Stochastic Permanent Income Model and Government Fiscal Policy

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1 Stochastic Permanent Income

1.1 Basic setup.

Linear State Space + Normal Shock:

- Let

$$x_{t+1} = Ax_t + Cw_{t+1} \tag{1}$$

$$y_t = G \cdot x_t \tag{2}$$

where A is $n \times n$ matrix, x is $n \times 1$ vector, C is $n \times m$ matrix, $w_{t+1} \sim N(0, I_{n\times m})$, i.i.d. normal shocks; G is $1 \times n$ vector, y_t is a scalar, which means "labor income"

- Consumer's Budget Constraint (assuming $\beta R = 1$):

$$F_{t+1} = \underbrace{\frac{1}{\beta}}_{\substack{\text{gross}\\\text{interest}\\\text{rate}}} (\underbrace{F_t}_{\substack{\text{Financial}\\\text{wealth}}} + y_t - c_t) \tag{3}$$

- Recall if $\{y_t\}$ is deterministic, and $R = 1/\beta$, then for any strictly concave $u(c)$ they achieved perfect consumption smoothing:

$$c_t = (1-\beta)(\underbrace{F_t}_{\substack{\text{Financial}\\\text{wealth}}} + \underbrace{\sum_{j=0}^{\infty} \beta^j y_{t+j}}_{\substack{\text{PDV of}\\\text{human}\\\text{wealth}}}) \tag{4}$$

- If y_t is stochastic, can we just replace the above equation with expected value?:

$$c_t = (1-\beta)(F_t + \underbrace{\mathbb{E}_t\left[\sum_{j=0}^{\infty} \beta^j y_{t+j}\right]}_{\substack{\text{expected PDV} \\ \text{of human wealth} \\ \text{with information at} \\ \text{time } t}}) \tag{5}$$

Note: if $u'(c)$ is not linear, then this is only an approximation

- Combine (3) and (5):

$$F_{t+1} = \frac{1}{\beta}\left[F_t + y_t - (1-\beta)\left(F_t + \mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j y_{t+j}\right]\right)\right] \tag{6}$$

$$= F_{t+1} = \frac{1}{\beta}\left[\beta F_t + y_t - (1-\beta)\mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j y_{t+j}\right]\right] \tag{7}$$

$$\Rightarrow F_{t+1} - F_t = \frac{1}{\beta}\left[y_t - (1-\beta)\mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j y_{t+j}\right]\right] \tag{8}$$

i.e. agents adds difference between y_t and permanent income. Now use (5) at t and $t+1$,

$$c_{t+1} = (1-\beta)\left[F_{t+1} + \mathbb{E}_{t+1}\left[\sum_{j=0}^{\infty}\beta^j y_{t+j+1}\right]\right] \tag{9}$$

$$c_t = (1-\beta)\left[F_t + \mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j y_{t+j}\right]\right] \tag{10}$$

$$\Rightarrow c_{t+1} - c_t = (1-\beta)(F_{t+1} - F_t) + (1-\beta)\left[\mathbb{E}_{t+1}\left[\sum_{j=0}^{\infty}\beta^j y_{t+j+1}\right] - \mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j y_{t+j}\right]\right] \tag{11}$$

Use (8) to find (after many steps):

Proposition:

$$\boxed{c_{t+1} - c_t = (1-\beta)\sum_{j=0}^{\infty}\beta^j(\underbrace{\mathbb{E}_{t+1}(y_{t+j+1})}_{\substack{\text{Forecast of } t+1, t+2, \cdots \\ \text{with time } t+1 \\ \text{information}}} - \underbrace{\mathbb{E}_t(y_{t+j+1})}_{\substack{\text{With time } t \\ \text{information}}})} \tag{12}$$

- Consumption only changes due to "surprise" of new information changing expected value

