# Theory of Stochastic Interest Rates

Jesse Perla
University of British Columbia

## 1 Risk Aversion

Previously we considered intertemporal smoothing. Here we will consider smoothing over stochastic states. With our welfare functions, the curvature of the $u(c)$ controls both, but they are distinct forces.

The role of curvature is through the $\mathbb{E}[f(X)] \neq f(\mathbb{E}[X])$ for nonlinear $f(x)$ and random variables $X$. This is called Jensen's inequality. When $f(x)$ is strictly concave, then $\mathbb{E}[f(X)] < f(\mathbb{E}[X])$.

To visualize this, consider the if the agent compares two "lotteries" The first is where

$$C = \begin{cases} c_L & \text{with probability } 1/2 \\ c_2 & \text{with probability } 1/2 \end{cases}$$

and the second is where $C = \frac{c_L + c_H}{2}$ with probability 1. The expected value of the two lotteries is the same, but the first is more risky.

A risk-neutral agent (i.e., one with linear utility) would be indifferent between them. What about those who are risk-averse?

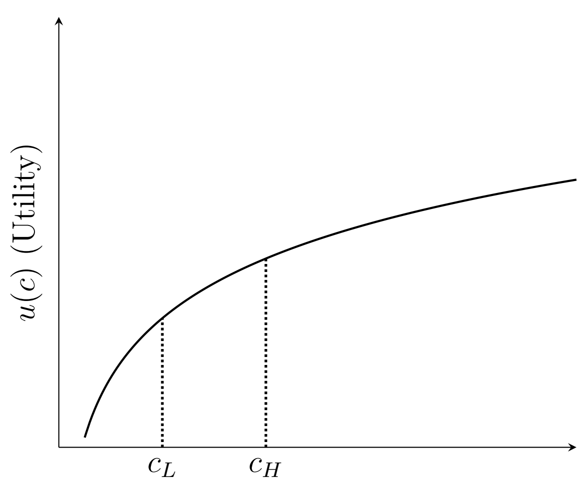


Figure 1: Utility of Consumption in both States for a Risk-Averse Agent

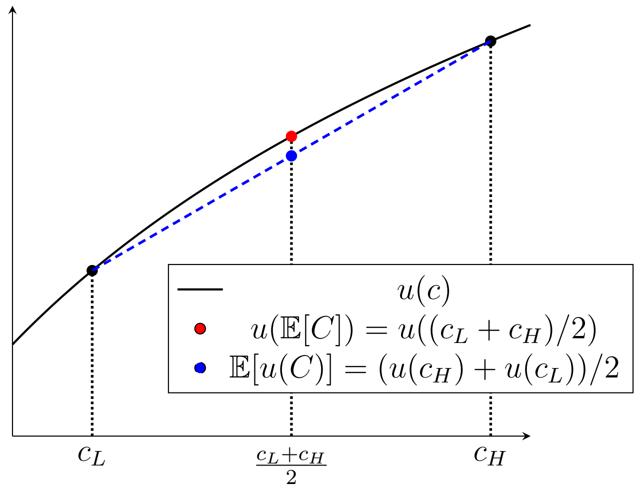


Figure 2: Average Utility of Consumption

# 2 Simple 2-period Stochastic model

## 2.1 Basic Setup

- $t = \{0, 1\}$, states$= \{A, B\}$, where state realized at $t = 1$

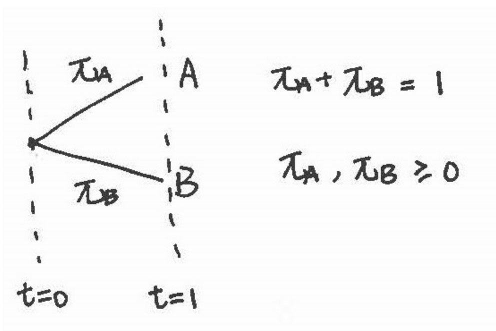


Figure 3: 2-period stochastic model

- Preferences (expected utility):

$$\mathbb{E}\left[u(c_0) + \beta u\left(c_1(s)\right)\right] \tag{1}$$
$$= u(c_0) + \beta u\left(c_1(A)\right)\pi_A + \beta u\left(c_1(B)\right)\pi_B \tag{2}$$

