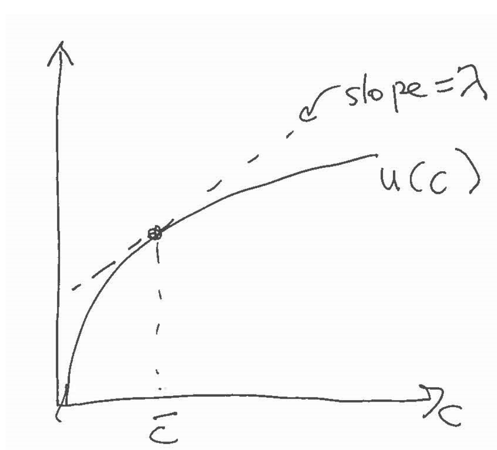

Figure 2: Lagrange Multiplier = Slope at Choice

- Take Budget constraint:

$$\sum_{j=0}^{\infty} R^{-j} c_{t+j} = F_t + \sum_{j=0}^{\infty} R^{-j} y_{t+j} \tag{6}$$

For example:
If $R = \frac{1}{\beta}$ solve for $\bar{c}$

$$\bar{c} = \underbrace{(1-\beta)}_{\substack{\text{marginal propensity}\\ \text{to consume}\\ \text{out of wealth}}} \left[\underbrace{\sum_{j=0}^{\infty} \beta^j y_{t+j}}_{\substack{\text{PDV of}\\ \text{human}\\ \text{wealth}}} + \underbrace{F_t}_{\substack{\text{Financial}\\ \text{wealth}}} \right] \tag{7}$$

1.3 Manipulating the budget constraint

$$\text{At time t:} \sum_{j=0}^{\infty} R^{-j} y_{t+j} + F_t = \sum_{j=0}^{\infty} R^{-j} c_{t+j} \tag{8}$$

$$\text{At time t+1:} \sum_{j=0}^{\infty} R^{-j} y_{t+j+1} + F_{t+1} = \sum_{j=0}^{\infty} R^{-j} c_{t+j+1} \tag{9}$$

Multiply the time $t+1$ equation by $\frac{1}{R}$, write out terms,
At time t :

$$F_t + y_t + R^{-1} y_{t+1} + R^{-2} y_{t+2} + \cdots = c_t + R^{-1} c_{t+1} + R^{-2} c_{t+2} + \cdots \tag{10}$$

At time t+1, then

$$R^{-1} F_{t+1} + ... + R^{-1} y_{t+1} + R^{-2} y_{t+2} + \cdots = R^{-1} c_{t+1} + R^{-2} c_{t+2} + \cdots \tag{11}$$

Subtracting equations, most terms were canceled out:

$$F_t - R^{-1} F_{t+1} + y_t = c_t \tag{12}$$

$\Rightarrow$

$$\boxed{\underbrace{F_{t+1}}_{\substack{\text{next}\\\text{periods}\\\text{wealth}}} = \underbrace{R}_{\substack{\text{gross}\\\text{interest}\\\text{rate}}} \left[\underbrace{F_t}_{\substack{\text{this}\\\text{period}\\\text{wealth}}} + \underbrace{y_t - c_t}_{\text{savings}} \right]} \tag{13}$$

Note: Sometimes define $1 + \underbrace{r}_{\substack{\text{net}\\\text{interest}\\\text{rate}}} \equiv \underbrace{R}_{\substack{\text{gross}\\\text{interest}\\\text{rate}}}$

So an equivalent form of the Permanent Income Hypothesis is:

$$\max_{\{c_t, F_{t+1}\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\}, F_0 \text{ is given} \tag{14}$$

$$\text{s.t. } F_{t+1} = R\left(F_t + y_t - c_t\right), t = 0, \cdots, \infty, \tag{15}$$

Note:

- In addition, it will require a variation of $\lim_{T\to\infty} \beta^{T+1} F_{T+1} \geq 0$, (Transversality Condition, No Ponzi-condition, etc.)

i.e. Agent cannot asymptotically have debt. We will ignore this constraint for now.

- There are period-by-period budget constraints.
- Chooses consumption and savings for next period.
- R is the gross rate of return on assets.

