(Static) General Equilibrium

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1 Basic Setup and Consumer Preferences

- Recall notation: $\boldsymbol{\partial}_c u \equiv \frac{\partial u(c,\ell)}{\partial c}$
- Commodities:
  - c: consumption good
  - ℓ: labor
  - k: capital, exogenously given for now
- Households:
  - Preferences over $\{c, \ell\}$: $u(c, \ell)$, where $0 \leq \ell \leq 1$ with $\boldsymbol{\partial}_\ell u(c, \ell) \leq 0$ and $\boldsymbol{\partial}_c u(c, \ell) > 0$
  - c is consumption, ℓ is hours of working (labor)
  - Constant total derivative gives indifference curves:$\boldsymbol{\partial}$

$$0 = \boldsymbol{\partial} u = \boldsymbol{\partial}_c u \boldsymbol{\partial} c + \boldsymbol{\partial}_\ell u \boldsymbol{\partial} \ell \tag{1}$$

    Reorganize

$$\frac{\boldsymbol{\partial} c}{\boldsymbol{\partial} \ell} = \frac{-\boldsymbol{\partial}_\ell u}{\boldsymbol{\partial}_c u} \tag{2}$$

    Interpreted as the "Marginal Rate of Substitution (MRS)" between consumption and labor supply

Example: $u(c, \ell) = \log c - B\ell$,
Here B is the disutility of labor and

$$\frac{\boldsymbol{\partial} c}{\boldsymbol{\partial} l} = \frac{-\boldsymbol{\partial}_\ell u}{\boldsymbol{\partial}_c u} = cB$$

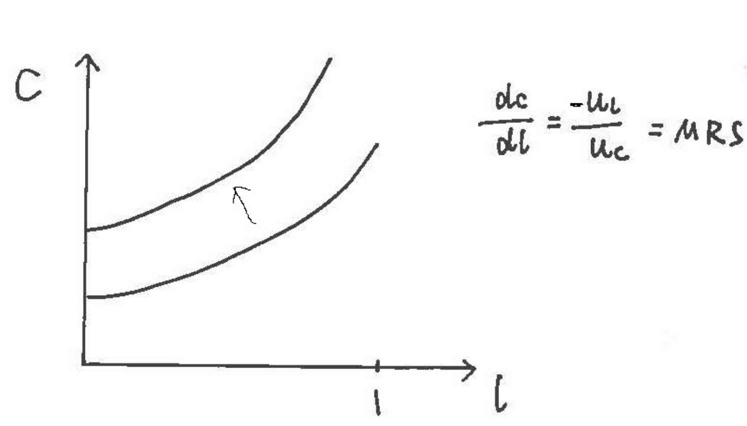

Figure 1: Indifference Curve

Note that if $B = 0$, indifferent to labor and the indifference curves are flat.

2 Production and Feasibility

A general production function using capital and labor

$$Y = F(k, \ell) \tag{3}$$

where Y is the total output, F is the production function, k is the capital inputs. ℓ is the labor inputs

Constant Returns to Scale Assume $F(k, \ell)$ has <u>constant returns to scale</u> (e.g. double inputs, double outputs) That is for any λ

$$\lambda F(k, \ell) = F(\lambda k, \lambda \ell) \tag{4}$$

Differentiate with respect to ℓ

$$\lambda \boldsymbol{\partial}_\ell F(k, \ell) = \lambda \boldsymbol{\partial}_\ell F(\lambda k, \lambda \ell) \tag{5}$$

Since we can use an arbitrary λ, let $\lambda = \frac{1}{k}$

$$\boldsymbol{\partial}_\ell F(k, \ell) = \boldsymbol{\partial}_\ell F\left(1, \frac{\ell}{k}\right) \tag{6}$$

So homogenous of degree 0 $\Rightarrow$ partials only depends on ratios of inputs[1]

[1] The choice of $\lambda = \frac{1}{k}$ is arbitrary. We could use $\lambda = \frac{1}{\ell}$ and do similar algebra in the capital-to-labor ratio. In fact, it will be more convenient when we do the neo-classical growth model with evolving capital.