- Endowment of Consumer (stochastic):
  - At $t = 0, y_0$
  - At $t = 1$, $\begin{cases} y_1(A) \text{ in state A} \\ y_1(B) \text{ in state B} \end{cases}$
- Commodities:

- $c_0$: consumption at $t = 0$
- $c_1(A)$; consumption in state $A$ at $t = 1$
- $c_1(B)$: consumption in state $B$ at $t = 1$

- Prices:
  - $q_0$: price of one unit of consumption at $t = 0$ (No risk or uncertainty to resolve)
  - $q_1(A)$: price of one unit of consumption at $t = 1$ in state $A$
  - $q_1(B)$: price of one unit of consumption at $t = 1$ in state $B$

## 2.2 Consumer's problem

- For consumer, solve:

$$\max_{\{c_0, c_1(A), c_1(B)\}} u(c_0) + \beta u(c_1(A))\pi_A + \beta u(c_1(B))\pi_B \tag{3}$$

$$\text{s.t. } q_0 c_0 + q_1(A)c_1(A) + q_1(B)c_1(B) \leq q_0 y_0 + q_1(A)y_1(A) + q_1(B)y_1(B) \equiv w_0 \tag{4}$$

- Form Lagrangian:

$$\mathcal{L} = u(c_0) + \beta u(c_1(A))\pi_A + \beta u(c_1(B))\pi_B + \underbrace{\lambda}_{L.M.} \underbrace{[w_0 - q_0 c_0 - q_1(A)c_1(A) - q_1(B)c_1(B)]}_{\text{Present value of consumption}} \tag{5}$$

- F.O.N.C:

$$[c_0] : u'(c_0) - \lambda q_0 = 0 \tag{6}$$

$$[c_1(A)] : \beta u'(c_1(A))\pi_A - \lambda q_1(A) = 0 \tag{7}$$

$$[c_1(B)] : \beta u'(c_1(B))\pi_B - \lambda q_1(B) = 0 \tag{8}$$

And budget constraint:

$$w_0 - q_0 c_0 - q_1(A)c_1(A) - q_1(B)c_1(B) = 0 \tag{9}$$

By using (6), (7) and (8), we have:

$$\lambda = \frac{u'(c_0)}{q_0} \tag{10}$$

$$\beta u'(c_1(A))\pi_A = u'(c_0) \cdot \frac{q_1(A)}{q_0} \tag{11}$$

$$\beta u'(c_1(B))\pi_B = u'(c_0) \cdot \frac{q_1(B)}{q_0} \tag{12}$$

where $q_0$ is an initial price level, just rescales all. Normalize $q_0 = 1$ using (10). Consumer's take prices $q_1(A)$, $q_1(B)$ as given

$$\boxed{\begin{aligned} q_1(A) &= \beta \underbrace{\frac{u'(c_1(A))}{u'(c_0)}}_{\substack{\text{Marginal}\\\text{utility}\\\text{of allocation}}} \pi_A && (13)\\ q_1(B) &= \beta \frac{u'(c_1(B))}{u'(c_0)} \underbrace{\pi_B}_{\text{probabilities}} && (14)\end{aligned}}$$

This is the Lucas '78 formulas for 2-period. If there is a representative consumer, then $c_1(A) = y_1(A)$, eats full endowment.

- Risk free:

$$q_1^{\text{RF}} = q_1(A) + q_1(B) = \frac{\beta}{u'(c_0)} \left[u'(c_1(A))\pi_A + u'(c_1(B))\pi_B\right] \tag{15}$$

$$= \underbrace{\beta \frac{\mathbb{E}\left[u'(c_1(s))\right]}{u'(c_0)}}_{\substack{\text{Ratio of average:}\\\text{Expected marginal utility}\\\text{to marginal utility today.}}} \tag{16}$$

where the risk free gross interest rate is: $R^{\text{RF}} = \frac{1}{q_1^{\text{RF}}}$

**Does this Extend to Infinite Horizon?** Without proof, yes. Assume that the state process $\{s_t\}$ is Markov, with transition probabilities $\pi(s_{t+1}|s_t)$ and $\pi(A|s_t) + \pi(B|s_t) = 1$, then (14) and (16) become

$$q_t(s_{t+1}) = \beta \frac{u'(c_{t+1}(s_{t+1}))}{u'(c_t(s_t))} \underbrace{\pi(s_{t+1}|s_t)}_{\substack{\text{Conditional}\\\text{Probability}}} \tag{17}$$

$$q_t^{\text{RF}} = \beta \frac{\mathbb{E}_t\left[u'(c_{t+1}(s_{t+1}))\right]}{u'(c_t(s_t))} \tag{18}$$

Section Section 3 and beyond will prove this in more generality, for claims on both period-by-period, and time-0 contingent claims.