- Only unanticipated changes in y_{t+j}, ... or other information which changes forecasts.
- Could be unanticipated changes in government policy or shock realizations.
- Finally, for a shock between $t \to t+1$ with our linear state space model:

$$
\begin{aligned}
c_{t+1} - c_t &= (1-\beta)\left[\sum_{j=0}^{\infty} \beta^j \left(\mathbb{E}_{t+1}(y_{t+j+1}) - \mathbb{E}_t(y_{t+j+1})\right)\right] && (13)\\
&= (1-\beta)\left[G(I-\beta A)^{-1} x_{t+1} - G(I-\beta A)^{-1} A x_t\right] && (14)\\
&= (1-\beta) G (I-\beta A)^{-1}\left[\underbrace{A x_t + C w_{t+1}}_{x_{t+1}} - A x_t\right] && (15)
\end{aligned}
$$

Solution with Linear Gaussian State Space

$$
\boxed{c_{t+1} - c_t = \underbrace{(1-\beta)}_{\substack{\text{Propensity to}\\ \text{Consume}}} \underbrace{G(I-\beta A)^{-1} \cdot C w_{t+1}}_{\substack{\text{PDV of impulse response}\\ \text{a shock to } x_{t+1}}}} \tag{16}
$$

i.e PDV of changes to forecasts from the realized shock.

1.2 Special case of Quadratic Preferences

- Recall Euler equation for Permanent Income Model:

$$
u'(c_t) = \beta(1+r) u'(c_{t+1}), \forall t = 0, \cdots, T-1 \tag{17}
$$

If stochastic consumption and $\beta = \frac{1}{1+r}$, just replace with expectation?

$$
\underbrace{u'(c_t)}_{\substack{\text{Marginal utility}\\ \text{this period}}} = \underbrace{\mathbb{E}_t\left[u'(c_{t+1})\right]}_{\substack{\text{Expectation of}\\ \text{marginal utility}\\ \text{next period}}} \tag{18}
$$

- Let $u(c) = \frac{a_1}{2} c^2 + a_2 c + a_3 \Rightarrow u'(c) = a_1 c + a_2$
  In Euler equation:

$$
\begin{aligned}
a_1 c_t + a_2 &= \mathbb{E}_t(a_1 c_{t+1} + a_2) && (19)\\
c_t &= \mathbb{E}_t(c_{t+1}) && (20)
\end{aligned}
$$

i.e., Euler equation implying perfect consumption smoothing with a deterministic process translates to consumption being a martingale if stochastic!

- Note:
  - In general, $\mathbb{E}_t(u(c)) \neq u(\mathbb{E}_t(c))$
  -Then, we can use the linear-stochastic state space model for forecasting $E_t c_{t+1}$ - Due to linearity, it simply forecasts mean.
  - This is a general result called "Certainty Equivalence" of optimizing a quadratic objective subject to a linear-gaussian state space model.
  - The decision is identical in a model with or without the certainty
  - However, the realized sequence contingent on the sequence, and utility, are not the same.

2 Examples

2.1 Pre-announced Tax Cut

- This will use a shock to knowledge about deterministic income processes, rather than a constant stream of shocks to income.

- Setup:
  - Government announce at $t = 0$ that at $t = 1$ it will borrow α from international markets at interest rate $(1 + r)$ per period and give it to each consumer.
  - They also announce that to eventually balance the budget, they will pay it back at $t = 2$ for simplicity by increasing taxation that period.
  - Assume consumers had deterministic y_{t+j}, which happens to consumption?

- $c_{t+1} - c_t = (1 - \beta) \sum_{j=0}^{\infty} \beta^j \left[\mathbb{E}_{t+1}(y_{t+j+1}) - \mathbb{E}_t(y_{t+j+1})\right]$

