Stochastic Asset Pricing and Expected Present Discounted Values

Jesse Perla
University of British Columbia

1 Asset Pricing with Markov Chains

1.1 Stochastic Asset Pricing with a Discrete States

- Setup:

  - Assume a discrete number, $1, \ldots N$, of possible states of the word

  - Let P be the transition matrix of the Markov chain for these states, and let $A \equiv P'$. That is, the transpose.

  - If π_t is the pmf as a row vector, let $x_t \equiv \pi_t'$ be distribution (pmf) of possible states for the random variable as a column vector.

- Take the standard forecast, $\pi_{t+1} = \pi_t P$ and take the transpose of both sides to get $x_{t+1} \equiv \pi_{t+1}' = (\pi_t P)' = P'\pi_t' \equiv Ax_t$. Then we see the forecast j into the future is

$$x_{t+j} = A^j \cdot x_t$$

- Let the payoff in each state be: $G = \begin{bmatrix} y_1 & \ldots & y_N \end{bmatrix}$, so $y_t = \begin{bmatrix} y_1 & \cdots & y_N \end{bmatrix} \cdot \begin{bmatrix} x_{1t} \\ x_{2t} \\ \vdots \\ x_{nt} \end{bmatrix} = G \cdot x_t$

- Given the possible payout states, the random variable Y_t is all of the possible payouts in G with probability x_t. So

$$y_t \equiv \mathbb{E}_t \left[Y_t \right] = Gx_t$$

- Compare to linear state space model: $x_{t+1} = Ax_t$ and $y_t = Gx_t$

- Example:

  $y_t = y_1$, if $x_{1t} = 1$; $y_t = y_N$, if $x_{Nt} = 1$ and if 50% change in each of the first 2 states,

  $$y_t = G \cdot x_t = \begin{bmatrix} y_1 & \cdots & y_N \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \tfrac{1}{2}y_1 + \tfrac{1}{2}y_2, \text{ i.e. can give expected dividends}$$

- Finally using markov chains,

  $x_{t+j} = A^j x_t$

  $y_t = G \cdot x_t$

  Using the forecast and weighting by the pmf:

  $y_{t+j} = \mathbb{E}_t \left[Y_{t+j} \right] = G x_{t+j}$

- Using these to find the asset price,

  $p_t(x_t) = \mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j Y_{t+j} \right] = G \left(\sum_{j=0}^{\infty} \beta^j A^j \right) x_t$

- This is close to our old form (note that we have the tranpose $A \equiv P'$:

$$\boxed{p(x_t) = G(I - \beta A)^{-1} x_t \rightarrow \text{Compare to deterministic formula!}} \tag{1}$$

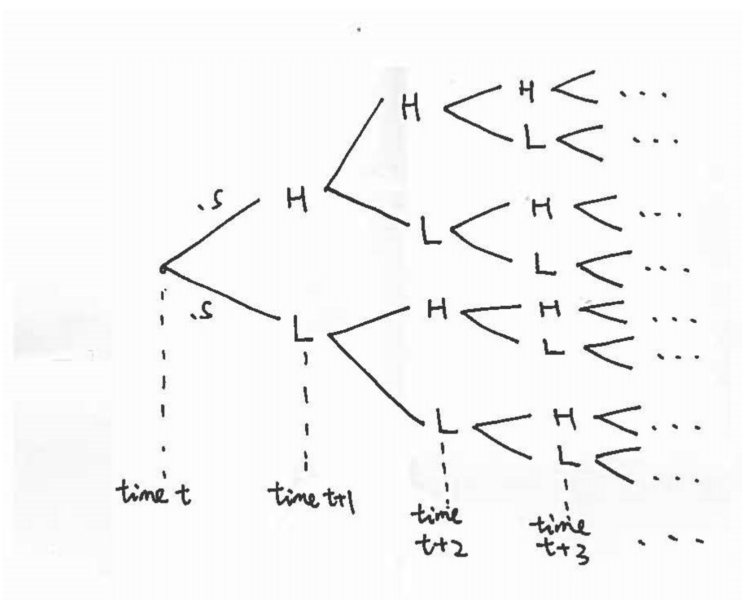

Figure 1: Expected PDV of dividends

Sequential vs. Recursive: An example is a H/L process of dividends.