# 3 Asset Trees

## 3.1 Introduction

- Consider a 2-state Markov chain with states $S = \{\bar{s_1}, \bar{s_2}\}$ and transition matrix:

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \text{ where } p_{ij} = \text{Prob}(s_{t+1} = \bar{s_j} \mid s_t = \bar{s_i}) \tag{19}$$

$$\text{And } \pi_0 = \begin{pmatrix} \pi_{01} \\ \pi_{02} \end{pmatrix} = \begin{bmatrix} \text{Prob}(s_0 = \bar{s_1}) \\ \text{Prob}(s_0 = \bar{s_2}) \end{bmatrix} \text{ as initial probability distribution over states} \tag{20}$$

where $\pi_{0,i} \geq 0$ and $\sum\limits_{i=1}^{2} \pi_{0,i} = 1$, $p_{i,j} \geq 0$ and $\sum\limits_{j=1}^{2} p_{i,j} = 1$ (i.e. probabilities)

- Denote $s_0^t$ as sequence of states from 0 to t. For example:

$$s_0^t = \{s_0, s_1, \cdots s_t\} \text{ for some } s_\tau \in S, \forall \tau = 0, \cdots t \tag{21}$$

This is an information set, a history.

- Example:

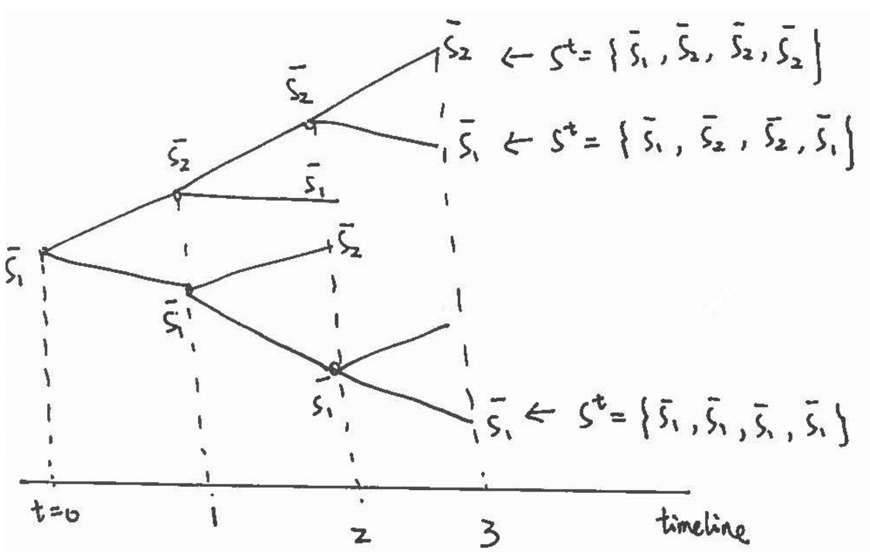


Figure 4: Asset Tree Example given $\bar{s}_1$ at $t = 0$

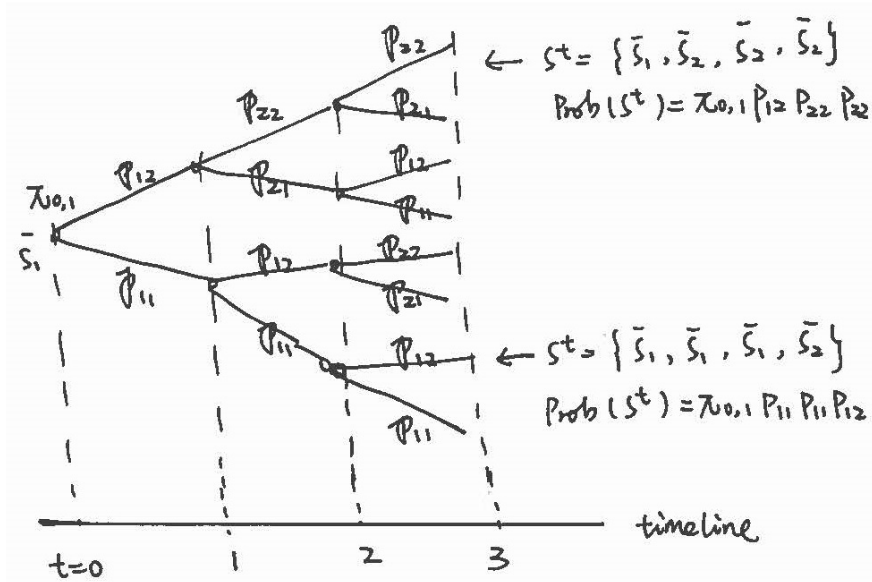


Figure 5: Jump Probabilities in Example

## 3.2 Probabilities of those jumps:

- In this way, build probabilities over histories $s_0$, which is denoted for simplicity as $s^t$.
- Denote the probability of a particular history as

$$\text{Prob}(s_t = \bar{s}^t) \equiv \pi_t(s^t) \text{ (e.g. with Markov chain)} \tag{22}$$

$$\pi_t\left(\{\bar{s}_i, \bar{s}_j, \bar{s}_k, \bar{s}_\ell, \cdots\}\right) = \underbrace{\pi_{0,i}}_{\substack{\text{Probability}\\ \text{start in } i}} \underbrace{p_{ij}}_{\text{Jump } i \text{ to } j} \underbrace{p_{jk}}_{\text{Jump } j \text{ to } k} p_{kl} \cdots \tag{23}$$

- Note that:

$$\sum_{\underbrace{s^t}_{\substack{\text{sum of all}\\ \text{possible}\\ \text{histories}\\ \text{to time } t}}} \underbrace{\pi_t(s^t)}_{\substack{\text{Probability}\\ \text{of history } s^t}} = 1, \forall t, \text{and } \pi_t(s^t) \geq 0 \tag{24}$$

## 3.3 Modelling risk

- At time 0, only knows $s_0$, does not know $s_1, s_2, \cdots$ but does know that at time $t$ the history $s^t$ <u>might</u> happen, and assigns probability $\pi_t(s^t)$ that it will.
- Consumer's utility for a particular sequence of $\{s_t\}$:

$$\sum_{t=0} \beta^t u(c_t(s^t)), \text{ where } c_t(s^t) \text{ is consumption at } \underline{t} \text{ for } \underline{\text{history } s^t} \tag{25}$$

Hence, consumption can depend on a particular history of the state.

- But at time 0, the agent only has probabilities over possible histories. Then consumer's expected utility at time 0 is:

$$\sum_{t}^{T}\sum_{s^t}\underbrace{\beta^t u(c_t(s^t))}_{\substack{\text{PDV given}\\ s^t\text{ happens}}}\ \underbrace{\pi_t(s^t)}_{\substack{\text{Probability}\\ s^t\text{ happens}}} \tag{26}$$

i.e. average discounted utilities: $\beta^t u(c_t(s^t))$ across histories $s^t$ at $t$, using probabilities $\pi_t(s^t)$ as weights.

- Generalize the notion of the risk-free bond maturing at time $t$ to be: History-Date Contingent Claim on Consumption

- Claim on consumption:

$$\underbrace{c_t}_{\text{time } t}\ \underbrace{(s^t)}_{\text{history } s^t} \tag{27}$$

for every $t$ every $s^t$ (Huge number of assets)

# 4 Complete Markets (all assets exist)

## 4.1 Basic setup

- At time 0, the consumer can buy or sell $c_t(s^t)$ at price $q_t^0(s^t)$, which is the claim to 1 unit of consumption at $t$ given history $s^t$ and there is no return in other states.
  - We could have assets which pay in several states, but they would be *spanned* by this simple set of assets.
- Consumer faces $\{q_t^0(s^t)\}$ as a price taker.