  Define: $\{\hat{y}_{t+1}\}_{j=0}^{\infty} = \left\{ y_t, \underbrace{y_{t+1} + \alpha, y_{t+2} - \alpha(1 + r)}_{\text{Only difference}}, y_{t+3}, \ldots, y_{t+j} \ldots \right\}$
  - Note that from t to $t + 1$, the agent has the news that $\{y_{t+j}\} \to \{\hat{y}_{t+j}\}$
  - This is a change in expectations:

$$c_1 - c_0 = (1 - \beta) \sum_{j=0}^{\infty} \beta^j \left[\mathbb{E}_1(y_{j+1}) - \mathbb{E}_0(y_{j+1})\right] = (1 - \beta) \sum_{j=0}^{\infty} \beta^j (\hat{y}_{j+1} - y_{j+1}) \quad (21)$$

$$\Rightarrow c_1 - c_0 = (1 - \beta) \sum_{j=0}^{\infty} \beta^j (y_{j+1} - y_{j+1}) + (1 - \beta) \left[\alpha - \beta(1 + r)\alpha\right] \quad (22)$$

- Notes: If $\beta = \frac{1}{1+r}$, then $c_1 - c_0 = 0$

  i.e. tax cut has <u>no effect</u> because of <u>anticipated rise</u> in taxes. Later, we will investigate cases why $\beta = \frac{1}{1+r}$ comes out of general equilibrium.

2.2 Timing of Tax Cuts

- Setup:
  - Between time 0 and 1, government announces that it will cut taxes to give α to each individual at a deterministic time $T \geq 1$
  - Assume they do not need to pay it back and taxes will not raise to compensate.
  - What happens to consumption at time $\{0, \ldots, T, T+1, \ldots\}$?
  - Assume y_{t+j+1} are deterministic.

- Solve:

$$c_1 - c_0 = (1-\beta) \sum_{j=0}^{\infty} \beta^j \left[\mathbb{E}_1(y_{j+1}) - \mathbb{E}_0(y_{j+1})\right] \tag{23}$$

$$= (1-\beta) \sum_{j=0}^{\infty} \beta^j \left[y_{j+1} - y_{j+1}\right] + (1-\beta) \cdot \beta^{T-1} \cdot \alpha \tag{24}$$

$$= \underbrace{(1-\beta)}_{\text{MPC out of wealth}} \quad \underbrace{\beta^{T-1} \cdot \alpha}_{\substack{\text{Chage in} \\ \text{permanent income}}} \tag{25}$$

- For $t \geq 1$:

$$\mathbb{E}_{t+1}(y_{t+j+1}) = \mathbb{E}_t(y_{t+j+1}) \tag{26}$$

$$\Rightarrow c_{t+1} - c_t = 0, \forall t \geq 1 \tag{27}$$

- That is:
  - Changes only happen at announcement, not at tax cut, T.
  - A similar approach with stochastic income would yield the same result.

<u>Variation:</u> The only reason that T enters the above is that PDV of the α delivery is discounted for the T period. If instead, the government announces they will set aside α, put it in the bank at R interest, and then deliver the α with interest at time T. In that case, interest compounds for $T-1$ period, which menas that

$$c_1 - c_0 = (1-\beta)\beta^{T-1}\left(R^{T-1}\alpha\right) = (1-\beta)\alpha$$

i.e., the tax break (no matter when it is actually implemented) adds α to the PDV of lifetime earning.

2.3 Example from Friedman-Muth

- Setup:

$$y_t = z_t + u_t \tag{28}$$
$$z_{t+1} = z_t + \sigma_1 w_{1t+1} \tag{29}$$
$$u_{t+1} = \sigma_2 w_{2t+1} \tag{30}$$

where y_t is income, z_t is the *persistent* or "permanent income", u_t is transitory changes in income;

  - Which one is a martingale (i.e., random walk here)?
  - Define the vector of shocks $w_{t+1} = \begin{pmatrix} w_{1t+1} \\ w_{2t+1} \end{pmatrix} \sim N(0_2, I_{2\times 2})$, i.e. iid normal distributed, mean 0, variance 1.