- Dividends are y_H with probability 0.5 and y_L with probability 0.5. iid. Can denote these as $\mathbb{P}(H) = \mathbb{P}(L) = 0.5$
- What is the expected present discounted value of payoffs? i.e. $p(Y_0) = \mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j Y_{t+j} \,\middle|\, Y_0 \right]$?

Figure 1 shows how complicated this is to think through sequentially. But can we write down a recursive version of this under the assumption that the price should be only a function of the current state?

Define p_H and p_L as the prices in state L vs. H. With this, we can write down a system of two equations and two unknowns.

$$p_H = y_H + \beta \mathbb{E}[p_i \mid H] = y_H + \beta [\mathbb{P}(H) p_H + \mathbb{P}(L) p_L] \tag{2}$$

$$p_L = y_L + \beta \mathbb{E}[p_i \mid L] = y_L + \beta [\mathbb{P}(H) p_H + \mathbb{P}(L) p_L] \tag{3}$$

Stack as vectors,

$$p \equiv \begin{bmatrix} p_H & p_L \end{bmatrix} \tag{4}$$

$$G \equiv \begin{bmatrix} y_H & y_L \end{bmatrix} \tag{5}$$

$$A \equiv \begin{bmatrix} \mathbb{P}(H) & \mathbb{P}(H) \\ \mathbb{P}(L) & \mathbb{P}(L) \end{bmatrix} \tag{6}$$

Then rewrite the system of equations

$$p = G + \beta p A \tag{7}$$

Rearrange, being careful with the commutative rules of matrices,

$$p(I - \beta A) = G \tag{8}$$

And assuming things are invertible,

$$p = G(I - \beta A)^{-1} \tag{9}$$

In the more general case of a markov chain, the A becomes the transpose of a markov chain—as it does in previous section. Here the columns are identical because the switches between L and H are iid.

To complete the solution, note that this p is a row vector, so if we set x_t as a column vector as above, $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ if H, $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ if L, then we can calculate the price as $p(x_t) = G(I - \beta A)^{-1} x_t$.

2 Stochastic Asset Pricing with Continuous State Spaces

2.1 Information Sets

- Conditional expectation is $\mathbb{E}(X|Y)$ means that in forming the expectation of X, can use anything in Y as if known with certainty. i.e., not a random variable.
- $\mathbb{E}_t(C_{t+1})$ is the abbreviation for $\mathbb{E}(C_{t+1} \mid C_t, C_{t-1}, C_{t-2}, \cdots$ and anything else we know at t)
  If first-order Markov, then $\mathbb{E}_t(C_{t+1}) = \mathbb{E}(C_{t+1} \mid \underbrace{C_t}_{\substack{\text{i.e. all info}\\ \text{in last}\\ \text{state}}})$
- What to choose for the state? Think through necessary information set of an agent.

2.2 Properties of Expectations

Key: Expectation is a *linear operator* and can be over scalars, vectors, or matrices. Some properties of expectations:

- Let a and b be scalar constants, and $\{x_t\}$ and $\{z_t\}$ be scalar random variables
- $\mathbb{E}_t\left[ax_{t+1} + bz_{t+1}\right] = a\mathbb{E}_t\left[x_{t+1}\right] + b\mathbb{E}_t\left[z_{t+1}\right]$
- But, be careful not to apply this for multiplication with other random variables. For example,
  - $\mathbb{E}_t\left[x_{t+1}z_{t+1}\right] \underbrace{\neq}_{\text{in general}} \mathbb{E}_t\left[x_{t+1}\right]\mathbb{E}_t\left[z_{t+1}\right]$. True if independent.
  - $\mathbb{E}_t\left[x_{t+1}^2\right] \underbrace{\neq}_{\text{in general}} \left(\mathbb{E}_t\left[x_{t+1}\right]\right)^2$. Note x_{t+1} and x_{t+1} are never independent.
  - As always, just be careful to keep the order (i.e., not commutative in general)
  - Of course, if the information is known then the expectation is the value itself, $\mathbb{E}_t\left[x_t\right] = \mathbb{E}\left[x_t | x_t\right] = x_t$
- Law of iterated expectations: $\mathbb{E}_t\left[\mathbb{E}_{t+1}\left[x_{t+2}\right]\right] = \mathbb{E}_t\left[x_{t+2}\right]$. Note: time t has less information than that of time $t+1$.