Example:

$$F(k,\ell) = Ak^{\alpha}\ell^{1-\alpha} \text{ for } \alpha \in (0,1) \tag{7}$$

Note: $\underbrace{\boldsymbol{\partial}_\ell F > 0, \boldsymbol{\partial}_k F > 0}_{\text{positive marginal products}}, \underbrace{\boldsymbol{\partial}_{kk} F < 0, \boldsymbol{\partial}_{\ell\ell} F < 0}_{\text{diminishing returns}}$

Also: $\boldsymbol{\partial}_\ell F(k,\ell) = (1-\alpha)Ak^{\alpha}\ell^{-\alpha} = (1-\alpha)A\left(\frac{\ell}{k}\right)^{-\alpha}$, $\boldsymbol{\partial}_k F(k,\ell) = \alpha A\left(\frac{\ell}{k}\right)^{1-\alpha}$

Feasibility: An allocation c, k, ℓ is <u>feasible</u> if

$$c + G \leq F(k,\ell) \tag{8}$$

Where c is consumption, G is exogenously given government expenditure in real goods, $F(k,\ell)$ is total output given choice ℓ. Key: feasibility and production functions involve "real" objects and allocations, think wheat, widgets, machines, and time rather than \$, conch-shells, taxes, paychecks, etc.

3 Planning Problem (Command Economy)

Dictators do not need prices! Benevolent dictators represent consumer preferences, tell people how much to work and consume, but are constrained by the "laws of nature" (i.e. feasibility and the production technology)

$$\max_{c,\ell} \{u(c,\ell)\} \tag{9}$$

$$\text{s.t. } c + G \leq F(k,\ell) \tag{10}$$

Setup the Lagrangian:

$$\mathcal{L} = u(c,\ell) + \lambda\left[F(k,\ell) - c - G\right] \tag{11}$$

Take the FOC and assume interior solutions,

$$\boldsymbol{\partial}_c u - \lambda = 0 \tag{12}$$

$$\boldsymbol{\partial}_\ell u + \lambda \boldsymbol{\partial}_\ell F(k,\ell) = 0 \tag{13}$$

Combine to form

$$\frac{-\boldsymbol{\partial}_\ell u}{\boldsymbol{\partial}_c u} = \boldsymbol{\partial}_\ell F \tag{14}$$

Note: MRS between consumption and labor = marginal product of labor!