Lagrangian for Lagrange Multipliers $\hat{\lambda}_t$ on budgets:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \hat{\lambda}_t [R(F_t + y_t - c_t) - F_{t+1}] \text{ where } \hat{\lambda}_t \text{ is LM on budget constraints} \tag{16}$$

Writing out portions of the sequence:

$$\mathcal{L} = \cdots + \beta^t u(c_t) + \beta^{t+1} u(c_{t+1}) + \hat{\lambda}_t [R(F_t + y_t - c_t) - F_{t+1}] + \tag{17}$$

$$\hat{\lambda_{t+1}} [R(F_{t+1} + y_{t+1} - c_{t+1}) - F_{t+2}] + \cdots \tag{18}$$

FOC(c_t): Use these terms,

$$0 = \beta^t u'(c_t) - \hat{\lambda}_t R \Rightarrow \hat{\lambda}_t R = \beta^t u'(c_t) \tag{19}$$

FOC(F_{t+1}):

$$0 = -\hat{\lambda_t} + \hat{\lambda_{t+1}} R \Rightarrow \frac{\hat{\lambda_{t+1}}}{\hat{\lambda_t}} = \frac{1}{R} \tag{20}$$

Take (19) at time t+1 and at time t:

$$\frac{\beta^{t+1}}{\beta^t} \frac{u'(c_{t+1})}{u'(c_t)} = \frac{\hat{\lambda_{t+1}} \cdot R}{\hat{\lambda_t} \cdot R} \tag{21}$$

Use (20)

$$\beta \frac{u'(c_{t+1})}{u'(c_t)} = \frac{1}{R} \Rightarrow \tag{22}$$

$$\boxed{u'(c_t) = \beta R u'(c_{t+1}) \text{ (Euler equation)}} \tag{23}$$

Note in (19), let $\lambda_t \beta^t \equiv \hat{\lambda_t}$ (just new definition for λ_t), then $\lambda_t \beta^t R = \beta^t u'(c_t) \Rightarrow u'(c_t) = R\lambda_t$.

These are present-value Lagrange multipliers and can be stationary instead of exponentially shrinking (i.e., if $c_t \to \bar{c}$).

Rescaling Lagrange Multipliers In general lagrange multipliers can be rescaled and redefined to make either math or interpretation easier. One way to think of this is to notice that

$$\begin{aligned} x^* &= \text{argmax}_x f(x) \quad \text{s.t. } g(x) = 0 && (24) \\ &= \text{argmax}_x f(x) \quad \text{s.t. } A\, g(x) = 0 && (25) \end{aligned}$$

For all $A \neq 0$ (which, crucially, cannot dependent on x).

In the above, what this means is that the FONCs using the two Lagrange multipliers would lead to the same x solutions. That is, taking the solving the FONC of

$$\mathcal{L}_1 \equiv f(x) + \lambda_1 g(x) \tag{26}$$

and

$$\mathcal{L}_2 \equiv f(x) + \lambda_2 A g(x) \tag{27}$$

wouldl lead to different $\lambda_1 \neq \lambda_2$ solution, but would have same x.

Because of this, we may sometimes skip rescaling steps like the above and write down the Lagrange multipliers directly as λ_2 from $\max_x f(x) \quad \text{s.t. } g(x) = 0$ directly, skipping the multiplication by A as above.

Alternative Lagrangian (Generally Preferred)

Directly group the λ_t inside sum,

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + \lambda_t [R(F_t + y_t - c_t) - F_{t+1}] \right], \text{ inside sum } \lambda_t \text{ is the same as } \lambda_t \beta^t \equiv \hat{\lambda}_t \tag{28}$$

$$\mathcal{L} = \cdots + \beta^t [u(c_t) + \lambda_t [R(F_t + y_t - c_t) - F_{t+1}]] + \tag{29}$$

$$\beta^{t+1} [u(c_{t+1}) + \lambda_{t+1} [R(F_{t+1} + y_{t+1} - c_{t+1}) - F_{t+2}]] + \cdots \tag{30}$$

$\frac{\partial}{\partial c_t}$:

$$\beta^t [u'(c_t) - \lambda_t R] = 0 \Rightarrow u'(c_t) = R\lambda_t \tag{31}$$

$\frac{\partial}{\partial F_{t+1}}$:

$$-\beta^t \lambda_t + \beta^{t+1} R \lambda_{t+1} = 0 \Rightarrow \frac{\lambda_{t+1}}{\lambda_t} = \frac{1}{\beta R} \tag{32}$$