## 4.2 Consumer Problem

- Consumers solve:

$$\max_{\{c_t(s^t)\}}\sum_{t=0}^{\infty}\sum_{s^t}\beta^t u(c_t(s^t))\pi_t(s^t) \tag{28}$$

$$\text{s.t.}\ \sum_{t=0}^{\infty}\underbrace{q_t^0(s^t)}_{\text{price}}\ \underbrace{c_t(s^t)}_{\text{quantity}} \le \underbrace{w_0}_{\text{time 0 wealth}} \tag{29}$$

- One budget constraint in a Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t(s^t))\pi_t(s^t) + \lambda \left( w_0 - \sum_{t=0}^{\infty} \sum_{s^t} q_t^0 c_t(s^t) \right) \tag{30}$$

where $\lambda$ is Lagrangian Multiplier, will be the "marginal utility of wealth".

- Take the FONC into $c_t(s^t)$

$$\beta^t u'(c_t(s^t))\pi_t(s^t) = \lambda \underbrace{q_t^0(s^t)}_{\substack{\text{given as}\\\text{price taker}}}, \text{ where } \lambda \text{ is to be determined from budget constraint} \tag{31}$$

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t q_t^0(s^t) c_t(s^t) = w_0 \tag{32}$$

## 4.3 Example with constant aggregate endowments

- Guess that $q_t^0(s^t) = \beta^t \pi_t(s^t)$
- Then by (31):

$$\beta^t u'(c_t(s^t))\pi_t(s^t) = \lambda \beta^t \pi_t(s^t) \tag{33}$$

$$\Rightarrow \lambda = u'(c_t(s^t)) \tag{34}$$

$$\Rightarrow q_0^0 = 1 \tag{35}$$

So if $c_t(s^t) = \bar{c}$, constant $\forall t$, $\forall s^t$

- And from (32):

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t(s^t) \bar{c} = w_0 \tag{36}$$

$$= \bar{c} \sum_{t=0}^{\infty} \beta^t \underbrace{\left[ \sum_{s^t} \pi_t(s^t) \right]}_{\substack{\text{must sum to 1}\\\text{since a probability}}} \tag{37}$$

$$= \bar{c} \sum_{t=0}^{\infty} \beta^t \tag{38}$$

$$\Rightarrow \boxed{\bar{c} = w_0(1-\beta)} \tag{39}$$

which means complete <u>consumption smoothing</u> across <u>time</u> and across <u>history</u>.

- What is price of 1 unit of consumption with certainty at time $t$?

$$\sum_{s^t} q_t^0(s^t) \equiv \bar{q}_t^{\,0} \tag{40}$$

$$= \sum_{s^t} \beta^t \pi_t(s^t) = \beta^t = \bar{q}_t^{\,0} \tag{41}$$

We can compare this to the risk-free interest rate with constant endowment.

# 5 Lucas 1978 Model

## 5.1 Basic setup

- Pure endowment representative agent economy:

$$c_t(s^t) = y_t(s^t), \text{ which is exogenous stochastic} \tag{42}$$

i.e. In equilibrium, the representative consumer will consume the entire endowment as a price taker with $q_t^0(s^t)$ prices. (42) is the <u>feasible</u> condition.

- Substitute (42) into FONC (32):

$$\beta^t u'(y_t(s^t))\pi_t(s^t) = \lambda q_t^0(s^t) \tag{43}$$

$$q_t^0(s^t) = \frac{1}{\lambda}\beta^t u'(y_t(s^t))\pi_t(s^t) \tag{44}$$

So if (44), then FONC hold.

- Budget constraint:

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)c_t(s^t) \leq \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)y_t(s^t) \tag{45}$$

this holds immediately for any $\lambda$ (i.e. price taker)

## 5.2 Competitive Equilibrium in Lucas 1978

- Given identical agents with exogenous stochastic endowment $y_t(s^t)$ and complete markets.

- A <u>feasible</u> allocation $\{c_t(s^t)\}$:

$$c_t(s^t) \leq y_t(s^t), \forall t, s^t \tag{46}$$

and a <u>price system</u> $\{q_t^0(s^t)\}$ is a competitive equilibrium if given $\{q_t^0(s^t)\}$, $\{c_t^0(s^t)\}$ solves the consumer's problem.