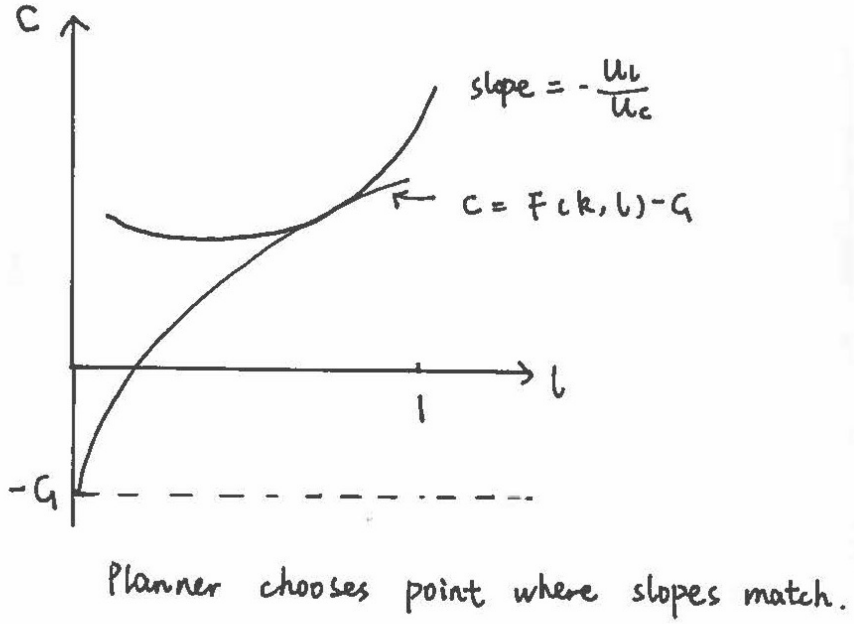

Figure 2: Planner

4 Competitive Equilibrium

A key question in "Welfare Economics" is whether you can "decentralize" the planner's economy with one where agents make individual decisions and prices/markets.

4.1 Households in Market Economy

Assumptions on the market structure

- Assume that consumers are price takers
- Assume that consumers own the capital (it doesn't matter if they do or the firms) and rent labor/capital at market prices.[2]
- Nominal prices mean denoted in \$, whereas real prices are relativ e to the consumption goods price.
- The nominal price of the consumption good is $\tilde{p}$, the nominal wage is $\tilde{w}$, and the nominal rental rate of capital is $\tilde{r}$.

[2]At this point, we are assuming that consumer's are identical. In the "Interest Rates" notes, we will prove conditions under which this doesn't matter.

- Let the real wage and rental rate be $w \equiv \tilde{w}/\tilde{p}$ and $r \equiv \tilde{r}/\tilde{p}$
- Governments pay for G (at market price $\tilde{p}$ through a marginal tax rate on labor, τ_ℓ and a per-capita tax $\tilde{p}\tau_h$

Consumer's Problem **Given prices**, the consumer's problem is

$$\max_{c,\ell,\geq 0} \{u(c,\ell)\} \tag{15}$$

$$\text{s.t.}\, \tilde{p}c \leq (1-\tau_\ell)\tilde{w}\ell + \tilde{r}k - \tilde{p}\tau_h \tag{16}$$

Note that unlike in planning problems, consumers use prices (i.e. budget constraints not feasibility)

For a more convenient formulation, put it in "real terms" by dividing the budget by $\tilde{p}$

$$\max_{c,\ell\geq 0} \{u(c,\ell)\} \tag{17}$$

$$\text{s.t.}\, c \leq (1-\tau_\ell)w\ell + rk - \tau_h \tag{18}$$

Form the lagrangian:

$$\mathcal{L} = u(c,\ell) + \lambda\left[w(1-\tau_\ell)\ell + rk - c - \tau_h\right] \tag{19}$$

Take the FONC (where the $c \geq 0$ and $l \geq 0$ multipliers are built in)

$$\boldsymbol{\partial}_c u - \lambda \leq 0, = 0 \text{ if } c > 0 \tag{20}$$

$$\boldsymbol{\partial}_\ell u + \lambda w(1-\tau_\ell) \leq 0, = 0 \text{ if } \ell > 0 \tag{21}$$

At equality (i.e., if both are interior):

$$\frac{-\boldsymbol{\partial}_\ell u}{\boldsymbol{\partial}_c u} = w(1-\tau_\ell) \tag{22}$$

Or, interpretting

$$\text{MRS} = \text{Real Wage} \tag{23}$$

So, labor is supplied until the marginal rate of substitution equals the real wages (after taxes)

In this figure $z = rk - \tau_h$ since they are additive.