Generalizing, let X_t and Z_t be vector random variables, and A and B be matrices or vectors,

- $\mathbb{E}_t\left[A \cdot X_{t+1} + B \cdot Z_{t+1}\right] = A \cdot \mathbb{E}_t\left[X_{t+1}\right] + B \cdot \mathbb{E}_t\left[Z_{t+1}\right]$
- These also all hold for any conditional expectation as well,
  $\mathbb{E}_t\left[A \cdot X_{t+1} + B \cdot Z_{t+1} \mid Z_t, X_t\right] = A \cdot \mathbb{E}_t\left[X_{t+1} \mid Z_t, X_t\right] + B \cdot \mathbb{E}_t\left[Z_{t+1} \mid Z_t, X_t\right]$

2.3 A Few Tricks with Normal Variables

- If a random variable, z is distributed as a normal random variable with mean μ and variance σ^2, it is denoted
$$z \sim N(\mu, \sigma^2)$$
- In terms of expectations, one can show that: $\mathbb{E}[z] = \mu$ and $\mathbb{E}[z^2] = \mu^2 + \sigma^2$
- Let $w \sim N(0, 1)$ be a normalized random variable. Then you can show that
$$z = \mu + \sigma w$$
- i.e., can convert any normal random variable to linear function of a normalized one
- With multivariate normal random variable, $q \in \mathbb{R}^n$ and denote its distribution,
$q \sim N(\mu, \Sigma)$
where the mean $\mu \in \mathbb{R}^n$ and $\Sigma \in \mathbb{R}^{n\times n}$ is the variance-covariance matrix.
- Keeping things simple, if the vector random variable is mean 0 and is independent (i.e., none of the components of the vector have any correlation) then we would write it in terms of vector mean and the identity for the covariance matrix
$q \sim N(0_n, I_{n\times n})$

2.4 Asset pricing in our state space model

Our Deterministic Model

- Recall: In the deterministic Linear state space, we have,
$$x_{t+1} = A \cdot x_t, \qquad \text{(Evolution)} \tag{10}$$
$$y_t = G \cdot x_t, \qquad \text{(Observation)} \tag{11}$$
- And asset pricing formula under risk neutrality is:
$$P_t = \sum_{j=0}^{\infty} \beta^j y_{t+j} = G(I - \beta \cdot A)^{-1} \cdot x_t \tag{12}$$

Making this a Stochastic Linear State Space

- Add randomness w_{t+1}, an $m \times 1$ vector, random variable:
$$x_{t+1} = Ax_t + C \cdot w_{t+1}, \qquad \text{(Evolution, stochastic)} \tag{13}$$
$$y_t = G \cdot x_t, \qquad \text{(Observation, still noise free)} \tag{14}$$

where A is $n \times n$ matrix, C is $n \times m$ matrix, w_{t+1} are $m \times 1$ matrices, x is $n \times 1$ vector; G is $1 \times n$ vector, y_t are scalars

- Note:
  w_{t+1} are independent, identically distributed variables; Gaussian of mean 0, covariance matrix $I_{m \times n}$. Hence, $\mathbb{E}(w_{it+1}) = 0$ for all $i = 1, \cdots m$, $\mathbb{E}(w_{it} w_{i't'}) = \begin{cases} 1, \text{ if } i = i', t = t' \\ 0, \text{ otherwise} \end{cases}$

- Notice that:

$$\mathbb{E}_t(x_{t+1}) = \mathbb{E}_t(A \cdot x_t + Cw_{t+1}) = A \cdot x_t + \underbrace{C \cdot \mathbb{E}_t(w_{t+1})}_{=0} = A \cdot x_t \tag{15}$$

$$\mathbb{E}_t(x_{t+2}) = \mathbb{E}_t \left(A \underbrace{(Ax_t + Cw_{t+1})}_{x_{t+1}} + C \cdot w_{t+2} \right) = \mathbb{E}_t(A^2 x_t + ACw_{t+1} + Cw_{t+2}) \tag{16}$$

$$= A^2 x_t + \underbrace{AC\mathbb{E}_t(w_{t+1})}_{=0} + \underbrace{C\mathbb{E}_t(w_{t+2})}_{=0} = A^2 x_t, \text{repeat for } t+3, \cdots \tag{17}$$