## 5.3 Example

- Two consumers ($i = 1, 2$) with identical preferences:

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t)) \pi_t(s^t) \tag{47}$$

  Note: no $i$ on $\beta^t$, $u$ and $\pi_t(s^t)$. Same probabilities for all consumers.

  - <u>State</u>: $S = \{0, 1\}$
  - <u>Endowments</u>:

$$y_t^1(s_t) = s_t \text{ and } y_t^2(s_t) = 1 - s_t \tag{48}$$

$$y_t^1(s_t) + y_t^2(s_t) = 1, \forall t, s^t \text{ (i.e. \underline{No aggregate risk})} \tag{49}$$

  - <u>Feasible allocation</u> $\{c_t^i(s^t)\}$:

$$c_t^1(s^t) + c_t^2(s^t) \leq y_t^1(s^t) + y_t^2(s^t), \forall t, s^t \tag{50}$$

  - <u>Price system</u> $\{q_t^0(s^t)\}$ is same for all $i$.

- A <u>competitive equilibrium</u> is a <u>price system</u> and <u>feasible allocation</u> such that given the price system, the allocations solve each households problem for each $i$:

$$\max_{\left\{c_t^i(s^t)\right\}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t)) \pi_t(s^t) \tag{51}$$

$$\text{s.t.} \sum_{t=0}^{\infty} \sum s^t q_t^0(s^t) \left(c_t^i(s^t) - y_t^i(s^t)\right) \tag{52}$$

- <u>How to solve</u>?

  - Difficult in general. Here, guess and verify:

$$q_t^0(s^t) = \beta^t \pi_t(s^t) \text{ (i.e. Same guess as a representative agent)} \tag{53}$$

– At guess, FONC for $i$:

$$\beta^t u'(c_t^i(s^t))\pi_t(s^t) = \lambda_i \beta^t \pi_t(s^t) \tag{54}$$
$$\Rightarrow u'(c_t^i(s^t)) = \lambda_i, \forall i = 1, 2 \tag{55}$$
$$\Rightarrow c_t^i(s^t) = c^i \text{ (Perfect smoothing. LM is } i \text{ independent)} \tag{56}$$

– Feasibility:

$$c^1 + c^2 = 1 \tag{57}$$

– From budget constraint:

$$c^i \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi_t(s^t) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t y_t^i(s^t)\pi_t(s^t) \equiv w_0^i \tag{58}$$
$$\Rightarrow c^i = (1-\beta) w_0^i \tag{59}$$

– Note:

$$w_0^1 + w_0^2 = \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t) \underbrace{\left[ y_t^1(s^t) + y_t^2(s^t) \right]}_{I} \tag{60}$$
$$= \sum_{t=0}^{\infty} \beta^t = \frac{1}{1-\beta} \tag{61}$$

This is as far as we can get without specifying a particular $\pi_t(s^t)$ process.

# 6 Complete vs Incomplete Market

## 6.1 Compete market

- Assume there exist assets for every possible history, i.e. $q_t(s^t)$, this is complete markets
- Consumer:

$$\max_{\{c_t(s^t)\}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t(s^t)) \cdot \pi_t(s^t) \tag{62}$$
$$\text{s.t.} \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) \left[ c_t(s^t) - y_t(s^t) \right] = 0 \tag{63}$$

where $y_t$ is endowment in history $s^t$

- FONC:

$$\beta^t u'(c_t(s^t))\pi_t(s^t) = \lambda q_t^0(s^t) \tag{64}$$

Divide for histories at $t, t+1$:

$$\frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)} = \beta \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \cdot \frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} \tag{65}$$

Note: $\frac{\pi_{t+1}(s^{t+1})}{\pi_t(s^t)} \equiv \pi_{t+1}(s^{t+1} \mid s^t)$, which is the <u>conditional probability</u> of $s^{t+1}$ <u>given</u> $s^t$.