- Setup in linear state space form:

$$\text{Since } x_t = \begin{pmatrix} z_t \\ u_t \end{pmatrix}, \text{we have:} \tag{31}$$
$$\underbrace{\begin{pmatrix} z_{t+1} \\ u_{t+1} \end{pmatrix}}_{x_{t+1}} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} z_t \\ u_t \end{pmatrix}}_{x_t} + \underbrace{\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} w_{1t+1} \\ w_{2t+1} \end{pmatrix}}_{w_{t+1}} \tag{32}$$
$$y_t = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{G} \cdot \underbrace{\begin{pmatrix} z_t \\ u_t \end{pmatrix}}_{x_t} \tag{33}$$
$$I - \beta A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1-\beta & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \tag{34}$$
$$(I - \beta A)^{-1} = \begin{pmatrix} \frac{1}{1-\beta} & 0 \\ 0 & 1 \end{pmatrix}, \text{ since diagonal matrix, its inverse is just 1 over each element} \tag{35}$$
$$G(I - \beta A)^{-1} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{1-\beta} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{1-\beta} & 1 \end{pmatrix} \tag{36}$$

- Recall:

$$c_t = (1-\beta)\left[F_t + \mathbb{E}_t(\sum_{j=0}^{\infty}\beta^j y_{t+j})\right] \tag{37}$$

$$= (1-\beta)\left[F_t + G(I-\beta A)^{-1}x_t\right] \tag{38}$$

$$\text{in this example} = (1-\beta)\left[F_t + \begin{pmatrix}\frac{1}{1-\beta} & 1\end{pmatrix}\cdot\begin{pmatrix}z_t\\ u_t\end{pmatrix}\right] \tag{39}$$

$$c_t = (1-\beta)\left[F_t + \frac{1}{1-\beta}z_t + u_t\right] \Rightarrow \tag{40}$$

$$c_t = (1-\beta)F_t + z_t + (1-\beta)u_t \tag{41}$$

Note: coefficient on u_t is $(1-\beta)$, the marginal propensity to consumer (MPC) out of transitory income: coefficient of z_t is 1, which is the MPC out of permanent income. The marginal propensity to consumer out of financial wealth F_t is the same as before.

- Recall:

$$c_{t+1} - c_t = (1-\beta)G(1-\beta A)^{-1}\cdot C\cdot w_{t+1} \tag{42}$$

$$= (1-\beta)\begin{pmatrix}\frac{1}{1-\beta} & 1\end{pmatrix}\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{pmatrix}\cdot\begin{pmatrix}w_{1t+1}\\ w_{2t+1}\end{pmatrix} \tag{43}$$

$$= \sigma_1 w_{1t+1} + (1-\beta)\sigma_2 w_{2t+1} \tag{44}$$

i.e.Consumes all of the permanent shock, and the MPC out of the transitory shock.

- What about savings?

Recall:

$$F_{t+1} - F_t = \frac{1}{\beta}\left[y_t - (1-\beta)\mathbb{E}_t \sum_{j=0}^{\infty} \beta^j y_{t+j}\right] \tag{45}$$

$$= \frac{1}{\beta}\left[G \cdot x_t - (1-\beta)G(I-\beta A)^{-1} x_t\right] \tag{46}$$

$$= \frac{1}{\beta} G\left[I - (1-\beta)G(I-\beta A)^{-1}\right] x_t \tag{47}$$

$$G \cdot I = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix} \tag{48}$$

$$G(I-\beta A)^{-1} = \begin{pmatrix} \frac{1}{1-\beta} & 1 \end{pmatrix} \text{from before} \Rightarrow \tag{49}$$

$$F_{t+1} - F_t = \frac{1}{\beta}\left[\begin{pmatrix} 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1-\beta \end{pmatrix}\right] \begin{pmatrix} z_t \\ u_t \end{pmatrix} \tag{50}$$

$$= \frac{1}{\beta} \begin{pmatrix} 0 & \beta \end{pmatrix} \begin{pmatrix} z_t \\ u_t \end{pmatrix} \tag{51}$$

$$= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} z_t \\ u_t \end{pmatrix} \Rightarrow \tag{52}$$

$$\boxed{F_{t+1} - F_t = u_t} \tag{53}$$

i.e. Consumer spends all of z_t, saves nothing but a fraction of transitory income (Note returns on savings to F_{t+1}). The fraction of u_t consumed is the annuity value $\frac{R-1}{R}u_t$ since $R(1 - \frac{R-1}{R})u_t = u_t$ for the rest of the income.