- Forecasting Formulas:

$$\mathbb{E}_t(x_{t+j}) = A^j x_t, \text{ and } \mathbb{E}_t \left(\sum_{j=0}^{\infty} \beta^j x_{t+j} \right) = (I - \beta \cdot A)^{-1} x_t \tag{18}$$

$$\mathbb{E}_t(y_{t+j}) = G \cdot A^j x_t, \text{ and } \mathbb{E}_t \left(\sum_{j=0}^{\infty} \beta^j y_{t+j} \right) = G \cdot (I - \beta A)^{-1} x_t \tag{19}$$

2.5 Price of Stochastic Dividend Stream

$$p_t = \mathbb{E}_t \left(\sum_{j=0}^{\infty} \beta^j \underbrace{y_{t+j}}_{\substack{\text{i.e. forecast of} \\ y_{t+j} \\ \text{given time t} \\ \text{information}}} \right) + \text{possible bubble} = G(I - \beta A)^{-1} x_t + \text{possible bubble} \tag{20}$$

$$\underline{Or}, p_t = \underbrace{y_t}_{\substack{\text{dividend} \\ \text{today}}} + \beta \cdot \underbrace{\mathbb{E}_t(p_{t+1})}_{\substack{\text{expectation} \\ \text{of price} \\ \text{tomorrow}}} \tag{21}$$

- Method (Guess and Verify):

Guess $p_t = H \cdot x_t$, H is $1 \times n$ vector to be determined, x is $n \times 1$ vector
Substitute into equation:

$$H \cdot x_t = y_t + \beta \cdot \mathbb{E}_t\left(Hx_{t+1}\right) \tag{22}$$

$$\Rightarrow H \cdot x_t = G \cdot x_t + \beta H \mathbb{E}_t(A \cdot x_t + C \cdot w_{t+1}) = G \cdot x_t + \beta H A x_t \tag{23}$$

To hold for any x_t,

$$H(I - \beta A) = G \Rightarrow \tag{24}$$

$$H = G(I - \beta A)^{-1} \Rightarrow \tag{25}$$

$$\boxed{p_t = G(I - \beta A)^{-1} x_t} \tag{26}$$

- Note:
  - This is consistent with EPDV calculation.
  - Same as formula without random w_{t+1}

2.6 Forecast Errors

How far off are the agent's forecasts of $t + 1$ given time t information? To do a simple example:

- Let $x_{t+1} = x_t + \sigma w_{t+1}$
- With $w_{t+1} \sim N(0, 1)$. i.e., $\mathbb{E}_t\left[w_{t+1}\right] = 0$ and $\mathbb{E}_t\left[w_{t+1}^2\right] = 1$.
- Trivial linear-Gaussian-state space. The expected forecast error is,

$$\mathbb{E}_t\left[FE_{t+1}\right] \equiv \mathbb{E}_t\left[x_{t+1} - \mathbb{E}_t\left[x_{t+1}\right]\right] = \mathbb{E}_t\left[x_{t+1}\right] - \mathbb{E}_t\left[x_{t+1}\right] = 0$$

- i.e., no systematic error. What about the variance of the forecast errors?
- The variance of a random variable, z_t is defined as $\mathbb{V}_t\left(z_{t+1}\right) \equiv \mathbb{E}_t\left[z_{t+1}^2\right] - \left(\mathbb{E}_t\left[z_{t+1}\right]\right)^2$
- So to find the variance of the forecast error:

$$\mathbb{V}_t(FE_{t+1}) = \mathbb{E}_t\left[FE_{t+1}^2\right] - \left(\mathbb{E}_t\left[FE_{t+1}\right]\right)^2 \tag{27}$$

$$= \mathbb{E}_t\left[\left(x_{t+1} - \mathbb{E}_t\left[x_{t+1}\right]\right)^2\right] - 0 \tag{28}$$

$$= \mathbb{E}_t\left[\left(x_t + \sigma w_{t+1} - \mathbb{E}_t\left[x_t + \sigma w_{t+1}\right]\right)^2\right] \tag{29}$$

$$= \mathbb{E}_t\left[\left(\sigma w_{t+1}\right)^2\right] = \sigma^2 \tag{30}$$

2.7 Linear Gaussian State Space Example

- On average, a worker's productivity, z_t, adds a random draw of $N(\alpha, \sigma^2)$ each period.