- <u>Note</u>:

  - Conditional probabilities are very easy to calculate here if markov, since it will only depend on <u>last</u> state. i.e. $\pi_{t+1}(s^{t+1} \mid s^t) = \pi(s_{t+1} \mid s_t)$, with markov chain $[\pi_{ij}] = p$, there are just transition probabilities $p_{ij}$
  - For example: $S = \{A, B\}$, $P = \begin{bmatrix} \pi_{AA} & \pi_{AB} \\ \pi_{BA} & \pi_{BB} \end{bmatrix}$, then:

$$\pi_{t+1}\left(\left\{s^{t+1} = s_2, s^t = s_1, \cdots\right\} \mid \left\{s^t = s_1, \cdots\right\}\right) = \pi_{s_1 s_2} \tag{66}$$

  - Define:

$$\frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)} \equiv q_{t+1}^t\left(s^{t+1} \mid s^t\right) \tag{67}$$

  as the one-step ahead "pricing kernel". i.e. price at time $t$ of $t+1$ consumption in state $s^{t+1}$ given state $s^t$ happened. So write pricing equation as:

$$\boxed{q_{t+1}^t\left(s^{t+1} \mid s^t\right) \equiv \beta \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \pi_{t+1}\left(s^{t+1} \mid s^t\right)} \tag{68}$$

  - Price, in history $s^t$, of a unit of consumption at time $t+1$ <u>with certainty</u>?

- Buy assets for every possible state

$$\underbrace{\sum_{s^{t+1} \mid s^t} q_{t+1}^t\left(s^{t+1} \mid s^t\right)}_{\substack{\text{price at node } s^t \\ \text{of a risk-free} \\ \text{claim to consumption} \\ \text{at } t+1}} \equiv \underbrace{(R_t(s^t))^{-1}}_{\substack{\text{reciprocal of gross} \\ \text{one period} \\ \text{interest rate} \\ \text{of node } s^t}} \tag{69}$$

With (68):

$$\boxed{1 = \beta R_t(s^t) \sum_{s^{t+1}|s^t} \frac{u'\left(c_{t+1}(s^{t+1})\right)}{u'\left(c_t(s^t)\right)} \cdot \pi_t(s^{t+1} \mid s^t)} \tag{70}$$

A little looser notation:

$$1 = \beta R_t \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)}\right] \tag{71}$$

Compare to permanent income hypthothesis with exogenous $R$. Shows $\beta R = 1$ is the natural solution if the consumer is able to achieve perfect income smoothing.

- Calculating risk free interest rates from the data:
  - Assume representative consumer with aggregate $y_t(s^t)$
  - Determine probabilities for process you believe $y_t(s^t)$ takes, then:

    $$\frac{1}{R_t} = \beta \mathbb{E}_t \left[\frac{u'(y_{t+1})}{u'(y_t)}\right] \tag{72}$$

    using the process, for $y_{t+1}$ from $y_t$ for expectations.
  - Note that for risk neutral consumers:

    $$u(c) = c \cdot A \Rightarrow u'(c) = A \tag{73}$$
    $$\Rightarrow \boxed{R\beta = 1}, \text{ for any stochastic process} \tag{74}$$

- Note that the complete markets pricing kernel have all the above but not vice versa. The above only holds "on average".

## 6.2 Incomplete Markets

- Motivation:
  - What if not all of the markets exist for all $s^t$? Then cannot smooth completely at time 0
  - Extreme version, can only buy a 1-period risk free bond paying interest rate $R_t(s^t)$, holdings $A_t(s^{t-1})$

- For consumer, given assets $\underbrace{A_t}_{\text{Asset today}} \cdot \overbrace{(s^{t-1})}^{\text{previous history } s^{t-1}}$

$$\max_{\{c_t(s^t), A_{t+1}(s^t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} u(c_t(s^t))\pi_t(s^t) \tag{75}$$

$$\text{s.t.} \quad \underbrace{A_{t+1}(s^t)}_{\substack{\text{bond holdings}\\ \text{tomorrow}}} = \underbrace{R_t(s^t)}_{\substack{\text{interest}\\ \text{paid on}\\ \text{holding}\\ \text{risk-free}}} \left[ \underbrace{y_t(s^t) - c_t(s^t)}_{\text{saving}} + \underbrace{A_t(\overbrace{s^{t-1}}^{\text{set from } s^{t-1} \text{ history}})}_{\text{previous assets}} \right] \tag{76}$$

Note: instead of 1 budget constraint, we have $\lambda_t(s^t)$ possible multipliers.

- Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t(s^t))\pi_t(s^t) \tag{77}$$

$$+ \sum_{t=0}^{\infty} \sum_{s^t} \lambda_t(s^t) \left[ A_t(s^{t-1}) + y_t(s^t) - c_t(s^t) - R_t^{-1}(s^t) A_{t+1}(s^t) \right] \tag{78}$$

- FONC:

$$[C_t(s^t)] : \beta^t u'(c_t(s^t))\pi_t(s^t) = \lambda_t(s^t) \tag{79}$$

$$[A_{t+1}(s^t)] : -\lambda_t(s^t) R_t^{-1}(s^t) + \underbrace{\sum_{s^{t+1}|s^t} \lambda_{t+1}(s^{t+1})}_{\text{any might show up}} \tag{80}$$

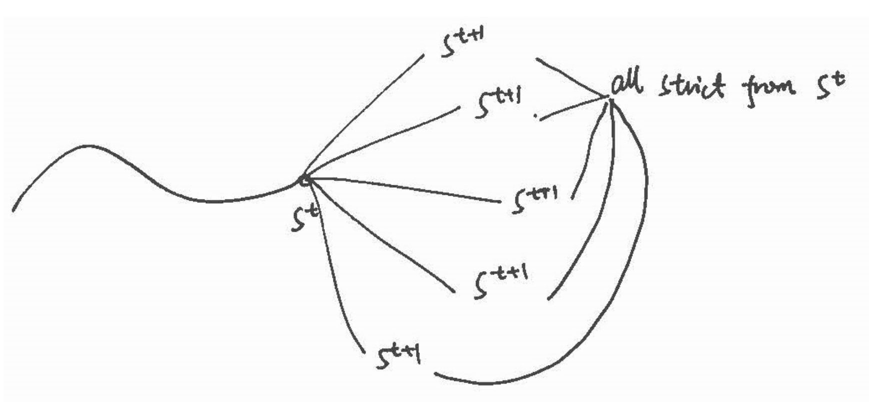


Figure 6: Example Path Conditioned on History $s^t$

- Substitute (80) to (79)

$$R_t^{-1}(s^t)\beta^t u'\left(c_t(s^t)\right)\pi_t(s^t) = \beta^{t+1}\sum_{s^{t+1}|s^t} u'\left(c_{t+1}(s^{t+1})\right)\pi_{t+1}(s^{t+1}) \tag{81}$$

$$\Rightarrow 1 = \beta R_t(s^t)\sum_{s^{t+1}|s^t}\frac{u'\left(c_{t+1}(s^{t+1})\right)}{u'\left(c_t(s^t)\right)}\cdot\pi_{t+1}(s^{t+1}\mid s^t) \tag{82}$$

$$\Rightarrow 1 = \beta\mathbb{E}_t\left[R_t\frac{u'(c_{t+1})}{u'(c_t)}\right],\ \text{same as the risk free calculated under complete market} \tag{83}$$

## 6.3 Punchlines:

- Under complete markets, intertemporal marginal rates of substitution:

$$\beta\frac{u'\left(c_{t+1}(s^{t+1})\right)}{u'\left((s^t)\right)}\pi_t(s^{t+1}\mid s^t) \tag{84}$$

  are equated for all consumers able to trade at relative prices $q_{t+1}^t\left(s^{t+1}\mid s^t\right)$

- Under incomplete markets with only a risk-free security with gross returns $R_t(s^t)$, only the average intertemporal rates of substitutes.

$$\beta\sum_{s^{t+1}|s^t}\frac{u'\left(c_{t+1}(s^{t+1})\right)}{u'\left(c_t(s^t)\right)}\pi_t(s^{t+1}\mid s_1^t) \tag{85}$$

  are equated across consumers

- Permanent Income Hypothesis in Incomplete markets
  - Stochastic $y_t(s^t)$ with incomplete markets:

$$\underbrace{\frac{1}{R_t}}_{\text{interest rate}} = \beta\mathbb{E}_t\left[\underbrace{\frac{u'\left(y_{t+1}(s^{t+1})\right)}{u'\left(y_t(s^t)\right)}}_{\text{can use aggregate endowment}}\right] \tag{86}$$

  - If markets were complete, consumers would eat a constant share of aggregate output.