Take (31) divided at t and $t+1$

$$\frac{u'(c_{t+1})}{u'(c_t)} = \frac{\lambda_{t+1}}{\lambda_t} \text{ (These are called "present-value lagrange multipliers")} \tag{33}$$

$$= \frac{1}{\beta R} \text{ from (32)} \tag{34}$$

$\Rightarrow$

$$\boxed{u'(c_t) = \beta R u'(c_{t+1}) \text{ (Identical Euler equation)}} \tag{35}$$

- Recall that Budget constraints need to be held. Could use either the ∞ number of period-by-period constraints, or lifetime budget constraint:

$$\boxed{\sum_{j=0}^{\infty} \left(\frac{1}{R}\right)^j c_{t+j} = F_t + \sum_{j=0}^{\infty} \left(\frac{1}{R}\right)^j y_{t+j}} \tag{36}$$

With this and Euler Equation, may not need to solve for intermediate F_{t+1} choices, though could at end.

2 Finite Horizon

2.1 Finite Horizon (Formally)

Die at $T > 0$, i.e. no utility from consumption.
Need additional constraint so they do not die in debt:

$$\max_{\{c_t, F_{t+1}\}_{t=0}^{T}} \left\{ \sum_{t=0}^{T} \beta^t u(c_t) \right\} \tag{37}$$

$$\text{s.t. } F_{t+1} = R(F_t + y_t - c_t) \text{ for } t = 0, \cdots T \tag{38}$$

$$F_{T+1} \geq 0 \text{ (weakly positive assets at death)} \tag{39}$$

$$F_0 \text{ given} \tag{40}$$

Multipliers: Use present value constraints:

$$\lambda_0 \cdots \lambda_T \text{ on first constraints} \tag{41}$$

$$\lambda_{T+1} \text{ on last constraint} \tag{42}$$

i.e. total of $T + 2$ constraints

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \left[u(c_t) + \lambda_t [R(F_t + y_t - c_t) - F_{t+1}] \right] + \underbrace{\beta^{T+1} \lambda_{T+1}}_{\substack{\text{terminal} \\ \text{constraint} \\ \text{present value} \\ \hat{\lambda}_{T+1} \equiv \beta^{T+1} \lambda_{T+1}}} F_{T+1} \tag{43}$$

FOC(c_t):

$$u'(c_t) = R\lambda_t, \text{ for t} = 0, \cdots T \tag{44}$$

$$\text{i.e. die at T+1, no utility} \Rightarrow \frac{u'(c_{t+1})}{u'(c_t)} = \frac{\lambda_{t+1}}{\lambda_t}, \forall t \leq T - 1 \tag{45}$$

FOC(F_{t+1}), for t < T:

$$0 = -\lambda_t + \lambda_{t+1} \beta R \Rightarrow \lambda_t = \beta R \lambda_{t+1} \tag{46}$$

FOC(F_{T+1}):

$$0 = -\beta^T \lambda_T + \beta^{T+1} \lambda_{T+1} \Rightarrow \lambda_T = \beta \lambda_{T+1} \tag{47}$$

With $\underbrace{\beta^{T+1} F_{T+1} \lambda_{T+1} = 0, \lambda_{T+1} \geq 0}_{\substack{\text{Since inequality constraint} \\ \text{Need complementarity slackness}}}$

Does terminal constraint bind? (By Contradiction)
Assume not, i.e. $\lambda_{T+1} = 0 (\Rightarrow F_{T+1} > 0)$. Then from (47), $\lambda_T = 0$, and from (46) $\lambda_t = 0$ for all t.
So if it does not bind at time T+1, it never binds. Only possible if $u'(c_t) = 0 \ \forall$ t, which is a contradiction. Hence, $F_{T+1} = 0$ (unless there is a budget feasible satiation point where $u'(c_t) = 0$ for all t. e.g. Quadratic utility with enormous budgets.)

2.2 Period Budget Constraints and Lifetime Budget Constraints

See a derivation of Appendix B when takes the

$$F_{t+1} = R(F_t + y_t - c_t), \forall t = 0, \cdots T \tag{48}$$

$$F_{T+1} = 0 \tag{49}$$

and finds

$$\boxed{\underbrace{\sum_{j=0}^{T-t} R^{-j} c_{t+j}}_{\substack{\text{PDV of} \\ \text{consumption}}} = \underbrace{\sum_{j=0}^{T-t} R^{-j} y_{t+j}}_{\substack{\text{PDV of} \\ \text{labor income}}} + \underbrace{F_t}_{\substack{\text{Financial} \\ \text{wealth}}}} \tag{50}$$

If $T \to \infty$, then we see that this is identical to (2). This result provides somewhat of the opposite direction to Section 1.3, where we show that the sequential constraints imply the lifetime budget constraint (under conditions on what happens to F_T as $T \to infinity$).