- Firm productivity q_t, adds γ each period, which is deterministic.

- Wages are a linear combination of: $W_t = \theta z_t + (1-\theta) q_t$
- Setup in Linear Gaussian form:

- Guess state: $x_t = \begin{bmatrix} z_t \\ q_t \\ 1 \end{bmatrix}$

- Note: if $w_{t+1} \sim N(0,1)$, then

$$\alpha + \sigma w_{t+1} \sim N(\alpha, \sigma^2)$$

- The state space model is then,

$$\underbrace{\begin{bmatrix} z_{t+1} \\ q_{t+1} \\ 1 \end{bmatrix}}_{x_{t+1}} = \underbrace{\begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} z_t \\ q_t \\ 1 \end{bmatrix}}_{x_t} + \underbrace{\begin{bmatrix} \sigma \\ 0 \\ 0 \end{bmatrix} w_{t+1}}_{C \cdot w_{t+1}} \tag{31}$$

$$W_t = \begin{bmatrix} \theta & 1-\theta & 0 \end{bmatrix} \begin{bmatrix} z_t \\ q_t \\ 1 \end{bmatrix} \tag{32}$$

$$W_t = G \cdot x_t \tag{33}$$

What is the expected PDV of human capital? (i.e., stochastic version of the permanent income calculations)

$$\mathbb{E}_t \left[\sum_{j=0}^{\infty} \beta^j y_{t+j} \right] = G(I - \beta A)^{-1} x_t \tag{34}$$

A Stochastic Bubbles

- To isolate the bubble term, consider the special case where $y_t = 0$ for all t.
- We want to solve $p_t = \beta \mathbb{E}_t(p_{t+1}), \beta = \frac{1}{1+r}$.

- Guess: $p_t = C_t\beta^{-t}$, where C_t is a random variable, and $\{C_t\}$ is a *martingale*, that is, satisfies $\mathbb{E}_t(C_{t+1}) = C_t$, i.e. best forecast of future value is today's value (e.g. random walk).

- To verify $p_t = \beta\mathbb{E}_t(p_{t+1})$, substitute our guess: $C_t\beta^{-t} = \beta \cdot \mathbb{E}_t(\beta^{-(t+1)}C_{t+1}) = \beta^{-t} \cdot \mathbb{E}_t(C_{t+1}) = \beta^{-t}C_t$, i.e. verified that $p_t = C_t\beta^{-t}$ satisfies equation.

- Example:

$$C_{t+1} = \begin{cases} \lambda^{-1}C_t \text{ with probability } \lambda \in (0,1) \\ 0 \text{ with probability } 1-\lambda \end{cases}$$

  - Note: $\mathbb{E}_t(C_{t+1}) = \lambda \cdot (\lambda^{-1}C_t) + 0 = C_t, \Rightarrow$ a martingale
  - Note that if at some $C_{t+j} = 0 \Rightarrow C_{t+j+1} = 0$, etc., i.e. the bubble has popped.
  - From any C_0,

$$C_t = \begin{cases} \lambda^{-t}C_0, \text{ if bubble has not popped} \\ 0, \text{ if the bubble has popped} \end{cases}$$

$$p_t = \begin{cases} \beta^{-t} \cdot \lambda^{-t} \cdot C_0 = (\beta\lambda)^{-t}C_0, \text{ until popped} \\ 0, \text{ after the bubble has popped} \end{cases}$$

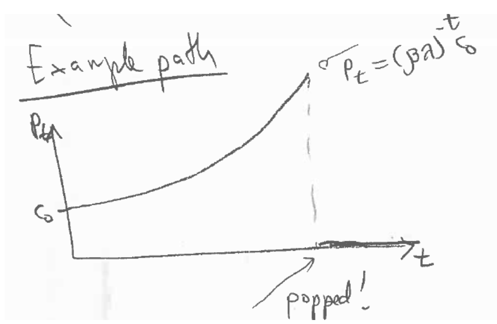

Figure 2: Stochastic Bubble