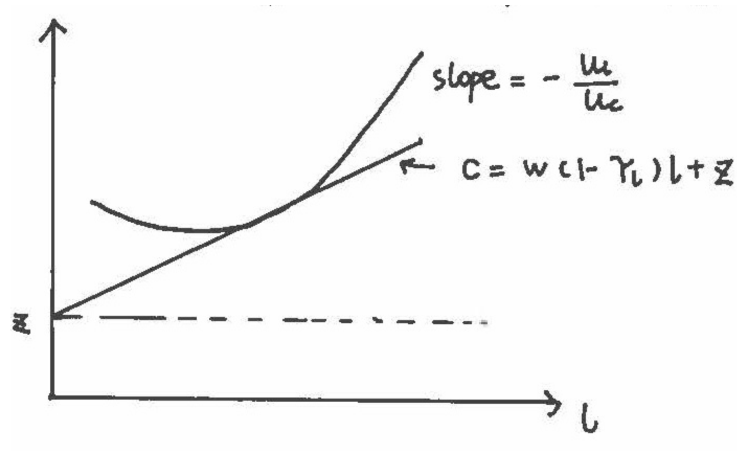

Figure 3: At equality, budget matches the indifference curve

4.2 Firm's Problem

Assume a large number of firms labeled (arbitrarily) labeled i renting capital and labor from the consumer, and selling them (and the government) the good.

Crucially firms are is competitive, i.e. price takers!

The objective of each firm is to maximize profits given the production technology and prices

$$\max_{\{Y(i),k(i),\ell(i)\}} \{\tilde{p}Y(i) - \tilde{r}k(i) - \tilde{w}\ell(i)\} \tag{24}$$

$$\text{s.t. } Y(i) = F(k(i), \ell(i)) \tag{25}$$

Substitute the production function into the objective

$$\max_{\{k(i),\ell(i)\}} \{\tilde{p}F(k(i), \ell(i)) - \tilde{r}k(i) - \tilde{w}\ell(i)\} \tag{26}$$

Take the FONCS

$$\tilde{p}\boldsymbol{\partial}_k F(k(i), \ell(i)) - \tilde{r} = 0 \quad \Rightarrow \quad \frac{\tilde{r}}{\tilde{p}} = \boldsymbol{\partial}_k F(k(i), \ell(i)) \tag{27}$$

$$\tilde{p}\boldsymbol{\partial}_\ell F(k(i), \ell(i)) - \tilde{w} = 0 \quad \Rightarrow \quad \frac{\tilde{w}}{\tilde{p}} = \boldsymbol{\partial}_\ell F(k(i), \ell(i)) \tag{28}$$

Or, in terms of "real prices"

$$\boldsymbol{\partial}_k F(k(i), \ell(i)) = r \tag{29}$$

$$\boldsymbol{\partial}_\ell F(k(i), \ell(i)) = w \tag{30}$$

If CRS, then the marginal products are,

$$\boldsymbol{\partial}_k F\left(1, \frac{\ell(i)}{k(i)}\right) = r \tag{31}$$

$$\boldsymbol{\partial}_\ell F\left(1, \frac{\ell(i)}{k(i)}\right) = w \tag{32}$$

Firm Size Distribution Up until now, we have taken no stand on the number of firms or their relative size.

- Note that the size of the particular firm, i, and the levels of $k(i)$ and $\ell(i)$, cannot be determined by these equations because the ℓ and k are never separable
- On the other hand, the ratio $\frac{k(i)}{\ell(i)}$, which must be identical for all firms. Because of this, drop the i index.
- With constant returns to scale, we can use the output of a single "representative" firm with these competitive prices.

This is an example of a proof of "aggregation" to a representative agent. We will do similar derivations for using a representative consumer.

4.3 Competitive Equilibrium (With $G = 0, \tau_\ell = 0, \tau_h = 0$)

In this economy, a competitive equilibrium is defined as

- A feasible allocation is a bundle of $\{k, \ell, c\}$ that satisfies $c \leq F(k, \ell)$ with k given
- A price system is a pair $\{w, r\}$
- A competitive equilibrium is a feasible allocation and price system such that:
  - (1) Given $\{w, r\}$, $\{c, \ell\}$ solves the household's problem.
  - (2) Given $\{w, r\}$, $\{\ell, k\}$ solves the firm's problem.

Note that the conditions for feasibility and the allocations exactly match those of a planner. No prices, and no way to avoid the "laws of nature".

We say that a competitive equilibrium **decentralized** a planning problem if it delivers identical allocations.