2.3 Summary of Equations

$$u'(c_t) = \beta R u'(c_{t+1}), \forall t = 0, \cdots T-1 \tag{51}$$

$$F_{t+1} = R(F_t + y_t - c_t), \forall t = 0, \cdots T \tag{52}$$

$$F_{T+1} = 0, F_0 \text{ given} \tag{53}$$

Given : $F_0, \{y_t\}_{t=0}^T$, find $\{c_t\}_{t=0}^T$. How?

In general, we need computers to solve this problem. See Appendix A for an example algorithm.

3 Examples

Assume we have utility as following:

$$u(c) = \log(c), T \leq \infty \tag{54}$$

$$\Rightarrow u'(c) = \frac{1}{c} \tag{55}$$

From Euler equation:

$$\frac{1}{c_t} = \beta R \frac{1}{c_{t+1}} \Rightarrow c_{t+1} = \beta R c_t, \text{ for } t = 0, \cdots T \tag{56}$$

$$\Rightarrow c_t = (\beta R)^t c_0 \tag{57}$$

Find c_0 from lifetime budget:

Example 1: $T = \infty$, $y_t = \delta^t y_0$, $F_0 = 0$
Use budget:

$$0 = \sum_{j=0}^{\infty} R^{-j} \left((\beta R)^j c_0 - \delta^j y_0 \right) \tag{58}$$

$$\Rightarrow c_0 \sum_{j=0}^{\infty} \beta^j = \sum_{j=0}^{\infty} (\delta R^{-1})^j y_0 \text{ (assume } |\frac{\delta}{R}| < 1) \tag{59}$$

$\Rightarrow$

$$\boxed{c_0 = (1-\beta) \cdot \left(\frac{1}{1 - \frac{\delta}{R}} \right) y_0} \tag{60}$$

Example 2: $T < \infty$, $y_t = \delta^t y_0$, $F_0 = 0$

$$0 = \sum_{j=0}^{T} R^{-j}\left((\beta R)^j c_0 - \delta^j y_0\right) \tag{61}$$

$$\Rightarrow c_0 \sum_{j=0}^{T} \beta^j = \sum_{j=0}^{T} (\delta R^{-1})^j y_0 \tag{62}$$

$$\text{Use partial geometric sums:} \frac{1-\beta^{T+1}}{1-\beta} c_0 = \left[\frac{1-\left(\frac{\delta}{R}\right)^{T+1}}{1-\frac{\delta}{R}}\right] y_0 \tag{63}$$

Example 3: $F_0 = 0, T = 70, y_t = \begin{cases} \delta^t y_0, t = 0, \cdots 50 \\ 0, t = 51 \cdots 70 \end{cases}$

i.e. stop working at 50, die at 70, where t is the age.

Put into budget:

$$c_0 \sum_{t=0}^{70} R^{-t} (\beta R)^t = \sum_{t=0}^{50} \left(\frac{\delta}{R}\right)^t \cdot y_0 \tag{64}$$

$$\Rightarrow c_0 \cdot \frac{1-\beta^{71}}{1-\beta} = \frac{1-\left(\frac{\delta}{R}\right)^{51}}{1-\frac{\delta}{R}} y_0 \tag{65}$$

$$\tag{66}$$

$\Rightarrow$ solution:

$$c_0 = \frac{1-\beta}{1-\beta^{71}} \cdot \frac{1-\left(\frac{\delta}{R}\right)^{51}}{1-\frac{\delta}{R}} \cdot y_0 \tag{67}$$

$$c_t = (\beta R)^t \cdot c_0 \tag{68}$$

$$F_{t+1} = R(F_t + y_0 \delta^t - (\beta R)^t \cdot c_0) \tag{69}$$

can substitute for c_0 to get lifetime savings.

Example in special case: $\beta R = 1 \Rightarrow c_t = c_0, \forall t.$ Constant consumption

A Shooting Method

An iterative approach: Iterative approach:

1. If we knew c_0, then since we know F_0
   (a) Use (51) to find c_1

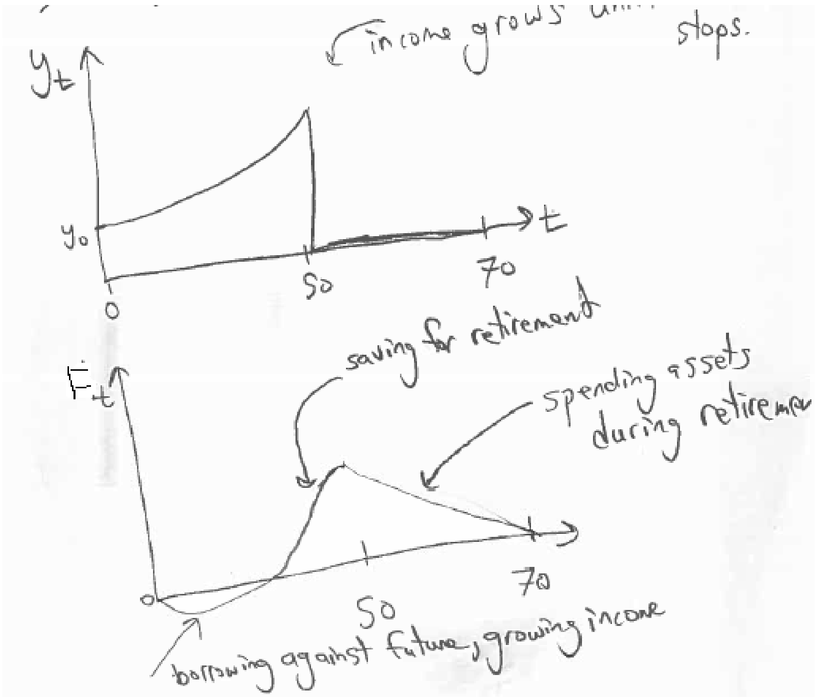

Figure 3: $\beta R = 1$

(b) Use (52) to find F_1

2. Repeat for $\{c_2, F_2\}$, etc. to $T-1$
3. Use (52) with c_T, F_T to get F_{T+1},
4. Check (53) satisfied or not.

Convergence of the Iterative solution:

1. Guess c_0
2. Find sequence using the above steps to get F_{T+1}
3. If $F_{T+1} > 0$, raise c_0 and try again If $F_{T+1} < 0$, lower c_0 and try again.
4. Stop when $F_{T+1} \approx 0$

B Lifetime Budget Constraint Derivation

We could use:

$$F_{t+1} = R(F_t + y_t - c_t), \forall t = 0, \cdots T \tag{B.1}$$

$$F_{T+1} = 0 \tag{B.2}$$

Alternatively, work backwards from T:

$$0 = R(F_T + y_T - c_T) \Rightarrow c_T - y_T = F_T \tag{B.3}$$

Then for $T - 1$:

$$F_T = R(F_{T-1} + y_{T-1} - c_{T-1}) \tag{B.4}$$

$$\Rightarrow \text{using previous result, } c_T - y_T = R(F_{T-1} + y_{T-1} - c_{T-1}) \tag{B.5}$$

$$\Rightarrow F_{T-1} = c_{T-1} - y_{T-1} + \frac{1}{R}(c_T - y_T) \tag{B.6}$$

At t $= T - 2$:

$$F_{T-1} = R(F_{T-2} + y_{T-2} - c_{T-2}) \tag{B.7}$$

$$\Rightarrow c_{T-1} - y_{T-1} + \frac{1}{R}(c_T - y_t) = R(F_{T-2} + y_{T-2} + c_{T-2}), \text{etc.} \tag{B.8}$$

Finally, repeat until F_0:

$$F_0 = \sum_{t=0}^{T} (\frac{1}{R})^t [c_t - y_t] \tag{B.9}$$

More generally, starting at any t,

$$\boxed{\underbrace{\sum_{j=0}^{T-t} R^{-j} c_{t+j}}_{\substack{\text{PDV of}\\ \text{consumption}}} = \underbrace{\sum_{j=0}^{T-t} R^{-j} y_{t+j}}_{\substack{\text{PDV of}\\ \text{labor income}}} + \underbrace{F_t}_{\substack{\text{Financial}\\ \text{wealth}}}} \tag{B.10}$$

Compare to the lifetime budget constraint