4.4 Example

4.4.1 Setup:

- Compute competitive equilibrium, where

$$u(c,\ell) = \ln c - B\ell \tag{33}$$

$$F(k,\ell) = Ak^{\alpha}\ell^{1-\alpha}, G = 0 \tag{34}$$

- Method:
  - (a) Solve planner's problem
  - (b) Reverse engineer required prices to support that equilibrium
  - (c) Verify competitive equilibrium conditions hold.

4.4.2 Steps:

- (a) Planning Problem:
  Recall from FONC for planner:

$$\boldsymbol{\partial}_{\ell} F = \frac{-\boldsymbol{\partial}_{\ell} u}{\boldsymbol{\partial}_{c} u} \Rightarrow (1-\alpha)A\left(\frac{\ell}{k}\right)^{-\alpha} = cB \tag{35}$$

And from feasibility:

$$c = Ak^{\alpha}\ell^{1-\alpha} \tag{36}$$

So we have:

$$(1-\alpha)A\ell^{-\alpha}k^{\alpha} = BAk^{\alpha}\ell^{1-\alpha} \Rightarrow \tag{37}$$

$$\boxed{\ell = \frac{1-\alpha}{B}} \tag{38}$$

Substitute into feasibility,

$$\boxed{c = Ak^{\alpha}\left(\frac{1-\alpha}{B}\right)^{1-\alpha}} \tag{39}$$

This is the feasible allocation(c, l, k)

- (b) Reverse engineer prices:

$$F(k,\ell) = Ak^{\alpha}\ell^{1-\alpha} \Rightarrow \tag{40}$$
$$\boldsymbol{\partial}_k F(k,\ell) = \alpha A k^{\alpha-1}\ell^{1-\alpha} \tag{41}$$
$$\boldsymbol{\partial}_\ell F(k,\ell) = (1-\alpha)Ak^{\alpha}\ell^{-\alpha} \tag{42}$$

  From Firm's FOC:

$$\boldsymbol{\partial}_k F(k,\ell) = \alpha A k^{\alpha-1}\ell^{1-\alpha} = \boxed{\alpha A k^{\alpha-1}\left(\frac{1-\alpha}{B}\right)^{1-\alpha} = r} \tag{43}$$
$$\boldsymbol{\partial}_\ell F(k,\ell) = (1-\alpha)Ak^{\alpha}\ell^{-\alpha} = \boxed{(1-\alpha)Ak^{\alpha}\left(\frac{1-\alpha}{B}\right)^{-\alpha} = w} \tag{44}$$

- (c) Verify: Since $\alpha \in (0,1)$, $A > 0$, $B > 0$, $k > 0$, then $r, w > 0$ are prices that are strictly positive

4.4.3 Verification of CE conditions:

1. Feasibility: Yes, since used same feasibility to solve planners problem.
2. FONC of Firms: Yes, used to reverse engineer prices directly.
3. FONC of household:
   Note, FONC used for planner:

$$\frac{-\boldsymbol{\partial}_\ell u}{\boldsymbol{\partial}_c u} = \boldsymbol{\partial}_\ell F \tag{45}$$

   FONC for household:

$$\frac{-\boldsymbol{\partial}_\ell u}{\boldsymbol{\partial}_c u} = w \tag{46}$$

   Plug in FONC of firm for $w = \boldsymbol{\partial}_\ell F$

$$\frac{-\boldsymbol{\partial}_\ell u}{\boldsymbol{\partial}_c u} = \boldsymbol{\partial}_\ell F \tag{47}$$

   Same FOC used as planner.

In nominal terms, this holds for any $\tilde{p} > 0$, but is unique in real terms.

Because the allocations are identical to the planning problem, this competitive equilibrium decentralized this planning problem.

- It will turn out that even if $G > 0$ with $\tau_h > 0$, it will still decentralize
- You can see that if $\tau_\ell > 0$, the allocations will be different since the labor supply equation is distorted. In that case, it would not decentralize the planning problem.